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An Intertemporal Pricing Model for CO2 Allowances: The Impact of the Clean Development Mechanism.

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Ai miei nonni,

perché prima di

un’intera tesi in inglese

si meritano una bella dedica in italiano.

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Abstract

The increasing global attention to greenhouse emissions and the recent creation of EU Emission Trading Scheme has clearly suggested the need of consistent methods to value projects aimed to reduce gases. This need particularly concerns companies that have to find a way to both remain profitable and conform to new legal requirements. Multiple ways of cutting emission costs are available nowadays: short term abatement measures, which primary involve switching production machinery from coal to gas; long term abatement measures, which envisage the implementation of new types of projects – e.g Clean Development Mechanism or Joint Implementation Mechanism suggested by Kyoto Protocol -. In this work we study the impact of the introduction of both kinds of policy in a pricing model for CO_2 allowances.

Keywords: *CO_2 emission certificates, EU-ETS system, CDM projects, Stochastic optimal control.*

Contents

1 Introduction

The Kyoto protocol, signed in December 1997 in the homonymous Japanese city, established the basis for the global fight against carbon emissions. Not all countries in the world have signed it – e.g. Afghanistan or Taiwan -, and some of the countries that subscribed the protocol haven't ratified it yet[1] – from now on "Non-Annex I countries"[2] – in opposition to those nations that have both signed and ratified it – from now on "Annex I countries" -. The original mechanisms introduced were mainly three:[3]

- International Emission Trading (IET): it permits the trade of CO2 allowances' credits – Assigned Amount Units (AAUs) – between Annex I countries;
- Joint Implementation (JI): it consists in projects implemented by an Annex I country into another Annex I country. Those projects give origins to carbon credit called Emission Reduction Units (ERUs) for the implementing country, while create carbon debits of AAUs that have to be deducted from the host country quota;
- Clean Development Mechanism (CDM): it involves the enforcement of projects by Annex I countries into Non-Annex I countries. The plan under analysis allows the Annex I country to achieve carbon credits called Certified Emission Reduction Units (CERs) that will be added to its own endowment of carbon certificates.

The European Union has been one of the first to create a trading scheme system, the European Union Emission Trading System - EU ETS -, which is nowadays the most developed in the field.

Meanwhile lots of studies have been focused on the ways of optimizing this relatively new system. Environmental finance is a branch of finance that has an important role in this sort of works. Within it, an even more innovative research front is the so-called carbon finance, whose main goal is to understand price dynamics of carbon permits. Two types of factors that could influence the evolution of CO2 prices are currently under investigation:[4] short-term abatement measures and long-term abatement measures. The main difference between the two is the time the measure needs to become effective and reduce GHG emissions. Short-

[1] See Borloo et al. (2008).
[2] The term "Annex I countries" is interchangeable with "Annex B countries" since it includes countries listed in Annex B of the Kyoto Protocol. "Annex I" is used just for simplicity but refers exactly to the same nations.
[3] See Carmona, Fehr and Hinz (2009).
[4] See Carmona, Fehr and Hinz (2009).

term measures are typically the ones whose results occur rapidly, already starting from the first compliance period – the period at the end of which a company is required to comply with the "cap and trade"[5] system depending on CO2 emitted throughout the period -. They are mainly represented by fuel switching processes – e.g. switching machinery from coal to gas – or production re-schedule. Long-term measures, on the contrary, become effectively carbon profitable only some years after their inception: they require high initial investments – which can be considered fixed costs – that will be recovered over the time of workability of the plan through the carbon returns collected during the entire project's horizon. JI and CDM belong in all the effects to this category. They depend critically on the availability of a long term horizon in order to amortize their initial consistent cost. It has been observed[6] that the number of these projects sharply fell in the final part of Phase II: they have become less relevant in this pre-2013 period since their validity was conditioned to the fact that, even if registered before 2013, they would have started to generate carbon emission reduction from 2013 onwards.[7] However their number started to grow again in these first months of Phase III, and it is forecasted to reach maximum peaks in the actual Phase due to its major length.[8]

In this work project a two-scenario finite horizon, continuous-time model is built in order to reproduce the EU-ETS taking in consideration the environment with and without the presence of CDM, in both models short term abatement measures are present. We focus only on CDM since they are the most interesting instruments to lower carbon reduction. JIs are only mechanisms to reallocate credits within countries that ratified the Protocol and, actually, do not generate new carbon allowances. Possible extensions of the research could try to insert this additional abatement measure in the model and study a more complete and realistic scenario. At the end a numerical simulation is implemented in order to verify the effects of the presence of CDM projects on carbon price.

[5] "The overall volume of GHG that can be emitted each year by the power plants, factories and other companies covered by the system is subject to a cap set at EU level. Within this Europe-wide cap, companies receive or buy emission allowances which they can trade if they wish", The EU-Emission Trading System (EU ETS) – European Commission Factsheet, 2013.

[6] See Kossoy and Guigon (2012).

[7] See Directive 2009/29/EC.

[8] Phase III will be 8 years long, the longest Phase since the creation of the EU-ETS.

2 Literature Review

An initial input for the body of literature today known as environmental finance has been given by Coase (1960) and Dales (1968). These authors were the first to propose the idea of tradable allowances as a way of endogenizing the social cost of pollution and make more effective the resolution of this increasingly analyzed environmental problem. After these publications a wide number of studies headed toward the search for the equilibrium price of emission allowances. The topic was particularly deepened by Cronshaw and Kruse (1996) and Rubin (1996) who demonstrated the equality between such price at equilibrium and the marginal cost of the cheapest available abatement strategy for pollution. These results apply only to situations without uncertainty, so their real implementation is quite difficult and they remain confined as more theoretical findings. Nonetheless they represent the basis for the future developments in the field. Carmona, Fehr and Hinz (2009) analyzed in a more realistic way the environmental problem, contextualizing the opportunity of reducing carbon emission in the newborn EU Emission Trading Scheme. They were the first to make a distinction between short-term and long-term abatement measures, available to firms after the implementation of the Kyoto Protocol to reduce carbon emissions. The paper suggests a model for pricing CO2 permits in case N firms decide to apply fuel switching, the cheapest short-term abatement process available. The analysis under consideration relates only to one trading period – in the case of EU ETS it is the year - that, even if divided in subperiods to account for within-period trading among firms, does not permit to consider multiperiod abatement strategies such as JI or CDM, explicitly introduced by Kyoto Protocol. Evaluating only short-term measures is not representative of the real future possibilities available to firms since, even if in the present they are the cheapest procedures, in the near future they could be considered obsolete: when a firm already switches its technologies it becomes harder to find new profitable opportunities to switch them again. Seifert et al. (2008) consider one representative agent/firm that can decide how to comply with the pollution restrictions either paying a penalty or reducing its emissions. The paper develops an interesting model that permits to analyze the spot price of CO2 allowances at the beginning of the compliance period. Starting from this equilibrium price a sensitivity analysis is conducted in order to understand which variables impact on this price. Like the previous paper also this one lacks a multiperiod view and is limited to short-term measures implemented in a compliance interval. Chesney and Taschini (2012) re-elaborate the preceding works introducing asymmetric information between participants in the carbon market. The main finding is that the carbon price reflects the probability of not complying with the regulation at the end of the period. They introduce the problem of long-term abatement projects but only as a matter whose value can be influenced by the carbon price path. Actually they want to predict future carbon spot prices in order to understand what could be the actual value of these projects, while in this paper we want to understand which impact the availability of these schemes can have on the decision to implement them from the firms' and regulator's point of view.

We have identified a gap in the literature history mainly concerning the non-inclusion of long term projects in decisions taken by firms relating to their emission schemes. Since we support the idea that those projects are an essential part of a company's decision making process, we want to include them in the analysis and see if their introduction is worth or not. Their presence should be profitable simultaneously[9] for the firms and the Policy Maker: a company in order to implement them should be better off, in terms of wealth, with their inclusion, while the Policy Maker should observe a diminution in the overall level of CO_2 emitted in the environment. In order to control for those two effects we used dynamic programming instruments respectively for evaluating the firms' wealth, function of both the emission policy of every company and the rules imposed by the EU-ETS,[10] and the aggregate level of emissions in the air, function of the emission policies only. At the end of this paper we elaborate a numerical simulation showing under which conditions the presence of long term projects is justified. In these cases CDM should be considered a basic instrument for the Policy Maker to influence carbon price preventing the Scheme to reach extreme peaks.

3 The Model

3.1 The main assumptions

Following the assumptions made by Seifert et al (2008) the model does not refer to the wealth-maximization of a single firm, but rather to a social wealth-maximization problem in which all firms that take part in the economic process are considered.[11] The social planner

[9]If only one part considers the project necessary it will not apport any positive effects in the world we are considering.

[10]In particular we will show that some EU-ETS's rules will impact directly on the level of emissions chosen by firms.

[11]For a discussion of the equivalence between this two maximization problems refer to Appendix B.

is called to make a social-optimum choice, considering the aggregate level of emission and the aggregate costs linked with emission cut. All the parameters presented thereafter should be intended as aggregate ones.

As already discussed in the introduction, the framework in which the model is inserted is a finite horizon one, with T representing the final period under analysis. We assumed T to be the duration of a EU-ETS Phase, presently it is equal to 8 years. Since in the European scheme a compliance period is one year long[12] we impose a penalty condition for every integer intermediate instant:

$$P\left(x_t^A\right) = Min\left[0, p_t(e_{t-1} - x_t^A)\right] \ , \ t = 1, 2, 3.., T \tag{1}$$

where x^A is the total accounted emissions for the firms in the Scheme - it will be defined later in Sections 3.2 and 3.3 -, e_t is the initial endowment of EUAs[13] allocated at the beginning of every compliance period to the companies by the regulator - for $t = 0, 1, 2, .., T-1$ - and p_t is the penalty charged for every additional emission unit.

