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IMMUNIZATION AND HEDGING OF FIXED-INCOME SECURITIES IN COMPARISON

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INTRODUCTION

At the basis of this thesis there is the analysis of two important strategies implemented by portfolio managers in order to cover the risk of the interest risk, that is the immunization theory and the hedging of fixed-income securities. In particular the two arguments are presented from a theoretical point of view by treating important theorem of the immunization and the basis of the use of the derivatives such as the interest rate future.

The motivations that allow to be interested in this subjects have two different aspects. First, I like to solve problem and face complexities: during my three-years undergraduate program I satisfy this aptitude in quantitative exams such as financial mathematics and markets and derivatives. Second, in a more and more globalized and at the same time more fragile and breakable economy, I have been interested in how to face the uncertainty technically.

The objective of the thesis is the introduction of two different strategies in facing the uncertainty in the fixed income market and, finally, a comparison between them in order to show the strength points and weaknesses. The thesis is divided into three chapters, each one divided into three paragraphs: the first paragraph aim to introduce the basic concepts of financial mathematics applied by the strategies. The first paragraph treats the interest rate and how to compute it, the second introduces to the structure of a bond and how the interest rate affects the value of a bond. The third paragraph is focus on the two indicators of the volatility of a bond, namely the duration and the convexity. The second chapter introduces with the first paragraph the concept of investment accumulation and investment duration, then the second paragraph is focus on the Fisher and Weil's theorem and on the Redington's theorem of the immunization. Finally, the third paragraph of the second chapter points out the Vasicek and Fong's discoveries, namely the general theorem of immunization and its theorization by not implying additive shifts but random shifts of the interest rate. The third chapter starts with an overview of the derivative world in the first paragraph. The second paragraph introduces the interest rate futures and, in conclusion the third one points out the theory of the hedging by applying the duration and the convexity in order to be cover by the interest rate volatility.

Thanks to this thesis it has been possible to analyze deeply two important strategies applied in the world of the fixed income and, in the conclusion chapter, to make some comparison in real world applications of the two strategies.

I. BASIC CONCEPT

1. Interest Rate

The "*Interest Rate*" is the ratio between the *Interest* and the *Present value*, in which the Interest is computed by subtracting the *Present value* to the *Future value*; and represents the percentage return than an investor obtains in a financial operation.

A financial operation is an exchange of two different amounts of money in different periods of time and it can be understood, in regard of who supplies or who demands, as "*Investment* operation" in the former case or as "*Financing* operation" in the latter case.

In an Investment operation, any investor that owns a given amount of money "*x*", decides to forgo this amount at time "t_0", in order to receive a greater amount of money "*y*" at time "t_1". This statement can be represented by the following expression:

$$\underset{t_0}{x} \quad \overline{\qquad\qquad\qquad\qquad} \quad \underset{t_1}{y}$$

where the value *x* is the "*Present value*" (or *Initial investment*), and the value *y* is the "*Future value*".

Hence, the following example expresses, given the features of a contract, how to compute the Interest rate:

- suppose a farmer has $10,000 dollars and invests this amount in a contract that will provide him $10,500 dollars back after one year: the interest rate will be 5%.

In a Financing operation, a borrower, that needs money to finance a project, obligates himself to return an amount of money back at t_1, in order to receive, at t_0, a necessary amount of money to finance a project. The difference between the Future value and the Present value is the *"Discount"*. The discount rate is defined as the ratio between the Discount and the Future value.

- For instance, a company needs to finance a positive-NPV project and decides to issue a Zero-Coupon Bond, whose maturity[1] is in one year with face value[2] of $1.000. The company receives $950 at t_0. So, the discount rate is 5%.

Summarizing the two financial operations discussed so far, what distinguishes them is only the point of view of the person that submits the financial transaction. In fact, there is a relation between the interest rate and the discount rate. Analytically it is stated by the following expressions:

$$d = \frac{i}{1+i} \Leftrightarrow i = \frac{d}{1-d}$$

[1] The maturity is the period of time that the investment lasts.
[2] The face value is the imprinted value on the Zero-coupon bond, that the investor has the right to receive at maturity.

where *d* is the discount factor and *i* is the interest rate.

The interest rate can be assumed in three different ways: in terms of the so-called "*Rate of return*", in terms of the "*Discount rate*", or in relation to the "*Opportunity cost*". The first (also called "*Yield to maturity*") represents the minimum rate of return that an investor should receive when accomplishing an investment. The second embodies the rate's capability of obtaining the discount factors when computing the present values. The third expresses the cost an investor forgoes by deciding to invest in a different option.

Understanding the entity of the interest rate is fundamental when analyzing the market. In fact, economic theory evidences that the interest rate is determined by the meet of demand and supply of funds within the marketplace. In case the demand of funds increases, the interest rate arises; on the other hand, when the interest rate decreases then the supply of funds arises.

"More into details, in analyzing the market interest rates, the interest rate is the outcome of 5 factors:

$$interest\ rate = (real)\ riskfree\ rate + inflation\ risk\ premium + default\ risk\ premium + liquidity\ risk\ premium + maturity\ risk\ premium$$[3]"

where:

- The *(real) risk-free rate* is the expression of the interest guaranteed for an investor for buying risk-free securities, such as governmental bonds;
- the *inflation risk premium* is the premium obtained by the investor to compensate the loss of purchasing power of a unit of currency;
- the *default risk premium* is the premium that rewards an investor for the likelihood of the borrower's default;
- the *liquidity risk premium* is the premium that compensates an investor for the difficulty of transforming the investment in money;
- the *maturity risk premium* is the premium that reward an investor for the increasing of the duration[4] of the investment

The computations in the previous examples have the assumption that there is only one unitary period of time. To introduce more periods, it is necessary to explain the compounding. The compounding consists of considering the future value at the end of a unitary period of time such as the present value of the next period. Analytically, it is:

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[3] Anonymous Author that cannot be quoted
[4] In this case, the duration is not the financial duration of a fixed-income security, but the period of time an investment lasts.

$$n_1 = n_0(1+i)$$
$$n_2 = n_1(1+i) = n_0(1+i)^2$$
$$n_3 = n_2(1+i) = n_0(1+i)^3$$
$$\dots$$
$$n_t = n_{t-1}(1+i) = n_0(1+i)^t$$

So, the relation between the future value and present value for many periods of time is:

$$FV = PV(1+i)^t$$

where FV is the future value, PV is the present value and t is the number of periods.
If an investment provides an annual interest rate of 5% the Principal is $1.000 and the maturity is 5 years. In order to compute the future value of the investment after 5 years, it is necessary to include in the computation the interest over 5 years.

$$FV = \$1.000 \ * (1+5\%)^5 = \$1.276.28$$

To clarify the example proposed, let's imagine an investor who deposits $1.000 in a bank in time 0. At the end of the first year, given an interest rate of 5%, the investor will have in his bank account a future value of $1.050. The investor will leave that amount for another year. So, the future value of $1.050 is the present value at the beginning of the second year. The amount at the end of the second years is computed as:

$$FV_2 = \$1.050 * (1+5\%) = \$1.102.50$$

The compounded interest can be computed in different intervals, every year, every 6, 4, 3 months or every day, in order to obtain the daily, trimestral or semiannual interest rate. Each m-period interest rate can be expressed as annual effective interest rate, by equalizing the m compounding formula to the easy formula of the annual compounding.

$$[1+i(m)]^m = 1+i(a) \quad \Leftrightarrow \quad i(a) = \ [1+i(m)]^m - 1$$

where i(m) is the interest of the m period, and i(a) is the effective annual interest rate.
If a consumer goes to a bank to finance the buy a house with a mortgage, the interest the bank offers is a monthly interest rate that compounds 12 times a year. That means that the consumer pays interest every month, but the interest that bank shows is not the periodic monthly interest rate, in fact the financial institution often quote an annual interest rate that we refer to as the *stated annual interest rate* or *quoted interest rate*. It is

denoted by j(m):

$$j(m) = \frac{i(m)}{\frac{1}{m}}$$

where i(m) is the effective rate for m periods and m is the number of periods.

For instance, the bank might state that the mortgage rate is the 10% compounded monthly. So, the monthly interest rate is $10\%/12 = 0.083\%$ and the monthly cash payment will be $PV * 0.083\%$. Finally, the annual effective rate is $(1 + 0.083\%)^{12} - 1 = 10.47\%$.

If the number of compounding periods per year becomes infinite, then interest is said to compound continuously[5]. It is obtained:

$$\lim_{n\to\infty} j(m) = \lim_{n\to\infty} \left((1 + i(a))^{\frac{1}{m}} - 1\right)/(1/m) = \lim_{\left(\frac{1}{m}\right)\to 0} \left((1 + i(a))^{\frac{1}{m}} - 1\right)/(1/m) = \ln\left(1 + i(a)\right) = \delta$$

where i(a) is the annual effective rate and the final step is justified because of the notable limitation:

$$\lim_{x\to 0}(a^x - 1)/x = \ln a$$

For example, an investor finds out an investment proposal whose interest continuously compounding is 6%. The investor decides to finance that proposal with \$3.000 for 1 year. The future value that the investor will gain is $FV = \$3.000e^{6\%*1} = \$3.185.51$. So mathematically, the formula for the interest continuously compounding is:

$$FV = PV\ e^{\delta * t}$$

where t is the maturity of the investment.

[5] The interest continuously compounded is the most famous and useful because of its simplicity of computation.

2. Bond

Bonds are the most traded debt securities in the World, which have standardized income streams. The cash streams depend on the face value "F" and the coupon rate "c" of the bond. The coupon rate is expressed as an annual rate of return: it means that the effective rate is computed by dividing the coupon rate by the number of payments done in one year "*n*". The face value is the value imprinted on the contract and it is included only in the last payment that the holder receives at maturity. Hence, each cash payment is $\frac{CF}{n}$ and the final payment is equal to $F + \frac{Fc}{n}$.

For example, an US 10 years T-bond with face value of $1.000 and a semiannual coupon rate of 10%, paid has the following cash flow:

$100	$100	...	$1,000
t_1	t_2	...	t_{20}

The cash payment is equal to $100 and is paid every 6 months. The last payment consists of the last coupon and the repayment of face value at maturity.

The evaluation of a bond is done using the yield to maturity: in fact, it is used to compute "*Discount function*" of each cash payment in order to compute the present value of the future cash payments:

$$v = \left(1 + \frac{r}{n}\right)^{-n*t}$$

where n is the number of payment at year and t is the number of years.

