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POSITION AUCTIONS:

DESIGN, INCENTIVES AND STRATEGIES

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TABLE OF CONTENTS

INTRODUCTION

The objective of this thesis is to provide a technical explanation about the mechanism through which position auctions are designed and implemented. A particular attention will be given to the section of online ad auctions. In fact, nowadays the most successful search engines, base their auctions formats on the consideration of some specific assumptions regarding agents' behavior and incentives.

Before analyzing the models provided by the famous economist Hal R. Varian, some concepts related to the world of second price auctions will be analyzed.

In principle, a classification of the different types of auctions existing in the marketplace is provided. Thereafter, the topic explored is the one of the Generalized Second Price (GSP) auctions. In fact, as it will be possible to see, position auctions indeed rely on some basic rules of this types of auctions. It is impossible to describe the Generalized Second Price auctions, without mentioning the Vickrey-Clarke-Groves (VCG) mechanism. In fact the latter, presents some peculiarities strictly linked to GSP auctions, even if it differs in some aspects. For example, truthful reporting, is a dominant equilibrium only in VCG auctions. Finally an overview about the Pay Per Click pricing will be given , in order to understand why eventually this was the winning kind of pricing chosen, and its deterrent action towards the so-called "skewed bidding".

After this first part, where the fundamental preliminary knowledge is provided to the reader, the work goes on by introducing the model on position auctions, elaborated in detail by Varian. The model carefully examines the process by which positions are assigned to agents by search engines. The economic aim pursued is an efficient allocation of positions in order to maximize welfare; analogously the advertisers tries to maximize its surplus. However there is a substantial divergence of incentives in advertising: the publisher owns space on its web page for an ad and it is willing to sell these ad impressions to the highest bidders. On the other hand, the advertiser does not care about ad impressions but it is interested about the number of visitors on its web site. This implies that the publisher wants to sell impression while the advertiser wants to buy clicks. The latter can be considered as a problem of exchange rate, which can be described by the predicted clickthrough rate. The clickthrough rate aligns the interests of buyers and sellers but creates other problems, e.g. if an advertiser pays only for clicks then it has no incentives to economize on impressions. Nevertheless the exchange rate has been standardized indeed to the clickthrough rate.

The description of the mechanism underneath position auctions, reserves a special part for its application to online ad auctions, where additional and interesting observations integrate the general framework.

A brief section is dedicated to the previous literature present before Varian's elaborates, from whom the author took some inspiration and insights.

Varian's studies provided a reference point for many subsequent works related to position auctions and it represents a solid and valid structure, seriously considered from the economic community. This implies that his model gave rise to a series of academic elaborates. The one that has been selected and analyzed in this thesis is a study made by Kuminov and Tennenholtz on competitive safety strategies in position auctions, in which the model of position auctions relied on is represented indeed by Varian's one, even if some slight changes were applied for research objectives.

Both Varian's elaborates and Kuminov and Tennenholtz' ones, focus on the notion of incentives with respect to agents' behavior.

The last section of the thesis reports some economic arguments related to: the market for internet advertising, the features of computer- mediated transactions and the logic beneath the so-called "Googleconomics".

In conclusion, the ultimate objective of this thesis is to explain and understand why search engines implement online auctions in order to allocate positions on their webpages. This is done also by exploring the effectiveness of alternative methods. Nowadays, especially in online markets, it is possible to notice the centrality of these themes for many companies and their respective revenue policies.

PRELIMINARY NOTIONS

AUCTIONS: CLASSIFICATION AND DESIGN

There exist two types of auction formats: open bid auctions and sealed bid auctions. Open bid auctions can either be ascending bid auctions or descending bid auctions. In the first case the price is raised until the point in which only one bidder remains; the latter wins and pays the final price. In the second case the price is lowered until somebody accepts it; he wins the objects and pays the current price. However this thesis will concentrate on sealed bid auctions. The latter are composed of two subgroups: first price and second price auctions. The main difference between the two is that in first price auctions the highest bidder wins and pays his bid, while in second price auctions the highest bidder wins but pays the second highest bid.

Auctions can differ also with respect to the valuation of the bidders. In private value auctions each bidder knows only his value while in common value auctions the value of the object is the same for everyone but bidders have different private information about that value.

Online advertisements via auction mechanisms are one of the main sources of income for many internet companies. When users make searches on Google they give start automatically to a position auction, which takes place among many different advertisers. This process allows the search engine to earn significant revenues per auctions so it is crucial to designing well these auctions (Ashlagi et al, 2010).

Good auction design is not “one size fits all”. Risk aversion affects the revenue equivalence result. Revenue equivalence is a concept that derives from the assumption that each of a given number of risk-neutral potential buyers of an object has a privately-known signal drawn independently from a common strictly-increasing distribution. Given these circumstances, then any auction mechanism where:

I. the object is always awarded to the buyer with the highest signal, and
II. any bidder characterized by the lowest-feasible signal expects zero surplus

yields the same expected revenue. The implied result is that each bidder makes the same expected payment as a function of his signal. However in second-price auctions risk aversion has no effect on a bidder's optimal strategy. Ascending auctions lead to higher expected prices than sealed-bid second price auctions, which in turn lead to higher expected prices than first- price auctions. The intuition is that the winning bidder's surplus is due to his private information. So the lower is the winners information, the higher the expected price (Klemperer, 2002).

ISSUES ON THE GENERALIZED SECOND-PRICE AUCTION

The Generalized Second-Price Auction (GSP) is tailored to the unique environment of the market for online ads. This novel mechanism has led to a spectacular commercial success (ex: Google's total revenue in 2005 was $6.14 billion).

GSP auctions work in the following way (Edelman et al, 2007): when a user enters a search term into a search engine he gets back a page with results containing both the links most relevant for the query and the sponsored links, namely paid advertisements. The user can clearly distinguish the two; advertisers thus target their ads based on search keywords. When a user clicks on the sponsored link, he is sent to the advertiser's webpage. The latter then pays the search engine for sending the user to its web page, the so called PPC pricing. Since the number of ads the search engine can display to users is limited , different positions on the search page have different desirability for advertisers (Edelman et al ,2007). In fact an ad shown on the top is more likely to be clicked than an ad shown at the bottom. This implies that auctions are a natural choice for search engines for the need of having a mechanism for allocating positions to advertisers.

In the simplest GSP auction, for a specific keyword, advertisers submit bids stating their maximum willingness to pay for a click. If a user clicks on ad in position i , that advertiser is charged an amount equal to the next highest bid, i.e. the bid of the advertiser in position i+1. If a search engine offered only one advertisement per result page, the mechanism would be equivalent to the Vickrey-Clarke-Groves(VCG) mechanism. However the GSP auction lacks some properties of the VCG one. For example GSP does not have an equilibrium in dominant strategies, and truth telling is not an equilibrium of GSP. However the difference between the GSP and the VCG mechanisms will be clearer in the next section.

VICKREY-CLARKE-GROVES AUCTIONS

The design of online auctions includes the consideration of Vickrey-Clarke-Groves (VCG) auction mechanism. One of the major goals is to design the right incentives such that the efficient outcomes will be chosen and implement the efficient outcome in dominant strategies. Efficiency can be maximized in two ways:

- Choose efficient outcomes given the bids
- Each player pays his "social cost"

DESCRIPTION OF THE GENERAL VCG DESIGN

Even if Vickrey's original research included both auctions of a single item and auctions of multiple identical items, the mechanism is often referred to as the second-price sealed bid auction, i.e. Vickrey auction is the one for single items. Bidders simultaneously submit sealed bid for the item. The highest bidder wins the item, but the winner pays the amount of the second highest bid. These rules imply that a winner bidder can never affect the price it pays , so there is no incentive for any bidder to misrepresent his value. This outcomes provides important results from the point of view of information asymmetries in the market. From bidder n's perspective it can be proven that the amount he bids determines only whether he wins and only by bidding his true value he can be sure to win exactly when he is willing to pay the price.

In Vickrey' original treatment of multiple units of homogeneous good (Ausubel and Milgrom):

1) Each bidder is assumed to have monotonic non- increasing marginal values for the good.
2) The bidders simultaneously submit sealed bid comprising demand curves.
3) The seller combines the individual demand curves to determine an aggregate demand curve and a clearing price for S units
4) Each bidder wins the quantity he demanded at the clearing price
5) However rather than paying the prices he bid or the clearing price for his units, a winning bidder pays the opportunity cost for the units won.

The mechanism can be used either as a mechanism to sell (standard auction) or as a mechanism to buy (reverse auction). In the first case the buyers generally pay a discount compared to the clearing price; in the second case the sellers generally receive a premium compared to the clearing price.

Since Vickrey original contribution his auction design has been melded with the Clarke-Groves design for public goods. The resulting auction design works both for homogeneous and heterogeneous goods and does not require that bidders have non-increasing marginal values. Still this mechanism (Ausubel and Milgrom):

1) Assigns goods efficiently
2) Charges bidders the opportunity cost of the items they win

The main difference is that the amounts paid cannot generally be expressed as the sum of bids for individual items.

Formally the VCG mechanism gives rise to the following result:

Theorem 1: Truthful reporting is a dominant strategy for each bidder in the VCG mechanism. Moreover when each bidder reports truthfully, the outcome of the mechanism is one that maximizes total value.

Ausubel and Milgrom show that agent n's payoff from truthful reporting, is always optimal and that no other reporting is optimal.

VIRTUES AND VICES OF THE VCG MECHANISM

One key element of strength is the dominant strategy property. This feature reduces the costs of the auction by making it easier for bidders to determine their optimal bidding strategies and by eliminating bidders' incentives to spend resources learning about competitors' values or strategies which is a pure waste from a social perspective since it is not needed to find the efficient allocation. This property has also the apparent advantage of adding reliability to the efficient prediction, because it means that the conclusion is not sensitive to assumptions about what bidders may know about each others' values and strategies (Ausubel, Milgrom). This feature is also reinforced by the following theorem. Before defining it is useful to explain the concept of *"smooth path connectivity"*, since it is an extra assumption made in the theorem. Namely, given any two functions in V, there exists a smoothly parameterized family of functions, {v(x, t)}, entirely lying in V and connecting the two functions.

Theorem 2: if the set of possible value functions, V, is *smoothly path connected* and contains the zero function, then there exists a unique direct revelation mechanism for which truthful reporting is a dominant strategy. This implies that the outcomes are always efficient, and there are no payments by or to losing bidders in the VCG mechanism.

The latter has also desirable properties from the point of view of the scope of its application because theorems 1 and 2 do not impose restrictions on the bidders' possible rankings of different outcomes. Lastly the average revenues are not less than that from any other efficient mechanism, even when the notion of implementation is expanded to include the Bayesian equilibrium as confirmed by the revenue equivalence theorem (Ausubel, Milgrom). Nevertheless the VCG mechanism does not exclude drawbacks. In fact it allows the seller revenues to be low or zero. In addition it presents vulnerability to collusion by a coalition of losing bidders and to the use of multiple bidding identities by a single bidder.

However by analyzing some conditions it is possible to spot the possibility of these weaknesses. In fact in economic environment where every bidder has substitutes preferences, the abovementioned weaknesses will never occur. When complexity is present or there is a single bidder whose preferences violate the substitutes condition, all the weaknesses are present. However VCG is implemented rarely also because often it excludes discussing auction revenues, that for private resellers are of primary importance.

WHY PAY-PER-CLICK PRICING

Online advertising is primarily priced using Pay Per Click (PPC): advertisers pay only when a consumer clicks on the advertisement. Slots for advertisements are auctioned and per-click bids are weighted by the probability of a click , the clickthrough rate (CTR) and other factors (N. Agarwal et al, 2009). The PPC method allows the advertising platform (Google) to bundle otherwise heterogeneous items for example impressions on different positions on a search page into more homogeneous units, simplifying the advertiser's bidding problem. However PPC presents drawbacks as well (N. Agarwal et al, 2009):

1) All clicks are not created equal; e.g. clicks on a Paris hotel website that is displayed for a search for Paris Hilton may result in lower profit conditional on the click.
2) For infrequently searched phrases it is difficult for the advertiser to accurately estimate the rate at which clicks convert into sales, thus increasing the risk and monitoring costs to advertisers and diminishing their incentives to advertise broadly.
3) A problem of "click fraud": when publishers receive a share of advertising revenue, advertisers place a single bid applying to many publishers and revenue is derived through clicks. So a small publisher could be tempted to click on ads on its pages anonymously in order to inflate its payments.

One possible solution is the Pay Per Action (PPA) advertising system. Here advertisers pay only when consumers complete predefined actions on their web site. The appeal is that they pay only when the valuable events occur (N. Agarwal et al). At a first glace it could seem that PPA and PPC are just two variants of the same system: a click versus an action. However there are a number of problem with PPA that do not arise with PPC systems:

- To maximize the value of the system to advertisers a PPA system would allow them to specify more than one action, since most advertisers sell products of varying value.
- The probability that an action is recorded can be controlled by an advertiser in more complex ways.

These two characteristics create incentives for advertisers to engage in strategic behavior that undermines the efficiency of risk reallocation. More in detail advertisers have the incentive to engage in the so called “skewed bidding”. This means that even if there are many ways to achieve the same aggregate bid there is the tendency of bidding high on actions that are overestimated and this minimizes the expected payment for a given aggregate bid. Advertisers are also prompted to combine skewed bidding with the strategic manipulation of the probabilities of different actions through destroyed links and artificial stock outs. The problem about the fact that the advertiser has an incentive to “overbid” on actions underestimated by the platform is augmented by the fact that advertisers have control over the reporting of actions.

Skewing leads to allocation inefficiencies with respect to sponsored links: bidders whose action probabilities have been misestimated most severely by the ad platform will be favored because those bidders perceive the largest gap between their bid as calculated as calculated by the ad platform and their payment, and thus can afford to place bids perceived to be advantageous by the ad platform. Another consequence is that firms that are willing to actively game the system can outbid those that are not. Further the potential gain from risk allocation is diminished as advertisers’ optimal strategies do not accurately report actions.

So it is difficult to solve the inefficiencies without losing some benefits of PPA pricing. We expect to see advertising platforms to restrict PPA systems to a single action or to place strong restrictions on changes in bids.

