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Bioeconomic modeling

An optimal control approach

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Abstract

This thesis is about a continuous time dynamic model for the economic exploitation of natural resources such as fish stocks essentially due to Clark (2010). To study this bioeconomic model, I first develop from the mathematical field of optimal control theory a set of necessary conditions for optimality known as Pontryagin's maximum principle. Applying this tool to the bioeconomic model, I can compare the model's predictions to those of a baseline scenario characterized by unregulated access to the resource, and I find that in the latter case the equilibrium level of the resource is lower. Finally, I study the policy implications of this result, presenting and analyzing different types of regulation for the fishing industry such as input controls, which restrict fishing gear, output controls, which limit fishing effort, taxes, and quotas.

The main contribution of this work is to develop a dynamic bioeconomic model in a more rigorous optimal control formulation, and to analyze different forms of regulations in the framework of the model.

Contents

1 Introduction

An important and common problem in economics is how to manage assets, meaning anything that produces economic value and can be owned or controlled in some way. A first categorization of assets that has important implications from an accounting perspective is, as usual, between tangible and intangible ones since intangible assets other than financial assets are very hard to define and evaluate. From an economic perspective, however, a perhaps more meaningful division is between *renewable* and *non-renewable* assets. It is fairly difficult to characterize what a renewable asset truly is: since it is an asset, it must provide economic value, meaning that it cannot be in infinite supply, and it must be possible to exclude, at least under some circumstances, others from its use; since it is renewable, it must be *naturally* replaced.

The key elements in this definition are *(i)* the natural occurring process that replaces the resource and *(ii)* the economic nature of the resource. For example, sunlight and wind are indeed renewable, but they are also in infinite supply: what is in finite supply is the equipment necessary to produce, store and channel electricity from solar and wind power. Oxygen and fresh water are also renewable, and it can no longer be safely assumed that they are in infinite supply or that the access to water cannot be restricted, and they should therefore be considered as renewable assets. However, it looks foolish as well as criminal to put a price on water or air and limit access to these vital resources.

Finally, the last sizable category that could and does, in fact, fit the characteristics of renewable assets are living biological organisms (other than humans), since all natural species are indeed renewable and some plants and animals have a long history of economic exploitation. The discipline that studies how to optimally manage these assets is called "Bioeconomics" and is the subject matter of this thesis. For clarity, I collect the remarks from the above discussion in the following two tentative definitions.

Definition 1 (Renewable asset)**.** *A renewable asset is an asset that can be naturally replaced. Examples of renewable assets are biological systems such as forests, marine and freshwater resources, grasslands and deserts, and wildlife populations.*

Definition 2 (Bioeconomics)**.** *Bioeconomics, a field at the interplay of economics and biology, is the discipline that aims at developing a theory of optimal management of renewable assets.*

While they are fairly restrictive and can certainly be improved, these working definitions manage to exemplify the *economic* nature of the problem: at least on a first reading, renewable resources are just another class, although fairly peculiar, of assets, and their management is subject to similar economic incentives and constraints. The definitions, however, give no information on whether the issue of managing renewable resources is also a *relevant* one, meaning that even serious mismanagement of the assets does not severely affect society as a whole.

To give an idea of the magnitude of the problem, just considering one type of renewable assets, namely marine and freshwater resources, in the European Union alone about 85 000 and 116 000 employees work, respectively, in the fishing and fish processing industry; moreover, these two sectors produced economic value for about EUR 3.5 billion and EUR 30 billion in 2013. Equally importantly, in some European coastal communities fisheries are the main employer accounting for more than half of local jobs (European Commission 2014a). These few figures about fisheries may seem minor compared to EU level employment or GDP, but they do not convey the complete story: fish as a source of protein are an important component of a healthy diet, and as part of the marine ecosystem are one of the drivers for the "blue economy" that generates about 5.6 million jobs with an economic value of about EUR 495 billion per year (European Commission 2014b).

If renewable assets are indeed highly valuable to society, then it remains the question of how to optimally manage them. Continuing with the fishery example, the European Union, recognizing the importance of the fishing industry, tackles this issue with "Regulation (EU) No 1380/2013" and "Regulation (EU) 2015/812" of the European Parliament and of the Council that lay down the legal principles behind the "Common Fisheries Policy (CFP)." Quoting from these regulations, the CFP

> should ensure that fishing and aquaculture activities contribute to long-term environmental, economic, and social sustainability [...] should contribute to increased productivity, to a fair standard of living for the fisheries sector including small-scale fisheries, and to stable markets, and it should ensure the availability of food supplies and that they reach consumers at reasonable prices

In the above quotation, the emphasis is on the *economic* and *environmental sustainability* of the fishery policy, which is a fundamental and widely shared principle by managers of renewable resources.

Sustainability, which for the moment I do not try to define precisely, conveys the idea that exploitation of a natural resource cannot occur without bounds of any sort much like it happened in the past. A classical example of unsustainable harvesting is represented by the so called "tragedy of the commons:" while it is individually rational to harvest as much as possible from a common resource, such a behavior is not socially optimal and may lead to the depletion of the resource if it is not stopped in time. Fueled by a growing global demand and improvements in harvesting technology, a development along the lines of the tragedy of the commons seemed to be the norm for renewable assets just until the recent past, when resource managers began to implement new regulations that tried to limit the furious competition for a dwindling resource.

These few lines about over-harvesting of natural resources stress also the very import role of *time* for the bioeconomic problem. If harvesters do not have the luxury to postpone their harvesting decision to a more favorable moment, for example when fish stocks have reproduced, then the final outcome will probably be close to the tragedy of the commons, with a depleted natural population and many harvesters forced to quit their jobs. But what if they can instead more or less freely allocate their harvesting decision in time? What is the "optimal" strategy in this case? And is it also a "sustainable" strategy?

To answer these questions in a general way, managers and harvesters cannot phrase the problem exclusively in *static* terms but should explicitly consider its *dynamic* dimension. Recalling the economic nature of renewable assets, the problem "should ideally be cast in capital-theoretic terms" (Clark and Munro 1975), where the capital stock is nothing else but the natural population that, like traditional capital, can yield a sustainable consumption flow through time. Hence, the objective becomes to find the optimal harvesting policy which, in turn, determines the optimal response of the population since, under perfect information, the harvest path (or investment in the case of traditional capital) entirely characterizes the law of motion of the capital stock. As it is for the rest of capital theory when no strategic interactions are present, the management of renewable resources is essentially a problem in the area of mathematics called optimal control theory, about which I will speak at length later on in the thesis.

Proceeding in this way it remains, however, to define in what sense the harvesting policy so chosen should be optimal and sustainable. From the point of view of harvesters, a harvesting policy that maximizes present value profits, the harvesting decision being an inter-temporal one, is an easy candidate for optimality; it is also a sustainable one if it does not lead to depletion of the natural population. An easy way to check sustainability of a policy is to compare it with what I have so far called the tragedy of the commons, because, as in the fishery example, this is the natural outcome of unregulated open access resource exploitation. As it turns out and is intuitively true, the equilibrium population level when harvesters can carefully weight the future against the present by picking an harvest path that maximizes their discounted profits is always higher than the one under unregulated open access resource exploitation. Moreover, profitability under the first set of conditions is much higher than under the second one.

These results are, in turn, a sort of theoretical justification for regulating renewable assets: if managers want to maximize social welfare, then they can certainly not forgo economic profitability coupled with resource preservation. To achieve their objective managers have at their disposal a wide array of instruments, from taxes to fixed quotas of harvest, from controls on harvesting techniques to a ceiling on the maximum amount of harvest. More practically, if not to achieve the highest possible social welfare from the resource, it is certainly desirable not to reach the tragedy of the commons equilibrium, and between the aforementioned tools the most effective one to this end has proved to be some form of restricted access to the harvesting grounds. By limiting the number of harvesters and curtailing their effort, managers can align the incentive of harvesters towards a more long-run horizon avoiding the very short run competition for a depleted resource.

Finally, the structure of this thesis is similar to the order in which I have thus far proposed the core topics in this introduction. Hence, in Chapter 2 I present the field of bioeconomics, its objectives and its main tools; I also introduce different models that are common in the literature and discuss why they may or may not be appropriate. First, I examine models that focus exclusively on the biological aspects of renewable assets and completely neglect the economic implications, then others that combine the two sides of the problem but do so only in a static way, and finally at the end of the chapter I arrive to explain a dynamic bioeconomic model in three different forms.

To study this dynamic bioeconomic model, in Chapter 3 I introduce the field of optimal control theory, briefly discuss its connection with calculus of variation, and present the key ideas behind (part of) the theory. The main results of this chapter are the proof of a reduced version of a necessary optimality condition known as Pontryagin's maximum principle, and two theorems for the existence of optimal controls.

Having developed the necessary tools, in Chapter 4 I apply them to the study of the dynamic bioeconomic model for the case of a single price-taker firm. I re-derive in an alternative way a famous result due to Clark and Munro (1975) known as the "Golden rule" of bioeconomics for its similarity with the golden rule in capital theory.

Finally, in Chapter 5 I present the past and current effort at regulation of fisheries as an example of a renewable resource, and discuss why they may or may not be successful. To study the suitability of different regulations, I first introduce a definition of a social optimal population level, with an unregulated open access resource as baseline scenario, and then look at if and how different forms of regulation can modify the incentives for harvesters in such a way as to reach the social optimal population level. In Chapter 6 I conclude.

When writing the thesis, I used the book of Clark: "*Mathematical Bioeconomics*" like a sort of handbook for the modeling part as the choice and order of the topics treated in this thesis show. As Clark does, the models presented are, while simple, general enough in scope to include many sorts of renewable assets such as fisheries, forestry, and other wildlife populations. The focus is, however, always on fish stocks for many reasons: they are one of the renewable assets with the longest history of commercial harvesting; they are a widespread and economically important resource; they are also a challenging problem since their distribution in space is not fixed as it is for trees, and at least some species are mobile enough that their sustainable exploitation requires international agreements. Notwithstanding the last point, this additional complexity of the fishery problem is not present in this thesis, because as I mentioned before, this work aims at presenting few results that should hold qualitatively true for a general renewable asset, and because the subject matter is already mathematically complicated in its basic form. In this spirit, I also do not consider more realistic models such as multi-species models, growth and aging models, predator-prey models, etc.

To conclude, in this work my main contribution is the effort to develop the dynamic bioeconomic model of Chapter 2 in a more rigorous optimal control formulation, and to analyze different forms of regulations in the setting of the model.

2 Bioeconomics

In this chapter I begin to study the bioeconomic problem of managing renewable assets. In Sections 2.1 - 2.2 I introduce the necessary terminology and tools from the fields of biology and economics. In Sec. 2.3 I define different optimality concepts that resource managers could follow, and finally in Sec. 2.4 I present a general dynamic bioeconomic model that is the focus of this work.

2.1 The bioeconomic problem

Bioeconomics is a field at the interplay of economics and biology that aims at developing a theory of how to "best" exploit renewable assets such as forests, wildlife populations, and marine and freshwater resources, where the "best" way is a mixture of economics, biological, and possibly ethical considerations. The main ingredients of a bioeconomic model are on the *biological* side the dynamics of the exploited population, e.g., how the natural population grows and how it responds to harvesting, and on the *economic* side the dynamics of the economic system of reference, e.g., the general price level, the degree of competition in the market, or the time preference of economic agents.

Following Clark (2010), a basic population model consists of a simple differential equation (2.1) that captures the population harvesting dynamics plus some non-negativity constraints (2.2) on the population and harvesting rate

$$\frac{\mathrm{d}y}{\mathrm{d}t} = G(y(t)) - h(t), \quad y(0) = y_0 \tag{2.1}$$

$$y(t) \geq 0, \quad 0 \leq h(t) \leq \bar{h} \tag{2.2}$$

where $t \in [0, T]$ is a time index with T possibly infinite, $y : [0, T] \to [0, +\infty]$ is the population level at time t, $G(y)$ is some function that describes the growth rate of the population when no harvesting occurs, and $h : [0, T] \to [0, \bar{h}]$ is the harvesting rate at time t with $\bar{h}$ the maximum harvesting rate. Hereafter, whenever it is clear from the context I drop the time variable t to simplify notation.

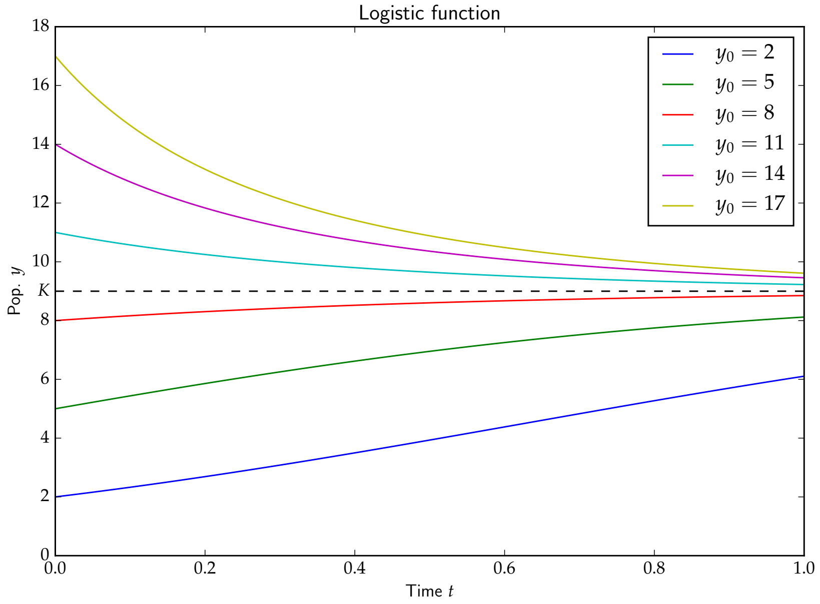

FIGURE 2.1: Plot of logistic function $y(t) = K/(1 + ce^{-rt})$

The simplest functional form for $G(y)$ that characterizes population dynamics and does not lead to unbounded growth is probably the logistic equation due to Verhulst

$$G(y(t)) = \frac{\mathrm{d}y}{\mathrm{d}t} = ry(t)\left(1 - \frac{y(t)}{K}\right) \tag{2.3}$$

with solution

$$y(t) = \frac{K}{1 + ce^{-rt}} \qquad \text{with } c = \frac{K - y_0}{y_0} \tag{2.4}$$

where r is the *intrinsic growth rate* of the population and K is the *carrying capacity* of the environment where the population lives. The introduction of this carrying capacity term K represents an important improvement over the exponential growth model $\dot{y}(t) = rt$, the dot denoting the time derivative of the population, because it bounds the growth of the system to some sort of natural threshold determined by the finite amount of resources available to the population. As Figure 2.1 shows, although the system may temporarily exceed the carrying capacity K if the initial population level y_0 is above K, the population level will eventually stabilize at the carrying capacity, which in this sense represents the long-term equilibrium of the system when no harvesting occurs. More generally, many functions that are suited to model population growth are solutions of the *generalized logistic equation*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ry^{\alpha}\left[1 - \left(\frac{y}{K}\right)^{\beta}\right]^{\gamma} \tag{2.5}$$

where α, β, γ are positive real numbers whose biological significance depends on the modeled population, and r, K have the same interpretation as before: the logistic equation (2.3) is a particular case of Eq. (2.5) with $\alpha = \beta = \gamma = 1$ (Tsoularis and Wallace 2002). As with the logistic function, a population that behaves according to a generalized logistic function asymptotically reaches its carrying capacity K

$$\lim_{t\to\infty} y(t) = K \tag{2.6}$$

attaining maximum growth rate at the point

$$y_{\text{inf}} = \left(1 + \frac{\beta\gamma}{\alpha}\right)^{-\frac{1}{\beta}} K \tag{2.7}$$

provided that $y_0 < y_{\text{inf}}$. Finally, the point of maximum relative growth rate $(\mathrm{d}y/\mathrm{d}t)(1/y)$ is given by the following expression

$$y_{\text{rel}} = \left(1 + \frac{\beta\gamma}{\alpha - 1}\right)^{-\frac{1}{\beta}} K \tag{2.8}$$

if $y_0 < y_{\text{rel}}$. For the logistic model of Eq. (2.3), the points of maximum absolute and relative growth rate are $y_{\text{inf}} = K/2$ and $y_{\text{rel}} = 0$.

The logistic framework introduced thus far, while allowing to mathematically model natural populations, ignores any harvesting decisions from private firms or public institutions. As outlined in the introduction, the bioeconomic question of managing renewable assets is essentially a problem in capital theory, and as such economic incentives and constraints shape harvesting policies. Adopting terminology from fishery bioeconomics, the choices of harvesters can be cast both in terms of harvest $h(t)$ or "effort" $E(t)$, where the unit of measure of the latter are, for fisheries, "Standard Vessel Units (SVU)," and the problem is perfectly equivalent under both formulations since to each effort level corresponds a harvest rate.

To formalize this relationship, Schaefer (1954) developed its famous equation

$$h(t) = qE(t)y(t) \tag{2.9}$$

where q is called the *catchability* coefficient and represents the proportion of the stock of the resource $y(t)$ harvested per unit of effort. According to this equation, for a given effort level E the harvest is directly proportional to the current population level y with q the constant of proportionality, or alternatively the fraction h/y of harvested population relative to total is directly proportional to the exerted effort E.

While Eq. (2.9) models a possible relationship between the "catch" or harvest h, the population level y, and the harvest level E, it alone cannot explain why harvesters would choose an effort level E_1 rather than E_2 or E_3. Obviously, the price level and the cost of effort matter when deciding how much to harvest, and for firms a simple profit function is of the type

$$\Pi(y,h) = Ph - cE = Ph - h\frac{c}{qy} = [P - c(y)]h \tag{2.10}$$

$$\Pi(y,E) = Ph - cE = PqEy - cE = [Pqy - c]E \tag{2.11}$$

with P the price of the harvested resource, c some constant marginal cost, and $c(y) := c/(qy)$. As I mentioned before, Eqs. (2.10) and (2.11) are perfectly equivalent since using the Schaefer equation (2.9) and as long as $y > 0$, I can always pass from effort E to harvest h according to convenience. The interpretation of the two equations are fairly straightforward, revenues for price-taker firms being simply the harvest times its price and costs being proportional to the effort level E.