The price of a bond maturing in T years is then:

$$P = \sum_{t=1}^{n*T} v^t * CFt = \frac{cF}{n} * \frac{1}{1 + \frac{r}{n}} + \frac{cF}{n} * \frac{1}{\left(1 + \frac{r}{n}\right)^2} + \cdots + \frac{cF}{n} * \frac{1}{\left(1 + \frac{r}{n}\right)^{n*T}} + \frac{F}{\left(1 + \frac{r}{n}\right)^{n*T}}$$

$$= \left(\frac{cF}{n}\right) \sum_{t=1}^{n*T} \left(1 + \frac{r}{n}\right)^{-t} + F\left(1 + \frac{r}{n}\right)^{-nt}$$

where the first of part of the of the formula is the sum of discount factors multiplied by cF/n, that is the present value of the sum of thee coupon payments, and where the second part is the present value of the face value F to be received on the maturity date. The factor to compute in an easy way the present value of the sum of the cash payment is obtained by the following demonstration:

$$\sum_{t=1}^{n*T} v^t = v + v^2 + \cdots + v^{n*t} = v * \frac{1 - v^{n*t}}{1 - v} = \frac{1}{1+i} * \frac{1 - v^{n*t}}{1 - \frac{1}{1+i}} = \frac{1}{1+i} * \frac{1 - v^{n*t}}{\frac{i}{1+i}} = \frac{1 - v^{v*t}}{i}$$

So, the value of the bond can be expressed as:

$$P = \frac{cF}{r} * \left(1 - \left(1 - \frac{r}{n}\right)^{-nT}\right) + F\left(1 + \frac{r}{n}\right)^{-nT}$$

In order to standardize in the market, the price of all the quoted bonds having different face values, the bond price is divided by F and it is expressed as a percentage by

$$p = \frac{P}{F} * 100 = 100\frac{c}{r}\left[1 - \left(1 - \frac{r}{n}\right)^{-nT}\right] + 100\left(1 + \frac{r}{n}\right)^{-nT}$$

For example, a bond has a quoted p of 78% and a face value of $10,000. Thus, the price of bond is:

$$P = 10000 * \frac{78}{100} = \$\,7800$$

Given the p for each bond, it's possible to differentiate the bond in par bonds, premium bonds and discount bonds. If $p = 100$, the bond is called "*par bond*"; if $p > 100$, the bond is a "*premium bond*" and it is a "*discount bond*" if $p < 100$.

The equation of "*p*" can be simply expressed as:

$$p = \frac{c}{r}100 + 100\left(1 - \frac{c}{r}\right)\left(1 + \frac{r}{n}\right)^{-nT}$$

this form lets to see that p = 100 when c = r. When c = r, the price of bond cannot vary with maturity, because the price of the bond is always equal to the face value.

In case $c < r$, the bond is a discount bond and it's possible to calculate the discount of the bond:

$$d = 100 - p = 100\left(\frac{c}{r} - 1\right)\left[\left(1 + \frac{r}{2}\right)^{-nT} - 1\right]$$

that is the number of dollars subtracted from the face value of 100 to determine the price.

If $c > r$, the bond is sold at premium. The premium is

$$p - 100 > 0$$

that is, the additional amount the coupon exceeds the interest earned. The excess can be regarded as a partial repayment of the borrowed initial amount, so that the price of the bond is reduced and the result is opposite of the case of discount bonds.

As the time passes, the value of the bond does not remain constant until the maturity because of two elements that are expressed by the equation of P: the time "t" and the yield to maturity "r". The coupon rate and face value remain fixed during all the maturity, because they are previously determined at the beginning of the contract (in fact, this type of security is called fixed income). So, any change in the value could be caused by the change of the time and by the change of yield to maturity. If the yield to maturity does not change, the value of the bond is called the "*amortized value*" of the bond. The changes differ if the bond is a par, premium or discount bond. In case of a premium bond, the price decreases with the increase of the time. In case of discount bond, the price arises with the time. Finally, the par bond value remains the same.

For example, the yield to maturity is 11%, there are three different bonds with face value of $1,000, maturity 10 years, semiannual payment and coupon rates of 6%, 11%, and 15%, So, the first coupon is a discount bond the third is premium bond. The table 1 shows the three different amortized values of the three bonds.

Table 1 – Amortized Value of 6%, 11%, 15%-coupon Bonds at different maturities.

Maturity	6% Bond	11% Bond	15% Bond
10	701.24 $	1,000.00 $	1,239.01 $
9.5	709.81 $	1,000.00 $	1,232.15 $
9	718.85 $	1.000.00 $	1,224.92 $
8.5	728.38 $	1,000.00 $	1,217.29 $
8	738.45 $	1,000.00 $	1,209.24 $
7.5	749.06 $	1,000.00 $	1,200.75 $
7	760.26 $	1,000.00 $	1,191.79 $
6.5	772.07 $	1,000.00 $	1,182.34 $
6	784.54 $	1,000.00 $	1,172.37 $
5.5	797.69 $	1,000.00 $	1,161.85 $
5	811.56 $	1,000.00 $	1,150.75 $
4.5	826.20 $	1,000.00 $	1,139.04 $
4	841.64 $	1,000.00 $	1,126.69 $
3.5	857.93 $	1,000.00 $	1,113.66 $
3	875.11 $	1,000.00 $	1,099.91 $
2.5	893.24 $	1,000.00 $	1,085.41 $
2	912.37 $	1,000.00 $	1,070.10 $
1.5	932.55 $	1,000.00 $	1,053.96 $

1	953.84 $	1,000.00 $	1,036.93 $
0.5	976.30 $	1,000.00 $	1,018.96 $
0	1.000.00 $	1,000.00 $	1,000.00 $

The amortized value of discount bond increases as it approached maturity. The amortized value of premium behaves in opposition to the trend of the discount bond value and the amortized value of the par bond in fixed to the face value.

The second way the price of a bond can change is because of the yield to maturity. Two bond with the same face value and the same coupon rate can have different values only because of yield to maturity. The prices, in table 2, is inversely related to the yield. The two bonds have the same face value of $1,000 and the same coupon rate of 8%.

Table 2 – Prices of 15-year and 7-yeear Maturity Bond at different Yield to Maturity.

YTM	15M Bond	7M Bond
12%	724.70 $	814.10 $
11.50%	739.75 $	819.92 $
11%	755.46 $	825.89 $
10.50%	771.87 $	831.99 $
10%	789.01 $	838.25 $
9.50%	806.93 $	844.65 $
9%	825.67 $	851.21 $
8.50%	845.27 $	857.94 $
8%	865.79 $	864.83 $
7.50%	887.28 $	871.89 $
7%	909.79 $	879.12 $
6.50%	933.39 $	886.54 $
6%	958.13 $	894.14 $
5.50%	984.08 $	901.94 $
5%	1,011.32 $	909.94 $
4.50%	1,039.92 $	918.14 $
4%	1,069.97 $	926.55 $

The second column bond has a maturity of 15 years, while the third column bond has a maturity of 7 years. The table 2 shows that, not only the price of the you bond is inversely related to the yield to maturity, but also that the volatility of the value is related to the maturity: the bigger is the maturity, bigger is the change of the value in case of the same change of the yield to maturity between different bonds.

3. Duration

In previous paragraph, it has been described the basic characteristics of the bond, that is the face value, coupon, maturity, the yield to maturity. The first is the value expressed on the bond, the second is the periodical cash payments, the third the whole time the investment lasts, and finally, the yield to maturity is the rate of return of the investment or the percentage improvement, gained by deciding to invest in that certain bond. At this point given the basic features, a saver who decides to buy a bond knows how much he must pay to make the investment, by calculating the price and by obtaining the relative valuation. Although at the time of trade the bond has a price "P_0", the price changes if the initial conditions change. The main elements that affect the bond price are not only the volatility of the yield to maturity, but the coupon rate and the maturity too. Each factor affects the valuation in a different and distinct way; so, in order to obtain a general law for the bond movement, it is introduced by different examples.

Given a 10-year 7% semi-annual compounding Coupon bond whose face value is $100, its cash flow is:

3.5	3.5	…	103.5
t_1	t_2	…	t_{10}

by considering a range of yield to maturity between the 5% to 9% with steps of 25 basis point[6] in the yield to maturity, the relation between the bond price and the yield to maturity is inversely related, the price change from $115.59 with a YTM of 5% to $86.99 with a YTM equal to 9%. As it is shown in the table below, there is also a further observation: the lower is the level of yield to maturity, the change in the price is larger.

Table 3 – Changes of the Bond Price at different Yield to Maturity

YTM	Price	ΔPrice
5%	115.59 $	-
5.25%	113.48 $	-2.11 $
5.50%	111.42 $	-2.06 $
5.75%	109.41 $	-2.01 $
6.00%	107.44 $	-1.97 $
6.25%	105.52 $	-1.92 $
6.50%	103.63 $	-1.88 $
6.75%	101.80 $	-1.84 $
7.00%	100.00 $	-1.80 $
7.25%	98.24 $	-1.76 $
7.50%	96.53 $	-1.72 $
7.75%	94.85 $	-1.68 $
8.00%	93.20 $	-1.64 $
8.25%	91.60 $	-1.61 $
8.50%	90.03 $	-1.57 $
8.75%	88.49 $	-1.54 $

[6] The one basis point is equal to 0.01%.

9.00%	86.99 $	-1.50 $

In fact, imagine the yield will decrease from 5.25% to 5%. The price gain is $ 2.11. On the other hand, if the reduction of the yield to maturity is from 9% to 8,75%, the price increase is only $ 1.50.
So, the general rule that the example points out as follow is:

"*Given a fixed income stream, the percentage change in its value for a given change in the yield to maturity is larger the smaller the initial yield to maturity.*"[7]

The procedure can be adopted also to understand the movement of the bond price, keep fixed the maturity dates and the yield to maturity and by leaving free to change the coupon rate, that is by studying different bond that differentiate themselves in the coupon rate. So, the general rule states:

"Given a fixed income stream with a given maturity and yield to maturity, the percentage change in its value is smaller the larger the coupon rate for any given change in the yield to maturity."[8]

Finally, it remains the change movement caused by the maturity. The percentage change of a bond price fluctuates with the maturity of the bond for a given change in the yield to maturity. So, the sensitivity of the bond price is greater the longer is the maturity to a given yield to maturity. In the table below, there are the prices of bond with maturity from 5 to 40, whose coupon rate is 6% semiannual compounding with face value of $ 1,000 and a yield to maturity equal to 4%. The 5-years bond price improves until a percentage of 8,89%, the 15-years one until $ 1,223.96. This inductive rule does not work for some discount bond. The trend of this discount bond has a maximum at some maturity and then, by tending the maturity to infinitive, the percentage increase, decrease at a limitation point.