In the absence of abatement policies the firms emit at a rate y_t at every instant for the entire duration of the period; y_t is not under the control of the planner, and it can be split in two parts: a deterministic component $\mu(t, y_t)$, and a volatility of $\sigma(t, y_t)$.

The emission process y_t follows the subsequent stochastic process, with y_0 equal to a given constant:

$$dy_t = \mu(t, y_t)dt + \sigma(t, y_t)dW_t \tag{2}$$

where dW_t represents the instantaneous increment of a standard Wiener process.[14] It appears reasonable to have an exogenous emission rate since in reality it can be affected by unexpected factors: changes in prices or structural changes in the sector.

In the models there are mainly three ways of reducing carbon emissions in every instant t:

[12] At the end of every year the firms have to comply with the restrictions of the Scheme, and in case their emissions overpass their permits they have to pay a penalty.

[13] EUAs stay for European Union Allowances that is the European equivalent of AAUs.

[14] A standard Wiener process, also called standard Brownian motion, is a continuous time stochastic process; on the interval $[0, T]$ it satisfies the following conditions: $W(0) = 0$ and for $0 \leq s < t \leq T$ $W(s) - W(t) \sim \sqrt{t-s} * N(0, 1)$ where $N(.)$ is the normal distribution with mean $\mu = 0$ and variance $\delta^2 = 1$; so the Wiener process has stationary and independent increments.

type of abatement strategy	cost of the strategy	payout of the strategy
buy CO2 permits on the market	$S(t)$	n
short term abatement strategies	θ_t	f
CDM projects	$\frac{C}{t+1}$	γ_t

(Table 1.)

The carbon price in every instant is $S(t)$, n is the quantity of CO2 embedded in every permit; θ_t is the price of implementing a short term abatement measure at every t it is considered exogenous, f is the quantity of CO2 abated by the short term strategy in every instant; $\frac{C}{t+1}$ is the fixed cost of implementing a CDM project - it is decreasing in time meaning that a full initial amount C is paid only at $t = 0$ while thereafter only lower amounts are paid to maintain the project operative - , while γ_t is the carbon return of the project in every t it is also taken as exogenous, it depends on two factors: g, the quantity of CO2 effectively saved by the project in every moment and α, the conversion rate adopted by the regulator to convert carbon saved in carbon reduction within the scheme. In particular γ_t can be modelled as:

$$\gamma_t = \alpha_t g. \tag{3}$$

For example if $\alpha = 1$ every unit of carbon saved by the project can be translated in a complete unit of carbon saved in the scheme, if $\alpha < 1$ the scheme accepts as carbon reduction a lower quantity than the one effectively cut, and if $\alpha > 1$ the scheme accepts as carbon reduction a higher quantity than the one effectively saved in the CDM. This α is an interesting variable to study since it can be chosen by the policy maker and directly affects the carbon price through the carbon return of the projects under consideration.

In order to simplify the problem we assume homogeneity in the economy: every firm implements in period 0 the same project with the same cost and the same carbon returns. So if N firms are considered in the economy the cost of a single CDM will result in $\frac{C}{N}$ while the return in every instant of a single project will be $\frac{\gamma_t}{N}$.

As previously explained trade of permits is explicitly considered as a way of reducing carbon emissions. Nevertheless, since we are considering an aggregate problem, it does not impact on the solution. In fact if $z_{i,t}$ represents the number of permits exchanged - positive if bought, negative if sold - by firm i at time t, we have to assume the market clearing condition $\sum_{i=1}^{N} z_{i,t}^{*} = 0$, and the exchange of permits does not impact anymore on the aggregate objective function of our problem.[15] This condition implies the assumption that permits are exchanged only between firms that take part to the Scheme; in reality we all know that the trading is open also to other institutions as well as to individual investors. So, using the above mentioned description, would give only a partial and incomplete description of the real market. For now we limit our analysis to this more abstract case leaving this problem to possible future expansion of the study.

At this point we have to mention some simplifications that help us to solve explicitly the maximization problem that will be introduced in Sections 3.2 and 3.3: in particular $\mu(t, y_t) = 0$ and $\sigma(t, y_t) = \delta$ so the volatility is independent of time, in this case y_t is assumed to follow a translated Brownian motion; in addition the time discount factor $r = 0$, there is no time preference in the model; finally we assume $\gamma_t = \gamma$ constant along the duration T of the Phase.[16] These conditions, even if reductive in terms of reality, help us to solve the model and understand the basic properties of the solutions.

3.2 Model 1 - without CDM

3.2.1 General Setup

Now that all the assumptions have been stated we can go deeply in the specifications of the two models. We start with the one in which CDM are not considered.

The total expected emission in the environment at period t is x_t: it has an uncontrolled component y_t,[17] and a component that can be influenced directly by behaviors of the companies that take part in the scheme, u_t:

$$x_t = y_t - \int_0^t u_s ds \ . \tag{4}$$

The expected emissions coincide with the actual ones because we assume the past to be the best prediction of the future:

$$E_t(y_T) = y_t. \tag{5}$$

[15] See Appendix B for the complete derivation.

[16] The stability of α-policy will rely, in reality, on the actual duration of the phase: for small T it is reasonable to assume constant γ, while for big T the Policy Maker could find more favourable to change α along time in order to better influence the carbon price.

[17] What impact on x_t is the expected value in instant t of the total emission over the period $[0, T]$: y_t.

Following directly from the aforementioned definition the final and initial expectations read respectively:

$$x_T = y_T - \int_0^T u_s ds \quad , \tag{6}$$

$$x_0 = y_0. \tag{7}$$

An infinitesimal increment in x_t will be:

$$dx_t = -u_t dt + \delta dW_t. \tag{8}$$

At this point it is useful to introduce a focal distinction between actual emissions reflected in Equation (4) and the emissions related to the account of the firms. The latter ones come directly from the rules of the Scheme that allow firms, with permits in excess, to keep those permits as credits for the next period, whereas firms with a lack of permits only have to pay the penalty but are not affected by the permits' account of next period. What we mean is that if in the first compliance period the firm pollutes less than the number of permits it has been provided with, $e_0 > x_1$, it will be able to transfer the excess of permits $(e_0 - x_1)$ to the second compliance period, while in the case $e_0 < x_1$ it will have to pay a penalty of $p_1 (x_1 - e_0)$, but no debt will be registered in her second period permits' account which will be cleared at the moment of the payment. We will call the account value of emission x^A to distinguish it from the actual value of total emissions x; all the choices, that the firms will take, will depend on x^A, while the policy maker will be interested mainly in x as an indicator of the overall pollution present in the environment. Notice that an infinitesimal increment in x^A will be the same of x - as expressed in Equation (8) -, except at integer t. Due to this "jump" at the end of every compliance periods in the account side of emissions, we will have a discontinuity between the final and initial value of emissions registered by firms, in particular the final value of accounted emissions for the first period will be:[18]

$$x_1^A = y_1 - y_0 - \int_0^1 u_s ds = x_1 - x_0, \tag{9}$$

while the initial value of emissions accounted by firms for the second period - to distinguish it from the previous one we indicate it with a '+' - will be:

$$x_{1+}^A = -\max\left[0, e_0 - x_1^A\right]. \tag{10}$$

[18]Notice that only for the first period we have a coincidence between x_1^A and x_1.

If we want to generalize this condition we have to better define it compliance period by compliance period; in particular considering a compliance period that starts in m and finishes in n,[19] we have the final condition:[20]

$$x_n^A = x_n - x_m - \sum_{i=1}^{m} \max\left[0, e_{i-1} - (x_i - x_{i-1})\right], \tag{11}$$

and the initial condition of the next compliance period:[21]

$$x_{n+}^A = -\sum_{i=1}^{n} \max\left[0, e_{i-1} - (x_i - x_{i-1})\right]. \tag{12}$$

This condition will apply also to all the subsequent intermediate x_t^A in the next compliance period $[n, o]$, with $t \in (n, o)$:

$$x_t^A = x_t - x_n - \sum_{i=1}^{n} \max\left[0, e_{i-1} - (x_i - x_{i-1})\right]. \tag{13}$$

Equation (4) is qualitatively different from both Equations (11) and (12) since it expresses the actual emissions in the environment and it is not influenced by firms' carbon credits. The final compliance conditions $P(\cdot)$ will be all expressed in terms of x_n^A if referred to the final period n.

The aggregate cost function, increasing in time, is assumed to satisfy the quadratic relation between cost and chosen abatement rate:

$$C(t, u_t) = -\frac{1}{2\theta_t} u_t^2 . \tag{14}$$

Every firm in reality will choose first to use the cheapest available measures and only after to use the more expensive ones. Consequently we will observe an increasing sequence of θ_t where $\theta_1 < \theta_2 < ... < \theta_T$ so that the marginal abatement cost $\frac{1}{\theta_t}$ will be decreasing in time. Also in this case it is assumed that all firms have access in every period to the same abatement technology. As already stated by Seifert et Al (2008) the simplification that all firms face an equal marginal abatement cost could be a very restrictive assumption since if all firms could use the same technology to abate a carbon trading scheme would become meaningless; in fact the scheme can be considered a valuable option for firms to find on the

[19] Remember that a compliance period in the EU ETS is 1 year long, therefore in order to contestualize our model in reality we should have $m - n = 1$.