2.2 The Schaefer brake and depensation

While the Schaefer equation seems a very convenient tool to model the catch-effort relationship, a caveat of Eq. (2.9) is that, since the level of effort E models how difficult it is for the harvester to screen a natural resource, Eq. (2.9) implicitly assumes that the population is always distributed uniformly over its living area. This assumption may not be reasonable if members of a population are able to move to a new and "better" foraging area, whereby I intend an area more abundant in food or other resources necessary to the survival of the population, when their old one has decreased in quality because of exogenous, such as natural predation and harvesting, or endogenous factors such as increased competition for food. More generally, according to the "Ideal free distribution" theory, if animals are aware of the resource concentration of each foraging area, are *free* to move between patches of resources, and can *ideally* pick the best one, then the distribution of the population between patches will be proportional to their resource concentration (Fretwell and Calver 1969). In the fishing industry, for example, Eq. (2.9) may not always hold, since harvesting by fishermen is seldom at random and instead targets the spots with the highest density of fish, which in turn are free to move to better patches. Hence, an "improved" form of the Schaefer equation is

$$h = q\rho(y)E \tag{2.12}$$

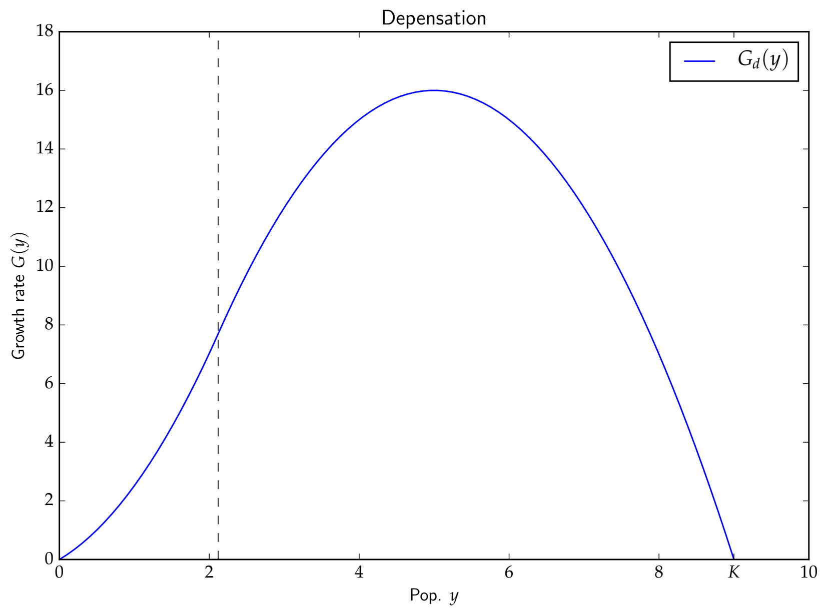

FIGURE 2.2: Depensatory growth

where $\rho(y)$ denotes the average concentration of the population in some patch when the overall stock level is y.

Furthermore, another implication of the Schaefer catch equation (2.9) is that no extinction of the population is possible since it is not economically profitable: the profits from harvesting are

$$\Pi(y,h) = P(h)h - cE = \left[P(h) - \frac{c}{qy}\right]h \tag{2.13}$$

and they become infinitely negative as the population stock approaches zero, meaning that a natural "brake" on harvesting exists for low level of the population. In reality, however, many species have been commercially harvested to extinction suggesting that there might exist a "threshold" which is not present in the simple models introduced thus far and below which population dynamics $G(y)$ change. This idea lead to the concept of "depensation," which according to Liermann and Hilborn (2001) I define as follows.

Definition 3 (Depensation)**.** *A population's dynamics are depensatory if the per-capita rate of growth decreases as the population density decreases to low levels. Denoting by $g(y)$ the per-capita rate of growth, depensation occurs whenever $\dot{g}(y) > 0$ where*

$$g(y) = \frac{\dot{y}}{y} = \frac{G(y)}{y}$$

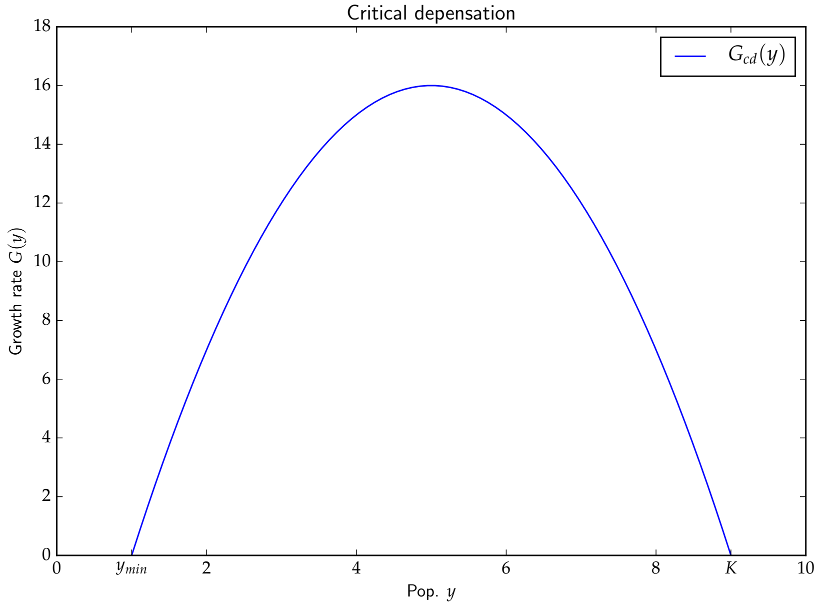

FIGURE 2.3: Critical depensation

and $G(y)$ is the natural growth rate of the population.

Hence, to have depensation at low level of the population y the growth function $G(y)$ cannot be convex, since

$$\dot{g}(y(t)) = \frac{\ddot{y}(t)y(t) - \dot{y}(t)^2}{y(t)^2} = \frac{\ddot{y}(t)}{y(t)} - g(y(t))^2$$

and $\dot{g}(y)$ can be positive only if $\ddot{y}(t) > 0$ as Fig. 2.2 shows for the curve $G_d(y)$ when the stock is low (non-convexity to right of the black dashed line). A practical implication of depensation is that since when population is low enough the per-capita growth rate does not kick up, then it may take quite some time before the stock can recover.

While depensation *per se* does not imply the extinction of the stock, for some populations once a critical value is reached there is no possibility of recovery. As Fig. 2.3 shows, if the stock falls below the threshold y_{min}, then the growth rate $G_{cd}(y)$ becomes negative eventually leading the population to extinction.

Definition 4 (Critical depensation)**.** *Critical depensation in a population occurs when the per-capita growth rate becomes negative at low levels of the stock.*

Finally, in view of the above discussion, it may seem doubtful to keep using the standard Schaefer equation (2.9) and the same logistic functional form for the growth function $G(y)$ that does not allow depensatory dynamics. While depensation is indeed a serious problem, there is no way at the moment to know if a population exhibits depensation before it occurs, meaning that a model that incorporates depensation is potentially as wrong as a model that does not. Hence, to keep the basic model as simple as possible, I continue assuming throughout the text a logistic growth function, and, always for convenience, I continue using the standard Schaefer equation (2.9) instead of the modified one (2.12).

2.3 Optimality in bioeconomics

Bionomic equilibrium

With the economic and biological dynamics at hand, it should be time to try defining what is the "optimum" that resource managers in bioeconomics should try to achieve, but before attempting this task I recall here an important point from Chapter 1. As the chapter briefly explains, renewable assets have traditionally been in a condition of *unregulated open access*, and in this case economic theory predicts that as long as the industry is profitable, the cost of entry is low enough, and incumbent firms cannot credibly deter entry, new firms will enter the market until revenues from harvesting are just enough to cover costs. From Eq. (2.10), economic profits are zero if

$$\Pi(h, y) = 0 \iff \left[P - \frac{c}{qy}\right] h = 0 \implies y_{BE} = \frac{c}{Pq} \tag{2.14}$$

leading to the following equilibrium concept due to Gordon (1954).

Definition 5 (BE)**.** *The "Bionomic equilibrium (BE)" of a harvestable resource is the harvesting rate h_{BE} such that for any $h' > h_{BE}$ harvesting becomes no longer economically profitable.*

In principle, bionomic equilibrium is not necessarily the worst outcome from the point of view of preservation of the natural resource: as Eq. (2.14) shows, unless prices P are high enough for a long period, costs c are low, or the catchability q of the population is high, the equilibrium level y_{BE} may be considerably far away from depletion of the population. In practice, however, for resource managers the bionomic equilibrium is a fairly bad state of affairs since the population stock y_{BE} is typically much lower than its sustainable level. Furthermore, as the theory shows, bionomic equilibrium is also a sort of natural and "unavoidable" outcome for renewable assets and especially fish stocks, since the conditions of initial high profitability and unregulated free entry have historically been the norm rather than the exception. Hence, avoiding the pitfalls of bionomic equilibrium seems like a truly daunting task for resource managers, but not all hope is lost for at least two reasons:

1. clearly, *if* the assumption of unregulated open access to the resource fail, then the prediction of bionomic equilibrium no longer holds;

2. more importantly, the BE concept does not take into account the dynamic dimension of the problem. If economic agents are not forced to a scramble competition and can instead plan ahead, their harvesting decision and corresponding population level may be very different from the ones under bionomic equilibrium.

Maximum sustained yield

A first candidate for optimality is the widely used "Maximum sustained yield (MSY)," which ignores the market value of the harvested resources, and simply looks at the highest harvest rate h that may be sustained indefinitely without depleting the population.

Definition 6 (MSY)**.** *The "Maximum sustained yield (MSY)" for a harvestable resource is the largest harvest rate h_{MSY} that can be sustained over an indefinite period and does not lead to depletion of the resource.*

As usual when looking for an interior maximum, at the MSY the first order necessary condition $G'(y_{MSY}) = 0$ must hold true: if $h < G(y_{MSY})$ then it is always possible to increase the harvesting to some h_1 with $h < h_1 \leq G(y_{MSY})$ without depleting the population; if instead harvesting continues at a rate $h_2 > h_{MSY}$ then this strategy will ultimately extinguish the population as $y(t) \to 0$ for large t. In Fig. 2.4 I plot the MSY for the logistic growth function (2.3) which is

$$h_{MSY} = \frac{K}{2} \tag{2.15}$$

and is just a particular case of the generalized logistic equation given by (2.7)

$$h_{MSY} = \left(1 + \frac{\beta\gamma}{\alpha}\right)^{-\frac{1}{\beta}} K \tag{2.16}$$

as remarked before.

As the figure shows for the logistic case, a first caveat of the MSY is that the optimal harvest rate h_{MSY} is an unstable equilibrium: inaccurate estimates of the population may produce an "optimal" harvest rate $h' > h_{MSY}$ which actually leads to over-harvesting of the resource stock and possibly to its extinction. A second caveat of the MSY approach is that it ignores the economic nature of the problem: what if firms (or the government) extract "too much," but still less than the MSY level, of some resource leading to supply in excess of demand? Evidently, it is sub-optimal to over-supply some commodity and a bioeconomics optimality concept must at least take into

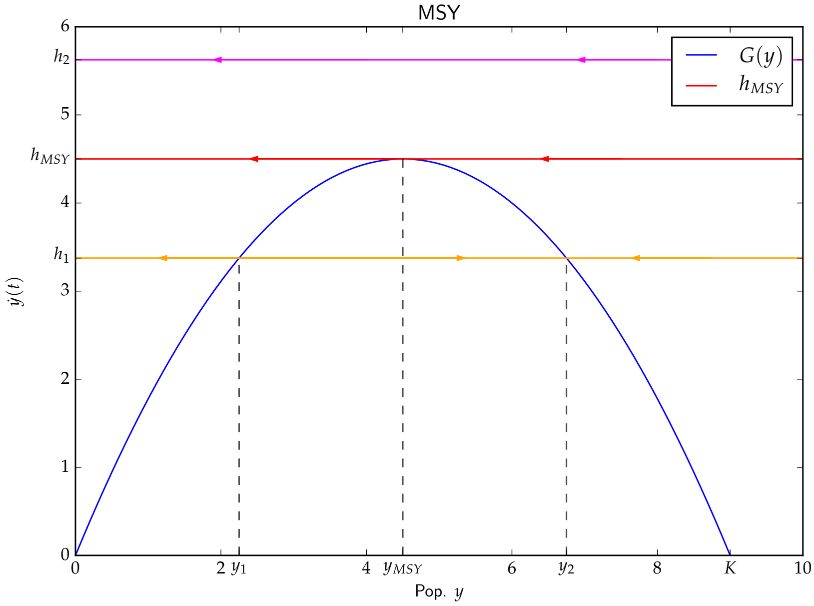

FIGURE 2.4: MSY with logistic growth

account the market value of the harvested resource. Finally, its last caveat is that, like the BE, it does not take into account the dynamic nature of the problem.

Maximum economic yield

Alternatively to the MSY, an approach that explicitly considers the economic aspect of the problem is to proceed as in Eq. (2.14)

$$\Pi(y,h) = Ph - cE = \left[P - \frac{c}{qy}\right] h = [P - c(y)]h \tag{2.17}$$

$$\text{with } \dot{y} = G(y) - h \tag{2.18}$$

and look for a population level y^* and harvest rate h^* that maximizes economic profits without causing to stock depletion. Imposing the steady state condition $\dot{y} = 0$ and substituting $h^* = G(y)$ in the profit equation yields

$$\Pi(y,h) = Ph - cE = \left[P - \frac{c}{qy}\right] G(y) = [P - c(y)]G(y) \tag{2.19}$$

suggesting that the population level y^* that maximizes Eq. (2.19) is a sort of "sustainable" economic equilibrium, which leads to the following definition.

Definition 7 (MEY). *The "Maximum economic yield (MEY)" is the population level y_{MEY} at which harvesting is sustainable, meaning that it does not lead to de-*

pletion of the stock, and that maximizes economic profits of the form

$$\Pi(y,h) = \left[P - \frac{c}{qy}\right] h \tag{2.20}$$

Looking at Eq. (2.19), the first and second order condition for an interior maximum are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}\Pi(y) &= -c'(y)G(y) + [P - c(y)]G'(y) = 0\\ \frac{\mathrm{d}^2}{\mathrm{d}y^2}\Pi(y) &= -\left[P - c(y)\right]G''(y) - G(y)c''(y) - 2G'(y)c'(y) < 0\end{aligned}$$

Assuming a logistic growth function, $G(y) = ry(1 - y/K)$, the MEY is

$$G'(y_{MEY}) - \frac{c'(y_{MEY})G(y_{MEY})}{[P - c(y_{MEY})]} = 0 \tag{2.21}$$

since the second order condition is $\mathrm{d}^2\Pi(y)/\mathrm{d}y^2 = -2Pr/K < 0$ and $P > c(y_{MEY})$.

While it may appear that the maximum economic yield strikes the right balance between preservation of the natural resource and economic profitability, the MEY concept, like the BE and MSY, ignores the very important dynamic dimension of the bioeconomic problem, both from a biological and economic perspective.

A dynamic model

If a static model, even if grounded in sound biological and economic theory, is lacking (but may be a good approximation in some cases) because it ignores the time dimension of the problem, a model that is dynamic both from a biological and economic standpoint could be a viable alternative.

On the economic side, time dynamics are in terms of profits or utility: recalling the capital-theoretic formulation of renewable assets, before acting (e.g., consuming, investing), rational economic agents must formulate a plan that specifies their decisions up to the relevant (possibly infinite) final time horizon T. As usual, agents *discount* the future generally preferring present rewards to future ones, leading to a present value computation that in the case of a harvesting firms reads

$$PV(\Pi) = \int_0^T e^{-\delta t}\Pi(y(t), h(t))\,\mathrm{d}t$$

where $\delta \in [0, +\infty)$ is the instantaneous discount rate that if close to 0 implies no "impatience" for consumption, $e^{-\delta t}$ is the continuous time discount factor, and $\Pi(y(t), h(t))$ are the instantaneous profits from harvesting. In this formulation the harvester should choose, if it exists, a harvesting strategy $h^*(t)$ that maximizes the present value of its (certain) future profits.

On the biological side, since firms must specify at each point in time their harvesting decision $h(t)$, they must carefully exploit population dynamics $G(y(t))$ because if they harvest too much today, the population growth $\dot{y}(t)$ may become negative leading to depletion of the stock.

To recap, unlike the MSY, BE, or MEY, this dynamic formulation considers, at least theoretically, *(i)* both the biological and economic side of the problem and *(ii)* the need to continuously adjust harvesting strategy $h(t)$ due to feedback effects with the population level $y(t)$.

2.4 Three dynamic bioeconomic models

If the dynamic model is the one that better captures the main aspects of the bioeconomic problem, various market structures may lead *ceteris paribus* to very different outcomes both from an economic and biological standpoint. While the in the rest of the thesis I focus mainly on a the case of a single firm operating under conditions of perfect competition (or better when the firm is a price taker), for the sake of completeness I present here three possible cases: *(i)* perfect competition, *(ii)* monopoly, and *(iii)* social planner.

