Vasiček Mathematical Model

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Dipartimento di Economia e Finanza

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SHORT RATE MODELS IN CONTINUOS TIME WITH FOCUS ON VASIĈEK MATHEMATICAL MODEL

Relatore:

Professoressa Sara Biagini

Candidato:

Matricola 197921



Libera Università degli Studi Guido Carli

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The short term interest rate is one of the most important values determined when making a market analysis. Many different models have been created for the purpose of explaining its behavior rather than for any other variable in finance.

The main aim of all those models is to outline a process for the short rate r in order to characterize changes related to the term structure of interest rates.

In particular the short rate r_t is the instantaneous rate continuously compounded at time t.

Clearly all the models that i'm going to describe and the model (Vasiĉek) that i'm going to use to make an empirical analysis of data are all time-continuous, practically this implies that the time framework that we are taking in consideration does not simply go from t to t_1 but from t to ∞ . Since there is the vacancy of a common framework and of an appropriate benchwark is very hard to evaluate the performances of those model with respect to the capacity of capturing short term rates.

The main scope of this thesis is to give a general description of what are short models starting by giving a general stochastic differential equation which solution is the rate we are looking for.

Moreover i'm going to highlight all the assumptions and all the conditions that have the necessity to hold to make the models valid, than i'll demonstrate that when there's no arbitrage all bonds have equal market price of risk independently of the maturity, this is necessary because one of the variable of the term structure equation is Λ the market price of risk. Than i'll define the general term structure equation for the short rates and explain the meaning of each variable present in the equation

Some necessary things to clarify, before we are going to use data and find empirical results, are martingales and martingale modelling in relation to the fact that the dynamics of the rate have a martingale measure Q.

The model i'm going to focus on is the one invented by Vasiček, first i will highlight how to estimate the parameters related to the model than discuss how to compute its term structure.

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Ultimately i will use some data, compute the price of a bond using the model mentioned before and comment the results obtained providing also a graphical representation.

2 GENERAL CHARACTERISTICS AND SOME PARTICULAR ASSUMPTIONS REGARDING SHORT RATE MODELS

We start by posing the problem on how to model arbitrage free zero coupon bond prices defined by the processes $\{p(x,N) ; N \ge 0\}$ of course N represents the time horizon where the price is defined.

Instinctively the price P (t,N) is probably influenced by the behavior of the short rate of interest over the interval [t,T], the incipit is to determine an a priori definition for the dynamics of the short rate of interest.

Indeed this has been the general path taken for interest rate theory, so we will model the interest rate under an objective probability measure P, defined by the resolution of a Stochastic Differential Equation of the form:

$dr(t) = \mu(t,r(t))dt + \sigma(t,r(t))d\hat{W}(t)$

- Where μ is the drift term
- σ is the diffusion term
- Ŵ outlines a Wiener Process also defined as Standard Brownian Motion

Another way to interpret this equation is to write:

$$\mathbf{r(t+s)} - \mathbf{r(t)} = \int_t^{t+s} \mu(r(u); u) du + \int_t^{t+s} \sigma(\mathbf{r(u)}; u) d\hat{W}(u)$$

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This equation represents a continuous time stochastic process composed by a Lebesgue integral and an Ito's integral, where the former is defined as the integral of a variable defined on a sigma algebra while the latter is outlines the dynamics of any Markovian function of the Brownian Motion of W.

Of course the only variable given is the short rate associated to the money account which price process X is defined by the following equation:

dX(t) = r(t)X(t)dt

2.1 Assumptions

First Assumption

To clarify what is the meaning of this equation we are going to make a particular assumption:

Assumption 1.1: "We assume the existence of one exogenously given (locally risk free) asset. The price, X, of this asset has dynamics given by equation (1.3) where the dynamics of r, under the objective probability measure P, are given by equation (23.1)."

The necessity to make such assumption is to outline the fact that this exogenously given risk free asset is considered as the underlying asset and the short rate associated to it as the underlying short rate.

Of course for this to being valid there is the need of a market rich of bonds otherwise the fact that we have only one asset, even though he is the underlying one, doesn't help us to model the short rate.

So we assume that:

Second Assumption

"We assume that exists a market for zero coupon T-bonds for each value of T."

The market existence help us to understand the relationship between all the price dynamics of the bonds at each different maturity, of course since T is undefined there is an infinite number of zero coupon bonds.