EVOLUTION OF MARKET INSTITUTIONS AND ASSESSING THE MARKET DEVELOPMENT

Sponsored search auctions represent a case study of whether, in which way and how quickly markets choose to address their structural failures. In fact, recently many mechanisms have been designed from the beginning, thus reversing and replacing old allocation mechanisms with much superior ones, as happened for example for radio spectrum auctions (Edelman et al, 2007). However the internet advertising market evolved much faster than any other market, probably because of higher competitive pressures, lower barriers to entry, improved technology and so on. The authors provide a synthetic chronological review of the development of sponsored search mechanisms, which included four phases:

Early Internet Advertising. This type of advertisements started to appear in 1994 and were sold on a per-impression basis. This implied that advertisers paid flat fees to show their ads a fixed number of times, which in general they went around 1000 showings, i.e. “impressions”. In this period contracts were negotiated and concluded on a case-by-case basis which implied that:

- Minimum contracts for advertising purchases were large
- Entry was slow

Generalized First Price Auctions. In 1997 a completely new way of selling internet advertising was introduced by Overture, a firm then become GoTo and now part of Yahoo!. The initial Overture auction design implied that each advertiser submitted a bid, which reported each advertiser’s willingness to pay on a per-click basis. In this way advertisers were allowed to target their ads: instead of paying for an ad shown to every kind of consumer visiting a website, advertisers are now provided with the possibility of choosing which keyword are relevant to their products and how much they valued each of those keywords. Moreover advertising was no longer sold per 1000 impressions, rather one click at a time. Each time a consumer clicked on a sponsored link, an advertiser’s account was directly billed the amount of his most recent bid. The highest bid was made most prominent by arranging links in descending order of bids. The success of Overture's paid search platform occurred thanks to the transparency of the mechanism, the ease of use and the low entry costs. However the perfection of the mechanism was far to happen because of the fact that bids could be changed very often and this gave instability to the system.

Generalized Second-Price Auctions. The Generalized First Price Auction created volatile prices and allocative inefficiencies by encouraging inefficient investments in the gaming system. In fact the mechanism implied that the advertiser who could react to competitors' moves fastest gained a big advantage. Google managed to address these problems by recognizing that an advertiser in position i would be never willing to pay more than one bid increment above the bid of the advertiser in bid (i+1). This principle was indeed adopted in its auction mechanism. In the simplest GSP auction, in fact, an advertiser in position i pays:

1) A price per click equal to the bid of an advertiser in position (i+1)
2) A small increment which typically corresponds to $0.01

The above structure made the market more user friendly and less vulnerable to the gaming. In fact these desirable properties made companies as Yahoo!/Overture to switch to GSP.

Generalized Second-Price and VCG Auctions. The two mechanisms present similarities in the sense that both set each agent's payments uniquely on the allocation and bids of other players and not based on the agent's own bid. Nevertheless GSP is different from VCG. In fact GSP doesn't have an equilibrium in dominant strategies and truth-telling is not an equilibrium in GSP. The two mechanisms would be identical if and only if there is only one slot. If there are more than one slot they would be different.

- GSP charges the advertiser in position i the **bid** of the advertiser in position i+1, while
- VCG charges the advertiser in position i the **externality** that he imposes on others, by removing one slot away from them. The advertiser in position i totally pays an

amount equal to the difference between the aggregate value of clicks that all advertisers would have received in the case i was not present and in the case he is present.

It is useful to notice that (Edelman et al, 2007):

1) An advertiser in position j<i, receives an externality equal to zero since he is not affected by i
2) An advertiser in position j>i, would have been awarded position (j-1) if agent i was absent. Here the externality corresponds to his value per click times the difference in the number of clicks in position j and (j-1)

The above described chronology shows three main steps in the development of the sponsored search advertising market. In the first one ads were sold manually, in large batches and on a cost-per-impression basis. In the second one, Overture began to streamline advertisement sales, having the drawback of instability. Finally Google implemented the GSP auction, later adopted by Yahoo!. It is peculiar to notice that Google and Yahoo! preferred for many years GSP to VCG. In fact the latter is hard to explain to standard buyers; switching to it may imply enormous transaction costs since VCG revenues are lower with respect to GSP ones for the same bids (Edelman et al, 2007). Switching costs can be high both for advertisers and for search engines because of the fact that the revenue outcomes of switching to VCG in not certain and simply testing a new system can be really expensive.

POSITION AUCTIONS

INTRODUCTION TO VARIAN'S MODELS

The analysis that will be here presented, consists in a theoretical analysis of Varian's studies on position auctions. His work provides an explanation of how search engines base the design of their auction formats. Game theory fits in a perfect way the need of understanding bidders behavior and incentives thanks to the right quantitative tools available in the discipline. For example, one key concept fully exploited relates to the notions of Nash Equilibrium and Symmetric Nash Equilibrium which have a strong link with incentives. Furthermore the author relies also on some achievements in the discipline as the one of Vickrey-Clarke-Groves mechanism and the Generalized-Second-Price auction. During his study on position auctions the author dedicates a special section for online ad auctions. However this specific topic is deepened and better reconsidered in a subsequent work made by Varian in 2009, namely “Online Ad Auctions”.

A THEORETIC APPROACH

Varian’s studies on position auctions were aimed at maximizing the allocation of slots. The basic design of ad auctions is simple. This is structured in a way that each advertiser has to choose a set of keywords which are related to the product it wishes to sell (Varian, 2007). Each of them makes a bid for each keyword which represents his willingness to pay if a user clicks on its ad . When a user search query matches a keyword, a set of ads is displayed. The latter are ranked by bids and the ad with the highest bid receives the best position; in fact it is the one most likely to be clicked by end customers. If the user clicks on the ad the advertiser is charged an amount that depends on the bid of the advertiser below in the ranking.

THE MODEL

Assume that the following conditions hold (Varian, 2007):

- There are agents a=1,…,A
- There are slots s=1,…,S
- The slots are numbered so that $x_1 > x_2 > \ldots > x_s$
- $x_s = 0$ for all s>S
- The number of agents is greater than the number of slots

The model provides the following two definitions:

- v_a is the value per click of the agent assigned to slot s and

- x_s is the correspondent clickthrough rate for slot s, i.e. a measure of the number of users that click on a link . In fact as previously mentioned higher positions receive more clicks.

In this way, agent a's valuation for slot s is given by :

$\mathbf{u_{as}=v_a x_s}$

According to the above assumptions, Varian considers the problem of assigning agents to slots. In this case agents are represented by advertisers while slots symbolize positions on a web page. This logic brings to conceive the equation $u_{as}=v_a x_s$ as the expected profit to advertiser a from appearing in slot s.

In position auctions slots are assigned and sold via an auctions. This implies that:

1) Each agent bids an amount b_a
2) The slot with the best clickthrough rate is awarded to the slot with the highest bid and assigned to the agent with the highest bid; the second-best slot is assigned to the agent with the second highest bid, and so on.

These concepts can be reshaped in game theoretic terms, by distinguishing them into two categories: definitions and assumptions. The assumption relates to the fact that the context is the one of second price auctions, which implies that:

- $p_s= b_{s+1}$, i.e. the price agent s faces is equal to the bid of the agent immediately below him

Given that, it is possible to provide the following two definitions:

- b_a is the amount that each agent bids
- v_s is the value per click of the agent assigned to slot s. The private value in the underlying model represents the utility that each agent derives from a single unit of CTR (clickthrough rate, i.e. the rate at which sponsored links are clicked by users).

The implication is that the expected profit from acquiring slot s for agent a is :

$(v_a-p_s)x_s= (v_a-b_{s+1})x_s$

This nice mathematical structure of position auctions is strongly related to the two-sided matching models, i.e. bilateral exchanges mechanisms between two disjoint parties.

NASH EQUILIBRIUM OF POSITION AUCTIONS

Assume there are S=4 available slots. It is known that (Varian, 2007):

I. **$x_s>x_{s+1}$**. This means that slots are numbered in decreasing order of clickthrough rate. The CTR can be interpreted as a publicly known property of a slot which does not depend on the player who is using it (Kuminov and Tennenholtz, 2007).
II. **$b_s>b_{s+1}$**. Players bids are conceived as the maximal price per unit of CTR they are ready to pay to the CTR provider. The just stated inequality implies that agents are ordered in decreasing number of bids (Kuminov and Tennenholtz, 2007).

The auction structure thus has as consequence that if agent number 3 wants to move up by one position, he would be forced to bid an amount at least as equal as b_2. However if agent number 2 is willing to move down by one position the amount he has to bid would be just at least equal to $b_4=p_3$,i.e. the bid of agent in position 4. The reasoning just described leads to achieve two conclusions (Varian, 2007):

1) To move to a higher slot it is necessary to beat the **bid** of the agent who currently occupies that slot.
2) To move to a lower slot, it is only necessary to beat the **price** of that agent who currently occupies the slot below.

This game can be modeled as a simultaneous move game with complete information since each agent a simultaneously chooses a bid b_a. Thereafter the bids are ordered and the price that each agent has to pay is determined, as implied by second price auctions, by the bid of the agent below him in the ranking.

Table 1 (Hal R. Varian, Position auctions, 2007)

POSITION	VALUE	BID	PRICE	CTR
1	v_1	b_1	$p_1=b_2$	x_1
2	v_2	b_2	$p_2=b_3$	x_2
3	v_3	b_3	$p_3=b_4$	x_3
4	v_4	b_4	$p_4=b_5$	x_4
5	v_5	b_5	0	0

A Nash equilibrium implies that in equilibrium each agent prefers his current slot to any other slot so that (Varian, 2007):

1) $(v_s-p_s)x_s \geq (v_s-p_t)x_t$ *for t>s* (1)
2) $(v_s-p_s)x_s \geq (v_s-p_t)x_{t-1}$ *for t<s* (2)

where $\mathbf{p_t=b_{t+1}}$

The inference is that a NE is a set of bids $b_1>b_2\ldots>b_n$ such that no agent strictly benefits by decreasing his bid and getting a lesser slot and no agent strictly benefits by increasing his bid and getting a better slot (Kuminov and Tennenholtz,2007).

It is important to observe that the inequalities are linear in prices so that given (x_s) and (v_s), it is possible to solve the maximum and the minimum equilibrium revenue attainable by the auction (Varian, 2007). Another observation relates to the fact that generally there is a range of bids and prices that satisfy the inequalities so that a slight change in the bid will not affect the agent's position or payment.

A symmetric Nash equilibrium (SNE) is a subset of Nash equilibria that can be defined as (Varian, 2007):

- $(v_s-p_s)x_s \geq (v_s-p_t)x_t$ *for all t and s*

An equivalent definition is:

- $v_s(x_s-x_t) \geq p_sx_s- p_tx_t$ *for all t and s*

It can be noticed that the inequalities characterizing an SNE are the same characterizing an NE for t>s. Since the above definitions assume fixed valuations, the game is essentially a complete information game.

At this point, the model goes on by temporarily suspending the auction argument, analyzed until now. Suppose that (Varian, 2007) :

- prices are given exogenously
- agents can purchase slots at these prices

The fact that in SNE each agent prefers to purchase the slot as it is rather than some other slot, makes possible to include the notion of competitive equilibrium in this description . The SNE prices thus provide supporting prices for the classic assignment problem (Varian, 2007). Despite these supporting prices can only be calculated by using a linear program, in this special case the prices can be computed using a simple recursive formula. Some arguments can be shown in this respect (Varian, 2007). Namely, the author went through the consideration of the following facts:

Fact 1: Non-negative surplus

In a SNE $v_s \geq p_s$

Proof: by using the inequalities that define a SNE,

$(v_s-p_s)x_s \geq (v_{s+1}-p_{s+1})x_{s+1}=0$

Since $x_{s+1}=0$

Fact 2: Monotone values

In a SNE $v_{s-1} \geq v_s$ for all s.

Proof: SNE definition leads to the following conditions:

1) $v_t(x_t-x_s) \geq p_t x_t - p_s x_s$ (3)
2) $v_s(x_s-x_t) \geq p_s x_s - p_t x_t$ (4)

the addition of these two inequalities provides as result:

$(v_t-v_s)(x_t-x_s)\geq 0$

This shows that v_t and x_t must be ordered using the same methodology. Moreover because of the fact that agents with higher values are assigned to better slots a SNE is an efficient allocation (Varian, 2007).

Fact 3: Monotone prices

In a SNE $p_{s-1}x_{s-1}>p_sx_s$ and $p_{s-1}>p_s$ for all s. If $v_s>p_s$ then $p_{s-1}>p_s$

Proof: the definition of SNE previously analyzes was:

$(v_s-p_s)x_s\geq(v_s-p_{s-1})x_{s-1}$

However the latter can be rearranged to get:

$p_{s-1}x_{s-1}\geq p_sx_s+v_s(x_{s-1}-x_s)>p_sx_s$

However this explanation just proves the first part. The second part can be proved by writing:

$p_{s-1}x_{s-1}\geq p_sx_s+v_s(x_{s-1}-x_s)\geq p_sx_s+p_s(x_{s-1}-x_s)=p_sx_{s-1}$

By erasing x_{s-1} it can be seen that $p_{s-1}\geq p_s$. In addition, if $v_s>p_s$ then the second inequality is strict, and this proves the last part of the fact.

Fact 4: NE Ɔ SNE

If a set of prices is SNE it is a NE.

Proof: the fact that $p_{t-1}\geq p_t$ implies that:

$(v_s-p_s)x_s\geq(v_s-p_t)x_t\geq (v_s-p_{t-1})x_t$ *for all s and t.*

The set of symmetric NE is attractive mainly for the following property: in order to verify if the entire set of inequalities is satisfied it is only necessary to verify the inequalities for one step up or down.

Fact 5: One step solution

If a set of bids satisfies the symmetric Nash equilibria inequalities for s+1 and s-1, then it satisfies the inequalities for all s.

Proof: the author provides this proof by implementing an example. Assume that the SNE relation holds for :

1) slot 1 and 2
2) slot 2 and 3

the aim is now to show that it holds also for slot 1 and 3. By exploiting the fact that $v_1 \geq v_2$,

$v_1(x_1-x_2) \geq p_1x_1-p_2x_2 \longrightarrow v_1(x_1-x_2) \geq p_1x_1-p_2x_2$

$v_2(x_2-x_3) \geq p_2x_2-p_3x_3 \longrightarrow v_1(x_2-x_3) \geq p_2x_2-p_3x_3$

By adding the left and the right columns, the result obtained is:

$v_1(x_1-x_3) \geq p_1x_1-p_3x_3$ as was to be shown. A similar argument can be proven in the other direction.

USEFUL INSIGHTS

The facts just described can be implemented to obtain an explicit characterization of equilibrium prices and equilibrium bids.

