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Modelling Energy Prices: Pricing Derivatives in Electricity Markets

SUMMARY

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Introduction

Electricity has often been considered a typical example of natural monopoly: as an indivisible, capital intensive product totally dependent on a network structure which constantly requires instantaneous balance between production and consumption, it seemed natural to entrust its production and distribution to state-owned, monopolistic companies. Only during the last two decades many countries around the world undertook a process of reforms for the liberalization of this sector, in order to introduce competition and foster investments, efficiency and price transparency.

It is probably too early to assess if this worldwide process of liberalization of the power sector has been a success or not, and certainly the judgement may differ depending on the market we consider. What is interesting and has inspired this work is the fact that with liberalization electricity prices have started to follow the laws of supply and demand rather than being decided by a central authority. So the particular nature of electricity has brought extremely high volatility in prices, far higher than the one observed in other commodity and financial assets markets, and the occasional occurrence of spikes in the process, introducing the necessity of modeling the dynamics of power prices in a proper way in order to meet the new-coming needs of risk management and derivative pricing. The problem hence consists in adapting the mathematical and statistical tools already widely used in financial markets to model stock prices and price derivatives, to the unique characteristics of electricity prices and of the particular derivatives used in such markets.

The contribution of this thesis is twofold. First, it provides the mathematical and statistical tools which lie at the basis of every modeling approach for dynamic stochastic processes; moreover, it gives an economical background for energy markets in a general fashion and for the most popular approaches in modeling energy prices. Secondly, it applies the previous arguments in order to analyse the spot price model proposed in Kluge [16] and in Kluge, Hambly and Howison [15] for electricity prices, trying to explain every passage as clearly as possible.

To this end, the thesis is organised as follows. In Chapter 1, we introduce the main mathematical and statistical tools used for modeling stochastic processes. It is an important chapter since it provides a fundamental theoretical background, necessary for the remainder of the work. In Chapter 2 we give a descriptive picture of energy markets in general and of the most widely derivative products used in those markets; in addition, we review the most basic approaches to modeling prices and pricing derivatives, and we provide the possible methods we may adopt to modify these models in order to implement the peculiar characteristics of energy commodities. Finally, in Chapter 3, we introduce jump-diffusion models for energy prices, and give a thorough analysis of the mean-reverting spot price model with spikes, proposed in Kluge [16] and in Kluge, Hambly and Howison [15] for electricity prices.

1. Probability Measures and Stochastic Calculus

The first chapter provides the theoretical framework which is at the basis of any kind of modeling in a stochastic context and hence of the analysis of financial and commodity markets. This is not a descriptive chapter, in the sense that it proposes many definitions and theorems which will often be used as a reference in the subsequent chapters. Therefore, here we introduce the main results exposed in this chapter.

As a starting point, we state the definition of $\sigma - algebra$ and *probability measure*, concepts which underpin our future analysis of stochastic calculus. A **sigma-algebra** can be defined in the following way:

Definition 1. (Sigma-algebra) Let Ω be a non-empty set, $\Omega \neq \emptyset$. A $\sigma - algebra$ $\mathcal{F}$ on Ω is a family of subsets of Ω such that:

1. **(Empty set)** $\emptyset \in \mathcal{F}$;
2. **(Complement)** if $F \in \mathcal{F}$ then $F^c := (\Omega \setminus F) \in \mathcal{F}$;
3. **(Countable unions)** If $F_1, F_2, \ldots \in \mathcal{F}$, *then* $\bigcup_{n=1}^{\infty} F_n \in \mathcal{F}$.

The notion of sigma-algebra is essential for the definition of a measure since it collects only a certain subclass of "non-pathological" subsets of Ω called **measurable sets**, which have properties that one may expect from a measurable set, namely that the complement of a measurable set is a measurable set and that a countable union of measurable sets is still a measurable set. In the framework of probability space, a sigma-algebra represents all the available information, meaning the collection of all the events that may happen with a certain probability.

We have the following definition of **measure**.

Definition 2. (Measure) A *measure* on the $\sigma-algebra$ $\mathcal{F}$ of Ω is an application

$$P : \mathcal{F} \to [0, +\infty]$$

such that:

1. $P(\emptyset) = 0$;
2. **(Countable additivity)** for all countable collections $(F_n)_{n\in\mathbb{N}}$of pairwise disjoint sets in $\mathcal{F}$, it holds

$$P(\bigcup_{n\geq 1}^{\infty} F_n) = \sum_{n\geq 1}^{\infty} P(F_n).$$

If $P(\Omega) < \infty$, P is called a **finite measure**. In addition, if

3. $P(\Omega) = 1$,

holds, then we say that P is a *probability measure*.

The couple composed by a non-empty set Ω and a sigma-algebra $\mathcal{F}$ on Ω is called a **measurable space**, and a triple $(\Omega,\ \mathcal{F},\ P)$ is called **measure space**, i.e. a measurable space endowed by a positive measure defined on the sigma-algebra of its measurable sets. If P is a **probability measure**, then the triple is a **probability space**. In this case, Ω is called **sample space** and can be thought of as the set of all possible outcomes of an experiment; an element E of $\mathcal{F}$ is called **event** and $P(E)$ is the probability of the event E. In this context, we introduce the following definition of **random variable**.