Also what we are interested in is to find a price for interest rate derivatives such as swaps and options.

From what we have stated up until now one can asses that bond prices are uniquely determined by the Probability dynamics of the short but is not true, since the bond price depends also on the underlying risk free assets.

The model we are trying to show is very similar to the Black-Scholes standard model where the stochastic process of the short rate is similar to the underlying stochastic process for the stock price S and the equation 1.3 which represents a money account which is also present on the Black-Scholes model.

So a person can say that since the assumption that the price process is determined only by the P-dynamics of the undelying asset is valid for the Black-Scholes than is also valid for this model.

To demonstrate that this is not true, we start by defining a meta theorem which states that "a market is incomplete when the number of traded assets is lower than the number of random sources".

Then we continue by taking in consideration the existence of a market where the only asset present is the money account (equation 1.3), the random part of the equation is the Wiener Process and since the number of traded assets taken in consideration (so without considering the money account) is 0, according to the meta theorem, the market is incomplete.

Another way to verify incompleteness is to realize that is pretty much impossible to form portfolios and the only possible profit is given by investing all the money in the bank and let them grow at a certain rate, this prevents the possibility of replicating derivatives. The impossibility of replicating derivatives makes us understand that the market is no arbitrage free.

Also let's consider a portfolio which follows Black-Scholes model and in particular it is formed by an underlying asset G and a derivative L, we choose weights to eliminate the Wiener process that will give us a riskless asset with rate of return **r** equal to the short rate, writing such equality will give us a Partial Differential Equation.

However this approach makes no sense since G is not the price of an asset traded in the market and so talking about process of G and talking about the possibility of creating a portfolio based on G has no purpose.

To summarize in a few steps:

- 1. The value of the derivative we have defined is not entirely determined by the condition that the market is free of arbitrage and by the dynamics of G.
- 2. Since there are not enough underlying assets, pricing a derivative in terms of them is not possible.

From the situation that we were analyzing before the digression made we can clearly see that in our model we have a situation of market incompleteness, since the number of random sources is bigger than the amount of traded assets.

If we compare the model we are building with Black-Scholes one main difference arise, since in the Black-Scholes we have that the underlying asset is the stock S which in our model would correspond to r, but in our model r is not the price of an asset so is impossible to form a portfolio based on r.

To summarize briefly what we have said:

• The short rate dynamics and the free of arbitrage condition of the market are not enough to define a price for the bond, this is because we do not have enough underlying assets to price a derivative in terms of them.

This is why we need to take in consideration more variables that I will explain later in the next chapters.

3 DEFINING A TERM STRUCTURE EQUATION FOR SHORT RATE

3.1 Term Structure Equation

We start by defining the Price of a bond with maturity t_n (t_n -Bond):

$$P(t,t_n) = F(t, r(t); t_n)$$

- Where t is the first year
- The short rate is defined as r(t)
- Differently t_n is the time to maturity

Also we need to define boundaries, the condition we pose is that at time to maturity the t_n -bond values 1 dollar, now our interest is to clarify and define one of the factors that is fundamental for outlining the Term Structure Equation for short rate models, is also fundamental since help us separate between the short rate model we are trying to define and the Black-Scholes model.

We start by defining two different time to maturity K and G according to the Ito's formula and the description we have given for price processes, we can define different equations for the dynamics of the K-Bond.

 $dF^{\kappa} = F^{\kappa} \alpha_{\kappa} dt + F^{\kappa} \sigma_{\kappa} d\hat{W},$

- Where **F**^K is defined as F(t,r; K)
- Where α_{κ} is the drift-term related to the performance of asset in a given market
- Where σ_{κ} is the volatility related to the asset took in consideration

We denote r and t as subindices for the partial derivatives,

$$\alpha_{K} = \frac{\sigma_{K}^{K} + \mu_{F_{r}}^{T} + \frac{1}{2} \sigma^{2} F^{T}_{rr}}{F^{K}}$$

$$\sigma_{K} = \frac{\sigma_{F_{r}}^{K}}{F^{T}}$$

We assume the portfolio (β_{K},β_{G}) , with value described by this equation :

$$dV = V \{\beta_{K} dF^{K}/F^{K} + \beta_{G} dF^{G}/F^{G}\}$$

Inserting the differential equation $dF^{\kappa} = F^{\kappa} \alpha_{\kappa} dt + F^{\kappa} \sigma_{\kappa} d\hat{W}$, and the equation for Bond G after some calculation we have this result

