*Department of* Economics and Finance *Chair* Mathematics

# Taming complex allocation

## Algorithms and insights applicable to FCC spectrum Combinatorial Auctions

SUPERVISOR

PhD Prof. Marco Dall'Aglio

CANDIDATE

Gianfranco Portaluri

ID. 187031

ACADEMIC YEAR 2015 / 2016

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*List of abbreviations*

| | |
|---|---|
| CA | *Combinatorial Auction* |
| EDP | *Edge-disjoint Path* |
| GA | *Greedy Algorithm* |
| GVA | *General Vickrey Auction* |
| NE | *Nash Equilibrium* |
| NP | *Non-deterministic Polynomial (Problem)* |
| OPT | *Optimal Allocation* |
| OR | *Exclusive Bidding Language* |
| PCS | *Personal Communication Services* |
| SAA | *Simultaneous Ascending Auction* |
| SM | *Single Minded* |
| SV | *Single Valued* |
| VCG | *Vickrey-Clarke-Groves (Auction)* |
| XOR | *Nonexclusive Bidding Language* |
| WDP | *Winner Determination Problem* |
| WSPP | *Weighted-set Packing Problem* |

*Abstract*

*This paper pursues an investigation over the importance to understand combinatorial algorithms in auction setting for the optimal allocation of an indivisible good and the subsequent maximization of welfare. In chapter 1 we are going to discuss why the theory behind the so called design of combinatorial auctions is fundamental in the problems that we bear in mind to solve, giving a sufficient mathematical background to our inquiry in the context of single minded bidders, with reference to case of public elicitation of preference or no elicitation. In this part we will examine a taxonomy of different algorithms, seeking for an approximation of the winner determination problem rather than an optimal allocation. In chapter 2 we are examining the most practical application concerning indivisible goods, the radiofrequency spectrum licenses, understanding strengths and weaknesses of different auction designs and choosing a best way of implementation.*

# INTRODUCTION

In today complex world, the need for overconsumption and excess ownership of a good or service is common knowledge. The neoclassical microeconomics theory gives way to a partial, approximated solution of the problem of finding equilibria and at the same time pursuing a welfare and profit maximization, identifying in Pareto optimality the main focus of its existence and self-enforcing a reason to this theory. Apart from the latest and very precious assumptions of bounded rationality and framing implemented by behavioural economic theory, there is one aspect clearly assumed to be true and incontestable in this setting; the items that are object of the economic problem – to realize how is better to act in allocation of scarce resources – are clearly non-atomic and divisible in a (theoretically) infinite number of subunits.

That is why our inquiry starts, with the question: what if the objects of the economic problem would be indivisible and impossible to reallocate a similar amount of them to a second person, when that unit has already been allocated to a determined individual? The social welfare that should derive from this reasoning should be less than optimal. Though, we can find some method to cope with this striking and apparently uncompromising dilemma with some algorithmic procedures, thanks to whom we succeed in modeling reality in a fairly approximated way. Moreover, Cramton et al. (2006) report that "what makes such allocation problems particularly interesting and challenging is that the buyers have complex preferences over packages of objects: the value of a package to a potential buyer is not necessarily equal to the sum of the values of individual items; the buyer may view some items as substitute for one another and some items as complements". The value of the whole package then is valued greater than the sum of two parts, especially if the items are considered perfect complements.

This type of allocation problem is becoming so frequent and relevant in many markets for goods ad services (for example in FCC radio spectrum licensing – on which we will focus our attention - operation research, procurement and transport optimization) that an increasing number of computer scientists, economists and mathematicians are putting much more effort than just grasping at it with mere words. The topic behind the findings that we are going to explain is commonly known in algorithmic game theory as *combinatorial auctions*, since it involves a deep analysis of auctioning methods, proofed by a strong mathematical background. We will refer to combinatorial auctions from now on as CA.

Conceptually, applying a repeatable algorithm to the solution of a combinatorial allocation problem is not difficult. Once you understand which is the right method, then it is possible to efficiently allocate to a certain degree of approximation the items to the right bidder. The algorithm behind it will ensure that, given that determined conditions are matched, these approximated allocations can be found again in other circumstances and with other data.

Hence in reality it does not exist any 'good' or general combinatorial auction algorithm that could fit allocation problems of every kind. There exist particular settings that work extremely well with CAs, depending over particular conditions, such as: the nature of items auctioned, their complementarity or substitutability with other objects traded (largely depending in turn to the agent's preferences) and finally on the amount of agents bidding in the auction. Anyway, one thing we must give for sure: a combinatorial auction mechanism should be treated in the same way as it is a game of incomplete information, where the weakly dominant strategies represent a "good enough" way to allocate goods and clear a payment scheme. We will prove where and for which reason combinatorial setting succeed in being an efficient market clearing mechanism due to a comparative advantage, and where modifications must be introduced in order to ameliorate the process or where totally different auction design are preferred.

## CHAPTER 1

This chapter begins with a first contact with the setting of combinatorial auctions, their related difficulties and peculiarities. Various definitions about computational complexity and winner determination problem in CAs are issued in section 1.1. A deep analysis of many allocating algorithms concerning complex items are given, restricting our point of view over the condition of single minded bidders in 1.2. A critical perspective is introduced in 1.3 through a comparison with the most classic auction setting, the Vicrey-Clarke-Groves auction, claiming its limits in market clearing. In section 1.4 further analysis over the unknown and known domains of bidders' preferences are investigated.

### *1.1 Facing the problem*

The setting that combinatorial auctions try to address could be formalized as follows. Suppose we do have a set of *m* items, which are auctioned to a number of *n* bidders. The auction takes place concurrently, i.e. all the bidders must make their bid at the same time. We suppose moreover that each of the bidders has a preference over a determined subset of items, defined as 'packages' and of course some or the totality of items contained in the *i*th preference relative to bidder *i* can be present in the *j*th preference relative to bidder *j*. That is to say, we cannot exclude the case in which:

*Let $N = \{i,j,k\}$ be the set containing three bidders participating in a transaction, with $i \neq j$, $j \neq k$, $i \neq k$.*

*Let $M = \{x,y,z\}$ be the set containing all the possible items traded in the market, and let $e_M$, $e_M'$, $e_M'' \subset M$ be three generalized elements of this set. Then, $\forall\ e_M, e_M', e_M'' \subset M$, with $e_M \neq e_M'$ and $e_M \neq e_M''$, $e_M' \neq e_M''$ we can always find in a combinatorial auction setting three sets I, J, K s.t.*

$$\begin{aligned} &\textit{Bidder i's preference set} \quad I = \{e_M, e_M', e_M''\} \\ &\textit{Bidder j's preference set} \quad J = \{e_M, e_M'', e_M''\} \qquad (1) \\ &\textit{Bidder k's preference set} \quad K = \{e_M, e_M', e_M''\} \end{aligned}$$

We can notice from (1) that in the extreme case bidder *i* and bidder *k* have a preference over the same identical bundle, while bidder *i* and bidder *k* have in common the first item and the second item. This is a very rare example, for $m = n$. In the majority of cases, we have that $m < n^2$. But that is not the end of the story. When allowing for combinatorial bidding, the participants to the resulting auction are entitled to place a value over multiple bundles of indivisible items. For the sake of illustration, let $M = \{m_1,\ldots,m_6\}$, with $M$ the set denoting all the items auctioned, $N = \{n_1,n_2,n_3\}$ being the set of all bidders in the auction. A scheme for a combinatorial auction arising from this market could be:

| | Bid | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ |
|---|---|---|---|---|---|---|---|
| $n_1$ | 25 | • | | • | • | | |
| $n_1$ | 18 | • | • | | | | |
| $n_1$ | 47 | | | | • | • | • |
| $n_2$ | 39 | | | • | • | | |
| $n_2$ | 24 | • | | • | | | |
| $n_3$ | 28 | | | | | • | • |

Figure 1. A combinatorial auction scenario

The dots represent the inclusion of the item $m_k$, with $k \in \{1,2,3,4,5,6\}$ in a certain bundle, blank spaces represent no inclusion. We see that bidders can *combine* their auction over multiple bundles. For instance, bidder $n_1$ can bid \$25 on the combination of items $\{m_1, m_3, m_4\}$. But at the same time she could bid \$18 over the bundle $\{m_1, m_2\}$. The same applies to the other bidders.

The combinatorial nature of our problem stems from the fact that bidders indeed have *valuation functions* relative to particular subsets or bundles of items. This valuation function is, for adherence to reality addressed to have a real value in money, i.e. each bidder is assumed to be a rational utility maximizer, and the valuation function quantifies her willingness to pay for the bundle of items. How can we represent such a value function, so as to quantify it?

*Definition 1.1*

*If we let A be the set of all the assets being auctioned, with* $A = |m|$. *and let* $S \subseteq A$ *s.t. S represents a particular combination of assets, then we can determine a value function* $v_i$ *,* $i = \{1,\ldots,n\}$*, s.t.*

$$v_i(S) \qquad \text{is the value function relative to bidder } i \text{ over bundle } S$$

This value function is addressed the scope of determining the willingness to enter into a new bid if the stake is sufficiently high, or to quit if it is sufficiently low. We need then to impose some assumption for the continuation or not of the game. The two main assumptions that we pose on this value function specifications are:

*Definition 1.2*

*Let S, T be two sets, with* $S \subseteq T$*, we assume:*

$$\textit{Monotonicity: for} \quad S \subseteq T \quad \rightarrow \quad v(S) \leq v(T) \tag{2}$$

$$\textit{Normalization:} \quad v(\emptyset) = 0$$

where monotonicity issue states the increasing behavior of valuation function for any superset *T* including bundle *S,* and normalization establishes the difference arising from being a winner in the auction or not.

It is quite implicit, but very important to stress again that each bidder may see, depending on her value function a particular set of items as *interrelated* with another. If the items would have been traded separately in fact, the problem would have been posed differently, and the setting of a normal ascending English auction would have been sufficient for our purposes. As a matter of fact, we should allow each bidder to confront with another one by means of a *bidding language*. This in turn could be of any nature, but for simplicity of scopes we will address a simple preference expression. If we account for a set G, where *n* bidders are contained, each of these bidder may have a strict preference or weak preference over a set of *m* alternatives. For instance:

*Definition 1.3*

*Let* $B = \{b_1, \dots, b_n\}$ *be a set containing n participants to an auction.*

*Let* $x, y \in S$ *be two different items or bundles that a bidder i is requested to rank.*

*Then a strict preference for bidder i between x and y is formalized as:*

$$x \succ_i y$$

*And a weak preference for bidder i between x and y is denoted:*

$$x \succcurlyeq_i y$$

*We can hence implement these definitions by saying that bidder i has a strict preference for bundle X over bundle Y, with* $X = \{x_1, x_2\}$ *and* $Y = \{y_1, y_2\}$ *iff.*

$$X \succ_i Y$$

This points out the next necessity of definition for a bidding valuation function regarding indivisible goods, i.e. to contest that the value of the sum of two separate objects is equal to the value of the union of the bundle formed. Two types of object can indeed arise:

*Definition 1.4*

*For sets S and T, with $S \cap T = \emptyset$, $S = \{s_1, \dots, s_m\}$ and $T = \{t_1, \dots, t_m\}$, then $\forall\ s_k$ , $t_k$ with $k = \{1, \dots, m\}$*

$$\begin{aligned} &S, T \text{ are complements} \quad \leftrightarrow \quad v(S \cup T) > v(S) + v(T) \\ &S, T \text{ are substitutes} \quad \leftrightarrow \quad v(S \cup T) < v(S) + v(T) \end{aligned} \tag{3}$$

If Marlene for example bids \$3 on *S*, \$9 on *T* and \$15 on $S \cup T$, then this situation denotes complementarity. The opposite applies in case of substitutability.