Perfect competition Under perfect competition, the bioeconomic model becomes

$$\Pi_c^d = \int_0^T e^{-\delta t}\Pi(y(t), h(t))\,\mathrm{d}t = \int_0^T e^{-\delta t}[P - c(y(t))]h(t)\,\mathrm{d}t \tag{2.22}$$

$$\dot{y}(t) = ry(t)\left(1 - \frac{y(t)}{K}\right) - h(t) \tag{2.23}$$

$$y(t) \geq 0, \qquad y(0) = y_0 \qquad 0 \leq h(t) \leq \bar{h} \tag{2.24}$$

with $y(0) = y_0$ and as usual the aim of the economic agent is to maximize Eq. (2.22) subject to the constraints (2.23) - (2.24). The profit function under perfect competition is equal to Eq. (2.10) with the same considerations.

Monopoly Under a monopolistic regime, the bioeconomic model becomes

$$\Pi_m^d = \int_0^T e^{-\delta t}[P(h(t)) - c(y(t))]h(t)\,\mathrm{d}t \tag{2.25}$$

where $P(h)$ is the inverse demand function $P(Q)$ evaluated at the harvest level h, and the maximization of Π_m^d is subject to the constraints (2.23)-(2.24) as under perfect competition.

Social planner With the social planner, the bioeconomic model becomes

$$\Pi_s^d = \int_0^T e^{-\delta t}[U(h(t)) - c(y(t))h(t)]\,\mathrm{d}t \quad \text{with } U(h) = \int_0^h P(Q)\,\mathrm{d}Q \tag{2.26}$$

where $U(h)$ is the social utility of consumption and $U(h) - c(y)h$ is the associated consumer surplus. As in the previous cases, the maximization of Π_s^d with respect to h is subject to the same constraints (2.23)-(2.24).

The interpretation of $U(h)$ and of the associated consumer surplus is linked to the inverse demand function $P(Q)$: assuming $P(Q)$ as downward-sloping and abstracting from the formal microeconomic derivation of a demand correspondence, a possible interpretation of $P(\cdot)$ is as the aggregation of the willingness-to-pay of a continuum of consumers with unit demand. At any price $P = P(Q)$ for some quantity Q, the marginal consumer with valuation $v = P(Q)$ will buy the good and retain zero surplus since he is paying exactly his valuation of the good, but also all the consumers with valuations $v' = P(Q')$ for any $0 < Q' < Q$ will buy the good and get instead a positive surplus $v' - P > 0$. Hence, in the consumer surplus expression $U(h) - c(y)h$, $U(h)$ measures exactly how much surplus consumers receive from the harvested resource if they could acquire the good at zero cost, while $c(y)h$ measures the costs of providing the good, remembering from Eq. (2.10) that $c(y)h = hc/(qy) = cE$, i.e., $c(y)h$ is exactly how much effort a harvester must exert under constant marginal costs.

Finally, to ease notation I summarize all three bioeconomic models as

$$\max_h \int_0^T e^{-\delta t}\,[M(h(t)) - c(y(t))h(t)]\;\mathrm{d}t \tag{2.27}$$

$$\dot{y}(t) = ry(t)\left(1 - \frac{y(t)}{K}\right) - h(t) \tag{2.28}$$

$$y(t) \geq 0, \qquad y(0) = y_0 \qquad 0 \leq h(t) \leq \bar{h} \tag{2.29}$$

where $M(h) = Ph$ for the perfect competition case, $M(h) = P(h)h$ for the monopolist, and $M(h) = U(h)$ for the social planner. As a concluding remark to this section, I must add that contrary to many models in economics I consider a finite horizon T for the maximization problem, reason being that a finite T greatly simplifies the theory developed in Chapter 3, and that for all practical purposes it suffices to consider a long enough (but not necessarily infinite) time period.

3 Control Theory

In this chapter I present the necessary tools from the field of optimal control theory to study the dynamic bioeconomic model of Chapter 2. In Sec. 3.1 I introduce preliminary definitions and cast the bioeconomic problem in a precise optimal control theoretic formulation. In Sec. 3.2 I state and prove a reduced version of a necessary condition for optimality known as the Pontryagin's maximum principle, and in Sec. 3.3 I present two theorems for the existence of optimal controls.

3.1 Optimal control formulation

For the bioeconomic model of Chapter 2, the mathematical field of control theory allows to find, under suitable assumptions, necessary and sufficient conditions for the optimal harvesting strategy $h^*(\cdot)$, if it exists. In some generality a typical optimal control problem in *Bolza form* is

$$\max_{u(\cdot)} J(u(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), u(t))\, \mathrm{d}t + K(t_1, x(t_1)) \tag{3.1}$$

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{3.2}$$

where $L : [t_0, t_1] \times \mathbb{R}^n \times U \to \mathbb{R}$ is the *running cost*, $K(t_1, x(t_1)) : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}$ is the *terminal cost*, $f : [t_0, t_1] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is the *state equation*, $J : \mathcal{U} \to \mathbb{R}$ is the *performance index* with $\mathcal{U}$ some function space, and U is a closed subset of $\mathbb{R}^m$. The function $u : [t_0, t_1] \to U \subseteq \mathbb{R}^m$ is the *control* of the system and $x : [t_0, t_1] \to \mathbb{R}^n$ is the *state* with given initial condition $x(t_0) = x_0$ and free terminal condition $x(t_1)$.

Two important special cases of the Bolza problem are the *Lagrange form*, where the terminal cost $K(t_1, x(t_1)) \equiv 0$, and the *Mayer form*, where the running cost $L(t, x, u) \equiv 0$. These three formulations are all equivalent, since to go from a Bolza problem to a Lagrange one, assuming that all the derivatives that appear in what follows exist, I can write

$$\begin{aligned}
K(t_1, x(t_1)) &= K(t_0, x(t_0)) + \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t} K(t, x(t))\,\mathrm{d}t \\
&= K(t_0, x(t_0)) + \int_{t_0}^{t_1} K_t(t, x(t)) + K_x(t, x(t))\dot{x}(t)\,\mathrm{d}t \\
&= K(t_0, x(t_0)) + \int_{t_0}^{t_1} K_t(t, x(t)) + K_x(t, x(t)) f(t, x(t), u(t))\,\mathrm{d}t
\end{aligned}$$

and, because $K(t_0, x(t_0))$ is a constant that does not depend on u and does not affect the maximization problem, denoting by

$$\tilde{L}(t, x, u) = L(t, x, u) + K_t(t, x) + K_x(t, x) f(t, x, u) \tag{3.3}$$

I can replace the original running cost $L(t, x, u)$ in Eq. 3.1 with the modified running cost $\tilde{L}(t, x, u)$ yielding a problem with no terminal cost $K(t_1, x(t_1))$. Similarly, to go from the Lagrange form to the Mayer form, I can define a new vector-valued state variable $x(t)' = (x_1(t), x_2(t))' \in \mathbb{R}^{n+1}$ as

$$\dot{x}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} f(t, x(t), u(t)) \\ L(t, x(t), u(t)) \end{pmatrix} =: g(t, x(t), u(t)) \tag{3.4}$$

with initial conditions $x(t_0) = (x_1(t_0), x_2(t_0))' = (x_0, 0)'$. In this formulation the performance index of the system is

$$\phi(x(t_1)) = x_{21} := x_2(t_1) = J(u(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), u(t))\,\mathrm{d}t$$

The Mayer form is especially convenient for deriving the necessary conditions for optimality of Sec. 3.2, but to develop a meaningful theory I must first introduce some preliminary definitions and results. Hence, following Liberzon (2012) I begin my brief voyage in the field of optimal control theory with a precise definition of control strategy and control set.

Definition 8 (Control function). *A control u is a piecewise continuous function $u : [t_0, t_1] \to U$. I denote by $\mathcal{U}_{PC}[t_0, t_1]$ the set of all piecewise continuous control functions.*

Lemma 1 (Control set). *The class $\mathcal{U}_{PC}[t_0, t_1]$ has the following property: if $u(\cdot) \in \mathcal{U}_{PC}[t_0, t_1]$, $v_i \in U$ for $i = 1, 2, \ldots, m$ and $\tau_i - h_i < t \leq \tau_i$ are non-overlapping intervals intersecting $[t_0, t_1]$, then $\tilde{u}(\cdot) \in \mathcal{U}_{PC}[t_0, t_1]$*

$$\tilde{u}(t) = \begin{cases} v_i & \text{if } \tau_i - h_i < t \leq \tau_i \\ u(t) & \text{if } t \in [t_0, t_1] \text{ and } \notin \bigcup_i (\tau_i - h_i, \tau_i] \end{cases}$$

Proof. Since the function $u : [t_0, t_1] \to U$ is piecewise continuous, then the sum $u(t) + \tilde{v}_i(t)$, where $\tilde{v}_i(t) : [t_0, t_1] \to U$ is a piecewise continuous function such that

$$\tilde{v}_i(t) = \begin{cases} v_i - u(t) & \text{if } \tau_i - h_i < t \leq \tau_i \\ 0 & \text{else} \end{cases}$$

$$\tilde{u}(t) = \sum_{i=1}^{m} \tilde{v}_i(t) + u(t)$$

is also piecewise continuous and belongs to $\mathcal{U}_{PC}[t_0, t_1]$. Hence, being the sum of piecewise continuous functions, $\tilde{u}(t)$ also belongs to $\mathcal{U}_{PC}[t_0, t_1]$. □

The first issue in deriving an optimality condition is that problem (3.1) - (3.2) needs to be *well posed*, meaning that for every admissible control $u(t)$ and initial conditions (t_0, x_0) the state equation (3.2) needs to have a unique solution $x(t)$ over some time interval $[t_0, t_1]$.

Definition 9 (Well posed system)**.** *An optimal control problem in Lagrange form is well posed if for every admissible control $u(t)$ and initial conditions (t_0, x_0) the state equation* (3.2) *has a unique global solution $x(t)$.*

For example, the following theorem, whose proof can be found in Hale (2009), with the accompanying definition provides sufficient conditions to have a well-posed system.

Theorem 1. *If $f(t, x, u)$ is continuous in t and u and C^1 in x, $f_x(t, x, u)$ is continuous in t and u, then there exists a unique local solution to Eq.* (3.2)*. If in addition the solution $x(t) = x(t; t_0, x_0, u)$ is bounded for each t such that $x(t)$ exists, then the problem* (3.1) - (3.2) *is well-posed.*

Definition 10 (State function)**.** *For a control $u(t)$ defined on $[t_0, t_1]$, the solution $x(t)$ of the differential equation $\dot{x}(t) = g(t, x(t), u(t))$ with initial condition $x(t_0) = x_0$ is called state, trajectory or response to the control $u(t)$ and initial condition x_0.*

While the aim of this chapter is to develop the necessary techniques from optimal control theory, the main objective of this thesis is to apply them to the bioeconomic model of Eqs. (2.27) - (2.29). As remarked before, it is notationally more convenient to state and prove the necessary conditions for an optimal control in Mayer form rather than in Lagrange or Bolza form. Hence, going back to the bioeconomic model of Eqs. (2.27) - (2.29)

$$\max_{h} \int_0^T e^{-\delta t} \left[M(h(t)) - c(y(t))h(t) \right] \mathrm{d}t \tag{3.5}$$

$$\dot{y}(t) = r y(t) \left(1 - \frac{y(t)}{K} \right) - h(t) \tag{3.6}$$

$$y(t) \geq 0, \qquad y(0) = y_0 \qquad 0 \leq h(t) \leq \bar{h} \tag{3.7}$$

in Mayer form (ignoring the non-negativity constraints) it becomes

$$\dot{x}(t) = \begin{pmatrix} \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = g(t, x(t), u(t)) \tag{3.8}$$

where

$$g(t, x, u) = \begin{pmatrix} f(x, u) \\ L(t, x, u) \end{pmatrix} = \begin{pmatrix} ry(1 - y/K) - u \\ e^{-\delta t} (M(u) - c(y)u) \end{pmatrix} \tag{3.9}$$

where y is the population level and $x = (y, z)' \in \mathbb{R}^2$ as in Eq. (3.4). The control is the harvesting rate $u \equiv h$; the set U is the interval $[0, \bar{h}]$ of maximum harvesting effort, and the time interval $[t_0, t_1]$ is $[0, T]$ with T fixed. It is clear from this formulation that $f(x, u), L(t, x, u)$ and $g(t, x, u)$ respect the continuity assumptions of Theorem 1 and since by Lemma 2 the solution $y(t; t_0, x_0, u)$ to Eq. (3.6) is also bounded, then the state equation (3.8) admits a unique global solution.

Lemma 2. *Let $u(t)$ be an admissible control and $y(t; u)$ the solution to $\dot{y}(t) = G(y(t)) - u(t)$, where $G(y)$ is the logistic equation* (2.3). *Then $y(t; u)$ is bounded.*

Proof. The solution $y(t; u)$ is bounded below by the non-negativity constraint $y(t) \geq 0$. Since

$$y(t; u) = y_0 + \int_0^t G(y(s)) \, \mathrm{d}s - \int_0^t u(s) \, \mathrm{d}s \leq x_0 + \int_0^t G(y(s)) \, \mathrm{d}s = y(t; 0)$$

then $y(t; u)$ is also bounded from above because

$$y(t; u) \leq y(t; 0) = \frac{y_0 K}{y_0 + (K - y_0) \exp^{-rt}} \leq \max\{y_0, K\}$$

□

3.2 Pontryagin's maximum principle

Preliminary results

The main result of this section is a set of necessary conditions for the optimal control problem in Mayer form called "Pontryagin's maximum principle (PMP)", and the key ideas in the proof are from the field of calculus of variations, now a part of the more general optimal control theory.

In classical calculus of variations the objective is to maximize a functional of the form

$$J(y) = \int_{t_0}^{t_1} L(t, y(t), \dot{y}(t)) \, \mathrm{d}t \tag{3.10}$$

without any state equation (3.2) or control $u(t)$. It is clear from this formulation that the calculus of variation problem is also an optimal control one by defining $u := \dot{y}$ and

$$J(u) = \int_{t_0}^{t_1} L(t, y(t), u(t))\, \mathrm{d}t \quad \text{with } \dot{y}(t) = u(t) \tag{3.11}$$

Before approaching the maximization of Eq. (3.10), I must define what is a "maximum" $y^* \in \mathcal{K}$ of the functional $J : \mathcal{K} \to \mathbb{R}$ where $\mathcal{K} \subseteq \mathcal{Y}$ and $\mathcal{Y}$ is some vector space where all the y functions live: for example, in the optimal control case by Def. 8 the vector space $\mathcal{Y}$ is the class of all piecewise continuous functions. As it is the case for a multivariable function $f : \mathbb{R}^n \to \mathbb{R}$, for the functional J both the notions of *local* and *global* maximum are possible and to make sense of them I must equip the space $\mathcal{Y}$ with a norm $\|\cdot\|$. The norm in turn induces the concept of *distance* or *metric* between two elements y_1, y_2 of $\mathcal{Y}$, $d(y_1, y_2) := \|y_1 - y_2\|$. I refer to the Appendix A for a more precise discussion.

Finally, for the optimal control case where $\mathcal{Y} = \mathcal{U}_{PC}[t_0, t_1]$ the norm I use is

$$\|u\|_\infty = \sup_{t_0 \le t \le t_1} |u(t)|$$

and I have the following definition of a *strong* local (and global) maximum.

Definition 11 (Local and global maximum)**.** *A function $y^* \in \mathcal{Y}$ is a local maximum of J over $\mathcal{Y}$ if there exists a $\epsilon > 0$ such that $J(y^*) \ge J(y)$ for all $y \in \mathcal{Y}$ satisfying $\|y - y^*\|_\infty < \epsilon$. If $J(y^*) \ge J(y)$ holds for all $y \in \mathcal{Y}$ then y^* is a global maximum.*

Returning to the calculus of variations problem of Eq. (3.10), the main intuition is that if y^* is a maximum (global or local), then for any small "perturbation" y_ϵ of y^* the value of $J(\cdot)$ should not change much. The strategy is similar to the multivariate case $f : D \subset \mathbb{R}^n \to \mathbb{R}$ with D an open set: let x^* be a local maximum of f, $d \in \mathbb{R}^n$ some arbitrary direction, and $\epsilon(d)$ be such that $x^* + \epsilon d$ is still in D for all $0 \le \epsilon < \epsilon(v)$. Then calling $g(\epsilon) := f(x^* + \epsilon d)$ and assuming f differentiable, it must be that

$$\lim_{\epsilon \downarrow 0} \frac{g(\epsilon) - g(0)}{\epsilon} = \frac{\mathrm{d}}{\mathrm{d}\epsilon} g(\epsilon)|_{\epsilon=0} \le 0 \tag{3.12}$$

since $g(\epsilon) \le g(0) = f(x^*)$ and $f(x^*)$ is a local maximum. Hence, following Fleming and Rishel (2012), the same argument yields a similar necessary condition for the maximum of a functional J for the case of a "general" perturbation of the type $u^\epsilon = \zeta(\epsilon)$ for some function ζ.

Theorem 2 (Conditions for a maximum). *Let $u^* \in \mathcal{K}$ be a maximum of the functional $J : \mathcal{K} \subseteq \mathcal{Y} \to \mathbb{R}$. Define a mapping $\zeta : [0, \eta] \to \mathcal{K}$ such that $\zeta(0) = u^*$. If the composite function $f(\epsilon) = J(\zeta(\epsilon))$ is differentiable, then*

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} J(\zeta(\epsilon))|_{\epsilon=0} \leq 0$$

Proof. By the same reasoning as in Eq. (3.12) it must be that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} J(\zeta(\epsilon))|_{\epsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\epsilon} f(\epsilon)|_{\epsilon=0} \leq 0$$

□

Through the machinery introduced so far plus the Fundamental lemma of calculus of variations, Euler and Lagrange developed their celebrated first-order necessary condition for optimality in calculus of variations

$$L_y(t, y(t), \dot{y}(t)) = \frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{y}}(t, y(t), \dot{y}(t))$$

which, however, does not directly apply to the optimal control problem (3.1) - (3.2). The main reason for this complication is that while in calculus of variations directly perturbing the trajectory $y(t)$ is a good strategy, in the more general framework of optimal control I do not know a priori if the perturbed trajectory $y^\epsilon(t)$ is still a solution of the state equation (3.2) and, even if it is, it may be very difficult to characterize which class of perturbations produces a *well posed* problem. Hence, a better strategy is to perturb instead *first* the control $u(t)$ and *then* study the perturbed state $x^\epsilon(t)$ in terms of the perturbed control $u^\epsilon(t)$. To this end, I introduce the following definition of a perturbed control.