Definition 3. (Random Variable) Let $(\Omega, \mathcal{F},\ P)$ be a probability space and $(E,\Im)$ a measurable space. Then an $(E,\Im)-valued\ random\ variable$ is a function $X : \Omega \to E$ which is $(\mathcal{F}, \Im)$-measurable. If $E = \mathbb{R}$ and $\Im = \mathscr{B}(\mathbb{R})$, then it is called a $real - valued\ random\ variable$. If $X = (X_1, \ldots, X_N)$ is defined on $(\mathbb{R}^N, \mathscr{B}(\mathbb{R}^N))$ we will refer to it as a $N - dimensional$ random variable.

The probability measure induced on a measurable space by a random variable X is called **distribution**. Knowing the distribution of a random variable allows us to calculate its expected value and variance. The **expected value** of a random variable X is the average of all possible values of X weighted by their probability P. The **variance** instead and measures the "width" of the distribution around its mean, i.e. it gives an estimate of how much a random variable spreads out on average around its expected value.

A **stochastic process** is a collection of random variables and it is often used to represent the evolution of some random value over time. More formally,

Definition 4. (Stochastic Process) A *stochastic process* is a family of random variables $X = (X_t)_{t\in T}$ of the form:

$$X = (\Omega,\ \mathcal{F},\ (\mathcal{F}_t)_{t\in T},\ (X_t)_{t\in T},\ P)$$

where:

— T (*set of times*) can be a subset of $\mathbb{N}$ or $\mathbb{R}^+$.
— $\mathcal{F}$ is a *sigma − algebra* of Ω.
— P is a *probability measure* on $(\Omega, \mathcal{F})$.
— $(\mathcal{F}_t)_{t\in T}$ is a *filtration*, i.e. a family of increasing (i.e. such that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for any n) *sub − σ − algebras* of $\mathcal{F}$.
— $(X_t)_{t\in T}$ is a family of random variables on $(\Omega, \mathcal{F})$ to values in a measurable space $(E, \mathcal{E})$ and such that, for every t, X_t is $\mathcal{F}_t$ *− measurable*. In such case X_t is said to be *adapted to the filtration* $(\mathcal{F}_t)_t$, or equivalently that $\mathcal{F}_n^X \subseteq \mathcal{F}_n$, for every $t \in T$.

Two important examples of stochastic processes are the Brownian motion and the Poisson process. A **Brownian motion** is a particular type of Gaussian process.

Definition 5. (Brownian Motion) A real-valued process $B = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (B_t)_{t\geq 0}, P\right)$ is a *Brownian motion* if

1. B_t is $\mathcal{F}_t$ *− measurable* for each $t \geq 0$;
2. $B_0 = 0$ *almost surely*;
3. for every $0 \leq s \leq t$ the random variable $B_t - B_s$ is independent from $\mathcal{F}_s$ **(independent increments property)**;
4. for every $0 \leq s \leq t$ the random variable $B_t - B_s$ has law $N(0, t-s)$ **(stationary increments property)**;
5. B_t has *continuous paths*.

The **Poisson process** is the prototype of a pure jump process and it is important to study its main features for our future analysis of electricity prices. The Poisson process ideally is collocated at the opposite extreme from Brownian motion, since it only changes values by means of jumps, and even then, the jumps are nicely spaced.

Definition 6. (Poisson Process) Let $\{\mathcal{F}_t\}$ be a filtration, not necessarily satisfying the usual conditions. A *Poisson process* with parameter $\lambda > 0$ is a stochastic process N with the following properties:

1. $N_0 = 0$, almost surely.
2. The paths of N_t are right continuous with left limits.
3. If $s < t$, then $N_t - N_s$ is a *Poisson random variable* with parameter $\lambda(t-s)$.
4. If $s < t$, then $N_t - N_s$ is independent of $\mathcal{F}_s$.

The parameter λ is called the **intensity** of the Poisson process.

Stochastic differential equations like the *Geometric Brownian motion* and the *Ornstein − Uhlenbeck process*, and *Ito's Lemma* are among the most important instruments for the study of a dynamic process and therefore for the analysis of financial markets.

Definition 7. We will say that the process $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, (\xi_t)_{t\in[u,t]}, (B_t)_t, P)$ is the solution of the *stochastic differential equation*

$$\begin{aligned} d\xi_t &= b(\xi_t, t)\,dt + \sigma(\xi_t, t)\,dB_t \\ \xi_u &= x \qquad\qquad\qquad x \in \mathbb{R}^m \end{aligned}$$

if

1. $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, (B_t)_t, P)$ is a *standard d − dimensional Brownian motion*;
2. for every $t \in [u, T]$ we have

$$\xi_t = x + \int_u^t b(\xi_s, s)\,ds + \int_u^t \sigma(\xi_s, s)\,dB_s.$$

σ is the **diffusion coefficient**, b is the **drift coefficient.**

The following one-dimensional equation is called the **Ornstein-Uhlenbeck process**,

$$\begin{aligned} d\xi_t &= -\lambda\xi_t dt + \sigma dB_t \\ \xi_0 &= x \end{aligned}$$

where λ, $\sigma \in \mathbb{R}$, i.e. we suppose a linear drift and a constant diffusion coefficient. In particular, λ represents the **speed of mean reversion**, in a sense which will be specified in the following chapters.

Let's now consider the one-dimensional equation

$$\begin{aligned} d\xi_t &= \mu\xi_t dt + \sigma\xi_t dB_t \\ \xi_0 &= x \end{aligned}$$

This is called a **Geometric Brownian motion**. Since it can only take positive values, this process is widely used in financial applications, expecially to describe the evolution of prices.