[20] In the following three equations, the summation starts from 1 if the firm never paid a penalty before compliance period n; otherwise it should start in the period after the last penalty payment. To keep notation lighter, we write as if the former is always the case. The modification to the general case is straightforward.

[21] Note that x_{n+}^A is a function of x_i^A, $i = 1, .., n$.

market cheapest ways to abate their carbon emissions levels, given that they face different costs of short term abatement.

At this point, remembering the sequence of final conditions for every compliance period in Equation (1), the objective function can be formulated as:

$$\max_{(u_t)_{t\in[0,T]}} E_0\left[\int_0^T C\left(t,u_t\right)dt+\sum_{i=1}^{T}P\left(x_i^A\right)\right], \tag{15}$$

with the final conditions expressed as

$$P(x_i^A)=\min\left[0,p_i(e_{i-1}-x_i^A)\right] \text{ for } i=1,..,T-1. \tag{16}$$

This problem can be characterized as partial equilibrium since it refers solely to carbon market, taking in consideration the "carbon" side of both costs and revenues: the cost function refers to costs related to emission abatement measures while the "carbon" revenues are considered for the state equation, in particular they contribute to decrease the overall level of emissions in the system.

3.2.2 Solution for T=2

We conducted the analysis for T=2 for ease of exposition, but an identical procedure can be applied to every T.

In this scenario the problem can be reformulated as follows:

$$\max_{(u_t)_{t\in[0,2]}} E_0\left[\int_0^2 -\frac{1}{2\theta_t}u_t^2dt+\sum_{i=1}^{2}\min\left[0,p_i(e_{i-1}-x_1^A)\right]\right] . \tag{17}$$

We approached our analysis through a dynamic programming backward approach solving first the optimization problem over the period $[1,2]$,[22] considering as final condition:

$$P(x_2^A)=\min\left[0,p_2(e_1-x_2^A)\right] , \tag{18}$$

and finding an explicit resolution for the firms' value function $V(\cdot)$, at the beginning of the considered interval: $V_1\left(1,x_{1+}^A\right)$;[23] we, then, rewrote our problem as:

$$\max_{(u_t)_{t\in[0,2]}} E_0\left[\int_0^1 -\frac{1}{2\theta_t}u_t^2dt+\min\left[0,p_1(e_0-x_1^A)\right]+V_1(1,x_{1+}^A)\right], \tag{19}$$

[22] This second part coincides with the solution found by Seifert et al (2008).

[23] We evaluated $V(\cdot)$ at x_{1+}^A and not at x_1^A because firms base their decision not on actual emissions but on accounted emissions, therefore taking in consideration eventual credits obtained in the precedent period.

taking into consideration as final condition:

$$P(x_1^A) = \min\left[0, p_1(e_0 - x_1^A)\right] + V_1(1, x_{1+}^A) \ , \tag{20}$$

we arrived at an implicit formulation of $V_0(0, x_0^A)$.

It is possible to generalize the problem, in particular the principle of optimality for stochastic optimal control requires that:

$$V(t, x_t^A) = \max_{u_t} E_t\left[C(t, u_t)dt + V(t + dt, x_t^A + dx_t^A)\right] \ . \tag{21}$$

This only holds on $(0,1) \cup (1,2) \cup ... \cup (T-1, T)$, but not on $t = 1, 2, ... T-1$ where we have the discontinuity analyzed before in relation with the rule imposed by EU ETS Scheme at every crossing point between two compliance periods.

Since from Equations (2) and (8) the expected change in $V(\cdot)$ is:[24]

$$E(dV) = V^{(t)}dt - u_t V^{(x)}dt + \frac{1}{2}\delta^2 V^{(xx)}dt \ , \tag{22}$$

the Hamilton-Jacobi-Bellman equation results in:

$$0 = \max_{u_t}\left[-\frac{1}{2\theta_t}u_t^2 + V^{(t)} - u_t V^{(x)} + \frac{1}{2}\delta^2 V^{(xx)}\right] . \tag{23}$$

Using the FOC of this maximization problem we arrive to the optimal abatement rate:

$$u_t^* = V^{(x)}\theta_t \ , \tag{24}$$

and remembering that the equilibrium carbon price equals the marginal abatement cost:

$$S^*(t, x_t^A) = \frac{u_t^*}{\theta_t} = -V^{(x)} \ . \tag{25}$$

The optimal carbon prices for the two compliance periods respectively S_1, S_0,[25] calling:

$$A(x_t^A, t) = \frac{e_1 - x_t^A + p_2\theta_1(2-t)}{\delta\sqrt{(2-t)}}, \tag{26}$$

$$B(x_t^A, t) = \frac{e_1 - x_t^A}{\delta\sqrt{(2-t)}} \ , \tag{27}$$

$$D\left(x_t^A, t\right) = \frac{p_2(p_2\theta_1 + 2(e_1 - x_t^A))}{2\delta^2} \ , \tag{28}$$

[24] See Seifert et al (2008), pag 184.
[25] For the explicit derivation of these expressions see Appendix A.

and:

$$\operatorname{erf}(k) = \frac{2}{\sqrt{\pi}} \int_0^k e^{-t^2} dt. \tag{29}$$

are:

$$S_1(t, x_t^A) = \frac{p_2 e^{D\left(x_t^A, t\right)} [1 - \operatorname{erf}(A\left(x_t^A, t\right))]}{1 + \operatorname{erf}(B\left(x_t^A, t\right)) + [e^{D\left(x_t^A, t\right)}(1 - \operatorname{erf}(A\left(x_t^A, t\right))]}, \tag{30}$$

and, defining:

$$\phi_-\left(t, x_t^A\right) = \frac{1}{2\delta\sqrt{(1-t)\pi}} \int_{-\infty}^{e_0} e^{\frac{-(x_t^A - x_0^A)^2}{2\delta^2(1-t)}} dx_0^A, \tag{31}$$

$$\phi_+\left(t, x_t^A\right) = \frac{1}{2\delta\sqrt{(1-t)\pi}} \int_{e_0}^{+\infty} e^{\frac{-(x_t^A - x_0^A)^2}{2\delta^2(1-t)}} e^{\frac{p_1(e_0 - x_0^A)\theta_t}{\delta^2}} dx_0^A, \tag{32}$$

$$\xi_-\left(t, x_t^A\right) = \frac{1}{2\delta\sqrt{(1-t)\pi}} \int_{-\infty}^{e_0} e^{\frac{-(x_t^A - x_0^A)^2}{2\delta^2(1-t)}} \frac{-2(x_t^A - x_0^A)}{2\delta^2(1-t)} dx_0^A, \tag{33}$$

$$\xi_+\left(t, x_t^A\right) = \frac{1}{2\delta\sqrt{(1-t)\pi}} \int_{e_0}^{+\infty} e^{\frac{-(x_t^A - x_0^A)^2}{2\delta^2(1-t)}} e^{\frac{p_1(e_0 - x_0^A)\theta_t}{\delta^2}} \frac{-2(x_t^A - x_0^A)}{2\delta^2(1-t)} dx_0^A, \tag{34}$$

together with:

$$F_1 = \frac{1}{2\delta}(1 + \operatorname{erf}(B(x_{0+}^A, 0)) + e^{D\left(x_{0+}^A, 0\right)}(1 - \operatorname{erf}(A\left(x_{0+}^A, 0\right)))). \tag{35}$$

$$S_0\left(t, x_t^A\right) = \frac{\delta^2}{\theta_t}\left(\frac{\xi_-\left(t, x_t^A\right) F_1 + \xi_+\left(t, x_t^A\right) F_1}{\phi_-\left(t, x_t^A\right) F_1 + \phi_+\left(t, x_t^A\right) F_1}\right). \tag{36}$$

3.3 Model 2 - with CDM

3.3.1 General Setup

The structure of this second Model resembles the one of Model 1, the principal difference is that now every firm implements a CDM project in period 0; the aggregate initial cost of these projects is C and they deliver an aggregate return of γ[26] in every instant t.

The total expected emissions x_t is now influenced also by the carbon returns of the projects in every instant of the aforementioned subperiod, namely γ:

$$x_t = y_t - \int_0^t (u_s + \gamma) ds. \tag{37}$$

Following directly from the aforementioned definition only the final expectation appears changed:

$$x_T = y_T - \int_0^T (u_s + \gamma) ds, \tag{38}$$

[26] For the specification of γ see Equation (3).

while the initial one remains the same as Equation (7).

Therefore an infinitesimal increment in x_t will be:

$$dx_t = -(u_t + \gamma)dt + \delta dW_t \ . \tag{39}$$

Also in Model 2 we have a differentiation between actual emissions - Equation (37) - and accounted emissions that, generalized over a compliance period $[m, n]$, now read at the end of the period:[27]

$$x_n^A = x_n - x_m - \sum_{i=1}^{m} \max\left[0, e_{i-1} - (x_i - x_{i-1})\right], \tag{40}$$

and at the beginning of the next compliance period:[28]

$$x_{n+}^A = -\sum_{i=1}^{n} \max\left[0, e_{i-1} - (x_i - x_{i-1})\right]. \tag{41}$$

This condition will apply also to all the subsequent intermediate x_t^A in the next compliance period $[n, o]$,with $t \in (n, o)$:

$$x_t^A = x_t - x_n - \sum_{i=1}^{n} \max\left[0, e_{i-1} - (x_i - x_{i-1})\right]. \tag{42}$$

As in the previous model, the aggregate cost function is assumed increasing-in time, but now it presents an additional component linked with the costs associated to CDM projects:

$$C\left(t, u_t\right) = -\frac{1}{2\theta_t} u_t^2 - \frac{C}{t+1} \ . \tag{43}$$

The objective function can be formulated as in Model 1, following Equation (15). With final conditions expressed as in Equation (16).