Table 4 – Changes due to Maturity

Maturity	Price 6%	Change
5	1,089.83 $	8.98%
10	1,163.51 $	16.35%
15	1,223.96 $	22.40%
20	1,273.55 $	27.36%
30	1,347.61 $	34.76%
35	1,374.99 $	37.50%
40	1,397.45 $	39.74%

The following rule:
"For a par or premium bonds and mortgages the percentage increase in the price for a given decrease in the yield to maturity increases with maturity. For some discount bonds the percentage increase in price first increases with maturity and then decreases with maturity once maturity is large enough."[9]

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[7] BIERWAG, Gerald O., *Duration Analysis: Managing Interest Rate risk*, (Cambridge, Mass.: Ballinger Publishing Company, 1987), p. 53.
[8] BIERWAG, Gerald O., *Duration Analysis: Managing Interest Rate risk*, (Cambridge, Mass.: Ballinger Publishing Company, 1987), p. 55.
[9] BIERWAG, Gerald O., *Duration Analysis: Managing Interest Rate risk*, (Cambridge, Mass.: Ballinger Publishing Company, 1987), p. 57.

Although the three rules must be never forgotten, they can be summarized in one indicator called Duration. It is the weighted average of the times on which cash flows are promised, whose weighted are the percentage of the present value of each cash payment at the time considered over the price of the bond.

$$D = \sum_{t=1}^{T} w_t * t$$

where the w_t is

$$\begin{cases} w_t = \dfrac{\dfrac{c * FV}{n} * \dfrac{1}{(1+i_n)^{t*n}}}{P} & t < T \\ w_t = \dfrac{FV * \left(1 + \dfrac{c}{n}\right) * \dfrac{1}{(1+i_n)^{T*n}}}{P} & t = T \end{cases}$$

Suppose there is a \$10,000 face value 5% semi-annual compounding bond with yield to maturity of 7% and maturity 3 years. The calculation of the duration is pointed out in the table below. The first column shows the semesters the coupons are paid. The second column shows the cash flows of the bond. The third one shows the present value of the cash flow at t_0. The second step is the computation of the weights: they are obtained by dividing each present value from the bond price. Finally, the last step consists of multiplying the present value and the weight and of summing all the factors, The computed duration is expressed in number of semesters; if the investment requires computation in years, it is necessary to divide by two because a semester are two part of one year.

Table 5 – Computation of the duration of 5%-coupon Bond with 10,000 face value and with 7% YTM

Time	Cash Flow	Present Value	w_t	$PV * w_t$
1	250.00 $	241.55 $	0.025514123	0.025514123
2	250.00 $	233.38 $	0.024651327	0.049302653
3	250.00 $	225.49 $	0.023817707	0.07145312
4	250.00 $	217.86 $	0.023012277	0.092049108
5	250.00 $	210.49 $	0.022234084	0.111170421
6	10,250.00 $	8,338.38 $	0.880770483	5.284622895
	9,467.14 $	9,467.14 $	1	
			Duration	2.81705616

The duration can also be expressed in a continuous time, by using the continuous compounding. So:

$$D_\delta = \frac{\sum_{t=1}^{T*n} t * CF_t e^{-\delta t}}{\sum_{t=1}^{T*n} CF_t e^{-\delta t}}$$

In addition to the function of the index of the characteristics of the bond, there are two more ways of how to use the duration: the first one is to consider the best moment in which to sell the bond in case of changing of the yield to maturity; the second as index of the volatility of the bond.

According the first function, in case of changing the yield to maturity, the duration is the best moment in which to sell the bond. Suppose the Fed improves the treasury rate by improving it by 50 basis point. The same investor knows that has suffered a loss because of the interest market movement. In fact, the yield to maturity is higher, it means that the discount factors are lower and so the price of the bond is lower than before.

To prove that suppose a bond, formed by two cash flow, the holder wants to know the price in the moment "*h*" between t_1 and t_2, it is represented below:

$$P_h = CF_1 e^{\delta*(h-1)} + CF_2 e^{-\delta*(2-h)} = CF_1 e^{-\delta*(1-h)} + CF_2 e^{-\delta*(2-h)} = (CF_1 e^{-\delta} + CF_2 e^{-\delta*2}) * e^{\delta h} = P_0 * e^{\delta h}$$

where t_1 is equal to 1 and t_2 is equal to 2.

Now, suppose that be the interest rate change of Δδ: the price will be:

$$P_h^* = CF_1 e^{(\delta+\Delta\delta)*(h-1)} + CF_2 e^{-(\delta+\Delta\delta)*(2-h)} = CF_1 e^{-(\delta+\Delta\delta)*(1-h)} + CF_2 e^{-(\delta+\Delta\delta)*(2-h)}$$

what is necessary to prove is that there is a moment "*h*", such that the value P_h^* is at least the same of P_h.

$$P_h^* = CF_1 e^{-(\delta)*(1-h)} e^{-\Delta\delta*(1-h)} + CF_2 e^{-(\delta)*(2-h)} e^{-\Delta\delta*(2-h)} = \left(CF_1 e^{-(\delta)} e^{-\Delta\delta*(1-h)} + CF_2 e^{-(2\delta)} e^{-\Delta\delta*(2-h)}\right) * e^{\delta h}$$

by computing the first derivative:

$$\frac{dP_h^*}{d\delta} = \left(-(1-h) * CF_1 e^{-(\delta)} e^{-\Delta\delta*(1-h)} - (2-h) * CF_2 e^{-(2\delta)} e^{-\Delta\delta*(2-h)}\right) * e^{\delta h} = \left(-(1-h) * PV_1 e^{-\Delta\delta*(1-h)} - (2-h) * PV_2 e^{-\Delta\delta*(2-h)}\right) * e^{\delta h}$$

and the second derivative:

$$\frac{dP_h^*}{d\delta^2} = \left((1-h)^2 * PV_1 e^{-\Delta\delta*(1-h)} + (2-h)^2 * PV_2 e^{-\Delta\delta*(2-h)}\right) * e^{\delta h} \geq 0$$

the second derivative is higher than zero, so the function is concave. In order to $P_h^* > P_h$, the function needs a minimum in $\delta = 0$. So, it means:

$$\frac{dP_h^*}{dy}(0) = 0$$

$$\frac{dP_h^*}{dy} = (-(1-h)*PV_1 - (2-h)PV_2)*e^{\delta h} = hPV_1 - PV_1 + hPV_2 - 2PV_2 = 0$$

$$h = PV_1 + 2PV_2/(PV_1 + PV_2)$$

where

$$P = PV_1 + PV_2$$

$$PV_1 + 2PV_2 = t_1 w_1 + t_2 w_2$$

So:

$$h = D$$

The third way to interpret the duration is to consider the duration as an index of the volatility of the bond price. The volatility is the relative variation of the price function, keeping fixed the yield to maturity

$$P = \sum CF_t e^{-\delta * t}$$

$$\frac{dP}{d\delta} = \sum -t * CF_t e^{-\delta * t} \cong \lim_{\Delta\delta \to 0} \frac{P(\delta + \Delta\delta) - P(\delta)}{\Delta\delta}$$

$$\frac{\frac{dP}{d\delta}}{P} \cong \frac{P(\delta + \Delta\delta) - P(\delta)}{\Delta\delta * P} \cong \frac{\sum -t * CF_t e^{-\delta * t}}{\sum CF_t e^{-\delta * t}} = -D$$

It is the elasticity, that is the variation expressed as ratio over the value of the price before the variation. So, the variation of a bond can be expressed as a function whose independent variable is the duration.

$$P(\delta + \Delta\delta) \cong P(\delta) - D * \Delta\delta * P(\delta)^{10}$$

$$\frac{P(\delta + \Delta\delta) - P(\delta)}{P(\delta)} \cong -D * \Delta\delta$$

This expression points out that the percentage variation of the bond price depends on the variation of the interest rate and on the value of the bond duration. "The larger the duration of a security, the larger the

[10] In discrete time, the formula is different: $P(i + \Delta i) \cong P(i) - \frac{D}{1+i} * \Delta i * P(i)$. The term $-\frac{D}{1+i}$ is called the Modified Duration.

percentage change in the security price for a given change in the yield to maturity.[11]" Moreover, the equation shows that the variation of the price is directly and linear related to the change of the yield to maturity. As argued before, that is not true because of the general rule between the price variation and the variation of the yield to maturity. The result of the equation is a good approximation in order to foresee the trend of the price, given small oscillation of the yield to maturity.

The result before can be improved by writing the other elements of the Taylor's polynomial[12]:

$$P(\delta + \Delta\delta) \cong P(\delta) - D * \Delta\delta * P(\delta) + \frac{\sum t^2 * CF_t e^{-\delta * t}}{\sum CF_t e^{-\delta * t}} * \left(P(\delta) * \frac{\Delta\delta^2}{2!}\right)$$

$$= P(\delta) - D * \Delta\delta * P(\delta) + C * \left(P(\delta) * \frac{\Delta\delta^2}{2!}\right)$$ [13]

where C is called Convexity of the bond. It is the second derivative of price function or the second duration, whose definition is the squared weighted average of the periods of time.

The duration is also an important tool to assess, not only the sensitivity of a bond, but also the bond portfolio one. There two different way to calculate the duration of a portfolio: the first, through the computation of the weighted average of the durations of the security; the second, by considering the cash flow of each bond as a unique bond and then by calculating the duration of the pseudo-bond.

Suppose the portfolio has two kind of bond: the short-time bond with maturity 3 years, face value \$100 and semiannual compounded 10% interest rate and the long-time semiannual compounded 10% interest rate with maturity 10 years and face value \$1,000.

The cash flows of the two bonds are:

5	5	...	105
t_1	t_2	...	t_6

50	50	...	1050
t_1	t_2	...	t_{20}

according the second method, the portfolio cash flow is:

55	...	155	50	...	1050
t_1	...	t_6	t_7	...	t_{20}

[11] BIERWAG, Gerald O., *Duration Analysis: Managing Interest Rate risk*, (Cambridge, Mass.: Ballinger Publishing Company, 1987), p. 70.

[12] Taylor's polynomial: $T_n(f, x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2(\frac{1}{2!}) + \cdots + f^{(n)}(x_0)(x - x_0)^n(\frac{1}{n!})$

[13] In discrete time, there is the Modified Convexity $C = \frac{1}{(1+i)^2} * \left(D(i) + C(i)\right)$

so, the duration of the portfolio with one 3-years bond and one 10-years bond is 12.38.