Theorem 8. ***(Ito's Lemma)*** *Let X_i, $i = 1, \ldots, m$, be processes which have the stochastic differential*

$$dX_i(t) = F_i(t)dt + G_i(t)dB(t) \qquad i = 1, \ldots, m$$

and let $f : \mathbb{R}_x^m \times \mathbb{R}_t^+ \to \mathbb{R}$ be a continuous function in (x, t), continuously differentiable once in t and twice in x. Then given $X_t = (X_1(t), \ldots, X_m(t))$, the process $(f(X_t, t))_t$ has the stochastic differential

$$\begin{aligned} df(X_t, t) &= \left(f_t(X_t, t) + \sum_{i=1}^{m} f_{x_i}(X_t, t) F_i(t) + \frac{1}{2} \sum_{i,j=1}^{m} f_{x_i x_j}(X_t, t) G_i(t) G_j(t) \right) dt \\ &+ \sum_{i=1}^{m} f_{x_i}(X_t, t) G_i(t) dB_t \\ &= f_t(X_t, t)\, dt + \sum_{i=1}^{m} f_{x_i}(X_t, t)\, dX_i(t) + \frac{1}{2} \sum_{i,j=1}^{m} f_{x_i x_j}(X_t, t) G_i(t) G_j(t) dt. \end{aligned}$$

Denoting with f' the gradient of f with respect to x, we can write Ito's Lemma more compactly as

$$df(X_t, t) = f_t(X_t, t) + f'(X_t, t)\, dX_t + \frac{1}{2} \sum_{i,j=1}^{m} f_{x_i x_j}(X_t, t)\, d\langle X_i, X_j \rangle_t .$$

Let $b(x,t) = (b_i(x,t))_{1 \le i \le m}$ and $\sigma(x,t) = (\sigma_{ij}(x,t))_{1 \le i \le m}^{1 \le j \le d}$ be measurable functions defined on $\mathbb{R}^m \times [0, T]$ to values in $\mathbb{R}^m$ and in $M(m, d)$[1] respectively.

2. Energy Markets and Electricity

Energy markets are commodity markets which deal specifically with the trade and supply of energy. They include commodities that are quite different in nature, for example fuels like oil, gas and coal, electricity, or emissions and weather products. It is clear that these kinds of markets present features which make them unique with respect to the other financial markets. In this chapter we try to describe, without claiming to be complete, the main characteristics and structures of fuels and power markets, and the most important and peculiar derivative products developed in these markets. Finally, in order to introduce the subject of the last chapter, we review the basic approaches to price modeling and to derivative pricing, and the possible solutions one may adopt to adapt these models to the unique price evolution processes of energy commodities.

2.1. Energy markets

For convenience, we break energy markets in two main categories: **fuel markets** and **electricity**, or **power**, **markets**. The two most important fuel

[1] $M(m, d)$ is the vector space of the real matrices $m \times d$.

markets are oil and gas markets. With the **oil market** we refer to the trade of two basically different products: crude oil and refined products. Physical **crude oil** markets are highly fluid, global and volatile, the quality of the oil being determined essentially by two factors, the density of the oil and its sulfur content. Through a complex refining procedure, it is possible to obtain many different **refined products** from crude oil. Refined product markets have a much smaller the scale of operations than crude oil markets and refined products must meet very stringent quality standards. **Gas** is nowadays the fastest growing energy commodity. A great part of its demand comes from industrial customers, residentials and commercial consumption, which together accrue for almost 75% of the demand (according to Geman [9]), but a fast-growing share is being used for power generation. Gas cannot be moved over large distances without incurring in considerable transportation costs; therefore, three regional markets have come into existence, with limited trade between them: one in the Americas, one in Europe and the last in Asia. There are three main types of physical trading contracts: **swing contracts**, **baseload contracts** and **firm contracts**.

Historically the **power** sector was extremely regulated worldwide with prices generally settled by regulatory authorities controlled by each individual country. However in the last two decades many countries have started to liberalize this sector in order to improve efficiency and reduce electricity prices. Electricity delivery contracts are now traded in many regular markets and in this case prices are determined by the mechanisms of supply and demand. Electrical energy cannot be stored efficiently and therefore is considered an instantaneous consumption good. For this reason, the equilibrium between supply and demand needs to be secured at any time. Demand is highly inelastic with respect to price and is characterized by daily, weekly and annual **seasonality**, which highly affects prices. In particular, during the day there is generally the distinction between **on-peak prices** during high demand hours, and **off-peak prices** during the rest of the day. Supply may suddenly change in case of plant outage or problems in the transmission network, and this, together with the inelasticity of demand and the absence of a buffering effect of inventory, may cause **spikes** in price trajectories. In general, electricity markets are structured as **day-ahead markets**, meaning that transactions are referred to the generation of the following day, or as **day-of markets**, where power generation for the rest of the day is transacted; generally these structures coexist with a **hour-ahead** or an **Ex-post market**. Deregulated spot power markets generally can be either pools or exchange markets.

2.2. Basic products and structures

Derivatives, according to Hull [11], can be defined as financial instruments whose value depends on, or derives from, the values of other, more basic, underlying variables. Such instruments are extensively used in financial markets to hedge positions, take advantage of arbitrage opportunities or speculate in the market, and they naturally migrated to commodity and energy markets to fulfill all the risk management needs that these markets create. The most **standard derivative** products, also called "**plain vanilla**", are forwards and futures, standard options and swaps; **non-standard products** are called **exotic** and a wide variety of them developed in energy markets to meet the needs of operators and the peculiarities of the underlying products.