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$dV = V \{\beta_{\kappa}\alpha_{\kappa} + \beta_{G}\alpha_{G}\} dt + V \{\beta_{\kappa}\alpha_{\kappa} + \beta_{G}\alpha_{G}\} d\hat{W}$

Of course for those portfolio equations to work we need those conditions to be valid:

- $\beta_{\kappa} + \beta_{G} = 1$ This says that the sum of the weights is equal to 1
- $\beta_{\kappa}\sigma_{\kappa} + \beta_{G}\sigma_{G} = 0$ This implies that portfolio risk is equal to 0 (?)

With those 2 conditions applied we can eliminate the term $d\hat{W}$ from the equation which now gives:

 $dV = V\{\beta_{K}\alpha_{K} + \beta_{G}\alpha_{G}\} dt$

By solving the system of equation given by this last equation and the other 2 we obtain

$$\beta_{K} = - \frac{\sigma_{K}}{\sigma_{G} - \sigma_{K}}$$
$$\beta_{G} = \frac{\sigma_{G}}{\sigma_{G} - \sigma_{K}}$$

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Those two equations represents the weight of each asset in the portfolio, also by simply substituting we can see that



This equation states that the value of a portfolio is function of its own return at a certain time t.

In case of a no arbitrage market the portfolio has the rate of return that corresponds to the short rate of interest.

$$\frac{\alpha_{\rm K}\sigma_{\rm G}-\alpha_{\rm G}\sigma_{\rm K}}{\sigma_{\rm G}-\sigma_{\rm K}}=r(t)$$

by manipulating a little bit this formula can be rewritten as:

 $\frac{\alpha_{K}(t) - r(t)}{\sigma_{K}(t)} = \frac{\alpha_{G}(t) - r(t)}{\sigma_{G}(t)}$

Where $\alpha_{\kappa}(t)$ is the return associated to bond K while r(t) is the return associated to the riskless asset, also $\sigma_{\kappa}(t)$ is the volatility associated to the bond K.

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Moreover the difference $\alpha_{K}(t) - r(t)$ is the risk premium associated to bond K. This is why for bond K we can define a variable that identify the risk premium weighted for the volatility.

$$\frac{\alpha_{\kappa}(t) - r(t)}{\sigma_{\kappa}(t)} = \lambda(t)$$

The variable just introduced represents the market risk premium weighted for the volatility of the particular bond taken in consideration, of course the market is considered to be arbitrage free because otherwise such condition would not be valid.

In particular this result is valid for both bonds K, G and for all the bonds that are in the same market regardless of their individual time to maturity.

The motivation of why we have done all this fatigue to demonstrate this existence of this variable is because this variable is fundamental to define the term structure equation associated to short rates.

Starting from the last equation that we have found we can substitute the values for α_{K} and for σ_{K} to obtain the **"Term Structure Equation "**

$$F_t^{K} + \{\mu - \lambda\sigma\} F_r^{K} + \frac{1}{2} \sigma^2 F_{rr}^{K} - rF^{K} = 0$$

$$F^{\kappa}(K,r) = 1$$

We can clearly state that the Term Structure Equation takes the form of partial differential equation very similar to the Black-Scholes one but with more factor since it

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has also the market price of risk which can't be determined by the model but is usually determined exogenously, the same reasoning applies to μ and σ .

We know that the objective probability measure P is not enough to measure correctly the short term rate and consequently the price of the bond, this is we will define a new equation that will respect a different measure.

 $F(t, r; G) = E^{Q}_{t,r} [e^{-\int_{t}^{G} r(t) dt}]$

Bond prices are now under the martingale measure Q also this measure varies for different values of λ .

The Black-Scholes differs from this, because its martingale measure it is completely determined. This is due to the fact that the market related to the Black-Scholes model is complete, while the market at which we are referring is not complete.

We know that the different Bond Prices are determined partly by the Q-dynamics and partly by the market, in particular for each different values of λ , which is the market price of risk, it exists a bond market consistent with the r-dynamics.

Specifically part of the bond price process will be determined by the demand and supply of the bonds in the market taken in consideration and factors such as risk aversion of investors.