According the first method, the first step is to compute the durations of each bond. (In this case they are 5.33 and 13.09.) For the second step, it is necessary the weight for each bond, that is:

$$w_{B_n} = p_n * \frac{n}{V}$$

where p is the price of the bond, n is the number of the bonds, and V is the value of the portfolio. So, the duration of the portfolio is $D = \frac{100}{1100} * 5.33 + \frac{1000}{1100} * 13.09 = 12.38$.

II. IMMUNIZATION

1. Investment Accumulation and duration

By investing in fixed-income security, the investor will face two types of interest risk: Price risk and Reinvestments risk. The price risk is the risk of changing the value of the security because of the change of the interest rate and the reinvestment risk is the "level of earnings on any reinvestments of cash flow during the period". In case of increasing the interest rate, the value of the fixed-income security decrease, but that improvement implies greater rate of returns for reinvestment of the future cash flows; on the other side, if the yield decreases, the bond value improves but the rate of return for the future reinvestments decreases too. So, the change of the yield affects the investment through two ways, and according to the different securities, one may prevail on the other. The price risk and the reinvestment risk can be described by the duration of the investment in the fixed-income. In fact, "the time required for investment accumulation to offset any capital gain or loss from any change in yield is exactly equal to the initial duration of the portfolio.[14]" The price risk change has been explained by the approximation of the percentage change, computed with the duration and the change of the yield: $\frac{P(\delta+\Delta\delta)-P(\delta)}{P(\delta)} \cong -D * \Delta\delta$.

Regarding the reinvestment risk, in the previous chapter, the duration is also the best time to leave the investment to obtain the yield, fixed before, in case the yield changes. To prove this result, the assumption was that the value in time "h" between the time 1 and time 2 was:

$$P_h = CF_1 e^{\delta*(h-1)} + CF_2 e^{-\delta*(2-h)}$$

that is, the cash flow in time 1 is capitalized until the time "h". The amount received in time 1 is reinvested at the continuously compounding interest rate. It means that duration can be used to analyze the reinvestment risk too.

This result can also obtain in a different way. Suppose the value of the investment is $V = V(r)$ where r is the yield to maturity at which all the securities are priced and the value of the investment is a function of r. Imagine that instantly at the time of pricing, the r increases or decreases so that the $V = V(r')$, where r' is the new yield to maturity. If $r' > r$, there will be an instant loss in the value of the investment; on the other side, if $r' < r$, the investment value will increase. Finally, in case the yield to maturity does not vary, the investment grows at the initial yield to maturity *r*: so, the value at time k is $V(r)_k = V(r) * (1+r)^k$. On the other hand, the change in the yield to maturity allows the investment to grows at a different rate of return. If the yield increases, the investment value will suffer an instant loss but, then it will grow at greater rate, whereas, in case the yield decreases, the investment arises instantly but, then it will grow at a lower rate of return: at t_k, the value is $V(r')_k = V(r') * (1+r')^k$. The two different functions are the mathematical

[14] BIERWAG, Gerald O., *Duration Analysis: Managing Interest Rate risk*, (Cambridge, Mass.: Ballinger Publishing Company, 1987), p. 91.

representation of the trend of the bond value in the case of the yield to maturity changes. The two functions have not the same features. In fact, the slopes are different because the grow rates are different and there are two intersection with the y-axis. Given this features, it's possible to imagine that, during the maturity the two functions intersect in a point in time. It means that, in spite of the change of the yield to maturity, the two functions have the same value. As demonstrated in the previous chapter, that point in time is the duration.
For example, suppose that the mutual fund wants to invest $1,000,000 in two kinds of bond with face value of $1,000 and coupon rate of 10%. The first bond maturity is 10 years and the second bond maturity is 20 years. The mutual fund wants that the duration of the investment is the 17 semesters that is 8 years and a semester. Otherwise, the bonds are par bonds at the purchase. The duration of the two bonds are 13.085 and 18.017, so the number of Bond A the fund purchases is given by the follow equation:

$$D = D_A * B_A + (1 - B_A) * D_B$$

where $B_A = P * n/V$, D is the duration objective and $D_A$and D_B are the durations of the two bonds.
The datum that is looked for is B_A;so,

$$B_A = \frac{D - D_B}{D_A - D_B}$$

By substituting:

$$B_A = \frac{D - D_B}{D_A - D_B} = P * \frac{n}{V}$$

So, the number of bonds A that the fund purchases is:

$$n_A = \frac{D - D_B}{D_A - D_B} * \frac{V}{P}$$

$$n_A = \frac{17 - 18.017}{13.085 - 18.017} * \frac{\$1{,}000{,}000}{\$1{,}000} = 206$$

$$n_B = 794$$

So, the value to invest in bond A is $206,000 and the value for bond B is $794,000. The table shows how the value of the portfolio changes when the yield to maturity changes. In K = 0 the value of portfolio decreases if the yield goes up and vice versa when the yield goes down, the value increases. Moreover, in k = 17, both the yield increases or decreases, the value of the portfolio does not decrease from the value fixed at the beginning of the investment. It means that any change in the yield does not affect the profit that was expected at the time of the duration.

Table 6 – Values of the Portfolio at different yield to maturity and at different Maturity

K/Yield	7%	7,50%	8%	8,50%	9%	9,50%	10%	10,50%	11%	11,50%	12%	12,50%	13,50%
0	1.298.231,20 €	1.239.732,64 €	1.185.136,78 €	1.134.135,24 €	1.086.446,41 €	1.041.813,06 €	1.000.000,00 €	960.792,13 €	923.992,59 €	889.421,07 €	856.912,39 €	826.315,06 €	770.309,96 €
1	1.343.669,29 €	1.286.222,61 €	1.232.542,25 €	1.182.335,98 €	1.135.336,50 €	1.091.299,18 €	1.050.000,00 €	1.011.233,72 €	974.812,18 €	940.562,78 €	908.327,13 €	877.959,75 €	822.305,88 €
2	1.390.697,72 €	1.334.455,96 €	1.281.843,94 €	1.232.585,26 €	1.186.426,65 €	1.143.135,89 €	1.102.500,00 €	1.064.323,49 €	1.028.426,85 €	994.645,14 €	962.826,76 €	932.832,23 €	877.811,53 €
3	1.439.372,14 €	1.384.498,06 €	1.333.117,70 €	1.284.970,14 €	1.239.815,84 €	1.197.434,85 €	1.157.625,00 €	1.120.200,47 €	1.084.990,32 €	1.051.837,24 €	1.020.596,37 €	991.134,25 €	937.063,81 €
4	1.489.750,16 €	1.436.416,74 €	1.386.442,41 €	1.339.581,37 €	1.295.607,56 €	1.254.313,00 €	1.215.506,25 €	1.179.011,00 €	1.144.664,79 €	1.112.317,88 €	1.081.832,15 €	1.053.080,14 €	1.000.315,61 €
5	1.541.891,42 €	1.490.282,37 €	1.441.900,11 €	1.396.513,58 €	1.353.909,90 €	1.313.892,87 €	1.276.281,56 €	1.240.909,07 €	1.207.621,36 €	1.176.276,16 €	1.146.742,08 €	1.118.897,65 €	1.067.836,92 €
6	1.595.857,62 €	1.546.167,95 €	1.499.576,11 €	1.455.865,40 €	1.414.835,84 €	1.376.302,78 €	1.340.095,64 €	1.306.056,80 €	1.274.040,53 €	1.243.912,04 €	1.215.546,60 €	1.188.828,75 €	1.139.915,91 €
7	1.651.712,64 €	1.604.149,25 €	1.559.559,15 €	1.517.739,68 €	1.478.503,46 €	1.441.677,16 €	1.407.100,42 €	1.374.624,78 €	1.344.112,76 €	1.315.436,98 €	1.288.479,40 €	1.263.130,55 €	1.216.860,24 €
8	1.709.522,58 €	1.664.304,85 €	1.621.941,52 €	1.582.243,62 €	1.545.036,11 €	1.510.156,83 €	1.477.455,44 €	1.446.792,58 €	1.418.038,96 €	1.391.074,61 €	1.365.788,16 €	1.342.076,21 €	1.298.998,30 €
9	1.769.355,87 €	1.726.716,28 €	1.686.819,18 €	1.649.488,97 €	1.614.562,74 €	1.581.889,28 €	1.551.328,22 €	1.522.749,19 €	1.496.031,10 €	1.471.061,40 €	1.447.735,45 €	1.425.955,97 €	1.386.680,69 €
10	1.831.283,32 €	1.791.468,14 €	1.754.291,95 €	1.719.592,25 €	1.687.218,06 €	1.657.029,02 €	1.628.894,63 €	1.602.693,53 €	1.578.312,81 €	1.555.647,43 €	1.534.599,58 €	1.515.078,22 €	1.480.281,63 €
11	1.895.378,24 €	1.858.648,20 €	1.824.463,63 €	1.792.674,92 €	1.763.142,87 €	1.735.737,89 €	1.710.339,36 €	1.686.834,94 €	1.665.120,02 €	1.645.097,15 €	1.626.675,55 €	1.609.770,61 €	1.580.200,64 €
12	1.961.716,48 €	1.928.347,50 €	1.897.442,17 €	1.868.863,61 €	1.842.484,30 €	1.818.185,44 €	1.795.856,33 €	1.775.393,77 €	1.756.701,62 €	1.739.690,24 €	1.724.276,08 €	1.710.381,27 €	1.686.864,19 €
13	2.030.376,56 €	2.000.660,54 €	1.973.339,86 €	1.948.290,31 €	1.925.396,10 €	1.904.549,25 €	1.885.649,14 €	1.868.601,94 €	1.853.320,21 €	1.839.722,43 €	1.827.732,65 €	1.817.280,10 €	1.800.727,52 €
14	2.101.439,74 €	2.075.685,31 €	2.052.273,45 €	2.031.092,65 €	2.012.038,92 €	1.995.015,34 €	1.979.931,60 €	1.966.703,55 €	1.955.252,82 €	1.945.506,47 €	1.937.396,61 €	1.930.860,11 €	1.922.276,63 €
15	2.174.990,13 €	2.153.523,50 €	2.134.364,39 €	2.117.414,09 €	2.102.580,67 €	2.089.778,57 €	2.078.928,18 €	2.069.955,48 €	2.062.791,73 €	2.057.373,09 €	2.053.640,41 €	2.051.538,86 €	2.052.030,30 €
16	2.251.114,78 €	2.234.280,64 €	2.219.738,97 €	2.207.404,19 €	2.197.196,80 €	2.189.043,05 €	2.182.874,59 €	2.178.628,15 €	2.176.245,27 €	2.175.672,04 €	2.176.858,83 €	2.179.760,04 €	2.190.542,34 €
17	2.329.903,80 €	2.318.066,16 €	2.308.528,53 €	2.301.218,86 €	2.296.070,66 €	2.293.022,60 €	2.292.018,32 €	2.293.006,12 €	2.295.938,76 €	2.300.773,19 €	2.307.470,36 €	2.315.995,04 €	2.338.403,95 €
18	2.411.450,43 €	2.404.993,64 €	2.400.869,67 €	2.399.020,67 €	2.399.393,84 €	2.401.941,17 €	2.406.619,23 €	2.413.388,94 €	2.422.215,39 €	2.433.067,64 €	2.445.918,58 €	2.460.744,73 €	2.496.246,22 €
19	2.495.851,20 €	2.495.180,90 €	2.496.904,45 €	2.500.979,04 €	2.507.366,56 €	2.516.033,38 €	2.526.950,20 €	2.540.091,86 €	2.555.437,24 €	2.572.969,03 €	2.592.673,70 €	2.614.541,28 €	2.664.742,84 €
20	2.583.205,99 €	2.588.750,19 €	2.596.780,63 €	2.607.270,65 €	2.620.198,06 €	2.635.544,96 €	2.653.297,71 €	2.673.446,69 €	2.695.986,29 €	2.720.914,75 €	2.748.234,12 €	2.777.950,11 €	2.844.612,98 €
21	2.673.618,20 €	2.685.828,32 €	2.700.651,86 €	2.718.079,66 €	2.738.106,97 €	2.760.733,35 €	2.785.962,59 €	2.813.802,64 €	2.844.265,53 €	2.877.367,35 €	2.913.128,16 €	2.951.571,99 €	3.036.624,36 €
22	2.767.194,83 €	2.786.546,88 €	2.808.677,93 €	2.833.598,04 €	2.861.321,78 €	2.891.868,18 €	2.925.260,72 €	2.961.527,28 €	3.000.700,14 €	3.042.815,97 €	3.087.915,85 €	3.136.045,24 €	3.241.596,50 €
23	2.864.046,65 €	2.891.042,39 €	2.921.025,05 €	2.954.025,96 €	2.990.081,26 €	3.029.231,92 €	3.071.523,76 €	3.117.007,46 €	3.165.738,64 €	3.217.777,89 €	3.273.190,81 €	3.332.048,07 €	3.460.404,26 €
24	2.964.288,29 €	2.999.456,48 €	3.037.866,05 €	3.079.572,06 €	3.124.634,92 €	3.173.120,44 €	3.225.099,94 €	3.280.650,35 €	3.339.854,27 €	3.402.800,12 €	3.469.582,25 €	3.540.301,07 €	3.693.981,55 €
25	3.068.038,38 €	3.111.936,10 €	3.159.380,69 €	3.210.453,87 €	3.265.243,49 €	3.323.843,66 €	3.386.354,94 €	3.452.884,49 €	3.523.546,25 €	3.598.461,13 €	3.677.757,19 €	3.761.569,89 €	3.943.325,31 €
26	3.175.419,72 €	3.228.633,70 €	3.285.755,92 €	3.346.898,16 €	3.412.179,45 €	3.481.726,23 €	3.555.672,69 €	3.634.160,93 €	3.717.341,30 €	3.805.372,64 €	3.898.422,62 €	3.996.668,01 €	4.209.499,76 €