A **futures** contract is in general an agreement between two parties to buy or sell a commodity or financial product at a certain time in the future for a certain price (Hull [11]). A commodity futures is a highly standardized exchange-traded contract which is typically physically settled and is defined by the following characteristics:

— Volume
— Price
— Delivery location

— Delivery period
— Last trading day or settlement date.
A **forward** contract is defined in the same way of a futures contract, since it is an agreement to buy or sell an asset at a certain future time for a certain price (Hull [11]). The main difference consists in the fact that a forward is an **over-the-counter** (**OTC**) product, which means that it is not traded in exchange markets and therefore it guarantees more flexibility to the parties than futures contracts. In commodity markets, a forward contract can be physically or financially settled and specifies some delivery details such as the total quantity of the commodity considered, the delivery time and location and the price agreed. Moreover, in energy markets it is frequent that a forward contract pays over a whole **delivery period**, say $[T_1, T_2]$. Then, the strike price of a zero-cost forward contract depends strictly on when the money is paid. In particular, if the forward pays $(S_t - f_t)\,\Delta t$ at time t, we say that it is **instantly settled**, while if the contract specifies the payment of the whole amount at the end of the delivery period, then we say that the forward is **settled at maturity**.

Hull [11] describes a **swap** in the following terms: "a swap is an agreement between two companies to exchange cash flows in the future. The agreement defines the dates when the cash flows are to be paid and the way in which they are to be calculated. Usually the calculation of the cash flows involves the future value of an interest rate, an exchange rate, or other market variable". The most popular swaps are the plain vanilla **interest rate swaps**, and **currency swaps**. A variety of swap structures appear in energy markets. We have for example **fixed-for-float** swaps, which involve an exchange of cash flows or commodities. In **differential swaps** instead the floating flow is calculated as the difference between the prices of two assets. One party can also choose to share with the other the profits deriving from a favourable movement of the price of the underlying asset with a **participation swap**, which allows the fixed leg holder to retain a certain percentage of the upside price movement. Finally, options (calls or puts) on swaps exist and are called **swaptions**.

As a general definition, an **option** is a derivative which gives its holder the right, but not the obligation, to buy (**Call option**) or sell (**Put option**) an underlying asset at a predetermined **strike price** on or before a **maturity** (**expiration**) **date**. The most popular standard options are call and put options. If the holder has the possibility of exercising the option only at the expiration date, we are at the presence of a so called **European option**, while if the option can be exercised at any time up to maturity, it is an **American option**. Based on the strike price, we can further distinguish between **at the money (ATM) options**, if the strike price is concentrated near the current market price of the underlying, **out of the money (OTM) options**, if the strike price is significantly below the market price for call option and above the market price for put options, and **in the money (ITM) options**, if on the contrary calls have their strike below current market price and put above market price.

Options in energy markets are defined in the same way as in financial markets, however, they usualy include also other specifications, like
— Location,
— Exercise time,
— Delivery conditions, and the type or quality of the underlying product,
— Strike,
— Volume.
In particular, it is frequent that in energy markets the right guaranteed by the option is applied to a whole period of time rather than only at the exercise date: we can in fact find **calendar-year**, **quarterly** and **monthly options** depending on the period of time considered. Often, options are defined on a whole period of time, but the exercise decision is established on a daily basis during that period. These are **options on the spot commodity** and are very common since they offer the possibility of managing price risk on a daily basis. **Daily options** are options which are exercised every day during a specified

period at a fixed strike price. **Index** or **cash options** instead can be exercised at a floating strike price based on a chosen index. In power markets **hourly options** also exist, with which it is possible to manage power prices risk on a real-time hourly basis.

Exotic options are options with a more complex structure than standard options. In energy markets they have great relevance and are widely used since they suit the characteristics of the transactions performed in these markets particularly well and are naturally embedded in most common energy contracts. The most important kinds of exotic options are

— **spread options**, where a spread is a price differential between two commodities. Indeed, power plants, refineries or transmission lines can be described as spreads between input fuels and output commodities or geographical spreads. Common classes of spreads are the **quality spread** between different qualities of the same commodities, **geographic spread** between different locations, **time** or **calendar spread** and **intercommodity spread**, between two different but linked commodities;

— **tolling agreements**, which can be considered as a leasing contract between the owner of a plant and a "toller" who holds raw materials and need to use the plant to process them. Then the agreement specify that the toller can use the plant to process his/her materials paying a premium (*capacity payment*) and take possession of the final output. In financial terms we can represent this agreement as a call option on the final output with floating strike linked to raw materials prices. **Calls-on-toll** are compund options which allow its holders to enter, if they want, into a tolling agreement at some future date T_0 in exchange of a specific payment K.

— **swings, recalls and nominations**, which are **volumetric options** that give the holder the right, but not the obligation, to adjust the volume of received or delivered commodity. **Swing options** can be defined as a options which allow the owner to exercise K times the right to vary the amount of commodity delivered within a certain range over N periods. **Nomination options** are similar to swing options since they offer the holder the right to change the volume of the underlying commodity delivered K times over N periods, but, unlike swings, now the level of the volume is adjusted for the remainder of the contract until next right is exercised. **Recalls** are like swing options, but are used to interrupt delivery under stressful circumstances, while swings are used for managing demand.

