Department: Business and Management   Chair: Advanced Financial Mathematics

"Negative interest rates: how the existent financial models can fit with this new scenario? A focus on Vasicek and CIR"

Supervisor:
Professor Gaia Barone
Co-supervisor: Professor Marco Pirra

Student:
Silvia Compagnone
N° 674821

ACADEMIC YEAR 2016/2017

INDEX

ABSTRACT

My project work has the purpose to question on the use of financial models with this new scenario of negative interest rates. Precisely, the two models I am going to study are the Vasicek and CIR.

First of all, the research will be focused on the analysis of them looking at both their limitations and strengths, with the aim of making some needed and essential adjustments for this shift in macroeconomic scenario.

The entire work, moreover, has the objective of understanding which are the economic agents affected and which are the future perspectives in this economic situation.

1. Introduction: negative interest rates and deflation

Karl Popper stated, "In so far as a scientific statement speaks about reality, it must be falsifiable, and in so far as it is not falsifiable, it does not speak about reality."

Negative interest rates have appeared in the last years, they have been set by different central banks, among which ECB, with the aim of helping the economy to restart. However, this possibility was not contemplated in economic and financial handbook, since it was considered a fictitious scenario, inconceivable in the real world.

Though, as Popper affirmed scientific statements must be falsifiable to be adapted to reality, otherwise they are not able to continue to work.

The choice to write a thesis on this financial topic, the presence in real world of negative interest rates, is driven by the deep interest towards this field of study and the will to understand if the already existent financial models are able to continue working with this new economic situation, proposing some developments of study.

It is moreover important, in my opinion, focusing the attention on this change, given its impact on both economic and monetary system and on citizens' everyday life.

My work wants to start from the general macroeconomic dynamics, which have forced the principal central banks to overpass their "classic" monetary policies choices, up to, as in the special case of ECB, enlarge its perspective in order to allow the employment of instruments never used before.

What has happened after the last economic crisis?

Europe has started experimenting a period of heavy deflation, and since the ECB's objective is to maintain price stability through the control of inflation; it can operate setting interest rates. If from one side, ECB used to increase interest rates in order to fight against too high inflation, on the other side, it can contrast deflation decreasing the rates. This has been the main objective for which negative interest rates have been set.

It is important to keep in mind, however, that sometimes tools considered necessary to heal the economy, as it can happen for medicines, can have much more serious side effects of the pathology that are called upon to treat.

In particular, just think if an error is done in two moments such as the intervention times and the doses methods of administration, and the virus that we would like to eradicate becomes immune to treatment, forcing the doctor, in the increasingly desperate attempt to save the patient, whether increasing the dosage of drugs or trying to experiment new ones, even more powerful. This vicious circle can lead to death of the virus, such as that of the sick.

I would like now explaining how the work is going to be structured.

In the **II** chapter, I am going to make a literature review, with the aim of building a framework of the existent literature. I will explore the Vasicek and the CIR model, assessing the main assumptions and the results provided by them.

I went through these two models with the two main papers; an equilibrium characterization of the term structure by Oldrich Vasicek, and a theory of the term structure of interest rates by John C. Cox, Jonathan E. Ingersoll, Jr., and Stephen A. Ross.

After that, I will go also through the Hull and White model to provide a sight also in a no-arbitrage model, thanks to the paper Pricing Interest-Rate-Derivative Securities.

In the **III** chapter, I will explain the development I have decided to study in this work. In the first paragraph, I will go through each step I followed during my research, explaining the procedure and each alteration needed (particularly the study will be divided in three different stages). First of all, I will start with the study of the Vasicek and CIR model to see if they can work in both economic scenarios of positive and negative interest rates; then, I will proceed testing how much error is produced by the model in the estimation of the term structure, i.e. comparing the models against themselves but with different inputs, once with the estimated term structure, and once with the current one. Finally, the last step of the development has been the comparison of the two models established in the first stage and the no-arbitrage model already proposed in the literature review, the Hull and White one factor.

Going ahead, I will focus on the data I used for applying and testing the models; and last but not least I will explain the results obtained, through some explanatory graphs.

In the **IV** chapter, furthermore, I will propose a study of the volatility (i.e. σ), to see how it is related to interest rates. The purpose is to understand if this parameter is a good predictor of the trend of interest rates. It is obvious that the variable I am studying cannot be explained only through one variable, and that to have a broader view other parameters should be added, such as for example the GDP. According to this concept, I will go through each data used and results obtained in the following paragraph of this part.

Finally, in the **V** and last chapter, I will close the project summing up all the main findings and making a final evaluation of the current scenario. I will go deeper in the future perspectives and I will try to explain both sides of the balance brought by difficult decisions that sometimes must be taken.

I will finish my project stressing on some crucial questions, which are the one that are being asked to institutional organizations by the entire pool of economic agents (financial institutions, corporations, private citizens, and so on).

1.1 A shift in the economic scenario

The macroeconomic crisis of 2008-2009, as it has already been discussed a lot of times until now, has led to a tremendous decline of economic activity and a sharp rise of unemployment in developed countries.[1]

It is well known that the entire world has been suffering from this last dramatic crisis, but it is possible to affirm with almost absolute certainty that European Union has been one of the most affected economic area (it is in fact enough to mention that in 2009 the GDP went down by 4.3%).

This decline has not affected only this important macroeconomic measure; as a matter of fact, one of the most crucial aspects the economy is fighting is deflation.

This general decline in prices has been especially caused by a little earlier drop in GDP, just mentioned above, which was in turn linked to the rise of unemployment and to the loss of consumers' purchasing power as obvious consequence. Given this scenario, with the worst economic crisis the globe has never experienced both in terms of financial consequences and length of the period of depression, it has been clear that intervention of central banks and regulatory bodies were needed, and as result, numerous central banks have decided to adopt expansionary accommodative policies to boost the economy and allow people to breath.

In a first moment (more or less at the end of 2013, period in which Europe was starting to experience a deflationary trend), the ECB president, Mario Draghi, showed not to be much worried about this issue, since in his view, that situation was considered not to be so dangerous. It is possible to attribute this decision, beyond of the analysis the bank has surely made, to the influence of countries that were not experimenting such a worst situation (too much economic diversity among European Union members).

Let's talk for example of Germany, which could boast an inflation rate at about 1.3%, against the mean of many other countries, which instead were fighting with a rate floating around 0.7%.

This scenario was, surely, not a positive signal. As a matter of fact, deflation could become a big problem for countries, especially for the ones characterized by a

---

[1] Welsch, H. and Kühling, J., "How has the crisis of 2008-09 affected subjective well-being? Evidence from 25 OECD countries", Bulletin of Economic Research, 2016

high sovereign debt such as numerous countries of the Eurozone; it is just possible to keep Italy, Spain, Portugal, or Greece as examples.

Even if at first sight, this economic phenomenon (deflation) could be seen as a positive event (given the deep decrease in prices), actually an extended deflation period is dangerous and unhealthy as a hyperinflation one (of which in Europe we have memories for what concerns the German case of 1948, after the war).

Going deeper, in a deflation period the decrease in goods and services prices does not come without, as I have already cited above, a reduction in salaries, which does not allow people and businesses to maintain their obligations towards financial intermediaries (such as mortgages or any other financing forms) and that cause the loss of consumers' purchasing power.

Moreover, another side effect is the tendency to postpone every kind of purchase, willing to wait until the relative price will reach the optimal level.

All these ways of behaving, together with the usually precarious scenery it is possible to find when struggling in this situation, lead to the demolition of consumption, which is considered the foundation of an economic system rebirth.

Therefore, after some months spent without any positive improvements in the scenario, ECB has decided to intervene, being on 4th June 2014 the first major central bank in the world to make one of its rates negative (i.e. the rate on deposits), with the aim of getting credits flowing into the troubled economies.[2] So, after a great effort and concern in developing expansionary monetary policies, in fact, in many European countries the low interest rate environment has been increasingly replaced by a negative interest rate environment.[3]

As it is commonly known, ECB, contrary to the US FED, which has the power of creating jobs, can only inject life in the European economy controlling inflation (being obsessed, always for historical reasons, by the target level of almost 2%, which never has to be overpassed).

---

[2] Debanjan, D., "Negative Interest Rate Policy by ECB: A Case Study", Skyline Business Journal, Volume X issue, 2014-2015

[3] Kerbl, S. and Sigmund, M., "From low to negative rates: an asymmetric dilemma", Financial stability report 32, 2016

The tools it can use to carry out this duty are setting and adjusting interest rates. Especially, ECB can act on three different rates: first, the marginal lending facility, the rate which allows banks to get overnight finance from the Euro system; the main refinancing operations (MRO), which is the one thanks to which liquidity can be injected in the banking system; and last the deposit facility, which is used by banks when they make overnight deposits with the Euro system.

The decision of ECB has been to lower all these three rates, caring about the fact of keeping the fair distance among them in order to not destroy the money market.

This financial measure has been used also few times ago in Nordic countries (such as Sweden and Denmark), but it is better not to look at this historical notoriety since the effects produced on expected results have been very low and far from the one expected.

Of course, given the heavy economic trouble that Europe was facing already by some years, this has not been the only expansionary monetary policy ECB has produced.

As a matter of fact, Quantitative Easing has been announced at the threshold of 2015. This second measure has concerned an expanded purchase program, with which to be more precise, ECB has promised to buy sovereign bonds of its state members to be incorporated in its existing portfolio of private assets. Initially, the purpose was to combine asset purchase, which has to be carried out until at least September 2016, to the amount of € 80 billion.[4]

What then has happened, given the low but constant economic growth, it has been the fact that QE has been extended until the end of 2017, contemporary reducing the amount of the monthly purchase from € 80 billion to € 60 billion.

It is to simple to state and judge whether the ECB has taken right or bad financial measures, since the problem is deeper and more complex than what sometimes citizens can or want demonstrate, but it is interesting comment on how real economy (taken into account commercial banks and private citizens) has reacted to this shifted scenario.

1.2 Thoughts and reactions of the economic agents

[4] European Central Bank. "ECB announces expanded asset purchase programme", January 22, 2015. https://www.ecb.europa.eu/press/pr/date/2015/html/pr150122_1.en.html

When dissatisfaction is the main sentiment in a society, it is very difficult if not impossible, I would say, to get a great consensus, whichever the decision taken is.

Moreover, considering the critical historical period the economy was experimenting, private citizens, private sector, and commercial banks were got stuck in a situation of suffocation.

Given this overview, a straightforward connection has been for these agents to look at these measures as a new risk, a bet, hence something that could be another time synonymous of uncertainty.

Will these policies push the economy, re-giving life to Europe and generally to developed countries, or will it worsen the already dangerous and unstable reality?
What will be the effect on private savings and why do I, as bank, have to pay central bank to keep my deposit, loosing earnings? Overall, which are the positive sides in this emblematic decision?

Mario Draghi, the Italian chief, together with the ECB board, has tried to transmit trust and calmness, explaining that there will be no direct impact on citizens' savings, even if the effect at the end of the day can be considered indirect.

It is undisputable, however, that the most hit agents have been and continue to be commercial banks, which not only are not gaining earnings from their deposits, but even worse have to pay ECB to keep their money stalled.

This has been, really, the starting point from which this decision has been structured and based. As a matter of fact, ECB had a positive hope that the banks stopped accumulating money and started lending more to consumers, businesses, or among banks, boosting the economy.[5]

However, as each vicious circle that is respected, there is always an even more negative consequence, which in this case has been identified in the possibility for banks to pass these major costs on customers, already in a difficult situation and averse to the banking system, bringing again the economy at a stagnant point.

As a matter of fact, this decision has brought many side effects and now it is possible to determine that banks are suffering a lot, being unable also to profit from savings accounts, and incurring in heavy loss.

---

[5] Debanjan, D., "Negative Interest Rate Policy by ECB: A Case Study", Skyline Business Journal, Volume X issue, 2014-2015.

Apart from this negative side, ECB's purpose was conceived in order to create an environment, which should benefit savers, seen as supporters of growth and as a foundation for the increase in rates once the monetary accommodation would be reach. Here, it is spontaneous questioning the reason behind the "punishment" of savers, given their important role, and instead the reward of borrowers. But fortunately, also this time, ECB has, without doubt, clarified its position, affirming that its core business is making more or less attractive for households and businesses to save or borrow money, but this is not done in the spirit of punishment or reward[6] of anyone.

Today, after some years of work, ECB, in one of the last bulletin between the end of 2016 and the beginning of 2017, has stated that it is not yet the time to make changes to this expansionary monetary policy, for both what concerns the level of interest rates and the quantitative easing measure. The central bank thinks that the economy in the euro zone is receiving the right stimulus, even if the results are arriving in a slow but constant way.

It is not the time of leaving the economy to grow alone, since the inflation dynamic is not yet able to sustain itself without the help of these tools. What it should be needed in this environment is a fiscal policy, which should boost the economy and helping the decisions taken on the monetary side.

2. Literature review

Many theories have been proposed to explain the construction of the term structure, describing the evolution during the time of the entire zero curve. Although the literature covers a wide variety of such theories, this review will be focused on three major models, two of which are equilibrium models and one is a no-arbitrage model.

These models are: "An equilibrium characterization of the term structure" by Oldrich Vasicek, "A theory of the term structure of interest rates" by Cox, Ingersoll, and Ross, and the "Pricing interest-rate-derivative securities" by John Hull and Alan White. Although the literature presents a full explanation of the doctrine, this work will primarily focus on the development of this model in order to fit them to the current scenario of negative interest rates, adopted by the main central banks during the last

[6] European Central Bank, "The ECB`s negative interest rate." https://www.ecb.europa.eu/explainers/tell-me-more/html/why-negative-interest-rate.en.html

economic crisis (e.g. ECB fixed for the first time in its history a negative interest rate in June 2014).

2.1 An equilibrium characterization of the term structure by Oldrich Vasicek

"This paper derives a general form of the term structure of interest rates"[7]. Before presenting the fundamental assumptions of the model and explain the results obtained, it is necessary to underline and expose some notations on the scenario in which the model operates.

Within the construction of the model, it is defined a market where "default free claims" (that are discount bonds on an established amount of money, which can be delivered at a given future date) are traded by investors.

It is, also, necessary to determine the key value of the model in order to better understand all the steps of the development.

Let's start with $P(t, s)$, which can be defined as the price at time t of a discount bond maturing at time s, with $t \leq s$ and a unit maturity value $P(s, s) = 1$. Proceeding, $R(t, T)$ is the internal rate of return at time t on a bond with maturity date $s = t + T$.

$$R(t,T) = -\frac{1}{T}\log P(t, t+T) \quad with\, T > 0 \tag{1}$$

$F(t, s)$, instead, represents the marginal rate of return given by investing in a bond for an additional instant, i.e. the forward rate:

$$R(t,T) = \frac{1}{T}\int_{t}^{t+T} F(t,\tau)d\tau \tag{2}$$

$$F(t,s) = \frac{\partial}{\partial s}[(s-t)R(t,s-t)] \tag{3}$$

The spot rate, i.e. the instantaneous rate at which is possible to borrow and lend, instead, is identified with $r(t)$.

$$r(t) = R(t,0) = \lim_{T\to 0} R(t,T) \tag{4}$$

To conclude, W is the amount of the loan, which is going to be borrowed and lent, and we can define:

$$dW = Wr(t)dt \tag{5}$$

---

[7] Vasicek O. A., "An equilibrium characterization of the term structure", Journal of Financial Economics, North-Holland Publishing Company, Berkeley, CA, U.S.A., 1997.

which means that at any time t, the current value $r(t)$ of the spot rate is the instantaneous rate of the increase of the loan value.

Given this scenario, now I would like to put in evidence the three main assumptions on which the model is built.

First, the spot rate is determined by a stochastic differential equation in the form of:

$$dr = f(r,t)dt + \rho(r,t)dz \qquad (6)$$

where the first term represents the drift and the second the variance.
dr follows a continuous Markov process, meaning that the instantaneous interest rate is characterized by a single state variable (i.e. the current value) and that the probability distribution of the segment $\{r(\tau), \tau \geq t\}$ is completely determined by the value of $r(t)$.