Complementarity issue deals most of the time to the reverse case of CA, named the *many-to-one* case. Imagine a situation in which there are multiple sellers to propose an auction where only one buyer is present (for example in government expenses for renovations of public goods, procurement or when the goods in object complements each other well)[1]. But many other applications, like the one that we are going to examine can be found in the *one-to-many* situation. In this case, the single bidder can place a value that is higher than the sum of each element alone, like stated in (3); for this reason it is also referred to as being the situation of *super-additivity*. On the other way around, substitutability can lead to a value placement for the most valuable between two packages, let's say *U, W* (we should therefore $max(v_i(U), v_i(W))$ or no comparative value placement at all (for example, which train ticket would you reasonably prefer between two that have the same price and let you make the same distance?); often it is named the situation of *sub-additivity.*

If we cannot account for neither substitutability nor for complementarity, then an auction designed as a set of independent simple auctions will be for sure a good design to work with. But if situation like (3) arises, and especially when bidders have both complementarity and substitutability issues in different measure among each other with regards to the same items, then things get acutely harsh.[2]

[1] Examples of a many-to-one case is the allocation of bus routes from Cantillon and Pesendorfer in Cramton et al. (2006).

[2] We should stress however that implicit in the definition of bidder preferences valuation functions should be *quasi linear* in their real specification. Indeed, if $v_i(S)$ is the value attached to a particular bundle *S* from agent *i* and $p(S)$ is the real valued price that *i* must pay for obtaining the bundle, then the right function is $v_i(S) - p(S)$. We also assume no

Lehmann et al. (2002) pinpoints this situation in few simple words: "[…]in the presence of complementarity […] (single item auctions) will lead to less than optimal results: for example a participant ending up with a left shoe and another one with the right shoe, or the left shoe auctioned for almost nothing because bidders fear not to be able to get the right shoe and the right shoe then auctioned for nothing to the buyer of the left shoe since no one is interested in just a right shoe".

Since our main goal anyway is an approximately optimal allocation of the scarce resources among bidders and the various uses, we should focus our attention over the maximization of the total welfare arising from the auction, allowing for its formal definition. We define a welfare maximizing allocation $A = \{S_1,...,S_n\}$ among $n$ bidders one satisfying these two conditions:

$$S_i \cap S_j = \emptyset \qquad , \forall\ i \neq j$$

$$\max \textstyle\sum_i v_i (S_i),\ \ for\ i = \{1,...,n\} \tag{4}$$

The first property ensures the pairs representing the bids of the $n$ participants, like $(S_i, v_i)$ are not in conflict one with the other, so each bidder does not care for the valuations of another bidder and there is an absence of externalities in the allocation method. The value functions are in fact kept private, and we will see in a few paragraphs that a dominant strategy is to continue bidding keeping faith to the original value (so not modifying it during the course of the actions).

During the allocating process we shall not forget of another important property that pertains to CAs. There is no need that each item or package is allocated during the process. That is to say, this discriminates also between an optimal and a suboptimal allocation. Formally, if we keep pace with (4), we let $q$ be a function or mapping $s.t.\ N \xrightarrow{q} M'$, where $M' = M \cup \complement(A)$ , and $M, N$ are taken from (1). This concept takes the name of *free disposal* in auction design.

Feasible solutions for the allocation problem can be derived either finding a real equilibrium point by modelling the problem as an integer program and then solving it with a search-based algorithm (this solution is more complex and less explicative, it is really interesting though and further analysis is in Sandholm, 2000), either searching for suboptimal solution relaxing determined assumptions, approximating the real value function to a determined asymptotic degree. Generally speaking, in economics the goal for an optimal allocation that would not imply any loss over the auctioneer revenues, as well as no waste of auctioned resources at disposal as well as welfare

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externalities between the bidders, i.e. each bidder focuses not on the preferences of the other bidders, but only on his personal ones (Nisan et al., 2007).

maximization for bidders is a gift to be desired, of course. But the truth is that it is very difficult to set a problem without taking into account for difficulties in calculations and bidders' preference in this case, so approximated solutions in the second way of implementation tend to be more realistic and computationally tractable. That is why we will mainly focus over auction designs and algorithms that retrieve a result in approximated fashion, and which principally deal with the problem of welfare maximization.

Because of the computational problem that we mentioned above, we do not immediately get into the jungle, but rather we try to assess the nature of such problems from the outside, in order to better know our enemy. Since it could be partially a new concept for the majority of the readers, for what concerns computational complexity we should better define now the concepts of *NP-hardness* and *NP- completeness* in definition 1.5 below.

*Definition 1.5*

*i)* *NP-completeness: a problem is defined as belonging to the family of NP-complete if it is a nondeterministic problem and if, once found an algorithm to resolve it in polynomial time then this algorithm could be implemented to efficiently and quickly resolve any other problem of the NP family. Practically* $\nexists$ *such an algorithm, but studies proved that hypothetically an NP-complete problem could be solved by a nondeterministic Turing machine, offsetting it with an algorithm providing a zero error of approximation, thus leading to the new category of ZPP-problems.*[3]

*ii)* *NP-hardness: a problem defined as belonging to the family of NP-hard ones is by definition difficult to resolve in an efficient way on the basis of polynomial time. More formally, a problem H is said to be NP-hard if* $\exists$ *a NP-complete problem L s.t.* $L \approx H$ *in a polynomial time.*

In a somehow rough way trying to quantify the polynomial time necessary to reduce the allocation problem by means of a specific algorithm is helpful in the same way of a nth-order Taylor expansion when approximating a determined function. We will deal in combinatorial setting always with NP class problems, since they involve not only decisional problems but optimization problems too. Trying to address the NP family classification by means of Venn diagrams, we can see a clear distinction in Figure 1, where problems belonging to the polynomial solvable family *P*, with

[3] See Lehmann, Mueller, Sandholm in Cramton et al. (2006) and Hastad (1999).

$P \subset NP$ are considered different or not from those of non-polynomial solvable (this distinction between $P = NP$ or $P \neq NP$ is still a controversial object of debate that goes beyond our scopes):

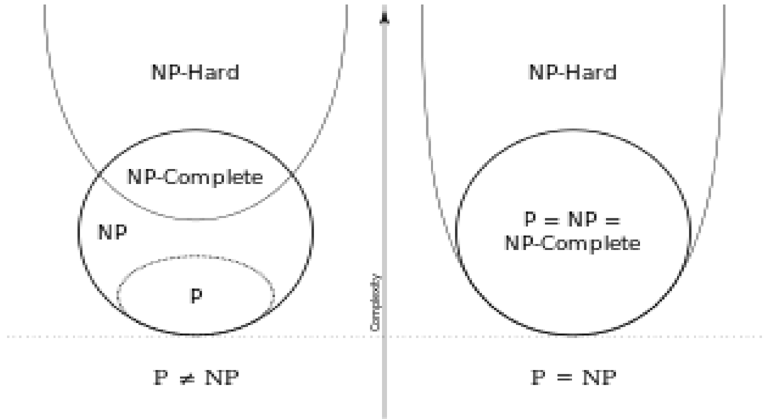
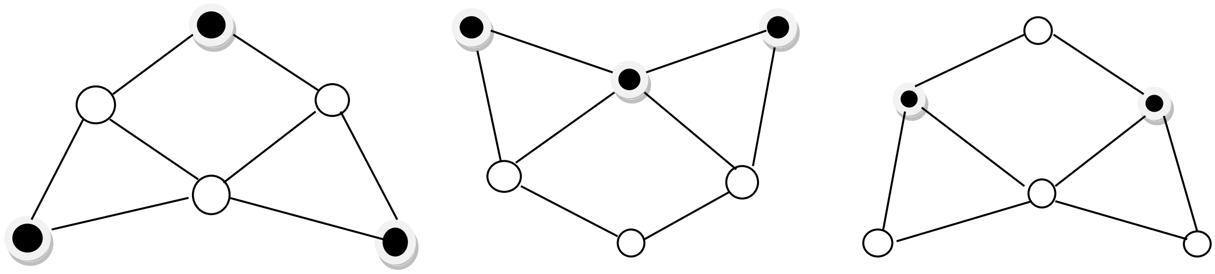



Figure 2: NP-problems classes

Unfortunately for us, exponential objects like combinatorial valuations are things to treat very cautiously when it comes to computing. They cannot simply be approximated by a nth-order polynomial expression since they tend to increase with a higher speed than polynomials. Apart the increasing power of computation by modern processors, it is quite cumbersome and counter-effective to try to address the optimized allocation problem in an absolute perspective. Procurement and industrial entities always try to model the problem of this type with an approximation, and that is why we chose to adhere to this point of view. But NP-problems can be more effectively understood if compared to a graphical scheme. We find extremely useful for our purposes to define the *weighted-set packing problem,* for the first time classified by Karp (1972) as one of the most important problems of NP class. The weighted packing problems can be formalized in first instance as linear integer program graphed in a . If we let for instance $M$ be a universe with size $|M|$ and $S$ be a family of subsets *s.t* $S \subseteq M$, we can define a *packing* $P$ a subfamily for which $P \subseteq S$ and s.t. all sets $\{p_1, \ldots, p_m\}$ are pairwise disjoint, i.e $\forall\ p', p'' \in P$, then $p' \cap p'' = \emptyset$ . The size of the packing set is $|p|$ and inputs are represented by pairs of the type $(M, S)$, and $k$ as a natural number. In the set packing *decision problem* we are asked to find in the inputs if $\exists\ P$ in $(M, S)$ s.t. $|k|$ is the size of $P$ or even a greater size. In the set packing *optimization problem* (that best suits our needs) we are asked if $\exists\ P$ s.t. $P$ has maximum size. The $P$ having maximum size is denoted as the maximum independent set, with $|k| = |p|$. In the figure below we see a *G(V,E) graph* where the vertex of the web are represented by the bidders, while the edges represent the items or bundles traded in the auction (hence the usual notation as graphs G(vertex, edge)). To each vertex are associated a weight $w_i$ indicating the valuation that each bidder $i$ poses over a determined collection of edges. Darkened nodes represent the maximum $k$ size of the independent subsets. But how to proceed for this set packing optimization problem? Intuitively, we should first match with a principle of cardinality a $p$ having a maximum weight $w$, and only then $\forall\ t \in \{0,1,\ldots,n\text{-}2, n\text{-}1\}$ we should repeat an algorithm such that, increasing $t$ to $t+1$ we create a set $Y_t = \{p \in P_{t-1}, w(p) = max\}$, and $A_t = \{p \in P_{t-1} \mid Y \cap P \neq \emptyset\}$ and we subtract $P_{t-1}$ from $A_t$ in order to update the $P_t$, until we remain with $P = \emptyset$. In few words, this graphical problem could be summarized by the question: "how many bidders can we count as winners in the auction, given a $n$ number of effective participants?".



a) b) c)

Figure 2: a G(V,E) graph with maximum independent subset $P$ with $|k_p| = 3 = p$. Figure a) indicates the largest independent subset $P$ with $|k_p| = 3$. Figure b) is not even an independent subset. Figure c) represents not the largest subset possible, with size $|k_p| = 2$.

Nevertheless, during our analyses four principal difficulties will arise, that will hamper recursively calculations and will impose boundaries on findings. These can be collected as:

*Definition 1.6*

*i)* *Computational complexity:* the allocation problem is computationally hard even for a small number of bidders, even for a small number of items auctioned.

*ii)* *Representation and communication:* how can we face the problem of value function computation and communication to the auctioneer? Especially, value function tend to increase exponentially when items are implemented in the bundle.[4] Given an amount of

[4] Lehmann et al., (2002) suggest that the size set of the allocation with at least two bidders is exponential in $m$, and the set of possible types of bids sent to the auctioneer is of size $2^m$, so it is practically unfeasible even for a couple of dozen of items auctioned.

$m$ auctioned items, the possible combinations that bid-taker could receive during the auction could reach up to the number of $2^m-1$.

iii) *Strategies:* can we shape bidders' strategies effectively? It strongly depends on the auction design, i.e over its bidding language and rules (how and when the bid can be transmitted to auctioneer), the market clearing rules (who is entitled to gain what), the information disclosure (what information are disclosed and when).

iv) *Mechanism:* how can we design an efficient auction in order to achieve a specific outcome representing a strategic behavior?