2.3. Modeling price processes

Modeling energy prices and electricity in particular, requires different instruments from the usual ones used in financial markets. In stock markets the most widely used model of stock price behaviour is the Geometric Brownian motion, since it resembles the path of stock prices quite well, it can only take positive values and its expected returns are independent of the value of the process. If S_t denotes as usual the spot price at time t and dB_t the increments of standard Brownian motion, the most standard form of the Geometric Brownian motion is:

$$\frac{dS_t}{S_t} = bdt + \sigma dB_t.$$

A frequently used equivalent form of (2.3.1) is obtained with the change of variables

$$Z_t = \ln S_t$$

and using Ito's lemma we obtain

$$dZ_t \quad = \quad \left(b - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t.$$

If the value of Z_t is known at some initial time t and b and σ are constant, then its solution is a Geometric Brownian motion, i.e., with $S_s = x$,

$$S_t^{x,s} = x \exp\left[\left(b - \frac{\sigma^2}{2}\right)(t-s) + \sigma\left(B_t - B_s\right)\right].$$

Equation (2.3.4) implies that for any future time t, $t \geq s$, the variable Z_t is normally distributed, since it is a linear combination of $B_t \sim N(s, t)$:

$$Z_t \sim \phi\left[Z_s + \left(b - \frac{\sigma^2}{2}\right)(t-s), \sigma\sqrt{t-s}\right].$$

Black and Scholes [6] demonstrated that it is possible to derive a pricing formula for European call and put options, based on specific hypotheses and the assumption that the market consists of two assets with dynamics given by

$$\begin{aligned} dR(t) &= rR(t)dt, \\ dS(t) &= S(t)bdt + S(t)\sigma dB(t). \end{aligned}$$

In particular, the Black and Scholes pricing formulas for **European calls** and **puts** are:

$$\begin{aligned} c(t) &= sN(d_1) - Ke^{-r(T-t)}N(d_2) \\ p(t) &= Ke^{-r(T-t)}N(-d_2) - sN(-d_1), \end{aligned}$$

where

$$d_2 = -\frac{\ln\frac{K}{s} - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_1 = d_2 + \sigma\sqrt{T-t}.$$

European options on **forwards** can be priced extending the results we obtained, as in Black [5]. If we assume that the forward price follows a lognormal process with zero mean

$$\frac{df_t}{f_t} = \sigma dB_t,$$

with σ constant, then equations (2.3.46) and (2.3.48) can be modified replacing s with f_t to obtain

$$\begin{aligned} c(t) &= e^{-r(T-t)}\left[f_t N(d_1) - KN(d_2)\right], \\ p(t) &= e^{-r(T-t)}\left[KN(-d_2) - f_t N(-d_1)\right], \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\frac{f_t}{K} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\ln\frac{f_t}{K} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}, \end{aligned}$$

and σ is now the volatility of the forward price.

Energy prices possess distinctive features which make Geometric Brownian motion in its standard form not suitable to describe properly their dynamics. Two essential attributes in particular seem to characterize energy prices, differently from stock prices: **mean-reversion** and **seasonality**. A quantity is said to be mean reverting toward a certain **long-term mean**, if the further it moves away from this level, the higher the probability that in the future it will move back towards it (Eydeland [8]). We can include **mean-reversion** in a process in several ways. For example, we can assume that spot prices on average tend toward their **long-term mean**. This means that the drift term changes sign depending on whether prices are above or below the long-term level. Namely, we have

$$\frac{dS_t}{S_t} = \kappa\left(S_\infty - S_t\right)dt + \sigma dB_t,$$

where S_∞ is the long-term mean of spot prices and κ is the **strength of mean reversion**.

Another possible solution is to assume mean reversion to the **long-term price logarithm**. The process now has the following form

$$\frac{dS_t}{S_t} = \kappa\left(\theta - \log S_t\right)dt + \sigma dB_t.$$

Seasonality is one of the most typical characteristics of commodity prices and it depends essentially on supply and demand factors. In order to account for seasonality while keeping mean reversion in the process, the model for commodity spot prices may be written as follows:

$$\ln S(t) = f\left(t\right) + X\left(t\right),$$

where $f\left(t\right)$ is a deterministic component accounting for the seasonality of prices, usually expressed as a sin or cos with annual or semi-annual periodicity, and

$$dX(t) = \left(\alpha - \beta X(t)\right)dt + \sigma dB_t$$

is a mean-reverting process.

Many statistical properties of energy prices, like pronounced skewness and kurtosis, invalidate the use of Geometric Brownian motion as a process to model their evolution, too. Two possible explanations for the inconsistency of the lognormal assumption are **stochastic volatility** or the presence of **jumps** in the price process.

A first approach to model volatility, instead of supposing that it remains constant as in Black and Scholes model, is to assume that it takes a determinate functional form which depends on some chosen variable. **Constant Elasticity of Volatility** models (CEV) assume that instantaneous volatility is a function of the prices themselves, that is,

$$\frac{dS}{S} = rdt + \sigma\left(S\right)dB.$$

We can also model volatility as a continuous time process. **Stochastic volatility models** usually have the following form:

$$\begin{aligned} \frac{dS}{S} &= \mu\left(S,\, t\right)dt + \sigma dB_1, \\ dm\left(\sigma\right) &= \gamma\left(\sigma,\, t\right)dt + \phi\left(\sigma,\, t\right)dB_2, \\ dB_1 dB_2 &= \rho dt. \end{aligned}$$

The variance of returns is hence modelled as some function of a stochastic process with its drift and diffusion coefficient with a correlation $\rho \geq 0$ between the two processes.