Because of the fact that the agent in position s is not willing to move down one slot (Varian, 2007):

1) $(v_s-p_s)x_s \geq (v_s-p_{s+1})x_{s+1}$

Analogously since agent in position s+1 does not want to move up one slot:

2) $(v_{s+1}-p_{s+1})x_{s+1} \geq (v_{s+1}-p_s)x_s$

Putting the previous two inequalities together we find:

$$v_s(x_s-x_{s+1})+p_{s+1}x_{s+1} \geq p_sx_s \geq v_{s+1}(x_s-x_{s+1})+p_{s+1}x_{s+1} \quad (5)$$

These inequalities can also be written in the following way, recalling that $p_s=b_{s+1}$:

$v_{s-1}(x_{s-1}-x_s)+b_{s+1}x_s \geq b_s x_{s-1} \geq v_s(x_{s-1}-x_s)+b_{s+1}x_s$ (6)

Let's assume that the following condition holds (Varian, 2007):

$\alpha_s=x_s/x_{s-1}<1$

then the inequalities can also be written as:

$v_{s-1}(1-\alpha_s)+b_{s+1}\alpha_s \geq b_s \geq v_s(1-\alpha_s) + b_{s+1}\alpha_s$ (7)

Therefore, the equivalent conditions (5)-(7) show that, in equilibrium, each agent's bid is bounded above and below respectively by a convex combination of the bid of the agent immediately below him and a value which can either be his own or the one of the agent immediately above him.

This is an attainment that allows to state that the pure strategy Nash equilibria can be merely found by recursively selecting a sequence of bids that satisfy these inequalities (Varian, 2007). The upper and lower bounds in inequalities (6) will be implemented in order to analyze the boundary cases. The recursions then become (Varian, 2007):

$b_s^U x_{s-1}=v_{s-1}(x_{s-1}-x_s)+b_{s+1}x_s$ (8)

$b_s^L x_{s-1}=v_s(x_{s-1}-x_s)+b_{s+1}x_s$ (9)

These recursions provide as solution:

$b_s^U x_{s-1}=\sum_{t \geq s} v_{t-1}(x_{t-1}-x_t)$ (10)

$b_s^L x_{s-1}=\sum_{t \geq s} v_t(x_{t-1}-x_t)$ (11)

The values implemented for the recursions derive from the fact that there are only S positions, so that $x_s=0$ when s>S. However if s=S+1, the outcome would be:

$b_{s+1}^L x_s=v_{s+1}(x_s-x_{s+1})$

$=v_{s+1}x_s$

It can thus be concluded that for the first excluded bidder, it is optimal to bid his own value. This argument recalls the Vickrey auction mechanism, that later will be explained. So if you are excluded, bidding lower than your value does not make sense; nevertheless if you do happen to be shown you will be able to get a payoff (Varian, 2007).

BOUNDS: THE UNDERLYING LOGIC

Sometimes agents find bidding at one end of the upper or lower bounds particularly attractive to the bidder, even if any bid comprised in the range described by equations (5) and (7) is a SNE and thus a NE bid (Varian, 2007). Suppose that:

1) I am in a position s and I am making a profit of: $(v_s-b_{s+1})x_s$.
2) In NE my bid is optimal given my beliefs with respect to the bids of other agents
3) I can change my bid in range specified by equation (6)
4) I can't change my payments or positions

The question that arises at this point relates to the utilities' maximizing behavior of agents. In fact any agent would find optimal to set the highest bid possible so that if it exceeds the agent above him and he moves up by one slot, he is sure to make at least as much profit as he is making now (Varian, 2007). In this respect the worst case occurs when I beat the advertiser above me only for a very small amount and I am anyway obliged to pay my bid b_s minus a tiny amount. The issue can be analyzed from two different perspectives. The first result derives from a reasoning that starts from the analysis of the break even situation and ends up with the computation of the lower bound recursion. The break even case satisfies the following equation (Varian, 2007):

$(v_s-b_s^*)x_{s-1} = (v_s-b_{s+1})x_s$

The latter represents a comparison between the worst case possible (where profit moves up) to current profit. Solving for b_s^*, the result is (Varian,2007):

$\mathbf{b_s^*x_{s-1}=v_s(x_{s-1}-x_s) + b_{s+1}x_s}$

This equation coincides to the lower-bound recursion described in equation (9).

On the other hand it is possible to think defensively in order to get the upper bound recursion. If an agent sets a bid too high, he will squeeze the profit of the player ahead of him at the point that he could prefer to move town to his position. The highest breakeven bid that would not induce the agent above to move down is (Varian, 2007):

His profit now= how much he would make in my position

$(v_{s-1}-b_s^*)x_{s-1}= (v_{s-1}-b_{s+1})x_s$

By solving the equation we obtain:

$\mathbf{b_s^* x_{s-1}=v_{s-1}(x_{s-1}-x_s) + b_{s+1}x_s}$, as previously found in equation (8).

So one might argue that setting the bid so that I am able to make a profit if a move up in the ranking is a reasonable strategy even if any bid in the range (5) is a reasonable strategy.

<u>REVENUES: NE, SNE and ADDITIONAL CONCERNS</u>

Varian's study continues by focusing on the concept of revenues, one of the main pillars in auction mechanisms. The topic of revenues is reconsidered also in 2009 starting from the same logic applied in 2007. However, rather than focusing on the NE and SNE aspects of the issue, in his recent work, the author's goal is more oriented at explaining the logic behind the upper and lower bounds of revenues and providing a way to compute auction revenues. As economic theory suggests, advertisers are interested in surplus maximization; in this case surplus can be intended as the value of clicks they receive minus the cost of those clicks. Here the conclusions achieved in the two elaborates will find a pattern of integration. In describing the role of revenue models , which are a pillar of companies' business strategy, the author recalls that in general search engines implement revenue models based on the following rules of the Generalized Second Price Auctions:

- Each advertiser a chooses a bid b_a
- The advertisers are ordered by bid times predicted clickthrough rate of advertiser a in slot s, i.e. $b_a e_a$.

- The price that advertiser a pays for a click is the minimum bid necessary to retain its position
- If there are fewer bidders than slots, the last bidder pays a reserve price r

In principle, Varian analyzes the topic of revenues from the following considerations. He argues that by summing equations (10) -(11) over s=1,…, S upper and lower bounds on total revenue in an SNE can be computed. For instance if the number of slots is S=3, the lower and upper bounds can be obtained by:

1) $R^L=v_2(x_1-x_2)+2v_3(x_2-x_3)+3v_4x_3$
2) $R^U=v_1(x_1-x_2)+2v_2(x_2-x_3)+3v_3x_3$

The process that allowed to arrive to these results will be later illustrated in the next paragraph.

Underlying assumption (Varian, 2009) : all advertisers are characterized by the same quality, so:

$\mathbf{e_a \equiv 1}$ for all advertisers.

In equilibrium the advertiser placed in slot s+1 doesn't want to move up to slot s, so that:

$(v_{s+1}-p_{s+1}) \times (x_{s+1}) \geq (v_{s+1} - p_s)\, x_s$.

By rearranging the equation it is possible to have an important result(Varian,2009):

(1) $\mathbf{p_s x_s} \geq p_{s+1}x_{s+1} + v_{s+1}\,(x_s - x_{s+1})$

The latter outcome is meaningful because it shows that the cost of slot s must be at least as large as the cost of slot s+1 plus the value of the incremental clicks attributable to the higher position, i.e. plus a premium. The relevant value is thus the one of s+1. In fact that is the bid that the advertiser in slot s must beat (Varian, 2009).

The price of the last ad shown on the page is either the reserve price or the bid of the first omitted ad. This price can be denoted by p_m . Now it is possible to solve the recursion in inequality (1) repeatedly to get the following inequalities (Varian, 2009):

$p_1x_1 \geq v_2(x_1\text{-}x_2) + v_3(x_2\text{-}x_3) + v_4(x_3\text{-}x_4) + \ldots + p_m x_m$

$p_2x_2 \geq \quad\quad +v_3(x_2\text{-}x_3) + v_4(x_3\text{-}x_4) + \ldots + p_m x_m$

$p_3x_3 \geq \quad\quad\quad\quad + v_4(x_3\text{-}x_4) + \ldots + p_m x_m$

By summing up the terms it is possible to get a lower bound on total revenue:

$\sum_s p_s x_s \geq v_2(x_1\text{-}x_2) + 2v_3(x_2\text{-}x_3) + \ldots + (m-1)p_m x_m$

By analogy, in equilibrium each advertiser prefers its slot to the slot above it. In this way it is possible to obtain an upper bound on total revenue as well:

$\sum_s p_s x_s \leq v_1(x_1\text{-}x_2) + 2v_2(x_2\text{-}x_3) + \ldots + (m-1)p_m x_m$

So now we have proven how the author arrived to the definition of upper and lower bounds provided previously. As it is possible to verify it was in fact deriving indeed from this reasoning. Namely (Varian, 2007):

- $R^L = v_2(x_1\text{-}x_2) + 2v_3(x_2\text{-}x_3) + 3v_4x_3$
- $R^U = v_1(x_1\text{-}x_2) + 2v_2(x_2\text{-}x_3) + 3v_3x_3$

Were equations defined considering the special circumstance were S=3.

Advertisers values can be thought as drawn from a distribution. In fact if we assume that S slots are available, the ads that are shown are the ones with the S largest values among the available ads in the set. In the contingency where there is a large number of advertisers competing for a small number of slots, the upper and the lower bounds will be closed together.

In conclusion this simple calculation allows to determine the auction revenue.

It can be shown that these bounds have a relationship with the bounds for the NE. Because of the fact that the set of NE contains the set of SNEs, it could be possible to conclude that the maximum and minimum revenues are larger and smaller, respectively, if compared to the SNE maximum and minimum revenue (Varian, 2007). Nevertheless this is confirmed only in part. In fact on the one hand, it is true that the upper bound for the SNE revenue coincides with the maximum revenue for the NE. But on the other hand, the lower bound on revenue from the NE is in general less with respect to the revenue bound for the SNE.

<u>**Fact 6:**</u> The maximum revenue NE yields the same revenue as the upper recursive solution to the SNE (Varian, 2007).

<u>Proof:</u> Assume the following conditions:

1) Let (p_s^N) be the prices related to the maximum revenue NE
2) Let (p_s^U) be the prices which solve the upper recursion for the SNE

Since NE Ɔ SNE, it must hold the condition for which the revenue associated with (p_s^N) must be at least equal to the one associated with (p_s^U). Given the definition of NE in equation (1), the result is that:

$p_s^N x_s V p_{s+1}^N x_{s+1} + v_s(x_s - x_{s+1})$

From equation (8) which provides the definition of upper bound recursion, we have:

$p_s^U x_s = p_{s+1}^U x_{s+1} + v_s(x_s - x_{s+1})$

it is important to highlight that the recursions start at s=S. Since $x_{s+1}=0$, we obtain:

$p_s^N V v_s = p_s^U$

By analyzing the recursions immediately above, it follows that (Varian, 2007):

$\mathbf{p_s^U \geq p_s^N}$ *for all s*

This result has an important meaning: the maximum revenue from SNE is at least as large as the maximum revenue from NE (Varian, 2007). Thus the revenue must be equal. By analogy it is possible to make examples in which the minimum revenue NE has less revenue with respect to the solution to the lower recursion for the SNE; however this is merely the consequence of the fact that the set of inequalities that define the NE strictly contains the one defining the SNE. Given these conclusions, some general relation can be finalized (Varian, 2007):

Maximum revenue NE = value of upper recursion of SNE

≥ value of the lower recursion of SNE ≥ min revenue NE

Underlying assumption: the inequalities are strict except in the degenerate cases.

AN INCENTIVE-BASED VIEW

Since until now optimal bids in position auctions have been conceived as being dependent on other agents' bids, it is interesting to explore the possibility that other auction structures let agent a's optimal bid to depend entirely on its value. Even if authors as Demange and Gale demonstrate that the answer is "yes", by implementing a variation of the Hungarian algorithm for the assignment problem, here the VCG mechanism will be taken into consideration, because it takes a simpler form.

Let's assume (Varian, 2007):

I. a central authority is going to choose some outcome z in order to maximize the sum of the reported utilities of agents a=1,…,A.
II. agent a's true utility function is denoted by $u_a(*)$
III. agent a's reported utility function is denoted by $r_a(*)$

The above assumptions imply that, in considering the VCG framework as an alternative to other types of auction models, Varian conceives a mechanism where (Varian, 2009):

- Each advertiser reports a value r_a
- Each advertiser pays the cost that it imposes on the other advertisers, using the values reported by other agents

For the purpose of aligning incentives, the center declares it will pay each agent the sum of the utilities reported by the other agents at the utility-maximizing outcome. So that the centre will maximize (Varian, 2007):

$r_a(z) + \sum_{b \neq a} r_b(z)$

However agent a cares about:

$u_a(z) + \sum_{b \neq a} r_b(z)$

It is not difficult to understand that in order for agent a to maximize his own payoff, the true utility function and the reported one should coincide so that:

$r_a(*) = u_a(*)$

In fact in this way the center will optimize exactly what agent a wants. By subtracting an amount from agent *a* that does not depend on its report, the size of the side payments can be reduced. A convenient choice could be maximizing the sum of reported utilities omitting agent a's report (Varian, 2007). This implies that the final payoff to agent a becomes:

$u_a(z) + \sum_{b \neq a} r_b(z) - \max_y \sum_{b \neq a} r_b(y)$

Agent a's payment can be conceived as the harm that his presence imposes on other agents, i.e. the difference between what they get when agent a is present and what they would get if agent a is absent. If the problem of assigning agents to positions is considered, if agent s-1 is omitted, each agent below him will move up by one position while agents above s-1 are unaffected. Hence the payment that agent s-1 must make is (Varian, 2007):

$$\text{VCG payment of agent s-1} = \sum_{t \geq s} r_t(x_{t-1} - x_t) \qquad (14)$$

In this case r_t is the reported value of agent t. It should be recalled that in the dominant strategy VCG equilibrium, each agent t will announce $r_t = v_t$ so that:

$$\text{VCG payment of agent s-1} = \sum_{t \geq s} v_t(x_{t-1} - x_t) \qquad (15)$$

The analogy with the lower bound for the symmetric Nash equilibria is clear, if compared to expression (11).

It can be demonstrated that this relationship is true in general even for arbitrary u_{as}. A wide range of authors (e.g. Demange and Gale (1985)) agree on the fact that the best (in terms of cost) equilibrium for buyers in the competitive equilibrium for the assignment problem is indeed the one given by the VCG mechanism.

In order to understand this process it is useful to report the practical example provided by Varian with respect to the abovementioned concepts.

Assume that there are 3 slots and 4 advertisers. On the one hand if advertiser 1 is present, the other 3 receive the reported values $r_2x_2 + r_3x_3$. As it is possible to observe, advertiser 4 is not present so it receives an amount equal to zero. On the other hand if advertiser 1 is absent, the other three advertisers would each move up by a position. This implies that their reported value would be $r_2x_1+r_3x_2+r_4x_3$.