Due to all these difficulties encountered, it is ever more hard for a bid-taker to match who exactly are the winners in the auction, and who are the loser. This concept is named *winner determination problem* or (WDP) and it is of fundamental importance to understand it for the forthcoming explanations. The definition of the WDP is deeply entrenched with the one made above for weighted-packing problems, since at the scope of what we proposing they are prone to the same difficulty of computation.

There is also the possibility to settle the WDP in a more analytical way. As for the weighted set packing problem, we could set our question starting from a collection of $n$ pairs of bidding declarations, thus defining a vector $D = \{(S_1, b_1), (S_2, b_2), \dots, (S_n, b_n)\}$. Given this, the WDP ends to find the subset of disjoint bids such that the sum of values $b_i$ is maximized. Following the notation of the weighted packing set above a straightforward definition of the integer program for WDP can be:

*Definition 1.7*

*Let $P \subseteq M$ be represented as a binary vector $[p_1, \dots, p_{|M|}]$ where $p_h = 1 \leftrightarrow p_h \subset D$ , and 0 otherwise. Thus a single bid, say $b_i$ can be represented by a vector $[p_{i1}, \dots, p_{i|M|}]$ and a value $b_i$ for the constant value assigned to bid i. Let furthermore $x_i$ be a binary variable s.t. $x_i = 1 \leftrightarrow b_i$ wins, with $x_i \in \{0,1\}$. Then the WDP can be posed as the following integer program:*

$$\max \sum_{i=1}^{n} b_i \ x_i \tag{5}$$

$$\text{s.t.} \qquad \sum_{i=1}^{n} p_{ih} \, x_i \leq 1 \qquad \forall \ 1 \leq h \leq |M|$$

Then the optimization reduces to find an optimal allocation of values *{0,1}* to variables $x_i$ *such that the sum of bidding values is maximized while keeping for the constraint that each bundle h is allocated to at most one bid.* We deduce that the free disposal concept is preserved, since there is the possibility some bundles are left unassigned.

### *1.2 Single minded bidders*

Since accounting for each of the three difficulties mentioned above is not the main goal of this work but rather to find practical algorithms that allow for a specified amount of approximation in the allocation process, as well as in welfare maximization, we will disregard for a moment the second difficulty in our reasoning, the one concerning the representation and communication of value functions from bidders to sellers, since this has for principal object the one to maximize revenues for auctioneers, and not the (4). Moreover, we leave out from our reasoning the possibility that the bidders may express their preferences through a complex or more expressive bidding languages, like OR and XOR combinations[5]. We leave analysis of this aspect to further research.

Then, if we take into account for *i)* and *iii)* in *Definition 1.6*, we may set the allocation in a way far easier than usual and model reality restricting the bidding subset of any participant in the auction to a unique set of items. That is, we do not have the uncertainty arising from (1) anymore. This type of bidders has a particular name in game theory, they are addressed as *single minded bidders.* Such bidders are interested only in a particular, specified bundle of items, and they associate to the purchase of this set a scalar value, while if they do not get that particular set then they associate to this bid the value of zero. More formally we can say:

*Definition 1.8*

*A bidder can be called single minded if, given* $v_i^* \in \mathbb{R}^+$ *, with* $i=\{1,\dots,n\}$*,* $S_i^* = \{s_1^*,\dots,s_m^*\}$*, with* $S_i^* \subseteq S$*, then*

[5] Bidding strategies can be complex too: an example are the OR and XOR bidding languages. For a brief introduction, given two bids A and B an OR of the two bids means that either bid A and B are accepted, or both; on the other hand, for XOR bidding only A or B separately could be accepted. For more details see Lehmann et al. (2002) and Wilenius (2009). For concrete application of complex bidding languages to sourcing see Sandholm (2007).

$$\exists\ S_i^*, v_i^* \quad s.t. \quad \begin{cases} v_i(S) = v_i^* & \forall\ \ S_i^* \subseteq\ S \\ v_i(S) = 0 & otherwise \end{cases} \tag{6}$$

*Where $v_i$* represents here the value bidder i is ready to pay for her optimal package $S_i$* . S is any superset including the specified optimal bundle of items.*

We notice that the allocation problem with single minded bidders easily reduce our input/output elements to a pair *(S*, v*).* From now on, unless otherwise stated we will reason only in terms of single minded (SM) bidding behaviour.

It is therefore clear that an optimal allocation in this setting would be to give every bidder the bundle that he exactly places as the most valuable, or nothing at all, i.e:

$$V_{sm} = \begin{cases} S = S^* \\ S\ = \emptyset \end{cases}$$

*where $V_{sm}$ represent the single minded optimal outcome.*

The winners at the end of the allocation should enter in a particular category. If we allow *W* be the set of all winning combinations, taking two bidders namely *i, j s.t.* $\forall\ i \neq j, with\ i, j\ \in W$ *,* our problem should now resolve:

$$\rightarrow S_i^* \cap S_j^* = \emptyset\ ,\ max \textstyle\sum_{i,j \in W} v_i^* \tag{7}$$

### *1.2.a Approximation, special cases and heuristics*

Once we have the defined the setting, we try to address analytically welfare optimization problem through a simple approximation mechanism, called *c-degree approximation*. This approximation method restrict the bidders' welfare saw in (4) to a reduced subset of allocations. While inputs for SM bidders to enter the auction are all the collected pairs $(S_i^*, v_i^*)\ \forall\ i = \{1,...,n\}$, the output is instead represented by all the subsets of winning bids (remember that we must always consider for non-frictional absence of externality among purchasers by means of disjoint preference sets) that maximizes the social welfare.

*Definition 1.9*

*Let $W = \{W_1, \ldots, W_n\}$ be the set of all winning bids s.t $W \subset S$. Let furthermore $c \in \mathbb{R}^+$, with $0 < c < 1$ in general.*

*We say that an allocation $\{S_1, \ldots, S_n\}$ is a c-degree approximation of the optimal one for n bidders if:*

$$\forall \{W_1, \ldots, W_n\} \in W \quad \rightarrow \quad \frac{\sum_i v_i(W_i)}{\sum_i v_i(S_i)} \leq c \tag{8}$$

We can better see how this approximation works by multiplying both sides of the (6) by its denominator, holding:

$$\sum_i v_i \ (W_i) \ \leq \ \sum_i v_i \ (S_i) \ c \qquad \forall \ i \in \{1, \ldots, n\} \tag{9}$$

We can see this finding in an even more general way:

*Theorem 1.1*

*Let f and g be two functions s.t. $\mathbb{N} \xrightarrow{f,g} \mathbb{N}$. If $\exists \ c > 0$, $n_0$* s.t. *$f(n) \leq cg(n) \ \forall \ n \geq n_0$, then we can say that g is an asymptotic upper bound (within a constant) for f. We denote such boundary with the notation $f = O(g)$. If in addition, $\forall \ c > 0$, we can find a $n_0 \leq n$, then $n_0 \rightarrow f = o(g)$ that means that*

$$f \approx g \leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \quad \text{with } g(n) \neq 0. \tag{10}$$

Can this simple approximation be our panacea to the resolution of complex allocation problem when the number of bidders and items in set is large? Of course not. The NP-hardness of *Definition 1.5* still would persist in computation, making our life uneasy. From section *1.1* we notice that not only the problem still is classified as NP-complete when searching for the maximum independent set, but even when approximated within a factor of $n^{1-\varepsilon}$ (for any fixed $\varepsilon > 0$) it remains a NP-complete problem.[6] As a rule of thumb, the problem should be classified as a polynomial deterministic problem if $\exists$ an algorithm that finds the solution in $O(n^k)$ steps of polynomial time, for some fixed *k*. Anyway, an approximated solution with an incentive compatible mechanism could be found, not only because of the approximation mechanism, but also if certain strategies of bidders are verified. We will discuss this topic in section *1.2.c.* Special situations were pinpointed by

[6] Nisan, Noam et al.(2007), 271-272 and Hastad (1999) suggest that since the welfare is exactly equal to the size of independent set, then we can express NP-hardness also from the side of items amount. Assuming that $m<n^2$ as in (1), finding a socially optimal allocation not being approximated within a factor $m^{1/2-\varepsilon}$ is still NP computationally hard.

Rothkopf et al. (1998) using geometric based structures capable to resolve WDP in *Definition 1.7* in $O(n^2)$ time.

One special case which require a rather simple explanation in algorithm resolution are the case in which single minded bidders demand a set that is composed of no more than two items, i.e. when $|S_i^*| \leq 2$. Another one is called the 'linear order case' so when bidder's preference falls over a contiguous set of items, rather than random items: if we define the allocation set as $S_i^* = \{q^i, q^i + 1, \ldots, r^i\}$ for some $q$, $r$ s.t. $1 \leq j^i \leq r^i \leq m$ , then a possible example could be an auction for the allocation of a strip of seashore where are displayed different businesses, like hotels or entertainment locations.

In the same way the position and speed of an atomic particle cannot be retrieved at the same time, remembering Planck's findings, optimal efficiency and rapidity of allocation cannot be found with normal linear heuristics, a method completely opposite from proceeding with an algorithm, unless we do not relax for linear programming. That is to say, in order to grasp socially optimal allocation condition and at the same moment doing that in computationally fast time, we should be able to find them even without repeatable methods as we are doing in this paper. But this goes beyond our scopes. As stated in the introduction, we will try to focus on allocating problems with a supposed compact number of participants and items traded, investigating if the algorithmic method behind is applicable to an ever greater number with a reasonable degree of approximation. But before doing this we need a further assumption of mechanisms eliciting truth inside the auction process.

### *1.2.b Incentive based approximation mechanism*

During all the last section we only focused over the computational feasibility of NP-complete problems, but now the time has come to face the third difficulty of precisely retrieving bidder's strategies. Since information about value function when bidding should not be elicited to the public by default, but be kept private information of each bidder, we must guess the strategy pursued by each agent in order to reach more quickly a welfare maximizing allocation.

The main goal in order to understand the process of a simple approximating algorithm is to search for an attracting way to collect right or almost right data out of the bids received. Imagine that we are addressed to be the auctioneer in the next combinatorial auction at our County Hall, where lost & found objects are sold in large carton packages, that bidders can inspect and make an idea of how much they would give for a specific bundle contained inside one of them. Imagine also, since bidder's preferences are private, that you can only make yourself an idea of how much a bidder could place over an item, but you cannot be entirely sure of that. Then your life will be far easier if any participant would be asked not to lie at their valuations, allowing this quasi-experiment to be effective. This aspect of auctioning is particularly essential for economists, we noticed. All of them assume that optimality conditions directly flow from the capacity of bidders to be consistent with their personal payoffs. On the contrary, computer scientists are more straightforward and mechanical, ignoring the possibilities of setting up a strategy like the algorithm we will see in few paragraphs.  A first attempt to define what makes a mechanism an incentive compatible one is:

*Definition 1.10*

*An incentive compatible mechanism represents an allocation algorithm and payment function s.t. each bidder has more incentive not to lie, but instead to fairly report his valuation function truthfully to the auctioneer.*

A well known theorem, named the *revelation principle* (Myerson, 1981) clearly states that:

*Theorem 1.2*

*if we are given any type of mechanism containing a Bayesian Nash equilibrium*[7]*, then there will always exist a mechanism leading to incentive compatibility with the same allocation and payment scheme of the previous one.*

In fact, bidding truthfully should be not of particular help, unless it would provide a higher payoff at the end of the game. If we reason in term of selfish utility maximization (as we supposed the bidders to reason) and if we allow $g_i(S)$ to be the pleasure $i$ retrieves from having declared her subset $S$, with $g_i(S) \in \mathbb{R}^+$, and $g_i(S')$ the payoff from having declared a truthful bundle of preference, then we should conclude in light of *Definition 1.4* that for a truthful mechanism there is the necessity:

$$g_i(S') - p_i(S') \geq g_i(S) - p_i(S) \qquad (11)$$

[7] Recall that a Bayesian NE is the extension of the NE for games with incomplete information. Each player plays a bets response given the strategies of the other player. Players maximize the expected utility given their private information, the joint distributions of the other players' private information and given the strategies of the other players. CA represent a very common example of what a Bayesian game means.

This appears to be in contrast anyway to the weakest concept of Nash Equilibrium, in a situation where the dominant strategy would be to retaliate and bid a different amount from the real one, i.e. from $v_i$.