3. A Jump Diffusion Model for Electricity Markets

A possible explanation for the inconsistency of the lognormality assumption for energy prices is the presence of **jumps** in the process. Indeed, the occurrence of **spikes** in the dynamic of prices is a characteristic feature of energy prices, in particular of electricity prices. Spikes are defined as a large upward movement in prices, immediately followed by a rapid downward movement to the normal level, or vice versa. **Jump-diffusion models** are a class of models which allows to incorporate jumps or spikes in the process of interest and hence are widely used to model energy prices. A jump-diffusion process is generally a combination of a diffusion process and a jump process, typically represented through a discontinuous Poisson process.

A typical jump diffusion process for price returns has usually the following representation

$$\frac{dS_t}{S_t} = (\mu - \lambda k)\, dt + \sigma dB_t + (J_t - 1)\, dN_t,$$

or, after the usual substitution $Z_t = \ln S_t$,

$$dZ_t = \left(\mu - \lambda k - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t + \ln (J_t)\, dN_t,$$

where $J_t - 1$, $J_t \geq 0$, is a random variable which represents the jump magnitude in price returns, $k = E(J_t - 1)$ is the expected jump magnitude and λ is the intensity of the Poisson process.

A mean-reverting spot price model with spikes is described in Kluge [16] and in Kluge, Hambly and Howison [15]. The authors propose a mean-reverting process for electricity spot prices, enriched with a seasonal component and a jump component, which could represent the occurrence of spikes in the price process. The model implements the mean-reversion, seasonality and jump properties defining the spot price process S as the exponential of the sum of three components: a **deterministic periodic function** f representing seasonality, an **Ornstein-Uhlenbeck** (OU) **process** X and a **mean-reverting process with a jump component** to incorporate spikes Y:

$$\begin{aligned} S_t &= \exp(f(t) + X_t + Y_t), \\ dX_t &= -\alpha X_t dt + \sigma dB_t, \\ dY_t &= -\beta Y_{t-} dt + J_t dN_t, \end{aligned}$$

where as usual N_t is a Poisson-process with intensity λ and J_t is an independent and identically distributed (i.i.d.) process representing the jump size. By assumption, B_t, N_t and J_t are mutually independent processes. In particular, the Ornstein-Uhlenbeck process represents the price path without the occurrence of spikes, i.e. the situation when the market is not under stress; the jump-diffusion process can represent the "spikey" nature of power prices by letting a very high level of mean reversion β.

We can compute the moment generating function of the whole process $S_t = \exp(f(t) + X_t + Y_t)$. Keeping in mind the fact that X_t and Y_t are mutually independent, and therefore the expectation of their product is simply equal to the product of their expectations, we have the following moment generating function of $f(t) + X_t + Y_t$.

Theorem 9. *Let[2] the spot process (S_t) be defined by (3.2.1-2-3) and let (Z_t) be its natural logarithm, i.e. $Z_t := \ln S_t = f(t) + X_t + Y_t$, with X_0 and Y_0 given. Then the moment generating function of Z_t is*

$$\begin{aligned} E\left[e^{\theta Z_t}\right] &= \exp\left(\theta f(t) + \theta X_0 e^{-\alpha t} + \theta^2 \frac{\sigma^2}{4\alpha}\left(1 - e^{-2\alpha t}\right)\right. \\ &+ \left.\theta Y_0 e^{-\beta t} + \lambda \int_0^t m_J\left(\theta e^{-\beta s}\right) - 1 ds\right). \end{aligned}$$

The expectation value of the spot process S at time T, given X_t and Y_t be the values at time t, immediately follows by setting $\theta = 1$. Hence[3]:

$$\begin{aligned} E[S_T \mid X_t, Y_t] &= \exp\left(f(T) + X_t e^{-\alpha(T-t)} + Y_t e^{-\beta(T-t)} + \frac{\sigma^2}{4\alpha}\left(1 - e^{-2\alpha(T-t)}\right)\right. \\ &+ \left.\lambda \int_0^{T-t} m_J\left(e^{-\beta s}\right) - 1 ds\right). \end{aligned}$$

[2] Kluge, Hambly and Howison [15], Theorem 2.4.
[3] See Kluge [16], Corollary 3.4.13.

This expectation is formed by the deterministic seasonal component, the initial terms, a contribution from the volatility of X_t and the jump term.

For very high mean reversion rates β and small jump intensities λ, the main contribution to the jump distribution comes from the last jump. Then, let's define the **truncated spike process** as

$$\tilde{Y}_t := \begin{cases} J_{N_t} e^{-\beta\left(t-T_{N_t}\right)} & N_t > 0, \\ 0 & N_t = 0. \end{cases}$$

Unlike Y_t, $\tilde{Y}_t$ is a process consisting only of the last jump occured. Now, it can be shown that the truncated spike process is identically distributed as

$$Z_t := \begin{cases} J_1 e^{-\beta T_1} & T_1 \leq t, \\ 0 & T_1 > t. \end{cases}$$

So, we have the following result.