Continuing, the second assumption is that the price $P(t, s)$ of a discount bond depends on the behavior, at time t, of this just mentioned segment over the term of the bond. Here, what come out are three main hypotheses: expectation, market segmentation, and liquidity preference hypothesis, and $R(t, T)$ can be defined as:

$$R(t,T) = E_t\left(\frac{1}{T}\int_t^{t+T} r(\tau)d\tau\right) + \bar{\pi}(t,T,r(t)) \qquad (7)$$

Finally, the last and third assumption states that the market is efficient, that is there are no transaction costs, information is available to all investors simultaneously, and evenly investors act rationally (they prefer more wealth to less, and use all available information).

$$P(t,s) = P(t,s,r(t)) \qquad (8)$$

Once set this scenario, it is possible to affirm that the value of the spot rate is the only state variable for the whole term structure.

Also the process of the bond price is determined by:

$$dP = P\mu(t,s)dt - P\sigma(t,s)dz.^{\text{i}} \qquad (8)$$

In this paper it is also studied the construction of the market price of risk, an essential measure in the use and application of these models. Let's define it $q(t, r)$, which represents how much the instantaneous rate of return on a bond increase with an additional unit of risk. For a bond of any maturity, we can define:

$$q(t,r) = \frac{\mu(t,s,r) - r}{\sigma(t,s,r)} \quad s \geq t \qquad (9)$$

which can be used to derive the equation to find the price of a discount bond.

To conclude, assumptions 1, 2, and 3, reported above, are used to show that the expected rate of return on any bond in excess of the spot rate is proportional to its standard deviation. "This property is then used to derive a partial differential equation for bond prices"[8] in the form of

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}\text{ii} \qquad (10)$$

As it can be observed, in this important scientific paper, there is no trace of the ability of this model to work in a negative interest rates environment. It is, however, from this fundament that my work wants to start.

2.2 A theory of the term structure of interest rates by John C. Cox, Jonathan E. Ingersoll, Jr., and Stephen A. Ross

The research done by Cox, Ingersoll, and Ross, instead, "uses an intertemporal general equilibrium asset pricing model to study the term structure of interest rates".[9] Here, the bond prices are determined by different concepts, which are anticipations, risk aversion, investment alternatives, and preferences about the timing of consumption.

Starting point: what is the term structure of interest rates? It measures the relationship among the yields on riskless securities that differ only in their term to maturity, explaining the market's anticipations of future events.

It is necessary here to take into account three main hypothesis. However, before listing all of them, it is important to be focused on one crucial aspect of this model, which is indeed linked to the proposed topic I decided to work on. As a matter of fact, the CIR model does not allow the use of negative interest rates in its application.

This is why, nowadays, it is necessary to question these models and propose new ways of working.

The first assumption to be mentioned is the expectations hypothesis; which states that the bonds are priced in a way such that the implied forwards rates are equal to the expected spot rates.

---

[8] Vasicek O. A., "An equilibrium characterization of the term structure", Journal of Financial Economics North-Holland Publishing Company, Berkeley, CA, U.S.A., 1997

[9] Cox, J. C., Ingersoll, J. E. and Ross, S. A., "A theory of the term structure of interest rates", Econometrica Vol. 53, No. 2, 1985

After only this introduction it is possible to derive two important postulates: first of all, the return provided by holding a long-term bond to maturity is equal to the expected return on repeated investment in a series of short-term bonds; then, the expected rate of return over the next holding period is equal for bonds of all maturities.
Going ahead with the second hypothesis, it is important to talk of the liquidity preference one. According to it, forward rates are pushed to be always greater than expected spot rates by risk aversion, moreover this difference between them represents the amount thanks to which investors are pushed to hold longer-term securities.

Last but not least, we need to speak about the market segmentation hypothesis, according to which individuals are driven in their choices by strong maturity preferences and there are different and separate markets in which it is possible to find bonds with different maturities.

After the analysis of the model, it is possible to come up with two results.

The equilibrium interest rate can be written as:

$$r(w,y,t) = \frac{\lambda^*}{WJ_w} = a^{*\prime}\alpha + a^{*\prime}GG'a^*W\left(\frac{J_{ww}}{J_w}\right) + a^{*\prime}GS'\left(\frac{J_{wy}}{J_w}\right) = a^{*\prime}\alpha - \left(\frac{-J_{ww}}{J_w}\right)\left(\frac{VarW}{W}\right) - \sum_{i=1}^{k}\left(\frac{-J_{wyi}}{J_w}\right)\left(\frac{CovWYi}{W}\right) \tag{11}$$

The equilibrium value of any contingent claim, F, must satisfy the following differential equation:

$$\Phi_w F_w + \Phi_y F_y \tag{12}$$

This equation represents the risk premium for a security that is in equilibrium.

To sum up, the bond prices depend only on one random variable, which serves as an instrumental variable for the underlying technological uncertainty.

$$P(r,t,T) = A(t,T)e^{-B(t,T)r} \tag{13}$$
[iii]

This formula, which identifies the price of a bond, can be defined a decreasing convex function of the interest rate, an increasing function of the time, and also a decreasing function of the maturity.

Specifically, a decreasing convex function of the mean interest rate level θ, which is the long term rate at which r tends in the future, and of the speed of adjustment parameter κ if the interest rate is less than θ.

Concluding, the dynamics of bond prices are given by the stochastic differential equation:

$$dP = r[1 - \lambda B(t,T)]Pdt - B(t,T)P\sigma\sqrt{r}dz_1 \qquad (14)$$

which means that the returns on bonds are perfectly negatively correlated with changes in the interest rates.

Usually, when we search on any terminal bond prices, we used to find yields rather than prices. In this specific case, the yield-to-maturity is defined as:

$$R(r,t,T) = \left[\frac{rB(t,T) - \ln A(t,T)}{(T-t)}\right] \qquad (15)$$

2.3 Pricing Interest-Rate-Derivative Securities – J. Hull and A. White

As I have already mentioned in the introduction, I decided to mention also the existent literature concerning the no-arbitrage model proposed by John Hull and Alan White.

This time the research wants to show "that the one-state-variable interest-rate models of Vasicek (1977) and Cox, Ingersoll, and Ross (1985) can be extended in a way that they can be consistent with both the current term structure of interest rates and either the current volatilities of all spot interest rates or the current volatilities of all forward interest rates."[10]

This model starts from the traditional process which includes the mean-reverting property, used in both the previous explanations, i.e. $dr = a(b - r)dt + \sigma r^{\beta} dz$. The difference is that in the two equilibrium models a, b, σ, and β were considered constants. Now, in this case, it is possible to have two outputs; when β is equal to 0 we are referring to Vasicek model, while when β equals 0.5, the CIR model is the one we are considering. Moreover, in this model, some time-dependent parameters are going to be added in order to develop the previous explained existent literature, and to pass from two equilibrium models to a no-arbitrage one.

These adjustments can be observed in this new process for *r*, just reported below:

[10] Hull, J. C. and White, A., "Pricing Interest Rate Derivative Securities, Review of Financial Studies", 1990

$$dr = [\theta(t) + a(t)(b - r)]dt + \sigma(t)r^{\beta}dz \qquad (16)$$

$$dr = a\left[\frac{\theta(t)}{a(t)} + (b - r)\right]dt + \sigma d(t)r^{\beta}z \qquad (17)$$

The formula (16) can explain two important things; first of all, it defines a model in which the drift rate $\theta(t)$ depends on time, instead of being considered at a constant level; and also, looking at the equation (17), it indicates a model in which also the reversion level is a function $\frac{\theta(t)}{a(t)}$ of time.

Let's now analyze both the case just mentioned before. As it has already been stated, in order to extend the Vasicek model, it is necessary to set β=0:

$$dr = [\theta(t) + a(t)(b - r)]dt + \sigma(t)dz \qquad (18)$$

In this circumstance, the market price of interest-rate risk is a function of *X*(*t*), and this means that the price of any discount bond, which depends on r, must satisfy:

$$f_t + [\phi(t) - a(t)r]f_r + \frac{1}{2}\sigma(t)^2 f_{rr} - rf = 0 \qquad (19)$$ [iv]

It can follow that the price of a contingent claim, which pays off 1 unit at time T can be found solving the previous equation (with *f*=1, and *t*=*T*):

$$f = A(t,T)e^{-B(t,T)r} \qquad (20)$$

This function satisfies the boundary condition when:

$$A_t - \phi(t)AB + \frac{1}{2}\sigma(t)^2 AB^2 = 0 \qquad (21)$$

And

$$B_t - a(t)B + 1 = 0 \qquad (22)$$

with

$$A(T,T) = 1 \; and \; B(T,T) = 0 \qquad (23)$$

What comes out is that if the two equations are solved taking into account the boundary conditions, the equation provides the price of a discount bond maturing at time T. When we are in the situation in which $a(t)$, $\phi(t)$, and $\sigma(t)$ are constant, it is possible to derive the Vasicek bond-pricing formula, in which $B(t, T)$ and $A(t, T)$ are:

$$B(t,T) = \frac{(1 - e^{-a(T-t)})}{a} \qquad (24)$$

$$A(t,T) = e^{\left[\frac{(B(t,T)-T+t)\left(a\phi-\frac{\sigma^2}{2}\right)}{a^2}-\frac{\sigma^2 B(t,T)^2}{4a}\right]} \qquad (25)$$

Instead, when β=0.5, the Hull and White model represents the extension of the CIR model:

$$dr = [\theta(t) + a(t)(b-r)]dt + \sigma(t)\sqrt{r}dz \qquad (26)$$

In this case, the market price of interest-rate risk is called $\lambda(t)\sqrt{r}$. The differential equation that must be satisfied by the price, *f*, of any claim contingent on *r*, this time, is given by:

$$f_t + [\phi(t) - \psi(t)r]f_r + \frac{1}{2}\sigma(t)^2 r f_{rr} - rf = 0^{\text{v}} \qquad (27)$$

Also this time the equation to be considered for the pricing of a discount bond is equal to formula (20), just mentioned at the beginning of this page.

This satisfies the differential equation when:

$$A_t - \phi(t)AB = 0 \qquad (29)$$

$$B_t - \psi(t)B - \frac{1}{2}\sigma(t)^2B^2 + 1 = 0 \qquad (30)$$

If formula (29) and (30) are the solutions to the ordinary differential equations, also this time under the boundary conditions as in the Vasicek model, the price equation gives the price at time t of a contingent claim maturing at time T. So, it follows the CIR bond-pricing formula, with the definition of $B(t, T)$ and $A(t, T)$:

$$B(t,T) = \frac{2\left(e^{\gamma(T-t)}-1\right)}{(\gamma+\psi)(e^{\gamma(T-t)}-1)+2\gamma} \qquad (31)$$

$$A(t,T) = \left[\frac{2\gamma e^{(\gamma+\psi)\frac{(T-t)}{2}}}{(\gamma+\psi)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{\frac{2\phi}{\sigma^2}} \qquad (32)$$

where:

$$\gamma = \sqrt{(\psi^2 + 2\sigma^2)} \qquad (33)$$

The model developed by the two scholars had the purpose, as it has already been highlighted, to prove how the Vasicek and the CIR interest-rate models can be adjusted and broaden to include the consistency and fit with both the current-term structure of spot and/or forward interest rates available in the market and the volatilities of interest rates always under the form of the current-term structure.

As a matter of fact, the model studied and developed by Hull and White can be defined as the Vasicek model with a reference level, which depends on time. At time t, the short term spot rate tends to revert towards $\frac{\theta(t)}{a}$ with a speed equals to a. In this model, the entire term-structure, which is fitted perfectly with the real one, is the input for the model.

In this research, the drift of the process for r at the time t is:

$$F_t(0,t) + a[F(0,t) - r] \qquad (34)$$

On average, *r* follows approximately the slope of the initial curve of the instantaneous forward rates. The equation of the ZCB prices is equal to:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}\text{vi} \qquad (35)$$

2.4 Conclusions

To conclude this part of the project work, it is important to keep in mind and have a clear view on the main contributions and assumptions given by these studies.

The most important thing to take into account is that all the financial models, including these three I have decided to study, are based on the main concept that in the reality it is not possible to have negative interest rates.

This is not what has happened in the real world, since after the last economic crisis a lot of central banks have started to set interest rates with a negative sign in order to give the possibility to the economy of restarting and being re-boost.

Given this framework, in which the state of the art is stopped in an environment different from the reality, the purpose of my study is to analyze these models and figure out new developments in order to try to fit the literature to the new economic scenario, seeing if these models with the needed adjustments can work.

3. Implementation of the models

The purpose of this chapter is to show the implementation of the models reviewed in the previous paragraph. In order to do so, I decided to proceed with the study in three different and separate steps.

The first one, which is also the most important given the main goal of my research work, concerns the study of the Vasicek and CIR model to see if they can work in both economic scenarios of positive and negative interest rates. What it will come out from this analysis is that a shift in the conditions brings some needed and essential changes in the models, in order to be adapted to the new conditions of the market. Without generating these alterations, in fact, it is not possible, as I will show, to apply the CIR model, due to one of its fundamental assumption.

After this first assessment and to confirm whether or not the two models provide the same results, it has been made a comparison between them, first calculating the bond prices with the estimated term structure and then making the same procedure but with the current-term structure, which is perfectly aligned with the market economy. In this part, the purpose has been to show the error produced by the model in the estimation of the term structure.

Finally, the last step of the development has been the comparison of the two models established in the first stage and the no-arbitrage model already proposed in the literature review, the Hull and White one factor.

The scope of this development has been to compute the trend of the error that comes out from the difference between the bond prices of Hull and White with both the two equilibrium models. In this case, the main objective of the work has been to present how much the two equilibrium models are far from being the better tool used by traders, with respect to the late adopted no-arbitrage models.

3.1 Analysis and construction of the first development

Besides every possible use and development of these models, the main purpose has been to question on the possibility for them to work in a reality that is by now so far from the one studied in all the manuals.

Each financial model is set on some important assumptions, which allows it to operate and provide the expected results. It is, therefore, impossible to disregard from them, and moreover it is essential go through them.

In this case, while for Vasicek model the possibility of allowing negative interest rates is contemplated, even if never applied until now (since some years ago, negative interest rates never appeared in the economy); for CIR model there is the strict ban regarding this prospect, since, first of all, it is stated in one of the fundamental assumptions of the model, and proceeding, mathematically speaking the formulas provided by the model are not able to make calculations when a negative sign appears.

As a matter of fact, while for the former the process of r is given (as it has already cited) by:

$$dr = a(b - r)dt + \sigma dz \qquad (36)$$

for the latter the process has a tiny difference, which though creates a relevant problem, i.e.:

$$dr = a(b - r)dt + \sigma\sqrt{r}dz \qquad (37)$$

Having the $\sqrt{r}$ in the last term of the equation (37), this does not allow the stochastic process for *r* to be calculated when the input of the model is a negative interest rate.

In order to fix this problem, the idea has been to add a variable in the determination of *r*. Let's call this parameter α, and let's define *r* equals to the rate (given by x, which in the case should be the negative rate observed in the market, plus α).

$$r(t) = x(t) + \alpha \qquad (38)$$

The parameter α needs to be adjusted every time, being equal at least to the maximum negative interest observed in the term structure and as it can be obviously understood greater than 0, in order to offset the negative sign of $x(t)$.

Once this alteration has been made, the next step in the analysis has been the construction of the term structure for both the Vasicek and the CIR model.

In both the cases, I used a Monte Carlo simulation to estimate the process for *r*, obtaining 500 dr trials. This procedure has been repeated for 10 years with a quarterly frequency.

The inputs given for this estimation have been $r(0m)$, the instantaneous three months' rate observed at time 0; the parameters *a* (i.e. the speed of the rate adjustment), *b* (i.e. the mean-reverting level of the rate in the long-term), *sigma* (σ, i.e. the volatility linked to the term structure observed in each year); and the variable Δt (in this case chosen to be equal to 3 months).

After completing the entire simulation, I took the average for each period and added this mean value of dr to the spot rate.

Let's explain it e.g. for the rate I used as input at time 0 (i.e. $r(0m)$). Once I simulated dr for the first 3 months, and I took the average of this value, let's call it dr_1 I found $r(3m)$ as:

$$r(3m) = r(0m) + dr_1 \qquad (39)$$

An important part of the work, in this stage, has been the estimation of the parameters a, b, and sigma used in the procedure previous explained.