If we reason in general terms, we may let $V_{sm}$ denote the set of all single minded bids over $m$ items, and $A$ set of all allocations of the $m$ items over the $n$ participants. Then we denote as single minded allocation mechanism a function or mapping which satisfies the following:

$$f : (V_{sm})^n \rightarrow A \qquad (12)$$

and we define a SM payment scheme as:

$$p_i : (V_{sm})^n \rightarrow \mathbb{R}, \text{ for } i = \{1,\dots,n\} \qquad (13)$$

The SM general mechanism is said to be computationally efficient if both $f$ and $p_i$ can be computed in polynomial time, following Rothkopf et al. (1998) and Definition 1.1.

The mechanism is also said to be *incentive compatible*[8] if , for every set of value functions $\{v_i, \dots, v_n\}$ with $v_i' \in V_{sm}$ , we have that $v_i(a) - p_i(v_i, v_{-i}) \geq v_i(a') - p_i(v_i', v_{-i}), \forall v_i'$[9]. The notation $v_{-i}$ indicates that the valuation function estimated always belongs to bidder $i$, but in the case is evaluated if she would not participate to the auction.

Sufficient conditions for the existence of an incentive based mechanism are explained by Lehmann et al. (2002), grounding the necessity for four main properties:

*Definition 1.11*

*i)* *Exactness: bidder i can take all the preferred set or nothing, i.e. if $g_i$ represents the payoff or pleasure derived after an allocation, then $g_i = S_i$ or $g_i = \emptyset$. The bidder is either granted or denied the bid.*

*ii)* *Monotonicity: any pair ($S_i$ ', $v_i$ ') cannot lead to lose for i, i.e. if $S \subseteq g_i$ , $S' \subseteq S$,*

$v' \geq v \rightarrow S' \subseteq g_i'$.

*iii)* *Critical payment: if $S \subseteq g_i \rightarrow p_i = v_{i\ critical}$ , that is the bidder will pay just up to the minimum amount for which otherwise, bidding less she would have lost the auction.*

*iv)* *Participation: if S is not contained in $S_i^*$, then the payment scheme relative to i is 0.*

[8] In dominant strategies.

[9] When $a = f(v_i, v_{-i})$ , $a' = f(v'_i, v_{-i})$ *and* $v_i(a) = v_i$ *if* i wins to obtain set a and 0 in all other cases.

All of these are deemed to be by the same paper as belonging to VGA auction type too. But from the moment that VGA proves to be an incompatible setting for high numbered CAs as we discuss later in 1.3, we need for sure another algorithm that is not linked to a VCG auction setting, and that can be reduced to an approximation of optimal allocation. Most of the scholars and literature about CA find in a particular algorithm, called *greedy mechanism* the solution to this dilemma, since it is one of the most elegant solutions not imposing restrictions over bidders' preferences.

### *1.2.c The greedy algorithm*

As a general rule, a mechanism pursuing optimal or approximately-optimal allocation in CAs starts retrieving a vector of bids and finishes mapping this stream of bids into a subset representing the allocation. The first allocation is the final one if we are in a one-shot game, otherwise the game continues with a repeated number of rounds, each allocating a different amount of the bundles depending on the bidder valuations. In any case, the allocation, temporary or not must entail a payment scheme that goes from the bidders to the auctioneer. If we place it in formulas:

The algorithm that we are going to propose is the same expressed in the form of Nisan et al. (2007), and constitute necessarily of steps. There is an initial situation in which the participants must be placed in order according to their normalized valuation function, and then the process binds to be classified in two subgroups, one belonging to the set of winners, one regarding to the set of non-winners. The peculiarity of this algorithm is that it maintains the characteristics of no externality $(S_i \cap S_j = \emptyset\ ,\ \forall\ i,j \in N)$ because the loser (we will refer with $j$ to her for simplicity) effectively would have won if bidder $i$ (the winner) would not have entered the transaction, but she does not have to pay anything for the transaction. The computation of the payment is performed simultaneously with the execution of the GA. We are giving a brief explanation of what is happening in this process right below. Let $S^* = [S_1^*,\ldots,S_n^*]$ be a vector of the optimal allocation for each bidder , $V^* = [v_1^*,\ldots,v_n^*]$ a vector representing all the scalar values associated with each SM bidder's optimal allocation, and finally let $W$ be the collection of all the winning combination pairs s.t. $[(S_1^*,v_1^*), (S_2^*, v_2^*),\ldots,(S_n^*, v_n^*)] \in W$ *for some* $n \in N$. We can thus trace our algorithm starting from:

$$\textit{Initialization: reorder and normalize the bids s.t. } \frac{v_1^*}{\sqrt{|S_1^*|}} \geq \frac{v_2^*}{\sqrt{|S_2^*|}} \geq \ldots \geq \frac{v_n^*}{\sqrt{|S_n^*|}} \textit{ and } W=\emptyset \quad (14)$$

Since there must be $n$ bids, the time involved for this sorting process should be about $n$ log$n$ dimension. Formally we assume that each bidder has a different norm at denominator, from the property of (5). Therefore $W$ becomes the set of winning allocation for individuals $i$ satisfying this property:

$$For\ i, j \in N, with\ i \neq j\ check\ that\ if\ S_j^* \cap \left( \bigcup_{j \in W} S_j^* \right) = \emptyset \rightarrow W = W \cup \{i\}. \tag{15}$$

In few words, the algorithm fetches the bid of agent $L$, *say,* and then it compares with all the other bids, checking for conflicts with the bid that are previously granted. If $L$ agent's bid is in contrast (it requires the bundle already allocated to, say, agent $M$), then the algorithm does not grant to $L$ the bid. This phase requires, obviously, $n$ periods to be developed.

The final allocation of winners is $W$. The associated payment mechanism (remember participants are utility maximizers) is different for each type of agent:

$$\forall\ i, j \in W \rightarrow \quad p_i = \frac{v_j^*}{\sqrt{\frac{|S_j^*|}{|S_i^*|}}} \tag{16}$$

where $j$ is considered being the *smallest index for which* $S_i^* \cap S_j^* \neq \emptyset$ . The choice to pose payment scheme in terms of agent $j$ at the numerator stems from the fact that $j$'s valuation function *could have won* if $i$ would have not participated to the auction. For any other factor $k < j,\ k \neq i$ there is again no externality , so the empty space occurs as a result.[10]

The process described above lead in theory to a list of greedy winners, depending on their willingness to reveal their true bid. In the sense that not to lie pays very much, because the first bidding with their respective true bid will have a comparative advantage w.r.t. the participants that decided to shadow their personal valuation function. The greedy mechanism is assumed to satisfy the rules of computational efficiency, because approximated solutions to the degree of $\sqrt{m}$ can be found in polynomial time. In this way a greedy algorithm provides an efficient answer at least for the winner determination problem.

For what concerns incentive compatibility, it can be proven that it is the best strategy to pursue in any other value function monotonically increasing in $v_i^*$ and monotonically decreasing in $|S_i^*|$.

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[10] If j does not exist, then we fall again on the single minded bidder case of a null valuation function, so $p_i = 0$.

Moreover, a mechanism for single minded bidders is confirmed to be incentive compatible if and only if the winner continues winning with no upper boundary valuation function, and when there is no room for a winner's curse after the allocation has taken place. In formal words two necessary conditions are required:

*Definition 1.12*

i) *Monotonicity: a single minded bidder i winning with ($S_i^*$, $v_i^*$) keeps winning* $\forall\ v_i' > v_i^*$ *and* $\forall\ S_i' \subset S_i^*$.

ii) *Critical payment: the winner should pay the bid-taker the least $v_i'$ payment, and still win. For example, the pair ($S_i^*$, $v_i'$) should still win w.r.t. other bids.*

Monotonicity constraint is ensured because of increasing valuation function or restricting the set for which the bidder participates in the auction makes him always uppermost in the greedy order. This finding is a proof for the monotonicity principle that we addressed in definition 1.4. The critical payment condition stems instead from the minimum distance at which the winner *i* is before *j* in the amount bid. So, as long as *i* keeps bidding truthfully, the disparity will take place in favour of her, allowing also to save money.

Concretely, the social welfare should be now approximated to a $\sqrt{m}$ quantity. If we describe with *OPT* a welfare optimizing allocation through single minded bidders, the welfare approximated maximization is:

*Theorem 1.3*

*Given* $\sum_{i \in OPT} v_i^*$ *as the welfare maximizing allocation and W the output of the greedy algorithm, we want to demonstrate that*

$$\sum_{i \in OPT} v_i^* \leq \sqrt{m} \sum_{i \in W} v_i^* . \qquad (17)$$

We give now a proof for this.

*Proof*

*For* $\forall\ i \in W$, *let OPT = {*$j \in OPT,\ j \geq i$ *given* $S_i^* \cap S_j^* \neq \emptyset$ *} be the set of people that were not allowed for a winning because of bidder i was before them in the greedy allocation*. That is, we proof by contradiction using the condition of absence of externalities in (4) that:

$$\sum_{j \in OPT_i} v_j^* \leq \sqrt{m} \; v_i^* \tag{18}$$

*For each* $j \in OPT_i$ *the reservation price will always be lower than the normalized* $v_i^*$*, that is:*

$$v_j^* \leq v_i^* \frac{\sqrt{|S_j^*|}}{\sqrt{|S_i^*|}} \tag{19}$$

From inequality (14) we can sum up all the $j \in OPT_i$ *:*

$$\sum_{j \in OPT_i} v_j^* \leq \frac{v_i^*}{\sqrt{|S_i^*|}} \sum_{j \in OPT_i} \sqrt{|S_j^*|} \tag{20}$$

And finally by means of the Cauchy-Schwarz inequality we can constrain the inequality to give us a solution that is approximating the true value of the welfare:

$$\sum_{j \in OPT_i} \sqrt{|S_j^*|} \leq \sqrt{|OPT_i|} \sqrt{\sum_{j \in OPT_i} |S_j|} \tag{21}$$

But every optimal allocation for the losers $S_j^*$ must intersect the set of the optimal allocation of winners $S_i^*$ in the $OPT_i$ *by definition.* Thus, if we allow for (4) to be enforced, the allocations $S_i^*, S_j^*$ should be disjoint, and $|OPT_i| \leq |S_i^*|$. Since also the allocation $OPT$, that we define as $\sum_{j \in OPT_i} |S_j|$ must be necessarily bounded by the maximum number of items $m$, then we replace (17) with

$$\sum_{j \in OPT_i} \sqrt{|S_j^*|} \leq \sqrt{|S_i^*|} \sqrt{m} \tag{22}$$

And plugging it into inequality (14), it gives

$$\sum_{j \in OPT_i} v_j^* \leq \sqrt{m} \, v_i^* \quad \blacksquare \tag{23}$$

We can therefore say that the optimal allocation is *asymptotically upper bounded* and the GA approximates the allocation to degree $O(\sqrt{m})$. This is in line with what we said before in *Theorem 1.1,* because we notice that in every iteration made over $i$, the contribution added to algorithm solution is an $\sqrt{m}$ *–th* fraction of what the $OPT_i$ could achieve.

Lehmann et al. (2002) suggest slightly considerable variations to normalization procedure in (9). For example, always sorting the elements in a descending ordered list by a criterion based on $a/|S|^l$ for some $l > 0$, and possibly depending on the number of items auctioned, then we confirm the monotonicity assumption in *Definition 1.5.* Many values of $l$ have been tested. For instance, if $l = 1$ then we can see it is the worst case of all: the ratio cannot be larger than the value of the items, so this skinny bound does not ensure us a great confidence. For $l < 1$ the situation becomes better and better. Recently, best results from Lehmann (2002) and Haldòrsson (2000) have proved to range around $l = 0.5$, since it guarantees an approximation even finer than $O(\sqrt{m})$ and we can hope for finding a polynomial time algorithm solving the computational feasibility problem even $\forall\ \alpha \in \mathbb{R}$ s.t.

$$\frac{a}{|S|^{\alpha l}}$$

Since we want to test if our proposed mechanism works, Lehmann et al. (2002) makes a very simple but specific counterexample for the greedy allocation mechanism, and we feel indebted to them when reporting it.[11] Assume there are two goods $a$ and $b$, with $a \neq b$ in general and three bidders, Red, Green and Blue. Red bids 10 for $a$, Green bids 19 for the set $\{a, b\}$ and Blue bids 8 for $b$ *only*. We sort as before the bids by a decreasing average amount and obtain: Red's bid for $a$ (average 10), Green's bid for the couple $\{a, b\}$ (average 9.5) and Blue's bid for $b$. The greedy algorithm grants Red's bid, denies Green's bid since it conflicts with Red's one and grants Blue's bid. The final allocation is not socially optimal then, since Green remains without anything, but it represents the best approximation to the socially optimal allocation here. We can see then our GR mechanism does not fail to work, at least with a non-complex preference for the bidders over the items auctioned.

