Different methods for pricing barrier options

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Presentation

The principal aim of this thesis is to revise different methods used to price barrier options and how they perform in different situations, providing some numerical examples.
Options are financial instruments which value depends on the value of another asset, called **underlyings**, that may be a stock, a future and so on. Their utilization is increased during these years since they can be useful for many different scopes, as hedging, in order to mitigate the risk taken in an investment with opposite position, and speculation, trying to predict the direction in which markets will move. Nowadays is possible to find a large variety of options thanks to the fact that the financial markets have evolved over time and many particular categories have been developed.
For example one very widespread and used are **path-dependent** options which value does not depend only on the final value of the underlyings but also from the values achieved during a predefined interval. Exotic options are part of this group and their computation is more diffult than normal vanilla also because their underlying instrument may be non-standard: among them we can count asian options, which value is given by the average value of the asset during the interval, then lookback options' value may depend on the maximum or minimum reached by the asset and finally barrier options which we will analyze in details later on.
Barrier options are similar to plain vanilla, but in the first one there is a barrier, which may be one constant or two constant or they may have also time dependent barriers. If during the contract time the underlying breaches the barrier the option will be valid or null depending on the nature of the option.

Among barrier options there are different categories, with different characteristics:

- *down-in*: when the option starts having value if the underlying touches the barrier going down;
- *down-out*: the same of a down-in but with the difference that if the underlying reaches the barrier the option becomes null;
- *up and in*: the options is validated if the underlying breaches the barrier going up;
- *up and out*: as an up-in, but in this case the option becomes null when the barrier is reached.

After this brief introduction is useful to make some theoretical assumptions in order to develop the scenario in which we will analyze these options. In general, the work will be structured as follows:

1. First of all will be discussed the mathematical and probabilities tools that we need to operate in options context, as probability spaces, stochastic processes, Black-Scholes model and its properties and so on.

2. Then we will introduce continuous monitored barrier options and some methods used to evaluate them, in particular Monte Carlo simulations (with some corrections), Merton analytics formula and trinomial model with the Ritchken correction.

3. In this chapter, instead, will be analyzed the case of discrete monitored barrier options using the correction applied to the closed Merton formula and developing a trinomial lattice discussed by Broadie, Glasserman and Kou.

4. In the conclusion there will be some considerations on the different methods used with the respective MatLab codes.

1 Probability Spaces

First of all we have to introduce the concept of probabilty space: it is defined by the following parameters $(\Omega, \mathcal{F}, \mathbb{P})$ where:

- Ω is the set of all possible outcomes, for example if we are dealing with the price of an asset Ω is a path space.
- $\mathcal{F}$ is a collection of subsets of Ω and is called *sigma-algebra* if it satisfies the following properties :
  1. Ω is in $\mathcal{F}$, and Ω is considered to be the universal set in the following context.
  2. if a subset A is contained in $\mathcal{F}$ then also A^C is in $\mathcal{F}$
  3. if A_i for $i = 1, 2, 3..$ are sets contained in $\mathcal{F}$ then also their *union* is contained in $\mathcal{F} \to A = \cup_{i=1}^{\infty} A_i \subset \mathcal{F}$
- $\mathbb{P}$ is a non-negative normalized measure on the events, for example if A is an event then $\mathbb{P}(A)$ represents the measure of chances that A will happen; it also has some properties:
  1. $\mathbb{P} : \mathcal{F} \to [0, 1]$
  2. $\mathbb{P}(\Omega) = 1$ (normalization property)
  3. $\mathbb{P}$ is countably additive: if $(A_n)_n$ is a sequence of disjoint events, then

$$\mathbb{P}(\sqcup_n A_n) = \sum_n \mathbb{P}(A_n)$$

These probability spaces are very important for our studies because is the place where our simulated processes will move, hence know their theoretical characteristics is fundamental. Let's now consider a real random variable X. X on $(\Omega, \mathcal{F})$ is a function on Ω which takes values in $\mathbb{R}$:

$$X : \Omega \to \mathbb{R}$$

and is $\mathcal{F}$-measurable, so the counter-image of any half line $(-\infty, x]$ is an event:

$$\{X \leqslant x\} \in \mathcal{F}$$

for all $x \in \mathbb{R}$. It may happen that $\mathcal{F}$ is too big: in this situation we may use $\mathcal{G}$ which is a subset of $\mathcal{F}$; for the sake of simplicity let's impose that $\mathcal{F}$ is small enough, generally speaking we can say that:

$$\sigma(X) := \sigma(\{X \leqslant x\} | x \in \mathbb{R})$$

so the sigma algebra (σ) generated by a random variable is a sub-sigma algebra of $\mathcal{F}$ and it is the smallest 'space' for which X is measurable on Ω.
In Finance is important to define our time horizion and it varies depending on what is the aim of the engaged activity, for example pension funds which usually have portfolios with low risk and longer maturity will have a higher T than investment banks assets which have different functions and usually adopt riskier strategies with smaller maturities.
Having said that, it's easy to understand that a *filtered* probability space is the space analyzed with information available till time $t < T$ $(\Omega, (\mathcal{F}_t)_{t<[0,T]}, \mathbb{P})$.

1.1 Stochastic Processes

Stochastic processes define the path followed by variables which change value over time. They can be classified in discrete time processes, when the variable changes value at predetermined istants of time, and continue time processes, when the variable may change the value in an interval of time (for example consider a contract which monitoring is over the manteinance period).
Given a filtered space $(\Omega, (\mathcal{F}_t)_{t<[0,T]}, \mathbb{P})$ we consider a stochastic process $S = S(t)_t$ a collection of real valued random variables measurable from $(\Omega, \mathcal{F})$ to $\mathcal{R}$.
If we want S to be random but non anticipative the process must respect some conditions in order to be adapted to the filtration, or $\mathcal{F}_t$-measurable:

- for any fixed time t,
$$S(t) : \Omega \longrightarrow \mathbb{R}$$
- for all fixed reals x, the set $\{S(t) \leqslant x\}$ belongs to $\mathcal{F}_t$

Computing the Cumulative Distribution Function of $S(t)$ we know that

$$P(S_t \leq x)$$

and in the case of path dependent options we need the joint distribution of $S(t_1), S(t_2)$:

$$P(S(t_1) \leq x_1, S(t_2) \leq x_2)$$

can be defined as the sets $\{S(t_1) \leq x_1\} \cap \{S(t_2) \leq x_2\}$ for any x_i that belongs to $\mathcal{F}_{t2} \subseteq \mathcal{F}_T = \mathcal{F}$. Actually computation in this case is very difficult because we should consider correlation between all the dates considered in the process.

Expectations The average value of a random variable X weighted according to the probability of happening is know as mean of X and is also called expected value.
For a discrete random variable X:

$$E[X] = \sum_i X_i \cdot p_i$$

if X is instead continuous, with density p_X then $P(x < X \leq x + dx) = p_X(x)dx$ and we have:

$$E[X] = \int x p_X(x) dx$$

The integral must be finite and it has to exist otherwise these definitions do not hold. Expectations are linear operation, hence the expectation of the linear combination of two random variable X and Y is computed without the need of the joint distribution:

$$E[aX + bY] = aE[X] + bE[Y]$$

The same cannot be said, for example, for the computation of $E[XY]$ because one variable may influence the other and we have to correct the combination with the correlation factor, or as we will do later we may state independence as assumption. Now imagine that we have a continuous random variable, X, and a deterministic function of such variable, Y:

$$Y = g(X)$$

and for example X is the price of a stock at time T and g represents the payoff function of this stock, so one way to compute the price of the option at time 0 is to calculate the expected value of function Y. But in the case that the function g is and indicator (for example it assumes discrete value 0 or 1, as we wil see later in the case that an event Σ happens) Y will not have a density. Nevertheless if g is invertible and differentiable with $g' \neq 0$ the density of y is given by:

$$p_Y(y) = p_X(g^{-1}(y)) \frac{1}{|g'(g^{-1}(y))|}$$

So the expected value of Y is :

$$E(Y) = \int y p_Y(y) dy$$

and when g is regular and invertible :

$$\int yp_X(g^{-1}(y))\frac{1}{|(g')g^{-1}(y)|}dy = \int g(x)p_X(x)dx$$

considering that, by definition, $x = g^{-1}(y)$. This last formula can be always used because it is based on the density of X so also if Y is an indicator and has discrete values as before it gives us the expected value useful to price the option.