Still the problem remains on which value of λ we choose, the only way to do this is by getting market data and see which is the value of λ .

Since the bonds described above are deterministic we can define a more general contingent N-claim:

$X = \phi(r(N))$

• Where ϕ is a real valued function.

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After having defined this clam, it is easy to see that the price of a bond in an arbitragefree market is equal to:

$\Pi(t;\phi) = F(t,r(t))$

This equation helps us to solve the problem we have with the boundary of F:

$$F_t + \{\mu - \lambda\sigma\} F_r + 1/2\sigma^2 F_{rr} - rF = 0$$
$$F(N,r) = \phi$$

So F has the stochastic representation:

$$F(t,r;N) = E^{Q}_{t,r}[exp\{-\int_{t}^{T} r(t)dt\} \times \varphi(r(N))]$$

Where the martingale measure Q and the variables t,r denote that the expectations respect the following dynamics :

$dr(s) = {\mu - \lambda \sigma} ds + \sigma dW(s)$

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The equation we have found is the general final term structure equation for short rate models.

4 DESCRIPTION OF MARTINGALE MODELS FOR THE SHORT RATES

We start by defining the terms of the term structure equation :

- Where μ is the drift term
- Where σ is the diffusion term
- And λ is the market price of risk

Let's start by considering σ to be given a priori. Than we notice that in the term structure equation what really matters is the part of the equation represented by μ - $\lambda\sigma$, which is the drift term of the short rate of interest associated with the martingale measure Q. So clearly both μ and λ are specified under the martingale measure Q, so by modelling the dynamics of the short rate under this measure we can write:

$dr(t) = \mu(t,r(t))dt + \sigma(t,r(t))dW(t)$

This is a general way of describing r-dynamics but there are a lot of different approaches in the literature used to outline this relation i'm going to list the most popular ones.

1. Vasiček

$$dr = (b - ar)dt + \sigma dW (a > 0)$$

- Where b ar is the drift term
- Where σ is the diffusion
- 2. Cox-Ingersoll-Ross (CIR)

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dr = a(b - r)dt +
$$\sigma \sqrt{r}$$
 dW

- Where a(b r) is the drift term
- And $\sigma \sqrt{r}$ is the diffusion
- 3. Dothan

$$dr = ardt + \sigma rdW$$

- Where ar is the drift term
- And or is the diffusion term
- 4. Black-Derman-Toy

$$dr = \Theta(t)rdt + \sigma(t)rdW$$

- Where $\Theta(t)$ r is the drift term
- Where $\sigma(t)r$ is the diffusion term

5 APPROACH TO THE ESTIMATION OF THE PARAMETERS OF MARTINGALE MODELS WITH PARTICULAR FOCUS ON VASYCEK MODEL

Now we will consider how to estimate the various variable that characterize those models, since we want to focus on Vasiček, I will estimate the parameters related to it. Basically we need to estimate a, b and σ , the problem with this situation is that we can't use a standard estimation for the parameters of an SDE because since we are modelling r under the Martingale Q measure, we can't make observation in the real world, because when we observe in the real world we are observing under the objective measure P.

In fact if we use standard statistical procedures we won't get Q-variables but something which is non sensical.

Even though we have no clue on how to estimate the parameters it is possible to demonstrate that the parameter σ is the same under both P and Q so it can be estimated by the data.

Differently the drift term is a total different story, first of all we can notice that the martingale measure is decided by the market, so to gather data about the drift term we can collect price information from the market by inverting the yield curve.

The process works like this:

• We define a model that possess several parameters and denote all the parameters under vector $\boldsymbol{\beta}$

$$dr(t) = \mu(t,r(t);\beta)dt + \sigma(t,r(t);\alpha)dW(t)$$

• We solve this for every time of maturity N and the term structure equation is

$$F_t^N \mu F_r^N + 1/2\sigma^2 F_{rr}^N - rF^N = 0$$

 $F^N(N,r) = 1$

So now we have found the theoretical term structure

$$P(t,N;\beta) = F^{N}(t,r;\beta)$$

- First of all we can collect price data from the market, infact we can observe the bond price at time 0 for all maturities N. Moreover we can denote the following term structure {p^{*}(0,N); N ≥ 0}.
- Moreover we will choose the value for β such that the theoretical term structure $\{p(0,N;\beta); N \ge 0\}$ is compatible with the empirical term structure $\{p^*(0,N); N \ge 0\}$. This allow us to find the value for the parameter β .
- Now if we insert β^* into μ and σ , we have found the particular martingale measure we were looking for.
- After having defined our martingale measure Q, we can price interest rate derivatives like for example Y = r(r(N)), so the price process is the result of Π(t; r) = K(t,r(t)) where K solves the term structure equation

Of course since some of the models are much easier to find than others, we have the necessity to define the subject called "affine term structures".