Definition 12 (Strong variation of a control). *Let $u(t)$ be a control and v a fixed element of U. For $\tau \in (t_0, t_1]$ the control u^ϵ*

$$u^\epsilon(t) = \begin{cases} v & \text{if } \tau - \epsilon < t \leq \tau \\ u(t) & \text{else in } [t_0, t_1] \end{cases}$$

is called a strong variation of the control u. The perturbed control $u^\epsilon(t)$ is still in the control set by Lemma 1.

Proof of the PMP

With the notion of strong variation of a control at hand, following Fleming and Rishel (2012) I split in several lemmas the proof of the Pontryagin's maximum principle for the general Mayer form system of Eq. (3.4).

- In Lemma 3 I study the perturbed state $x^\epsilon(t)$

- In Lemma 4 I introduce the carefully chosen *adjoint* variable $p(t)$
- In Lemma 5 I look at the value of the functional $J(u^\epsilon)$

Collecting all these results will finally yield the Pontryagin's maximum principle in the formulation of Theorem 4 for the *fixed time, free endpoint* problem.

Lemma 3. *If x^ϵ are solutions of Eq.* (3.4) *corresponding to the strong variation u^ϵ of Def. 12 with the same initial condition $x^\epsilon(t_0) = x_0$, then*

$$x^\epsilon(t) = x(t) + \epsilon\delta x(t) + o(t,\epsilon)$$

where $\delta x(t) = 0$ if $t_0 \leq t < \tau$ and

$$\delta x(t) = g(\tau, x(\tau), v) - g(\tau, x(\tau), u(\tau)) + \int_\tau^t g_x(s, x(s), u(s))\delta x(s)\, \mathrm{d}s$$

if $\tau \leq t \leq t_1$. The term $o(t,\epsilon)$ denotes a function such that $o(t,\epsilon) \to 0$ as $\epsilon \to 0$ for all t.

Proof. Let $x(t)$ be the solution to the state equation $\dot{x}(t) = g(t, x(t), u(t))$ with control $u(t)$ and $x^\epsilon(t)$ be the solution to $\dot{x}^\epsilon(t) = g(t, x^\epsilon(t), u^\epsilon(t))$ with respect to the strong variation $u^\epsilon(t)$ of the original control $u(t)$. By Def. 12 $u^\epsilon(t) = u(t)$ if $t \in [t_0, \tau - \epsilon]$ and therefore $x^\epsilon(t) = x(t)$. Hence, the claim is true with $\delta x(t) = 0$ for $t_0 \leq t < \tau$.

For $\tau \leq t \leq t_1$ taking a first-order Taylor expansion around $t = \tau$ of $x(t)$ I have

$$x(\tau - \epsilon) = x(\tau) - \dot{x}(\tau)\epsilon + o(\tau, \epsilon)$$

and rearranging terms and using $\dot{x}(t) = g(t, x(t), u(t))$ gives

$$x(\tau) = x(\tau - \epsilon) + g(\tau, x(\tau), u(\tau))\epsilon + o(\tau, \epsilon) \tag{3.13}$$

Taking again a first-order Taylor expansion around $t = \tau - \epsilon$ of the perturbed response $x^\epsilon(t)$ yields

$$x^\epsilon(\tau) = x^\epsilon(\tau - \epsilon) + \dot{x}^\epsilon(\tau - \epsilon)\epsilon + o(\tau, \epsilon)$$

where $\dot{x}^\epsilon$ denotes the right-sided derivative. Since at $t = \tau - \epsilon$ the solution $x^\epsilon(t)$ is equal to the unperturbed solution $x(t)$ by construction of $u^\epsilon(t)$, then I have

$$x^\epsilon(\tau) = x(\tau - \epsilon) + g(\tau - \epsilon, x(\tau - \epsilon), v)\epsilon + o(\tau, \epsilon) \tag{3.14}$$

Finally a new first-order Taylor expansion of $g(\tau - \epsilon, x(\tau - \epsilon), v)\epsilon$ around $x(\tau)$ gives

$$\begin{aligned} g(\tau - \epsilon, x(\tau - \epsilon), v)\epsilon = g(\tau, x(\tau), v)\epsilon + g_x(\tau, x(\tau), v)[x(\tau - \epsilon) + \\ - x(\tau)]\epsilon + o(|x(t) - x(\tau)|)\epsilon \end{aligned} \tag{3.15}$$

where the last two terms of Eq. (3.15) are of higher order and can be omitted in a linear approximation. Hence, substituting Eq. (3.15) into Eq. (3.14) I have

$$x^{\epsilon}(\tau) = x(\tau - \epsilon) + g(\tau, x(\tau), v)\epsilon + o(\tau, \epsilon)$$

and using Eq. (3.13) it yields

$$x^{\epsilon}(\tau) - x(\tau) = [g(\tau, x(\tau), v) - g(\tau, x(\tau), u(\tau))]\,\epsilon + o(\tau, \epsilon) \qquad (3.16)$$

and denoting $\zeta_{\tau}(v) := g(\tau, x(\tau), v) - g(\tau, x(\tau), u(\tau))$ gives

$$x^{\epsilon}(\tau) - x(\tau) = \zeta_{\tau}(v)\epsilon + o(\tau, \epsilon) \qquad (3.17)$$

Writing

$$x^{\epsilon}(t) = x(t) + \delta x(t)\epsilon + o(t, \epsilon) \qquad (3.18)$$

for some function $\delta x : [\tau, t_1] \to \mathbb{R}^2$, I know from Eq. (3.17) that $\delta x(\tau)$ exists and $\delta x(\tau) = \zeta_{\tau}(v)$. Passing to the state equation (3.4) in integral form for the perturbed response $x^{\epsilon}(t)$ yields

$$x^{\epsilon}(t) = x^{\epsilon}(\tau) + \int_{\tau}^{t} g(s, x^{\epsilon}(s), u(s))\,\mathrm{d}s$$

and differentiating both sides of this equation with respect to ϵ at $\epsilon = 0$ and from Eq. (3.18) with $t = \tau$, I obtain

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} x^{\epsilon}(t)|_{\epsilon=0} = \delta x(\tau) + \int_{\tau}^{t} g_x(s, x^{\epsilon}(s), u(s)) \frac{\mathrm{d}}{\mathrm{d}\epsilon} x^{\epsilon}(s)|_{\epsilon=0}\,\mathrm{d}s$$

Using the fact that the derivative of $x^{\epsilon}(t)$ with respect to ϵ at $\epsilon = 0$ is just $\delta x(t)$ and that $\delta x(\tau) = \zeta_{\tau}(v)$, then for $\tau \leq t \leq t_1$ I have

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} x^{\epsilon}(t)|_{\epsilon=0} = \delta x(t) = \zeta_{\tau}(v) + \int_{\tau}^{t} g_x(s, x(s), u(s))\delta x(s)\,\mathrm{d}s$$

or equivalently

$$\delta x(t) = g(\tau, x(\tau), v) - g(\tau, x(\tau), u(\tau)) + \int_{\tau}^{t} g_x(s, x(s), u(s))\delta x(s)\,\mathrm{d}s$$

□

Theorem 3 (Adjoint equation). *Let $A(t)$ be a $n \times n$ matrix, y_0 a n-dimensional vector, and $b(t)$ a n-dimensional vector of piecewise continuous functions defined on an interval $[t_0, t_1]$. If $\tau \in [t_0, t_1]$, then there exists a unique piecewise continuously differentiable solution of the differential equation*

$$\dot{y}(t) = A(t)y(t) + b(t)$$

on $[t_0, t_1]$ which satisfies the condition $y(\tau) = y_0$.

The differential equation $\dot{p}(t) = -A(t)p(t)$ is called the adjoint differential equation of $\dot{y}(t) = A(t)y(t) + b(t)$

Lemma 4. *If $y(t)$ is a solution of $\dot{y}(t) = A(t)y(t) + b(t)$ and $p(t)$ a solution of its adjoint equation, then for any two τ_1, τ_2*

$$p(\tau_2)'y(\tau_2) - p(\tau_1)'y(\tau_1) = \int_{\tau_1}^{\tau_2} p(t)'b(t)\,\mathrm{d}t$$

Proof. By the product rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p(t)'y(t)\right) = -p(t)'A(t)y(t) + p(t)'A(t)y(t) + p(t)'b(t)$$

and integrating from τ_1 to τ_2

$$p(\tau_2)'y(\tau_2) - p(\tau_1)'y(\tau_1) = \int_{\tau_1}^{\tau_2} p(t)'b(t)\,\mathrm{d}t$$

□

Lemma 5. *For $0 \leq \epsilon \leq \eta$ let*

$$J(u^\epsilon) = \int_{t_0}^{t_1} L(t, x^\epsilon(t), u^\epsilon(t))\,\mathrm{d}t =: \phi(x^\epsilon(t_1))$$

where $u^\epsilon(t)$ is a perturbed control as in Def. 12, and let $p(t)$ be the solution of the adjoint equation $\dot{p}(t)' = -p(t)'g_x(t, x(t), u(t))$ with boundary condition $p(t_1)' = -\phi_x(x^\epsilon(t_1))$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}J(u^\epsilon)|_{\epsilon=0} = -p(\tau)'[g(\tau, x(\tau), v) - g(\tau, x(\tau), u(\tau))]$$

Proof. By Lemma 4 with $y(t) = \delta x(t)$, $a(t) = g(t, x(t), v) - g(t, x(t), u(t))$, $b(t) = 0$ and $\tau_1 = \tau$ and $\tau_2 = t_1$, I have that

$$p(t_1)'\delta x(t_1) - p(\tau)'\delta x(\tau) = \int_{\tau}^{t_1} p(t)'b(t)\,\mathrm{d}t = 0$$
$$\Longrightarrow \phi_x(x^\epsilon(t_1))\delta x(t_1) = -p(\tau)'[g(\tau, x(\tau), v) - g(\tau, x(\tau), u(\tau))]$$

since $p(t_1) = -\phi_x(x^\epsilon(t_1))$ and Lemma 3. Finally, since $J(u^\epsilon) = \phi(x^\epsilon(t_1))$ by the chain rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}J(u^\epsilon)|_{\epsilon=0} &= \phi_x(x^\epsilon(t_1))\frac{\mathrm{d}}{\mathrm{d}\epsilon}x^\epsilon(t_1) = \phi_x(x^\epsilon(t_1))\delta x(t_1) \\ &= -p(\tau)'[g(\tau, x(\tau), v) - g(\tau, x(\tau), u(\tau))]\end{aligned}$$

where the penultimate equality is because of Lemma 3. □

Finally, to simplify notation, I redefine the adjoint as $\tilde{p}(t) = -p(t)$ and I introduce the Hamiltonian defined as

$$\mathcal{H}(t, x(t), u(t), p(t)) := \langle \tilde{p}(t), g(t, x(t), u(t)) \rangle = \tilde{p}(t)'g(t, x(t), u(t)) \tag{3.19}$$

where $\langle \cdot, \cdot \rangle$ denotes the dot product. Using the Hamiltonian, at last I can state the Pontryagin's maximum principle for the fixed time - free endpoint case

Theorem 4 (PMP). *Let $\mathcal{H}(t, x(t), u(t), p(t)) = \langle p(t), g(t, x(t), u(t)) \rangle$ be the Hamiltonian for the Mayer problem 3.4. If u^* is an optimal control with x^* its response, then*

$$\mathcal{H}(t, x^*(t), u^*(t), p(t)) \geq \mathcal{H}(t, x^*(t), v, p(t)) \tag{3.20}$$

for each $v \in U$ and $t \in (t_0, t_1]$, where $p(t)$ is the solution of

$$\dot{p}(t)' = -\mathcal{H}_x(t, x^*(t), u^*(t), p(t)) \tag{3.21}$$

with boundary condition

$$p(t_1)' = \phi_x(x(t_1)) \tag{3.22}$$

Proof. By Lemma 5

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} J(u^\epsilon)|_{\epsilon=0} = -p(t)'[g(t, x^*(t), v) - g(t, x^*(t), u^*(t))]$$

and by Theorem 2 if u^* is a maximizer, then it must be that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} J(u^\epsilon)|_{\epsilon=0} \leq 0 \implies -p(t)'[g(t, x^*(t), v) - g(t, x^*(t), u^*(t))] \leq 0$$

and recalling that in the Hamiltonian definition (3.19) I inverted the sign of the adjoint $\tilde{p}(t) = -p(t)$ it yields

$$\mathcal{H}(t, x^*(t), u^*(t), p(t)) \geq \mathcal{H}(t, x^*(t), v, p(t))$$

□

Recalling that $g(t, x(t), u(t))' = (f(t, x(t), u(t)), L(t, x(t), u(t)))'$, to pass from the Mayer to the Lagrange form I have

$$\begin{aligned} g_x(t, x(t), u(t)) &= \begin{pmatrix} f_{x_1}(t, x(t), u(t)) & f_{x_2}(t, x(t), u(t)) \\ L_{x_1}(t, x(t), u(t)) & L_{x_2}(t, x(t), u(t)) \end{pmatrix} \\ &= \begin{pmatrix} f_{x_1}(t, x(t), u(t)) & 0 \\ L_{x_1}(t, x(t), u(t)) & 0 \end{pmatrix} \end{aligned}$$

and therefore

$$\begin{pmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \end{pmatrix}' = -\begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}' \begin{pmatrix} f_{x_1}(t, x(t), u(t)) & 0 \\ L_{x_1}(t, x(t), u(t)) & 0 \end{pmatrix}$$

with end conditions

$$\begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}' = \phi_x(x(t_1)) = \left(\frac{\mathrm{d}x_{21}}{\mathrm{d}x_1}, \frac{\mathrm{d}x_{21}}{\mathrm{d}x_2} \right) = (0, 1)$$

Then it must be that $p_2(t) = 1$ for $t \in (t_0, t_1]$ and rewriting the Hamiltonian (3.19) as

$$\mathcal{H}(t, x(t), u(t), p(t)) = p(t)f(t, x(t), u(t)) + L(t, x(t), u(t))$$

and collecting the previous remarks it gives the following version of Theorem 4.

Corollary 1 (PMP - Lagrange)**.** *The Hamiltonian for the Lagrange problem* (3.1) - (3.2) *is*

$$\mathcal{H}(t, x(t), u(t), p(t)) = L(t, x(t), u(t)) + p(t)f(t, x(t), u(t)) \tag{3.23}$$

If u^* is an optimal control for the problem, then

$$\mathcal{H}(t, x^*(t), u^*(t), p(t)) \geq \mathcal{H}(t, x^*(t), v, p(t)) \tag{3.24}$$

for each $v \in U$ and $t \in (t_0, t_1]$, where $p(t)$ is the solution of

$$\dot{p}(t) = -\mathcal{H}_x(t, x^*(t), u^*(t), p(t)) \tag{3.25}$$

with boundary condition

$$p(t_1) = 0 \tag{3.26}$$

The adjoint as shadow price

Using the PMP in the form of Corollary 1 for the Lagrange problem (3.1) - (3.2), let

$$\begin{aligned} V(t, x) &= \int_t^{t_1} L(s, x(s), u(s))\, \mathrm{d}s & (3.27)\\ &= \int_t^{t_1} L(s, x(s), u(s)) + p(s)f(s, x(s), u(s)) - p(s)\dot{x}(s)\, \mathrm{d}s & (3.28) \end{aligned}$$

since $\dot{x} = f(t, x, u)$. Integrating by parts the term $p(s)\dot{x}(s)$ yields

$$\begin{aligned} \int_t^{t_1} p(s)\dot{x}(s)\, \mathrm{d}s &= p(t_1)x(t_1) - p(t)x(t) - \int_t^{t_1} \dot{p}(s)x(s)\, \mathrm{d}s & (3.29)\\ &= -p(t)x(t) - \int_t^{t_1} \dot{p}(s)x(s)\, \mathrm{d}s & (3.30) \end{aligned}$$

because $p(t_1) = 0$ by Eq. 3.26 of the PMP. Differentiating Eq. (3.27) with respect to x and bringing the derivative inside the integral I have

$$\begin{aligned} \frac{\partial V(t, x)}{\partial x} &= \left(\int_t^{t_1} L_x(s, x(s), u(s)) + p(s)f_x(s, x(s), u(s)) + \dot{p}(s)\, \mathrm{d}s \right) + p(t)\\ &= \left(\int_t^{t_1} \mathcal{H}_x(s, x(s), u(s), p(s)) + \dot{p}(s)\, \mathrm{d}s \right) + p(t) \end{aligned}$$

and by Eq. 3.25 of the PMP the last integrand is zero. Hence, the marginal contribution of the state x to the performance index of the system V is exactly equal to the adjoint p

$$\frac{\partial V(t,x)}{\partial x} = p(t) \tag{3.31}$$

or, in economic terms, the adjoint p is the *shadow price* of the asset x, which for the bioeconomic problem (3.8) is the resource to be harvested.

3.3 Existence theorems

While the Pontryagin's maximum principle provides a set of necessary conditions for optimality, in some optimal control problems a maximizer u^* of the functional J may not exist. The following theorem assures that under a convexity assumption at least an optimal control exists.