We are again in front of a partial-equilibrium problem as it takes in consideration only the carbon market.

3.3.2 Solution for T=2

The same methodology as before is applied and the analysis is conducted for $T = 2$ for ease of exposition.[29]

[27] In the following three equations, the summation starts from 1 if the firm never paid a penalty before compliance period n; otherwise it should start in the period after the last penalty payment. To keep notation lighter, we write as if the former is always the case. The modification to the general case is straightforward.

[28] Note that x_{n+}^A is a function of x_i^A, $i = 1, .., n$.

[29] Remember that this procedure can be applied to every desired T.

In this scenario the problem can be reformulated as follows:

$$\max_{(u_t)_{t\in[0,2]}} E_0 \left[\int_0^2 \left(-\frac{1}{2\theta_t} u_t^2 - \frac{C}{t+1} \right) dt + \sum_{i=1}^{2} \min \left[0, p_i(e_{i-1} - x_i^A) \right] \right]. \tag{44}$$

Since the fixed cost C is not time dependent, we can subtract it from the expectation and add it back only at the end when explicit solutions for $V(\cdot)$ are found. Therefore Equation (44) can be rewritten as:

$$\max_{(u_t)_{t\in[0,2]}} E_0 \left[\int_0^2 -\frac{1}{2\theta_t} u_t^2 dt + \sum_{i=1}^{2} \min \left[0, p_i(e_{i-1} - x_i^A) \right] \right] - C \ln 3. \tag{45}$$

Equation (21) applies also to Model 2, together with the already discussed discontinuity of $V(\cdot)$. On account of the resulting expected change in $V(\cdot)$ from Equations (38) and (39):

$$E(dV) = V^{(t)}dt - (u_t + \gamma)V^{(x)}dt + \frac{1}{2}\delta^2 V^{(xx)}dt \ , \tag{46}$$

the Hamilton-Jacobi-Bellman equation results in:

$$0 = \max_{u_t} \left[-\frac{1}{2\theta_t} u_t^2 + V^{(t)} - (u_t + \gamma)V^{(x)} + \frac{1}{2}\delta^2 V^{(xx)} \right] . \tag{47}$$

Using the FOC of this maximization problem we arrive to the optimal abatement rate in Equation (24) which coincides with the one found in Model 1; this is due to the fact that the presence of CDM projects does not impact directly on the abatement rate that has to be chosen by the firm, not modifying consequently the optimal rate. Furthermore, remembering that the equilibrium carbon price equals the marginal abatement cost, we obtain S^* as in Equation (25); once more this optimal level is not impacted directly by the presence of CDM projects, since it ultimately relies on the optimal abatement rate.

We conduct our analysis through a dynamic programming backward approach following exactly the same steps previously faced. The second part of Equation (44) coincides with the problem stated by Seifert et al (2008), so we achieve an easy derivation of the solution which coincides with solution (3.2) of the paper[30] modified for the presence of γ in the expected emission function. Finding an explicit value for $V_1\left(1, x_{1+}^A\right)$ allows us to rewrite the maximization problem (44) as in Equation (19) with a new final condition expressed in Equation (20). Solving this new maximization problem lead us to $V_0(0, x_0^A)$ which now appears to be implicitly formulated.

[30] See Seifert et al (2008), pag. 185, Equation (3.2).

The optimal carbon prices for the two compliance periods respectively S_1, S_0,[31] and calling:

$$A_{CDM}(x_t^A, t) = \frac{e_1 - (x_{t+}^A + \gamma t) + \gamma + p_2\theta_1(2-t)}{\delta\sqrt{(2-t)}}, \tag{48}$$

$$B_{CDM}(x_t^A, t) = \frac{e_1 - (x_{t+}^A + \gamma t) + \gamma}{\delta\sqrt{(2-t)}}, \tag{49}$$

$$D_{CDM}\left(x_t^A, t\right) = \frac{p_2(p_2\theta_1 + 2(e_1 - (x_{t+}^A + \gamma t) + \gamma))}{2\delta^2}. \tag{50}$$

are now:

$$S_{1,CDM}(t, x_t^A) = \frac{p_2 e^{D_{CDM}\left(x_{t+}^A, t\right)}[1 - \mathrm{erf}(A_{CDM}\left(x_{t+}^A, t\right))]}{1 + \mathrm{erf}(B_{CDM}\left(x_{t+}^A, t\right)) + e^{D_{CDM}\left(x_{t+}^A, t\right)}[1 - \mathrm{erf}(A_{CDM}\left(x_{t+}^A, t\right)]}, \tag{51}$$

and, defining:

$$\hat{C} = C \ln\frac{3}{2}, \tag{52}$$

$$\phi_{-,CDM}\left(t, x_t^A\right) = \frac{e^{-\hat{C}\frac{\theta_1}{\delta^2}}}{2\delta\sqrt{(1-t)\pi}} \int_{-\infty}^{e_0} e^{\frac{-(x_t^A + \gamma t - x_0^A)^2}{2\delta^2(1-t)}} dx_0^A, \tag{53}$$

$$\phi_{+,CDM}\left(t, x_t^A\right) = \frac{e^{-\hat{C}\frac{\theta_1}{\delta^2}}}{2\delta\sqrt{(1-t)\pi}} \int_{e_0}^{+\infty} e^{\frac{-(x_t^A + \gamma t - x_0^A)^2}{2\delta^2(1-t)}} e^{\frac{p_1(e_0 - x_0^A)\theta_t}{\delta^2}} dx_0^A, \tag{54}$$

$$\xi_{-,CDM}\left(t, x_t^A\right) = \frac{e^{-\hat{C}\frac{\theta_1}{\delta^2}}}{2\delta\sqrt{(1-t)\pi}} \int_{-\infty}^{e_0} e^{\frac{-(x_t^A + \gamma t - x_0^A)^2}{2\delta^2(1-t)}} \frac{-2(x_t^A + \gamma t - x_0^A)}{2\delta^2(1-t)} dx_0^A, \tag{55}$$

$$\xi_{+,CDM}\left(t, x_t^A\right) = \frac{e^{-\hat{C}\frac{\theta_1}{\delta^2}}}{2\delta\sqrt{(1-t)\pi}} \int_{e_0}^{+\infty} e^{\frac{-(x_t^A + \gamma t - x_0^A)^2}{2\delta^2(1-t)}} e^{\frac{p_1(e_0 - x_0^A)\theta_t}{\delta^2}} \frac{-2(x_t^A + \gamma t - x_0^A)}{2\delta^2(1-t)} dx_0^A, \tag{56}$$

together with:

$$F_{1,CDM} = \frac{1}{2\delta}(1 + \mathrm{erf}(B_{CDM}(x_{0+}^A, 0)) + e^{D_{CDM}\left(x_{0+}^A, 0\right)}(1 - \mathrm{erf}(A_{CDM}\left(x_{0+}^A, 0\right)))). \tag{57}$$

$$S_{0,CDM}\left(t, x_t^A\right) = \frac{\delta^2}{\theta_t}\left(\frac{\xi_{-,CDM}\left(t, x_t^A\right) F_{1,CDM} + \xi_{+,CDM}\left(t, x_t^A\right) F_{1,CDM}}{\phi_{-,CDM}\left(t, x_t^A\right) F_{1,CDM} + \phi_{+,CDM}\left(t, x_t^A\right) F_1}\right). \tag{58}$$

4 Numerical validation

4.1 Simulation A

Now that we have the quantitative tools to analyze the problem, we want to put them in practice with some numerical simulation attempts. As already stated in the introduction, it

[31] For the explicit derivation of all the following expressions see Appendix A.

is our intention to verify numerically the relevance of the CDM presence. Choosing, like in Seifert (2008),[32] some numbers for the parameters included in the analysis, we were able to compare Model 1 and Model 2 scenarios. In particular we imposed in both Models:

δ	y_0	e_0	e_1	p_1	p_2
288	6240	150	100	70	130

(Table 2.)

and an increasing[33] sequence of θ_t :

$$\begin{aligned}\theta_t &= 0.24 \ \text{ for } t \in [0,1]\,,\\ \theta_t &= 0.35 \ \text{ for } t \in \left(1, \frac{1}{2}\right),\\ \theta_t &= 0.40 \ \text{ for } t \in \left[\frac{1}{2}, 2\right].\end{aligned}$$

while only for the case with CDM:

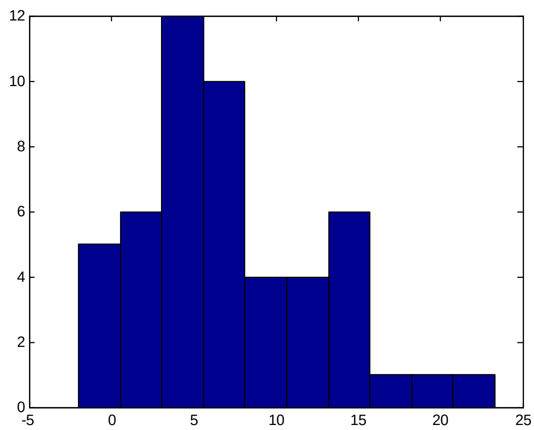
C	γ
25	2.5

(Table 3.)

with $\alpha = 1$ and $g = 2.5$.