Independence Suppose we have two random variable X and Y, which respectively have probability density p_X, p_y, and joint density probability $p(x, y) = P_X(x)p_Y(y)$. If they are uncorrelated then the expected value of their product will be:

$$E[XY] = E[X]E[Y]$$

and their correlation:

$$E[(X - E[X])(Y - [Y])] = 0$$

Dealing with a general bivariate Gaussian function and given that ρ is the correlation coefficient between X and Y we know that its density is :

$$p_{X,Y}(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}exp(-\frac{1}{2(1-\rho^2)}[\frac{(x-\mu)}{\sigma_1^2} - 2\rho\frac{(x-\mu)}{\sigma_1}\frac{(y-\nu)}{\sigma_2} + \frac{(y-\nu)^2}{\sigma_2^2}])$$

when both function are distributed as normal functions respectively with mean μ and ν and variances σ_1^2 and σ_2^2. When the correlation is equal to zero, so they are indipendent functions, we can rewrite the joint density function as the product of the marginal densities. So generalizing the formula when $\rho = 0$ and $\underline{\mu}$ is the vector of means and Σ is the variance-covariance matrix we know that n-variate Gaussian distribution has probability density funtion:

$$p(x_1, ...x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{det(\Sigma)}}exp((\underline{x} - \underline{\mu})'\Sigma^{-1}(\underline{x} - \underline{\mu}))$$

As demonstrated above independence always implies uncorrelation but the opposite is not true.

1.2 Conditional expectation and martingales

The sigma algebra $\mathcal{F}_T$ of a filtered space $(\Omega, \mathcal{F}_t, \mathbb{P})$ represents the information we possess relative to a particular event at time $t < T$, so if we have a set of information $\mathcal{F}_{t1}$ we are

interested in finding the conditional expectation of a process given a set of information we already have.
Consider for example the case of a process Y which value is know at t_2, the conditional expectation of Y at $t_1 < t_2$ is expressed as:

$$E[Y \mid \mathcal{F}_{t_1}]$$

so it is the best prediction we can make at t_1 given the information we have.
We define now the trivial sigma algebra as $\mathcal{F}_0$ when we don't have relevant information about our process Y and it is equal to:

$$\mathcal{F}_0 = \{0, \Omega\}$$

given that the information in the bracket are constants we can say:

$$E[Y \mid \mathcal{F}_0] = E[Y \mid c] = E[Y]$$

Martingales An adapted process M is defined as a martingale if:

$$E[M(t)|\mathcal{F}_s] = M(s)$$

for all $0 \leqslant s < t \leqslant T$.
To introduce this concept we will make an example of a game. At time T the process $S(T)$ will have payoff $S(T) = \Phi$. If S is a martingale the conditional expectation of the future payoff Φ at time t is equal to the current price $S(t)$ at time $t \in [0, T]$, therefore:

$$[S(T) \mid \mathcal{F}_t] = S(t)$$

1.3 Standard Brownian Motion

First of all, in order to talk about Brownian Motions, we have to introduce Markov processes

Markov processes These are stochastic processes for which the current price explains all the relevant information about the stock's history. Given this definition, the future value of the stock does not depend from the past and we just need the current price to make predictions.
A Markov process S is an adapted process such that, for every deterministic function $g = g(x)$, and for any arbitrary dates $t < T$, the conditional expectation of $g(S(t))$

satisfies

$$E[g(S(t)) \mid \mathcal{F}_t] = E[g(S(t)) \mid \mathcal{S}_t] = \overline{g}(S(t))$$

and only the information contained in the present value (t) of the process is needed to make the best prediction on the future value of $g(S(t))$

Brownian Motion Stocks prices follow particular paths called Brownian motion, or Wiener process, because their movement is the same used to describe the pattern realized by particles which have many molecular shocks. These processes moves with zero drift and unit variance, but we need also to explain the properties more in details.
More formally, in a filtered space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, taken t as a continuous time parameter, $W = (W(T))_{t\leqslant T}$ is a Brownian Motion if

- $W(0) = 0$
- W is adapted to the filtration
- for any $s < t$, the increment$W(t) - W(s)$ is independent of $\mathcal{F}_s$, and has distribution $N(0, t-s)$
- the paths $W(*, \omega)$ are continuous

from this definition we can say that:

- marginal distributions are Gaussian, for any t we can write $W(t) - W(0)$ which is normally distributed with $N(0, t)$
- Brownian Motions have increments identically independent: for any $u < s$ we can conclude that $W(u), W(t) - W(s)$ are independent and therefore have a joint normal distribution $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & t-s \end{pmatrix}\right)$

In general fixing $0 \leqslant t_1 < t_2 < ... < t_n \leqslant T$ we obtain n increments $W(t_1), W(t_2) - W(t_1), ..., W(t_n) - W(t_{n-1})$, independent and jointly Gaussian distributed, with:

$$N\left(\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} t_1 & 0 & \cdots & \cdots & 0 \\ 0 & t_2 - t_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t_n - t_{n-1} \end{pmatrix}\right)$$

In addition to being Markov processes, Brownian Motions are also martingales.

Proof : consider two different dates $t \leq T$ and write $W(T) = W(T) - W(t) + W(t)$. The conditional expectations $E[W(T) \mid \mathcal{F}_{\sqcup}]$ is:

$$E[W(T) \mid \mathcal{F}_{\sqcup}] = E[W(T) - W(t) + W(t) \mid \mathcal{F}_{\sqcup}] = W(t) + E[W(T) - W(t)] = W(t)$$

Geometric Brownian Motion Consider a Brownian Motion with drift μ and volatility σ which describes the path of a process Y. The exponential transform is

$$Y(t) = exp(X(t)) = exp(\mu t + \sigma W(t))$$

given that $X(t) = \mu t + \sigma W(t)$ and $W(t) \sim N(0, t)$. This expression is called Geometric Brownian Motion and, since it is the exponential of a Gaussian variable, its marginal distributions are lognormals (we will explain this concept in a moment)
then is easy to demonstrate that Geometric BM are also Martingales: the story is the same, if we write $W(t) = W(t) - W(s) + W(s)$, for $s < t$ we get the expectation

$$E[exp(\mu t + \sigma W(t))|\mathcal{F}_s] = exp(\mu t + \sigma W(s))E[exp(\sigma(W(t) - W(s))]$$

and the expectation is a Gaussian variable and has a normal distribution $N \sim (0, t - s)$, therefore is equal to $exp(\frac{\sigma^2}{2}(t - s))$, infact we obtain:

$$E[exp(\mu t + \sigma W(t))|\mathcal{F}_s] = exp(\mu t + \frac{\sigma^2}{2}(t - s) + \sigma W(s))$$

we can notice that the GBM is a Martingale if and only if the drift $\mu = -\frac{\sigma^2}{2}$ because

$$E[e^{\sigma(\sqrt{t-s})x}]$$

where x is an extraction from a random normal distribution and

$$\int e^{\sigma(\sqrt{t-s})x} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\Pi}} dx$$

$$\int \frac{e^{-\frac{1}{2}(x^2 - 2\sigma(\sqrt{t-s}) + (\sigma(\sqrt{t-s}))^2) + \frac{(\sigma(\sqrt{t-s}))^2}{2}}}{\sqrt{2\Pi}} dx$$

$$e^{\frac{(\sigma(\sqrt{t-s}))^2}{2}} \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(x - \sigma(\sqrt{t-s}))^2}}{\sqrt{2\Pi}} d(x - \sigma(\sqrt{t-s}))$$

given that the argument of the integral is the density function of a normal variable

$(x - \sigma(\sqrt{t-s}))$ it simplies to 1 given that the interval of validity is $[-\infty, +\infty]$ and we get that:

$$E[e^{\sigma(\sqrt{t-s})x}] = e^{\frac{(\sigma(\sqrt{t-s}))^2}{2}}$$

and is demonstrated that the GBM is a Martingale since all term depending on t disappears.

1.4 Itō's Formula

An Ito process is a generalized Wiener process in which parameters that describe its pattern are functions of underlying value x and time t; the following equation explains us the dynamics of the process:

$$dx = a(x,t)dt + b(x,t)dz$$

where a and b are respectively drift and standard deviation.

Itō's Lemma Price of options written on stocks is a function of stock price S and time t, in words it is in function of the stochastic variables which determine the underlying value.