Theorem 5 (Filippov)**.** *Consider the problem 3.1 - 3.2. Assume that at least a successful control exists, and that successful responses satisfy a bound:* $|x(t)| \leq M$ *for all admissible controls. If the set of points* $(f,L) = \{(f(t,x,u), L(t,x,u))|u \in U\}$ *is a convex set, then there exists an optimal control in the class* $\mathcal{U}_m[t_0,t_1] = \{u : [t_0,t_1] \to U | u \text{ measurable}\}$*.*

Proof. See Fleming and Rishel (2012). □

This last theorem is, however, not applicable to the general bioeconomic model of Eqs. (2.27) - (2.28). While at least a successful control exists in the form of $u(t) \equiv 0$ and the state $x(t)$ is bounded, the set (f,L) may not be convex as I show below. For all $u \in U$, the set (f,L) has the following structure for the bioeconomic model

$$(f,L) = \begin{pmatrix} f(t,x,u) \\ L(t,x,u) \end{pmatrix} = \begin{pmatrix} ry(1-y/K) - u \\ e^{-\delta t}\,(M(u) - c(y)u) \end{pmatrix} = \begin{pmatrix} \alpha - u \\ \beta M(u) + \gamma u \end{pmatrix} \tag{3.32}$$

where $x = (y,z)'$ as in Eq. (3.8), and $\alpha := ry(1-y/K)$, $\beta := e^{-\delta t}$ and $\gamma := -e^{-\delta t}c(y)$ do not depend on u. If $M(u)$ is not a linear function, then this set may very well fail to be convex as it is the case with $M(u) = P(u)u$ and $P(u) = a - bu$ a simple linear demand function. If, however, $M(u)$ is a linear function such as in the perfect competition case with $M(u) = Pu$, then the set (f,L) becomes

$$(f,L) = \left\{ v : v = \begin{pmatrix} \alpha - u \\ \beta u \end{pmatrix} \text{ for } u \in U \right\} \tag{3.33}$$

where $\alpha := ry(1-y/K)$ and $\beta := e^{-\delta t}[P - c(y)]$ do not depend on u. The set in Eq. (3.33) is indeed convex and therefore Theorem 5 guarantees the existence of an optimal control for the linear case.

Another approach to an existence theorem which does not need a convexity assumption is to focus on a smaller class of controls than $\mathcal{U}_{PC}[t_0, t_1]$. Following Macki and Strauss (1982), to prove a "restricted" existence theorem I consider the set of measurable functions $u : [t_0, t_1] \to U$ having Lipschitz constant λ as control class leading to the following definition.

Definition 13 (Lipschitz control class)**.** *$\mathcal{U}_\lambda[t_0, t_1]$ is the set of all controls that are measurable functions $u : [t_0, t_1] \to U$ having Lipschitz constant λ.*

I state below a key result from real analysis that I use in the proof of the existence of an optimal control $u^* \in \mathcal{U}_\lambda[t_0, t_1]$ for problem 3.1 - 3.2. I refer to the Sec. A.1 of the Appendix for a precise definition of the terms used in the statement of the theorem.

Theorem 6 (Arzelà-Ascoli)**.** *If a sequence of real-valued functions $\{f_n : \mathbb{R} \to \mathbb{R}\}$ is uniformly bounded and equicontinuous, then there exists a subsequence $\{f_{n_k}\}$ that converges uniformly.*

Proof. See Rudin (1964). □

Finally, before stating and proving in some generality Theorem 7 I introduce the following compactness lemma.

Lemma 6. *The Lipschitz control set $\mathcal{U}_\lambda[t_0, t_1]$ is a compact subset of $\mathcal{C}[t_0, t_1]$ in the sup norm $\|\cdot\|_\infty$.*

Proof. By definition of $\mathcal{U}_\lambda[t_0, t_1]$ any sequence $\{u_n(t)\} \subset \mathcal{U}_\lambda[t_0, t_1]$ is uniformly bounded and equicontinuous and by the Arzelà-Ascoli theorem there exists a subsequence $\{u_{n_k}\}$ that converges uniformly to some u. Since $u_{n_k} \to u$ uniformly, for every $\epsilon > 0$ and $t \in [t_0, t_1]$ there exists a N such that if $n_k \geq N$, then $\|u(t) - u_{n_k}(t)\|_\infty \leq \epsilon/2$. Moreover, by Lipschitz continuity of u_{n_k} for every $\tau_1, \tau_2 \in [t_0, t_1]$ it follows that $\|u_{n_k}(\tau_1) - u_{n_k}(\tau_2)\|_\infty \leq \lambda|\tau_1 - \tau_2|$. Finally,

$$\begin{aligned}\|u(\tau_1) - u(\tau_2)\|_\infty &\leq \|u(\tau_1) - u_{n_k}(\tau_1)\|_\infty + \|u_{n_k}(\tau_1) - u_{n_k}(\tau_2)\|_\infty + \\ &+ \|u_{n_k}(\tau_2) - u(\tau_2)\|_\infty \\ &< \frac{\epsilon}{2} + \lambda|\tau_1 - \tau_2| + \frac{\epsilon}{2} = \epsilon + \lambda|\tau_1 - \tau_2|\end{aligned}$$

and since this is true for every $\epsilon > 0$, then u is Lipschitz continuous; furthermore, since u is the pointwise limit of measurable functions it is also measurable. Hence, $u \in \mathcal{U}_\lambda[t_0, t_1]$ and the space $\mathcal{U}_\lambda[t_0, t_1]$ is compact. □

Theorem 7 (Restricted existence theorem)**.** *Consider the problem 3.1 - 3.2. Assume that at least an admissible control exists, and that successful responses satisfy a bound: $|x(t)| \leq M$ for all $u(t) \in \mathcal{U}_\lambda[t_0, t_1]$. If $f(t, x(t), u(t))$ and $L(t, x(t), u(t))$ are continuous, then there exists an optimal control.*

Proof. Since $|x(t)| \le M$ for any admissible control and $L(t, x, u)$ is continuous on the compact set $K := [t_0, t_1] \times [-M, M] \times U$, then $L(K) \subset \mathbb{R}$ is compact and therefore bounded. Hence, $L(t, x(t), u(t))$ is uniformly bounded for any admissible control $u \in \mathcal{U}_\lambda[t_0, t_1]$ and

$$|J(u)| = \left| \int_{t_0}^{t_1} L(t, x(t), u(t))\, \mathrm{d}t \right| \le \int_{t_0}^{t_1} |L(t, x(t), u(t))|\, \mathrm{d}t \le B\Delta T$$

where $\Delta T = t_1 - t_0$ and B is the uniform bound of $L(t, x(t), u(t))$ for all admissible controls. Since $J(u)$ is also bounded, then the infimum of $J(u)$ is finite, and so there exists a minimizing sequence $\{u_n(t)\}$ in $\mathcal{U}_\lambda[t_0, t_1]$ such that

$$\lim_{n\to\infty} J(u_n) \downarrow c := \inf_{u \in \mathcal{U}_\lambda[t_0,t_1]} J(u), \quad u_n(\cdot) \text{ defined on } [t_0, \tau(n)]$$

with associated response $\{x_n(t)\}$. Since the sequence $\{u_n\}$ is in $\mathcal{U}_\lambda[t_0, t_1]$ and this class is uniformly bounded and equicontinuous by definition, then also $\{u_n\}$ is uniformly bounded and equicontinuous. Hence, by the Arzelà–Ascoli theorem there exists a subsequence $\{u_{n_k}\}$, with associated response x_{n_k}, that converges uniformly to some u^* with $u^* \in \mathcal{U}_\lambda[t_0, t_1]$ by Lemma 6. If also the associated sequence of responses $\{x_{n_k}\}$ is uniformly convergent to some x^*, then passing to limits (re-labeling u_{n_k} as u_k and x_{n_k} as x_k) I obtain the claim

$$\begin{aligned}\inf J(u) &= \lim_k J(u_k) = \lim_k \int_{t_0}^{t_1} L(t, x_k(t), u_k(t))\, \mathrm{d}t \\ &= \int_{t_0}^{t_1} \lim_k L(t, x_k(t), u_k(t))\, \mathrm{d}t = \int_{t_0}^{t_1} L(t, x^*(t), u^*(t))\, \mathrm{d}t = J(u^*)\end{aligned}$$

and for the state dynamics

$$\begin{aligned}x^*(t) &= \lim_k x_k(t) = x_0 + \lim_k \int_0^t f(s, x_k(s), u_k(s))\, \mathrm{d}s \\ &= x_0 + \int_0^t \lim_k f(s, x_k(s), u_k(s))\, \mathrm{d}s = x_0 + \int_0^t f(s, x^*(s), u^*(s))\, \mathrm{d}s\end{aligned}$$

where I switch limits and integrals thanks to uniform convergence of x_k and u_k, and continuity of L and f.

If instead the associated sequence of response is not uniformly convergent, another application of Theorem 6 produces the claim as follows. Keeping the re-labeled x_k, u_k, since $f(t, x_k(t), u_k(t))$ is continuous on the compact set $[t_0, t_1] \times [-M, M] \times U$ and $\dot{x}_k(t) = f(t, x_k(t), u_k(t))$, then $\{|\dot{x}_k(t)|\}$ is uniformly bounded and so is the sequence $\{x_k(t)\}$. By the Mean Value theorem for any τ_1, τ_2 in $[t_0, t_1]$ there exists a τ^* in (τ_1, τ_2) such that $|x_k(\tau_1) - x_k(\tau_2)| =$

$|\dot{x}_k(\tau^*)||\tau_1 - \tau_2|$. Since the sequence $|\dot{x}_k(t)| \leq K$, then $|x_k(\tau_1) - x_k(\tau_2)| \leq K|\tau_1 - \tau_2|$ and therefore $\{x_k\}$ is equicontinuous. Hence, by the Arzelà–Ascoli theorem there exists a subsequence $\{x_{k_l}\}$, hereafter $\{x_l\}$, that converges uniformly to some $\bar{x}$. But since to x_l corresponds a u_l, which is a sub-sequence of the original convergent sub-sequence u_k, then also u_l will be a uniformly convergent subsequence. Hence, the claim follows as above. □

It is clear from Eqs. (2.27) - (2.28) that the bioeconomic model satisfies the assumptions of Theorem 7: $f(t, x(t), u(t))$ and $L(t, x(t), u(t))$, as defined implicitly in the Mayer form of Eq. (3.8), are clearly continuous; at least an admissible control exists, namely $u(t) \equiv 0$ for $t \in [0, T]$, and all successful responses $x(t)$ satisfy a bound because of Lemma 2.

4 An application of the maximum principle

In this chapter I apply the tools of optimal control theory developed in Chapter 3 to study the dynamic bioeconomic model of Sec. 2.4 for the case of a single price taker firm where the linearity of the problem allows an analytical solution. Coupled with the existence result of the Chapter 3, in Sec. 4.1 I derive necessary conditions for an optimal control, and in Sec. 4.2 I analyze the sensitivity of the solution with respect to the various parameters.

4.1 Analysis of a dynamic bioeconomic model

Following Liberzon (2012), I re-write in Mayer form the bioeconomic model of Eqs. (2.22) - (2.23) for the case of a single price-taker firm with a logistic growth function $G(y) = ry(1 - y/K)$ as

$$\dot{x}(t) = f(x(t)) + g(t, x(t))u(t) \tag{4.1}$$

where $u \equiv h \in U = [0, \bar{h}]$ is the new control, $x = (y, z)' \in \mathbb{R}^2$ is the new state with

$$\begin{aligned} \dot{y}(t) &= ry(t)\left(1 - \frac{y(t)}{K}\right) - u(t) \\ \dot{z}(t) &= e^{-\delta t}\left[P - c(y(t))\right]u(t) \end{aligned}$$

and

$$f(x) = \begin{pmatrix} ry(1 - y/K) \\ 0 \end{pmatrix} \qquad g(t, x) = \begin{pmatrix} -1 \\ e^{-\delta t}\left(p - c(y)\right) \end{pmatrix}$$

To simplify calculations, I assume that the instantaneous discount rate δ is equal to zero, implying that there is no explicit time dependence in g, that is $g(t, x) \equiv g(x)$. Finally, Eq. 4.1 shows that the bioeconomic model belongs to a special class of problems that are *affine* in the control of the form

$$\dot{x} = f(x) + g(x)u \tag{4.2}$$

for a general state $x \in \mathbb{R}^n$ and control $u \in U \subset \mathbb{R}$, and in this case an application of the maximum principle is especially simple. The Hamiltonian $\mathcal{H}(t,x,u,p) : [t_0,t_1] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ for Eq. 4.2 is

$$\mathcal{H}(t,x,u,p) = \langle p, f(x) + g(x)u \rangle \tag{4.3}$$

the adjoint equation is

$$\dot{p}(t) = -\mathcal{H}_x(t,x(t),u(t),p(t))' = -f_x(x(t))'p(t) - g_x(x(t))'p(t)u(t) \tag{4.4}$$

and the maximum condition from Theorem 4 is

$$\max_{u \in U} \mathcal{H}(t,x^*(t),u,p(t)) = \langle p(t), f(x^*(t)) \rangle + \langle p(t), g(x^*(t)) \rangle \, u \tag{4.5}$$

Denoting $\varphi(t) = \langle p(t), g(x^*(t)) \rangle$ and recalling that for the bioeconomic problem $U = [0,\bar{h}]$, the maximization in Eq. (4.5) leads to the following candidate optimal control

$$u^*(t) = \begin{cases} \bar{h} & \text{if } \varphi(t) > 0 \\ 0 & \text{if } \varphi(t) < 0 \\ ? & \text{if } \varphi(t) = 0 \end{cases} \tag{4.6}$$

Some of the interesting features of this control u^* are that *(i)* it switches value according to the function $\varphi(t)$ called the *switching function*, and *(ii)* it attains the two extremes 0 and $\bar{h}$ of the set U where it belongs switching abruptly between them according to $\varphi(t)$. Forgetting for the moment the case $\varphi(t) = 0$, I have the following definition.

Definition 14 (Bang-bang control). *A control strategy $u \in \mathcal{U}_{PC}[t_0,t_1]$ is called bang-bang if $u(t) \in U_0$ for all $t \in [t_0,t_1]$, where U_0 is the set of all the extreme values of U and defined as*

$$U_0 = \{u_0 \in U | u_0 \text{ is not a strict convex combination of points of } U\} \tag{4.7}$$

Nevertheless, when $\varphi(t) = 0$ the maximum principle gives no indication to what could be the value of u^* in Eq. (4.6): if the switching function vanishes for a finite length of time $\tau_1 < t < \tau_2$, then the optimal control in that interval is no longer *bang-bang* and is instead called *singular*. Hence, the question becomes how to approach this last case.

Definition 15 (Singular control). *Let $u^*(t) \in \mathcal{U}_{PC}[t_0,t_1]$ satisfy the conditions of the PMP, $x^*(t)$ be the its response, and $p(t)$ the solution to the adjoint equation. Then the strategy u^* is called singular over an open interval I if the first-order condition $\mathcal{H}_u(t,x^*(t),u^*(t),p(t)) = 0$ holds for $t \in I$ and the matrix of second-order partial derivatives $\mathcal{H}_{uu}(t,x^*(t),u^*(t),p(t))$ is singular on I. If u^* is a singular control, the associated response x^* is called a singular arc.*

Following Schättler and Ledzewicz (2012), if $\varphi(t)$ vanishes over some time interval (τ_1, τ_2), then also its derivative(s) must vanish

$$\varphi(t) = \langle p(t), g(x^*(t))\rangle = 0 \tag{4.8}$$

and

$$\begin{aligned}\dot\varphi(t) &= \langle \dot p(t), g(x^*(t))\rangle + \langle p(t), g_x(x^*(t))\dot x^*(t)\rangle \\ &= -\left\langle f_x(x^*(t))' p(t), g(x^*(t))\right\rangle - \left\langle g_x(x^*(t))' p(t), g(x^*(t))\right\rangle u^*(t) \\ &+ \langle p(t), g_x(x^*(t)) f(x^*(t))\rangle + \langle p(t), g_x(x^*(t)) g(x^*(t))\rangle u^*(t) \\ &= \langle p(t), g_x(x^*(t)) f(x^*(t)) - f_x(x^*(t)) g(x^*(t))\rangle = 0 \end{aligned} \tag{4.9}$$

The vector field $g_x(x^*)f(x^*) - f_x(x^*)g(x^*)$ in the last expression is called the *Lie bracket* and defined as

$$[f,g](x) := g_x(x)f(x) - f_x(x)g(x)$$

and plays an important role in assessing whether the control u will be singular or not. Extending the above calculation for a general vector field $h(x(t))$ it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle p(t), h(x(t))\rangle = \langle p(t), [f,h](x(t))\rangle + \langle p(t), [g,h](x(t))\rangle u(t) \tag{4.10}$$

and rewriting Eq. (4.9) using the Lie bracket $[f,g](x)$, for $\varphi(t)$ to vanish over an interval (τ_1, τ_2) it must be that

$$\varphi(t) = \langle p(t), g(x^*(t))\rangle = 0 \tag{4.11}$$

$$\dot\varphi(t) = \langle p(t), [f,g](x^*(t))\rangle = 0 \tag{4.12}$$

along the optimal trajectory $x^*(t)$. Using Eqs. (4.11) - (4.12) to rule out the existence of a singular control, it is enough that $g(x^*(t))$ and $[f,g](x^*(t))$ span $\mathbb{R}^2$ since $p(t) \neq (0,0)'$ and the 0-vector is the unique element of $\mathbb{R}^2$ that could be orthogonal to two linearly independent vectors in $\mathbb{R}^2$.