If an investor should invest in some fixed-income security, it must decide the period his money are fixed in that investment. The *planning period* is the period in which the investment is active. A portfolio of fixed-income securities is said to be immunized when the portfolio return is immune from changes in interest rate, namely the rate of return can fall below the level of the return: so, the planning period should be equal to the duration. This results can state the follow definition and rules:

"If a portfolio of securities is selected so that its duration is exactly equal to the length of the planning period, the portfolio is immunized so that the annual realized rate of return can never fall below the initial yield to maturity at which the securities were purchased.[15]"

"If the portfolio duration exceeds the length of the planning period, capital gain or losses incorporated into the annual realized rate of return and resulting from initial yield changes will dominate the reinvestment return over the planning period.[16]"

"If the portfolio duration is less than the planning period, the reinvestment return incorporated into the realized rate of return will dominate any initial capital gain or loss resulting from yield changes.[17]"

[15] BIERWAG, Gerald O., *Duration Analysis: Managing Interest Rate risk*, (Cambridge, Mass.: Ballinger Publishing Company, 1987), p. 96.
[16] BIERWAG, Gerald O., *Duration Analysis: Managing Interest Rate risk*, (Cambridge, Mass.: Ballinger Publishing Company, 1987), p. 96.
[17] BIERWAG, Gerald O., *Duration Analysis: Managing Interest Rate risk*, (Cambridge, Mass.: Ballinger Publishing Company, 1987), p. 97.

2. Immunization Fisher and Weil and Redington

The immunization refers to the problem of the financial equilibrium, formed by complex portfolios of positive and negative cash flows whose describes the investor's behavior.

The variable of the problem are the cash flows of the portfolio, that are known for certain at time of the problem; and the structure of the interest rate that is forecasted by the market information at the time of the valuation. A portfolio is in a financial equilibrium if the present value of the positive cash flows and the negative ones are equal at time of valuation, forecasted the market interest rate structure.

So, the immunization is the methods that allows to express the distribution of the positive and negative cash flows similar in order to be equally vulnerable to any shift in the interest rate structure.

The starting assumption of the classic approach to this problem is that the interest rate is subjected to only additive shift. It means that, given the interest rate structure, the trend of the interest rate remains constant and it suffers parallel shift. Formally, the hypothesis is:

$$\delta(t', s) = \delta(t, s) + Z(t, t') \quad \forall\, t' \geq t \quad \forall\, s \geq t'$$

where Z is a random variable that means the additive shift, suffered in the time interval between t and t'. So, the interest rate in t' will assume the same value that is assumed in t with the addition of the term Z that is independent by the interest structure and it do not modify the form of curve $\delta(t, s)$.

The formal definition of the financial equilibrium is the second tools to introduce then the immunization theorems. Given positive cash flow x with flow ($x_1\, x_2 \ldots x_n$) and negative cash flow y with payment ($y_1\, y_2 \ldots y_n$) that are in financial equilibrium in time t:

$$W(t, x) = W(t, y)$$

where the W means that the present value of the positive cash flow is equal to negative one; the two cash flows are called immunized if the post-additive shift value of the positive cash flow x is not minor of the post-additive shift value of the negative cash flow y:

$$W(t^{+}, x) \geq W(t^{+}, y)$$

where t+ is the instant in which the shift is effective, or equally, if the net value of the portfolio is not negative at the time of the shift:

$$W_N(t^{+}) = W(t^{+}, x) - W(t^{+}, y)$$

where $W_N(t^{+})$is the net value of the portfolio.

At this point, the dissertation will point out the different strategies to achieve the financial equilibrium. The research in this field aims to find a different solution to the maturity matching, that is the selection of the matching of the positive and negative cash flows at the same time t for each different flow.

The first theorem is the Fisher and Weil's theorem. Given that $\delta(t,s)$ is the continuous interest rate, that $L > 0$ is a payment owed at time H > t, and x is the positive cash flow at time (t_1 t_2 …t_n); it states that a portfolio is in financial equilibrium at time t if the investment flow present value is equal the value of the only negative payment of the liability:

$$W(t,x) = W(t,L)$$ [18]

and it is immunized if the post-shift value of x is not minor of the present value of L:

$$W(t^{+},x) \geq W(t^{+},L)$$

if and only if the duration of the cash flow x, calculated in time t, is equal to the maturity of L:

$$D(t,x) = H - t$$ [19]

The first theorem gives an operative rule in order to choose portfolio with immunized value at time t with effect in time t^{+}. Moreover, this results is a useful arbitrage strategy[20] that can be applied to bond with similar features such as the all the types of the US-government bond.

The improve the result of the first theorem by analyzing the result during the time, it necessary to introduce the following theorem whose objective is to guarantee that the immunization condition, in case of no change, will remain valid until the next movement of the interest rate structure.

Suppose the $\delta(t,s), s \geq t$ is the continuous interest rate, that L > 0 is the payment in time H > t and that the x is the cash flow at time (t_1 t_2 …t_n). Suppose that the budget constrain and the duration condition are satisfied. Then, if the interest rate structure does not suffer any random perturbation until the instant $t' < t_1$, for all instant $t \in [t_0, t']$, the budget constraint condition and the duration condition are still satisfied.

The result is an important rule of managing the immunized portfolio for discrete periods[21].

According to Fisher and Weil, an example of immunization is the following

[18] This condition is called Budget constraint condition

[19] This is the duration condition

[20] It is the practice of taking advantage of a price difference between two or more markets: striking a combination of matching deals that capitalize upon the imbalance, the profit being the difference between the market prices.

[21] The assumption in the dissertation is that any result is in continuous time, that is by considering instant periods whose interval length is infinitesimal.

The investor wants to build a portfolio to cover an payment L= \$1000 at maturity 5 years, by choosing two zero-coupon bond with maturity $t^1 = 4$ and $t^2 = 7$. Given a flat yield curve, whose continuously compounding interest is $\delta = 15\%$, the condition that must be followed are:

$$\begin{cases} \alpha_1 v(t,t_1) + \alpha_2 v(t,t_2) = L\, v(t,H) \\ (t_1 - t)\alpha_1 v(t,t_1) + (t_2 - t)\alpha_2 v(t,t_2) = (H - t)L\, v(t,H) \end{cases}$$

in order to know the proportion of the zero-coupon bonds:

$$\begin{cases} \alpha_1 = L\dfrac{v(t,H)(t_2 - H)}{v(t,t_1)(t_2 - t_1)} \\ \alpha_2 = L\dfrac{v(t,H)(H - t_1)}{v(t,t_2)(t_2 - t_1)} \end{cases}$$

So, the values are: $\alpha_1 = 573.80\ and\ \alpha_2 = 449.95$.