Consider a general variable x which follows an Ito process and a deterministic smooth function F of (t,x). F varies only in response to changes in (t,x). Then F varies according the following equation:

$$dF(t,x) = F_t(t,x)dt + F_x(t,x)dx$$

If we make a second-order approximation we get:

$$dF(t,x) = F_t(t,x)dt + F_x(t,x)dx + \frac{1}{2}(F_{xx}(t,x)(dx)^2 + 2F_{tx}(t,x)dtdx + F_{tt}(t,x)(dt)^2)$$

Usually second order elements of an approximation are negligible and are rarely considered.

We now take t as the time paramater and consider a function Y which depends on time and on a Brownian motion W. We consider the following:

$$Y(t) = F(t, W(t))$$

Using our second order approximation we consider the following variation for Y in terms of t and W, our $dF(t, W(t))$ is therefore equal to:

$$F_t(t,W(t))dt+F_x(t,W(t))dW(t)+\frac{1}{2}(F_{xx}(t,W)(dW(t))^2+2F_{tx}(t,W(t))dtdW(t)+F_{tt}(t,W(t))(dt)^2)$$

In this case the second order approximations are important for the purpose of our study. Following the intuition that $dW(t) = W(t + dt) - W(t) \sim N(0, dt)$, we can approximate the square increment $(W(t))^2$ with its mean:

$$(W(t))^2 \sim dt$$

and Ito's Lemma gives us the dynamics of the function F:

Let $F(t, x)$ be a smooth funciton (the minimal regularity required is $C^{1,2}(t, x)$). The Markov process defined by:

$$F(t, W(t))$$

has dynamics given by the following stochastic differential equation:

$$dF(t, W(t)) = (F_t(t, W(t)) + \frac{1}{2}F_{xx}(t, W(t)))dt + F_x(t, W(t))dW(t)$$

Diffusion processes A diffusion, which is another name given to an Itō process, is any adapted process Y whose dynamics may be written as:

$$dY(t) = \alpha(t)dt + \beta(t)dW(t)$$

where α and β are two coefficients. The first one, α, is referred to as the drift of the process. In reality though, in Finance the practice is to call drift the fraction $\frac{\alpha(t)}{Y(t)}$. The second coefficient, β, is the diffusion of the process.
In the case of Brownian motions (B) with drift and Geometric Brownian motions (S) we have the following conditions. B verifies:

$$dB(t) = \mu dt + \sigma dW(t)$$

while the Geometric Brownian motion, $S(t) = exp(bt + \sigma dW(t))$ satisfies:

$$dS(t) = \left(b + \frac{\sigma^2}{2}\right) S(t)dt + \sigma S(t)dW(t)$$

where sometimes we can call $\mu = (b + \frac{\sigma^2}{2})$.

1.5 Black-Scholes-Merton model

The Black-Scholes-Merton model is very famous in finance and it represents the base for pricing derivatives. In this model we have only two assets, a risk free asset that will pay an interest rate r and a risky stock that evolves as a geometric brownian motion. The bond pays continuously an interest $r \geqslant 0$ and $B(t) \geqslant e^{rt}$, and has the following characteristics:

$$\begin{cases} dB(t) = rB(t)dt \\ B(0) = 1 \end{cases}$$

The risky stock satisfies instead a stochastic differential equation with an initial condition (Cauchy's Problem); and it is defined as:

$$\begin{cases} dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \\ S(0) = S_0 \end{cases}$$

here, S_0 is the market price of the risky stock at time zero and the terms μ and σ are constants, with $\sigma \geqslant 0$, which are called drift and volatility.
Solving the above Cauchy problem we get that the unique solution of S is given by:

$$S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$$

and because of the lognormal property of the Geometric Brownian motion, we know that the marginals of the process $S(t)$ satisfy:

$$ln \frac{S(t)}{S_0} \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$$

meaning that the mean and the variance of the stock logreturns grow over time linearly. Given these properties μ is defined as the exponential growth of the average stock price while volatility is the standar deviation of the annual logreturn.