### *1.3 Why GVAs fail to be an incentive compatible mechanism*

A very useful and generally known method to cope with allocation mechanism has always been found in the Vickrey-Clarke-Groves (VCG) and General Vickrey Auction (GVA) designs. Initially studied by Vickrey (1961) as a second price open auction where each bidder is asked to report publicly her own reservation price[12], it was then implemented and modified by Clarke (1971) and Groves (1973), mainly in the context of second price *sealed bid* auctions (i.e when the bids are submitted in a private fashion, by means of closed envelopes). We will prove how these mechanisms reveal to be truthful and socially welfare optimizing only in a compact-package

[11] Lehmann et al. (2002), 13.

[12] With the words that we used before, a reservation price is the value function $v_i$ .

number context, while unfeasible when the auction entails large amount of items. Cramton et al. (2006) named this type of auction design "lovely but lonely" principally referring to VCG, since from the point of view of formalization, they have a very simple and mathematically elegant design explanation and also a rather 'fast' time for the solution to the WDP (it is proven that only *mn* time periods are required to optimally allocate single items), but they entail very serious drawbacks too. VCG represents the very classic auction design for art galleries (single objects) or at maximum for single identical items that everyone being accustomed to Sotheby's or Christie's has encountered at least once. In this setting, each bidder simultaneously submits sealed bids (valuation function is not revealed to public) for the item. The highest bid wins, though the winner is addressed a payment scheme equal to the bid of the second highest bidder (hence the name second-price auctions). That is, if bidder *a* bids \$5 and bidder *b* bids \$2, then the winner is *a* but she pays just \$2. For what concerns the multiunit identical item auction, the layout is a little bit more complicated, but this gets out of our scopes, since the concept of a homogeneous good traded in CAs is rather trivial, so we will skip this aspect.

We notice that this setting is not in contrast with paragraph 1.2.b. The dominant strategy here is not to misrepresent the own value function, since each bidder cannot have power to change the price he will have to pay after the allocation. A way to concretely represent this setting is where each bidder not participating values each bundles at 0. In this manner she will not influence the outcome. The money 'saved' by each bidder are by contrast more when she reported truthfully her own values: thus the items are allocated to the people who value them most, and if social welfare is not guaranteed to be optimum, at least *local welfare* (consumer surplus) is. Waste of money could be very harmful from a societal perspective. The modified version of VCG design is the *general Vicrey auction* (GVA). This represents a situation where the final allocation chosen *A* "maximizes the sum of the declared valuations of all other bidders and pays the auctioneer the sum of such valuations that would have been obtained if she had not participated in the auction"[13]: sometimes this aspect of GVAs is called *Vickrey discount*. Holstrom's theorem in Cramton et al. (2006) reports the principal virtues of GVAs. Furthermore, we want to give now a proof saying that we are in good faith to address GVA as a truthful mechanism indeed.

Firstly, we must mathematically formalize how the auction design in GVA works. If we recall the definition of a general outcome allocation *A* in (4), the definition of the generic vector of declarations $D = [d_1, d_2, \dots, d_n]$ where the hypothetic $d_i = (S_i, b_i)$. Then if we find a function *a* working on *D* the allocation within GVA design is

[13] Lehmann et al. (2002), 4.

$$a(D) = argmax_{S \in A} \quad \sum_{i=1}^{n} d_i \, (S_i^{-1}) \tag{24}$$

And therefore a payment scheme from the point of view of the loser $j$ and keeping $g_{i,j}$ the quantifier of utility or pleasure for bidders $i, j$ *with* $i \neq j$ could be formalized as

$$p_j(D) = - \sum_{i=1}^{n} d_i \, (g_i \, (D)) \; + \sum_{i=1}^{n} d_i \; (g_i(Z)) \tag{25}$$

where $Z_i = D_i \; \forall \; i \neq j$ and $Z_j(S) = 0 \; \forall \; S \subseteq M$ because $d_j(g_j(Z)) = 0$.

Then we may discuss the following theorem:

*Theorem 1.4*

*The GVA is a truthful mechanism.*

*Proof*

*Assume* $j \in W$, $t \in \mathbb{R}^+$, *D is a vector of declaration with* $D_i' = D_i$ *for any* $i \neq j$ *and* $D_j' = t$ *representing the declaration agent j would have made in case she bid truthfully. It is true that*

$$\sum_{i=1}^{n} d'_i \, (g_i(D')) \geq \sum_{i=1}^{n} d'_i \, (g_i(D)) \tag{26}$$

*But it is also true that, for E = D, D' we have*

$$d_i'(g_i(E)) = d_i(g_i(E)) \quad if \quad i \neq j \, , \, d_j'(g_j(E)) = t(g_j(E)). \tag{27}$$

*Therefore, splitting on both sides of inequality*

$$t(g_j(D')) - p_j(D') + \sum_{i=1}^{n} d_i \, (g_i(Z)) \geq t(g_j(D)) - p_j(D) + \sum_{i=1}^{n} d_i \, (g_i(Z)) \tag{28}$$

*and cancelling summations of agent i we remain with*

$$t(g_j(D')) - p_j(D') \geq t(g_j(D)) - p_j(D) \, . \qquad \blacksquare \tag{29}$$

We demonstrated that the payment of j does not depend on i's declaration, and it is therefore irrelevant to j's declaration to consider bidder i when making her own declarations. Moreover, we propose that if $j$ is indeed truthful, her utility $u_j$ in the auction would be nonnegative. Taking steps from what we just said we can construct a utility function for bidder $j$ as

$$u_j = d_j(g_j(D)) + \sum_{i=1}^{n} d_i \, (g_i(D)) - \sum_{i=1}^{n} d_i \, (g_i(Z)) \tag{30}$$

But since by assumption we defined utility for $j$ as exactly the first element of the second term of inequality, and we impose $u_j \in [0, \infty]$ then we have

$$\sum_{i=1}^{n} d_i \left(g_i(D)\right) - \sum_{i=1}^{n} d_i \left(g_i(Z)\right) \geq 0. \qquad \blacksquare \qquad (31)$$

Thus if each bidder would declare truthfully her own valuation, the consequent GVA allocation would maximize the social welfare, leading in a quasilinear setting to Pareto optimality.

Despite the striking advantage of optimally maximizing welfare by GVAs in comparison to a tentative of approximation by algorithm, there are potential weaknesses which must be considered. The most harmful in our inquiry on CAs is that GVAs do not maximize for auctioneer's revenues, neither in a total quantity neither in a marginal quantity. Even if the payment scheme for losers are 0 (participation rule is satisfied), it is not trivial calculation for who wins, and it is requested to compute $p_i = v_i - (v^* - v_{-i})\ \forall\ i \in \{1,\ldots,n\}$, where $v^*$ is the reservation price of the auctioneer. Sometimes the revenue can be so low that the auction does not even accounts for a great value of the items traded. Wilenius (2009) reports even that GVA can lead to reduced revenue from an increase in bid, because of a non-monotonicity behavior. Other not pleasant aspects are for example that GVAs allow collusion by a coalition of losing bidder (the heterogeneity item case), non-monotonicity of seller's revenues in the set of bidders and the amounts bid, and auctioneer vulnerability to the use of multiple bidding identities by a single bidder (this point is explained in a clear way by Yokoo in Cramton et al., 2007). Especially regarding online auctions, where the identity of the bidders cannot be exactly proven to be different, this constitutes a serious hampering result, since there is all the incentive for participants to *shill bidding,* i.e. to vote or bid with different identities but being indeed the same person. This is the reason why VCG it is not used by FCC spectrum auction, where the possibilities that are open to cartelization of losers in the auction can be quite dangerous to economic activity, and where the synergistic effect from holding different licenses is fundamental to revenue maximization and social welfare optimization. We will investigate this aspect in chapter 2. But before turning to next section we would empirically demonstrate how GVA can suffer of a lower inefficient revenue with respect to an increase in bidding.

Two bidders, Katie (bidder 1) and Eleanor (bidder 2) must bid over two elements, let's say a pear and an apple. Katie values pears 10, apples 5 and the whole bundle 15. Eleanor values instead pears 1, apples 6 and the entire bundle 12. For a dominant strategy perspective, if we disregard for a moment the possibility to bid over package, we'll have an optimal allocation based on Katie taking all the pears and Eleanor taking all the apples, so the $v^*$ for the auctioneer will be 16. The bundle allocations 12 and 15 are in this situation only possible if the rival does not participate at the auction: a dominant strategy is to pursue the fruit each girl likes most. Thus we have:

$$V^* = 16 \;\left|\; \begin{matrix} V_{-1}^* = 12 \\ \\ V_{-2}^* = 15 \end{matrix} \;\right|\; \begin{matrix} V_1 = 10 \\ \\ V_2 = 6 \end{matrix}$$

The payments given from Katie and Eleanor are respectively:

$$p_1 = v_1 – (v^* - v_{-1}) = 10 – (16 -12) = 6$$

$$p_2 = v_2 – (v^* - v_{-2}) = 6 – (16 - 15) = 5$$

From this GVA setting the bid-taker receives the sum of payment schemes, 11. Repeating instead the example with Eleanor bidding 10 instead of 6 for apples. Now the new payment scheme, following the same procedures will be 2 for Katie and 5 for Eleanor, resulting in a 7 of total revenue for the auctioneer. Thus we proved the GVA auction design is not monotonic.

### *1.4 Suboptimality-pursuing algorithms for single value domains*

In this section we will address the combinatorial allocation problem from a broader perspective, comparing the case where we do allow a partial knowledge of some bidders' aspects by mechanism designer, and the case where everything is totally private to mechanism designer. Extensions from the Lehmann et al. (2002) greedy algorithm are inferred in order to account for this two chances.

#### *1.4.a Known single value domains*

In this section we generalize the concept of single minded bidder in 1.2, still applying it to *n* utility maximizing bidders over *m* items; the only difference in that context is that bidders are addressed as being *single–value bidders* (SV). Each single value bidder retrieves the same payoff from a whatever nonzero outcome, otherwise he retrieves zero. An even more general definition is included, in which a SAT setting provides a reservation value for the value functions associated to the allocation; if we denote as $\pi_{SV}$ the payoff relative to the SV setting we have:

$$\pi_{SV} = \begin{cases} TRUE \quad or \quad 1 \quad \leftrightarrow \quad \forall\, Si \,\in S \\ FALSE \quad or \quad 0 \;\leftrightarrow\; otherwise \end{cases} \qquad (32)$$

For settings different than SV case, values retrieved from $\pi_{SV} \neq 0$ are real valued numbers greater than zero.

In this way, "a single value agent may desire many different subsets, the point is that she assigns the same value to all of them" states Babaioff et al. (2005). Keeping constant for the privacy of bidder's value functions, two further settings are denoted: one in which the mechanism designer for the CA does not know about the specific partition of *ith* bidder's outcome space, referred to as *known single value domains* (KSV), while another informational assumption is the opposite *unknown single value domains* (USV).[14] Both these settings are analyzed and compared below.