For the bioeconomic model the adjoint is $p(t) = (\lambda(t), \lambda_0(t))' \in \mathbb{R}^2$ with equation

$$\dot p(t) = \begin{pmatrix} -\frac{cu(t)}{qy^2(t)} - r\lambda(t) + \frac{2r}{K}\lambda(t)y(t) \\ 0 \end{pmatrix} \tag{4.13}$$

and the switching function is

$$\varphi(t) = [P - c(y(t))] - \lambda(t) \tag{4.14}$$

Similarly, the matrix $M \in \mathbb{R}^{2\times 2}$ whose first and second column are g and $[f,g]$ is

$$M = \begin{pmatrix} -1 & \frac{r}{K}\left(K - 2y(t)\right) \\ \left(-\frac{c}{qy(t)} + P\right) & \frac{cr(K-y(t))}{Kqy(t)} \end{pmatrix} \tag{4.15}$$

and its determinant is

$$\det(M) = \frac{r}{Kq}\left(-KPq + 2Pqy(t) - c\right) \tag{4.16}$$

To have $g(x^*(t))$ and $[f,g](x^*(t))$ linearly dependent along the optimal trajectory $x^*(t)$ the determinant of M must be zero and this is true if and only if

$$y^*(t) = \frac{K}{2} + \frac{c}{2Pq} \tag{4.17}$$

The meaning of this last equation is that if a singular control exists, then its singular arc must be as in Eq. (4.17) but I still have to find at least a candidate singular control. A way to proceed is to look once again at the switching function: using Eq. (4.10) the second derivative of the switching function is

$$\ddot{\varphi}(t) = \langle p(t), [f,[f,g]](x^*(t))\rangle + \langle p(t), [g,[f,g]](x^*(t))\rangle\, u^*(t) \tag{4.18}$$

and since $\ddot{\varphi}(t) = 0$, then a candidate singular control is

$$u_{\sin} = -\frac{\langle p, [f,[f,g]](x^*)\rangle}{\langle p, [g,[f,g]](x^*)\rangle} \tag{4.19}$$

if *(i)* $\langle p, [g,[f,g]](x^*)\rangle \neq 0$ and *(ii)* the control is admissible, that is $u \in U$. Neglecting for the moment the admissibility of the control, the following theorem gives a necessary condition for a singular control.

Theorem 8 (Generalized Legendre-Clebsch condition)**.** *If $u^*(t)$ is an optimal control with response $x^*(t)$ for the maximization problem* (4.2) *and $u^*(t)$ is singular on an open interval $I \subset [t_0, t_1]$, then there exists an adjoint $p(t)$ with the property that*

$$\begin{aligned}\langle p(t), [g,[f,g]](x^*(t))\rangle &= \frac{\partial}{\partial u}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{H}_u(t, x^*(t), u^*(t), p(t)) \\ &= \frac{\partial}{\partial u}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varphi(t) \geq 0 \qquad \textit{for all } t \in I\end{aligned}$$

For the bioeconomic system (4.1), the term $\langle p(t), [g,[f,g]](x^*(t))\rangle$ is

$$\langle p(t), [g,[f,g]](x^*(t))\rangle = \frac{2cr}{Kqy(t)} + \frac{2r}{K}\lambda(t) \tag{4.20}$$

and because along the singular arc $x^*(t)$ the switching function vanishes, $p - c(y^*(t)) = \lambda(t)$, then Eq. (4.20) becomes

$$\langle p(t), [g,[f,g]](x^*(t))\rangle = \frac{2P}{K}r > 0 \tag{4.21}$$

which is strictly positive. Hence, I can conclude that the control (4.19) satisfies the necessary Legendre-Clebsch condition and has a non-zero denominator. To finally compute the singular control $u_{\sin}$ and check whether it is admissible, I can simplify its expression if the vector fields $[f, g]$ and $[g, [f, g]]$ form a basis of $\mathbb{R}^2$, meaning that for some functions $a(x)$ and $b(x)$ I have

$$[f, [f, g]](x^*) = a(x^*)[f, g](x^*) + b(x^*)[g, [f, g]](x^*) \tag{4.22}$$

In this case the numerator of Eq. (4.19) becomes

$$\begin{aligned} \langle p, [f, [f, g]](x^*)\rangle &= a(x^*) \langle p, [f, g](x^*)\rangle + b(x^*) \langle p, [g, [f, g]](x^*)\rangle \\ &= b(x^*) \langle p, [g, [f, g]](x^*)\rangle \end{aligned}$$

because $\langle p(t), [f, g](x^*(t))\rangle = \dot{\varphi}(t) = 0$ from Eq. (4.12), and the singular control is

$$u_{\sin} = -\frac{b(x^*) \langle p, [g, [f, g]](x^*)\rangle}{\langle p, [g, [f, g]](x^*)\rangle} = -b(x^*) \tag{4.23}$$

For this bioeconomic model, it is indeed the case that $[f, g]$ and $[g, [f, g]]$ are a basis of $\mathbb{R}^2$ since their matrix

$$N = \Big[[f, g], [g, [f, g]]\Big] = \begin{pmatrix} \frac{r}{K}\,(K - 2y(t)) & \frac{2r}{K} \\ \frac{cr(K-y(t))}{Kqy(t)} & \frac{2cr}{Kqy(t)} \end{pmatrix}$$

has rank 2 because its determinant

$$\det(N) = -\frac{2cr^2}{K^2 q}$$

is never zero. Therefore, I can write $[f, [f, g]]$ as a linear combination of $[f, g]$ and $[g, [f, g]]$ as in Eq. (4.22) with

$$\begin{aligned} a(x(t)) &= a(y(t), z(t)) = a(y(t)) = -r + \frac{2r}{K} y(t) \\ b(x(t)) &= b(y(t), z(t)) = b(y(t)) = -\frac{r}{K}\,(K - y(t))\, y(t) \end{aligned}$$

and the singular control in feedback form is

$$u_{\sin}(t) = -b(x^*(t)) = \frac{r}{K}\,(K - y^*(t))\, y^*(t) \tag{4.24}$$

or, noting that $y^*(t) = K/2 + c/(2pq)$ along the singular arc

$$u_{\sin}(t) = \frac{Kr}{4} - \frac{c^2 r}{4KP^2 q^2} = \frac{r}{4KP^2 q^2}\left(K^2 P^2 q^2 - c^2\right) \tag{4.25}$$

and from this last expression the singular control $u_{\sin}$ is admissible if

$$\sqrt{KPq\left(KPq - \frac{4Pq}{r}\bar{h}\right)} < c < KPq \tag{4.26}$$

The economic interpretations of these bounds is that costs cannot be too high compared to the profitability of the resource, which positively relates to the carrying capacity K of the environment, the price level P, and the catchability coefficient q, nor can they be too low otherwise there is no need for a singular control.

To recap, I know that for the bioeconomic problem (4.1) an optimal control u^* must exist because of Theorem 5 albeit in the bigger class of measurable functions $\mathcal{U}_m[t_0, t_1] \supset \mathcal{U}_{PC}[t_0, t_1]$. The Pontryagin's maximum principle and the generalized Legendre-Clebsch condition give, respectively for the bang-bang and singular case, a set of necessary conditions that the optimal control must satisfy

$$u^*(t) = \begin{cases} \bar{h} & \text{if } \varphi(t) > 0 \\ u_{\sin}(t) & \text{if } \varphi(t) = 0 \\ 0 & \text{if } \varphi(t) < 0 \end{cases} \tag{4.27}$$

This last case $u_{\sin}(t)$, which is true only if a singular control exists, is especially relevant from a bioeconomic standpoint that Clark (2010) calls the corresponding singular arc $y^*(t)$ the "Golden rule." In its general formulation, which I treat in Sec. A.2 of the Appendix, the "Golden rule" is defined implicitly by

$$\delta = G'(y^*(t)) - \frac{G(y^*(t))c'(y^*(t))}{P - c(y^*(t))} \tag{4.28}$$

or explicitly, for the logistic case $G(y) = ry(1 - y/K)$, as

$$y_g := y^*(t) = \frac{K}{4}\left[\left(\frac{c}{PqK} + 1 - \frac{\delta}{r}\right) + \sqrt{\left(\frac{c}{PqK} + 1 - \frac{\delta}{r}\right)^2 + \frac{8c\delta}{PqKr}}\right] \tag{4.29}$$

that is the same as the singular arc in Eq. (4.17) if the instantaneous discount rate δ is equal to zero, which I assumed at the beginning of the section to ease computations.

The name "Golden rule" is because of the similarity of Eq. (4.28) with the standard condition in economic theory that the marginal productivity of capital should be equal to its cost, i.e., the interest rate. To see the correspondence, since in the bioeconomic model the objective rate of discount, i.e., the interest rate, is equal to the subjective rate of discount δ, and assuming that $c = 0$, Eq. (4.28) becomes

$$\delta = G'(y^*(t)) \tag{4.30}$$

meaning that the marginal productivity of the renewable asset is exactly equal to its opportunity costs. The extra term in Eq. (4.28) is a "stock effect" caused by the increased costs of harvesting $c(y) = c/(qy)$ for a decreased population.

The importance of discounting

More than simplifying calculations, the instantaneous discount rate δ has a very important economic meaning in the golden rule. What happens if, like in the bionomic equilibrium, entry in a particular resource extraction industry is almost costless? A reasonable consequence of free-entry is that harvesters will care almost exclusively about the present, since they do not expect to reap any economically significant future profits when free entry is possible, or in other worlds they will tend to heavily discount the future $\delta \to +\infty$. Hence, letting $\delta \to +\infty$ in the golden rule Eq. (4.28)

$$\begin{aligned}\lim_{\delta\to+\infty} \delta = +\infty = G'(y_g) - \frac{c'(y_g)G(y_g)}{P - c(y_g)} &= \frac{r}{K}(K - 2y_g) + \frac{r}{K}\frac{c}{q(y_g)^2}\frac{y_g(K - y_g)}{P - \frac{c}{qy_g}} \\ &= \frac{r}{K}(K - 2y_g) + \frac{cr}{K}\frac{K - y_g}{Pqy_g - c}\end{aligned}$$

and since the population $y(\cdot)$ is bounded, the only way for the golden rule expression to grow unbounded is that the last term goes to $+\infty$, and therefore

$$\frac{cr}{K}\frac{K - y_g}{Pqy_g - c} \to +\infty \iff Pqy_g - c \to 0 \implies y_g \to \frac{c}{Pq} = y_{BE} \tag{4.31}$$

where y_{BE} is the bionomic equilibrium from Eq. (2.14). Hence, under free entry and a high discount rate δ the dynamic bioeconomic model for the perfect competition case is qualitatively similar to the standard bionomic equilibrium.

4.2 Comparative statics

While for the logistic growth model, $G(y) = ry(1 - y/K)$, the golden rule Eq. (4.28) has an explicit solution and therefore I can directly compute comparative statics using Eq. (4.29), for a different form of the growth function $G(y)$ an explicit expression for the golden rule may not exist and I need to use the Implicit function theorem.

Theorem 9 (Implicit Function Theorem)**.** *Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a continuously differentiable function and (a, b) a point in $\mathbb{R}^{n+m}$ such that $f(a, b) = 0$. If the matrix $[\partial f_i/\partial y_j]_{ij}$ evaluated at (a, b) is invertible, then there exists an open set U*

containing a, an open set V containing b, and a unique continuously differentiable function $g : U \to V$ such that

$$\{(x, g(x))|x \in U\} = \{(x,y) \in U \times V | f(x,y) = 0\}$$

Hereafter, I use Theorem 9 to study the dependence of the golden rule solution y_g on the instantaneous discount factor δ.

Lemma 7. *Let $G(y)$ be a generalized logistic function as in Eq.* (2.5)

$$G(y) = ry^{\alpha}\left[1 - \left(\frac{y}{K}\right)^{\beta}\right]^{\gamma} \tag{4.32}$$

Then for some values of the parameters α, β, γ, the golden rule population level y_g, where y_g is a solution of Eq. (4.28)*, is a decreasing function of the instantaneous discount factor δ.*

Proof. From the golden rule equation, I define a function $F(\delta, y) : \mathbb{R}^2 \to \mathbb{R}$

$$F(\delta, y) = G'(y) - \frac{c'(y)G(y)}{P - c(y)} - \delta \tag{4.33}$$

with $F(\delta, y_g) = 0$. The partial derivative $F_\delta(\delta, y)$ of $F(\delta, y)$ with respect to δ is

$$F_\delta(\delta, y) = -1 < 0 \tag{4.34}$$

and the partial derivative $F_y(\delta, y)$ of $F(\delta, y)$ with respect to y is

$$\begin{aligned} F_y(\delta, y) = \ & G''(y) - \frac{[c''(y)G(y) + c'(y)G'(y)](P - c(y))}{(P - c(y))^2} + \\ & - \frac{c'(y)^2 G(y)}{(P - c(y))^2} \end{aligned} \tag{4.35}$$

Assuming the sufficient condition $G''(y) < 0$ for the population not to display depensation as defined in Sec. 2.2, the first term of Eq. (4.35) is negative. The third term is also positive (respectively negative if I consider the $-$ sign in front) since $G(y) > 0$ for $0 < y < K$. Finally, the numerator second term for a generalized logistic function is

$$\begin{aligned} & c''(y)G(y) + c'(y)G'(y) = \\ & \qquad = \frac{cr}{K^{\beta\gamma} q}\left(K^{\beta} - y^{\beta}\right)^{\gamma - 1}\left[(2-\alpha)K^{\beta} + (\alpha + \beta\gamma - 2)y^{\beta}\right] y^{\alpha - 3} \end{aligned}$$

and for $0 < y < K$, I have two possible cases for the second term of Eq. (4.35) to be positive: *(i)* the part $(\alpha + \beta\gamma - 2)$ is zero and $0 < \alpha < 2$, which is the case for the logistic function where $\alpha = \beta = \gamma = 1$, and therefore the whole sum is positive; *(ii)* the part $(\alpha + \beta\gamma - 2)$ is nonzero and for the whole sum to be positive it must be that

$$y > \left(\frac{\alpha - 2}{\alpha + \beta\gamma - 2}\right)^{\frac{1}{\beta}} K \tag{4.36}$$

The second term is positive (respectively negative if I consider the $-$ sign in front) because for profits $\Pi(y, h) = [P - c(y)]h$ to be positive the condition $P > c(y)$ must hold. Hence, $F(\delta, y)$ satisfies the hypotheses of Theorem 9 and in an open set near the solution y_g I have

$$\frac{\mathrm{d}y}{\mathrm{d}\delta} = -\frac{F_\delta(\delta, y)}{F_y(\delta, y)} < 0$$

since $F_y(\delta, y)$ is the sum of three negative terms and $F_\delta(\delta, y) < 0$. □

Finally, recalling that the bionomic equilibrium Eq. (2.14) is the limiting case of the golden rule as $\delta \to +\infty$, Lemma 7 implies that the golden rule population level y_g is higher than the bionomic equilibrium y_{BE} yielding the following proposition.

Proposition 1. *The golden rule population y_g is higher than the bionomic equilibrium y_{BE}*

$$y_g > y_{BE}$$

For the sake of completeness I also compute the sensitivity of the golden rule equilibrium y_g to small changes in price P. As one would expect, it turns out that in equilibrium higher prices imply a lower level of the population.

Proposition 2. *Let $G(y)$ be a generalized logistic function as in Eq.* (2.5)

$$G(y) = ry^\alpha \left[1 - \left(\frac{y}{K}\right)^\beta\right]^\gamma \tag{4.37}$$

Then for some values of the parameters α, β, γ, the golden rule population level y_g, where y_g is a decreasing function of the price level.

Proof. Proceeding as in Lemma 7, from the golden rule equation I define a function $F(P, y) : \mathbb{R}^2 \to \mathbb{R}$

$$F(P, y) = G'(y) - \frac{c'(y)G(y)}{P - c(y)} - \delta$$

with $F(P, y_g) = 0$ for any price P, and y_g a solution of Eq. (4.28). The partial derivative $F_P(P, y)$ of $F(P, y)$ with respect to P is

$$F_P(P, y) = \frac{c'(y)G(y)}{(P - c(y))^2} < 0$$

since $c'(y) = -c/(qy^2)$ is negative and $G(y)$ is positive for any $0 < y < K$ (and this is the case for the golden rule solution). The partial derivative $F_y(P,y)$ of $F(P,y)$ with respect to y is

$$F_y(P,y) = G''(y) - \frac{[c''(y)G(y) + c'(y)G'(y)](P - c(y))}{(P - c(y))^2} + \\ - \frac{c'(y)^2 G(y)}{(P - c(y))^2} \tag{4.38}$$

which is the same as in Lemma 7 and is negative under the same assumptions. Hence, $F(P,y)$ satisfies the hypotheses of Theorem 9 and therefore in an open set near the solution y_g

$$\frac{\mathrm{d}y}{\mathrm{d}P} = -\frac{F_P(P,y)}{F_y(P,y)} < 0$$

since $F_y(P,y)$ is the sum of three negative terms and $F_P(P,y) < 0$. □

5 Regulation of a bioeconomic system

In this chapter I study the effect of regulation on the bioeconomic system. In Sec. 5.1 I present a brief story of regulation of renewable assets, especially fisheries, and in Sec. 5.2 I define a notion of social optimality that , according to the model, should be the objective of resource managers. In Sec. 5.3 I outline the role of fixed costs, which I have thus far ignored, how they change the model and how they could be relevant from a policy perspective. Finally, in Sec. 5.4 I analyze the effect of different regulations on harvesting, and why they may or may not be effective.

5.1 A brief history of regulation

Before tackling the problem of regulating a fishery in the setting of the bioeconomic model so far developed, following Anderson and Seijo (2011) I propose here a brief history of the regulation introduced in the U.S. to shed some light on the main issues of modern commercial fisheries.

Until the end of the 20th century, in the U.S. fisheries were entirely open access but for the nominal cost of a fishing permit, required mainly for record keeping by the government, since at the time it was unthinkable to limit access to what was considered *terra nullius* and not subject to any form of sovereignty. Free entry and the apparently endless bounty of the sea attracted, in turn, an ever increasing number of fisherman that, as the bionomic equilibrium predicts, lead to over-harvesting and depletion of the fish population. Recognizing the dangers of such dynamics, the managers of the fisheries started to regulate fishing by putting restrictions either on inputs or output.