In order to perform this estimation, I used the time-series method; which is the procedure explained and used by Hull in the last published edition of Options, futures, and other derivatives (the 10th edition), where it has been dedicated one paragraph precisely to the estimation of the parameter for the Vasicek model.

This procedure makes use of the historic time series of a zero rate and try to estimate the process pursued by r in the real world through a linear regression.

The first thing has been to take the real world process for the short term rate on a daily basis and calculate the change as (r_{t+1} - r_t). Subsequently, I have performed a regression of the change in the rates (r_{t+1} - r_t) against the rates itself (r_t), and a, b, and sigma have been calculated from the regression results. Precisely, *a* has been computed as:

$$-coefficient\ of\ variable\ X1 * 250 \qquad (40)$$

$b*$ has been set equal to:

$$-\frac{coefficient\ of\ the\ intercept}{coefficient\ of\ the\ variable\ X1} \qquad (41)$$

and *sigma* as:

$$standard\ error * sqrt(250) \qquad (42)$$

Since it is possible to count about 250 observations per year, and Δt is equal to $\frac{1}{250}$, the computations made above show how the data obtained from the regression need to be annualized.

Until now, through this procedure, I have been able to calculate the parameters in the real world. Now, in order to transform them in risk neutral parameters, it has been necessary to insert the market price of risk, named λ.

Using a trial value of lambda, I proceed using the Vasicek model equations of $A(t, T)$, $B(t, T)$, (Appendix i), and the formula of the zero rate:

$$R(t,T) = -\frac{1}{T-t} lnA(t,T) + \frac{1}{T-t} B(t,T)r(t) \qquad (43)$$

In this way, I have obtained the zero-coupon rates as function of maturity and I had the possibility to compare it with the market zero-coupon rate.

At this point, for the last stage, I used the solver to determine the value of λ that minimizes the sum of squared errors between the zero-coupon rates given by the model and those taken from the market. In this way, the real world parameters can be converted to risk-neutral parameters; they can be used to apply the formulas and to derive the bond prices.

In order to confirm whether the parameters obtained are significant, it is possible to check the results using the maximum-likelihood method. As a matter of fact, this procedure consists in maximizing the likelihood function in relation to the values assumed by the parameters, which are the object of the estimation. In my case, all the estimations have been confirmed.

The likelihood has been computed as function of different parameters:

$$Likelihood = \sum - \ln\left(\frac{\sigma^2}{250}\right) - \left(\Delta r - \left(\frac{a(b^* - r)}{250}\right)\right) \qquad (44)$$

Switching now the focus on CIR model, I tried to apply the same procedure just mentioned above for Vasicek, but in this case the method did not work for the entire period taken into account since as I have already mentioned above, CIR model does not allow negative interest rates.

Indeed, in the computation, when a negative interest rate has appeared, the result gave an error; therefore, the parameters can be computed only for a shorter period, in which rates taken in consideration were always positive. However, in my specific and particular case, I decided to use the same parameters found for Vasicek also for CIR model, since this provides consistency when comparing the models between them. The only variation needed, from a structural point of view, has been the volatility, given that:

$$\sigma(vas) = \sigma(CIR) * \sqrt{r(0m)} \qquad (45)$$

$$\sigma(CIR) = \frac{\sigma(vas)}{\sqrt{r(0m)}} \qquad (46)$$

Everything I have explained until this stage has been done to apply the two models and perform the first test I supposed to do, which consisted in demonstrating that a shift in the conditions of the market brought some needed and essential changes in the models. Without generating these alterations, in fact, it should not be possible to apply the CIR model, due to one of its fundamental assumption.

Proceeding with the stream of thought anticipated in the introduction of this chapter, now I will introduce the second part of the analysis, which concerns the comparison between the two equilibrium models applied first with the estimated term structure and then with the current-term structure.

In this way taking all the other parameters equal to the previous step (i.e. ceteris paribus), the purpose has been to analyze the error provided by the models. Particularly, how much these equilibrium models stand apart from the real world.

Indeed, using the same models and changing only the input (i.e. the different term structure) has produced as result the error given by the estimation of the term structure, in other words, how far it stands from the current one.

The difference with respect to the first part of the research has been the absence of the computation of the process for r. Especially, I decided to take one of the main assumption of the no-arbitrage model, which is having the entire current-term structure as an input of the model and put it in both the equilibrium models, to give an outlook of the error committed when pricing bonds with these one factor equilibrium models.

Finally, in the third stage of my analysis has been introduced another model, the no-arbitrage one factor model of Hull and White. The purpose has been making a second comparison also this time with the two models studied above, which are the main topic of my research.

Here, the work has been constructed confronting the Vasicek and CIR model, applied as they have been developed (i.e. with the estimated term structure) with the Hull and White model, which expects the computation of bond prices through the current-term structure. As a matter of fact, since this is considered the input of the model, there is the perfect fit with the real world data.

Moreover, in this model it is necessary to introduce also the parameter lambda, which is needed to determine the parameters $\theta(t)$ and $\phi(t)$, which then are linked to the computation of $A(t, T)$.

In order to compute this parameter, I used the VBA tool on excel.

Since lambda is a constant that should be fixed by the user, I decided to use the goal seek command planned with a macro, with the aim to find the value of $\lambda(t)$ that makes $A(t, T)$ equals to the ones found applying Vasicek and CIR with the real term structure (this has been done for simplicity and consistency reasons).

Obviously also in this case, the other parameters (which are *a, b* and σ) have been taken equal to the ones I have calculated in the first part, with the time-series procedure explained and used by Hull, (also this time for consistency purpose).

This time, the main result I expected from the analysis was not as before a proper study of the error committed by the two equilibrium models in estimating the term structure, but more accurately a real investigation on the operation of these models in pricing bonds against the no-arbitrage model.

Concluding, I would like to highlight that once the entire analysis has been structured I proceeded testing the procedure for some different years, first for that years, going back in the past, where interest rates were positive and right away for the more recent years in which negative interest rates have occurred.

I will show all these results in the third and last paragraph of this chapter, only after having explained which data I have used to perform all the model just proposed.

3.2 Data

The data used for the computation in the different steps of the analysis have been taken from different sources and comprehend different indexes of the European market.

First of all, in order to proceed with the application of the time-series method, I used the OIS zero rates at 1 month, which are market rates and not theoretical ones. The features of this rate have driven my choice.

OIS, indeed, are popular among financial institutions because they are considered a good indicator of the interbank credit market and less risky than other well-known interest rates. Moreover, they are generally short-term rates, and this is useful to see the evolution of the yield curve during the years.

The time frame taken has been from September 2010 until February 2017, with 1631 observations used for performing the linear regression of the change in rates against the rates itself.

Looking at the data I kept, it is possible to see how the rates have been lowered increasingly between 2013 and 2014, up to becoming negative precisely on 1st September 2014 (r = - 0.01%). Once the regression has been performed, as I have already explained in the first paragraph, the output generated consisted in the real world parameters a, b^*, and σ.

$$a = 0.1772; b^{*} = -0.00575; and\ \sigma = 0.279\%$$

The following step has been the conversion of the real world parameters into risk neutral ones. The method used has been the one explained before, which consists in the use of the Solver in order to minimize the sum of squared differences between the rates observed in the market and the one estimated by the model.

For what concerns, instead, the market rates, I took the EONIA[11], which is the daily interbank interest rates. I decided to use this rate since it is the one I took also as input for the estimation of the term structure in the subsequent steps. Then, I computed the sum of squared differences between these rates and the one provided by the Vasicek Model for a trial lambda. Finally, from the minimization of this sum I found the optimal value of λ to transform the real world parameter in the risk neutral one. Precisely, a and *sigma* have kept their initial value, while $b*$ has been transformed in *b*, through the following formula:

$$b = \frac{b^{*} - \lambda * \sigma}{a} \qquad (47)$$

In the table reported below, there are all the parameters found for each year of the study. The method has been always the same for all the years taken in consideration, what has changed every time has been the input *r*(*0m*), while the regression has been based always on the same set of data mentioned above (which therefore includes all the years studied).

"Table 1- Real world and risk neutral parameters per year"

		Risk-neutral					
	Real-world	**2010**	**2011**	**2012**	**2015**	**2016**	**2017**
a	0.1772	0.1772	0.1772	0.1772	0.1772	0.1772	0.1772
b*	-0.575%	6.245%	6.204%	6.979%	1.810%	2.587%	2.147%
σ	0.279%	0.279%	0.279%	0.279%	0.279%	0.279%	0.279%
Trial λ		-4.3272	-4.3010	-4.7929	-1.5130	-2.0065	-1.7269

Source: results found from my personal elaboration

Here, one thing I noticed has been that the trials λ are smaller than the one estimated by Ahmad and Wilmott for the long-term period and for the American market (i.e. *λ*= -

[11]Bloomberg

1.2)[12]. This demonstrates significance and consistency, first because of the different market, the European one, which is riskier; and also due to the fear factor that has appeared when the conditions of the market became critical (as it has been during these years taken for the analysis).

The same data used for the linear regression (OIS rates) have been also used for the check made with the maximum-likelihood method, which has provided the same results just showed above.

Proceeding with the next stage of the analysis, now it is the time to speak about the proper application of the two equilibrium models. In this part, I have insert in the models the parameters found above, using the same constants of Vasicek also in CIR. What I have added, this time, has been the instantaneous rate $r(0m)$. I used the EONIA rates related to each year, and taken from Bloomberg.

"Table 2 - EONIA rates per year"

2010	2011	2012	2013	2014	2015	2016	2017
0.34%	0.42%	0.39%	0.06%	0.15%	-0.08%	-0.24%	-0.36%

Source: Bloomberg

From these data, I started the computation of the process for r, following the process explained in the paragraph one through the use of Monte Carlo simulation.

At this point, when willing to estimate the error produced by the two models in the estimation of the term structure and for the comparison between them and a no-arbitrage model, I needed another term structure observable in the market. Since, the OIS and EONIA are only instantaneous and short term rate, while now I needed a long-term yield curve, I decided to opt for the ECB yield curve, which is a theoretical term structure estimated with some financial models, but in my opinion the closest to the current term structure, I would call it the best available alternative observable in the market.

The data to which I am referring have been taken from ECB website[13]. There is, in fact, a section called "Statistics", in which it is possible to find all data related to Euro area yield curves under the tag "Financial markets and interest rates". The yield curves selected are the ones which include all euro bonds and not only the AAA rated bonds.

[12] Hull, John. *Options, futures, and other derivatives*, ninth edition. Boston: Prentice Hall, 2014
[13] https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/index.en.html

I would like to spend at this point some words on how this yield curve has been computed. For what concerns this explanation; I have looked, also this time, at the ECB website, where there is a document intended to explain the methodology used in this estimation. All the data (i.e. bonds and prices) are taken from EuroMTS Ltd[14], while the source of ratings is Fitch Ratings[15]. The curve is estimated through a modeling algorithm that minimizes the sum of the quadratic difference between the yields computed from the curve and the one actually measured. Yields are calculated according to the International Securities Market Association formula.[16]

For what concerns the time frame, I kept 3 years with positive interest rates (precisely January 2010, January 2011, and January 2012) and three years with negative sign rates (March 2015, January 2016, and January 2017).

At first I have used this curve to examine how much it deviates from the one I have forecasted, and hereafter I used it in the no arbitrage model of Hull and White.

3.3 Results

In this last paragraph related to the explanation of the model, I would like to provide some insights into the results I have obtained.

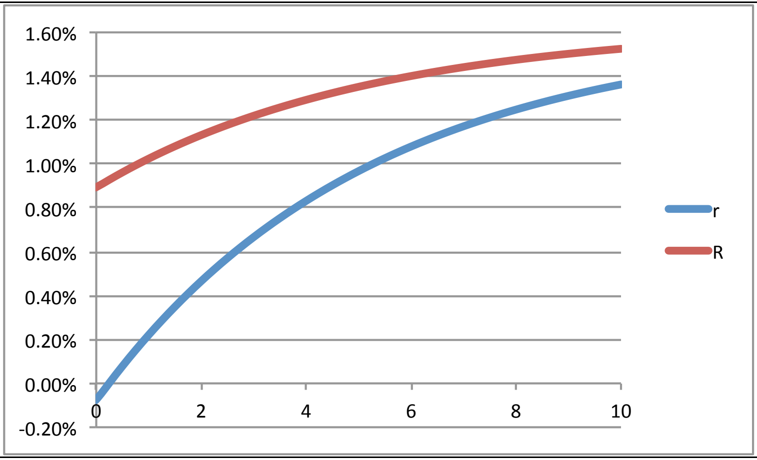
Following the same structure of the passage 3.1, I would like to start from the ground, talking about the application of the models in presence of negative interest rates. As I explained above, it has been possible thanks to some adjustments and here I would like to display, to prove their application, the term structures obtained for the three negative interest rates years (respectively 2015, 2016, and 2017).

As I have already explained, since for Vasicek model there were no problem (in fact it contemplates the possibility to have negative interest rates), I have worked a little bit on the adjustments needed for CIR, and for this reason I would like to show results of it (the Vasicek term structure, instead, can be found on the Appendix vii).

---

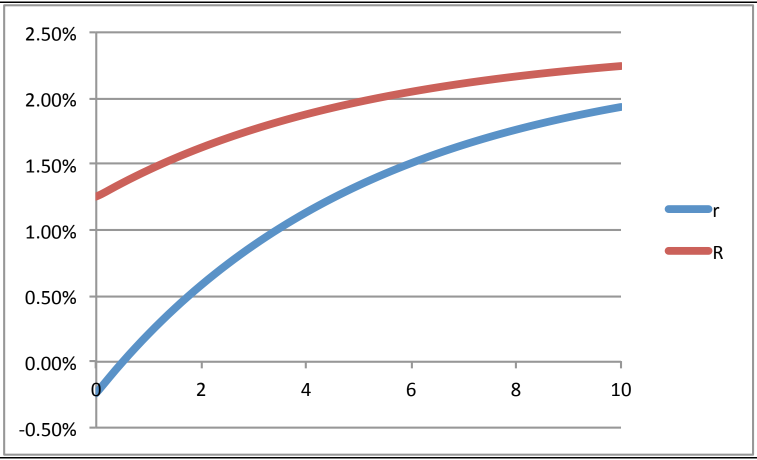
[14] www.euromts-ltd.com
[15] www.fitchratings.com
[16] European Central Bank. "The ECB's Directorate General Statistics releases euro area yield curves every TARGET working day at 12 noon Central European Summer Time", https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/technical_notes.pdf

“Figure 1 – The CIR process for r and R in 2015”



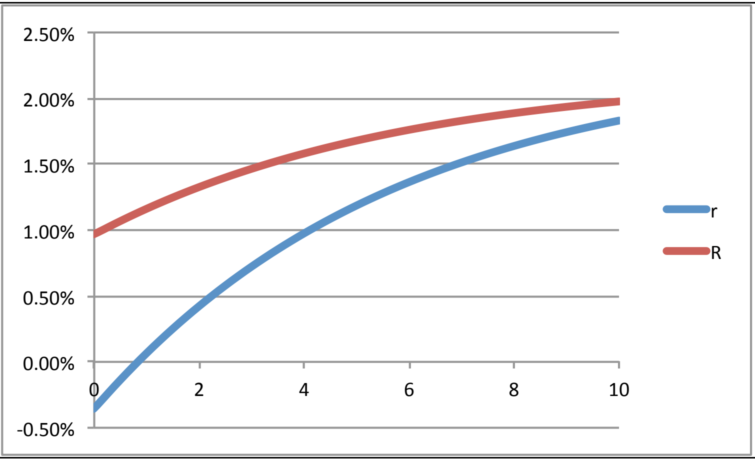
Source: personal elaboration

“Figure 2 – The CIR process for r and R in 2016”



Source: personal elaboration

“Figure 3 – The CIR process for r and R in 2017”



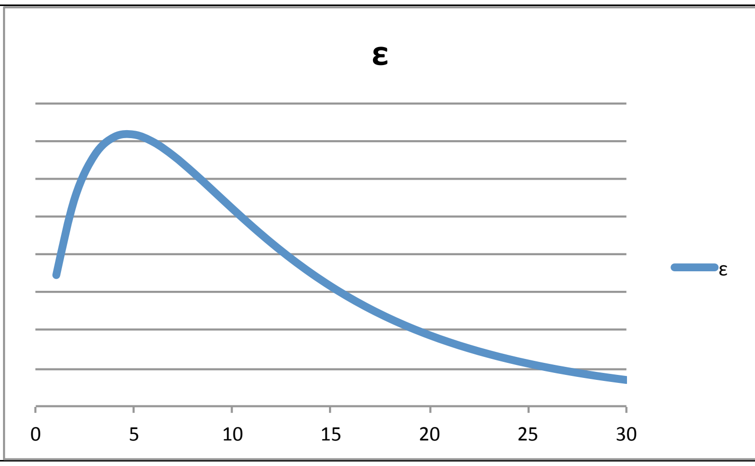
Source: personal elaboration

As it is shown in the graphs and as I have explained in the first part of this chapter the models can be applied even when the r(0m), input of the model, has a negative sign. As it is highlighted looking at the EONIA rates, we have in 2015, 2016, and 2017 respectively –0.08%, -0.24%, and -0.36%.