The difference between these two amounts represents the required payment by advertiser 1 and it corresponds to:

$r_2(x_1-x_2) + r_3(x_2-x_3) + r_4x_4$

Since the dominant strategy equilibrium in the VCG auction is for each advertiser to report its true value, advertiser 1's payment becomes:

$\mathbf{v_2(x_1-x_2) + v_3(x_2-x_3) + v_4x_4}$

This last result coincides with the lower bound of equilibrium payments previously described. The same calculations can be made for other bidders, so we can conclude that:

The revenue for the VCG auction is the same as the lower bound of the price equilibrium described above (Varian, 2009).

This is a special case of the two sided market.

Despite apparently the implementation of VCG auctions requires exact knowledge of the expected number of clicks in each position, this is not true. This concept can be understood by considering the following algorithm:

1) each time there is a click on position 1, charge advertiser 1 r_2
2) each time there is a click on position s > 1 pay advertiser 1 $r_s - r_{s+1}$

in the 3-advertisers example considered, the net payment made by advertiser 1 will be (Varian, 2009) :

$r_2x_1 - (r_2 - r_3)x_3$. The argument also extends for other advertisers.

BOUNDS ON VALUES

By using the observed equilibrium prices it is possible to derive useful bounds on the unobserved values of the agents in the symmetric Nash equilibrium case. Consider (Varian, 2007):

$$\mathbf{p_s=b_{s+1}}$$

this is the equilibrium price paid by agent s in a specific symmetric NE. Then, it must be true the following condition:

$$(v_s-p_s)x_s \geq (v_s-p_t)x_t$$

By rearranging the equation the result changes in this way:

$$v_s(x_s-x_t)\geq p_sx_s-p_tx_t$$

By dividing by x_s-x_t Varian arrives at this point:

$$\min_{t>s} \frac{p_sx_s-p_tx_t}{x_s-x_t} \geq v_s\geq \max_{t<s} \frac{p_sx_s-p_tx_t}{x_s-x_t}$$

Fact 5 stated that the maximum and the minimum are attained at the neighboring positions. For this reason it is possible to rewrite it in this way (Varian, 2007):

$$\frac{p_{s-1}x_{s-1}-p_sx_s}{x_{s-1}-x_s} \geq v_s\geq \frac{p_sx_s-p_{s+1}x_{s+1}}{x_s-x_{s+1}}$$

The interpretation of these inequalities is simple and straightforward: in fact the ratio represent the incremental cost of moving up by one position. Moreover if the latter inequalities are recursively applied it is possible to obtain (Varian, 2007):

$$v_1 \geq \frac{p_1x_1 - p_2x_2}{x_1 - x_2} \geq \qquad (16)$$

$$v_2 \geq \frac{p_2x_2 - p_3x_3}{x_2 - x_3} \geq \qquad (17)$$

So that:

$$\mathbf{v_s \geq p_s} \qquad (18)$$

The main conclusion that can be inferred is that the incremental costs have to decrease as one moves to lower positions. This result yields three important consequences(Varian, 2007):

- Each of the intervals should be non-empty. This is a necessary condition for the existence of a pure strategy NE. It is also sufficient since non-emptiness allows to find a set of values consistent with the equilibrium
- A bidding rule for the agents: if your value exceeds the marginal cost of moving up by one position, then bid higher until this is no longer true
- An intuitive characterization of SNE: the marginal cost of a click must increase as you move to higher positions. This happens because if it never decreased there would be an advertiser who passed up cheap clicks in order to buy expensive ones.

The author shows in a parallel manner that the same computation can be done also for the NE inequalities, in this way (Varian, 2007):

$$\min_{t>s} \frac{p_sx_s - p_{t-1}x_t}{x_s - x_t} \geq v_s \geq \max_{t<s} \frac{p_sx_s - p_tx_t}{x_s - x_t}$$

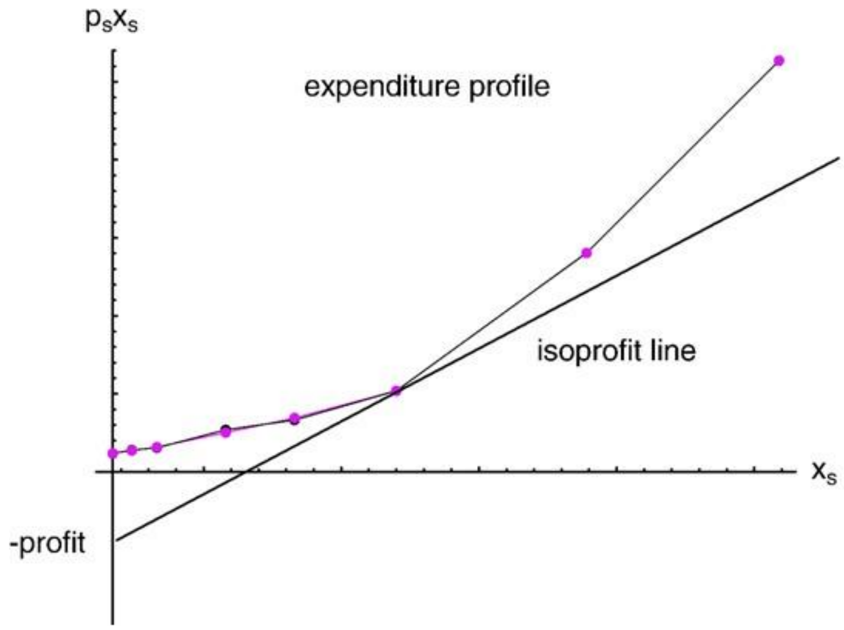
However there are two differences with respect to the other situation; namely:

1) The upper bounds for the NE (when t>s) are looser than the case of SNE
2) The upper bounds now comprise the entire set of bids, and are not just limited to the neighboring bids.

GEOMETRIC INTERPRETATION

The following figure represents the so called *"expenditure profile"*. It is possible to identify on the horizontal axes the clickthrough rates, while on the vertical axes the SNE expenditure, i.e. $p_s x_s = b_{s+1} x_s$. The slope of the segments that connect each vertex represent the marginal costs previously analyzed. Indeed these marginal costs have been shown to necessarily bound the agents' values (Varian, 2007). The previous discussion implies that if the observed choices are an SNE then the graph should be an increasing, convex function.

Graph 1 (Hal R. Varian, 2007)



The profit accruing to agent s is (Varian, 2007):

$$\boldsymbol{\pi = v_s x_s - p_s x_s}$$

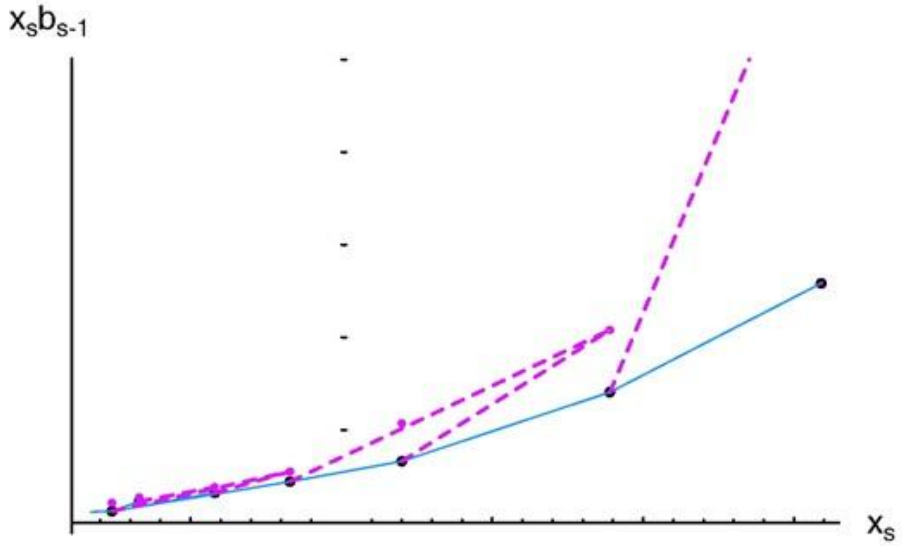
This implies that the iso-profit lines are expressed by:

$$p_s x_s = v_s x_s - \pi_s$$

these lines are straight and have slope equal to v_s, while the vertical intercept is $-\pi_s$. The figure shows that a profit-maximizing bidder is willing to select the position having to the lowest associated profit. So the range of slopes of the supporting hyperplanes at each point are indeed the range of values associated with the equilibrium (Varian, 2007).

Similarly the NE bounds can be illustrated by implementing the same sort of diagram. In the next figure the lower bounds show the SNE bounds along with the NE bounds from inequalities (19). Even if for the NE the lower bounds are the same, the upper bounds are looser and thus steeper than the SNE (Varian, 2007).

Graph 2 (Hal R. Varian, 2007)



<u>APPLICATIONS TO AD AUCTIONS</u>

In order to understand Google's auction design some refinements must be added to the abstract strategic structure of the position auction until now analyzed. Google ranks the ads not only on the basis of the bid alone, rather by the product of a measurement of ad quality and advertiser bid. Varian concentrates on this topic both in 2007 and in 2009. His study begins with the description of a set of auction rules, oriented at outlining the way search engines establish which , where and at which cost ads are shown. The points in common in the two elaborates can be identified in the following considerations:

1) Let *a*=1,…,A indicate advertisers, which in this case represent agents
2) Let *s*=1,….,S indicate slots
3) z_{as} is the advertiser s's observed clickthrough rate
4) Let (v_a, b_a, p_a) indicate the value, the bid, and the price per click of advertiser a for a given keyword

Underlying assumption: the observed and expected clickthrough rate for advertiser a in position s, i.e. $z_{as,}$ can also be written as the product of this “quality effect”,(e_s) and a “position effect”, (x_s). This formula has been chosen by Varian because it leads to simpler outcomes.

so that $\mathbf{z_{as}=e_s x_s}$

Nash equilibrium requires that each agent prefers his position to any other position by realizing that the cost and clickthrough rate of the other position is dependent on his ad quality.

If a specific auction is considered and advertisers are identified by their slot position, the rules imply that $b_1e_1>b_2e_2>,\ldots,>b_me_m$

It is useful to specify that m should be less than or equal to the number of possible slots. Since the price paid by advertiser in slot s is the minimum necessary to retain its position (Varian, 2009) : $p_se_s=b_{s+1}e_{s+1}$. By rearranging the terms of the equation the price results to be:

$\mathbf{p_s=b_{s+1}e_{s+1}/e_s}$

Similarly in his previous work Varian defines q_{st} as the amount that advertiser s needs to pay to be in position t. Since by construction we have: $q_{st}e_s=b_{t+1}e_{t+1}$, it yields the same result. In fact, solved for q_{st} becomes:

$$\mathbf{q_{st}=b_{t+1}e_{t+1}/\ e_s} \qquad (20)$$

It can thus be concluded that the price paid per click by the last advertiser is (Varian, 2009):

1) the reserve price if m<S
2) determined by the bid of the first omitted advertiser if m=S.

So it is possible to write (Varian, 2007):

$(v_s-q_{ss})e_sx_s \geq (v_s-q_{st})e_sx_t$

Varian substitutes equation (20) in the latter expression in order to find:

$(e_s v_s - b_{s+1} e_{s+1}) x_s \geq (e_s v_s - b_{t+1} e_{t+1}) x_t$

Letting the following assumptions of the model to hold:

- $p_s = b_{s+1} e_{s+1}$
- $p_t = b_{t+1} e_{t+1}$

yields:

$(e_s v_s - p_s) x_s \geq (e_s v_s - p_t) x_t$

If the same logic of equations (16) and (18) is applied then we have (Varian, 2007):

$$e_1 v_1 \geq \frac{p_1 x_1 - p_2 x_2}{x_1 - x_2} \geq \quad (21)$$

$$e_2 v_2 \geq \frac{p_2 x_2 - p_3 x_3}{x_2 - x_3} \geq \quad (22)$$

Then the outcome would be:

$$\mathbf{e_s v_s \geq p_s} \quad (23)$$

The previous inequalities are to be considered testable and stemming from the symmetric NE model. For the sake of completeness the category of "non-fully sold pages" should be described. The latter are auctions in which the number of ads displayed on the right hand side is less than 8 (Varian, 2007). In this situation the bottom ad on the page pays a reserve price that has been standardized at 5 cents.

Coming back to the topic of online ad auctions in 2009 , Varian remodeled some terms in the previous study while maintaining a sense of continuity with his antecedent work. His achievements provide a framework that search engines implement to design their auction formats. The reasoning will be here presented by trying to integrate and compare the insights got in 2007 and the ones got in 2009 for the sake of completeness and understanding. Even if the notation has experienced some slight changes , an homogenous pattern of description will be attempted to be achieved. The main inferences to be underlined are that: in his later work Varian reconsidered the topics of bounds and equilibrium revenues, while adding some interesting considerations on advertisers' surplus and bidding behavior.

A FOCUS ON BIDDING BEHAVIOUR

Notation: in the previous section x_s was the number of clocks in position s while here x_a is the number of clicks received by advertiser a in a given time period.

The just described model assumes that advertisers can choose their bids on an auction-by-auction basis. However in reality they choose often one bid that will apply to many auctions. Let' s assume that there is a stable relationship between an advertiser's bid and the number of clicks it receives during some time period (Varian, 2009). This can be included in the following equation:

$$\mathbf{b_a = B_a(x_a)}$$

The cost function $c_a(x_a)$ represents the cost that advertiser a must pay to receive x_a clicks during a given period. Even if both the bid function and the cost function depend on the interaction with the other advertisers in the ad auction, we take that behavior as fixed.

This framework shapes the advertiser's surplus as:

$$\mathbf{v_a x_a - c_a(x_a)}$$

it is possible to observe that the previous equation has been called "surplus" rather than "profit" because profit generally includes fixed costs. However conceptually surplus maximization is equivalent to profit maximization .

The profit-maximizing number of clicks, is suggested by conventional price theory as the point where :

Value= Marginal Cost

Given the optimal number of clicks it is possible to determine the average cost per click. This implies that once the advertiser conquers awareness about the cost-per-click and the bid-per-click function, it can determine its optimal behavior (Varian, 2009). Nowadays these relationships can be estimated only through experimentation.

ADVERTISER SURPLUS

The previous relationships can be used to construct the so called "surplus ratio", i.e. a bound on the ratio of aggregate value to aggregate cost.