Starting from KSV, that is easier from a computation point of view since we get rid of the second difficulty already counted for in Lemma 1.2 many algorithm models are brought. The first is the already known SAT domain model, where some variables attached to a certain bidder *I* get a binary value as stated before. Due to private valuation function we should build a mechanism for it that elicit the information. The algorithm in this case is best defined as a randomized association to each variable (read: bundle) in equal probability to TRUE or FALSE. The expected value E[*m*] = 1- $½^{m}$ represents a fraction of the optimal value, when each auction contains at least *m* bundles to be chosen. This expected value represents an approximation of an m-based SAT in polynomial time. Afterwards the algorithm proceeds ranking the values elicited by the bids in an ascending fashion, very close to the GA for single minded bidder seen in section 1.2.c, with the only difference that now the ratios to compare are of the dimension $v_i / \sqrt{S_i^m}$ . We must say that, in opposition to the previous GA, the one working for single value bidders is only incentive based for known value domains. The GA for single minded bidders here cannot be used, since the results give numbers too far from the optimal allocation, and the help that we wished to have from an approximation disappears.

### *1.4.b Single value CAs*

The third family of algorithms that we study always belongs to the greedy is one in which CAs are made by single value bidders. That is, each bidder's set is a singleton[15], and the EDP cannot make a distinction between any of the paths starting from a source node to a target node. It is exactly the one that we encountered in 1.2.c but now it is generalized at the case where bidder *i* is *k-minded*, i.e.

[14] KSV have information over preferred set that is public, while bundle valuations are private domains. USV have both preferred set and bundle valuations private.

[15] We recall a singleton is a set composed by just one element or member, like S = {e}. Doubletons are sets composed by two elements like D = {a,b} and so on and so forth.

when she has a set of desired bundles of size *k.* These bundles are known so we fall on the KSV again. Babaioff et al.(2005) state that the solution to this problem can be found in a polynomial time of amount $\sqrt{m}$ +1 if we follow the GA, claiming that it leads to truthful bidding too, according to monotonicity rule. If we allow *W* to be the set of winners' allocation and *OPT* the set of optimal allocation, we can make a two distinctive conditionally optimal subsets:

$$OPT_1 = OPT \cap W$$

$$OPT_2 = OPT/W$$

If we suppose that all single value bidders accept the preferred bundle in allocated them by $W_2$ allocation, because this is valid, and if we directly follow the $\sqrt{m}$ approximation of the original greedy mechanism, then we can say:

$$\sum_{i \in W_2} v_i \quad \leq \quad \sqrt{m} \sum_{i \in W_1} v_i \qquad (33)$$

From this follows:

$$(\sqrt{m} + 1) \sum_{i \in W_1} v_i \;\geq\; \sum_{i \in OPT} v_i \qquad (34)$$

Since the time implied to solve the allocation by means of this method depends on the total number of bundles that the agents desire, we can 'shorten the queue' directly asking the bidders (remember this is the KSV case) for a minimal size subset that they would prefer, given that that bundle exists. Talking with the language of EDP, we are addressed to find the shortest path between two nodes, given a bidder's preference. It is more or less as if we cut the original web into subgraphs corresponding to bidders' paths, and we may be very cautious in checking that these multiple routes do not encounter or overlap (remember equation (4)). Thanks to this restriction, we can make our purpose lighter and computable in a polynomial time approximation.

### *1.4.c Unknown single value domains*

What if either the preference set $S_i$ either the value function $v_i$ is private domain of the bidders? This entails a further difficulty, i.e. the one of get designer informed about the preferred set by each bidder. We want to show that in such case a greedy mechanism design like the one mentioned in 1.4.a does not lead to truthful declaration of intent by bidders to auctioneers, therefore the outcome cannot be approximately optimal.

Let's set a graph containing five nodes, where each bidder has a different preference (not revealed) over different paths (this could be for instance applied to the case of bus routing choice that we will see afterwards). Each path is undirected, so the $s_n$ (source node) and the $t_n$ (target node) can be wherever in this graph. For now let's refer to nodes with the simple notation of $n_r$, with $r = \{1,2,3,4,5\}$. A possible plot could be:

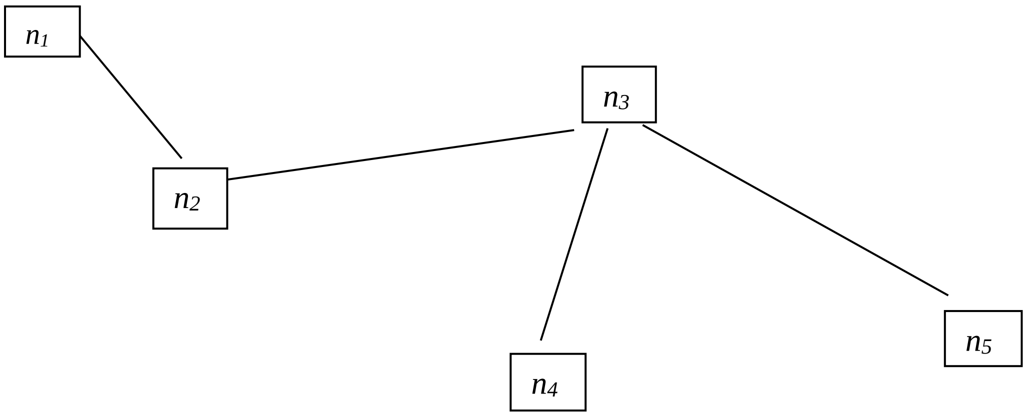


Figure 3: a tree-branched graph for USV

Suppose Marie has a valuation (\$10, ($n_1$, $n_2$)) – which means she values ten dollars the route starting from node 1 and terminating to node 2. Suppose Rachel has instead (\$5, ($n_1$, $n_5$)), so she loves walking through all the park promenade, although she could find the same pleasure going to the zoo. Suppose they bid truthfully to auctioneer. Another bidder, Sylvia has (\$100, ($n_1$, $n_2$)) because she extremely needs that route in order to carry heavy weights more quickly from her shop to her home. If Sylvia bids truthfully, then she will get the path ($n_1$, $n_2$) without a word, just paying \$10 to beat Marie, because this was the critical payment addressed in definition 1.4. In this situation Sylvia is satisfied (she has gained what she wanted). But what if Sylvia bids not truthfully? For example, with \$100 she can beat any other bidder in this competition, having all the park for her own purposes. But now the critical payment dropped to 0, since she could have won declaring any quantity higher than 0. We mean, she is still satisfied of course, but not because of her true type It is no surprise that Sylvia could increase his utility by declare a false bid and making the lion of the situation.

In light of this example we can of course say: *differently from the KSV, in USV it is not true that a bidder is satisfied iff. she is a winner. Given the winning outcome W, a USV algorithm can say if the bidder is a winner, but it cannot say if she is exactly satisfied with her starting optimal set.* In order to be sure that the bidding truthfully is in the best interest of participants, we should take the steps as suggested from section 1.2.b that the bidders have a monotonic behaviour and a critical payment to which refer. There are two concrete possibilities arising in the route example then:

*Definition 1.13*

*Let $S'_i$ be the final allocation given to bidder i, generally different from her preferred optimal allocation $S_i^*$.*

*We claim the allocation $(S'_i, v_i)$ represents a winner's satisfying lie for I with respect to not participating if there exists a value function $v_i$ s.t. the greedy mechanism is reached also with this special allocation, i.e. if $G(S_i, v_i) \in S'_I$, or better if $G(S_i, v_i) \in (S'_i \cap S_i)$.*

*We claim instead a loser's satisfying lie for being $G(S_i, v_i) \in S'_i \setminus S_i$, that is the loser remains satisfied if she is given anyway the allocation $S'_i$ that is different from her preferred one, because if she pointed at that one she would have gained nothing.*

In an ideal single minded CA as reported in 1.2 there is no room for loser's satisfying lie (loser have not the possibility to be given anything, they simply end up with empty space) and any winner's satisfying lie must be a superset of the agent's desired bundle. The assumptions for monotonicity imply here that to lie during a USV bid is a strategy reporting

*Definition 1.14*

i) *Minimal payments to seller: given the $S_i^*$ allocation that should be placed rather than $S'_I$, this allocation is a winner's satisfying lie with respect to not participating, since $g_i(S'_i) \geq g_i(S_i^*)$ so the winner is better off anyway;*
ii) *Winning encouragement: given the non-preferred $S'_i$ allocation, this represents a loser's satisfying lie with respect to not participating if $g_i(S_i^*) = 0$.*

We notice that for what concerns the USV a greedy algorithm cannot stand for optimality, since either losers and winners will tend to bid untruthful values in order to gain something. This contradicts the concept of incentive compatibility issued in 1.2.a.

We then have proven that assumptions regarding truthfulness only survive in a known domain setting, while they are not guaranteed by a USV environment. We conclude that apart from the various and helpful algorithms found to cope with the optimal allocation for single minded bidders, incentive compatibility issue is a sufficient condition for the establishment of an approximately optimal allocation in CAs with single minded actors, especially when we allow for such actors to set a specific identical value over each other set not being their preferred one.

# CHAPTER 2

In this chapter we are addressing CAs not only in light of their computational background, but rather on their concrete, applicative perspective. We will discuss in the first paragraph why auctions in general became the most preferred tool for allocating single items or packages of items, substituting older and more inefficient methods not eliciting truthful preference from bidders. In paragraph 2.2 we are showing the most striking example of a one-to-many combinatorial auction, the one issued by Federal Communication Commission in the US in order to sell rights of use for the electromagnetic and radio spectrum, since July 1994. We provide a brief recap of the advantages these setting represented for the nation in terms of revenues. Practical computational and strategic issues brought by FCC spectrum auctions are reported and assessed in 2.3, along with a comparison for other settings judged suitable for such licenses, like the simultaneous ascending auction. Here we will establish if the greedy allocation mechanism or other restrictive algorithm is effectively of help to allocate in an efficient manner the items and packages.

## *2.1 The role of auctions as the most efficient mean of allocation*

Starting from the theoretical framework expressed so far, we should of course find a concrete scope, a reason that help us to understand why auctions for indivisible goods are so relevant in a modern market mechanism.  Think over a market allocation in which auction are not feasible, in first instance. Which other methods are we left us to distribute a scarce resource (let’s forget for a moment that it should be complex and non-divisible as mentioned before)? Only state centralized distribution administrative or lotteries remain. But this has been proven not satisfactory in terms of revenue for the seller and social welfare maximization, since the former contains a strong distortionary component due to taxation – the main channel through which Governments usually finance their expenditures – as well as a lack of transparency and a time consuming list of screening costs both for the State who has to fetch the right bidder for the right quantity of items auctioned, both for the bidder themselves who try to influence regulator’s decisions, most of the time ending in lobbies and corrupted market mechanisms[16]. The latter instead is not allowing for an ever increasing

[16] The term ‘beauty contest’ is plainly addressed to this type of State-bidder transaction and it is explained in Cramton, (2001)

number of bidders to enter transactions, especially when the object of auction is extremely valuable as we will see in next section, so revenue for the auctioneer will be not maximized. In any case the efficiency won't be pursued: that is why, starting from recent decades auction setting has found to be the most effective over a cost-benefit analysis in a major number of transactions. Furthermore, the ease with whom we approach nowadays to the electronic commerce triggered the necessity of placing auction setting as the most preferred channel of transaction even for complex items allocations.

The primary advantage of an auction is its tendency to assign a certain valuable item to the bidder who places the highest value on it, depending of course on its strategy of assignment. Auctions usually foster innovation and competition in all economic setting they were at work on average, and they are universally recognized to lower bidders' uncertainty, increase their bidding aggression (although the variety of strategies as a whole is reduced), and finally to give transparency in the allocation, since everyone knows who has won what after the last bid and the consequent allocation.

## *2.2 FCC spectrum and its impact on market*

In light of what we discussed just above, it is not surprising that auctions started to get applied on an ever increasing rate for many types of items. Here we introduce one case that is the most relevant, and that is still on debate because of its strong political drawbacks, as well as for the Pareto optimality result that tries to achieve.