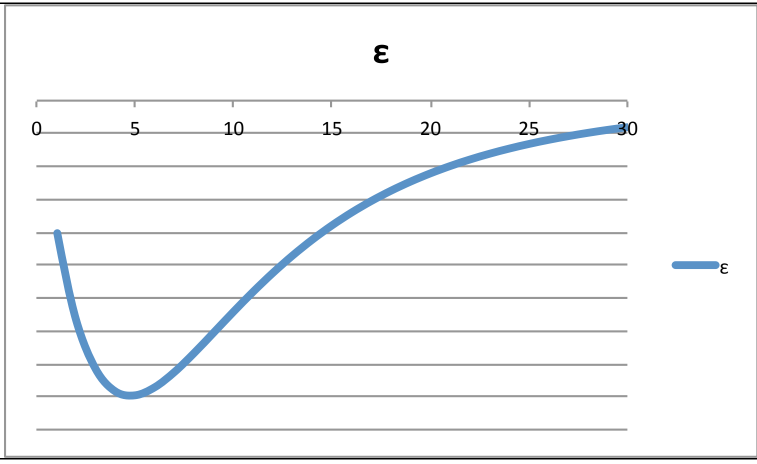
For what concerns, instead, the three years with positive interest rates; I will not display the results, since it is undisputed that under those assumptions the models work. Another thing I would like to bear in mind is that the term structure of CIR tends to lie always under the yield curve provided by the Vasicek model, in both the scenario of positive and negative interest rates.

Going ahead with the second part of the study I would like now to show the trend of the error estimated between the price of the bonds calculated through the two equilibrium models first with the term structure originated by the Monte Carlo Simulation, in which it is important to remember that the rate used as input of the model corresponds to the EONIA instantaneous rate found for each single year, and then, in a second moment, with the current-term structure, perfectly fitted with the reality. In this way, as it has been explained in the previous paragraph, it is possible to measure the error done by the model in the estimation of the term structure (i.e. $R_{vas} – R_{MKT}$).

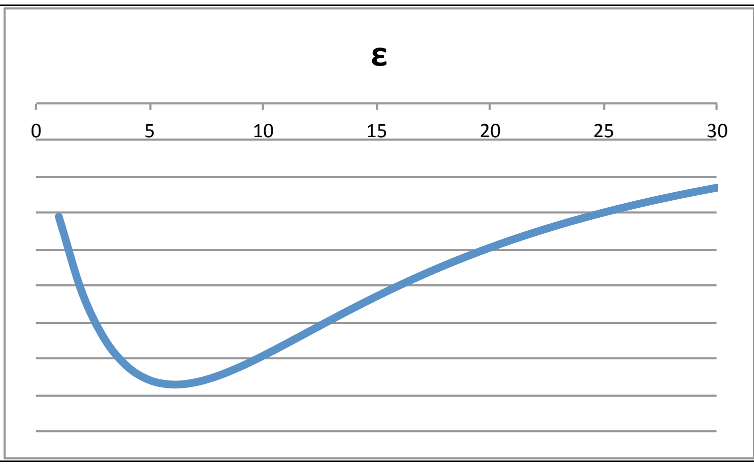
"Figure 4 – The error trend for Vasicek model in 2010"

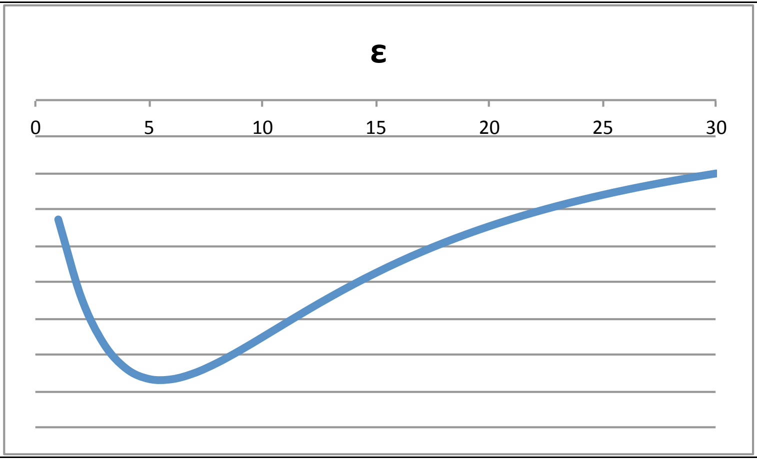


Source: personal elaboration

“Figure 5 – The error trend for CIR model in 2010”

Source: personal elaboration

The two error trends proposed above are only two examples I have decided to put here (since for space problem it is not possible to display all the graphs obtained in the study, which will be then showed in the appendix viii). Precisely, these are referred to year 2010, while the next I will show are referred to year 2015, to provide a view also on the negative sign scenario.

“Figure 6 – The error trend for Vasicek model in 2015”

Source: personal elaboration

“Figure 7 – The error trend for CIR model in 2015”

Source: personal elaboration

As it can be observed from the graphs, the trend found has been always the same, with a huge perception of the error always between 0 and 20 years (with a peak circa at 7 years).

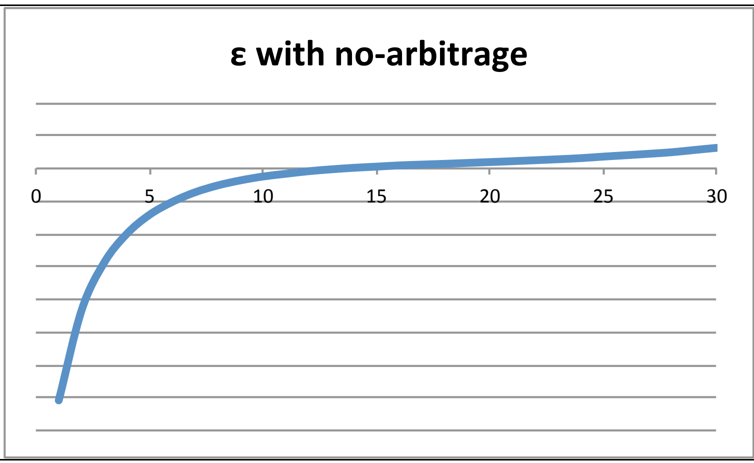
What is also important to mention, in my opinion, is that the two equilibrium models, which were already not reliable with positive interest rates, seem to be even less reliable when negative interest rates appear, giving a higher level of the estimation of the error. For this reason, it is well known among traders that these approaches are not satisfactory; as a matter of fact, they believe it is not possible to trust in the bond price when it is used one of these models.

Finally, for the last part of the study I would like to show the error estimation given by the difference between the equilibrium model with the “fictitious” term structure and the no-arbitrage model of Hull and White, where, as it has already been said, there is perfect fit with the real world term structure.

From the previous analysis, it has seemed that in the long term the process followed by r estimated with a Monte Carlo simulation can be considered a quite good estimation of the real term structure, while in the short term the error is quite big.

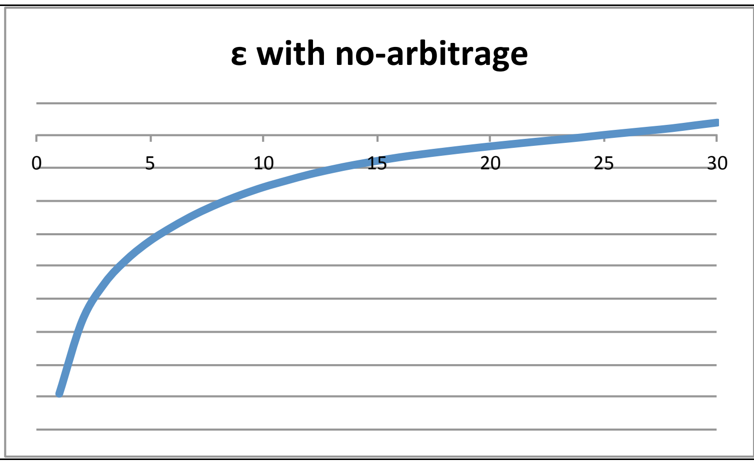
The difference form the previous analysis is that in the first case the error was in absolute value, meaning that it is given by the application of the same models with two different inputs, that is with two different term structures (precisely, one estimated and one that is the current-term structure); instead in this case, the error is more related to the outcome of the two different models, i.e. the prices of the bond (i.e. while in the first scenario I have applied the same model, in the latter one I have compared two different models based on different assumptions).

"Figure 8 – The error between Vasicek model and Hull & White in 2010"



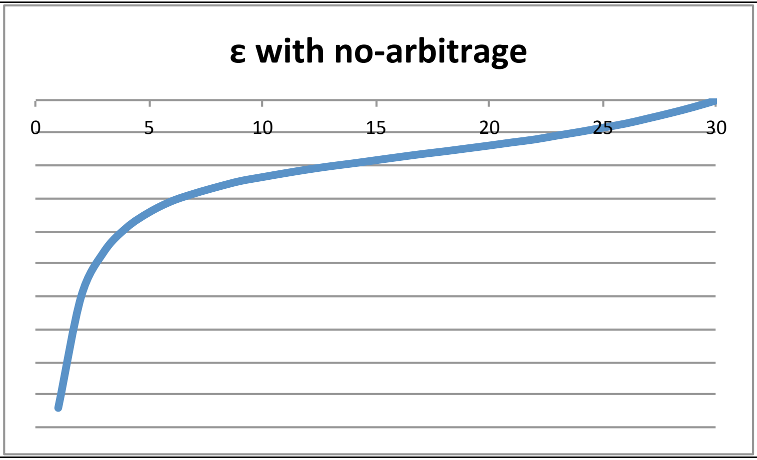
Source: personal elaboration

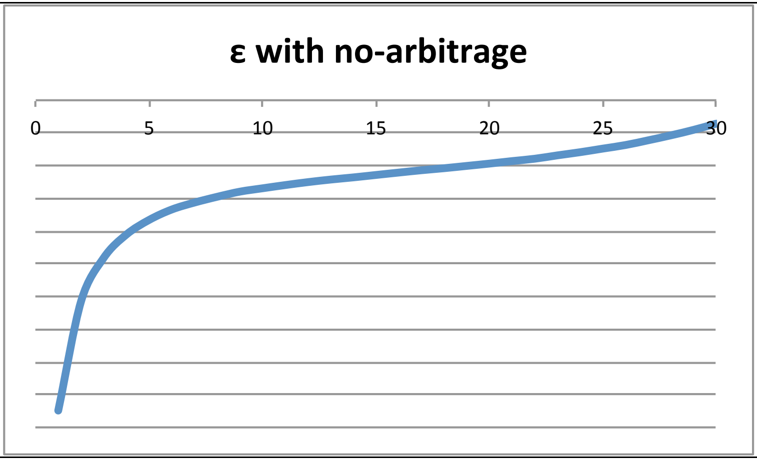
"Figure 9 – The error between CIR model and Hull & White in 2010"



Source: personal elaboration

These first two graphs relates to one of the year with positive interest rates, the next I will propose, instead, are for a negative sign interest rate year.

“Figure 10 – The error between Vasicek model and Hull & White in 2015”

Source: personal elaboration

“Figure 10 – The error between CIR model and Hull & White in 2015”

Source: personal elaboration

Also for this last part of the analysis, I have displayed only two years for space purpose (as in the cases above, it is possible to find the graphs for the remaining years in the appendix ix). Overall, the trend throughout the entire period taken into account has been pretty similar and homogeneous. As a matter of fact, as it has been found in the previous results obtained, the two equilibrium models seem to generate a huge problem in the first years, while going ahead in the time, they converge towards the same results obtained applying the Hull and White no arbitrage model.

4. A study of the volatility (σ)

In this chapter I will proceed talking about the relationship between the volatility of interest rates (let’s define it as σ) and the current-term structure.

The purpose of this analysis is to try to find evidence of the existence of this connection and in case of an affirmative response the point is understanding how they are linked and related.

My question behind this analysis has been if the volatility can be considered a good predictor of the trend of interest rates. It is obvious that this trend cannot be explained only through one variable, and that a deeper study would request the addition of other parameters to prove more significance and relevance.

As a matter of fact, it is not possible that the current-term structure is explained only by one variable, since the entire macroeconomic scenario and a lot of economic measures affect it, given its important in both financial markets and real economy.
In any case, the entire analysis on this topic will be exposed later on.

4.1 Explanation of the procedure used

The study has started with the computation of the volatility on an annual basis for each time frame of the current-term structure (i.e. 3m, 6m, 9m, 1y, and then continuing on a yearly basis until year 30).

The formula I used in excel, starting from daily bond yields has been:

$$\sigma = Standard\ deviation(daily\ yield_t) * \sqrt{365} \quad (48)$$

In this way I obtained 33 observations for each year, and I have repeated this procedure always for the same years taken into consideration in the previous chapter for the estimation of the parameters part (precisely 2009, 2010, 2011, 2012, 2013, and 2014 for positive interest rates and 2015, 2016, and 2017 for the negative ones).

The next stage has concerned the decision of which statistical tool use in order to perform this analysis. What I have decided to perform has been the linear regression (always performed on excel), since it is considered, in any statistics handbook, the best tool to find out whether two numerical variables are related, dependent, independent, or associated.

The inputs inserted in the regression have been the vector of current-term structure for Y variable, and the vector of the volatility for the X parameter.

The purpose of performing this regression has been the one of running out with a function of the type:

$$\hat{Y} = \alpha + \beta * \hat{X} + \varepsilon \quad (49)$$

where α represents the value assumed by the variable Y when X is equal to 0 (i.e. the intercept), β is the estimation of the parameter that help to code the relationship between the two variables, and the error represents the remaining variation in Y that cannot be explained by the model.

Once the regression has been performed, what I looked for, with the aim of finding significant results for my questioning, have been different variables.

First of all, I took into account the coefficient of determination, R^2 (i.e. the squared of the estimated correlation coefficient of parameters, related to the sample in question). It represents the variation of the dependent variable that is explained by the independent/explanatory variable using the model. This measure is expressed as a percentage, and consequently ranges between 0 and 1. Being it squared, the parameter loses the sign of connection; however, this can be explained by the sign of β.

Going ahead, the second output given by the regression that I decided to examine has been t-stat. This parameter is computed as:

$$t - stat = \frac{\hat{\beta}}{standard\ error\ of\ X} \qquad (50)$$

The t-value measures the size of the difference relative to the variation in the sample data, i.e. the calculated difference represented in units of standard error. In order to get significance from these data, I looked for, in each regression performed, the absolute value of t.

Sure enough, if $|t| > 1.65$ the value can be judged with one star, with $|t| > 1.95$ the stars associated are two, and last when $|t| > 2.64$ we end up having three stars.

The number of asterisks (that would be the stars mentioned before) represents the level of significance of the parameters; the more the asterisks, the greater the significance.

As I was expecting before performing the analysis, the volatility is somehow linked to interest rates term structure, and it can explain part of their trend and behavior. Once I have made these analysis, I have deduced my results and conclusions, which I will show further on, in the next paragraphs.

4.2 **Data**

The data I have chosen for performing the analysis just explained above were taken from Bloomberg.