Definition 16 (Input controls)**.** *Input controls are measures designed to curtail fishing effort, such as closing access to fisheries, restricting gear and methods, or limiting fishing area or season.*

Definition 17 (Output controls)**.** *Output controls are measure designed to manage the overall catch by fishermen. They include setting total allowable catch (TAC),* *which is the amount of fish that may be harvested by the entire fleet in a given fishing season, bycatch limits, meaning the amount of non-target species taken, or bag limits, meaning the number of fish that may be landed in a day.*

While input controls alone may seem effective to well manage a fishery and were historically the first ones deployed, they introduced in the fishing industry perverse incentives that were incompatible with the preservation of the stock, employment, and the overall welfare of society. If an input control were to limit the horsepower of boats, then fishermen would use secondary boats to empty their catch and continue fishing. If instead managers were to limit the fishing season, then fishermen would look for ways to increase their fishing effort in the shorter time period.

At some point this game between managers, which sought to prevent over-harvesting through increasingly restrictive regulations, and fishermen, which sought to sidestep managers' controls, developed into a true "race for the fish:" in a shorter and shorter fishing season, fishermen had to harvest as much as possible since whatever they did not it was for others to claim. This race generated in turn at least two unintended and negative consequences (other than continue depletion of the fish stock): the development of excess capacity and a general disregard for safety measures by fishermen that tried to take advantage of the very short fishing window they had available, working conditions notwithstanding. Finally, to try to end the race for the fish the focus began to shift from keeping access to the fishery open but with input controls in place, to limit the number of new entrants to the fishery through a new form of output controls called "Dedicated access privileges," which the US Commission on Ocean Policy (2004) defines as

Definition 18 (Dedicated access privilege)**.** *A dedicated access privilege is a form of output control that grants to a fisherman or other entity the privilege to harvest a given fraction of the total allowable catch.*

Very important examples of dedicated access privilege are "Individual fishing quotas (IFQs)", which allow a fishermen to catch a specified quota of the TAC, and "Individual transferable quotas (ITQs)", which are IFQs that the owner can sell or transfer to other fishermen. Broadly speaking, with a personal quota TAC-based scheme in place fishermen can harvest up to their assigned quota at their leisure, and once each quota has been fished and the TAC reached, managers close the fishery for the rest of the season.

Hence, dedicated access privilege seem to be the right tools that could end the race for the fish by aligning managers' and fishermen's interests: since their quota is fixed, fishermen no longer have an incentive to game the system by developing excess capacity to catch a lot of fish early on, and can instead focus on reducing costs and selling high quality fish at the best possible price to boost their profits.

5.2 Social optimality in the model

Sec. 5.1 outlined the main challenges when regulating resource exploitation, but while the need for regulation is evident from the collapse of many modern day fisheries, I still have not clearly defined what should be the objective(s) of the regulator. Should a manager only prevent the depletion of the population perhaps at the expense of economic profitability, or should it carefully weight the needs of all the stakeholders and strive for a common sustainable harvest policy? And if yes, where is the right balance between harvesting, preservation of the natural resource, and society at large?

Recalling the discussion in Sec. 2.3, a principle of optimality must carefully balance between preservation of the natural resource and economic efficiency. In this regard, managers should not excessively rely on the MSY and MEY because they either neglect the economic implications of harvesting decisions or they simply do not consider the dynamic nature of the problem. Hence, Sec. 2.3 presented the dynamic bioeconomic model as the preferable alternative, and for this reason I identify the golden rule solution of Eq. (4.28) for the case of a single price taker firm as my optimality concept.

This choice may seem somewhat subpar given that the standard way to assess socially optimality in economics is to look at the social planner of Eq. (2.26) and its welfare function. There are, however, at least two major difficulties with the social planner approach: first, the nonlinearity of Eq. (2.26) does not allow an analytical solution as in the case of perfect competition; and second, there is no immediate way to compare a *time-dependent* optimal control $u^*(t)$ and response $x^*(t)$ with the *static* bionomic equilibrium without introducing any type of long-term behavior of the system and analyzing its convergence. Picking the golden rule as a socially optimal equilibrium permits to sidestep these issues, since it has an analytic form in Eq. (4.28) or (4.29), and, at least for the logistic growth function $G(y) = ry(1 - y/K)$, has a constant solution y_g.

Furthermore, if agents are enough forward looking, meaning their instantaneous discount factor δ is close to zero, and the time horizon T is long enough, then even in the setting of perfect competition there can be no overharvesting of natural resources without severely affecting future profits, and profit maximization coupled with species' preservation seems to me as close as to what a social planner should strive to obtain. Hence, I have the following definition.

Definition 19 ("Social optimum")**.** *I define an equilibrium E as socially optimal if its population level y_E is the same as the one under the golden rule*

$$\delta = G'(y_g) - \frac{G(y_g)c'(y_g)}{P - c(y_g)} \tag{5.1}$$

where y_g is the golden rule stock level.

A caveat of this approach is that to considerably simplify the problem I have thus far omitted from my analysis any kind of fixed costs an harvester may face. On theoretical grounds, my choice is justified assuming that capital is perfectly "malleable," meaning that there exists no "constraint upon its disinvestment" (Clark, Clarke, and Munro 1979). Since, however, assets are never in practice perfectly malleable, and it is actually far more common for an investment decision to be quasi-irreversible, in the next section I briefly consider what are the effects of capital on harvesting.

5.3 The role of capital costs

If capital costs are not perfectly malleable, then they could have different effects depending on the type of access, open or restricted, to the natural resource. Under a restricted access scheme, e.g., an IFQ, fishermen do not have to compete for a common resource up to a zero profit lower bound, and their optimization problem of Eqs. (2.22) - (2.23) changes as follows

$$\max_{h,I} \int_0^T e^{-\delta t} \left([P - c(y(t))]\, h(t) - c_f I(t) \right) \mathrm{d}t \tag{5.2}$$

$$\dot{y}(t) = ry(t) \left(1 - \frac{y(t)}{K} \right) - h(t) \tag{5.3}$$

$$\dot{\kappa}(t) = I(t) - \gamma \kappa(t) \tag{5.4}$$

$$y(t) \geq 0, \qquad I(t) \geq 0, \qquad 0 \leq h(t) \leq \kappa(t) q y(t) \tag{5.5}$$

where $I(t)$ is the non-negative investment decision and additional control, c_f is the cost of investing, and $\kappa(t)$ is the capital stock and additional state. The law of motion of capital, Eq. (5.4), takes the usual form of investment minus depreciated capital, with γ the depreciation rate. Finally, using Eq. (2.9) to rewrite the constraint $0 \leq h(t) \leq \kappa(t)qy(t)$ as $0 \leq E(t) \leq \kappa(t)$, it should be clear that in this new capital-based formulation the ceiling on effort (or harvesting) is no longer exogenous but instead determined by the available capital stock $\kappa(t)$. The rest is unchanged from Eqs. (2.22) - (2.23).

The main prediction of the extended model is that, under completely non-malleable assets, the fishery goes through two different phases: a short run period, when an initial high profitability leads to development of excess capacity (from a long run perspective) and when only operating costs matter, and a long run period with a new optimal equilibrium that is biological sustainable and incorporates both operating and capital costs, and where excess assets have depreciated (Clark, Clarke, and Munro 1979).

Similarly, if access to the fishery is not restricted and is instead open, overcapacity develops in response to the initial high profitability and of the "race for the fish" that follows. As Sec. 5.1 shows, when fishermen have to compete with one another for a dwindling resource in a short fishing season, they try to deploy as much as capacity as they can to harvest as much as possible. While individually rational, this scramble competition is obviously economically inefficient and causes a perhaps significant welfare loss for society as a whole.

Hence, according to the model over-capacity, especially in the short run (the model predicts, at least for restricted access fishery, a long-run equilibrium with no excess capacity), is certainly problem for resource preservation, but only as a *consequence* of the "race for the fish." The natural (and a bit simplistic) conclusion is that if managers succeed in halting the race, then fishermen will no longer have an incentive to over-invest in capacity compared to what they need for a sustainable harvest. Still, it remains open the question of what to do with the existing over-capacity and I will briefly talk about it when discussing IFQs and ITQs in Subsec. 5.4

5.4 Regulation in the model

Open access unregulated harvesting

Having defined the "social optimum," I begin my analysis by considering the case of an *open access unregulated* harvesting industry. As in Subsec. 2.3, when there is no restriction on the number of participants to harvest and entrance costs are negligible, the prediction of the model is bionomic equilibrium. Is bionomic equilibrium y_{BE} optimal according to Definition 19? The first easy way to compare y_{BE} with the golden rule solution y_g is to look at the extremes. Recalling that the bionomic equilibrium is the limit case of the golden rule when agents infinitely discount the future from Eq. (4.31)

$$y_g \to y_{BE} = \frac{c}{Pq} \qquad \text{as } \delta \to +\infty \tag{5.6}$$

I compare Eq. (5.6) with the no discounting case, $\delta = 0$, of Eq. (4.17) which I report below

$$y_{g^*} = \frac{K}{2} + \frac{c}{2Pq} \tag{5.7}$$

Looking at Eqs. (5.6) - (5.7) I can immediately conclude that the bionomic equilibrium is suboptimal since $y_g > y_{BE}$ as long as $KPq > c$ which is one of the assumptions for the admissibility of the singular control u_{sin} in Chapter 4. To proceed more generally, I can use Proposition 1 to conclude that an open access unregulated fishery always reaches a suboptimal equilibrium from a social standpoint.

Open access with input controls

Recalling the discussion in Sec. 5.1, the next step from a completely open access unregulated fishery has historically been to put input controls in place.

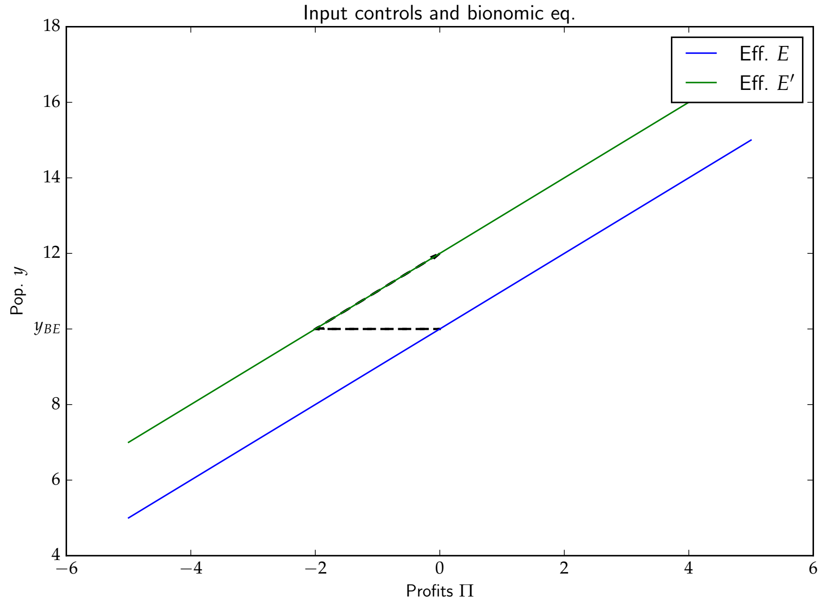

FIGURE 5.1: Effect of input controls on bionomic eq.

To analyze the effect of one such type of input control, namely restrictions on gears and fishing methods, I proceed with a simple graphical analysis using Fig. 5.1. I assume that before managers introduce input controls the fishery is at bionomic equilibrium y_{BE} with zero economic profits for fishermen. I also assume that to harvest at bionomic equilibrium fishermen are using an effort level $E \equiv E_{BE}$ which is close to the maximum effort $\bar{E}$, so that, once the new regulation reduces the maximum effort $\bar{E}$, and corresponding harvest level $\bar{h}$, to $\bar{E}'$ and $\bar{h}'$, the bionomic effort level is no longer feasible, $\bar{E}' < E_{BE} < \bar{E}$. In Fig. 5.1 the old effort level determines the blue curve: it is upward sloping because, for the same level of effort, profits Π are an increasing function of the fish density qy

$$\Pi(y, h) = Ph - cE(t) = (Pqy - c)\, E \tag{5.8}$$

where $h = qyE$ in the Schaefer catch equation. Eq. (5.8) also shows that since fishermen can no longer exert efforts greater than $\bar{E}'$, then for the same level of stock they will reap lower profits, causing in Fig. 5.1 a shift from the old effort curve (in blue) to a new effort curve (in green). Finally, under the assumptions of the model, I predict the following dynamics:

1. Once input controls are in place, the stock does not recover immediately from y_{BE} to $y' > y_{BE}$. Since in the short run the stock cannot increase very much and effort is curtailed at $\bar{E}'$, profits will fall from

the bionomic level Π_{BE} to $\Pi' < \Pi_{BE} = 0$. In Fig. 5.1 there is a shift from the blue to the green curve along the horizontal dotted arrow.

2. At some time depending on the population growth function $G(y)$ the stock will begin to recover *if overall effort is still below* $\bar{E}'$. In Fig. 5.1 the "equilibrium path" is moving along the green curve following the diagonal arrow.

3. If the stock continues to recover along the green curve, then in the medium to long run the population will stabilize around a new equilibrium level y'_E. Since the objective of the input controls were to cease over-harvesting, the new stock level y'_E should be higher than y_{BE} (and new profits $\Pi' > \Pi_{BE} = 0$). But then, *if entrance to the fishery is still open*, the bioeconomic system will tend once again towards bionomic equilibrium since the newfound profitability will attract new entrants until economic profits are driven down to zero and the stock level to y_{BE}.

Hence, in the framework of the above analysis, for input controls to succeed it must be that *(i)* fishermen do not find any way to raise their effort levels above $\bar{E}'$, and that *(ii)* entrance to the fishery cannot be completely open. In reality, however, fishermen have always found ways to legally circumvent regulations that limits fishing effort as Sec. 5.1 explains: for example, if fishing boats could not exceed a given length, then fishermen in some cases increased the width of boats. Even more unpractical is the proposal to impose restrictions both on inputs *and* on access to the fishery: often, resource intensive industries are the main employers in rural areas, and, as the model shows, present (certain) pain for future (uncertain) gain is simply not an appealing (and sometimes affordable) message in this case.

To recap, input controls in an open access fishery do not seem to be a solution to the over-harvesting problem: as I show in Subsec. 5.4, harvesting in a completely open access and unregulated fishery is suboptimal according to Def. 19, but input controls seem as either ineffective if implemented alone, or as politically unfeasible if coupled with restrictions on access.

Open access with taxes

In this subsection I still consider the case of an open access fishery, but instead of more traditional input or output controls I examine the effect of taxes. According to Defs. 16 - 17, taxes are, strictly speaking, neither an input nor an output control. The government could in principle impose them both on inputs, such as a tax on boats or fishing gear, or on output, such as a tax on harvest, with the declared intent of managing either effort or total catch. Since here I am considering a tax on harvested fish I would probably include it in the output controls category.

Going back to the model of an open access fishery, if the government were to impose a tax τ on all harvesters, profits would become

$$\Pi(y,h,\tau) = \left[(P-\tau) - \frac{c}{qy}\right] h \tag{5.9}$$

Since access to the fishery is still open, fishermen will continue to enter until economic profitability is driven down to zero leading to a modified after-tax bionomic equilibrium

$$y_{BE} = \frac{c}{q(P-\tau)} \tag{5.10}$$

As before, I use Definition 19 to compare the new bionomic equilibrium (5.10) to the extreme case of the golden rule (5.7) when $\delta = 0$. For a social optimum to exist, it must be that $y_{BE} = y_g$ or

$$\frac{K}{2} + \frac{c}{2Pq} = \frac{c}{q(P-\tau)} \tag{5.11}$$

and solving for the tax τ yields

$$\tau^* = \frac{P\,(KPq - c)}{KPq + c} \tag{5.12}$$

which is positive since $KPq > c$ by assumption. While τ^* is the optimal tax on harvesting for the extreme case when forward-looking agents value the future as much as the present, for a more general result I can use either Proposition 2, recalling that the bionomic equilibrium is the limit case of the golden rule for $\delta \to +\infty$, or proceed directly as follows.

Defining a new price level $P' := P - \tau$ the after-tax bionomic equilibrium (5.10) becomes

$$y_{BE} = \frac{c}{q(P-\tau)} = \frac{c}{qP'} \tag{5.13}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}P} y_{BE}(P) = -\frac{c}{qP^2} < 0 \tag{5.14}$$

and since y_{BE} is a decreasing function of the price, if prices decreases from P to P', then the after-tax bionomic equilibrium will increase from y_{BE} to y'_{BE} and get closer to the golden rule solution y_g, which is an equilibrium level bigger than y_{BE} by Proposition 1. Hence, the qualitative result from the above discussion is that to approximate the golden rule solution the tax rate τ must be positive. To get a more quantitative result, in the case of the golden rule solution with logistic growth $G(y) = ry(1 - y/K)$ I can actually compute the optimal tax rate τ^* using Eqs. (4.29) and (5.10)

$$\frac{K}{4}\left(-\frac{\delta}{r}+\sqrt{\left(-\frac{\delta}{r}+1+\frac{c}{KPq}\right)^2+\frac{8\delta c}{KPqr}}+1+\frac{c}{KPq}\right)=\frac{c}{q\,(P-\tau)} \tag{5.15}$$

and solving for τ.

A big difficult of the tax argument is that the optimal tax could depend on a lot of parameters, six for the general case of Eq. (5.15), that managers need to precisely estimate. Moreover, while in the model those parameters are time-invariant, in practice a bioeconomic system is much more non-autonomous in nature: prices, costs and future expectations change as do the natural growth rate and the carrying capacity of the fishery. Hence, it is unlikely that managers could truly compute an optimal tax, but the qualitative result that taxes are an effective instrument for steering harvest levels towards the social optimal equilibrium is still an important conclusion.