Lemma 10. ***(Moment generating function of the truncated spike process)*** *The[4] random variable $\tilde{Y}_t$ of the truncated spike process at time t with initial condition $\tilde{Y}_0 = 0$ has the moment generating function*

$$m_{\tilde{Y}}\left(\theta,\, t\right) = 1 + \lambda \int_0^t \left(m_J\left(\theta e^{-\beta s}\right) - 1\right) e^{-\lambda s} ds.$$

3.1. Option pricing

First of all, let's see to price an option whose payoff depends exclusively on the value of the underlying asset at maturity date T. If the contingent claim of such an option is given by $g\left(S_T\right)$, then its arbitrage free price at time t is given by

$$V\left(x,\, y,\, t\right) = e^{-r(T-t)} E^Q\left[g\left(S_T\right) \mid X_t = x,\, Y_t = y\right],$$

i.e. it is the expected value of the contingent claim at time T under the risk-neutral probability measure Q, discounted by the constant risk-free rate r and assuming that the mean-reverting and spike processes are individually observable at time t. The following general theorem puts the fact that the distribution function of a random variable may be recovered from its moment generating function.

Theorem 11. ***(Levy's Inversion Formula)*** *Let[5] $m\left(\theta\right) : \Theta \subset \mathbb{C} \to \mathbb{R}$ be the moment generating function of a random variable Z*

$$m\left(\theta\right) := E\left[e^{\theta Z}\right] = \int_{\mathbb{R}} e^{\theta Z} dF_Z\left(x\right),$$

then the cumulative distribution $F_Z : \mathbb{R} \to [0,\, 1]$ is given by

$$F_Z\left(x\right) = \frac{1}{2} - \frac{1}{\pi} \int_0^{+\infty} \frac{\Im\left(m\left(0 + i\nu\right) e^{-i\nu x}\right)}{\nu} d\nu,$$

where $\Im\left(z\right)$ denotes the imaginary part of any value $z \in \mathbb{C}$.

The inversion formula can be generalised to truncated moment generating functions (see Kluge [16], Proposition 4.3.3)

Proposition 12. *Let Z be a random variable and its truncated moment generating function be defined by*

$$G_\nu\left(x\right) := E\left[e^{\nu Z} \mathbf{1}_{\{Z \leq x\}}\right] = \int_{-\infty}^x e^{\nu y} dF_Z\left(y\right).$$

[4] Kluge, Hambly and Howison [15], Lemma 2.6.
[5] Kluge [16], Theorem 4.3.1.

If the moment generating function $m(\nu + i\theta)$ exists for some $\nu \in \mathbb{R}$ and all $\theta \in \mathbb{R}$, then

$$G_\nu(x) = \frac{m(\nu)}{2} - \frac{1}{\pi}\int_0^{+\infty} \frac{\Im\left(m(\nu + i\theta)\, e^{-i\theta x}\right)}{\theta} d\theta.$$

We can use this result to price put options because we have

$$E\left[(K - S_T)^+\right] = KE\left[\mathbf{1}_{S_T \le K}\right] - E\left[S_T \mathbf{1}_{S_T \le K}\right] = KG_0(\ln K) - G_1(\ln K),$$

and we can obtain the price of a call option from put-call parity.

Considering forward contracts, since electricity is a flow variable, they are in general specified over a whole delivery period, rather than one single maturity date. Let $f_t^{[T_1,T_2]}$ denote the strike price of a zero-cost forward contract with delivery period $[T_1, T_2]$ at time t. Formally, it is defined as

$$f_t^{[T_1,T_2]} = \int_{T_1}^{T_2} w(T; T_1, T_2)\, f_t^T dT,$$

where $w(T; T_1, T_2)$ is the weighting factor and is equal to $\frac{1}{T_2 - T_1}$ for settlement at maturity T_2 and to $\frac{re^{-rT}}{(e^{-rT_1} - e^{-rT_2})}$ for instantaneous settlement. We notice that such forward contracts are similar to Asian options[6]. An approximated method to price such options is to assume that the average price of the underlying is lognormally distributed and calculate its first and second moment. Then, using the Black [5] formula one obtains the approximated result (see Haug [10], for example). In our case, the first moment is given by:

$$M_1 = f_t^{[T_1,T_2]},$$

while the second moment is given by

$$\begin{aligned} M_2 &= E^Q\left[\left(\int_{T_1}^{T_2} w(T)\, f_{T_1}^{[T]}\right)^2 dT \mid \mathcal{F}_t\right] \\ &= \int_{T_1}^{T_2}\int_{T_1}^{T_2} w(T)\, w(T^*)\, E^Q\left[f_{T_1}^{[T]} f_{T_1}^{[T^*]} \mid \mathcal{F}_t\right] dT dT^*. \end{aligned} \tag{0.0.1}$$

Based on Black [5] formula for options on forwards, we give the following approximated price of a European call on a forward with delivery period:

$$c(t) = e^{-r(T-t)}\left[f_t^{[T_1,T_2]} N(d_1) - KN(d_2)\right],$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(f_t^{[T_1,T_2]}/K\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}, \\ \sigma^2 &= \frac{1}{T-t}\ln\left(\frac{M_2}{M_1^2}\right). \end{aligned}$$

The pricing of **swing options** is a more difficult problem. As already told, swings are volumetric options which allow the holder to exercise N times the right to vary the amount of commodity delivered, subject to daily and periodic constraints, over a predetermined period. In this way swing options provide for the flexibility necessary to deal with the complex patterns of consumption and the limited storability of energy.

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[6] **Asian options** are options where the payoff depends on the average price of the underlying asset during some specified period (Hull [11]).