Hypothesize that the term structure $\{p(t,N); 0 \le t \le N, N > 0\}$ is defined as

$$p(t,N) = F(t,r(t);N)$$

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Where F is

$F(t,r(t);T) = e^{A(t,N)-B(t,N)r}$

A and B are deterministic functions, then the model is said to have an affine term structure (ATS).

Since having an affine term structure makes things easier from an analytical and computational point of view so we need to find some signals that help us to recognize the presence of affine term structures.

Our goal is to find values for μ and σ for the affine term structure.

We start by assuming the following dynamics for r

$dr = \mu(t,r(t))dt + \sigma(t,r(t))dW(t)$

Also let's assume that this equation possesses an ATS which implies that the bond prices respect the following form $F(t,r(t);T) = e^{A(t,N)-B(t,N)r}$, so from this condition we can compute the partial derivatives of F, which is the variable that solves the term structure equation, consequently we find

$$A_t(t,N) - \{1 + B_t(t,N)\}r - \mu(t,r)B(t,N) + 1/2\sigma^2(t,r)B^2(t,N) = 0$$

- Where A and B are deterministic functions
- μ is the drift term
- σ^2 is the diffusion term

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The boundary value for the term structure equation F(N,r) = 1 tells that:

The equation outlines the relation necessary to hold between B, σ , μ and A for an ATS to exist but there are a lot of different values that μ and σ not for all of the exist A and B such that the equation above is satisfied.

So we want μ and σ such that A and B are functions able to solve the equation above, in general it is possible to observe that if μ and σ are affine functions, with respect to r, then the equation can be separated into two different differential equations for the unknown functions A and B.

We assume that μ and σ have the following form

 $\mu(t,r) = \rho(t)r + \kappa(t)$ σ(t,r) = $\sqrt{\lambda(t)r + v(t)}$

After collecting terms the equation becomes

 $A_{t}(t,N) - \kappa(t)B(t,N) + \frac{1}{2} v(t)B^{2}(t,N) - \{1 + B_{t}(t,N) + \rho(t)B(t,N) - \frac{1}{2} \lambda(t)B^{2}(t,N)\} = 0$

This holds for all values of t,N and r, so we fix the time variables and since the equation holds for all values of r than all the coefficient of r have to be 0.

So the equation becomes

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$B_t(t,N) + \rho(t)B(t,N) - \frac{1}{2}\lambda(t)B^2(t,N) = -1$

This is the equation necessary used to find the value of B that respect the conditions of Affine Term Structures.

Giving that the r term is zero it is clear that the other term must also be eliminated giving the equation

$A_{t}(t,N) = \kappa(t)B(t,N) - \frac{1}{2}\lambda(t)B^{2}(t,N)$

This equation is used to find the value of A that respect the conditions of Affine Term Structures.

Now we will sum up all the assumptions and equations for ATS to give a general framework.

We assume that μ and σ , as we already said, are defined like this

$$μ(t,r) = ρ(t)r + κ(t)$$
 $σ(t,r) = \sqrt{\lambda(t)r + v(t)}$

Then the model possess an ATS of the form $F(t,r(t);T) = e^{A(t,N)-B(t,N)r}$, consequently A and B satisfy the system

$$B_t(t,N) + \rho(t)B(t,N) - \frac{1}{2}\lambda(t)B^2(t,N) = -1$$

B(N,N) = 0

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$\begin{aligned} \mathsf{A}_{\mathsf{t}}(\mathsf{t},\mathsf{N}) &= \kappa(\mathsf{t})\mathsf{B}(\mathsf{t},\mathsf{N}) - \frac{1}{2}\,\lambda(t)\mathsf{B}^2(\mathsf{t},\mathsf{N})\\ \mathsf{A}(\mathsf{N},\mathsf{N}) &= \mathbf{0} \end{aligned}$

We see that equation $B_t(t,N) + \rho(t)B(t,N) - \frac{1}{2}\lambda(t)B^2(t,N) = -1$ is a Riccati equation, which is an ordinary differential equation quadratic in the unknown variable, in this situation related to the determination of B without using A.