1.6 Lognormal Property

If we consider a function Y of S such that:

$$Y = \ln S$$

and we want to derive the process of the $\ln S$, where $dS = \mu dt + \sigma dW(t)$, using Itō's Lemma we can state that the process followed by Y is:

$$dY = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW(t)$$

because

$\frac{\delta G}{\delta S} = \frac{1}{S} \qquad \frac{\delta^2 G}{\delta S^2} = -\frac{1}{S^2} \qquad \frac{\delta G}{\delta t} = 0$

It implies that:

$$\ln S_T - \ln S_0 \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right) T, \sigma^2 T\right)$$

and

$$\ln S_T \sim N\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T, \sigma^2 T\right)$$

Since $ln(S(t))$ has a normal ditribution and $S(T)$ is log-normal, it holds that the stock price at time T has a log-normal distribution.

1.7 Risk neutral valuation

For the lognormal property of stock prices we know that the mean and the variance grow linearly with time and we can say that :

$$E[S(t)] = S_0 e^{\mu T}$$

so μ represents the exponential growth of the average stock price. The variance instead is given by

$$Var[S(t)] = S_0^2 e^{2\mu T}(e^{2\sigma T} - 1)$$

One of the most important concept understanding BSM model is the risk neutral valuation that has great importance in the context of pricing options. Under this concept we assume that when we price derivatives we are risk neutral, and so we are ideally working in a 'risk neutral world'.

Naturally in the real world this is not the typical situation because investors ask for higher return when they are engaging risky activities, anyway this is just a pricing tool and in the real world prices are the same of those obtained from the 'neutral' world. There two fundamental characteristics in the risk neutral world:

1. first of all the expected return of stocks is the equal to the risk free rate;
2. then the risk free rate is also used to discount the expected payoff of the options.

As said before the stock prices follows the process:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

and considering with F the price of an option which depends on S for the Ito's Lemma F is a function of S and t. Hence the general differential equation:

$$dF(t,S) = F_t(t,S)dt + F_S(t,S)dS + \frac{1}{2}(F_{SS}(t,S)(dS)^2 + 2F_{tS}(t,S)dtdS + F_{tt}(t,S)(dt)^2$$

becomes:

$$dF(t,S(t)) = (F_S(t,S(t))\mu S + F_t(t,S(t)) + \frac{1}{2}F_{SS}(t,S(t))\sigma^2 S^2)dt + F_S(t,S(t))\sigma S dW(t)$$

where the last term of the equation $F_S(t,S(t))\sigma S dW(t)$ is the same of the differential equation of the stock price
Given that the option is influenced by the same source of uncertainty ($W(t)$) of the stock, it can be eliminated in a portfolio made by a short position on the option F and a long position on a certain quantity of stocks. The quantity needed for replication is:

$$F_S(t,S(t))$$

such that in a certain way the portfolio Θ is balanced.
The value of such portfolio will be:

$$\Theta = -F(t,S(t)) + F_S(t,S(t))S(t)$$

and the variation of Θ is:

$$d\Theta = -dF(t,S(t)) + F_S(t,S(t))dS(t)$$

Replacing respectively $dF(t,S(t))$ and $dS(t)$ with the respective formulas above we get

$d\Theta = F_S(t, S(t))\mu S + F_t(t, S(t)) + \frac{1}{2}F_{SS}(t, S(t))\sigma^2 S^2)dt + F_S(t, S(t))dW(t) + F_S(t, S(t))(\mu S(t)dt + \sigma S(t)dW(t))$

then, simplyfing this expression, we get

$$d\Theta = (-F_t(t, S(t)) - \frac{1}{2}F_{SS}\sigma^2 S^2)dt$$

in which the Wiener process (W(t)) has been eliminated.
Since now our portfolio is void of risk, its return must be equal to the return of risk-free assets in order to avoid arbitrage opportunities. Consider for example a bond that pays interest r, we now impose:

$$d\Theta = r\Theta dt$$

which can be written as

$$(F_t(t, S(t)) + \frac{1}{2}F_{SS}\sigma^2 S^2)dt = r[F(t, S(t)) - F_S(t, S(t))S(t)]dt$$

This is the Black-Scholes-Merton differential equation which has multiple solutions depending on which type of options we are considering. Later on we will demonstrate how is obtained the analytical formula used to price standard vanilla call and put options and barrier options.