Assume that advertiser *a* chooses a number of clicks x_a at a cost of c_a. However this advertiser is contemplating to change its bid so that it receives some smaller number of clicks $\hat{x}_a$ in return for a smaller cost, $\hat{c}_a$ (Varian, 2009). The assumption is that the surplus at (x_a, c_a) exceeds the one at ($\hat{x}_a$, $\hat{c}_a$), so that: $v_a x_a - c_a \geq v_a \hat{x}_a - \hat{c}_a$ that rearranged becomes: $x_a - \hat{x}_a$

$$\mathbf{v_a} \geq \frac{\mathbf{ca - \hat{c}a}}{\mathbf{xa - \hat{x}a}}$$

After some calculations the final expression is (Varian, 2009):

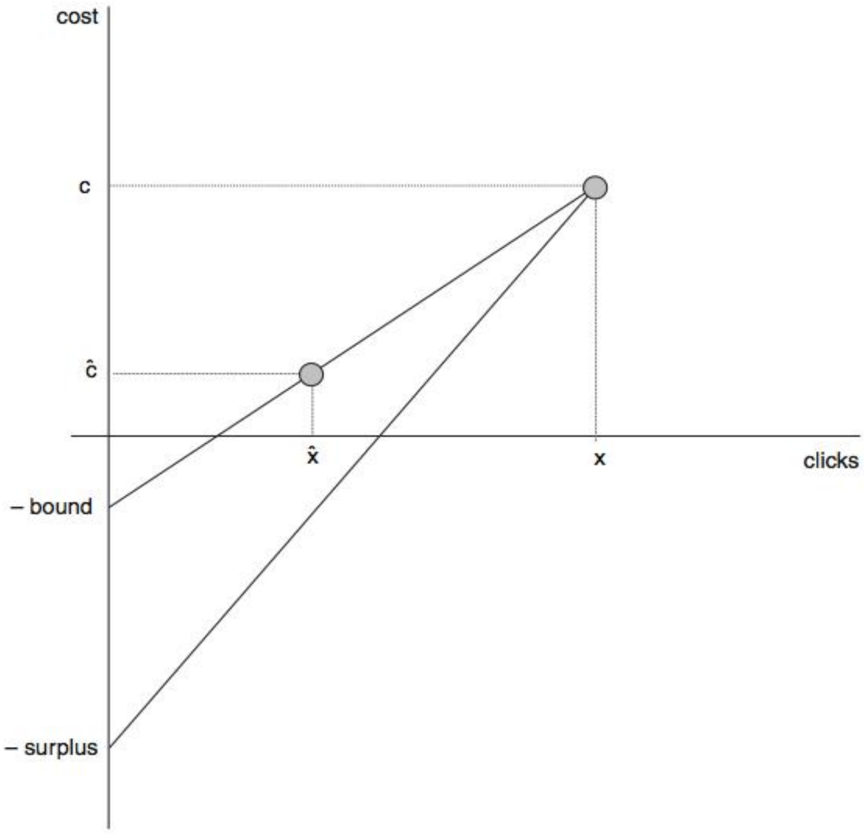
$$\frac{value}{cost} \geq \sum_{a=1}^{A} \frac{\text{ca} - \hat{c}\text{a}}{\text{xa} - \hat{x}\text{a}} \frac{\text{xa}}{\sum_b cb}$$

A GEOMETRIC INTERPRETATION

The function $\pi = v_a x_a - c_a$ describes the isosurplus line through x_a. The same function can be used to conclude that (Varian, 2009):

$$\mathbf{c_a = v_a x_a - \pi_a}$$

Graph 3 (Hal R. Varian, 2009)



the point ($\hat{x}_a, \hat{c}_a$) must lie above this line since it has a lower surplus by assumption (Varian, 2009). This implies that the chord that links (x_a, c_a) to ($\hat{x}_a, \hat{c}_a$) must intersect the vertical axis at a point above - π_a.

This reasoning demonstrates that the equation above gives a lower bound on the surplus ratio. By analogy it is possible to apply the same logic for the upper bound and this can be done by identifying a lower surplus point with a larger number of clicks and costs (Varian, 2009). The figure indicates in a clear way that the tightness of the bounds will depend on the curvature of the cost function. If the latter is affine, then the bounds will coincide and will be equal to the true surplus, while for highly convex functions, the bounds will be wider (Varian, 2009).

<u>MAIN RESULTS FROM THE STUDY OF ONLINE AD AUCTIONS</u>

The observations that can be clearly inferred are related to the number of clicks advertiser a is getting currently and to the cost of these clicks. However, the analysis does not cover the issue of how many clicks it would receive in a different position. In trying to address the question algorithm 1 is used, and it entails three steps (Varian, 2009):

Algorithm 1:

I. Cut advertiser a’s bid in half.
II. Establish the position and the willingness to pay in the ad auction with this lower bid.
III. Make an estimate on how many clicks it would receive being at this lower position.

The first step could appear extreme at a first glance, however a 50% decrease in bids implies only a 12% reduction in costs; while a small cut in bids determines a small reduction in clicks and costs so that $\partial c/\partial x$ could be of significant importance for small choices of ∂x.

Step two implies the implementation of auction rules.

The third step is the more complex calculation since it requires of how clicks of a given ad change as ad position changes, but it is simplified by the assumption that the actual number of clicks is represented by the product of the advertiser-specific effect and the position-specific effect.

An interesting result is that ad configurations that are “fully sold”, i.e., have all the slots occupied, tend to have a lower surplus than those which are “undersold” (Varian, 2009). Finally the previous estimates are applicable only to the value of “paid” clicks; by adding and including also the value of search results clicks the number obtained would be substantially larger.

INFORMATION REQUIREMENTS IN POSITION AUCTIONS

The Nash equilibria of the position auction game is a full-information solution concept. Even if the assumption that advertisers have full information could seem unrealistic, in real-world ad auctions is very likely to apply it. In fact Google reports clicks and impression data on an hour by hour basis (Varian, 2007). The search engine itself offers a “Traffic estimator" that provides an estimate of the number of clicks per day and the cost per day associated with the advertiser's choice of keywords. Moreover third-party companies known as "Search Engine Managers" (SEM's) offer a wide range of services related to managing bids (Varian, 2007). All these conditions imply that the full-information assumption is a reasonable approximation.

EMPIRICAL ANALYSIS

In order to see if the expenditure profile is increasing and convex it is possible to plot x_t versus the expenditure $b_t x_t$, given the fact that it is available a set of position effects, quality effects and bids. However if it happens that the graph is does not show the previously mentioned properties, a perturbation of the data can be made. So the main point to address is: what to perturb? The first variable to perturb is represented by the ad quality, e_s, because this variable is the most difficult to observe and for this reason has the highest level of uncertainty. Now consider these facts (Varian, 2007):

1) let ($d_s e_s$) the value of the perturbed ad quality
2) let (d_s) be the set of multipliers that indicate how much each ad quality has to be perturbed in order to satisfy the inequalities (21)-(23)

Because of the fact that prices, p_s, are linear functions of e_s, perturbations can be thought as applying to prices. The model can thus motivate the following quadratic problem (Varian, 2007):

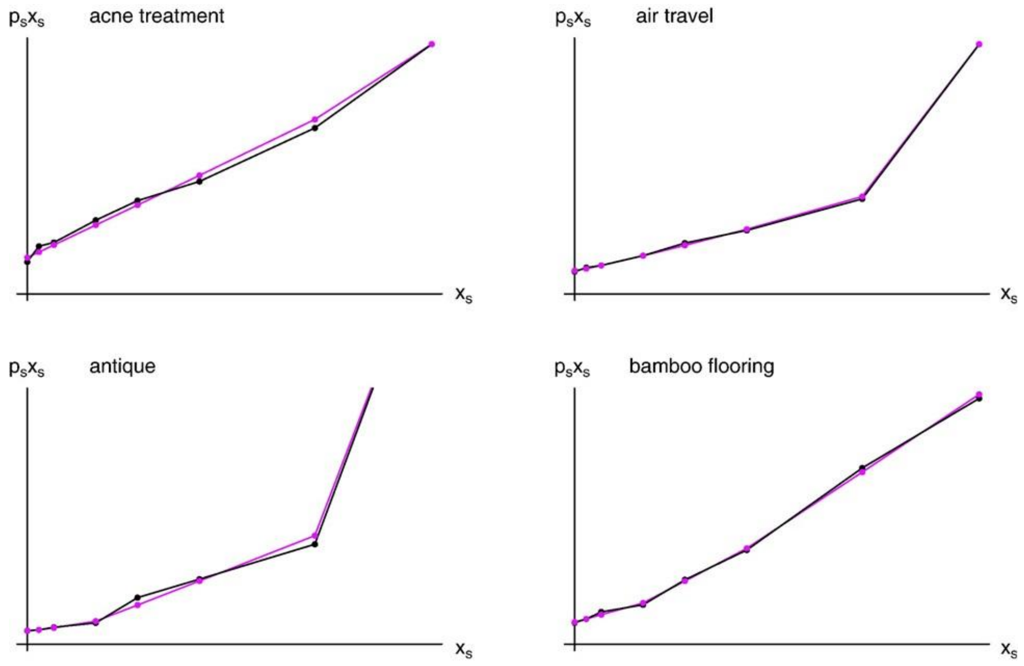
$$\min_d \sum_s (d_s - 1)^2$$

such that:

$$\frac{d_{s-1} p_{s-1} - x_{s-1} - d_s p_s x_s}{(x_{s-1} - x_s)} \geq \frac{d_s p_s x_s - d_{s+1} p_{s+1} x_{s+1}}{(x_s - x_{s+1})}$$

The latter is a simple quadratic problem easily solvable because the constraints are linear in (d_s) (Varian, 2007).

Graph 4 (Hal R. Varian, 2007)



In figure 4 examples of expenditures profiles using actual data are shown. As theoretically predicted the shape of the expenditure profile tends to be increasing and convex. Moreover is frequently flat. One possible reason is that Google will promote ads in these slots to the top-of-the-page position given that certain conditions hold. Varian made finally a statistical study where he concludes that relatively small perturbations are required to make the observations consistent with the SNE models. It is interesting to notice that since the NE inequalities are weaker than the SNE inequalities, the required perturbation for consistency with NE would be even smaller.

A COMPARISON WITH THE PREVIOUS LITERATURE

There are mainly three elaborates belonging to the previous literature in which the topic of position auctions is examined. The first one encountered in chronological order is a work made by Shapley and Shubik (1972). They describe an assignment game where agents are allocated objects with at most one object being assigned to an agent. In game theory terms agent a's valuation of object s is given by u_{as}. The goal is to find the assignment of objects to agents that maximizes value. This is possible to solve by using linear programming or other specific algorithms. The main result is that an optimal assignment can be decentralized by implementing price mechanism. This means that at an optimal assignment there will be a set of numbers (p_a) that can be interpreted as the price of the object assigned to agent a so that:

u_{as}-p_a≥ u_{as}-p_b for all a and b

the logical consequence is that at the prices (p_a) each agent will find himself in a condition where the object assigned to him is weakly preferred over any other object. The comparison of these concepts with the ones of symmetric Nash inequalities leads to notice that the definitions are the same with:

1) u_{as}=$v_a x_s$ and
2) p_a=$b_{a+1} x_s$

The intuition is that the position auction game described by Varian is merely a competitive equilibrium of an assignment game having a specific and special structure for utility. The special structure of u_{as} allows to solve for the largest and the smallest competitive equilibrium.

The second reference point can be identified in the work performed by Demange et al (1986). They construct an auction which establishes a competitive equilibrium. However the latter presents some differences with the position auction.

Finally the third work is represented by Edelman et al (2005). The concept they introduce is the one of “locally envy free equilibria”. They recognize the fact that advertisers bidding on search engines can change their bids very frequently. For this reason sponsored search auctions can be thought as continuous time or infinitely repeated games where advertisers originally have private information about their types, gradually learn the values of others, and can adjust their bids repeatedly. In principle the set of equilibria in these games can be very large , with players potentially punishing each other for deviations. Since the strategies required to support such equilibria can be very complex we focus on simple strategies by imposing some restrictions and assumptions:

- All values are common knowledge: over time advertisers learn all relevant information
- Bids can be changed at any time. **Stable bids must be best response to each other.** Otherwise they have an incentive to change it. The assumptions is thus that the bids form an equilibrium in the simultaneous move, one-shot game of complete information
- Establish the set of simple strategies that can be used by an advertiser to increase its payoff, beyond simple best responses to other players' bids.

Definition 1: an equilibrium of the simultaneous move game induced by GSP is locally envy-free if a player cannot improve his payoff by exchanging bids with the player ranked one position above him.

However their study concluded that each player can't increase his payoff by exchanging bids with the player ranked one position above him. This yields the same bids as the lower bound of the SNEs previously described. The model also assumes that advertisers have converged to a long run steady state, have learned about each other's values, and no longer have incentives to change their bids. These useful insights rely on the notion of "Generalized English Auction", where there is a clock showing the current price, which continually increases over time. At the beginning the price is zero and all advertisers are in the auction. An advertiser can drop out at any time, and his bid is the price on the clock at the time when he drops out. The auction is over when the next-to-last advertiser drops out. The price at which he dropped out is the payment of the last remaining advertiser and so on. So the English auction turned out to be a useful approximation.

This concept indeed demonstrates that the unique perfect equilibrium of this game is the same as the locally-envy free outcome.

Beyond the above mentioned studies there exist several papers that developed auctions yielding competitive equilibria for the assignment game. However their relationship with this paper is only marginal.

POSITION AUCTIONS: APPLICATION TO COMPETITIVE SAFETY STRATEGIES

One of the main challenges of game theory relates to the provision of an advice to the decision maker on how to choose an action in a given multi-agent encounter. However, many notions falling under the so-called "prescriptive agenda" face not fully satisfying answers. The concept of NE for instance is a very useful concept from a descriptive point of view but does not address the issue of how an agent *should* choose his action in a game. The fact that a NE strategy can only be justified by assuming that the other agents are committed to a specific action profile is an extremely strong assumption about their rationality(Kuminov and Tennenholtz, 2007) . There are mainly two approaches to the issue in order to solve the above mentioned problem. The first one does not consider any guarantee to the agent, but in one case, i.e. when the class of opponents he may face is strongly restricted. The second one, goes under the name of " competitive safety analysis" and deals with guarantees the agent can be provided with.

INTRODUCTION TO KUMINOV AND TENNENHOLTZ STUDIES

This model on competitive safety strategies has been elaborated by Kuminov and Tennenholtz and it relies on Varian's work on position auctions discussed in the previous sections. However the authors have considered opportune to apply some slight changes to the original framework in order to get some interesting insights on safety strategies; in some cases their decision was also related to the realness of the assumptions. However, these concepts will be clearer in the next paragraphs.