Briefly, I needed daily data for the European market, and since on the ECB website I could find only the annualized term structure; I searched on Bloomberg an Index which could include all the traded bonds in the European market.
What I looked for in Bloomberg has been the EUSR Index, and I downloaded this index from 01/01/2009 up to 16/02/2017.

I took daily data for each index, and precisely I took always the EUSR Index but with different maturities; the ones I have decided to consider have been the maturities equal to the ones available for the current-term structures taken from ECB website (as I mentioned in the previous paragraph I took 3m, 6m, 9m, 1y, 2y, and so on until year 30).

Let's take for example the EUSR3M Index, in which I have daily data for a maturity of 3-months, for it I computed the volatility for each year analyzed using the standard deviation formula and annualizing the computations, lastly I filled the table I created for volatility outputs completing the line for 3-months maturity and for each year (from 2009 up to 2017).

$$\sigma_{3m}^{2009} = standard\ deviation\ (yield_{01.01.2009}^{2009}: yield_{31.12.2009}^{2009}) * \sqrt{365} \quad (51)$$

Once I have used this formula for each year and for each time frame, I have obtained nine (numbers of years) vectors (30 x 1) representing the volatility vectors. These values have then been considered the X of the linear regression.

The volatilities computed are reported in the Table 3:

"Table 3 – Volatility term-structure per year"

Time	2009	2010	2011	2012	2013	2014	2015	2016	2017
0.25	5.92%	2.87%	6.73%	0.70%	0.54%	1.34%	1.65%	2.44%	0.31%
0.50	5.33%	2.65%	7.02%	1.01%	0.53%	1.34%	1.37%	2.35%	0.29%
0.75	4.70%	2.54%	7.35%	1.40%	0.66%	1.38%	1.26%	2.29%	0.28%
1	4.19%	2.65%	7.68%	1.81%	0.84%	1.47%	1.24%	2.25%	0.27%
2	3.31%	3.95%	8.66%	3.35%	1.60%	2.09%	1.24%	2.14%	0.27%
3	3.06%	4.95%	9.04%	4.62%	2.22%	2.99%	1.27%	2.12%	0.28%
4	2.96%	5.52%	9.04%	5.57%	2.71%	3.98%	1.56%	2.27%	0.30%
5	2.91%	5.86%	8.84%	6.21%	3.07%	4.94%	2.02%	2.58%	0.32%

6	2.90%	6.08%	8.55%	6.58%	3.32%	5.82%	2.52%	2.96%	0.34%
7	2.90%	6.25%	8.25%	6.73%	3.46%	6.58%	3.01%	3.35%	0.37%
8	2.91%	6.39%	7.96%	6.74%	3.52%	7.21%	3.48%	3.70%	0.38%
9	2.93%	6.54%	7.70%	6.64%	3.52%	7.70%	3.90%	4.01%	0.40%
10	2.95%	6.69%	7.47%	6.47%	3.45%	8.07%	4.28%	4.27%	0.41%
11	2.96%	6.84%	7.28%	6.28%	3.35%	8.33%	4.63%	4.49%	0.41%
12	2.97%	7.01%	7.11%	6.07%	3.23%	8.50%	4.93%	4.68%	0.42%
13	2.97%	7.17%	6.96%	5.87%	3.10%	8.59%	5.19%	4.84%	0.43%
14	2.97%	7.33%	6.84%	5.68%	2.96%	8.62%	5.42%	4.98%	0.43%
15	2.97%	7.49%	6.73%	5.51%	2.83%	8.59%	5.61%	5.10%	0.44%
16	2.96%	7.63%	6.63%	5.36%	2.71%	8.53%	5.77%	5.20%	0.44%
17	2.95%	7.75%	6.55%	5.23%	2.61%	8.44%	5.91%	5.29%	0.45%
18	2.93%	7.85%	6.48%	5.12%	2.52%	8.33%	6.02%	5.37%	0.45%
19	2.92%	7.92%	6.41%	5.03%	2.45%	8.20%	6.11%	5.45%	0.45%
20	2.91%	7.98%	6.35%	4.96%	2.41%	8.08%	6.18%	5.51%	0.46%
21	2.89%	8.00%	6.30%	4.90%	2.38%	7.95%	6.24%	5.57%	0.46%
22	2.88%	8.01%	6.25%	4.86%	2.37%	7.82%	6.28%	5.63%	0.46%
23	2.88%	7.99%	6.21%	4.83%	2.37%	7.70%	6.31%	5.67%	0.47%
24	2.89%	7.96%	6.17%	4.82%	2.40%	7.59%	6.33%	5.72%	0.47%
25	2.90%	7.90%	6.14%	4.82%	2.44%	7.48%	6.35%	5.76%	0.47%
26	2.92%	7.83%	6.12%	4.83%	2.50%	7.39%	6.37%	5.80%	0.47%
27	2.96%	7.75%	6.10%	4.85%	2.57%	7.30%	6.38%	5.84%	0.48%
28	3.01%	7.65%	6.09%	4.89%	2.65%	7.23%	6.39%	5.87%	0.48%
29	3.08%	7.55%	6.09%	4.94%	2.75%	7.17%	6.40%	5.90%	0.48%
30	3.16%	7.45%	6.09%	4.99%	2.86%	7.12%	6.41%	5.93%	0.48%
Source: Data on which volatility is computed are taken from Bloomberg									

For what concerns, instead, the Y used for the regression, I took the current-term structures of each year considered from ECB website, which are the same yield curves used in one of the step of the first development of my work.

I decided to use these data because they are coherent with what I have used for the computation of the volatility.

“Table 4 – ECB term-structure”									
Time	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**
0.25	1.75%	0.46%	0.98%	0.77%	0.34%	0.24%	-0.01%	-0.30%	-0.54%
0.50	1.75%	0.50%	1.13%	1.30%	0.55%	0.35%	-0.02%	-0.26%	-0.51%
0.75	1.83%	0.69%	1.27%	1.68%	0.69%	0.42%	-0.03%	-0.24%	-0.49%
1	1.95%	0.91%	1.41%	1.94%	0.78%	0.47%	-0.04%	-0.22%	-0.48%
2	2.39%	1.59%	1.93%	2.47%	1.03%	0.66%	-0.04%	-0.16%	-0.47%
3	2.72%	2.08%	2.38%	2.75%	1.30%	0.96%	0.02%	-0.07%	-0.39%
4	2.99%	2.48%	2.76%	3.03%	1.62%	1.34%	0.12%	0.08%	-0.24%
5	3.21%	2.81%	3.09%	3.31%	1.94%	1.73%	0.24%	0.27%	-0.04%
6	3.41%	3.10%	3.37%	3.57%	2.24%	2.07%	0.37%	0.47%	0.17%
7	3.57%	3.35%	3.60%	3.80%	2.50%	2.37%	0.51%	0.67%	0.38%
8	3.71%	3.57%	3.80%	4.00%	2.71%	2.62%	0.64%	0.86%	0.57%
9	3.83%	3.76%	3.97%	4.16%	2.89%	2.83%	0.76%	1.04%	0.74%
10	3.93%	3.92%	4.11%	4.30%	3.03%	3.00%	0.87%	1.19%	0.89%
11	4.02%	4.07%	4.23%	4.42%	3.15%	3.14%	0.97%	1.33%	1.01%
12	4.09%	4.19%	4.32%	4.51%	3.26%	3.26%	1.07%	1.46%	1.13%
13	4.15%	4.29%	4.40%	4.60%	3.34%	3.36%	1.15%	1.56%	1.22%
14	4.19%	4.37%	4.46%	4.67%	3.42%	3.45%	1.22%	1.66%	1.31%
15	4.22%	4.44%	4.51%	4.73%	3.48%	3.52%	1.29%	1.74%	1.38%
16	4.25%	4.50%	4.56%	4.79%	3.54%	3.59%	1.35%	1.81%	1.45%
17	4.26%	4.54%	4.59%	4.84%	3.59%	3.65%	1.40%	1.88%	1.51%
18	4.27%	4.57%	4.61%	4.88%	3.64%	3.70%	1.45%	1.94%	1.56%
19	4.27%	4.60%	4.63%	4.92%	3.68%	3.75%	1.50%	1.99%	1.61%
20	4.27%	4.61%	4.65%	4.95%	3.71%	3.79%	1.53%	2.04%	1.65%
21	4.25%	4.62%	4.66%	4.98%	3.75%	3.83%	1.57%	2.08%	1.69%
22	4.24%	4.62%	4.66%	5.01%	3.77%	3.86%	1.60%	2.12%	1.72%
23	4.22%	4.61%	4.67%	5.04%	3.80%	3.90%	1.63%	2.16%	1.75%
24	4.19%	4.59%	4.67%	5.06%	3.83%	3.92%	1.66%	2.19%	1.78%
25	4.16%	4.58%	4.67%	5.08%	3.85%	3.95%	1.69%	2.22%	1.81%
26	4.13%	4.55%	4.67%	5.10%	3.87%	3.98%	1.71%	2.25%	1.83%

27	4.10%	4.53%	4.66%	5.12%	3.89%	4.00%	1.73%	2.27%	1.85%
28	4.06%	4.50%	4.66%	5.14%	3.91%	4.02%	1.75%	2.30%	1.88%
29	4.02%	4.46%	4.66%	5.16%	3.93%	4.04%	1.77%	2.32%	1.89%
30	3.98%	4.43%	4.65%	5.17%	3.94%	4.06%	1.79%	2.34%	1.91%
Source: ECB website									

Finally, the regression has been performed with these data and I am going to show and comment the results in the next paragraph.

4.3 Results

As in any analysis worthy of respect, it is necessary to dwell on the subsequent results. The first two things I want to highlight are the R^2 and the t-stat obtained for all the years:

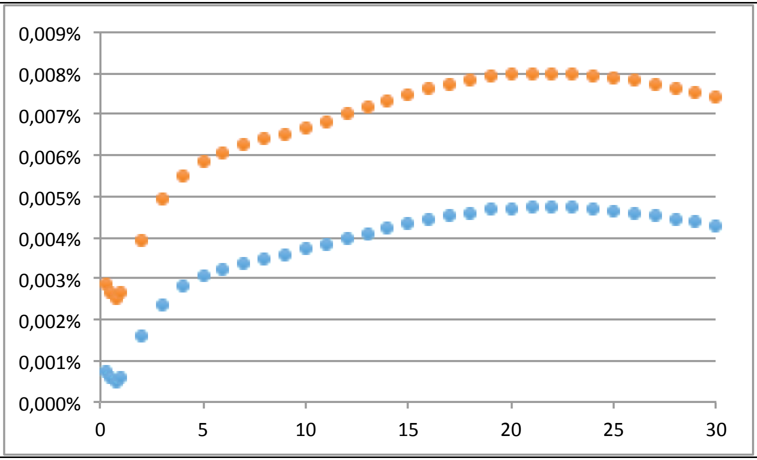
"Table 5 – Results obtained from the regression"									
Year	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**
R^2	67.92%	98.52%	33.25%	45.82%	38.48%	86.12%	98.85%	99.03%	98.70%
t-stat	-8.10	45.42	-3.93	5.12	4.40	13.87	51.53	56.16	48.48
Source: results found from my personal elaboration									

Looking at this table, it is possible to say that, in each year in this model, a big percentage of the dependent variable's variation is explained by the independent/explanatory variable.

What I want to say, however, is that even if I can affirm that the current-term structures' trend depends on the volatilities, in order to have a better model and a best explanation of the yield curve more variables are needed.

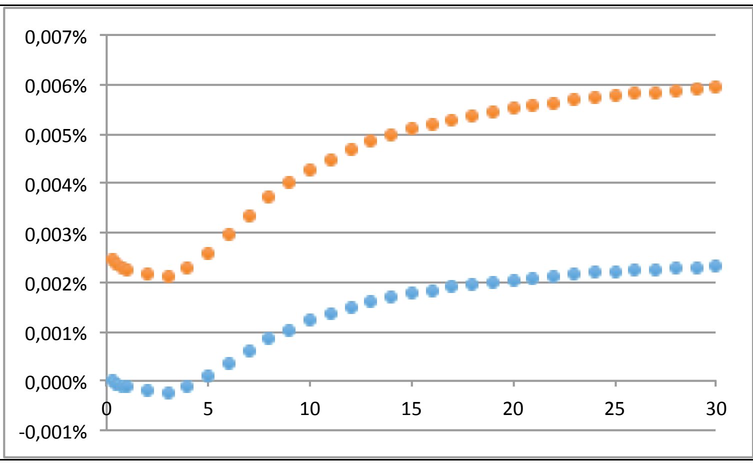
I am going to show now two graphs, one for a negative interest rates year and one for a positive interest rate year. In these charts, it is shown the function $\hat{Y}$, which shows the trend of the interest rates estimated through the vector of volatilities found above, in comparison with the trend of volatilities itself.

“**Figure 11 – Yield curve and volatility term-structure for 2010**”



Source: personal elaboration

“**Figure 11 – Yield curve and volatility term-structure for 2016**”



Source: personal elaboration

The charts for the other years are shown, as for the previous chapter, in the appendix x. The blue line represents the curve of interest rates, while the orange one concerns the volatility. As it is possible to see interest rates curve follows the same trend of the one of volatility.

Therefore, first conclusion to be drawn is that volatility and current-term structure are somehow linked, and that the former can explain the latter. So, given the trend of volatility, the behavior of the yield curve can be forecasted.

Once the current-term structure is set, it can be possible to predict also other important macroeconomic variables. One of them, for example, is the GDP.

Speaking about this constant, I would like to introduce an important paper written by Andrew Ang, Monika Piazzesi, and Min Wei named "What does the yield curve tell us about GDP growth?". It is not the only one research done about this topic; indeed there is by now an extended literature.

Even if my work has not been based precisely on this issue, having introduced it, I would like to mention the most important points concerning this analysis.

The main findings highlight the fact that the rule of thumb states that an inverted yield curve (short rate above long rates) indicates a recession in about a year, and yield curve inversions have preceded each of the last seven recessions.[17] This has happened, for instance, for the most recent recession, when the yield curve has seen an inverted shape in August 2006, just before the start of the recession in December 2007.

Going back to the analysis I performed in this part of my study, the second parameter I took into account has been the t-stat, which is also reported in the table shown at the beginning of this paragraph.

Also in this case, it is possible to state that the results can be interpreted positively and have some degree of significance.

As it is possible to observe in the table, for each year, the t-stat in absolute value is always greater than 2.64, which means that the significance of the model is high (***), and that the results are reliable.

Therefore, also in this case, second conclusion concerns the meaningfulness of the model, just tested through R^2, and now confirmed by t-stat.

5. Conclusions

It is clear, after all this analysis, that reality has changed and financial models need to be adapted in order to fit this new scenario and work in all possible directions. This is an important and crucial objective for economists, since this trend in interest rates has changed the traditional monetary policies and, since this choice, made by central banks in order to give renaissance to the economy, has influenced also citizens making very difficult in this period savings (or to say better not convenient), but, on the other side,

[17] Haubrich, J. G., Millington, S., "Yield Curve and Predicted GDP Growth", Economic Trends, July 2014

easing borrowers. At the same time, banks, especially the commercial one, would really appreciate and be happy to restart lending money and giving rebirth to real economy; but if from one side they are pushed in doing so by regulatory bodies through all these macroeconomic policies, on the other hand, they are experiencing a life of prohibitions and strict regulations, always brought by the same economic actors, which make really difficult to lend and give credit to retail customers, as well as to private and corporate clients.

Monetary policy makers often think in terms of a concept known as the real equilibrium rate or the “natural” rate of interest. This equilibrium rate is the interest rate that is consistent with stable inflation and output at its potential level.[18]

Setting short-term interest rates below this r has the effect of pushing upward economy and inflation, with the hope of reaching the target imposed by ECB. This is the reason why the bigger central banks in the world, among which ECB, have started handling accommodative monetary policies.

From a different point of view, there are, as usual, other consequences concerning this new macroeconomic scenario.

First of all, private citizens and businesses, in the long term, will prefer to retire their deposits, preferring to hold cash on which they have not to pay any interest (of course with other risks annexed). However, this can lead to financial instability.