1.8 Monte Carlo simulation

Monte Carlo is a numerical method used to evaluate options. It consists in a simulation of the path followed by the underlying stock which, using risk neutral valuation, will determine the value of the option.
This procedure is computed many times in order to have a large number of simulated paths (for example N = 10000) and each time it will obtained a final value for the option. Then it is computed the average of this final values and is determined the current value of the option using the risk-free rate discount factor.
Suppose that the asset price moves following the process :

$$dS = \mu S dt + \sigma S dW(t)$$

for a contract with maturity T, in the interval $[t, t+\Delta t]$, where $\Delta t = \frac{T}{J}$ S will be simulated as:

$$S(t + \Delta t) - S(t) = \mu S(t)\Delta t + \sigma S(t) Z\sqrt{\Delta t}$$

where Z has a normal standard distribution.
We do the above procedure for each Δt to get a simulation of the asset price which at time T will be

$$S(T) = S(0)e^{(\mu-\sigma^2/2)T+\sigma Z\sqrt{T}}$$

Depending on the type of option, it is calculated the value of the option at time T and then actualized at time 0, for example considering a standard Europen call we get:

$$C_i(T) = max(S(T) - K, 0)$$

for $i = 1, 2....N$ where N is the number of simulations computed. Taking the sum of all the payoffs and the computing the average we obtain:

$$C(T) = \frac{\sum_{N}^{i=1} C_i}{N}$$

and finally the price of the call at time 0 is equal to:

$$C(0) = e^{-rT}C(T)$$

2 Pricing methods for continuous monitored barrier options

2.1 Merton formula for pricing options

When we talk about path-dependent options it is possible to use different ways for pricing, for many of them it exists a closed formula which give us the exact price. In the context of barrier options Broadie, Glasserman and Kou developed some closed analytical forms starting from the BLack-Scholes-Merton model.
In this chapter we are going to see how the formula for continuous monitored barrier options is computed.
First of all we will see how Merton develop the closed formula for european call and put options. Assuming that we are in a risk neutral world, just as tool to price these options, we know that the option will move over time with drift $\mu = r - \frac{1}{2}\sigma^2$ and variance σ^2, and

$$dS_t = (r - \frac{1}{2}\sigma^2)dt + \sigma S_t dZ_t$$

and at time T, the stock value will be

$$E(S_T) = S_0 e^{\mu T}$$

in this case changing the drift and adding 'noise' to the process we get

$$E(S_T) = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}$$

2.1.1 Closed formula for Standard European options

Consider now a european call option which will pay $S_T - K$ if the stock price is above the strike price, or 0 otherwise. Its payoff is:

$$C_t = E[(S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z} - K)^+]$$

knowing that Z is a brownian motion and its jumps are independent we know that it distribution is normal, and also its density function, so rewriting the above expectation as an integral differentiated by Z we obtain:

$$e^{-rT}\int (S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z} - K)^+ \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\Pi}}dz$$

since only the positive value of the term in the bracket will give us a payoff higher than zero we restrict the validity of our integral in the interval [a,+∞) where $a = \frac{1}{\sigma\sqrt{T}}(ln(\frac{K}{S} - (r - \frac{1}{2}\sigma^2)T)$; hence:

$$e^{-rT}\int_a^{+\infty}(S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z} - K)^+ \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\Pi}}dz$$

$$e^{-rT}\int_a^{+\infty} S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\Pi}}dz - e^{-rT}K\int_a^{+\infty}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\Pi}}dz$$

$$e^{-rT}\int_a^{+\infty} S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\Pi}}dz - e^{-rT}KN(\frac{ln\frac{S}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}})$$

calling $\frac{ln\frac{S}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = y$ and proceeding with the first integral:

$$e^{-rT}S_0 e^{(r-\frac{1}{2}\sigma^2)T}\int_a^{+\infty} e^{\sigma\sqrt{T}Z}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\Pi}}dz - e^{-rT}KN(y)$$

$$e^{-rT}S_0 e^{(r-\frac{1}{2}\sigma^2)T}\int_a^{+\infty}\frac{e^{\sigma\sqrt{T}Z-\frac{1}{2}z^2}}{\sqrt{2\Pi}}dz - e^{-rT}KN(y)$$

adding and subtracting $\frac{\sigma^2 T}{2}$ to the exponent of the integrand we have:

$$e^{-rT}S_0 e^{(r-\frac{1}{2}\sigma^2)T}\int_a^{+\infty}\frac{e^{\sigma\sqrt{T}Z-\frac{1}{2}z^2+\frac{\sigma^2 T}{2}-\frac{\sigma^2 T}{2}}}{\sqrt{2\Pi}}dz - e^{-rT}KN(y)$$