Suppose that in $T = 0.25$ the yield curve changes and there is an additive shift Y = -3%. The portfolio net value and the duration condition change:

$$W_N = \alpha_1 v(t-T,t_1) + \alpha_2 v(t-T,t_2) - L\, v(t-T,H) = 0.060872$$

$$D_N = \frac{\sum_{k=1}^{2}(t_k - T)\alpha_k e^{-\delta(t_k - T)}}{\sum_{k=1}^{2}\alpha_k e^{-\delta(t_k - T)}} - (H - T) = 0.514178$$

The change in the yield curve improves the value of the portfolio and mismatch the duration and liability maturity, too. In order to rematch the condition the new proportion of the portfolio are:

$$\alpha_1 = 591.28\ \ \alpha_2 = 423.75$$

By introducing the Redington's theorem, the immunization strategy, achieved by Fisher and Weil is only an simplified case, in fact the assumption is that the negative cash flow consists of only one payment. Generally, the banks and institutional funds build portfolio with numerous assets and liabilities in order to maintain the structure of the portfolio as more flexible as possible, for example a bank can enter in a short position[22] of a mortgage backed security[23]or apply for a loan to the central bank and so it is obligated to a sequence of payment. In those cases, the yield risk can generate losses of millions so the bank really need to immunized their portfolio effectively.

[22] A short position is taken when the bond is sold
[23] A financial strument in which are pooled a totality of mortgage in order to form a diversified portfolio of mortgage.

In Redington's theorem, suppose that the $\delta(t,s)$ is the continuous interest rate, that x and y are the positive and negative cash flows at time (t_1 t_2 …t_n) and their present value are equal in time t:

$$W(t,x) = W(t,y)$$ [24]

If the yield curve suffers an additive shift in time t^+, then the post shift positive value of x is not minor than the post-shift negative value of y:

$$W(t^+,x) \geq W(t^+,y)$$

if the duration of x is equal to the duration of y

$$D(t,x) = D(t,y)$$ [25]

and the convexity of the x is not minor than the convexity of y:

$$C(t,x) \geq C(t,y)$$ [26]

[24] This condition is called Budget constraint condition.
[25] This is the duration condition.
[26] This is the convexity condition. It is the second derivative of price function or the second duration, whose definition is the squared weighted average of the periods of time.

3. Master Classic Immunization

The Fisher and Weil's theorem states the condition to immunize a portfolio of a positive cash flow and of a payment at maturity. The Redington's theorem states the general condition to immunize a portfolio formed by positive and negative cash flows. The first theorem can be considered as a special case of the second one: the duration of a single payment or of a zero-coupon bond is the maturity. Stated this observation, the problem of the immunization of a negative cash flow can be divided in many operations, formed by a single payment of the starting cash flow. The sufficient condition for the immunization is that the starting portfolio can be decompose in m portfolios that immunize the different payments of the flow y such as the Fisher and Weil's theorem. It means that the following properties are valid:

$$\sum_{j=1}^{m} c_{kj} = x_k \qquad k = 1, 2, 3, \dots, m$$

$$\sum_{k=1}^{m} c_{kj} v(t, t_k) = y_j v(t, t_j) \qquad j = 1, 2, 3, \dots, m$$

$$\frac{\sum_{k=1}^{m} t_k c_{kj} v(t, t_k)}{y_j v(t, t_j)} = t_j \qquad j = 1, 2, 3, \dots, m$$

where the first expression means that the generic positive cash flow x is expressed as the sum of the c components; where the followed two expressions represent the budget constraints and the duration conditions that must be satisfied simultaneously. This approach is not specified in the Redington's theorem and, hence, the in order to reformulate the decomposition criterion in a formal and mathematical way, the two mathematicians Oldrich Vasicek and Gifford Fong introduced the general immunization theorem for additive shift in 1982.

Suppose that the $\delta(t, s)$ is the continuous interest rate, that x and y are the positive and negative cash flows at time (t_1 t_2 …t_n) and their present value are equal in time t. If the yield curve suffers at time t^+an additive shift with random change, then the post-shift value of the flow x is not minor than the post-shift value of y:

$$W(t^+, x) \geq W(t^+, y)$$

if the following conditions are valid:

$$W(t, x) = W(t, y)$$ [27]

[27] This condition is called Budget constraint condition.

$$D(t,x) = D(t,y)$$ [28]

$$\sum_{k=1}^{m} |t_j - t_k| x_k v(t,t_k) \geq \sum_{k=1}^{m} |t_j - t_k| y_k v(t,t_k) \qquad j = 1,2,3,\dots,m$$

this theorem has two important conclusions that allow the theorem to be the general immunization theorem. The first one is the fact that the results are valid to for convex shifts. These shifts are a general class of shifts between which there are the additive shift. The proof is given by the Karamata's theorem. Given a shift equal to Y(s), it is a convex shift if the function:

$$f(s) = e^{-\int_t^s Y(u)du}, \qquad s \geq t$$

has the second derivative not negative:

$$\frac{d^2}{ds^2} f(s) = \frac{d^2}{ds^2} e^{-\int_t^s Y(u)du} = [Y^2(s) - Y'(s)] e^{-\int_t^s Y(u)du} \geq 0$$

So,

$$Y^2(s) \geq Y'(s)$$

The second conclusion is that the Redington's theorem is contained in the general immunization theorem which is demonstrate by the equivalence theorem. So, by satisfying the third condition of the general immunization theorem, is also satisfied the convexity condition of Redington's theorem.

Vasicek and Fong did not stop only at the previous point: the first assumption of the classic immunization theory is that any change in yield curve is additive. During the years, many mathematicians built models in order to improve the immunization theory and introduced model based on different yield trend. The modelling was not improved but there was added only different models that used only different condition to explain the different movement. Vasicek and Fong in 1982-83 published a paper which points out a new model that explains any yield curve movement based on any random shift.

Suppose that the $\delta(t,s)$ is the continuous interest rate, that x and y are the positive and negative cash flows at time (t_1 t_2 …t_n) and their present value are equal in time t.

[28] This is the duration condition.

$$W(t, x) = W(t, y)$$[29]

If the yield curve suffers at time t^{+}a shift such that:

$$\delta(t^{+}, s) = \delta(t, s) + Y(s)$$

with Y(s) function provided by first continuous and superiorly limited derivative, and if the following condition are valid:

$$D(t, x) = D(t, y)$$[30]

$$\sum_{k=1}^{m} |t_j - t_k| x_k v(t, t_k) \geq \sum_{k=1}^{m} |t_j - t_k| y_k v(t, t_k) \qquad j = 1, 2, 3, \dots, m$$

then between the post-shift value of the flow x and the post shift value of the flow y there is the following relation:

$$W(t^{+}, x) \geq W(t^{+}, y) + W(t, y) K [C(t, x) - C(t, y)]$$

where K is random variable with real values:

$$K = \frac{1}{2} inf_{s \geq t} \left(\frac{d^2}{ds^2} e^{-\int_t^s Y(u) du} \right)$$

Generally, the factor $[C(t, x) - C(t, y)]$ is substituted by $M_N^{(2)}(t)$.

$$M_N^{(2)}(t) = [C(t, x) - C(t, y)]$$

This value can be considered as a dispersion measure that is a measure that points out the volatility or the distribution of value of shifts. In other terms, it is the portfolio risk that the portfolio is immunized by a convex shift variations. A portfolio that is immunized by convex shifts is called optimally immunized at minimum risk. So, in order to build a portfolio of fixed-income securities, the minimization of $M_N^{(2)}(t)$ is a way to improve the lower bound[31] and the optimal strategy to protect the portfolio also if the lower bound can be negative.

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[29] This condition is called Budget constraint condition
[30] This is the duration condition
[31] The lower bound is the minimum level of value the portfolio can achieve given a random shift of the yield curve.

The results point out here are join under the nomination of Classic immunization theory or Semi-deterministic immunization. Thanks to the joint with the probability theory and thanks to important results such as the ito's lemma, the Black and Schools option pricing theory and the improvement of the general stochastic theory, to the classic immunization it is added a new theory called the stochastic immunization. It will not be discussed but it only cited so that the dissertation keeps on following its purpose.

III. HEDGING FIXED-INCOME SECURITIES

1. Introduction to Hedging

The hedging is a method with which an investor takes an investment position to offset the potential losses and gains that may be incurred by a companion investment, that is the investor reduces any risk related to a financial position by avoiding losses or limiting the gains. The main instruments used in finance are all the totality of derivatives and all the complex financial instruments the mathematical finance has been able to create: the most famous are futures, forwards, options, future options, swaps but asset-backed securities and Asian options too. The list of the different derivatives always become longer and longer, and the features of each one are different between them.

In order to give a general definition, a derivative is a financial instrument whose value depends on the value of an underlying one[32].A derivative is traded in two different markets: the first market is the clearing house and the second one is the over-the counter market. The most famous clearing house are the Chicago Board of Trade and the Chicago Mercantile Exchange, a clearing house is a market in which derivatives are traded as standardized contracts that have been defined by the exchange. Instead, the over the counter market is market formed by the totality of dealers[33] that participate to the market.

Two important derivatives are the futures and the forward. A forward contract is an agreement to buy or sell an underlying asset at certain future time for a certain price[34]. The part that agrees to buy the underlying asset is said that has a long position in the future contract; instead the part that will sell the underlying asset has a short position. So, in case the price of the asset is greater than the fixed price agreed in the contract, the long position can make a profit by selling the asset; instead, if the price is smaller than the agreed price, the short position can buy the asset at the market price and sell it at the bigger price fixed by the contract.

A future is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price[35]. Reading the definition, the two are the same contract but there are few differences between them. The futures are traded on the exchanges, they are standardized contract, settled daily and there is a fixed range of delivery dates; generally, they are closed out[36] prior to maturity and the exchange assure that there is no credit risk. The forward contract is a private contract between two part and it is not standardized and it is settled at the end of the contract. There is usually one delivery date and delivery or final cash settlement usually takes place at maturity. Moreover, the forward suffers the counterpart risk, namely the risk that the other part do not deliver or settle the contract.

The basic principle of hedging a risk by using the future markets is to take a position that neutralizes the risk. By the way, there are two different types of hedging: the short hedge and long hedge.

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[32] The underlying could be any kind of product: such as assets, commodities, currency, even the weather.

[33] A dealer is a financial intermediary that build his own portfolio by buying and by selling securities.

[34] HULL, John C, *Options Futures and Other Derivatives*, 7th Edition, (Toronto: Pearson Education International, 2009), p.3-4.

[35] HULL, John C, *Options Futures and Other Derivatives*, 7th Edition, (Toronto: Pearson Education International, 2009), p. 6.

[36] closing out a position means to enter into the opposite trade to the original one.

A short hedge is a hedge that involves a short position in a future contract. “A short hedge is appropriate when the hedger already owns the asset and expect to sell it at some time in the future”[37]. A classic example is the one about the farmer. A farmer will have his crop in May, but he does not know if the price in May is different from the time in which is done the valuation. So, the farmer can enter in a future contract to lock the selling price at time 0.