Moreover, another issue regards the way people used to value things, in nominal terms rather than in real ones. The belief that bigger is better takes shape also and especially in this case; as a matter of fact, valuing things in absolute way allows people to look at negative interest rates as an unnatural event.

Last issue, negative rates can be linked to something irrational because of institutional problems and the lack of knowledge on tax and legal discipline concerning this subject.

Arrived at this point, after the initial promise of only few months of negative signs, but given the extended period, it is necessary and essential ask some important questions on this situation.

First, how much lower can we go? And, do the persistence of low and/or negative interest rates pose particular challenges to the stability of the financial

---

[18] https://www.ecb.europa.eu/press/key/date/2016/html/sp160728.en.html

system?[19]

For the first question, it has been explained that the lower bound is charged taking into account the opportunity cost of holding cash. However, this rate of which I am speaking about is the physical one, whereas if we want to take into consideration the economic lower bound, this needs to be set to the length negative effects for banking system are outweighed by benefits of having this negative rates. The name that has been given to this bound is reversal rate.

Some studies found out that a reference rate close to -2% would pose a substantial burden on banks' profitability.

However, even if nothing is impossible, no one is expecting the rates to be lowered until that level; as a matter of fact, the rates have worked around values few lower than the zero level, and from now on, given the expectations of both ECB and other central banks, they should increase followed by the slowly economic recovery.

Even if the monetary policies adopted are giving some results, these are far from the one expected, which were thought to be faster and more painless.

One issue that could be considered one of the reasons for struggling in this situation is the inefficiency of the fiscal policies each state member's government should structure. Precisely, a fiscal policy can be done through an increase or a decrease of the public expense level (obviously linked to the same increase or decrease in the public debt), of the investments, or of the tax rate on income.

Concluding, ECB monetary policy seems starting to work and providing results, targeting the inflation level set by the agreement within euro zone. The effort now should be towards financial, and fiscal policies, in order to update them to the new scenario and allow the almost perfect work of the system.

Explaining better, how can the fiscal policy be helpful for the monetary policy?

Taking, for example, the just cited public expense, it is possible to say that in a stagnant economy, such as the one we are experiencing, an increase in this macroeconomic measure realized by the state produces a direct increasing effect on the per capita income. This improvement is then positively correlated to an increase in the monetary demand and, again, this is linked to the required increase of interest rates.

So, at this point, it seems spontaneous questioning on why this mix is not being

[19] https://www.ecb.europa.eu/press/key/date/2016/html/sp160728.en.html

used by economic agents given the catastrophic scenario and the difficulties Europe is having in the recovery phase of the recession.

However, even if the issue is so crucial and dramatic, the answer to this question seems really easy, for what concerns European Union and its members: the monetary union is not more enough at this economic stage.

ECB summits should focus on the stipulation of a serious plan regarding the fiscal unification; in this way it can start adopting different and helpful economic policies, allowing then to each single state member a space where act on this policies side for what concerns any possible structural adjustment.

Summing up what I would have discussed, there is a major negative trend around Europe’s ability to recover, since markets expectations on average inflation rate are circa at 1.28% in 10 years, estimation which is much less than the one predicted by Draghi (i.e. already from 2018 an average inflation rate at 1.6%).

As he said in one of his last speech to investors audience, inflation is still a “luxury” for Europe, even if the first quarter of 2017 positive results are starting to be showed with still some level of shyness, and this evidence has been confirmed also by a recent study, done precisely by ECB itself, in which it is demonstrated that future inflation expectation is strongly determined from the level of the current inflation.

I know that in this last part of my project work I went a little bit out of the main focus of my study, but I think it is not possible to speak about one topic so ancient for the economy (as well as fundamental), but at the same time rounded by a lot of modernity, given everything is happening and of which I have spoken until now.

It has been really interesting studying this world and going deeper on a topic which every agent in the economy should understand, or at least be aware of.

It is not clear what will happen in the long-term, as well in the short one, but one thing should be clear; ever new actions are needed and both financial and economics model need to be ready to be adjusted repetitively given the volatility in which we are getting used to live in.

Some indiscretions make us think that in a near future interest rates will grow up again, both for the already increased inflation and for the strong pressures made by countries, among which especially Germany seems to be alarmed and discontented.

Bibliography

Hull, John. *Options, futures, and other derivatives*, 9th edition. Boston: Prentice Hall, 2014

Hull, John. *Options, futures, and other derivatives*, 10th edition. Boston: Prentice Hall, 2017

Vasicek O. A., "An equilibrium characterization of the term structure", Journal of Financial Economics North-Holland Publishing Company, Berkeley, CA, U.S.A., 1997

Cox, J. C., Ingersoll, J. E. and Ross, S. A., "A theory of the term structure of interest rates", Econometrica Vol. 53, No. 2, 1985

Hull, J. C. and White, A., "Pricing Interest Rate Derivative Securities", Review of Financial Studies, 1990

Debanjan, D., "Negative Interest Rate Policy by ECB: A Case Study", Skyline Business Journal, Volume X issue, 2014-2015

Giugliano, Ferdinando. "Draghi: l'inflazione resta bassa per ora i tassi non si muovono E chiede pazienza ai tedeschi." Iusletter. January 20, 2017. Accessed February 17, 2017. http://iusletter.com/draghi-linflazione-resta-bassa-ora-tassi-non-si-muovono-chiede-pazienza-ai-tedeschi/

Kerbl, S. and Sigmund, M., "From low to negative rates: an asymmetric dilemma", Financial stability report 32, 2016

Peshev, P. and Beev, I., "Negative Nominal Interest Rates on Loans: The Newly-Established Normal Practice?", Economic Alternatives 2, 2015

Zaman, P. and Brown, C., “How Negative Interest Rates Affect Commodity Derivatives and Collateral Transactions in the European Market”, Journal of taxation and regulation of financial institutions

Li, J., Clemons, C. B., Young, G. W. and Zhu, J., “Solution of two-factor models with variable interest rates”, Journal of Computational and applied mathematics 222, 2008

Zhou, N. and Mamon, R., “An accessible implementation of interest rate models with Markov-switching”, Expert systems with applications 39, 2012

Marianito, R. R. and Rogemar, S. M., “An alternative approach to the calibration of the Vasicek and CIR interest rate models via generating functions”, Quantitative Finance, 2014

Zeytun, S. and Gupta, A., “A comparative study of the Vasicek and the CIR Model of the Short Rate”, Berichte des Fraunhofer ,2007

Ahmad, F., Hambly, B. and Ledger, S., “A stochastic partial differential equation model for the pricing of mortgage-backed securities”, 2016.

European Central Bank, "The ECB`s negative interest rate." https://www.ecb.europa.eu/explainers/tell-me-more/html/why-negative-interest-rate.en.html

European Central Bank. "Assessing the implications of negative interest rates.". https://www.ecb.europa.eu/press/key/date/2016/html/sp160728.en.html

Ang, A., Piazzesi, M. and Wei, M., “What does the yield curve tell us about GDP growth?”, Journal of Econometrics, 2006

Haubrich, J. G., Millington, S., “Yield Curve and Predicted GDP Growth”, Economic Trends, July 2014

Welsch, H. and Kühling, J., "How has the crisis of 2008-09 affected subjective well-being?", Evidence from 25 OECD countries, Bulletin of Economic Research, 2016

Massarelli, C., "Breve storia triste: la deflazione nell'Europa Unita (?) post crisi", 2016

European Central Bank. "Key ECB interest rates". http://www.ecb.europa.eu/stats/policy_and_exchange_rates/key_ecb_interest_rates/html/index.en.html

European Central Bank. "ECB announces expanded asset purchase programme", January 22, 2015. https://www.ecb.europa.eu/press/pr/date/2015/html/pr150122_1.en.html

Merli, A., Draghi resiste alle pressioni: " Non è il momento di cambiare linea sui tassi", 6 Aprile 2017, Il Sole 24 Ore, http://www.ilsole24ore.com/art/finanza-e-mercati/2017-04-06/draghi-non-e-momento-cambiare-linea-tassi-102213.shtml?uuid=AE38FX0

European Central Bank. "The ECB's Directorate General Statistics releases euro area yield curves every TARGET working day at 12 noon Central European Summer Time", https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/technical_notes.pdf

Appendix

[i] $\mu(t,s,r) = \frac{1}{P(t,s,r)}\left[\frac{\partial}{\partial t} + f\frac{\partial}{\partial r} + \frac{1}{2}\rho^2\frac{\partial^2}{\partial r^2}\right]P(t,s,r)$

$$\sigma(t,s,r) = -\frac{1}{P(t,s,r)}\rho\frac{\partial}{\partial r}P(t,s,r)$$

[ii] $B(t,T) = \frac{1-e^{-a(T-t)}}{a}$ and $A(t,T) = e^{\frac{[B(t;T)-T+t]\left(a^2b-\frac{\sigma^2}{2}\right)}{a^2}-\frac{\sigma^2B(t;T)^2}{4a}}$

[iii] $B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\kappa+\lambda)*(e^{\gamma(T-t)}-1)+2\gamma'}$,where $\gamma = [(\kappa+\lambda)^2+2\sigma^2]^{\frac{t}{2}}$ and

$$A(t,T) = \left[\frac{2\gamma e^{\left[\frac{(\kappa+\lambda+\gamma)(T-t)}{2}\right]}}{(\gamma+\kappa+\lambda)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{\frac{2\kappa\theta}{\sigma^2}}$$

[iv] $\phi(t) = a(t)b + \theta(t) - \lambda(t)\sigma(t)$

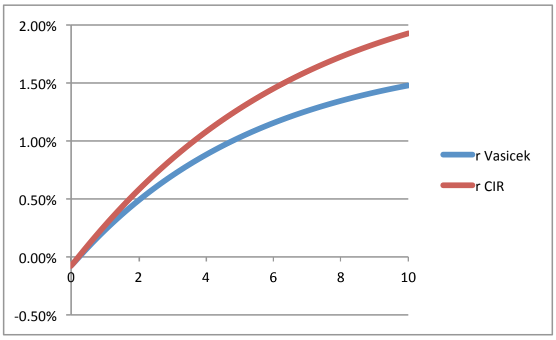
[v] $\phi(t) = a(t)b + \theta(t)$ and

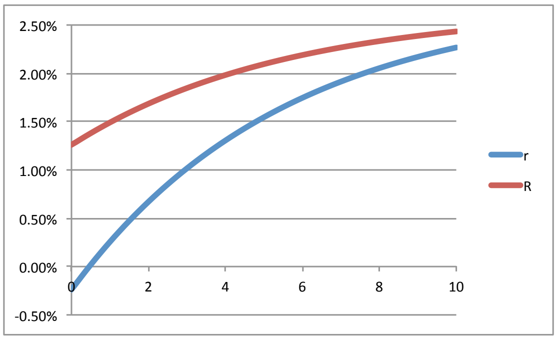
$$\psi(t) = a(t)b + \lambda(t)\sigma(t)$$

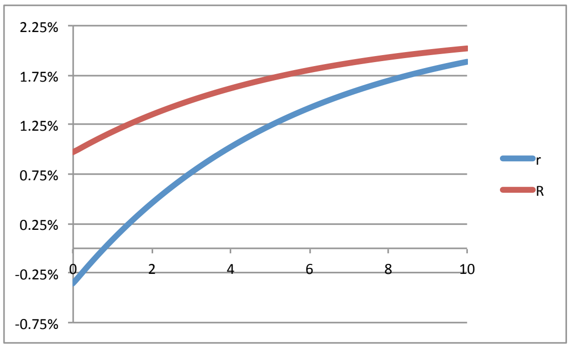
[vi] $B(t,T) = \frac{1-e^{-a(T-t)}}{a}$ and

$$\ln[A(t,T)] = \ln\left[\frac{P(0,T)}{P(0,t)}\right] + F(0,t)B(t,T) - \frac{\sigma^2(e^{-aT}-e^{-at})^2(e^{2at}-1)}{4a^3}$$