Dedicated access privileges

In this subsection I consider the case of a fishery where managers have implemented some form of dedicated access privileges. Since, according to Def. 18, they generally consist in granting of a fixed number of quotas to eligible participants, I can assume that after managers distribute quotas access to the fishery becomes restricted. In particular, in what follows I assume that managers are introducing an IFQ or ITQ.

The first issue when setting up a quota system is what should be the total allowable catch of the fishery. According to the optimality concept of Def. 19, the fairly obvious answer is that if fishermen harvest at the TAC then the corresponding population level must be the golden rule equilibrium level y_g. Denoting the TAC as $\bar{Q}$ and imposing the equilibrium condition $\dot{y} = 0$ it must be

$$\dot{y}_g = G(y_g) - \bar{Q} = 0 \implies \bar{Q} = G(y_g) \tag{5.16}$$

The second issue is, for a given quota distribution, if fishermen respond optimally to the IFQ or ITQ. Since access to the fishery is no longer free, the prediction of zero economic profits at bionomic equilibrium does no longer hold. Assuming for the sake of simplicity that there are N harvesters with the same cost structure and same quota $Q = \bar{Q}/N$, the optimization problem for anyone of the N firms becomes close to the one of the single price taker firm of Eqs. (2.22) - (2.23), namely

$$\max_h \int_0^T e^{-\delta t}[P - c(y(t))]h(t)\,\mathrm{d}t \tag{5.17}$$

$$\dot{y}(t) = ry(t)\left(1 - \frac{y(t)}{K}\right) - Nh(t) \tag{5.18}$$

$$y(t) \geq 0, \qquad y(0) = y_0 \qquad 0 \leq h(t) \leq Q \equiv \frac{\bar{Q}}{N} \tag{5.19}$$

Proceeding in the same way as in Chapter 4 with the same assumption of a zero discount rate, $\delta = 0$, the Hamiltonian for the problem in Lagrange form, which is linear in the control, and the switching function are

$$\mathcal{H}(t, y, h, \lambda) = [P - c(y)]h + \lambda\,[G(y) - Nh] \tag{5.20}$$

$$\varphi(t) = [P - c(y(t))] - \lambda(t) \tag{5.21}$$

leading to a similar combination of bang-bang and singular controls as in Eq. (4.27)

$$h^*(t) = \begin{cases} Q & \text{if } \varphi(t) > 0 \\ h_{\text{sin}} & \text{if } \varphi(t) = 0 \\ 0 & \text{if } \varphi(t) < 0 \end{cases} \tag{5.22}$$

As expected, for the N harvesters problem the singular arc and control are equal, respectively, to the golden rule and to a fraction of the singular control for the one firm case of Eqs. (4.17) and (4.24)

$$y^*(t) = y_g = \frac{K}{2} + \frac{c}{2pq} \tag{5.23}$$

$$h_{\text{sin}}(t) = \frac{1}{N}\frac{r}{K}\,(K - y^*(t))\,y^*(t) = \frac{Kr}{4N} - \frac{c^2 r}{4KNP^2q^2} \tag{5.24}$$

Hence, according to the model fishermen respond optimally since the quota system succeeds in steering their harvesting towards the golden rule solution which is the social optimum according to Def. 19

Finally, the third issue for a dedicated access privilege plan (IFQ or ITQ) is how to efficiently allocate the quotas between participants. Should managers simply grant quotas or rather sell them to fishermen? Since a natural resource like a fishery should, after all, accrue to all society, freely granting quotas appears to be a give away a valuable public asset with no return for the collectivity, but harvesters would naturally oppose to pay for something that they used to freely exploit. Currently, there is no clear answer in practice.

If quotas are sold, then it remains to determine what should be their value: should their price maximize revenues for the government while avoiding depletion of the fish stock, or should it achieve some other objective? In the setting of the N harvester IFQ problem of Eqs. (5.17) - (5.19), if quotas are sold at some price F, then the present value profits for a single harvester become

$$PV(\Pi) = \int_0^T e^{-\delta t}[P - c(y(t))]h(t)\,\mathrm{d}t - F \tag{5.25}$$

and the maximization does not change. If the government wants to maximize revenues from quotas it could charge a F such that when fishermen follow the optimal harvesting policy h^* with population response y^* their profits are barely positive, but while perhaps theoretically sound, this strategy is practically unfeasible. Alternatively, the government could set up an auction for quotas, or freely allocate a portion of quotas to eligible fishermen and auction or sell the rest.

If quotas are sold and are *transferable* as it is the case with an ITQ system, then, assuming that quotas are indeed exchanged on a more or less regular basis, their value should be the market price. While facilitating the price discovery both by managers (that can look at quota prices in similar markets before setting up a quota scheme in the fishery they manage) and fishermen, an ITQ system can also foster innovation in harvesting technology by fishermen. For example, fishermen in an ITQ have an incentive to invest in cost reduction more than they would do in an IFQ since they can not only harvest the same quota at a lower cost, but, if they find it profitable, can also expand their quota by buying from high marginal cost fishermen.

If quotas instead are not sold, then the question becomes how to distribute them between eligible fishermen. In this case, quotas are usually assigned on the basis of some sort of track record of active members of the fishery in the years just before the IFQ (or ITQ), but such an assignment strategy is not without trouble. If fishermen, both incumbent and entrant, can anticipate such a policy, then they have an incentive to fish even more in the hope of a larger future quota, leading to a more severe over-harvesting problem that can possibly nullify the effect of an IFQ (Clark 2006). In practice, this expectation issue is a serious problem because managers usually lack the political power to enforce the selling of quotas to fishermen that are accustomed to have a (financially) free access to the resource.

A possible solution to this conundrum could be to implement a double regime of ITQs (or IFQs) and taxes. Its first benefit is that, while no one likes taxes, public opinion could be much more sensitive (and supportive) of the problem of no taxation for a natural asset rather than to the technical issue of setting an appropriate price for IFQs. A second advantage of a double system is that, even if quotas are freely given away, fishermen should not have an incentive to invest in over-capacity since excessive gains could be taxed away. Moreover, managers could use revenues from taxes to compensate, at least partially, fishermen excluded from the quota system, according to a sort of Kaldor-Hicks efficiency principle. If coupled with some mandatory buy-back program, it is perhaps possible that this monetary compensation could be enough to offset a large fraction of the losses of the ineligible fishermen that deployed excessive capacity for the reasons outlined in Sec. 5.3. Finally, the combination of ITQs with taxes could reduce the estimation problem for taxes outlined in Subsec. 5.4, since to the estimate the TAC and then the ITQs managers need only to model the biological population and can safely ignore prices, costs, discount factors, etc.

6 Conclusion

The bioeconomic task of managing renewable assets such as forests, fish stocks, or grasslands is surely becoming a necessity rather than a "luxury affordable only by rich nations" if humanity is to learn from its past and finally look for a sustainable growth path (Clark 2010). The problem is economic in nature, renewable assets being just another form of capital capable of generating consumption flows, and economic theory should suggest possible solutions.

Unfortunately, the prevalent economic incentives and constraints push firms towards unsustainable harvest levels that risk depleting and ultimately extinguishing the exploited populations. In the literature, such a dynamic is called bionomic equilibrium, and, especially for fisheries, it is a far too common state of affairs: attracted by the initial high profitability of a natural resource, harvesters continue entering the industry until economic profits for entrant firms are driven down to zero, and the population close to depletion. It is widespread because an open access unregulated resource, which is a necessary condition for bionomic equilibrium to occur, has traditionally been the case for fisheries and other renewable assets.

Resource managers can and should introduce regulations to avoid the bionomic equilibrium outcome, but what objective should they strive for? They could, for example, wish to implement the "Maximum sustained yield (MSY)," meaning the highest harvest rate which can be sustained over an indefinite period and which does not lead to depletion of the population. But the MSY has at least one major deficiency because it ignores the economic rationale behind harvesting such as prices and cost of harvesting completely.

Alternatively to the MSY, managers could look at the "Maximum economic yield (MEY)," meaning the highest sustainable rate that maximizes economic profits. Compared to the MSY, the MEY does explicitly consider economic incentives, but, like the MSY, it neglects the fundamental dynamic nature of the problem. When deciding harvest levels, economic agents select an entire harvest path that stretches into the future and that may or may not be optimal depending on the response of the exploited population. Hence, rather than the MSY or MEY, managers should adopt a dynamic bioeconomic model of harvesting as benchmark for their decisions.

For the above reasons, following Clark (2010) in this thesis I focus mainly on a linear dynamic bioeconomic model of a single price taker firm. Through the tools of optimal control theory, I first derive an existence result for an optimal harvest level, and henceforth, thanks to the Pontryagin's maximum principle, necessary conditions for the optimal harvest and population level. The first important result is the so called modified golden rule which asserts that the marginal productivity of a renewable asset is exactly equal to its opportunity cost minus a "stock effect," and is the best population equilibrium that a single profit maximizer firm wants to achieve.

The second important result is that the golden rule solution always has a higher population stock than the bionomic equilibrium, implying that the latter is suboptimal with important consequences for policy. In fact, if managers want to escape from the trap of bionomic equilibrium and enact resource preservation coupled with economic efficiency, then the golden rule seems a good compromise, and it is indeed what I define as a social optimum.

To achieve the golden rule solution I consider different types of regulations: input and output controls, which mean respectively restrictions on effort and total catch, taxes and quotas. What I find is that, in the model, classical input and output controls alone cannot prevent a bioeconomic system from reaching bionomic equilibrium, since they do not manage to significantly modify harvesters' economic incentives.

Taxes seem a good instrument, because they succeed in implementing the golden rule solution even if managers do not limit access to the resource. However, to be effective they rely on the precise estimation of a large number of potentially time-varying parameters, where even a minor mistake could determine under- or over-harvesting situations that could cause, respectively, large economic losses to private firms or depletion of the resource stock.

Like taxes, quotas, or more specifically "Individual fishing quotas (IFQs)" and "Individual transferable quotas (ITQs)" manage to steer harvesting towards the golden rule solution, but, unlike taxes, they rely only on the right biological modeling of the population to determine the total allowable catch, thus reducing the magnitude of the estimation problem. However, quotas are controversial because there is no right way to implement them, since, unlike taxes, quotas restrict access to the resource, and, especially in fisheries, firms are strongly opposed to buying the right to harvest when access to the resource had traditionally been unrestricted. Furthermore, if quotas are freely granted, there is no way to compensate, at least partially, losers from the policy. A possible solution is to implement a double regime of taxes and quotas, where the revenue from taxes could in part offset the losses of ineligible participants to the quota system.

A Appendix

A.1 Mathematical definitions

Definition 20 (Norm). *A norm on a vector space V is a function $\|\cdot\| : V \to \mathbb{R}$ satisfying the following properties: for all $a \in \mathbb{R}$ and $u, v \in V$*

$$\|av\| = |a| \, \|v\|$$
$$\|u + v\| \leq \|u\| + \|v\|$$
$$\|v\| = 0 \iff v = 0$$

Definition 21 (Metric space). *A metric space is a pair (X, ρ) where X is a set and $\rho : X \times X \to \mathbb{R}$ is a function such that for all $x, y, z \in X$*

$$\rho(x, y) \geq 0$$
$$\rho(x, x) = 0 \iff x = 0$$
$$\rho(x, y) = \rho(y, x)$$
$$\rho(x, z) \leq \rho(x, y) + \rho(y, z)$$

Definition 22 (Lipschitz continuity). *A function $f : X \to Y$ with (X, d_X) and (Y, d_Y) two metric spaces is Lipschitz continuous with constant λ if $d_X(x_1, x_2) \leq \lambda d_Y(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.*

Definition 23 (Equicontinuity). *A family of functions $\mathcal{F} = \{f : X \to Y\}$ with (X, d_X) and (Y, d_Y) two metric spaces is equicontinuous at $x_0 \in X$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x_0), f(x)) < \epsilon$ for all $f \in \mathcal{F}$ and all x such that $d_X(x_0, x) < \delta$. $\mathcal{F}$ is equicontinuous if it is equicontinuous at every point $x \in X$.*

Definition 24 (Uniform boundedness). *A family of functions $\mathcal{F} = \{f : X \to \mathbb{R}\}$ is uniformly bounded if there exists a number M such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and all $x \in X$.*

Definition 25 (Uniform convergence). *A sequence of real-valued functions $\{f_n : X \to \mathbb{R}\}$ converges uniformly to a limit $f : X \to \mathbb{R}$ if for every $\epsilon > 0$, there exists a natural number N such that for all $x \in X$ all $n \geq N$ it is true that $|f_n(x) - f(x)| < \epsilon$.*

Definition 26 (Compactness)**.** *A space K is compact if each of its open covers contains a finite subcover. An open cover of K is a sequence of open sets A_i such that $K \subset \bigcup_{i \in I} A_i$.*

A.2 Golden rule

Recalling the discussion in Sec. 4.1, for the general case with $\delta \geq 0$ the bioeconomic model (4.1) in Mayer form is

$$\dot{x}(t) = f(x(t)) + g(t, x(t))u(t)$$

and using the familiar conditions $\varphi(t) = 0$ and $\dot{\varphi}(t) = 0$ it yields

$$\varphi(t) = \langle p(t), g(t, x^*(t)) \rangle = 0 \tag{A.1}$$
$$\dot{\varphi}(t) = \langle p(t), [f, g](x^*(t)) \rangle + \langle p(t), g_t(t, x^*(t)) \rangle = 0 \tag{A.2}$$

since as in Eq. (4.9)

$$\begin{aligned}
\dot{\varphi}(t) &= \langle \dot{p}(t), g(t, x^*(t)) \rangle + \langle p(t), g_x(t, x^*(t))\dot{x}^*(t) + g_t(t, x^*(t)) \rangle \\
&= -\left\langle f_x(x^*(t))' p(t), g(t, x^*(t)) \right\rangle - \left\langle g_x(t, x^*(t))' p, g(t, x^*(t)) \right\rangle u^*(t) \\
&+ \langle p(t), g_t(t, x^*(t)) \rangle + \langle p(t), g_x(t, x^*(t)) f(x^*(t)) \rangle \\
&+ \langle p(t), g_x(t, x^*(t)) g(t, x^*(t)) \rangle u^*(t) \\
&= \langle p(t), g_x(t, x^*(t)) f(x^*(t)) - f_x(x^*(t)) g(t, x^*(t)) \rangle + \langle p(t), g_t(t, x^*(t)) \rangle
\end{aligned}$$

where the extra term $\langle p, g_t(t, x^*) \rangle$ is because of the explicit time dependence of g. Therefore, proceeding in the same way as in Eqs. (4.11) - (4.15), to have a singular control the vector fields g and $[f, g] + g_t$ must be linearly dependent. Hence, the matrix $M \in \mathbb{R}^{2\times 2}$ whose first and second column are, respectively, g and $[f, g] + g_t$ must have a zero determinant

$$M = \begin{pmatrix} -1 & G(y^*(t)) \\ [P - c(y^*(t))]e^{-\delta t} & -G(y^*(t))c'(y^*(t))e^{-\delta t} - \delta e^{-\delta t}[P - c(y^*(t))] \end{pmatrix}$$

where its determinant is given by the following expression

$$\begin{aligned}
\det(M) &= G(y^*(t))e^{-\delta t} c'(y^*(t)) + \delta e^{-\delta t}[p - c(y^*(t))] \\
&- G'(y^*(t))e^{-\delta t}[p - c(y^*(t))]
\end{aligned}$$

The condition $\det(M) = 0$ implies that the state $y^*(t)$ along the singular arc must satisfy the following implicit equation

$$\delta = G'(y^*(t)) - \frac{G(y^*(t))c'(y^*(t))}{P - c(y^*(t))} \tag{A.3}$$

and, recalling that as Clark (2010) I assumed a logistic growth function $G(y) = ry(1 - y/K)$, the explicit solution of Eq. (A.3) is

$$y^*(t) = \frac{K}{4}\left[\left(\frac{c}{PqK} + 1 - \frac{\delta}{r}\right) + \sqrt{\left(\frac{c}{PqK} + 1 - \frac{\delta}{r}\right)^2 + \frac{8c\delta}{PqKr}}\right] \tag{A.4}$$

which is exactly the modified "Golden rule" of Eq. 4.29.

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List of Symbols and Abbrev.

c	Marginal costs per unit of effort
$c(y)$	Cost function per unit of harvest
δ	Instantaneous discount rate
$E(t)$	Effort level
$\bar{E}$	Maximum effort level
$G(y)$	Logistic growth function
$h(t)$	Harvesting rate at time t
$\bar{h}$	Maximum harvesting rate
K	Carrying capacity of the population
$P(\cdot)$	Price level or inverse demand function
Q	Individual quota in an IFQ or ITQ
q	Catchability coefficient
$\bar{Q}$	TAC level
r	Intrinsic growth rate of the population
τ	Tax on harvesting
$U(h)$	Social utility of consumption
$y(t)$	Population level at time t
y_{BE}	Bionomic equilibrium population level
y_g	Golden rule population level

Mathematical Symbols

$\mathcal{U}_\lambda[t_0, t_1]$	Class of functions in $\mathcal{U}_m[t_0, t_1]$ with Lipschitz constant λ
$\mathcal{U}_m[t_0, t_1]$	Class of measurable functions $u : [t_0, t_1] \to U$
$\mathcal{U}_{PC}[t_0, t_1]$	Class of all piecewise continuous functions $u : [t_0, t_1] \to U$
$p(t)$	Adjoint variable in PMP
U	Control set
$u(t)$	Control function in optimal control formulation
$u^\epsilon(t)$	Strong variation of a control
$x(t)$	State function in optimal control formulation
$x^\epsilon(t)$	Response to a perturbed control

Acronyms

BE	Bionomic Equilibrium
CFP	Common Fisheries Policy
IFQ	Individual fishing quota
ITQ	Individual transferable quota
MEY	Maximum economic yield
MSY	Maximum sustained yield
PMP	Pontryagin's maximum principle
TAC	Total allowable catch