Let the maturity date be a fixed T and the payoff at time t be for simplicity $(S_t - K)^+$, and let's assume that it is only possible to exercise one unit of the underlying at a time. Then, the value of the swing contract, $V(n, s, t)$, depends only on the price $S_t = s$ and the number n of exercise rights left. Kluge [16] formalises the optimization problem with the following equation:

$$V(n, s, t) = \max \begin{cases} e^{-r\Delta t} E^Q \left[V(n, S_{t+\Delta t}, t + \Delta t) \mid S_t = s \right] & , \\ e^{-r\Delta t} E^Q \left[V(n-1, S_{t+\Delta t}, t + \Delta t) \mid S_t = s \right] & + (s-K)^+ . \end{cases} \tag{0.0.2}$$

In other words, today's value is the highest between the expected tomorrow's value with n exercise rights left, and the expected tomorrow's value with $n - 1$ exercise rights plus the payoff from the exercise. According to Kluge [16], assuming that $V(k, s, t + \Delta t)$ is known for every k, s, then the expectation can be expressed as

$$E^Q \left[V(n, S_{t+\Delta t}, t + \Delta t) \mid S_t = s_i \right] \approx \sum_j V(n, s_j, t + \Delta t) \, p_{i,i+j}.$$

So, using the notation $V_{i,k}^n = V(n, s_i, t_k)$, the dynamic optimization can be written as

$$\begin{aligned} V_{i,k}^n &= \max \left\{ e^{-r\Delta t} \sum_j V_{j,k+1}^n p_{i,j}, \; e^{-r\Delta t} \sum_j V_{j,k+1}^{n-1} p_{i,j} + (s_i - K)^+ \right\}, \\ V_{i,k}^0 &= 0, \\ V_{i,m}^n &= 0, \end{aligned}$$

where the two boundary conditions mean that a swing option with no exercise rights left, and the one at maturity date T respectively, are both worth zero. If we assume that the mean-reverting process X_t and the jump process Y_t are both individually observable, then the value of the swing contract depends on both of them and we can write

$$V(n, x, y, t) = \max \begin{cases} e^{-r\Delta t} E^Q \left[V(n, X_{t+\Delta t}, Y_{t+\Delta t}, t + \Delta t) \mid X_t = x, \, Y_t = y \right] & , \\ e^{-r\Delta t} E^Q \left[V(n-1, X_{t+\Delta t}, Y_{t+\Delta t}, t + \Delta t) \mid X_t = x, \, Y_t = y \right] & \\ + \left(e^{f(t)+x+y} - K \right)^+ . & \end{cases}$$

In order to calculate the expectations, we need to create a non-uniform grid to perform the approximation and estimate the transition probabilities. Since X_t and Y_t are independent, the probability of arriving at the node $(X_{t+\Delta t}, Y_{t+\Delta t}) = (x_k, y_l)$ from a starting node of $(X_t, Y_t) = (x_i, y_j)$ is approximately given by

$$p_{i,j,k,l} \approx f_{X_{t+\Delta t} | X_t = x_i}(x_k) \cdot f_{Y_{t+\Delta t} | Y_t = y_j}(y_l) \, \Delta x \Delta y.$$

The conditional density of X_t is known, since $X_{t+\Delta t}$ following equation (3.2.8) is a normal variable with distribution

$$X_{t+\Delta t} \sim \mathcal{N}\left(x_i e^{-\alpha \Delta t}, \; \frac{\sigma^2}{2\alpha} \left(1 - e^{-2\alpha \Delta t} \right) \right).$$

As for the density of the spike process, we can use the approximations of the truncated spike process that we developed previously. A computer simulation of this method then gives a numerical value for the price of the swing option.

Conclusions

This thesis has tried to provide a picture of how mathematical and statistical instruments used in financial markets can be applied in energy markets and in particular in the case of electricity prices. Energy markets are interesting for many reasons:

— Their price evolution processes are unique: energy prices possess many unique characteristics. As commodities, their prices are mainly driven by supply and demand and often exhibit mean reversion. In addition, a distinctive feature of almost every energy commodity is the exceptional volatility of prices, higher than any other asset, and the frequent occurrence of spikes in the process.

— Energy derivatives are unique: we have seen that a number of energy derivative products are specific features of energy markets. Volumetric options, like swings or nominations, spread options or tolling agreement are all examples of products specifically developed to manage the risks associated with meeting the demand of natural gas or electricity. Even plain-vanilla products, for example forwards, exhibit unique features like delivery periods.

— Energy derivatives require special modeling methodologies: well-established models have to be adapted to energy markets providing them with particular features like stochastic volatility or a jump process, in order to address the specific characteristics of energy markets.

The model of [16] and [15] we analysed in the third chapter is an example of such efforts. The stochastic process introduced in the model is in fact capable of representing the main properties of electricity prices, namely, mean-reversion, seasonality and the occurrence of spikes. The model has the advantage of being analytically tractable, since it provides closed-form solutions of the expectation values for pricing path-independent options and an approximated pricing formula for options on forwards with a delivery period. Finally, the model proposes a grid-based method for pricing swing options, using the approximations of the probability distributions of the jump process.

We could have extended this thesis in several ways. Firstly, a computer simulation of the model would have been worthwhile, to test the performance of the model: in particular, we could have given a numerical implementation of the method for pricing swing options in order to perform a qualitative analysis of the prices at the variation of the parameters. In addition, the issue of the incompleteness of power markets and the consequent absence of a unique martingale measure, introduced at the beginning of section 3.3, would have deserved a deeper analysis.

Finally, we could have modified the model including alternative stochastic processes. For example, a natural development would have been to include a stochastic volatility process or a stochastic seasonality component. However we must notice that, even if such modifications would probably improve the ability of the model to fit historical data, they would make parameter calibration and option pricing further more difficult.

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