Than by substitution and integration we can find A.

We can prove that if μ and σ are time independent, than the necessary condition to have an ATS is that those variables are affine, moreover almost all the model listed before have an ATS apart from the Dothan and the Black-Derman-Toy.

In particular some models are surely easier than others to deal with, for example Vasiček and Hull-White mark out the short rate using a linear SDE.

Such equations characterized by r-processes are normally distributed and bond prices are defined by equations similar to this:

p(0,N) = E[exp{-
$$\int_0^N r(s) ds$$

In fact to compute bond prices for a model with such distribution, is similar to compute the expected value of a log-normal stochastic variable.

6 COMPUTATION OF VASIĈEK MODEL TERM STRUCTURE

The focus of this chapter is to derive the term structure for Vasiček model, in particular we will use the ATS theory developed before to compute what we are looking for. We know that short rates for Vasiček are given by the following equation

$dr = (b - ar)dt + \sigma dW$

It can be seen that this model possess the characteristic of being mean-reverting, which means that the equation will tend to revert to the mean b/a.

So equations used to find the values for A and B when we were taking in consideration ATS now become:

 $B_t(t,N) - aB(t,N) = -1$ B(N,N) = 0

 $A_t(t,N) = bB(t,N) - \frac{1}{2} \sigma^2 B^2(t,N),$ A(N,N) = 0

By fixing N for the first equation, we have a simple linear ordinary differential equation in the t-variable. Which can be solved as:

$$B(t,N) = 1/a \{1 - e^{-a(N-t)}\}$$

Differently integrating the second equation we get:

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$$A(t,N) = \sigma^2/2 \int_t^N B^2(s,N) ds - b \int_t^N B(s,N) ds$$

If we substitute for B the equation above we obtain the following equations :

 $\mathbf{p(t,N)} = \mathbf{e}^{A(t,N) - B(t,N)r(t)}$

Where

$$B(t,N) = 1/a \{1 - e^{-a(N-t)}\}$$

$$A(t,N) = \frac{\{B(t,N) - N + t\}(ab - 1/2\sigma^2)}{a^2} - \frac{\sigma^2 B(t,N)}{4a}$$

In case we are talking about bond options there's a specific formula to deal with them, if we are willing to use Vasiček model and Hull-White model

$$C(t,N,K,S) = p(t,S)F(d) - p(t,N) \cdot K \cdot F(d - \sigma_p)$$

Where:

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p(t,N)K

$$\sigma_{p} = 1/a \{ 1 - e^{-a(S-T)} \} \cdot \sqrt{\sigma^{2}/2\{1 - e^{-2a(N-t)}\}}$$

The data we are going to use to estimate the price are the Euribor with maturity one month with monthly granularity from 1st of January of 1999 to the 3rd of April of 2017. Those data are based on 220 observations.

We start by recalling the equation of the Vasiček model :

$$dr(t) = \alpha(\beta - r(t))dt + \sigma dW$$

$r(0) = r_0$

- Where α is the speed of mean reversion
- β is the long run mean
- σ is the instantenous volatility, or also called the diffusion term
- W is the Weiner process

7 EMPIRICAL EVALUATION OF BOND PRICE THROUGH VASIĈEK MODEL

7.1 Necessary theoretical assumptions

The focus of this chapter is to gather some data in order to calculate the bond price using Vasiček model.

Before calculating the bond price there is a question that is needed to be answered, Which is related to the estimation of the parameters b and a which are necessary to find the bond value.

We know that the general equation for Vasiček model is:

$$dr = (\beta - \alpha r)dt + \sigma dW(t)$$

Where W(t) is a random Wiener process modelling under the risk neutral measure $(\alpha>0)$.