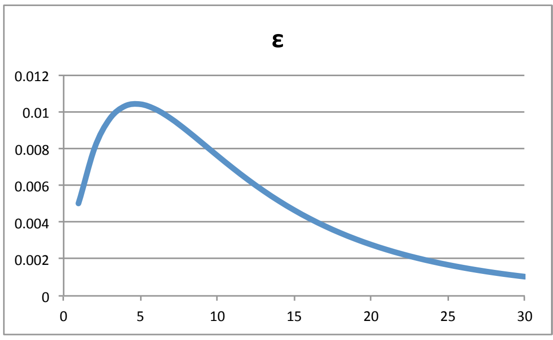
vii

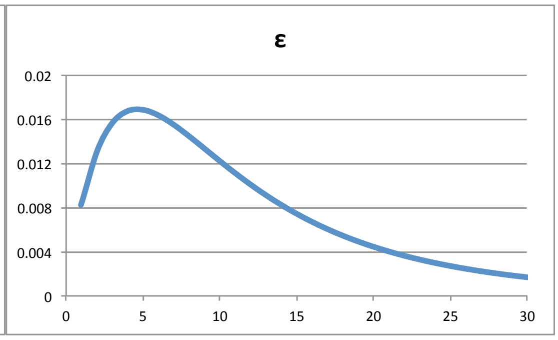

2.00%
1.50%
1.00%
0.50%
0.00%
-0.50%
0 2 4 6 8 10
r Vasicek
r CIR

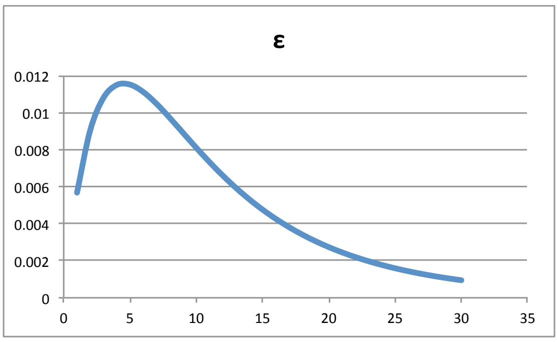

2.50%
2.00%
1.50%
1.00%
0.50%
0.00%
-0.50%
0 2 4 6 8 10
r
R

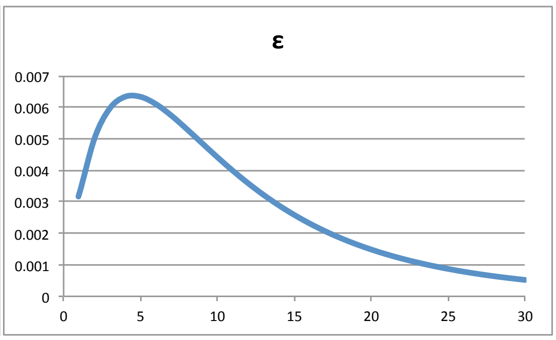

2.25%
1.75%
1.25%
0.75%
0.25%
-0.25%
-0.75%
0 2 4 6 8 10
r
R

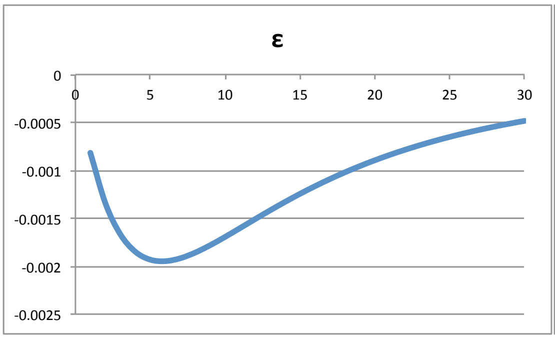
viii

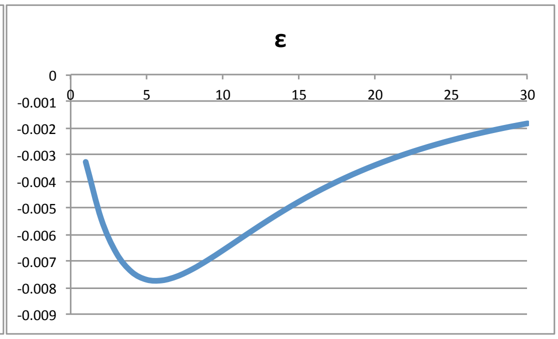

ε
0.012
0.01
0.008
0.006
0.004
0.002
0
0 5 10 15 20 25 30

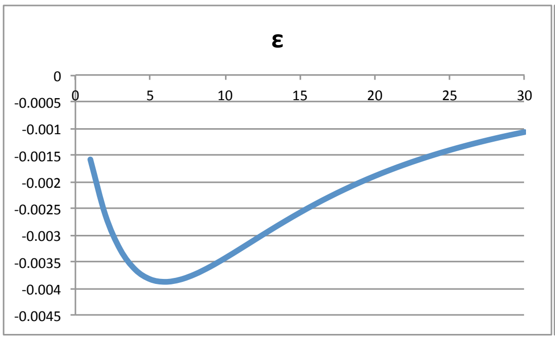

ε
0.02
0.016
0.012
0.008
0.004
0
0 5 10 15 20 25 30

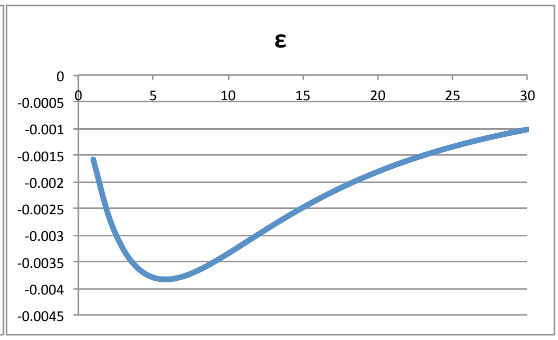

ε
0.012
0.01
0.008
0.006
0.004
0.002
0
0 5 10 15 20 25 30 35


ε
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
0 5 10 15 20 25 30

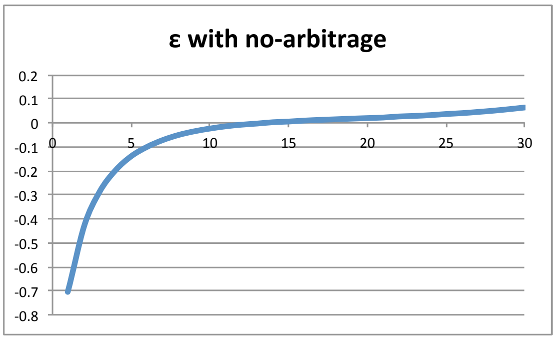

ε
0
-0.0005
-0.001
-0.0015
-0.002
-0.0025
0 5 10 15 20 25 30

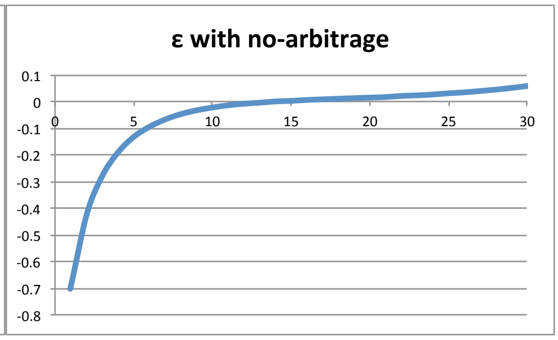

ε
0
-0.001
-0.002
-0.003
-0.004
-0.005
-0.006
-0.007
-0.008
-0.009
0 5 10 15 20 25 30

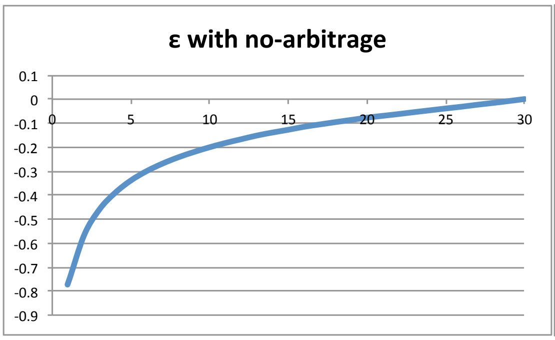

ε
0
-0.0005
-0.001
-0.0015
-0.002
-0.0025
-0.003
-0.0035
-0.004
-0.0045
0 5 10 15 20 25 30

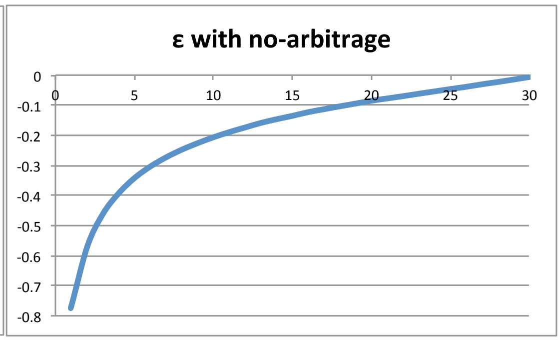

ε
0
-0.0005
-0.001
-0.0015
-0.002
-0.0025
-0.003
-0.0035
-0.004
-0.0045
0 5 10 15 20 25 30

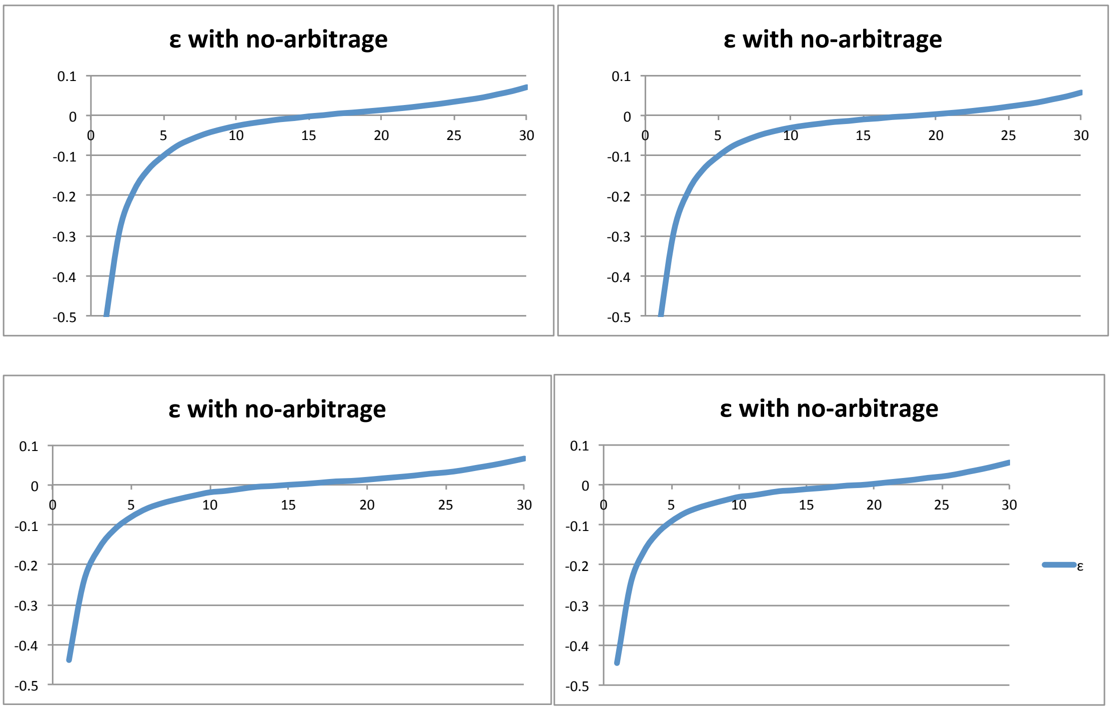
ix


ε with no-arbitrage
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
0 5 10 15 20 25 30

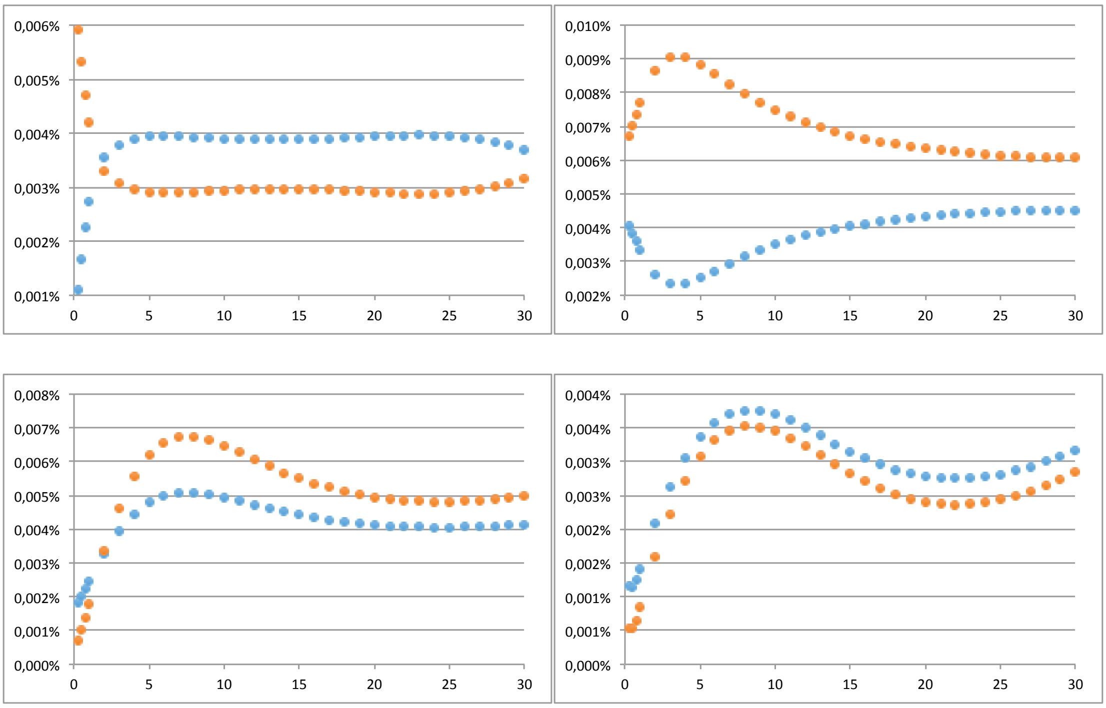

ε with no-arbitrage
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
0 5 10 15 20 25 30

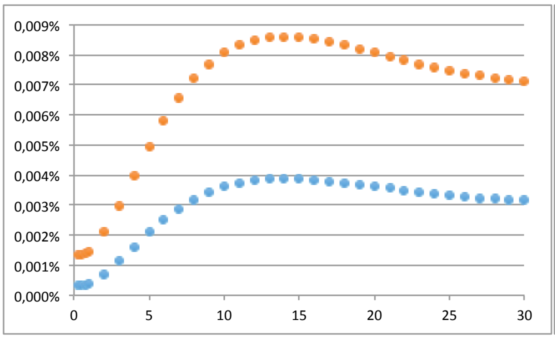

ε with no-arbitrage
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
0 5 10 15 20 25 30

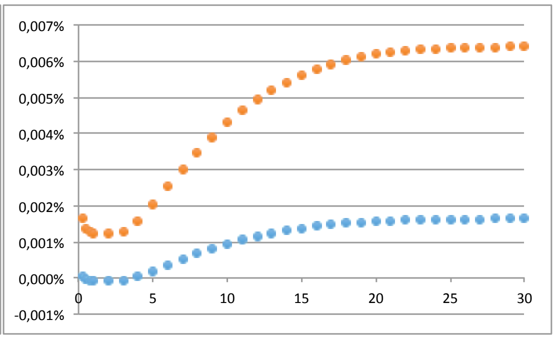

ε with no-arbitrage
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
0 5 10 15 20 25 30

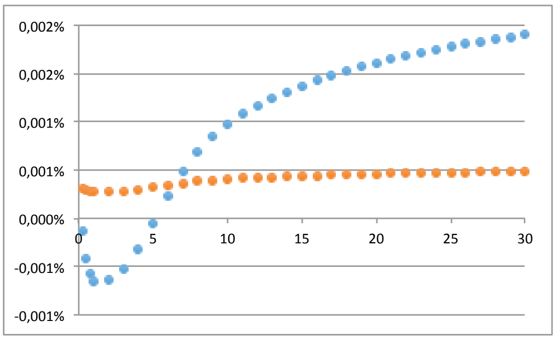

ε with no-arbitrage
ε with no-arbitrage
ε with no-arbitrage
ε with no-arbitrage
ε

x




0,009%
0,008%
0,007%
0,006%
0,005%
0,004%
0,003%
0,002%
0,001%
0,000%
0
5
10
15
20
25
30


0,007%
0,006%
0,005%
0,004%
0,003%
0,002%
0,001%
0,000%
-0,001%
0
5
10
15
20
25
30


0,002%
0,002%
0,001%
0,001%
0,000%
-0,001%
-0,001%
0
5
10
15
20
25
30

6. Summary

The purpose of this summary is to make clear in advance the path and the method I followed, showing how I structured my work.

The first thing I did, has been a broad overview of the current scenario and of all the consequences it has brought in the economy.

Subsequently, I proceeded making a literature review, with the aim of building a framework of the existent and important studies that have been made on this topic. I went through the Vasicek and the CIR model, assessing the main assumptions and the results provided by them. Particularly, I analyzed the two main papers; which are "An equilibrium characterization of the term structure" by Oldrich Vasicek, and "A theory of the term structure of interest rates" by John C. Cox, Jonathan E. Ingersoll, Jr., and Stephen A. Ross.

Moreover, I decided to not leave out, in the literature review part, the Hull and White model, in order to provide a sight also in a no-arbitrage model, analyzing also the paper "Pricing Interest-Rate-Derivative Securities".

After this first part, I started with the explanation of the development I performed. In the first paragraph, I went through each step I followed during my research, explaining the procedure and each alteration needed (particularly the study has been divided in three different stages). I started with the study of the Vasicek and CIR models to see if they could work in both economic scenarios of positive and negative interest rates; then, I proceeded testing how much error is produced by the model in the estimation of the term structure, precisely comparing the models between themselves but with different inputs; once with the estimated term structure, and then with the current one. Finally, the last step of the development has been the comparison of the two models established in the first stage with the no-arbitrage model already proposed in the literature review, the Hull and White one factor.

Once having concluded this first part of the research explaining why I decided to use some data instead of others, and showing all the results obtained, I proceeded with a study of the volatility (i.e. σ), to see how it is related to interest rates.

The purpose has been to understand if this parameter can be considered a good predictor of the trend of interest rates.

It is obvious that the dependent variable I decided to study cannot be justified only through one explanatory variable, and that to have a more precise and efficient result other parameters should be added, such as for example the GDP. Also concerning this part, I went through each data used and results obtained.

Finally, in the last part of my project work, I decided to conclude summing up all the main findings and making a final evaluation of the current scenario, and going deeper in the future perspectives to explain the different sides of the balance brought by difficult decisions that sometimes must be taken.

Now, I would like to go deeper into my elaboration in order to touch all the important steps of this study.

The purpose of my work would like to be focused on the study and analysis of the new and unusual economic scenario we are experiencing. As a matter of fact, the last years have been characterized by words and feelings such as volatility, uncertainty, deflation, low if not even negative growth, fear, and lack of trust.

Among different and "strange" actions, negative interest rates have appeared in the last years for the first time in the economic history, being set by different central banks, including ECB. Even if this decision seemed to appear as something negative, actually the purpose was to find a tool for helping the economy to restart.

It has been, in fact, the first time, real economy has faced this kind of accommodative policy, since negative interest rates have never used before, and, moreover it was a scenario not conceivable neither in financial handbook, being considered fictitious and impossible to become reality.

The choice to write a thesis on this financial topic, has been driven by the deep interest towards this field of study and by the will to understand if the already existent financial models are able to continue working with this new macroeconomic scenario, or if not (as I expected) trying to propose some future developments.

It is, moreover, interesting, in my opinion, focusing the attention on this change, given the impact they are having on some important economic agents such as commercial and investment banks, but also on private citizens' everyday life.

I decided to begin with the general macroeconomic dynamics as starting point. It is, in fact, important to understand which are the reasons that have forced the principal central banks to overpass their "classic" monetary policies choices, up to, as in the special case of ECB, enlarge its perspective (in 2014) in order to allow the employment of instruments never used before.

At this stage, I tried to answer to simple but crucial considerations, that all the agents that play a role in the markets are asking in these days.

What has happened after the last economic crisis? Why are we not able to come out from this difficult period? Will we be able to act and survive with these new economic policies?