A long hedge is a hedge that involves a long position in a future contract. “A long hedge is appropriate when the company knows it should purchase a certain asset in the future and wants to lock in the price”[38]. Moreover, “the long hedges are used to manage an existing short position”[39] Imagine a company should purchase a fixed amount of iron in time 1, the company can decide to enter in a future contract in time 0 to lock in the price in order to avoid the implied risk of the price movement.

Given the theory of the hedging, it seems any investor is able to remove any risk related to his existing position. In practice, there are same complexities that allows the hedge not to be perfectly safe. “The reason are as follows:

1. The asset whose price is to be hedge may not be exactly the same as the asset underlying the future contract.
2. The hedger may be uncertain as to the exact date when the asset will be bought or sold.
3. The hedge may require the future contract to be closed out before the delivery month.”[40]

All these problems can be summarized through the concept of basis risk.

The basis is the difference between the spot price of the contract to be hedged and the future price of the contract used at any given time.

$$Basis = Spot\ Price - Future\ price$$

if the asset to be hedged and the underlying asset of the future contract is the same, then the basis is equal to zero at the expiration of the future contract. As time passes, the basis changes and so, there is a strengthening of the basis if the basis increases or a weakening of the basis if the basis decreases.

Consider a situation in which the investor takes a short future position in time 1 to sell an asset in time 2. The effective price that is obtained for the asset with hedging is therefore:

$$S_2 + F_1 - F_2 = F_1 + b_2$$

[37] HULL, John C, *Options Futures and Other Derivatives*, 7th Edition, (Toronto: Pearson Education International, 2009), p. 46.
[38] HULL, John C, *Options Futures and Other Derivatives*, 7th Edition, (Toronto: Pearson Education International, 2009), p. 47.
[39] HULL, John C, *Options Futures and Other Derivatives*, 7th Edition, (Toronto: Pearson Education International, 2009), p.47
[40] HULL, John C, *Options Futures and Other Derivatives*, 7th Edition, (Toronto: Pearson Education International, 2009), p. 51

where b_2 is the basis at time t_2, $S_2$is the spot price in time t_1, F_1 is the price of the future contract at time t_1 and F_2 is the price of the future contract at time t_2.

Defining S_2^* as the price of the asset underlying the future contract at time 2, the previous formula can be written:

$$(S_2 - S_2^*) + F_1 + (S_2^* - F_2)$$

The terms $S_2 - S_2^*$ and $S_2^* - F_2$ represent two components of the basis risk. "The $S_2^* - F_2$ term is the basis that would exist if the asset being hedged were the same as the asset underlying the future contract. The $S_2 - S_2^*$ term is the basis arising from the difference between the two assets"[41].

[41] HULL, John C, *Options Futures and Other Derivatives*, 7th Edition, (Toronto: Pearson Education International, 2009), p. 53.

2. **Interest Rate Future**

A future contract is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price. Because the future is a contract, the exchange must specify in some detail the exact nature of the agreement between the two parties: in particular, the asset, the contract size, where and when the delivery will be made in case the last information are necessary. The asset can be any tradable asset in the market: for instance, commodities, currencies, assets, indexes and even the interest rate. The contract size is the most important feature to specify. It is the amount of the assets that should be delivered under one contract. If the size is too large the smaller investor is unable to hedge small risk or portfolio. If the size is too small, the transaction costs related to de hedge arise because the investor needs to buy more contracts.

The delivery arrangement is the clause about where the delivery will be made. It is particularly important for the commodities that involve significant transportation costs. Finally, the delivery months, that is the clause about the time in which the contract will be set. Beyond these four feature, the exchange specifies three features more: the price quote, that is how the price will be quoted for example the treasury bond are quoted in dollars and thirty-second of dollar; the limit up/down, that is the limit the exchange fixes from the previous day's close; and finally, the position limit that is the maximum number of contracts that an investor may hold.

The future discussed in this paragraph is the interest rate future. It is the future used to hedge the risk derived by the interest rate movements. The interest rate future that is traded in the Chicago Board of Trade is the treasury bond future.

First it is important to introduce the treasury bond convention about the day count and quotation.

"The day count defines the way in which interest rate accrues over time"[42]. An investor knows the interest fixed in a reference period (for example semi-annually) but the day count is useful to know the interest earned in different periods. The interest earned between two periods is

$$\frac{Number\ of\ days\ between\ periods}{Number\ of\ days\ in\ reference\ periods} * Interest\ earned\ in\ reference\ period$$

The day count convention that are usually used are:

- Actual/Actual
- 30/360
- Actual/360

The Actual/Actual conversion is the day count used for treasury bonds in the United States. Suppose an investor wants to know the interest earned between March 1 and July 3 of a $1,000 face value bond with semi-annual coupon rate of 8% and the reference period is March 1 – September 1. The computation of the interest is so:

[42] HULL, John C, *Options Futures and Other Derivatives*, 7th Edition, (Toronto: Pearson Education International, 2009), p. 129.

$$\frac{124}{184} * 40 = 26.957$$

Imagine that the bond traded has the same characteristics of the previous Treasury Bond, but is either a corporate or a Municipal bond whose day count is 30/360. The interest earned is:

$$\frac{122}{180} * 40 = 27.111$$

Finally, the Actual/360 convention is used for money market instruments such as the treasury bills, and so:

$$\frac{124}{180} * 40 = 27.556$$

The price quotations differ between the treasury bills and the treasury notes. Although they are two fixed-income securities issued by the United States, the treasury bill is a money-market contract whereas the treasury note is more complex bond.

The quotation of the treasury bill is represented by the annual rate of interest. So, the price is computed by:

$$Q = \frac{360}{n}(100 - P)$$

where Q is the quotation and P is the effective price the buyer must pay to be the owner of the bill.

On the other side, the Treasury Bond prices are quoted in dollars and thirty-seconds of a dollar. So, the quote of a $1,000 face value treasury bond could be 950-13[43]. The price of the bond is $950 + \frac{13}{32} = 950.41$ $.

The price an investor pays for purchasing a treasury bond is not only the quoted price. The cash price is the sum of the quoted price and the accrued interest since last coupon date:

$$Cash\ Price = Quoted\ Price + Accrued\ Interest$$

where the accrued interest is computed through the Actual/Actual convection explained early.

The most popular interest rate futures are: Treasury Bond future, Treasury Note future and 5-years Treasury Notes. "In the Treasury Bond future, any government bond that has more than 15 years from that day can be delivered. With Treasury Note futures, any government bond with maturity between $6\frac{1}{2}$ and 10 years can be delivered. In the 5-years Treasury Note futures contract, the bond delivered has a remaining life that is about 4 or 5 years"[44].

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[43] It is the general nomenclature to price Treasury Bonds

[44] HULL, John C, *Options Futures and Other Derivatives*, 7th Edition, (Toronto: Pearson Education International, 2009), p. 132

"The price of a Treasury bond future is quoted in the same way as the Treasury Bond prices"[45]. It is:

$$Cash\ Price = (Recent\ settlement\ Price * Convension\ factor) + Accrued\ interest$$

where the conversion factor is the ratio between the price of the underlying asset and the face value
Imagine a 7% semi-annual coupon rate and $100 face value bond with maturity 10 years, whose yield to maturity is equal to 8%. The conversion fact is

$$conv.fact. = \frac{\sum_{t=0}^{20} \frac{3.5}{(1+4\%)^t} + \frac{100}{(1+4\%)^{20}}}{100} = 0.932$$

The quoted future price of the interest rate future is a function of the underlying asset. According to the arbitrage condition, the future price should be equal to the present value of the price the investor will pay for the price. So, the future price function is:

$$F_0 = (S_0 - I)e^{rT}$$

"where *I* is the present value of the coupons during the life of the future contract, *t* is the time until the futures contract matures and r is he risk-free rate applicable to a time period of length *t*"[46].

[45] HULL, John C, *Options Futures and Other Derivatives*, 7th Edition, (Toronto: Pearson Education International, 2009), p. 132.
[46] HULL, John C, *Options Futures and Other Derivatives*, 7th Edition, (Toronto: Pearson Education International, 2009), p. 135.

3. Hedging with Duration and Convexity

In order to hedge, an investor needs to buy a number of interest future contracts that covers any change of the starting portfolio. The given number of such contracts sold per unit of the cash portfolio is called the hedging ratio[47]. If the hedging ratio is rightly determined the portfolio does not suffer any losses because any change in the cash portfolio will offset by the change in the future contracts.

The duration of the portfolio and the duration of the underlying assets of the future contracts can be used to compute the hedging ratio properly, but to the investor it is required to respect specific condition.

"Let $V(r)$ be the value of a cash portfolio held where r id the yield to maturity of the security yield. Let $F(r)$ be the future price for the delivery of a unit of some security at some future date. Let h be the number of units or contracts bought or sold for future delivery[48]. So, any change in the future price can be expressed by:

$$\Delta F = h[F(r) - F(r_0)]$$

where the r_0 is the initial yield to maturity of the underlying fixed income security of the future contract and r is the yield to maturity at time of the valuation. If the investor has a long position any increase of the price of the underlying assets determines an increase of value of the future contract; so, the h must be positive. On the other side, if the investor has a short position any decrease of the price of the underlying assets determines an increase of value of the future contract; so, the h must be negative.

If an investor builds a portfolio and decides to hedge it, the aggregate value of the investor's portfolio is:

$$P(r) = V(r) + h[F(r) - F(r_0)]$$

the variation of the yield to maturity affects the value of the portfolio. In fact, an increase of the yield to maturity reduces the value of fixed income securities portfolio, whereas the value of the future contracts arises given that the investor has taken a short position $(h < 0)$. On the other side, if the investor takes a long position in future contracts, that is $(h > 0)$, and the yield to maturity decreases, then the value of the fixed income securities portfolio arises and the future position decreases.

To analyze how the hedged portfolio reacts to any movement of the yield to maturity and so the sensitivity of the portfolio the evaluation is done through the derivative of the portfolio function. So:

$$P'(r) = V'(r_0) + hF'(r_0)$$

[47] BIERWAG, Gerald O., *Duration Analysis: Managing Interest Rate risk*, (Cambridge, Mass.: Ballinger Publishing Company, 1987), p. 178.

[48] BIERWAG, Gerald O., *Duration Analysis: Managing Interest Rate risk*, (Cambridge, Mass.: Ballinger Publishing Company, 1987), p. 178.

where the derivative of the portfolio is equal to minus the product between the duration of the portfolio and initial value of the portfolio. Given that the periods of time are discrete, the duration used is the modified duration; hence the derivative is:

$$V'(r_0) = -\frac{DV(r_0)}{1+r_0}$$

where $D$is the duration of the portfolio.