We start to resolve the following differential equation by taking the derivative of $e^{at}r_t$ which gives the following result:

$$d(e^{\alpha t}r_t) = r_t \alpha e^{\alpha t} dt + e^{\alpha t} dr_t$$

Substitute the first equation into the second to obtain the following expression:

$$d(e^{\alpha t}r_{t}) = r_{t}\alpha e^{\alpha t}dt + e^{\alpha t} \left[\alpha \left(\beta - r_{t}\right) dt + \sigma dW_{t}\right]$$

Rearranging this equation:

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$$d(e^{\alpha t}r_t) = e^{\alpha t}[\alpha\beta dt + \sigma dW_t]$$

If we take the integral from s to t for both sides of the equation:

$$\begin{aligned} \int_{s}^{t} d(e^{\alpha t} r_{u}) &= \int_{s}^{t} e^{au} \left[\alpha \beta du + \sigma dW_{u}\right] \\ e^{\alpha t} r_{t} - e^{\alpha s} r_{s} &= \alpha \beta \int_{s}^{t} e^{\alpha u} du + \sigma \int_{s}^{t} e^{\alpha u} dW_{u} \\ e^{\alpha t} r_{t} - e^{\alpha s} r_{s} + \beta \left(e^{\alpha t} - e^{\alpha s}\right) + \sigma \int_{s}^{t} e^{-\alpha (t-u)} dW_{u} \end{aligned}$$

By multiplying both sides of the equation for e^{-at}, we have the solution of the differential equation:

$$\mathbf{r}_{t} - \mathbf{e}^{-\alpha(t-s)} \mathbf{r}_{s} + \beta (\mathbf{1} - \mathbf{e}^{-\alpha(t-s)}) + \sigma \int_{s}^{t} \mathbf{e}^{-\alpha(t-u)} d\mathbf{W}_{u}$$

r_t is described by this equation which follows a Gaussian distribution with mean:

$$\mathsf{E}[\mathsf{r}_{\mathsf{t}}] = \beta + (\mathsf{r}_{\mathsf{s}} - \beta) \, \mathrm{e}^{-\alpha(\mathsf{t} - \mathsf{s})}$$

And Variance

 $Var(r_t) = \frac{\sigma^2}{2\alpha (1-e^{-2\alpha(t-s)})}$

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To estimate the parameters that we use to calculate the price we need to use the SDE found before so:

$$\mathbf{r}_{t} = \mathbf{e}^{-\alpha(t-s)} \mathbf{r}_{s} + \beta \left(\mathbf{1} - \mathbf{e}^{-\alpha(t-s)}\right) + \sigma \int_{s}^{t} \mathbf{e}^{-\alpha(t-u)} d\mathbf{W}_{u}$$

Where r_t is a random variable with mean and variance:

Mean: $e^{-\alpha(t-s)}r_s + \beta (1 - e^{-\alpha(t-s)})$

It is assumed that rt follows this process:

$$\mathbf{r}_{t1} = \mathbf{a} + \mathbf{b}\mathbf{r}_{t-1} + \mathbf{\delta}\mathbf{\varepsilon}_{t1}$$

To estimate the coefficient from the process we need to assume the SDE to be discrete on the following time horizon:

 $\Lambda t = t_t - t_{t-1}$

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$\Lambda r_t = \alpha (\beta - r_{t-1}) \Lambda t + \sigma \Lambda \epsilon_t$

Where ε_t is a Gaussian white noise ($\varepsilon \approx N(0,1)$)

Parameters of the equation defining the process are:

a = b (1 - e<sup>-
$$\alpha \wedge t$$</sup>)
b = e^{- $\alpha \wedge t$}
 $\delta = \sigma \sqrt{\frac{1 - e^{-2\alpha \wedge t}}{2\alpha}}$

Rearranging for the three parameters a, b and σ we have the equations necessary to find them:

b Ina

$$\beta = -----; \sigma = \delta \sqrt{\frac{-2ln(a)}{\Lambda t(1-a^2)}}$$

1-a Λt

According to the OLS theory, the least square equation for a, b and ϵ_t are the following:

$$b = \frac{\sum_{i=1}^{n} r(t) - \mathbf{a} \sum r(t-1)}{n}$$

$$n\sum_{i=1}^{n} r(t-1)r(t) - \sum r(t-1)\sum r(t)$$

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$$d = \frac{1}{n\sum r(t-1)^2 - (\sum r(t-1))^2}$$

$$\epsilon_{ti} = \sqrt{n \sum r(ti)^2 - (\sum r(ti))^2}$$

7.2 Calculation of the bond price

The data we are going to use to estimate the price of a bond are the Euribor with maturity 1 month and monthly granularity, the dataset is based on 220 observations starting from the 1st of January of 1999 and finishing to the 3rd of April of 2017. To find the bond value related to this data, we use R the statistical software and run the following commands:

Vasicek.OLS = function (euribor_1m_monthly_csv, dt)

This command is fundamental because it sets the function that we are going to use and how we are estimating the parameters of the model (OLS estimation). To define the length of the data, which in our case is denominated as "N" we use the following command:

N = length(euribor_1m_monthly_csv)

To isolate the only numeric variable in the data, which are the rates, we run this command:

euribor_1m_monthly_csv\$rate

For the sake of our thesis we need to mark out respectively the rate in t and the rate in t-1, which in our situation since the length of the dataset is N, the rates will be r_N and r_{N-1} , so:

r_N = rate = euribor_1m_monthly_csv\$rate[N]

r_{N-1} = lagrate = euribor_1m_monthly_csv\$rate[(N-1)]

Now to find the parameters for the Vasiček equation we will use the OLS method with the formulas associated to it for the estimation, we know that our $\Lambda t = 1/12$ since the Euribor rates have all maturity one month.

a = (N*sum(rate*lagrate)-sum(rate)*sum(lagrate))/ (N*sum(lagrate^2)sum(lagrate)^2)

b = sum(rate)-a*(sum(lagrate))/N

Recall that the Vasiček model is defined as:

$$dr(t) = \alpha \left(\beta - r(t)\right) dt + \sigma dW(t)$$

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In this chapter we have called a and b respectively α and β to make things easier to understand since the formula of the OLS estimation use this denomination for the parameters. Finding α require use the following command:

$\alpha = -\log(a)/dt$

Where α is the speed of mean reversion, to find β we use the following formula:

$\beta = \alpha/1-b$

Where β is the long run mean, by some manipulation of the previous formulas we can define a new function v2, for the purpose of finding σ , which is equal to:

v2=sum((rate-lagrate* α - β *(1-a)) ^2)/N

While σ is equal to

$\sigma = sqrt(2*\alpha*v2/(1-a^2))$

Where σ is the diffusion term associated with the Weiner Process (W).

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The values obtained for the 3 parameters are:

- α= 0.6272586
- β= -1.5949913
- σ= 1.9302522

From those parameters we can calculate the price of the bond, recall from the chapters before that the price of a bond is equal to:

$$\mathbf{p(t,N)} = \mathbf{e}^{A(t,N) - B(t,N)r(t)}$$

Where A and B are equal to:

$$B(t,N) = 1/a \{1 - e^{-a(N-t)}\}$$

$$A(t,N) = \frac{\{B(t,N) - N + t\}(ab - 1/2\sigma^2)}{a^2} - \frac{\sigma^2 B(t,N)}{4a}$$

In those two equations a and b are our α and $\beta,$ consequently the values for A and B are:

- A(t,N) = 4.5427003
- B(t,N) = 0.08119280

After having calculated those values the price of the bond is:

P(t,N) = 73.856646

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8 CONCLUSIONS

The main focus of this thesis is to give a general framework for short rate models, in particular by highlighting the fundamental assumptions necessary to make them valid. In addition we denoted a general term structure equation for those models, which is necessary to understand what is explained in the chapters ahead.

After that we have listed the most famous short rate models, focusing particularly on Vasiček, we outlined the importance of using a martingale measure for those rates by giving clear motivations.

The next steps were to define analytically and theoretically the parameters that were characterizing Vasiček model and to derive its term structure which is fundamental to find the equation for the price.

Finally after having gathered data on EURIBOR with maturity 1 month and after having estimated through the OLS approach the equations necessary to calculate Vasičeks model parameters ; we used Rstudio to find the price of a bond associated with those EURIBOR rates.

9 BIBLIOGRAPHY

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10 APPENDIX

In this appendix we will give a specific definition for the variable W; we start by saying that W is a "standard (one-dimensional) Wiener Process", such process is stochastic defined as {Wt}t>=0 indexed by non-negative values of t.

Also W is possess the following properties:

- W₀ = 0
- With probability equal to 1, the function t -> W_t is continuos in t
- The process {W_t}_{t>=0} has "stationary independent increments"
- This increment W_{t+k}-W_k has this Normal distribution (0,t)

One of the main reasons why the Brownian Motion is fundamental in probability theory is because it is similar to a limit of random rescaled random walks.