The results obtained are the following:[34]

Final Emissions $\mathbf{x}_2$ (Model 1- Model 2)



Variability Indicators	
Average	6.89
St. Deviation	5.67
Min. Value	-1.98
Max. Value	23.31
1st Quartile	3.21
Median Value	5.99
3rd Quartile	10.81

(Exhibit 1)

With a positive percentage change in $V(x_0, 0)$ from Model 1 to Model 2 of 3.15%.

As it can be noticed:

$$\bar{x}_2\,(with\ CDM) < \bar{x}_2\,(without\ CDM)\,,$$

[32] The analysis was mainly inspired by Seifert et. al (2008) Table 2, pag 186.

[33] This reflects the assumption of increasing marginal costs for short-term abatement measures.

[34] These numbers are the results obtained with 50 realizations of the infinite number of Brownian Motion that we can simulate. The simulations are performed in Excel with the Brownian Motion discretized over 100 points - each period -, and integrals are computed numerically by the rectangles' method.

meaning that Model 2 scenario will be preferred to the one of Model 1 by the Policy Maker because of the lower level of emissions in the environment; and

$$V(x_0, 0)\,(with\ CDM) > V(x_0, 0)\,(without\ CDM)\,,$$

meaning that the firms are better off in Model 2, if compared to Model 1, because they experience a higher level of wealth there. Therefore the two requirements - one imposed by the Policy Maker and the other by the firms taking part to the Scheme - are simultaneously satisfied, and CDM projects are always implemented under the aforementioned conditions.

4.2 Simulation B

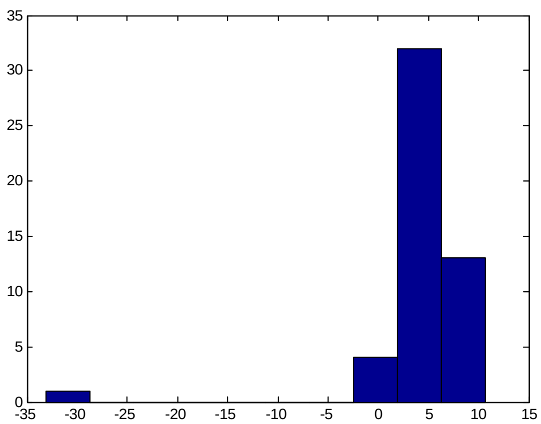
Now a static setting - the same elaborated by Seifert and extended in a multiperiod scenario- is taken into consideration, with non-changing number of permits and level of penalty:

δ	y_0	e_0	e_1	p_1	p_2
288	6240	150	150	70	70

(Table 5.)

and a constant $\theta = 0.24$. Maintaining unaltered the condition for Model 2, we obtain:

Final Emissions $\mathbf{x}_2$ (Model 1 - Model 2)



Variability Indicators	
Average	4.58
St. Deviation	5.91
Min. Value	-33.08
Max. Value	10.68
1^{st} Quartile	3.91
Median Value	5.29
3^{rd} Quartile	6.35

(Exhibit 2)

With a positive percentage change in $V(x_0, 0)$ from Model 1 to Model 2 of 3.20%.

As it can be expected, in the static case the introduction of CDM projects is still worth for the Policy Maker and the firms because it ensures lower carbon emissions and higher wealth level. Nevertheless, through a direct comparison with simulation A juxtaposing absolute values instead of differences and percentual changes, we realize that, while firms prefer Simulation B case - the reason is straightforward: in this scenario the cost of short-term abatement measures remains constant, together with penalty and number of permits immitted in the market, all throughout the Phase therefore the total cost paid by firms to reduce carbon emissions is overall lower -, the Policy Maker favors Simulation A case where absolute lower levels of emissions in the environment are reached.

To sum up the effectiveness of CDM projects' introduction is verified in all the Model specifications since it positively impacts both on the firms' wealth and on the overall carbon emissions' level, but we mostly agree with the fact that, in order to maximize their social value, they should be used in combination with a progressively increasing penalty system - as the one in Simulation A, which mostly resembles the real world case - where short-term abatement measures' cost increases with time, the penalty becomes more stringent year by year and the number of carbon permits immited on the market is similarly reduced. The regulator should consider, at all the effects, these projects an alternative way - additional to the number of carbon permits emitted - of impacting on the carbon market and possibly influencing carbon price's fluctuations.[35]

5 Conclusion and Remarks

The aim of this work has been to study the introduction of CDM projects in the European carbon market through a theoretical model. Using dynamic programming tools, we were able to derive the intertemporal choices of the social planner about emission processes of firms included in the Scheme. We derived such an intertemporal analysis for both the cases in which CDM were present or not. The dynamic problem was solved for a 2-period scenario that allowed for a simpler manual resolution, but it can be extended to whichever Phase's length with the help of specific mathematical software.

We concluded that the presence of long term projects is justified and can be exploited, as a policy-making instrument, at most in the cases where the regulation parameters reflect a punishment mechanism that penalizes more emissions at the end of the Phase than the ones at the beginning. We strongly think that such a structure mirrors the one actually in use in the EU-ETS, therefore we consider this conclusion valid and applicable to the European reality.

Recently the European Trading Scheme has faced some challenges relating to the drop of carbon price. In April 2013 this price reached a minimum peak of 3.05 euro, putting

[35]In Appendix C we will show carbon price reactions to changes in the different parameters under analysis.

in danger the very survival of the overall System. In fact if the price would continue to decrease, reaching the minimum admissible threshold of 0, there will be no more need of a carbon market: every market is useless if the good traded does not have a price. It is now clear that the Policy Maker should find new and more effective ways to influence the carbon price and take under control the market's tendencies.

Following the results of our study we recommend the Regulator to take in consideration CDM projects as a way of directly influencing the aforementioned variables. Our feeling is that those projects have been disregarded in the previous Phases of the Scheme, a fact that could be explained by the short duration of such Phases and the consequent complexity in developing complete projects. However, due to the length of the actual Phase, we consider the reappraisal of CDM projects, and macro-policies related to them, as a fundamental and unavoidable choice for the Policy Maker. All throughout the paper we underlined diverse factors that can be manipulated in order to achieve predefined goals: the use of the conversion rate between permits originated in a project and permits accepted in the market - the so called α-variable -; the cost of the CDM project; the recognized carbon returns of the project - strictly related to the α-policy -. Appraising these instruments could be a possible way of escaping the actual unwanted situation, bringing the Scheme back to a healtier and more effective functioning.

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7 Appendices

7.1 *Appendix A - Extensive Solution of the Maximization Problem in 2 Periods*

Since Model 1 and 2 are very similar for what concerns the objective function to maximize and the procedure used to solve the problem, we give the explicit resolution for Model 2 conscious that the same approach has been used for Model 1.

In Section 3 we already stated the initial problem in Equation (45);[36] making use of Dynamic Programming, we take the final part considering only the problem:

$$\max_{(u_t)_{t\in[1,2]}} E_1\left[\int_1^2 -\frac{1}{2\theta_t}u_t^2 dt + \min\left[0, p_2(e_1 - x_2^A)\right]\right] . \tag{59}$$

the final condition is espressed at the final period $T = 2$. Through the HJB Equation (47) we are able to compute the optimal abatement rate (24), and adding it back to the HJB Equation we arrive at the characteristic Partial Differential Equation - from now on PDE -:

$$V^{(t)} = -\frac{1}{2}\delta^2 V^{(xx)} - \frac{1}{2}\theta_t V^{2(x)} + \gamma V^{(x)} \ , \tag{60}$$

which is very similar to the one found in Seifert et. al (2008) with the exception of a new term in $V^{(x)}$ linked with γ.

Our goal is to reduce this PDE to a standard Heat Equation of the type:

$$\begin{gathered} f^{(t)} = \varphi f^{(xx)}, \\ f\left(x_1^A, 1\right) = g\left(x_1^A\right) . \end{gathered} \tag{61}$$

to be able to solve it through the standard solution expressed as:

$$f\left(x, t\right) = \frac{1}{\sqrt{4\pi\varphi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4\varphi t}} g(y)dy \ . \tag{62}$$

In order to follow this structure we need to apply to Equation (60) three transormations.

Transformation 1:

$$V(x_t^A, t) = \frac{\delta^2}{\theta_t}\ln(v(x_t^A, t)) \ , \tag{63}$$

with the inverse:

$$v(x_t^A, t) = e^{\frac{\theta_t}{\delta^2}V(x_t^A, t)} \ . \tag{64}$$

Transformation 2:

$$v(x_t^A, t) = u(z_t, t) \ , \tag{65}$$

[36] We take as given the already discussed - in Section 4.1 - question of the removal of $\frac{C}{t+1}$ from the maximization problem.

where $z_t = x_t + \gamma t$; with the inverse:

$$u(z_t, t) = v(z_t - \gamma t, t) \ . \tag{66}$$

Transformation 3:

$$u(z_t, t) = \tilde{u}(z_\tau, \tau) \ , \tag{67}$$

where $\tau = T - t$; with the inverse:

$$\tilde{u}(z_\tau, \tau) = u(z_t, T - \tau) \ . \tag{68}$$

From Transformation (63) we obtain the following values for the derivatives of V with respect to t and x_t - using the fact that θ_t is taken as exogenous - , respectively $V^{(t)}, V^{(x)}, V^{(xx)}$:[37]

$$\begin{aligned} V^{(t)} &= \frac{\delta^2}{\theta_t}\frac{v^{(t)}}{v}, \\ V^{(x)} &= \frac{\delta^2}{\theta_t}\frac{v^{(x)}}{v}, \\ V^{(xx)} &= \frac{\delta^2}{\theta_t}[\frac{v^{(xx)}}{v} - (\frac{v^{(x)}}{v})^2]. \end{aligned}$$

which plugged into PDE (60) gives:

$$v^{(t)} = -\frac{1}{2}\delta^2 v^{(xx)} + \gamma v^{(x)} \ . \tag{69}$$

From Transformation (65) we obtain the following values for the derivatives of v with respect to t and x_t, respectively $v^{(t)}, v^{(x)}, v^{(xx)}$:

$$\begin{aligned} v^{(t)} &= \gamma u^{(z)} + u^{(t)}, \\ v^{(x)} &= u^{(z)}, \\ v^{(xx)} &= u^{(zz)}. \end{aligned}$$

which plugged into PDE (69) gives:

$$u^{(t)} = -\frac{1}{2}\delta^2 u^{(zz)} \ , \tag{70}$$

which is almost Equation (61).