Analogously, the derivative of the future position:

$$F'(r_0) = -\frac{D_F F(r_0)}{1+r_0}$$

where $D_F$is the duration of the future position.

Hence, the sensitivity of the portfolio is:

$$P'(r) = -\frac{D^* P(r_0)}{1+r_0} = -\frac{DV(r_0)}{1+r_0} - h\frac{D_F F(r_0)}{1+r_0}$$

where $D^*$is the duration of the portfolio

$$D^* P(r_0) = DV(r_0) + hD_F F(r_0)$$

Given that $P(r_0) = V(r_0)$.Then:

$$D^* = D + \frac{hD_F F(r_0)}{V(r_0)}$$

If the variation of the yield to maturity does not affect the value of the hedged portfolio, it means that the duration D^* is zero. Hence, the previous expression is equal to zero and the hedging ratio can be expressed such as:

$$h = -\frac{D}{D_F}\frac{V(r_0)}{F(r_0)}$$

So, the hedging ratio is the product between the ratio between the duration of the fixed-income securities portfolio and that of the underlying assets of the future position, and the ratio between the initial value of the portfolio and the initial value of the future position.
The assumptions of this formula are the assumption that the yield to maturity of the portfolio is the same of the yield to maturity of the future position and the assumption that the two yields to maturity change by the same amount. Suppose that $r^c$is the yield to maturity of the portfolio and $r^f$is the yield to maturity of the future position, the value of the hedge portfolio is:

$$P(r^c, r^f) = V(r^c) + h[F(r^f) - F(r_0^f)]$$

the derivative is:

$$P'(r^c, r^f) = V'(r^c) + hF'(r^f)\ \frac{dr^f}{dr^c}$$

By substituting $a = \frac{dr^f}{dr^c}$ and the value $V'(r_0^c) = -\frac{DV(r_0^c)}{1+r_0^c}$ and $F'(r_0^f) = -\frac{D_F F(r_0^f)}{1+r_0^f}$; then:

$$-\frac{DV(r_0^c)}{1+r_0^c} - ah\frac{D_F F(r_0^f)}{1+r_0^f} = 0$$

where $D^M = \frac{D}{1+r_0^c}$ and $D_F^M = \frac{D_F}{1+r_0^f}$ are the modified duration[49] of the portfolio and future position. By substituting, the hedging ratio is:

$$h = -\frac{D^M V(r_0^c)}{D_F^M\ F(r_0^f)a}$$

The formula without the previous assumption maintain the same characteristics but "the prices and the duration of the portfolio and the future position are determined at different yield levels and the variable a appears in the formula to describe the relative motion of the yields to maturity"[50].
The analysis developed at this point is not free of assumption. In the previous chapter, the duration of the security is used to obtain an approximation of the value in case of any variation of the yield to maturity. The application of the duration is a linear approximation of the trend of a security. Practically, it means that the

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[49] For Modified duration retake the chapter I.3
[50] BIERWAG, Gerald O., *Duration Analysis: Managing Interest Rate risk*, (Cambridge, Mass.: Ballinger Publishing Company, 1987), p. 183.

trend of the security is well-explained only in the case the variation of the yield is small, because the linear approximation does not capture the curvature of the price trend. In order to hedge effectively, it is necessary to introduce in the analysis the convexity. Hence, in addition to the duration condition, it is necessary to introduce the convexity condition of a hedge portfolio, that is:

$$P''(r) = V''(r_0) + hF''(r_0)$$

where the second-order derivatives are $V''(r_0) = C$ and $F''(r_0) = C_F$; so:
Given that the conditions are two, it is necessary so introduce another unknown variable, and hence, the assets used to hedge are two. Moreover, the hedge aims to not only the duration neutrality, but also the convexity neutrality so the duration and the convexity of the portfolio must be equal to zero. Hence:

$$P'(r) = V'(r_0) + h_1F_1'(r_0) + h_2F_2'(r_0) = 0$$
$$P''(r) = V''(r_0) + h_1F_1''(r_0) + h_2F_2''(r_0) = 0$$

$$h_1F_1'(r_0) + h_2F_2'(r_0) = -V'(r_0)$$
$$h_1F_1''(r_0) + h_2F_2''(r_0) = -V''(r_0)$$

where it is a system with two expression and two unknown variables, that is $h_1$and h_2.
Finally, it is necessary to improve the analysis from the assumption of parallel shift. The yields to maturity for different maturity differ each other. The theory that explains this difference is the liquidity theory of Keigns. Moreover, the volatilities of the short-term yield and the one of the long-term one is different too. If the investor forecast that the portfolio is hedged if the yield to maturity arrive to 2%, he supposes that the yield curve suffers a parallel shift.
Let $V(r)$ the price of the portfolio of fixed income securities, then:

$$V(r) = \sum \frac{CF_{t_n}}{\left(1 + r(t_0, t_n)\right)^{t_n - t_0}}$$

To construct the global hedged portfolio:

$$P'(r_0) = \sum V'(r_0)dr(t_0, t_n) + \sum h_n \sum F_n(r_0)$$

that is equal to:

$$P'(r_0) = \sum [V'(r_0)dr(t_0, t_n) + \sum h_n F_n(r_0)]dr(t_0, t_n)$$

“A sufficient and necessary condition to have $P'(r_0) = 0$ up to a first-order approximation for any set of (small) variations $dr(t_0, t_n)$, for each $r(t_0, t_n)$ is”[51]:

$$V'(r_0)dr(t_0, t_n) + \sum h_n F_n(r_0) = 0$$

where “solving this linear system for h_n at each trading date gives the optimal hedging strategy”[52].

[51] MARTELLINI Lionel, PRIAULET Philippe & Stephane, *Fixed-Income Securities: Valuation, Risk Management and Portfolio Strategies*, (Chichester, West Sussex: Wiley, 2003), p. 189.

[52] MARTELLINI Lionel, PRIAULET Philippe & Stephane, *Fixed-Income Securities: Valuation, Risk Management and Portfolio Strategies*, (Chichester, West Sussex: Wiley, 2003), p. 189.

CONCLUSION

The three chapters written until this point are descriptions of the two important methods broadly used in the financial market. This paragraph focuses on salient differences and annotations of the two strategies in the real financial market. The immunization consists of the asset-liability matching in order to prevent the interest rate risk and the Hedging of the fixed-income securities is a technique used for the same reason, but it is applied by trading different products, that is the derivatives. The two are both meticulous strategies that require to be applied good knowledge not only of finance but also of mathematics and statistics; so, they cannot be applied by financial dabblers.

Immunization is a method mainly used by the banks to managed the inflow and the outflow of cash without be affected by interest rate risk. The aim of the banks is to be the place where the people can lend money to finance investment projects and where they can make money safe by any risk, by gaining an interest on the deposit. For these two functions, they are used two means: the mortgages and the credit deposits. These behave like fixed-income securities and they are fixed inflows and outflows for the bank. So, by matching any contract that has a fixed stream, the bank is immune to any change in the interest structure.

Instead, the hedging of fixed-income securities is a technique more used by the portfolio managers and mutual funds. A manager invests in fixed-incomes in order to balance the risk and return because of the needs of his clients. In case he forecasts any bad event that affect the value of his fixed-income portfolio, he faces two choices: to sell the portfolio or to buy a financial derivative that hedge the risk. If the duration has not passed yet, to sell the portfolio is not the optimal strategy; so, the hedging become a useful strategy to face the risk. In fact, the manager can enter in a short position in an interest rate future, in order to balance the profit and the loss obtained by the interest shifts.

By keeping on going the previous example, the transaction cost for the first strategy are lower. In other words, the transaction costs of buying future contracts are lower than selling and buying the security to avoid the risk. So, in case of the immunization, the portfolio immunized needs to be adjusted at each shift of interest rate. This is more expensive than buying future contracts.

A problem with hedging is contract size of the futures. Generally, each contract can provide a numerous amount of short or long position. This creates a difference between the investors that can apply the hedging strategy. In fact, the purchase of future contract is not suitable for smaller investors

One important difference between the immunization and the hedging of fixed-income securities is the factors that affect the profitability of the strategy. A future contract depends not only on the value of the spot price of the underlying asset but also on the interest rate considered. The interest rate used to compute the future contracts is the Inter Bank Offered Rates, such as the LIBOR (London), EURIBOR (Europe), SIBOR (Singapore), etc. The Inter Bank Offered Rates is the interest rate at which a bank borrows or lends money to another bank. It is determined by the supply and the demand of capitals by the banks.

On the other side, the immunization strategy depends on the interest of the portfolio bonds: that is the portfolio value depends only on the risk of interest changing of each bond that forms the portfolio.

Hence, in the immunization strategy, the risk analysis consists on the creditworthy and on the other fundamentals[53] related to the borrower, whereas the hedging risk analysis should include the trend of the Inter Bank Offered Rate used to compute the future contract.
Another problem related to the hedging of fixed-income strategy is that not all the bonds have the relative future contract, and so, not every bond is used as underlying in a derivative contract in a clearing house. That can limit the application of the strategy, because an investor who wants to hedge a specified bond should address to an investment bank in order to build the contract in the over-the-counter market. Given the features of the bond, the bank decides to enter into the contract and if it agrees, it decides how much to charge as transactions costs. For example, in case of illiquid market, the risk of not finding a counter part is high; so, the bank cannot be inclined to form the contract. Hence, the hedging depends on the liquidity of the bonds.

[53] The fundamentals are the qualitative and quantitative information that contribute to the economic well-being and to the financial valuation of a company.

REFERENCES

BIERWAG, Gerald O. *Duration Analysis: Managing Interest Rate risk*. Cambridge, Mass.: Ballinger Publishing Company, 1987.

BORTOT Paolo, MAGNANI, Umberto, OLIVIERI Gennaro. *Matematica Finanziaria: con esercizi* (Financial Mathematics: with exercises). 2nd Edition. Milano: Monduzzi Editore,1998.

DE FELICE, Massimo. and MORICONI, Franco. *La Teoria dell'Immunizzazione Finanziaria: Modelli e Strategie* (Theory of the Financial Immunization: Models and Strategies). Bologna: Il Mulino, 1991.

HULL, John C. *Options Futures and Other Derivatives*. 7th Edition. Toronto: Pearson Education International, 2009.

MARTELLINI Lionel, PRIAULET Philippe & Stephane. *Fixed-Income Securities: Valuation, Risk Management and Portfolio Strategies*. Chichester, West Sussex: Wiley, 2003.

SCANDALO Giacomo. *Matematica Finanziaria* (Financial Mathematics). Verona, Italy: Amon Editore, 2013.