THE REVISED AD-AUCTION SETTING

The model of ad auctions on which the analysis is based is build on the formal model discussed in the previous sections, made by Varian. However minor changes will be here applied. Even if the model has been originally presented for the complete information setting, its adaptation to the incomplete information one is straightforward. The ad-auction setting relies on the following facts (Kuminov and Tennenholtz, 2007):

- It is assumed that N=S; this means that there are N players competing for S slots.
- The clickthrough rate of a slot is denoted by x_i, $i \in \{1 \ldots N\}$. It is useful to recall that the CTR (Clickthrough rate) represents the publicly known property of a slot and is not depending on the player who is using it. Moreover $\forall i$: $x_i \geq x_{i+1}$, so slots are numbered in decreasing number of CTR. For the sake of simplicity, $x_i=0$ is defined for all i>N.
- In this model the private value is represented by the utility derived by each agent from a single unit of CTR. The latter is assumed to be the same, notwithstanding the slot from which it originates. In addition, $v_i, i \in \{1 \ldots N\}$ and $\forall i \in \{1 \ldots N\}$: $v_i>0$. Always for simplicity $v_i=0$ for all i>N
- The players' bids represent the maximal price per unit of CTR they are disposed to pay to the CTR provider. They can be denoted by: $\tilde{b}_i, i \in \{1 \ldots N\}$. Analogously of what we said before, $\tilde{b}_i \geq \tilde{b}_{i+1}$, which means that agents are numbered in decreasing order of bids. The rules of the auction imply that $\forall i$: $\tilde{b}_i \geq 0$; however for ease of analysis in this elaborate, $\tilde{b}_i=0$ for all i>N
- Varian's model assumes slots are assigned to users according to the decreasing number of bids. In that case the highest bidder is awarded the slot with the highest CTR; the second highest bidder, the second best slot, and so on. For the sake of simplicity in this model ties are broken. The price that must be paid by an agent per unit of CTR is equal to the bid of the agent immediately below him by following this ordering. The price paid by agent i is denoted by p_i so that:

  $\mathbf{p_i = \tilde{b}_{i+1}}$
- When agents' bids are: $\tilde{b}_1 \geq \tilde{b}_2 \geq \ldots \tilde{b}_I \geq \tilde{b}_{i+1} \ldots \geq \tilde{b}_N$, the utility of agent i, having private value v_i corresponds to:

  $(v_i - \tilde{b}_{i+1})x_i$

So it is possible to notice that Kuminov and Tennenholtz gave a sense of homogeneity to their study by posing:

I. $\mathbf{x_i=0}$ for all i>N

II. **v_i=0** for all i>N

III. **b_i=0** for all i>N

The definitions of NE and SNE in the complete information setting are already provided in (Varian, 2007). The SNE presents several nice properties, as(Kuminov and Tennenholtz, 2007) :

1) In a SNE,$\forall s : v_s > v_{s+1}$. This means that agent i is willing to bid higher than agent k only if i's true type is really higher than that of k. This notion is central because in this work it is assumed that agents are indexed in decreasing order of *valuations;* moreover they use indeed this property to state that this order coincides with the order of their bids in a SNE
2) If a numbered sequence of bids is a SNE it must be that:

   $\tilde{b}_s x_{s-1} \geq v_s (x_{s-1}-x_s) + \tilde{b}_{s+1} x_s$
3) Property number 2 implies the fact that the bid of agent i in a SNE is bounded in this way:

   $$\tilde{b}_i \geq \frac{1}{x_{i-1}} \sum_{t=i}^{N+1} v_t (x_{t-1} - x_t)$$

It is important to highlight the fact that a bidding profile where all bids are equal to their respective lower bounds shown above, is a SNE. The term that will be used here to denote this bidding profile is:*"the best SNE".*

Moreover in the best SNE agents never overbid and the payoff of agent i in the best SNE is $v_i x_i - \sum_{t=i+1}^{N+1} v_t (x_{t-1} - x_t)$. The latter is equal to what his equilibrium payoff would be in a VCG auction with the same valuation. In fact his payment corresponds indeed to the externality it imposes on other agents. The expression provided in (Varian, 2007) for the bid of agent i and the utility of agent i can be rewritten in the best SNE:

$$\tilde{b}_i \geq \frac{1}{x_{i-1}} \sum_{t=i}^{N+1} v_t (x_{t-1} - x_t) = \frac{1}{x_{i-1}} [v_i x_{i-1} - \sum_{t=1}^{N} x_t (v_t - v_{t+1})]$$

$$\widetilde{U}_i = (v_i - \widetilde{b}_{i+1})x_i = v_i x_i - v_{i+1} - v_{i+1}x_i + \sum_{t=i+1}^{N} x_t\,(v_t - v_{t+1}) = \sum_{t=i}^{N} x_t\,(v_t - v_{t+1})$$

One important thing is to underline that this model presents some differences with Varian's one. In primis, the previous work assumes that N>S, instead here N=S. This change brings two important consequences: firstly the situation is altered with respect to the auctioneer, who has enough ad slots for all the agents; moreover the only motivation for agents' bidding is the willingness to get a better slot. Perhaps this assumption is more realistic because of the fact that the nature of online advertisement does not present a practical limit on the number of ad slots, so the auctioneer has no reasonable motives to deny agents' requests for slots(Kuminov and Tennenholtz, 2007) .

Secondly in this model it is assumed that there is no reserve price. In this way the agent with the lowest bid gets the N^{th} slot for free. Differently from the previous case, here the decision is not linked with real-world similarities, rather it is considered a good approximation because the reserve price is negligible with respect to other agents' valuations.

In addition, even if the formulae for the bid and the utility in the best SNE derive from (Varian, 2007), they are valid in this model as well, where $v_{N+1}=0$. In fact it is possible to reduce this model from that of Varian by adding a fictitious player having fixed valuation equal to zero. It can be noticed that in the best equilibrium this player does not affect slot allocation and expected utility since it always bids truthfully. However recall that this reduction does not function for other kind of equilibrium where the fictitious agent could overbid.

<u>ANALYZING THE COMPLETE INFORMATION SETTING</u>

The assumptions in this case are that (Kuminov and Tennenholtz, 2007):

- Agents' valuations are fixed
- Agents' valuations are common knowledge

The aim is to find the payoff that an agent can guarantee to himself regardless of other agents' behavior. If there are no assumptions about other agents' rationality, they can always force the agent to take the N's slot, by bidding higher than the agent's valuation. The latter case implies that the payoff loss that the agent suffers relative to his payoff in an equilibrium can be defined unbounded. Even in the case where other agents are constrained not to bid above their valuation , the agent's payoff in best SNE could be N-ϵ times bigger with respect to his safety level ($0 < \epsilon \ll 1$). This can be understood in the following example:

- There are N agents and N slots
- $0 < q < 1$
- $x_1 > 0$
- the valuation is: $v_i = \sum_{t=1}^{N} q^{N-t} = \frac{1-q^{N-i+1}}{1-q}$
- The CTR's is: $x_i = x_1 q^{i-1}$

The consequence is that:

I	x_i	v_i
1	x_1	$1-q^N/1-q$
2	x_1q	$1-q^{N-1}/1-q$
…	…	…
N-1	x_1q^{N-2}	q+1
N	x_1q^{N-1}	1

The next objective is to determine the utility of agent 1. For the time being, let's assume that all other agents bid truthfully; this implies that a bid made by agent 1 capturing slot i gives a payoff of (Kuminov and Tennenholtz, 2007) :

U_1= (v_1-v_{i+1})x_i= $x_1q^{i-1}\sum_{t=1}^{i} q^{N-t}$ = x_1q^{i-1} · $q^{N-i}\frac{1-q^i}{1-q}$ = $x_1q^{N-1}\frac{1-q^i}{1-q}$

The expression is increasing in i and monotone. For this reason agent 1 prefers to:

1) Bid 0
2) Take slot N

Get a payoff that equals to = $x_1q^{N-1}\frac{1-q^N}{1-q}$

On the one hand, it is thus clear that the safety level payoff of agent 1 ($\widehat{U}_1$) cannot be higher that this value. On the other hand, the utility of agent 1 in the best SNE is(Kuminov and Tennenholtz, 2007) :

$\widetilde{U}_1$=$\sum_{t=1}^{N} x_t(v_t-v_{t+1})$= $\sum_{t=1}^{N} x_1q^{t-1}q^{N-t}$=$Nx_1q^{N-1}$

The **competitive safety ratio** is the ratio between the expected payoff in the best SNE induced by the valuations' realizations and the expected payoff guaranteed by the safety level strategy, given that other agents do not overbid. In this case it is:

$$R(N)=\frac{\widetilde{U}_1}{\widehat{U}_1} \geq \frac{Nx_1q^{N-1}}{x_1q^{N-1}\frac{1-q^N}{1-q}} = \frac{N(1-q)}{1-q^N}$$

The inference is that for every N and 0< $\epsilon \ll 1$, is possible to construct an example where the competitive safety ratio is at least N-ϵ by choosing q≤ $\frac{\epsilon}{N}$. The following theorem shows indeed that N is an upper bound on the competitive safety ratio:

Theorem 3: In the complete information setting with N slots and N players, the competitive safety ratio, i.e. the ratio between an agent's payoff in a best SNE and the payoff guaranteed by a safety level strategy, given that the agents do not overbid, is at most N.

Proof: when calculating the pure safety level of an agent, the underlying assumption is that other agents know the bid of the agent under consideration plus all the valuations. They try to maximize the utility of the agent considered, who will be here simply denoted as *the* agent (Kuminov and Tennenholtz, 2007).

<u>*We will adopt the following notation:*</u>

- v_i represents the i^{th} valuation in the ordered sequence of all agents' valuations
- $\hat{b}_i$ represents the i^{th} bid in the ordered sequence of all agents bids
- v' represents the valuation of the agent
- $\hat{b}_i'$ represents the bid of the agent
- *v-index(x):* [0,1] → {0…N-1} stands for the number of adversarial agents characterized by valuations strictly higher than x (e.g. the valuation of the agent under consideration is $v_{v\text{-index}(v')+1}$; the one of the agent immediately below is $v_{v\text{-index}(v')+2}$)
- *b-index(x)*: [0,1] → {0…N-1} stands for the number of adversarial agents characterized by bids strictly higher than x

the **utility of the agent** is (Kuminov and Tennenholtz, 2007):

$(v'-\hat{b}_{b\text{-index}(b')+2})x_{b\text{-index}(b')+1}$

So, in order to maximize the agent's payoff, it is necessary that other agents choose their bids in way that they maximize:

1) $\hat{b}_{b\text{-index}(b')+2}$

 and

2) b-index ($\hat{b}'$) +1

However the previous two are conflicting objectives. In fact since agents aren't allowed to overbid by assumption. This suggests that in order to maximize the price paid by the agent, some of the agents with valuation higher than his bid will have to bid lower than him, in order to make him get a better slot. Because of the fact that there is no need to let the agent go up more than one slot, an optimal adversarial strategy can be identified in the following threefold solution:

I. All adversarial agents having valuations smaller than the agent's bid must bid truthfully
II. The ones with valuations higher than the agent's bid, except one, should bid truthfully
III. One of the players having higher valuation can choose between bid truthfully or submit a bid equals to that of the agent.

This implies that (Kuminov and Tennenholtz, 2007):

- The agent's utility if all adversarial players bid truthfully would be:

  $x_{v\text{-index }(b')+1}\ (v'-v_{v\text{-index }(b')+2})$

- The agent's utility when one of the adversarial players with higher valuation chooses to bid an amount equal to him is:

  $x_{v\text{-index}(b')}(v'-\hat{b}')$

So, taking into consideration all valuations and his bid, the utility of an agent is:

$$\min\{x_{v\text{-index}(b')+1}\ (v'-v_{v\text{-index }(b')+2}),\ x_{v\text{-index}(b')}(v'-\hat{b}')\}$$

It will be soon explained why among all possible bids that guarantee slot k, i.e. all $\hat{b}'$ so that v-index ($\hat{b}'$) =k-1, the agent weakly prefers to submit the smallest bid possible. This is clear by recalling that:

- all the adversarial agents implement the strategy described above
- that if there is complete information the agent knows all the valuations
- Ties are decided in favor of the agent. This means that the agent can, without incurring a loss of utility, consider only N-v-index (v')+1 strategies – the valuations of the adversarial agents with valuations lower than his own

So, his safety level payoff is:

$\widehat{U}$= $\max_{0\leq b'}$ min {$x_{v\text{-index}(b')+1}(v'-v_{v\text{-index}(b')+2})$, $x_{v\text{-index}(b')}$ $(v'-\hat{b}')$}

Which yields as result:

$\max_{i:\ v\text{-index}(v')+1\leq i\leq N} x_i(v'-v_{i+1})$

From this stems the fact that for any index j such that v-index (v')+1 ≤j≤N:

$\frac{x_j}{\widehat{U}}\leq 1/\ v'-v_{j+1}$

This implies that the competitive safety ratio is:

$$\frac{\widetilde{U}}{\widehat{U}}=\frac{\sum_{t=v-\mathrm{index}(v')+1}^{N} x_t(v_t-v_{t+1})}{\widehat{U}} \leq \sum_{t=v-index(v')+1}^{N} \frac{v_t-v_{t+1}}{v'-v_{t+1}}\leq N$$

In the special circumstance in which the CTR's are linear, the following theorem provides the worst-case bound:

Theorem 4: In an ad auction setting with complete information where there are N slots and N players, when the CTR's are given by $x_i=d(N-i+1)$ for some d>0, the competitive safety ratio is at most:

$$\sum_{t=1}^{N}\frac{1}{t}\leq 1+\ln N$$

Proof: in the best SNE the payoff of the agent corresponds to (Kuminov and Tennenholtz):

$\widetilde{U}$= $(v'-b_{v\text{-index}(v')+2})x_{v\text{-index}(v')+1}=v'x_{v\text{-index}(v')+1}-\sum_{t=v-index(v')+2}^{N+1} v_t(x_{t-1}-x_t)$ = $v'd(N-v\text{-index}(v'))-d\sum_{t=v-index(v')+2}^{N+1} v_t=$

$=d\sum_{t=v-index(v')+2}^{N+1}(v'-v_t)$

Moreover the bound on the safety level payoff provided in theorem 1 implies that for all i such that v-index(v')+2 ≤i≤N+1, the following condition holds (Kuminov and Tennenholtz, 2007):

(v'-v)≤ $\frac{\hat{U}}{d(N-i+2)}$

The above statement implies that the competitive safety ratio is at most:

$$\frac{\tilde{U}}{\hat{U}} \leq \sum_{t=v-index(v')+2}^{N+1} \frac{1}{N-t+2} = 1+ \sum_{t=2}^{N-v-index(v')} \frac{1}{t} < 1+ \ln N$$

It can be shown that $\sum_{t=1}^{N} \frac{1}{t}$ is a tight bound, therefore it corresponds to the competitive safety ratio.