We are talking about the sales of the radio and electromagnetic spectrum for communication, one of the most exemplar topics brought in action by deregulation policies in 80's and 90's especially in US countries, after a long period of centralized regulation that found a break initially with President Jimmy Carter and with Ronald Reagan afterwards, bringing deregulation in transport and electricity provision respectively. But it was only with Bill Clinton's presidential era that licenses on broadcasting spectrum were highly deregulated (even a federal law was enacted in 1996, named the 'Telecommunications Act'). The first economist who proposed auctioning spectrum was Ronald Coase (1959), but it was not until July 1994 that spectrum auctions became a reality.[17] The American institution representing the role of the auctioneer in all the following 33 spectrum auction conducted since 1994 up to 2006 has been the Federal Communication Commission, that is where

[17] For more information about the history of spectrum auctions, see Hazlett (1998).

this type of transaction setting took its name from. In recent years most of the research over spectrum auction has been conducted over the so called Personal Communication Services (PCS) in the US, also to bring forth a package bidding setting that was found to be unreliable in first times.

Concretely, these auctions are designed to sell licenses for the use of a certain band of the electromagnetic/radio spectrum over a determined area of the country. The actors in this auctions that are the fiercest at bidding are of course mobile phone companies. We must not forget what we have told indeed in (3)*:* the value that is associated to a complementary bundle of object can be higher than the value of the single, independent objects. Rothkopf et al. (1998) make a clear example saying “a license for the Philadelphia region may be much more valuable to a company if that company also has licenses for the New York and /or the Washington/Baltimore regions”. Indeed, the vicinity of all these areas make a roaming much less costly and much more possible, thus increasing profits for mobile phone companies and social welfare for citizens.[18]

Nisan et al. (2007) reports an example of spectrum auction dated in August 2006 in which 1122 licenses were sold, each covering a 10 or 20 MHz spectrum band over a geographic area where a population ranging from 0.5 to 50 million lived. We can see therefore that in comparison to the small amount of items that we supposed to be sold under chapter 1, the 1122 licenses are a massive amount! And it is not all. There has been a case in which FCC sold, each time in a single simultaneous auction 99 spectrum radio licenses, 493 licenses, 1020 licenses and 1472 licenses.[19] There is clearly the necessity, as we mentioned in the single minded setting, that the number of items take into account the computational complexity issue mentioned in Definition 1.1 and Appendix 1.1. Remember that keeping constant our goal for auctioneer revenue and social welfare maximization, the bid-taker selling $n$ assets may handle bids over $2^n - 1$ different combinations of assets, so the problem must be addressed with the most extreme care.

### *2.3 A clash of design for FCC: SAA and CA to confront*

Since the beginning, not only US, but a variety of different countries chose to adopt spectrum auctions, and together with the increase of countries undertaking this aspect of the market, multiple views of the best auction setting were discussed. Of course, if the items were sold independently, then there would be no more the problem: a simple VCG auction could suffice our scopes

[18] Like roaming privileges, also general economies and diseconomies of scale may apply. Imagine the example of a simultaneous auction regarding off-shore oil leases: the extreme variability of returns and the large money involved in the process of sale impose sometimes a constraint over the managements in order to control risks.
[19] Rothkopf et al. (1998), 1131-1132.

immediately. It could also be of help to us because it allows losing bidders to cooperate, fixing a badly allocation. This would be a condition devotedly to be wished if it would be leading to an increase in the social welfare. But actually this happens only in a perfect Coasian world with transaction costs. It is fundamental then focusing right from the first moment to the best way to set the auction, so to allocate in the most effective manner the items right at the first time.

### *2.3.a The simultaneous ascending auction and free riders*

The majority of economists, including Cramton (2001) agree that an ascending simultaneous bid-based auction could be one of the best ways to cope with the problem of indivisible assets. But actually neither this kind of auction is one having by default a combinatorial setting, since they try as GVA to sell single items separately and not in packages. Allowing bidders to set preferences on packages can trigger a potential problem that seems apparently without solutions, if we allow for considerations in chapter 1. One of the fiercest adversary of combinatorial setting was Paul Milgrom, who always proved the inefficiency due to combinatorial setting because of capacity to manifest a *free rider problem*, i.e. the arising behavior of a heavy bidder that make such big offers whenever packages of items and single items are sold together in the same auction. Especially, this problem arises when bidders are given *first* the possibility to bid over entire bundles, and afterwards the auction follows the normal procedures of a simultaneous ascending auction. We describe now how this can lead to a very misleading result due to free rider agents, where bidders who should offer less are instead found to bid heavily, while bidders that were supposed to bid heavily get stuck in a situation of indecision. We can reason over the free rider problem as being a reverse (positive) externality gained from a bidder thanks to some other bidders, but this externality should not exist anyway for what we set in *(5).*

Imagine three bidders, named Dorothy, Helen and Susie must bid to obtain two different licenses, A and B. Let's relax the investigation for now on whether these two licenses are complementary or substitutes in their context. Dorothy and Helen are willing to pay up to 4 for licenses A and B respectively, and we suppose they are mutually ineligible to acquire the license of the other since a restriction for avoiding too much competition was imposed by FCC authority. On the contrary, Susie is free to buy both licenses in a bundle, but she has a lower valuation over them, say something greater by $\varepsilon$ than 1 and 2 respectively , with $\varepsilon > 0$. Let's recap the valuations of the three bidders, adding budget constraint too[20]:

| | A | B | AB | Budget constraint |
|---|---|---|---|---|
| Dorothy | 4 | - | - | 3 |
| Helen | - | 4 | - | 3 |
| Susie | $1+\varepsilon$ | $1+\varepsilon$ | $2+\varepsilon$ | 2 |

Table 1: CA auction triggering free rider

It is clear that Dorothy and Helen, since their reservation value is strictly higher than Susie's will adopt a *low-bid – low-bid strategy* in each round of SAA, also in a game with incomplete information, if we restrict our problem to integer number bidding with respect to their unique license. Susie will quit afterwards because she cannot bid an amount greater than $1+\varepsilon$ for each amount. But is Susie waits for bidding on single amounts and instead starts bidding on the bundle, then she could compete with Dorothy and Helen if $3 < 2+\varepsilon$ i.e. if $\varepsilon > {}^{3}/_{2}$ . In this case, if Dorothy would raise a little bit her bid over A, this would help Helen to acquire license B, and the symmetric would apply to Helen. Each of them would wait for the other to raise her own bid in order to defeat Susie. Even in a complete information setting, this game leads to inefficient mixed strategy equilibria. During the auction, if for any chance the prices for licenses are 1 for A, 1 for B and 2 for AB, Dorothy will be eager to raise her stake with probability ${}^{2}/_{3}$. The same happens to Helen, but her preference is focused on license B. Even if at the equilibrium Susie has only $\left(\frac{1}{9}\right)$ of possibility to acquire the bundle AB, then she will continue having a value function for each license that is one-fourth of Dorothy's and Helen's. This simple example shows how hampering can be this particular setting for auction, although Milgrom reserves a word of respect towards combinatorial allowance for what regards the allocation of complex resources using a linear programming problem and the Walrasian equilibrium. In next section we will try to contest this narrow view, searching for examples in which an optimal CA setting is possible, using single minded agents. We

[20] Budget constraints are kept very narrow in order to stress the free rider problem.

motivate our inquiry on the fact that Milgrom (2000) did not mentioned the case in which bidders can place the same value over determinate licenses.

### *2.3.b How combinatorial setting gathered new attention*

In the real world, when preferences are really heterogeneous the one with respect to the other and stakes at work are just not represented by mere bargaining for a unique item at a time (consider transaction costs for a firm that wants to conquer an entire area spectrum, it will require ages if transactions happen one by one, and in the meanwhile other participants can perform the market better), conclusions to draw are more stringent, and package bidding is for sure a necessity. If we fail not to take into account the possibility to reserve a preference over more than one items at a time, individual bidding exposes bidders to tradeoffs between searching to acquire a complementary item that was not possible to take at the same time (remember the example of Philadelphia region above), or to retaliate from the auction if the fear to lose more money than what valuation function allows. To package bidding have been addressed instead two main virtues, flowing from the mathematical background in chapter 1:

*Definition 2.1*

*i)* *Not allowing package bids can create inefficiencies for welfare.*

*ii)* *Because of lack of competition inside a non-package bidding scheme, the revenue for seller is lower.*

For what concerns *i)* Cramton (2001) provides a very interesting example, in which two contiguous car park allots, one for a motorcycle and the other for a car are sold together or separately. Distinct results are of course found. Suppose Sarah values both the allots as perfect complements, placing a \$100 over them: one spot only is worth nothing to her. The second actor, Melanie searchs for a car parking, but she does not own a motorcycle, so the combination for her is not valuable. Suppose each spot is valued \$75, as it is the combination of the two. From an outsider point of view, the spots are perfect substitutes. Any attempt from Sarah to win all the spaces leads to a failure, since Melanie would pay at least 75 for each one, making Sarah’s life uncomfortable and obliging her to arrive at minimum to pay \$150 for both allots. If Melanie hopes that Sarah drops out early from an ascending auction, then she will probably win, but she will have placed a too large amount of money for the bargain. The only equilibrium in this situation is where Melanie buys one of the allots for the smallest bid possible. If we allow instead for package bidding to exist, then Sarah wins both parkings at $150.

Unfortunately for us, package bidding solves some problems but create of others. Apart from the problem of complexity analyzed in definition 1.1 and by Rothkopf (1998)[21] by means of restrictions imposed on preference elicits and other successfully laboratory results like the CAMUS[22] used by Caltech, there are other problems concerning the fact if the values are public or only privately known. Milgrom (2000) calls the following problem the *threshold problem.* Suppose that, continuing with the last example a newcomer, Betty decides to enter into the auction, and since she is very inexpert of the real estate pricing environment she values either allot at $40. Thus the efficient outcome for each bidder to win both spots is now $(75+40) = $115. But this outcome may not happen when, as in our main case, the preferences are kept private. If Betty and Melanie place a value of $35 over each allot, but Sarah has already made her proposal of $100 for both, then there will be no incentive for the second and third bidder to reveal their own real valuations, hoping one by one that the other will break up the main bid of $100. In this case, a dominant strategy is confirmed to be a *retaliate-retaliate* NE, and therefore our findings over an incentive based mechanism as in 1.2.b is not supported at all. In this case both parking slots go to Sarah for $100.

Package bids were thus regarded as non-supporting the FCC spectrum auctions, at least in the first years of attempts, because of the threshold problem, the free rider problem and the increased complexity of bids (in part exacerbated we think because of the ease with whom actors address the auctions through ICT devices: remember that technologies can be a great benefit, as long as the scopes and permissibility of actions inside remain within a range of control). However, since the initial PCS auctions Cramton (2001) reveals that the combinatorial setting was rapidly gaining perspective of implementation and amelioration, and after an accurate research they were intended to be used in the 700 MHz spectrum auction. As a general fact, CA setting for FCC spectrum auctions were found to work quite well, if remaining in the range for not large complementarities across licenses, or whenever the complementarities would have been substantial, then CAs would have worked *just if valuation were similar across bidders*. So, keeping other factors constant, this setting reduces for the computational complexity (the similarity of preferences would have made $S_i^* \cap S_j^* \neq \emptyset$, thus breaking the tie with what said in *(4)* about absence of externalities and reducing the problem to a very small dimensioned winner set *W*.

[21] Remember the proposal for restrictions over amount of bidding in Rothkopf (1998).
[22] This stands for Combinatorial Auction Multi Unit Search, a mechanism that is exemplified in Leyton-Brown in Cramton et al. (2006).

However, when Alabama has preference over Georgia and Florida spectrum auction while West Virginia prefers to obtain Tennessee, North Carolina and Florida too, then the preferences are getting too much different the one from the other and if licenses continue to get sold one by one like in GVA singularly or simultaneous ascending auction (a method for allocating such type of licenses that was assumed to be the best in first times), then things can only get worse: there is the necessity to sell licenses in packages, so to help reduce the incentives for a demand reduction that happens in the other contexts, as well as to reduce threshold problem or improve efficiency.