In order to obtain exactly the Heat Equation we apply Transformation (67), from which we obtain the following values for the derivatives of u with respect to t and z_t, respectively $u^{(t)}, u^{(x)}, u^{(xx)}$:

$$\begin{aligned} u^{(t)} &= -\tilde{u}^{(\tau)}, \\ u^{(x)} &= \tilde{u}^{(z)}, \\ u^{(xx)} &= \tilde{u}^{(zz)}. \end{aligned}$$

[37] $V^{(xx)}$ represents the second derivative of $V(\cdot)$ with respect to x^A.

which plugged into PDE (70) gives:

$$\tilde{u}^{(\tau)} = \frac{1}{2}\delta^2 \tilde{u}^{(zz)} \ , \tag{71}$$

which is exactly Equation (61), with $\varphi = \frac{1}{2}\delta^2$.

Since, in order to solve the Heat Equation we need an initial condition, we use our bundary condition (18). To be consistent we apply to this condition all the transformations applied to the PDE. After Transformation (63) it reads:

$$P(x_2^A, 2) = e^{\min\left[0, p_2(e_1 - x_2^A)\right]\frac{\theta_2}{\delta^2}} . \tag{72}$$

Applying, then, Transformation (65) we obtain:

$$P(z_2, 2) = e^{\min[0, p_2(e_1 - (z_2 - 2\gamma))]\frac{\theta_2}{\delta^2}} . \tag{73}$$

The final condition is altered also by Transformation (67), since it is expressed in T and we know that:

$$t = T \Leftrightarrow \tau = 0 \ , \tag{74}$$

therefore it should be expressed at the beginning of the period under consideration; since we are analyzing separately the period $[1, 2]$, as it would be a unique one $\tau = 0$ is actually translatable in $\tau = 1$.Equation (73) becomes:

$$P(z_1, 1) = e^{\min[0, p_2(e_1 - z_1 + \gamma)]\frac{\theta_2}{\delta^2}} \tag{75}$$

Combining Equation (71) with the initial condition (75) we arrive at the solution:

$$\tilde{u}(z_\tau, \tau) = \frac{1}{2\delta\sqrt{\tau}} \left[1 + \mathrm{erf}(B'(z_\tau, \tau)) + e^{D'(z_\tau, \tau)}(1 - \mathrm{erf}(A'(z_\tau, \tau)))\right] \ , \tag{76}$$

where:

$$A'(z_\tau, \tau) = \frac{e_1 - z_\tau + \gamma + p_2\tau\theta_2}{\delta\sqrt{\tau}} \ , \tag{77}$$

$$B'(z_\tau, \tau) = \frac{e_1 - z_\tau + \gamma}{\delta\sqrt{\tau}} \ , \tag{78}$$

$$D'(z_\tau, \tau) = \frac{p_2(\theta_2 p_2 + 2(e_1 - z_\tau + \gamma))}{2\delta^2} \ . \tag{79}$$

Now we need to apply all the inverse transformations to express the solution in $V(x_t^A, t)$. We first apply to Equation (76) the Transformation (68) obtaining:

$$u(z_t, t) = \frac{1}{2\delta\sqrt{2-t}}[1 + \mathrm{erf}(B''(z_t, t)) + e^{D''(z_t, t)}(1 - \mathrm{erf}(A''(z_t, t)))], \tag{80}$$

where:

$$A''\left(z_t,t\right)=\frac{e_1-z_t+\gamma+p_2\theta_2(2-t)}{\delta\sqrt{(2-t)}}, \tag{81}$$

$$B''\left(z_t,t\right)=\frac{e_1-z_t+\gamma}{\delta\sqrt{(2-t)}}\ , \tag{82}$$

$$D''\left(z_t,t\right)=\frac{p_2(\theta_2p_2+2(e_1-z_t+\gamma))}{2\delta^2}\ . \tag{83}$$

Then we have to apply Transformation (66) to Equation (80) which gives:

$$v(x_t^A,t)=\frac{1}{2\delta\sqrt{2-t}}[1+\mathrm{erf}(B_{CDM}(x_t^A,t))+e^{D_{CDM}(x_t^A,t)}(1-\mathrm{erf}(A_{CDM}(x_t^A,t)))]\ , \tag{84}$$

Finally it remains to apply Transformation (64) to Equation (84) in order to get the final solution:[38]

$$\begin{aligned} V(x_t^A,t) &= \frac{\delta^2}{\theta_t}\ln\left[\begin{array}{c}\frac{1}{2\delta\sqrt{2-t}}[1+\mathrm{erf}(B_{CDM}(x_t^A,t))+\\ e^{D_{CDM}(x_t^A,t)}*\\ (1-\mathrm{erf}(A_{CDM}(x_t^A,t)))]\end{array}\right] \\ &\quad -C[\ln 3-\ln(t+1)], \end{aligned} \tag{85}$$

where $A(\cdot),B(\cdot)$ and $D(\cdot)$ are specified in Equations (48), (49) and (50). At this point, in order to obtain the optimal solution (25) we need to differentiate (85) with respect to x_t^A. It can be easily shown that this solution coincides with Equation (51), which gives us the instantaneous price in every moment of the interval $[1,2]$.

In order to proceed with dynamic programming we need the value function evaluated at $t=1$; from the previous discussion we know that this is a discontinuity point for $V(\cdot)$ which can take two different values depending on if it is evaluated at x_1^A or x_{1+}^A. Since we are using a dynamic programming approach and this final condition comes directly from the maximization over the period $[1,2]$ we should use $V\left(x_{1+}^A,1\right)$:

$$V(x_{1+}^A,1)=\frac{\delta^2}{\theta_1}\ln\left[\begin{array}{c}\frac{1}{2\delta}[1+\mathrm{erf}(B_{CDM}(x_{1+}^A,1))+\\ e^{D_{CDM}(x_{1+}^A,1)}*\\ (1-\mathrm{erf}(A_{CDM}(x_{1+}^A,1)))]\end{array}\right]-\hat{C}.^{39} \tag{86}$$

Once obtained this value we need to plug it back in Equation (59) in place of the last part of the Equation. The maximization problem can be rewritten as:

$$\max_{(u_t)_{t\in[0,1]}} E_0\left[\int_0^1-\frac{1}{2\theta_t}u_t^2dt+\min\left[0,p_1(e_0-x_1^A)\right]+V\left(x_{1+}^A,1\right)\right]\ , \tag{87}$$

[38] Remember that we have previously removed from the maximization problem the parameter -$\frac{C}{t+1}$, so now we have to add back to the value function $\int_t^2-\frac{C}{x+1}dx=-C[\log 3-\log(t+1)]$.

where the new terminal condition is:

$$P(x_1^A, 1) = \min\left[0, p_1(e_0 - x_1^A)\right] + V\left(x_{1+}^A, 1\right) \ . \tag{88}$$

Notice that this final condition contains both x_1^A and x_{1+}^A: the compliance condition is expressed in terms of x_1^A, while the maximum value $V(\cdot)$ depends on x_{1+}^A; we should not forget that x_{1+}^A is a function of x_1^A.

Given the fact that from Equation (59) nothing has changed, except for the final condition, also the PDE will remain the same as in Equation (60). Following exactly the same passages as before we only have to recompute the different final condition under all the transformations. After Transformation (63) we will obtain:

$$P(x_1^A, 1) = e^{\min\left[0, p_1(e_0 - x_1^A)\right]\frac{\theta_1}{\delta^2}} \begin{bmatrix} \frac{1}{2\delta}[1 + \text{erf}(B_{CDM}(x_{1+}^A, 1))+ \\ e^{D_{CDM}(x_{1+}^A, 1)} * \\ (1 - \text{erf}(A_{CDM}(x_{1+}^A, 1)))] \end{bmatrix} e^{-\hat{C}\frac{\theta_1}{\delta^2}}. \tag{89}$$

After Transormation (65) it will read as:[40]

$$P(z_1, 1) = e^{\min[0, p_1(e_0 - z_1 + \gamma)]\frac{\theta_1}{\delta^2}} \begin{bmatrix} \frac{1}{2\delta}[1 + \text{erf}(B_{CDM}(z_{1+} - \gamma, 1))+ \\ e^{D_{CDM}(z_{1+} - \gamma, 1)} * \\ (1 - \text{erf}(A_{CDM}(z_{1+} - \gamma, 1)))] \end{bmatrix} e^{-\hat{C}\frac{\theta_1}{\delta^2}}. \tag{90}$$