WHAT HAPPENS IN THE INCOMPLETE INFORMATION SETTING

The decision problem of an agent in an auction with incomplete information, will be here analyzed by considering the following six circumstances(Kuminov and Tennenholtz, 2007):

I. The agent which is being considered has valuation v' ∈ [0,1]; the latter is known to him
II. The agent makes an assumption about other agents' distributions: they are distributed according to a known distribution
III. The agent is characterized by risk neutrality
IV. The agent faces a choice between two available courses of action. The first one is to select to "play for the best SNE". This could for instance reflect the fact that the auction is repeated with the same agents and the same valuations; here the play sequence is conceived to converge at the best SNE. The value that the agent assigns to this action is corresponds to his expected payoff in the best SNE, induced from the realizations of the players valuations in which the expectation is conceived in relation to the distribution of the other agents' valuations. It should be noticed that this corresponds also to his expected payoff, given his valuation, present in the corresponding Bayes-Nash equilibrium of this auction. The second course of action is to implement a "safety level strategy". The latter implies that the agent chooses an action that guarantees him the best expected payoff possible during the auction, against any reasonable action taken by the other players. Notice that the expectation is computed with respect to the distribution of other agents' valuations.

The value assigned by the agent to this action is represented by the guaranteed expected payoff. In particular, the model of interaction assumed is the following one (Kuminov and Tennenholtz, 2007):

- All agents are assigned by taking into consideration their private values, which are fixed
- The agent considered, chooses his bid $\hat{b}'$, by considering his valuation only. In fact he is not aware about other agents' valuations, beyond their distribution
- The other agents choose their bids by considering $\hat{b}'$ plus the realizations of all valuations; they include the agent's one. It is assumed that: they can freely communicate, they cannot overbid (the bid should be at most equal to their valuations); they select the joint action that maximizes the agent's utility

This model of interaction implies that agents are not able to gain by using mixed strategies. For this reason **only pure strategies** will be here considered.

V. All things being equal, the intuition is that the second course of action is preferred to the first one because of the fact that it does not require neither elaborate and hard-to-justify assumptions with respect to the rationality of other agents' behavior nor additional structure on the basic auction format

VI. The agent is willing to know how much of his utility will be lost by selecting action 2 instead of action 1. However the answer depends on: the distribution of the valuations and the CTR values of the ads slots. Here uniformly distributed valuations will be considered and two central types of CTRs: exponentially decreasing and linearly decreasing.

EXPONENTIALLY DECREASING CTRs

Theorem 5: In the incomplete information ad auction setting characterized by the following parameters (Kuminov and Tennenholtz, 2007):

1) Agents' valuations are independently and uniformly distributed over [0,1]
2) The CTR's can be obtained by $x_k = x_1 q^{k-1}$ for $0<q<1$ and $x_1>0$

The ratio between the expected payoff in the best SNE stemming from the valuations' realizations and the expected payoff guaranteed thanks to the safety level strategy is :

At most e.

Underlying assumption: other agents do not overbid

Proof: let :

- v-index be a random variable, representing the number of adversarial agents with valuations strictly higher than v'.
- the goal be finding the agent's expected utility in the best SNE

$\widetilde{U}= \sum_{k=1}^{N} E\ (\widetilde{U}|$ v-index (v')+1=k) P(v-index (v')+1=k)

P(v-index(v')+1=k) = $\binom{N-1}{k-1}$ $(1\text{-}v')^{k\text{-}1}\ v'^{N\text{-}k}$

E($\widetilde{U}$| v-index(v')+1=k) = E ($\sum_{t=k}^{N} x_t$ (v_t-v_{t+1}) | v-index(v')+1=k) =

= x_k(v'- E(v_{k+1}|v-index(v')+1=k) +$\sum_{t=k+1}^{N} x_t$ E(v_t-v_{t+1} |v-index(v')+1=k)

From the above calculation emerges that the condition:

v-index(v')+1=k

implies that the valuations {v_t: t=k+1,…,N} represent the order statistics of N-k independent random variables, uniformly distributed over [o,v']. For this reason:

1) $v_t \sim$ v' Beta(N-t+1, t-k)
2) E(v_t)= v' $\frac{N-t+1}{N-k+1}$

So:

E(v_t-v_{t+1})=v'$\frac{N-t+1}{N-k+1}$ -v' $\frac{N-t}{N-k+1}$= $\frac{v'}{N-k+1}$

This implies that:

E ($\widetilde{U}$|v-index(v')+1=k) =x_k (v'-v'$\frac{N-k}{N-k+1}$) +$\sum_{t=k+1}^{N} x_t$ $\frac{v'}{N-k+1}$=

$= \frac{v'}{N-k+1} \sum_{t=k}^{N} x_t$

The consequence is that in best SNE the expected utility is (Kuminov and Tennenholtz, 2007):

$\widetilde{U} = \sum_{k=1}^{N} E$ (U|v-index(v')+1=k) P(i=k) which equals

$\frac{1}{N} \sum_{k=1}^{N} (\sum_{k=1}^{N} x_t) \binom{N}{k-1} (1-v')^{k-1} v'^{N-k+1}$

Because of the fact that:

$\sum_{i=k}^{N} x_i = x_k \frac{1-q^{N-k+1}}{1-q} = \frac{x_1}{1-q} (q^{k-1} - q^N)$

At the equilibrium, the expected utility is equal to:

$\widetilde{U} = \frac{1}{N} \sum_{k=1}^{N} (\sum_{k=1}^{N} x_t) \binom{N}{k-1} (1-v')^{k-1} v'^{N-k+1} =$

$\frac{x_1}{N(1-q)} [\sum_{k=1}^{N} \binom{N}{k-1} ((1-v')q)^{k-1} v'^{N-k+1} - q^N \sum_{k=1}^{N} \binom{N}{k-1} (1-v')^{k-1} v'^{N-k+1}]$

It is possible to reformulate the equations by recalling that the second sum corresponds to a particular case of the first sum when q=1. In this way the authors achieve the following result:

$\widetilde{U} = \frac{x_1}{N(1-q)} [((1-v')q + v')^N - q^N]$

However, when calculating the safety level the assumptions are that:

- The other agents know the bid of the agent and all the valuations
- The other agents seek to minimize the utility of the agent, which stems from all the valuations and his bid.

The agent instead (Kuminov and Tennenholtz, 2007):

- Knows his valuation
- Assumes other players' valuations are i.i.d. (independently identically distributed)

This implies that he will select a bid that maximizes:

$\widehat{U}= \sum_{k=1}^{N} E(\widehat{U}\,|\,\text{v-index}(\hat{b}')+1=k)P(\text{v-index}(\hat{b}')+1=k)$

$P(\text{v-index}(\hat{b}')+1=k) = \binom{N-1}{k-1} (1-\hat{b}')^{k-1}\hat{b}'^{N-k}$

$E(\widehat{U}\,|\,\text{v-index}(\hat{b}')+1=k) \geq x_k(v'-\hat{b}')$

For this reason:

$\widehat{U} \geq \max_{0\leq b'\leq 1} \sum_{k=1}^{N} x_k (v'-\hat{b}') \binom{N-1}{k-1} (1-\hat{b}')^{k-1}\hat{b}'^{N-k}=$

$=\max_{0\leq b'\leq 1} x_1 (v'-\hat{b}') ((1-q)\,\hat{b}'+q)^{N-1}$

In order to find the maximum it is possible to apply the first order condition, i.e. to find the first derivative and compare it to zero:

$\frac{d}{d\widehat{b'}}\widehat{U}=0$ which gives as ultimate result the following one:

$\hat{b}'=\frac{v'(N-1)(1-q)-q}{N(1-q)}= v'\frac{N-1}{N} - \frac{q}{N(1-q)}$

Since the above expression is positive if $v' \geq \frac{q}{(N-1)(1-q)}$

The maximum can be achieved using this strategy:

$$\hat{b}' = \begin{Bmatrix} v'\frac{N-1}{N} - \frac{q}{N(1-q)} & if\ v' \geq \frac{q}{(N-1)(1-q)} \\ 0 & otherwise \end{Bmatrix}$$

The safety ratio is computed in the model for both cases. After the due calculation the outcome achieved in the first case is (Kuminov and Tennenholtz, 2007):

$\widehat{U} \geq \frac{x_1 c^N}{N(1-q)} \frac{1}{e}$

Because of the fact that as previously seen:

$\widetilde{U} \leq \frac{x_1 c^N}{N(1-q)}$

It is possible to observe that the relative payoff loss from the implementation of the safety level strategy is bounded by e.

In the second case since the agent bids $\widehat{b'}$=0, he obtains the last slot for free, and so:

$\widehat{U} \geq v' x_1 q^{N-1}$

Recalling that:

$\widetilde{U} = \frac{x_1}{N(1-q)} \sum_{i=1}^{N} \binom{N}{i} ((1-q)v')^i q^{N-i}$

The relative payoff loss from the usage of the safety level strategy is bounded by:

$\frac{\widetilde{U}}{\widehat{U}} \leq 1/\, N(1-q)v'q^{N-1} \sum_{i=1}^{N} \binom{N}{i} ((1-q)v')^i q^{N-i} =$

$= \left(\frac{(1-q)v'+q}{q}\right)^{N-1}$

It is possible to replace v' with this bound in the expression above, which yields:

$\frac{\widetilde{U}}{\widehat{U}} < \left(\frac{\frac{q}{N-1}+q}{q}\right)^{N-1} = \left(1+\frac{1}{N-1}\right)^{N-1} < e$

In conclusion, it can be stated that the payoff loss ratio caused by using a safety level strategy rather than playing for SNE is bounded by e.

LINEARLY DECREASING CTRs

The following theorem provides a solution for linearly decreasing CTR's and uniformly distributed valuations.

Theorem 6: In the complete information setting with the following parameters (Kuminov and Tennenholtz, 2007):

I. Agents' valuations are independently and uniformly distributed over [0,1]
II. The CTR's can be obtained by $x_k = d(N-k+1)$ for d>0

III. The ratio between the expected payoff in the best SNE stemming from the valuations' realizations and the expected payoff guaranteed thanks to the safety level strategy is :

At most 2.

Underlying assumption: other agents do not overbid

Proof: In the best SNE, the utility of agent i is the following one:

$\widetilde{U}_i = (v_i - \widetilde{b}_{i+1})x_i = v_i x_i - \sum_{t=i+1}^{N} v_t\,(x_{t-1} - x_t)$

The latter can be compared with the expected utility of the agent in the best SNE, which is (Kuminov and Tennenholtz, 2007):

$\widetilde{U} = \sum_{k=1}^{N} E\,(\widetilde{U}\,|\,\text{v-index }(v')+1=k)\ P(\text{v-index}(v')+1=k)$

Since: $P(\text{v-index}(v')+1=k) = \binom{N-1}{k-1}\,(1-v')^{k-1}v'^{N-k}$

Then:

$E(\widetilde{U}\,|\,\text{v-index}(v')+1=k) = E\,(v_k x_k - \sum_{t=k+1}^{N} v_t\,(x_{t-1} - x_t)\,|\ \text{v-index }(v')+1=k) =$

$= v'd\,(N-k+1) - d\,\sum_{t=k+1}^{N} E(v_t\,|\ \text{v-index}(v')+1=k)$

It is important to recall that (Kuminov anf Tennenholtz, 2007):

- Given v-index(v')+1=k the valuation of the N-k agents having valuations lower than v' is uniformly distributed over [o,v']
- For the above reason the expected value of each of them, not constraining any order of them, is $\frac{v'}{2}$
- The expected utility at the equilibrium level is:

$$E(\widetilde{U}\,|\,\text{v-index}(v')+1=k) = v'd(N-k+1) - \frac{dv'(N-k)}{2} = dv'\,\frac{N-k+2}{2}$$

Therefore :

$\widetilde{U} = \frac{dv\prime}{2}\Big[(N+1)\sum_{k=1}^{N}\binom{N-1}{k-1}(1-v')^{k-1}\ v'^{N-k} - \sum_{k=1}^{N}(k-1)\binom{N-1}{k-1}(1-$

$v'k-1\ v'N-k$

By doing the right computations it results that:

$\widetilde{U} = \frac{dv\prime}{2}$ ((N-1)v'+2)

On the other hand, the previous theorem stated that the safety level of the agent is:

$\widehat{U} \geq \max_{0\leq b'\leq 1} \sum_{k=1}^{N} x_k (v'-\hat{b}') \binom{N-1}{k-1} (1-\hat{b}')^{k-1}\hat{b}'^{N-k}$

To find the maximum the same process is applied. Namely it is necessary to find the first derivative and compare it to zero (Kuminov and Tennenholtz, 2007):

$\frac{d}{d\widehat{b\prime}}\widehat{U} = 0$, which yields as outcome:

$\hat{b}' = \frac{v\prime}{2} - \frac{1}{2(N-1)}$

Here it is possible to recognize two cases:

1) $\frac{v'}{2} - \frac{1}{2(N-1)} \geq 0$, i.e. $v' \geq \frac{1}{N-1}$

In this specific case the safety level payoff is:

$\widehat{U} \geq d(\frac{v\prime}{2} + \frac{1}{2(N-1)})\left((N-1)\left(\frac{v'}{2} - \frac{1}{2(N-1)}\right) + 1\right) =$

$= \frac{d}{4}\left(v'^2(N-1) + 2v' + \frac{1}{N-1}\right)$

So, in this case the competitive safety level is represented by (Kuminov and Tennenholtz, 2007):

$\frac{\widetilde{U}}{\widehat{U}} \leq 2\,\frac{(N-1)v'^2+2v\prime}{v'^2(N-1)+2v'+\frac{1}{(N-1)}} < 2$

2) $\frac{v'}{2} - \frac{1}{2(N-1)} < 0$, i.e. $v' < \frac{1}{N-1}$

Here, in order to guarantee the maximal value, the agent should:

- Bid 0
- Take the lowest slot
- Get $\widehat{U}$=v'd

On the other hand in the best SNE the expected payoff is:

$\widetilde{U}=\frac{dv'}{2}$ ((N-1)v'+2) < $\frac{3dv'}{2}$

So the **competitive safety ratio** is:

$\frac{\widetilde{U}}{\widehat{U}}<\frac{3}{2}$

So, the payoff loss ratio stemming from the implementation of a safety level strategy and the abandonment of the SNE strategy is bounded by 2.