Actually, what is involved in a SAA? It is quite different from a GVA, since SAA is settled as a *multiple round* auction, and prices valuating the objects can only raise in parallel during the rounds. In this way each bidder can continue to bid from the point where she left: this technique is known as *pay-as-bid setting.* It is universally recognized that, although it reveals a very easy interpretative model, the SAA entails considerable distortions and theoretical limitations. The most dangerous is again its evidence of demand reduction: when applied, this method leads to an auctioneer's revenues marginal or even total decrement. In fact the tendency of the bidder in this case would be Consider a simple example where there are two companies, XYZ and QOR contending two identical licenses. The setting can be represented in this table:

| | License 1 | License 2 | License package {1,2} |
|---|---|---|---|
| Firm XYZ | 3 | 3 | 6 |
| Firm QOR | 2 | 2 | - |

Table 2: an SAA for two identical licenses (all scale in $US billion)

Since the minimum estimated from the value of bidder (let's imagine the reservation prices are public domain of the auctioneer) the bid-taker prospects to make a payoff $\pi$ = US$ 2bill + $\varepsilon$ for each license, and hopes XYZ will take both actually. But instead, XYZ will prefer in this case withhold on one of the licenses in order to obtain only one of them at a price equal to zero, rather than paying for two licenses. Thus the auction will conclude at a price of 0, causing inefficiency of allocation and a lower revenue for the seller. This reasoning is an extension from the concept of *free disposal* that we introduced in chapter 1. Even a simple VCG auction setting can guarantee revenue maximization and welfare maximization (provided single item auctioning is matched of course). For this reason modification to the SAA setting have been studied, like for example the *simultaneous proxy auction,* where a *proxy agent,* who is different from the real participant to the auction is requested to submit straightforward decisions over bidding, following some simple directive set by the real bidder. This setting in particular was found to extract socially optimizing results for what concerned welfare, and assessed at least a higher than minimum revenue for the seller.

But actually the method that was proven to be the most effective in recent times, and the one that we are going to describe more accurately, mainly regards offsetting Milgrom's free rider problem by reverse application of a SAA first, and only then allowance for package bidding. We give now a brief resume of what normally happens during such settings.

After having sponsored a conference on the topic a test issued by FCC involved 12 licenses comprehending 2 bands each (10 and 20 MHz) in 6 different regions. Nevertheless some bidders waited also for a 30 MHz fixed speed proposal to be auctioned, in order to add capacity. Some other wanted even a nationwide package to be auctioned. We see that the money involved in this process were not negligible.

The auction started as a simultaneous ascending auction for a singleton bid, but in the course of action, before the bidding decision to be taken (remember that the FCC spectrum auctions were made on a multiple round basis) anyone could have extended its own preference to the entire 12 range licenses. In this way the exposure problem[23] could have been reduced. The 12 licenses were in reality an extension of a package made of 9 licenses that was felt too restrictive, and with twelve there was a greater flexibility for synergistic powers. Milgrom and Wilson proposed a feasibility rule which tries to keep pace with the course of action by each bidder. This rule is called *activity rule* and it suggest that each bidder does not remain idle without bidding for a stage. In few words, the activity requirement increases in each stage of the auction, and it is better that bidders continue to be active, otherwise her eligibility will fall. This is thought to discourage small bidders to continue bidding when the game 'starts to be hard', and it is inherited from SAAs indeed. Sometimes it is even required an automatic stimulus, called *waiver* or *elicitor,* that checks if participants are too 'lazy' in their actions. Eventually, it obliges the participants to make their offer or quit.

Accounting for this rule, retrieved from the simultaneous ascending auction, FCC required for packages a minimum level of activity of 50% of bids. An assumption made by FCC was that an *activity* should have been identified with a consistent bid.

[23] With this term Cramton (2001) refers to the threshold problem. This problem was there limited also because of *consistent bids,* i.e. bids not overlapping one with the other and that were renewed during the course of the same round. For different rounds the bids were assumed instead to be mutually exclusive, but not in the same round.

The spectrum auction was conducted adopting a two round simultaneous stopping rule, so it ended after two consecutive bidding rounds and /or when there was no new bid on any license. Since two rounds are a very little time to decide for the right amount to bid, participants were obliged not to be idle *during* the process: all the advantage went to auctioneers' revenues. FCC specifies the minimum bidding *x*, but it is totally free to modify it during the process over a license by license or package by package basis. The activity rule was stated from FCC in three principal conditions:

*Definition 2.2*

*The minimum bid on a license must be greater than:*

i) the minimum opening bid, or reserve price;
ii) the bidders' own previous high bid plus a x%;
iii) the number of the bidding units of the package multiplied by the lowest bidding on any package in the last round[24].

The second statement of lemma 2.1 directly comes from the fact that each new round requests a completely mutually exclusive: a rival bidder's bid on a bundle can be *provisionally winning* even if it is lower than the highest bid of the previous round. The first and third part are stating that the bid must be at least higher than the starting minimum bid or the cheapest package controlling for the items contained in it. A provisional bid is considered therefore as a set of bid maximizing the total price, subject to the constraint that at most *m* items of type *q* could be auctioned, and each bidder could be associated to at maximum one provisional bid. The auction then continues for all the five rounds until they finish or until there is no provision of bidding by any participant: in that case the auction is declared closed, and a final allocation is drafted.

A nice table for the sake of illustration about how an FCC spectrum auction is conducted is contained in Ausubel and Milgrom (2002), where only 3 rounds are set and where two items, let's imagine they are a license for a 10MHz range in West Virginia (A) and another license for 30 MHz in New Jersey (B). These two items can be sold either alone, either in package. Recall that a bid which was 'defeated' in a round R can be provisionally winning in the following R+1 round, since the bid are assumed mutually exclusive in each round. We know that for a SAA the bid can never descend due to activity rule, while for SAA with package bidding bid can have a random walk in their nature of winning, provisionally winning or losing bids. There are three bidders to contend the licenses. We report the table as it is from the text, since we find it extremely clear:

[24] Cramton (2001) specifies "the last five rounds", but actually FCC spectrum auctions could be declared concluded just in two rounds if there are no elicited bidding.

| | Item A | Item B | Package AB |
|---|---|---|---|
| **Round R** | | | |
| Bidder X | 4 | 0 | 0 |
| Bidder Y | 5 | 0 | 0 |
| Bidder Z | 0 | 0 | 6 |
| Prov. Winning Bids | - | - | 6 |
| **Round R+1** | | | |
| Bidder X | 4 | 2 | 0 |
| Prov. Winning Bids | 5 | 2 | - |
| **Round R+2** | | | |
| Bidder Y | 5 | 6 | 0 |
| Prov. Winning Bids | 4 | 6 | - |

Table 3: a 3-round FCC spectrum auction

It is evident that during first round X and Y fight one each other for license A, while Z wants to acquire both licenses A and B in a package in order to have a synergy effect. The result is that for this round provisional winning bids here exist only for Z (SAA does not account for provisional bidding). During the second round X splits her strategy over A and B taken alone rather than in package. Since no other bid is submitted in this round apart from X, then the provisional winner is only X, with a bid of 4 for A, and with a bid of 2 for B. In round R+2, provisional bidder is Y, with 5 for A and 6 for B. Notice that X's provisional bid was defeated in round R+2 by Y, as well as concerning item B. For what regards package bidding, the provisional winner, Z is removed from the list when she is inactive for more than two rounds.

Modifications introduced when FCC started relying more and more over CAs are represented by *click box auctioning*: a bidder must specify a number between 1 and 9 as a multiplier to a percentage of the minimum bid. Another variant is to allow for *best and last bids:* it works just like an ultimatum game with imperfect information, and after submitting a last and best bid the bidder cannot make another offer. Since bids in each following rounds are independently distributed from the bids made in precedent rounds, then there is few importance if for two different bundles bidder *i* will try to obtain them separately in two rounds or all in a row in one round. This break the straight tie there was between simultaneous ascending auctions and combinatorial ones. With this adaptation any bidder can pursue a strategy in different rounds that is different from the precedent (maintaining always for the minimum activity rule), without the fear or the doubt they are going to fall in the winner’s or loser’s satisfying lie faced in 1.4. They get exactly what they want. Moreover, this setting elicits competitive bidding, because of the first mover advantage seen in the greedy algorithm. There is no need for impediment to withdrawals from the auctions, a problem already encountered in the SAA and GVA, while here there is no cartelization of losers to fight: bidders have an incentive to bid truthfully each round, and if they do not maintain for activity, they are ruled out. This entails that the package-bidding setting encourages participants to be large bidders, while the classic setting for SAA (Milgrom (2000) and Ausubel (2004)) encouraged small bidders to participate only for a small preferred subset contained in the general list for licenses.

Besides all these important techniques working on the argument of selling radio spectrum licenses *after* having been programmed over a certain area, we think it would be more essential to set the allocation of spectrum first, *rather than adapting a mechanism which may fail to work in part over the auction design.* As appendices attached to the body grow in a uniform and proportionate way during the growth process, it cannot be ignored the fact that setting an optimal roadmap of spectrum bandwith, time period, restrictions for the use and modalities of bidding *a priori* will put a positive spell over the entire revenue and welfare maximization. This can be done imposing *spectrum caps* (ceilings of ownership for the same bidder) for example at least in initial rounds and then leave them when competition gets intense enough, in order to limit anticompetitive concentration among bidders and to distinguish whether the situation would be ameliorated by the introduction of caps (a situation where there are some incumbent firms for instance). Another way is to give *designated entities* (small businesses, non-incumbent firms, women) participating in the auction the possibility to enter in the transaction without fearing to lose, through installment payments, set-aside policies and bidding credits. This spillover of permission is proven to foster revenue for Government as well as technological innovation. We see that all these hypotheses are not only based over something of theoretical as discussed in chapter, but rather involve exogenous variables which require time and effort to be analyzed, and in most cases they do not lead to an approximately optimal allocation of indivisible resources because they are part of a more complex environment.

## CONCLUSION

We gathered in this paper sufficient insights to say that mechanisms proceeding throughout combinatorial auctions have a serious potential in clearing the market and permit major improvements in computational feasibility for objects that were not optimally priced and allocated when considered as indivisible entities. In chapter 1 we gave particular attention to the difficulties that combinatorial calculation poses, weighting the computational complexity issues with examples and strategy implementation. Having described the particular condition for bidder single mindedness and how a truthful eliciting mechanism works; we have proved that the greedy algorithm not only can elicit truthful bidding strategy, but it also approximates the optimal allocation to a $O(\sqrt{m})$ degree, if incentive compatibility is matched. Comparing CAs with a simple unique-item auction like GVAs, we discovered that the latter, although requiring a less complex auction design, leads also to less than approximated competitive environment and hence to no revenue maximization for the seller, and no welfare maximization. We discovered that the greedy algorithm approximates to an asymptotic degree and it is a truthful eliciting mechanism in the case of known single value strategies from the bidders, while the unknown single value setting always retrieves a not optimal and neither a c-approximated optimization. Moreover, we discovered that the USV is not incentive compatible either. In chapter 2 we discussed over the serious impact CA had in last decades for markets, especially in the sales for radio spectrum licenses. A brief summary over the history of FCC spectrum auctions was given, along with comparison in hypothetical examples between simultaneous ascending auctions with and without package bidding allowance. It was found that effectively package bidding leads to more flexibility in revenue maximization for the auctioneer and social welfare maximization, though not for merit of the algorithms mentioned in chapter 1, since they are not implemented in reality. Since we did not found any strong evidence of the effective support from the greedy algorithm into real FCC spectrum sales, concluding that benefits in FCC combinatorial auctions from the real perspective arise from exogenous factors like political, engineering and economic compromises which are beyond our scopes this work. On the other side, the effectiveness of truthful bidding estimated in 1.2.b has been proved also in reality, since it is prerogative of simultaneous ascending auctions. We address further research for the application of the greedy algorithm in a real combinatorial auction context.

*Acknowledgements*

The author is especially grateful to the conscientious collaboration of his thesis supervisor and Professor of Mathematics Course PhD Prof. Marco Dall'Aglio and to PhD Prof. Marco Scarsini in quality of Professor of Game Theory, because without their suggestions and precious help this paper would have never been drafted. Thanks are also needed to the author's university colleagues and friends, in particular Federico Marcelli from Università degli Studi Roma Tre and Andrea Bianchi from Collegio di Sant'Anna of Pisa, who supported him in continuing and deepen the knowledge of the argument.

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