After Transformation (67), remembering the condition (74), it will become:

$$P(z_0, 0) = e^{\min[0, p_1(e_0 - z_0)]\frac{\theta_1}{\delta^2}} \begin{bmatrix} \frac{1}{2\delta}[1 + \text{erf}(B_{CDM}(z_{0+}, 0))+ \\ e^{D_{CDM}(z_{0+}, 0)} * \\ (1 - \text{erf}(A_{CDM}(z_{0+}, 0)))] \end{bmatrix} e^{-\hat{C}\frac{\theta_1}{\delta^2}}. \tag{91}$$

We are now ready to compute the solution of the Heat Equation, neverthless this time we are unable to obtain it explicitly; in particular we arrive at:

$$\begin{aligned} \tilde{u}(z_\tau, \tau) \quad = \quad & \frac{e^{-\hat{C}\frac{\theta_1}{\delta^2}}}{2\delta\sqrt{\pi\tau}}\Big(\int_{-\infty}^{e_0} \left\{ e^{\frac{-(z_\tau - z_0)^2}{2\delta^2\tau}} \begin{bmatrix} \frac{1}{2\delta}[1 + \text{erf}(B(z_{0+}, 0))+ \\ e^{D_{CDM}(z_{0+}, 0)} * \\ (1 - \text{erf}(A(z_{0+}, 0)))] \end{bmatrix} \right\} dz_0 + \\ & \int_{e_0}^{+\infty} \left\{ e^{\frac{-(z_\tau - z_0)^2}{2\delta^2\tau}} e^{p_1(e_0 - z_0)\frac{\theta_1}{\delta^2}} \begin{bmatrix} \frac{1}{2\delta}[1 + \text{erf}(B_{CDM}(z_{0+}, 0))+ \\ e^{D_{CDM}(z_{0+}, 0)} * \\ (1 - \text{erf}(A_{CDM}(z_{0+}, 0)))] \end{bmatrix} \right\} dz_0\Big) \ . \end{aligned} \tag{92}$$

In order to obtain a solution for $V(x_t^A, t)$, we need to apply all the inverse transormations; we start with Transformation (68),[41] obtaining:

$$\begin{aligned} u(z_t, t) \quad = \quad & \phi_{-,CDM}\left(t, z_t^A\right) \frac{1}{2\delta}[1 + \text{erf}(B_{CDM}(z_{0+}, 0)) + e^{D_{CDM}(z_{0+}, 0)}(1 - \text{erf}(A_{CDM}(z_{0+}, 0)))] \\ & + \phi_{+,CDM}\left(t, z_t^A\right) \frac{1}{2\delta}[1 + \text{erf}(B_{CDM}(z_{0+}, 0)) + e^{D_{CDM}(z_{0+}, 0)}(1 - \text{erf}(A_{CDM}(z_{0+}, 0)))]. \end{aligned} \tag{93}$$

[40] Since we want to mantain the distinction between x_1^A and x_{1+}^A we will use z_1 for the transformed x_1^A and z_{1+} for the transformed x_{1+}^A.

[41] Remeber that this Transformation applied on the final condition in T gave us the condition expressed in $(z_0, 0)$; now we should bring them back into (z_T, T), which in this case is equivalent to say $(z_1, 1)$.

Then we apply Transformation (66), obtaining:

$$v(x_t^A, t) = \phi_{-,CDM}\left(t, x_t^A\right) F_{1,CDM} + \phi_{+,CDM}\left(t, x_t^A\right) F_{1,CDM}. \tag{94}$$

Finally, applying Transformation (64) we arrive at the solution[42]:

$$V(x_t^A, t) = \frac{\delta^2}{\theta_t} \ln[\phi_{-,CDM}\left(t, x_t^A\right) F_{1,CDM} + \phi_{+,CDM}\left(t, x_t^A\right) F_{1,CDM}] - C[\ln 2 - \ln(t+1)]. \tag{95}$$

At this point, in order to obtain the optimal solution (25) we need to differentiate (95) with respect to x_t. It can be easily shown that this solution coincides with Equation (58), which gives us the instantaneous price in every moment of the interval $[0, 1]$.

7.2 *Appendix B - Equivalence between Single and Aggregate Problems*

This Appendix is mainly inspired by Seifert et al (2008), Appendix A. We want to give just a brief overview of the main ideas behind the equivalence of joint cost problem and individual cost problem, which are also recalled in this work project. Nevertheless we recommend to refer to Seifert's work to deepen the topic.

Trading among firms is allowed, but when considering the aggregate problem it doesn't impact anymore on the aggregate objective function, because of the market clearing condition. In particular if N firms are considered in the market we would have N different individual objective functions, each associated to the peculiar parameters of the firm i:

$$\max_{(u_{i,t}; z_{i,t})_{t\in[0,T]}} E_0 \left[\int_0^T e^{-rt} C_i\left(t, u_{i,t}\right) dt - \int_0^T e^{-rt} S(t) z_{i,t} + e^{-rt} P\left(x_{i,T}^A\right) \right], \tag{1b}$$

where $z_{i,t}$ represents the number of permits exchanged - positive if bought, and negative if sold - of firm i in time t. In this individual formulation each firm has to decide in addition to the optimal level of abatement rate for each period also the optimal level of trading for each period. The market clearing condition associate to this trade of permits reads:

$$\sum_{i=1}^{N} z_{i,t}^* = 0 \ . \tag{2b}$$

As far as only the firms in the scheme participate to the trading the cumulative level of

[42] Remember that we had previously removed from the maximization problem the parameter -$\frac{C}{t+1}$, so now we have to add back to the value function $\int_t^1 -\frac{C}{x+1} dx = -C[\log 2 - \log(t+1)]$

exchange should be 0.[43] Once we aggregate the problem we obtain:

$$\max_{(u_{i,t};z_{i,t})_{t\in[0,T]}} E_0 \left[\sum_{i=1}^{N} \int_0^T e^{-rt} C_i\left(t, u_{i,t}\right) dt - \sum_{i=1}^{N} \int_0^T e^{-rt} S(t) z_{i,t} + \sum_{i=1}^{N} e^{-rt} P\left(x_{i,T}^{A}\right) \right], \tag{3b}$$

which together with Equation (2b) becomes:

$$\max_{(u_{i,t})_{t\in[0,T]}} E_0 \left[\sum_{i=1}^{N} \int_0^T e^{-rt} C_i\left(t, u_{i,t}\right) dt + \sum_{i=1}^{N} e^{-rt} P\left(x_{i,T}^{A}\right) \right], \tag{4b}$$

which can be rewritten as Equation (15) if we assume:

$$C(t, u_t) = \sum_{i=1}^{N} C_i(t, u_{i,t}), \tag{5b}$$

and:

$$P\left(x_{i,T}^{A}\right) = \sum_{i=1}^{N} P\left(x_{i,T}^{A}\right). \tag{6b}$$

7.3 *Appendix C - Sensitivity Analysis*

We want to devote this paragraph to the analysis of which impact the change in the main parameters of the Model can have on the path of CO_2 price. Inspired by the sensitivity analysis conducted by Seifert,[44] we want to study the reaction of the price sequence $S(x_t, t)$ to changes in fundamental parameters when CDM projects are taken into account.[45] We used as benchmark the numerical case of Model 2 already discussed in previous Sections, and we let the parameters change to understand the carbon price reactions to these changes - our aim was to elaborate just an illustrative exercise, not an exhaustive comparative statics study -.

Changing Parameter	Benchmark Case	Study Case
α	1	1.5
θ	$0.24 \in [0,1], 0.35 \in (1,2]$	$0.2 \in [0,1], 0.31 \in (1,2]$
p_1	70	100
p_2	130	150
e_0	150	160
e_1	100	110

(Table 1c)

In order to have a broader view on the topic we analyzed for every case 50 scenarios - given by 50 different simulations of the Brownian Motion -:[46]. The study was conducted in

[43] This condition is not very close to reality since admits to participate to the trade only firms belonging to the scheme, we all know that in reality also other actors can buy or sell permits: for example individual investors with speculative aims. In this case the market clearing condition does not hold anymore.

[44] Seifert et al (2008), Table 2 pag. 186.

[45] Remember that, in Seifert work, long-term projects were not included in the analysis.

[46] The Brownian Motions were simulated and chosen randomly with the help of a dedicate software.

the same two-periods scenario - as in Section 3 -, but it can be extended to every desired horizon. To follow Seifert's structure we underlined the impact of a parameter's increase on $S(x_0, 0)$ and $S(x_1, 1)$ which are the carbon prices at the begging respectively of the first and the second compliance period.

We obtained the following results:

Changing Parameter (increase)	$S(x_0, 0)$ **change**	$S(x_1, 1)$ **change**
α	$+$	$-$
$\frac{1}{\theta}$	$-$	$+$
p_1	$+$	$+$
p_2	$-$	$+$
e_0	$-$	$+$
e_1	$+$	$-$

(Table 2c)

As it can be noticed, C is not included in the parameters to be varied. This derives from the fact that, even if there are policies that can influence C - for example public funding incentivizing CDM projects can decrease the cost of those projects -, this parameter does not directly influence the abatement level chosen and consequently the CO_2 price. It influences only the firms' wealth level giving an incentive, or viceversa a disincentive, to implement the projects. In the case of "low-cost" projects firms will face an higher level of initial wealth, therefore deciding to implement a CDM.

In analyzing Table (2.c) it is clear that an increase in the numbers of permits immitted in the market has the same negative effect as in Seifert's analysis; the only difference is that here we have a multiperiod model therefore the increase of permits will negatively impact only on the carbon price of the subsequent compliance period and not on the overall price.