It is necessary, given this overview, to keep in mind some important key aspects, to understand where this situation has come from.

We all know that the entire globe has been suffering from this last dramatic crisis, but it is possible to affirm with almost absolute certainty that European Union has been one of the most affected economic area, it is in fact enough to mention that in 2009 the GDP went down by 4.3%.

This decline has not affected only the classical real economic measures just mentioned above; as a matter of fact, one of the most crucial aspects the economy is fighting, since then, has been deflation.

This, in fact, can be considered the characterizing factor of this historical period, with a drop in prices caused by a little earlier GDP's downturn, which was in turn linked to the rise of unemployment and to the loss of consumers' purchasing power as obvious consequence.

Contrary to what should be thought, with deflation, consumers' purchasing power should increase, but if, in a period of general crisis, people loose their jobs, this cannot happen.

So, given this overview, the decision of regulatory bodies, after some initial years of waiting, has been to intervene using its tools, which are interest rates. The main objective of ECB, since ever, is to maintain price stability through the control of inflation, contrary to the US FED, which has the power of creating jobs. As a matter of fact, it can operate setting and adjusting interest rates. If from one side, it used to increase interest rates in order to fight against too high inflation; on the other side, it can contrast deflation decreasing the rates (and this can be considered the first reason for the introduction of negative interest rates).

Especially, ECB can act on three different rates: the marginal lending facility, which is the rate that allows banks to get overnight finance from the Euro system; the main refinancing operations (MRO), which allows the injection of liquidity into the banking system; and the deposit facility, which are used by banks when they make overnight deposits with the Euro system. ECB has lowered all these three rates, caring about the fact of keeping the fair distance among them in order to not destroy the money market.

Lowering interest rates, this will allow more people to borrow more money, resulting in an increase of the private spending, which in turn will cause the economy to grow and inflation to increase. As the opposite is considered true, when interest rates are increased, consumers will prefer to save as returns from savings will be higher. With less money in their pocket, the economy slows and inflation decreases.

So, in this scenario, characterized by the worst economic crisis the globe has ever experienced, both in terms of financial consequences and length of the period of depression, it has been clear that intervention of central banks and regulatory bodies were needed, and as result, numerous central banks have decided to adopt expansionary policy to boost economy and allow people to breath. Particularly ECB has decided to take action, being on 4th June 2014 the first major central bank in the world to make one of its rates negative (i.e. the rate on deposits), with the aim of getting credits flowing into the troubled economies.

Actually, in a first moment (more or less at the end of 2013, period in which Europe was starting to experience a deflationary trend), the ECB president, Mario Draghi, showed not to be much worried about this issue, since in his view, that situation was a transitory phase. It has been possible to attribute this decision, beyond of the analysis the bank has surely made, to the influence of countries that were not experimenting such a worst situation (too much economic diversity among European Union members, which does not allow to keep the right decision for all the members of the community).

Let's talk for example of Germany, which could boast an inflation rate at about 1.3%, against the mean of many other countries, which instead were fighting with a rate floating around 0.7%.

This scenario was, surely, not a positive signal. In fact, even if Europe, and in particular Germany, were obsessed by inflation (for historical reasons), also deflation should not have been undervalued, especially for those countries characterized by a high sovereign debt such as numerous countries of the Eurozone (it is just possible to keep Italy, Spain, Portugal, or Greece as examples).

As a matter of fact, even if at first sight, this economic phenomenon could be seen as a positive perspective (given the low prices), actually an extended deflation is dangerous and unhealthy as a hyperinflation (of which in Europe we have memories for what concerns the German case of 1948).

Going deeper, in a deflation period the decrease in goods and services prices is linked, as I have already cited above, to a reduction in salaries, which does not allow people and businesses to maintain their obligations towards financial intermediaries (such as mortgages or any other financing forms).

Moreover, another side effect is the tendency to postpone every kind of purchase, willing to wait until the relative price will reach the optimal level.

All these ways of behave has lead to the demolition of consumption, which is considered the foundation of an economic system rebirth.

This financial measure has been used also few times ago in Nordic countries (such as Sweden and Denmark), but it is better not to look at this historical notoriety since the effects produced on expected results have been very low.

Of course, given the heavy economic trouble that Europe was facing already by some years, this has not been the only expansionary monetary policy ECB has produced and put in action.

As a matter of fact, Quantitative Easing, which has been announced at the threshold of 2015, has concerned an expanded purchase programme. ECB has promised to buy sovereign bonds in order to be added to its private assets' portfolio. Initially, the purpose was to combine asset purchase, which has to be carried out until at least September 2016, to the amount of € 80 billion.

What then has happened, given the low but constant economic growth, it has been the fact that QE has been extended until the end of 2017, contemporary reducing the amount of the monthly purchase from € 80 billion to € 60 billion.

It is to simple to state and judge whether the ECB has taken right or bad financial measures, since the problem is deeper and more complex than what sometimes citizens can or want demonstrate, but it is interesting comment on how real economy (taken into account commercial banks and private citizens) has reacted to this shifted scenario.

When dissatisfaction is the main sentiment in a society, it is very difficult if not impossible, I would say, to get a great consensus, whichever the decision taken is. Clearly speaking, there will be always someone happy and someone discontent.

Moreover, considering the critical historical period the economy was experimenting, private citizens, private sector, and commercial banks were got stuck in a situation of suffocation.

With this dramatic situation, a straightforward connection has been for these agents to look at these measures as a new risk, a bet, hence something that could be another time synonymous of uncertainty.

Confusion has arisen, bringing with itself a lot of deep and emblematic queries, such for example: will these policies push the economy, re-giving life to Europe and generally to developed countries, or will it worsen the already dangerous and unstable reality? What will be the effect on private savings and why do I, as bank, have to pay central bank to keep my deposit, loosing earnings; which are the positive sides in this emblematic decision?

Mario Draghi, the Italian chief, together with his crew, has tried to transmit trust and calmness, explaining that there will be no direct impact on citizens' savings, even if the effect could be indirect.

It is undisputable, however, that commercial banks can be considered the most hit agents, which not only are not gaining earnings from their deposits, but also even worse, have to pay ECB to keep their money stalled.

This has been, really, the expectation on which this decision has been based. As a matter of fact, ECB had a positive hope that the banks stopped accumulating money and started lending more to consumers, businesses, or among banks, boosting the economy.

However, as each vicious circle that is respected, there is always a consequence, which in this case is identified in the willingness for banks to pass this major costs of these deposits on customers, already in a difficult situation and averse to the banking system, bringing again the economy at a stagnant point.

At the same time, banks would be really willing to do so, but if from one side they are pushed by the regulatory bodies through all these macroeconomic policies, on the other hand they are experiencing a life of prohibitions and strict regulations, always brought by the same actors, which make really difficult to lend and give credit to retail customers, as well as to private, and corporate clients.

As a matter of fact, this decision has brought many side effects and now it is possible to determine that banks are suffering a lot, being unable also to profit from savings accounts, and incurring in heavy loss.

Other consequences regarding this scenario are linked to the borrowers and savers dilemma. As a matter of fact, ECB's purpose was conceived in order to create an environment that should benefit savers, seen as supporters of growth and as a foundation for the increase in rates once the monetary accommodation will be reach. Here, it is spontaneous questioning the reason behind, in this scenario, the "punishment" of savers, given their important role, and instead the reward for borrowers. Fortunately, also this time, ECB has, without doubt, clarified its position, affirming that its core business is making more or less attractive for households and businesses to save or borrow money, but this is not done in the spirit of punishment or reward.

Today, after some years of work, ECB, in one of the last bulletin of 2017, has stated that it is not yet the time to make changes to this expansionary monetary policy, for both what concerns the level of interest rates and the quantitative easing measure, at least until the end of year 2017, which coincides with the end of QE. The central bank thinks that the economy in the euro zone is receiving the right stimulus, even if the results are arriving in a slow but constant way.

It is not the time of leaving the economy to grow alone, since the inflation dynamic is not yet able to self-sustain without the help of these expansionary policies. What it should be needed in this environment is a fiscal policy, which should boost the economy and help the decisions taken on the monetary side.

However, contrary to what ECB thought, there are in this period a lot of controversial parties, which are pressing Mr. Draghi in order to take some actions.

Continuing and entering in the crucial part of the work, as I have already briefly explained, the purpose of the study has been to show the implementation of the models reviewed in the literature review. I decided to proceed with the study in three different and separate steps.

The first one, which is also the most important given the main goal of my research work, has concerned the study of the Vasicek and CIR models to see if they can work in both economic scenarios of positive and negative interest rates. What it has come out from this analysis is that a shift in the conditions brings some needed and essential changes in the models, in order to be adapted to the new conditions of the market. Without generating these alterations, in fact, it is not possible, as I will show, to apply the CIR model, due to one of its fundamental assumption and to a math problem in the dr computation (proper the square root of r in the formula).

So, to fix this problem, the idea has been to add a variable α in the determination of r, which ends up being equal at least to the maximum negative interest observed in the term structure and obviously greater than zero.

Another important step, at this level, has been the estimation of the parameters needed for the computation of the process for r. This procedure has been done with the time series method, using the OIS at 1 month, and subsequently double-checking the results with the maximum-likelihood method. Through this stage, I tried to estimate the process pursued by r in the real world with the use of a linear regression.

The first thing has been to take the real world process for the short-term rate on a daily basis and calculate the change as (r_{t+1} - r_t). Subsequently, I have performed a regression of the change in the rates against the rates itself, and a, b, and sigma have been computed from the regression results.

Proceeding, I used a Monte Carlo simulation to estimate the process for r, obtaining 500 dr trials, and this procedure has been repeated for 10 years with a quarterly frequency. Lastly, after completing the entire simulation, I took the average for each period and I added this mean value of dr to the spot rate.

After this first assessment and to confirm whether or not the two models provide the same results, it has been made a comparison between them, first calculating the bond prices with the estimated term structure and then making the same procedure but with the current-term structure, which is perfectly aligned with the market economy. In this part, the purpose has been to show the error produced by the model in the estimation of the term structure.

Finally, the last step of the development has been the comparison of the two models established in the first stage and the no-arbitrage model already proposed in the literature review, the Hull and White one factor.

The difference from the previous analysis is that in the first case the error is in absolute value, meaning that it is given by the application of the same models with two different term structures; instead in this case, the error is more related to the outcome of the two different models, i.e. the prices of the bond.

The scope of this development has been to compute the trend of the error that comes out from the difference between the bond prices of Hull and White with both the two equilibrium models. In this case, the main objective of the work has been to present how much the two equilibrium models are far from being the better tool used by traders, with respect to the late adopted no-arbitrage models.

In the second half of my project work I proceeded talking about the relationship between the volatility of interest rates (let's define it as σ) and the current-term structure.

The purpose of this analysis has been to find some evidences of the existence of this connection and in case of an affirmative response the point has been to understand how they are linked and related.

My question behind this analysis has been if the volatility can be considered a good predictor of the trend of interest rates. It is obvious that this trend cannot be explained only through one variable, and that a deeper study would request the addition of other parameters to prove more significance and relevance.

As a matter of fact, it is not possible that the current-term structure is explained only by one variable, since the entire macroeconomic scenario and a lot of economic measures affect it, given its important in both financial markets and real economy.

Concluding, it has been clear, after all this analysis, that reality has changed and financial models need to be adapted in order to fit with this new scenario and work in all possible directions. This is an important and crucial objective for economists, since this trend in interest rates has changed the traditional monetary policies and, since this choice, made by central banks in order to give renaissance to the economy, influences also citizens making very difficult in this period savings (or to say better not convenient), but, on the other side, rewarding borrowers.

From another perspective, there are also other consequences concerning this scenario. First of all, private citizens and businesses, in the long term, will prefer to retire their deposits, preferring to hold cash, on which they have not to pay any interest (obviously with other risks annexed). However, this will bring to more financial instability.

Moreover, another issue regards the way people used to value things, in nominal terms rather than in real ones. The belief that bigger is better takes shape also and especially in this case; as a matter of fact, valuing things in absolute way allows people to look at negative interest rates as an unnatural event.

Last issue, negative rates can be linked to something irrational because of institutional problems and the lack of knowledge on tax and legal discipline on this subject.

Arrived at this point, after the initial promise of only few months of negative signs, but given the extended period, it is necessary and essential to ask some important questions on this situation.

First, how much lower can we go? And, do the persistence of low and/or negative interest rates pose particular challenges to the stability of the financial system?

For the first question, it has been explained that the lower bound is charged taking into account the opportunity cost of holding cash. However, this rate of which I am speaking about is the physical one, whereas the economic lower bound needs to be set to the length negative effects for banking system are outweighed by benefits of having this negative rates. The name that has been given to this bound is reversal rate and some studies found out that a reference rate close to -2% would pose a substantial burden on banks' profitability.

It is really crucial to have this concept clear in mind, since the physical lower bound is the rate at which disintermediation risk will materialize. At this rate, investors will prefer to withdraw their funds from banks and financial institutions, investing alone and in riskier financial instruments.

However, even if nothing is impossible, no one is expecting the rates to be lowered until that level; as a matter of fact, the rates have worked around values few lower than the zero level, and from now on, given the expectations showed by ECB, they should increase followed by the slowly economic recovery.

For the second question, instead, it is possible to say that, even if the expansionary policies adopted are giving some results, these are far from the one expected, which were thought to be faster and more painless.

One issue that could be considered one of the reasons for struggling in this situation is the inefficiency of the fiscal policies each state member's government should structure. Precisely, a fiscal policy can be done through an increase or a decrease of the public expense level (obviously linked to the same increase or decrease in the public debt), of the investments, or of the tax rate on income.

Concluding, ECB monetary policy seems starting to work and providing results, targeting the inflation level set by the agreement within euro zone. The effort now should be towards financial, and fiscal policies, in order to update them to the new scenario and allow the almost perfect work of the system.

Explaining better, how can the fiscal policy be helpful for the monetary policy?

Taking, for example, the just cited public expense, it is possible to say that in a stagnant economy, such as the one we are experiencing, an increase in this macroeconomic measure realized by the state produces a direct increasing effect on the per capita income. This improvement is then positively correlated to an increase in the monetary demand and, again, this is linked to such required increase of interest rates.

So, at this point, it seems spontaneous questioning on why this mix is not being used by economic agents given the catastrophic scenario and the difficulty Europe is having in the recovery phase of the recession.

However, even if the issue is so crucial and dramatic, the answer to this question seems really easy, for what concerns European Union and its members: the monetary union is not more enough at this economic stage.

ECB summits should focus on the stipulation of a serious plan regarding the fiscal unification; in this way it can start adopting different and helpful economic policies, allowing then to each single state member a space where act on the fiscal policies for what concerns any possible structural adjustment.

Summing up what I would have discussed, there is a major negative trend around Europe's ability to recover, since markets expectations on average inflation rate are circa at 1.28% in 10 years, estimation much less far from the one predicted by Draghi (i.e. already from 2018 an average inflation rate at 1.6%).

As he said in one of his last speech to investors' audience, inflation needs to be considered still a "luxury" for Europe. In fact, in the first quarter of 2017 positive results are starting to be showed with still some level of shyness, and this evidence has been confirmed also by a recent study, done precisely by ECB itself, in which it is demonstrated that future inflation expectation is strongly determined from the level of the current inflation.

I know that in this last part of my project work I went a little bit out of the main focus of my study, but I think it is not possible to speak about one topic so ancient for the economy (as well as fundamental), but at the same time rounded by a lot of modernity, given everything is happening and of which I have spoken until now.

It has been really interesting studying this world and going deeper on a topic which every agent in the economy should understand, or at least be aware of.

It is not clear what will happen in the long-term, as well in the short one, but one thing should be clear; ever new actions are needed and both financial and economics model need to be ready to be adjusted repetitively given the volatility in which we are getting used to live in.

The biggest battle in these times is considered to be the one of transforming every difficult moment into a new opportunity. Banks need to see this impossibility in remunerating the deposits and in getting margins from this operation as a way to explore new sources from which to improve the gross operating margin.

In Italy, this can be very difficult due to the poor knowledge citizens have on investments topic, but we have the chance to educate the Italian savers, which are considered one of the best savers in the world, in starting to be friendly with managed savings.