ADDING QUALITY EFFECTS TO THE LINEARLY DECREASING CTRs

In his model, Varian used an additional parameter, the "ad quality" property of an advertiser. This reflects the actual ad auction mechanism used by Google. As discussed in the previous sections, Varian's model that the amount of clicks received by an ad allocated to slot i, is the product of the measurement of the ad quality and the advertiser's bid. So, the model implied that the price per click paid by each advertiser was the minimum to retain its position (Kuminov and Tennenholtz, 2007).

Because of the fact that the model here considered assumed instead that all advertisers are of the same quality, it considered only this special case. It should be remembered that the negative results related to the complete information setting, are applicable here as well.

In order to analyze the incomplete information model, it is possible to rely on the observation that this model corresponds to a variation of the basic one, in which players' valuations are said to be the product of their "core" valuations and the ad quality parameter of the corresponding player. Given that, it will be here assumed that:

1) Both the valuations and the ad quality of advertisers are independently and uniformly distributed over [0,1]
2) The resulting equivalent basic model implies that the valuations are distributed according to the PDF (probability density function): f(x)= - lnx
3) There are linearly decreasing CTRs

So, the competitive safety ratio is given by the next theorem:

Theorem 7: In the incomplete information setting with the following parameters (Kuminov and Tennenholtz, 2007):

1) Agents' valuations are i.i.d over [0,1] with PDF f(x)= -lnx
2) The CTR's are represented by: x_k=d(N-K+1) for d>0

Then the ratio between the expected payoff in the best SNE induced by the valuations' realizations and the expected payoff stemming from the safety level strategy:

Is at most $1+\frac{3}{4}(N-1)$

Underlying assumption: other agents do not overbid

Proof: The main passages that prove the above theorem can be summarized in the following conclusions:

$F(a)= P(Z<a)= \frac{a(2\ln a-1)}{4(\ln a-1)}$

In the best SNE the utility of agent a is:

$\widetilde{U}_i= (v_i-\widetilde{b}_{i+1})x_i = v_ix_i - \sum_{t=i+1}^{N} v_t(x_{t-1}-x_t)$

So, the corresponding expected utility associated is:

$\widetilde{U}=\sum_{k=1}^{N} E(\widetilde{U}\,|$ v-index(v')+1=k) P(v-index(v')+1=k)

Since: P(v-index(v')+1=k) = $\binom{N-1}{k-1}$ $(1- F(v'))^{k-1} F(v')^{N-k}$

Then: E($\widetilde{U}\,|$ v-index(v')+1=k)= v'd(N-k+1)-d $\sum_{t=k+1}^{N} E(v_t)\,|$ v-index(v')+1=k)

Given that:

1) v-index(v')=k
2) there is no imposition on any ordering of players

the expected value of each member among the N-k players is:

E(v|v<v') = $\frac{\mathrm{v}'(2\ln \mathrm{v}'-1)}{4(\ln \mathrm{v}'-1)}$, which implies that at the equilibrium the expected utility is:

E($\widetilde{U}$|v-index(v')+1=k) = $\frac{\mathrm{dv}'}{4(\ln \mathrm{v}'-1)}$ [2N ln v'-3N+2ln v'-1 (k-1) (2ln v'-3)]

Kuminov and Tennenholtz continue their study with a series of computations, aimed at showing that the expected utility in best SNE is:

$\widetilde{U}$ = dv' $\left[1+\frac{v'}{4}(3-2\ln v')\,(N-1)\right]$

The previous theorem provided the safety level of the agent:

$\widehat{U}\geq \max_{0\leq b'\leq 1} \sum_{k=1}^{N} x_k\,(v'-\hat{b}')\binom{N-1}{k-1}(1-F(\hat{b}'))^{k-1}\,F(\hat{b}')^{N-k}=$

$=\max_{0\leq b'\leq 1} d(v'-\hat{b}')\,((N-1)\,\hat{b}'(1-\ln\hat{b}')+1)$

To find the maximum the same process applied before is implemented; thus the computation of the first derivative and the comparison of it to zero:

$\frac{\mathrm{d}}{\mathrm{d}\hat{b}'}\widehat{U}=$ d(N-1) (2$\hat{b}'$ ln$\hat{b}'$ +v' ln$\hat{b}'$ -$\hat{b}'$)-d

It should be noticed that the derivative is always negative. This implies that in order to guarantee the maximal value, the agent should:

1) bid 0
2) take the lowest slot
3) get $\widehat{U}$= v'd

The above considerations imply that the **competitive safety ratio** is:

$\frac{\widetilde{U}}{\widehat{U}} \leq 1+\frac{v'}{4}$ (3-2 ln v') (N-1)

It is possible to verify that the previous expression is monotonic and increasing over v'∈ [0,1]. The implication is that the maximal competitive safety ratio is obtained when v'=1. Then:

$\frac{\widetilde{U}}{\widehat{U}} \leq 1+\frac{3}{4}$ (N-1) which stands for the competitive safety ratio in this case.

ACCOUNTING FOR SPECIFIC VALUATION VALUES OF THE AGENT

While until now Kuminov and Tennenholtz concentrated on bounds on the competitive safety ratio that hold in an homogeneous way for all the valuations of the agent, now the possibility that those bounds can be enhanced by limiting the valuation values of the agent is explored.

Moreover, the decision problem formulated before implies that showing a good bound for a subset of possible valuations considers the fact that if the agent's valuation falls in that subset, he should analyze the possibility of using a safety level strategy. Differently for other kinds of valuations he could improve his situation by using alternative strategies. The following theorem motivates the next section:

Theorem 8: In the incomplete information ad auction setting, in which the agents' valuations are distributed over [0,1] with a PDF which is strictly positive in [0,1] and well-defined then (Kuminov and Tennenholtz, 2007):

∀N: $\lim_{v' \to 0^+} R$ =1, where v' represents the valuation of the agent

Underlying assumption: R is the ratio between the expected payoff in the best SNE induced by the valuations' realizations and the expected payoff guaranteed by the safety level strategy, given that other agents do not overbid.

Proof: let

1) x_N denote the CTR of the last slot
2) $\widehat{U}$ denote the safety level payoff of the agent

Then for any valuation it is known that v>c of the agent:

$\hat{U} \geq vx_N$

In fact the agent has always the possibility to bid 0 and guarantee at least the last slot.

On the other hand from the notion of expected equilibrium payoff it is possible to obtain that:

$$\lim_{v' \to 0^+} R \leq \lim_{v' \to 0^+} \frac{vx_N + x_1 vP(v - index(v') + 1 < N)}{vx_N} = 1$$

In fact as the valuation of the agent approaches 0, the probability that all other agents' valuation are greater than his one approximates 1.

However the above explained theorem has only motivational importance, while from a practical point of view it does not present considerable concerns. In fact even if it arrives at the point that for any competitive safety ratio R>1 there exists c>0 so that for every v'∈ $[0, c]$ the ratio is at most R, it fails in demonstrating how to compute this c or to show that it has a value large enough to be practically useful (Kuminov and Tennenholtz, 2007). So in order in order to study the behavior of the competitive safety ratio when v' assumes small values additional assumptions are required. This regard the CTR values of the slots and the distribution of the valuations.

IMPORTANT INFERENCES

In the next paragraphs, there will be presented some observations stemming directly from the previous analysis.

Observation 1: In the incomplete ad auction setting characterized by the following parameters (Kuminov and Tennenholtz, 2007):

- The agents' valuations are uniformly and independently distributed over [0,1]
- The CTRs are represented by $x_k = x_1 q^{k-1}$ for 0<q<1 and x_1 is positive

When the agent's valuation is: $v' < \frac{q}{(N-1)(1-q)}$

The ratio between the expected payoff in the best SNE stemming from the valuations' realizations and the expected payoff guaranteed thanks to the safety level strategy given that other agents do not overbid is obtained by:

$\left(\frac{(1-q)v'+q}{q}\right)^{N-1} \cong e \wedge \frac{(N-1)(1-q)}{q} v'$

It can thus be inferred that the competitive safety ratio diminishes exponentially for small valuations of the agent, as v' gets close to zero. However, the relevant range is very small. In fact for values: N=10 and q=0.5 the valuation of the agent should be lower than 0.01 in order to have a safety ratio of 1.1.

Observation 2: In the incomplete ad auction setting characterized by the following parameters (Kuminov and Tennenholtz, 2007):

- The agents' valuations are uniformly and independently distributed over [0,1]
- The CTRs are given by $x_k=d(N-k+1)$ for $d>0$

In the case where the agent's valuation corresponds to $v'<\frac{1}{N-1}$

Then the ratio between the expected payoff in the best SNE stemming from the valuations' realizations and the expected payoff guaranteed thanks to the safety level strategy given that other agents do not overbid is obtained should be:

At most: $\frac{1}{2}((N-1)v'+2)=1+\frac{N-1}{2}v'$

The above statement communicates that the competitive safety ratio decreases linearly for small valuations of the agent, as v' gets close to 0. In fact if N=11, to obtain a safety ratio of 1.1 the agent's valuation must be lower than 0.02.

Observation 3: In the incomplete information ad auction setting characterized by the following parameters (Kuminov and Tennenholtz, 2007):

- The agents' valuations are uniformly and independently distributed over [0,1] over a PDF f(x)= -lnx

- The CTRs are given by x_k=d(N-k+1) for d>0

It is possible to observe that the ratio between the expected payoff in the best SNE stemming from the valuations' realizations and the expected payoff guaranteed thanks to the safety level strategy given that other agents do not overbid is obtained is:

At most: $1+\frac{v'}{4}(3-2\ln v')(N-1)$

In fact in the case where N=11, in order to obtain a competitive safety ratio of 1.1, the agent's valuation should be lower than 0.0027. Even if this value might seem too low, the safety strategy turns to be very useful if the agent's valuation is assumed to have the same distribution as the opponents'.

INTUITIONS AND RESULTS

In this model the assumptions that the slots values are either decreasing exponentially or linearly are believed to be real-world assumptions. For the latter setting, in the complete information model, achieving a constant competitive safety ratio is not possible. Differently in the incomplete information setting having uniformly distributed valuations, a competitive safety ratio of e can be obtained for exponentially decreasing CTRs, while for linearly decreasing CTRs it is equal to 2 (Kuminov and Tennenholtz, 2007).

The intuition is that the divergent result obtained respectively for the complete and incomplete information settings is rooted in the fact that while a specific profile of valuations arbitrarily bad for the agent is shown, the probability that these "bad" profiles actually occur is negligible. Moreover the profiles that occur with a high probability, show a constant competitive safety ratio.

CONCLUSIVE REMARKS

PECULIARITIES OF THE MARKET FOR INTERNET ADVERTISING

The first thing that distinguishes the market for Internet advertising is that bids can be changed at any time. In fact an advertiser bid for a particular keyword will apply every time that keyword is entered by a search engine user, until the advertiser changes or withdraws that bid. For this reason the order ads are displayed to the user could be different any time since bids could have changed in the meantime.

Second search engines sell flows of perishable advertising services rather than storable objects: if there are no ads for a particular search term during some period of time, the “capacity” is wasted (Edelman et al,2007).

Third, unlike other markets where it is clear how to measure what is sold, there is no “unit” of Internet advertisement that is natural from the points of view of all the involved parties. In fact from the advertiser’s perspective, the relevant unit is the cost of attracting a customer who makes a purchase; the corresponding pricing model is a one where the advertiser pays only when a customer actually completes a transaction. Instead from the search engine’s perspective, the relevant unit is what it collects in revenues every time a user performs a search for a particular keyword; here the corresponding pricing model implies that the advertiser is charged every time its link is shown to a potential customer. PPC is a compromise between the two models: the advertiser pays every time a user clicks on the link (Edelman et al, 2007).

GSP insists that for each keyword, advertisers submit a single bid even if different items are for sale. This one-bid requirement makes sense in this setting. In fact the value of being in each position is proportional to the number of clicks associated with that position; the benefit of placing an ad in a higher position is proportional to the number of clicks associated with that position. Even if the benefit of placing an ad to a higher position is that it is clicked more, the users who click on ads in different positions are assumed to have the same values to advertisers, for example the same purchase probabilities. For this reason despite the GSP environment is multi-object, buyer valuations can be represented by one-dimensional types (Edelman et al, 2007).

It is important to remember that different search engines implement different mechanisms. In fact Yahoo! Ranks advertisers purely in decreasing order of bids, thus ignoring the possible different dimensions among them, while Google multiplies each advertiser's bid by its "quality score", which is based on CTR and other factors to compute its "rank number", ranks the ads by rank numbers and then charges each advertiser the smallest amount sufficient to exceed the rank number of the next advertiser.

COMPUTER MEDIATED TRANSACTIONS

The internet and the web represent a case of "combinatorial innovation", where the component parts of these technologies can be combined many times by innovators in order to create new devices and applications. Nowadays a wide portion of economic transaction takes place over the Internet. The impact of computer mediated transactions can be classified in 4 main categories (Varian, 2010):

- Facilitate new forms of contact
- Facilitate data extraction and analysis
- Facilitate controlled experimentation
- Facilitate personalization and customization

There is a crucial divergence of incentives in advertising: the publisher has space on its web page for an ad and it wants to sell these ad impressions to the highest bidders. The advertiser does not care about ad impressions but it does care about visitors on its web site. This implies that the publisher wants to sell impression while the advertiser wants to buy clicks. This is a problem of exchange rate (Varian,2010). This is equal to the predicted clickthrough rate. The latter aligns the interests of buyers and sellers but creates other problems. For example if an advertiser pays only for clicks then it has no incentives to economize on impressions.

THE ROLE OF POSITION AUCTIONS IN GOOGLECONOMICS

Google hired Varian as a chief economist. Googleconomics is two-folded: has a micro and a macro aspect.

The macroeconomic side involves some of the company's apparent altruistic behavior. Google frees out products as Google Chrome, Android Os and so on. According to Varian anything that increases Internet use ultimately enriches Google; and since using Internet without using Google is not possible, we are speaking about a competition for "eyeballs" (Levy,2009).

The microeconomics of Google is a little bit more complex. Selling ads generates both profits and huge amounts of data about users tastes and habits. These data are analyzed by the search engine in order to predict future consumer behavior. In fact Googleconomics is a system of constant self-analysis.

As the business grew Kamangar and Veach decided to price the slots on the side of the page by means of an auction. It was a huge marketplace of virtual auctions in which sealed bids were submitted in advance and winners are determined algorithmically in fractions of a second (Levy, 2009).

When Varian was hired in 2002, a bit after the implementation of the auction-based version of Adwords, he found out that the mathematical structure of the Google auction was the same was the same as those two-sided matching markets and his contribution to the firm was really decisive.

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