Another analogy, found with Seifert's work, is the positive impact on carbon price of an increase in the penalty; in particular the increase of the first period penalty will push up carbon prices of the following periods, while the increase of the second period penalty will push up prices only from the second period on.

An increase in the marginal cost of short-term abatement measures has the same positive effect as in Seifert's analysis, but only in the second period. The explanation remains the same: once that short-term measures become more expensive to implement, there will be a natural shift to the implementation of long-term projects and to the purchase of permits from the market; this will push carbon price up particularly in the second period because in the first one short terms abatement measure are still considered affordable and the real shift occurs only thereafter.

Finally the main originality of this work is represented by the presence of CDM projects that, also through the so-called $\alpha-$policy, impact on carbon price. The reaction of this price is ambiguous: it increases - even though very lightly - in the first period, and decreases in the second one. This could be due to a general inclination of firms of buying more permits in the first compliance period than in the second one for different reasons: the carry-on mechanism that allows companies to bring permits in excess from the first to the second period; the fact that CDM projects are not effectively productive[47] from the initial period and start bring their carbon contribution only afterwards, therefore firms recognize their beneficial effect only with a time-lag. Anyway, what really matters is that after a first adjustment period the increase of α pushes down carbon prices; the reason is very straightforward: with a bigger α, leaving all the other parameters unchanged, a bigger number of permits is present on the market and carbon prices will naturally go down due to this allowances' overabundance.

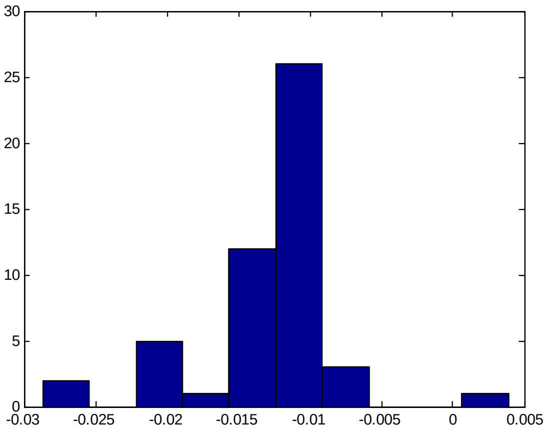
In succession all the histograms with the frequency density of percentage changes in $S(x_1, 1)$ registered in the passage from the benchmark case - all parameters being unchanged - to the study case when one parameter per time is left vary, as already underlined before the sample is composed by 50 observations obtained through as much simulations of the Brownian Motion.[48]

[47]With productive we mean that they need some periods to recover the big initial investment needed for their implementation.

[48]Notice that $S(x_0, 0)$ does not depend on the realization of the Brownian Motion - it is based on the expectation at time 0 -, therefore its values - both in the benchmark and in the study scenarios - remain the same.

Increase of α :

$\mathbf{S}(x_1, 1)$ deviation from benchmark

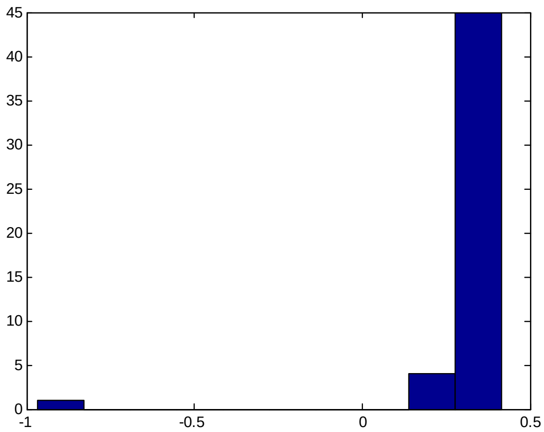


Percentage change from benchmark	
$S(x_0, 0)$	$+0.59\%$
$S(x_1, 1)$	
Average	-1.26%
St. Deviation	$+0.49\%$
Min. Value	-2.87%
Max. Value	$+0.39\%$
1^{st} Quartile	-1.38%
Median Value	-1.14%
3^{rd} Quartile	-0.98%

(Exhibit 1c)

Increase of p_1 :

$\mathbf{S}(x_1, 1)$ deviation from benchmark

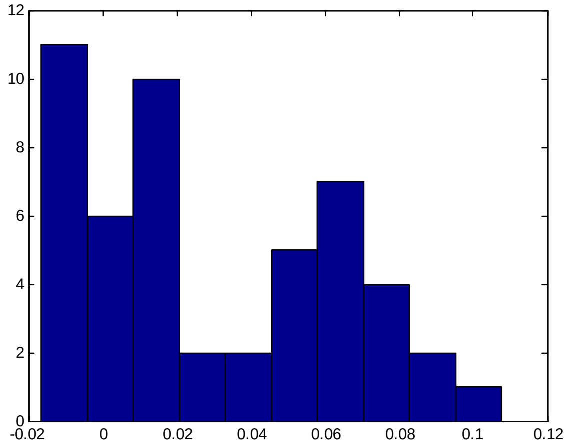


Percentage change from benchmark	
$S(x_0, 0)$	$+37.32\%$
$S(x_1, 1)$	
Average	$+30.47\%$
St. Deviation	$+18.70\%$
Min. Value	-96.66%
Max. Value	$+41.45\%$
1^{st} Quartile	$+31.62\%$
Median Value	$+33.39\%$
3^{rd} Quartile	$+34.84\%$

(Exhibit 2c)

Increase of p_2 :

S $(x_1, 1)$ **deviation from benchmark**



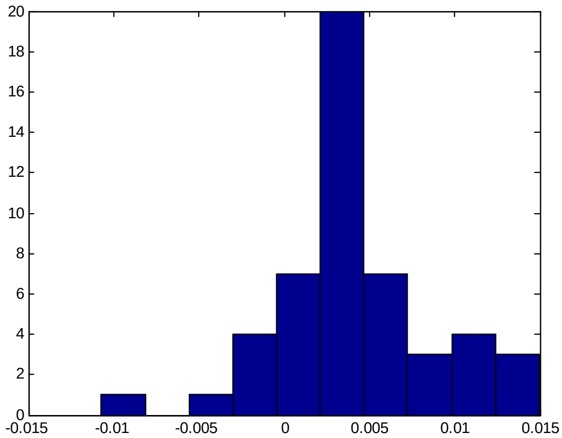
Percentage change from benchmark

$S(x_0, 0)$	-26.52%
$S(x_1, 1)$	
Average	$+2.93\%$
St. Deviation	$+3.37\%$
Min. Value	-1.65%
Max. Value	$+10.77\%$
1^{st} Quartile	-0.01%
Median Value	$+1.48\%$
3^{rd} Quartile	$+6.03\%$

(Exhibit 3c)

Increase of e_0 :

S $(x_1, 1)$ **deviation from benchmark**



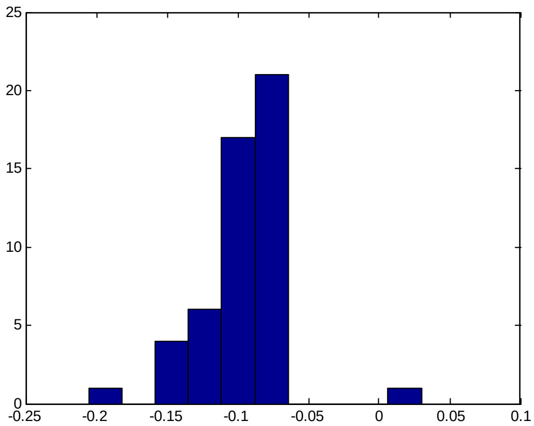
Percentage change from benchmark

$S(x_0, 0)$	-4.92%
$S(x_1, 1)$	
Average	$+0.40\%$
St. Deviation	$+0.45\%$
Min. Value	-1.07%
Max. Value	$+1.50\%$
1^{st} Quartile	$+0.21\%$
Median Value	$+0.34\%$
3^{rd} Quartile	$+0.56\%$

(Exhibit 4c)

Increase of e_1 :

S $(x_1, 1)$ **deviation from benchmark**

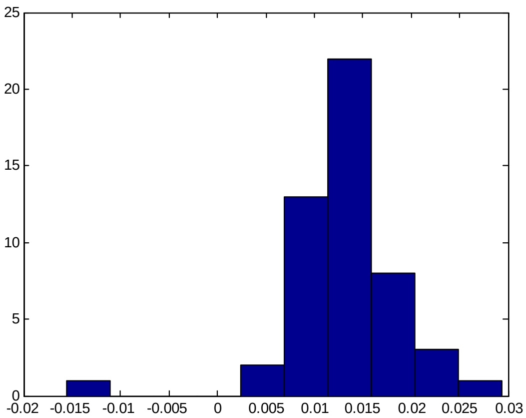


Percentage change from benchmark	
$S(x_0, 0)$	$+0.41\%$
$S(x_1, 1)$	
Average	-9.31%
St. Deviation	$+3.41\%$
Min. Value	-20.56%
Max. Value	$+2.98\%$
1^{st} Quartile	-10.35%
Median Value	-9.02%
3^{rd} Quartile	-7.02%

(Exhibit 5c)

Decrease of θ :

S $(x_1, 1)$ **deviation from benchmark**



Percentage change from benchmark	
$S(x_0, 0)$	-20.00%
$S(x_1, 1)$	
Average	$+1.31\%$
St. Deviation	$+0.61\%$
Min. Value	-1.55%
Max. Value	$+2.94\%$
1^{st} Quartile	$+1.04\%$
Median Value	$+1.31\%$
3^{rd} Quartile	$+1.50\%$

(Exhibit 6c)