since using this 'trick' we get the square of a binomial we can rewrite this exponent as $\frac{1}{2}\sigma^2 T - \frac{1}{2}(z - \sigma\sqrt{T})^2$ and, focusing only on the region where the option will have value:

$$e^{-rT}S_0 e^{(r-\frac{1}{2}\sigma^2)T+\frac{1}{2}\sigma^2 T}\int_a^{+\infty}\frac{e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2}}{\sqrt{2\Pi}}dz$$

with $z - \sigma\sqrt{T} = b$ and simplifing the term outside the integral

$$S_0\int_{\frac{1}{\sigma\sqrt{T}}(ln\frac{K}{S}-(r-\sigma^2/2)T)-\sigma\sqrt{T}}^{+\infty}\frac{e^{-\frac{1}{2}b^2}}{\sqrt{2\Pi}}db$$

$$S_0 N(\frac{1}{\sigma\sqrt{T}}(ln\frac{S}{K} + (r - \sigma^2/2)T) + \sigma\sqrt{T})$$

$$S_0 N(\frac{ln(S/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}})$$

so finally imposing $\frac{ln(S/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = x$ we get the Merton formula for an European call option:

$$C = S_0 N(x) - e^{-rT} K N(y)$$

To understand in deep this formula we can rewrite it as follows:

$$C = e^{-rT} [S_0 e^{-rT} \frac{N(x)}{N(y)} - K] N(y)$$

where e^{-rT} is the discount factor, $S_0 e^{-rT} \frac{N(x)}{N(y)}$ is the expected value of S at time T conditioned by $S_T > K$ and $N(y)$ is the probability that the call will be exerted.
Then, in the case of a put option, the formula will be:

$$P = e^{-rT} K N(-y) - S_0 N(-x)$$

2.1.2 Closed formula for Barrier options

Running maximum and minimum In this section the Merton formula for standard optiond will be extended to barrier options.
First of all we have to underline that is possible to compute the value of these options because it can be calculated the running maximum and minimum processes on which depends a general stochastic process X and their density functions.
Consider an asset $S(t)$ with $0 \leq t \leq \infty 4$ and a costant barrier H, we define the event τ_H as the hitting time of H, and for a general down-in we can say that:

$$\tau_H = inf t \geq 0 \mid S(t) = H$$

The S-process absorbed at H is defined by

$$S_H(t) = S(t \wedge \tau)$$

we denote with $\varphi(z; 0, 1)$ the general density function for a variable z and the cumulative distribution function $N(z)$ is:

$$N(z) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}z^2} dz$$

and, as we said before, considering a Wiener process:

$$\begin{cases} dS(t) = \mu S(t) dt + \sigma S(t) dW(t) \\ S(0) = S_0 \end{cases}$$

what we are searching for is the distribution of $S_H(t)$, that is the marginal distribution of the S-process, absorbed at the point H (in barrier options context we mean the region in which the price of the underling stock touch the barrier H)
The density function $f_H(S;t,S_0)$ which is the density of the absorbed process $S_H(t)$ is given by:

$$f_H(S;t,S_0) = \varphi(S;S_0+\mu t,\sigma\sqrt{t}) - e^{-\frac{2\mu(S_0-H}{\sigma^2}}\varphi(S,\mu t-S_0+2H,\sigma\sqrt{t})$$

naturally if we are dealing with a upper barrier option the interval for this density will be $(-\infty,H)$, otherwise if $H<S_0$ the interval will be (H,∞).
The distribution functions of the running maximum and minimum of the absorbed process, respectively to the two cases in which $S(t)\geq S_0$ and $S(t)\leq S_0$ will be, for the maximum:

$$f_{Max(t)}(S(t)) = N(\frac{S(t)-S_0-\mu t}{\sigma\sqrt{t}} - e^{2\frac{\mu(S(t)-S_0}{\sigma^2}}N(-\frac{S(t)-S_0+\mu t}{\sigma\sqrt{t}}$$

and for the minimum:

$$f_{min(t)}(S(t)) = N(\frac{S(t)-S_0-\mu t}{\sigma\sqrt{t}} + e^{2\frac{\mu(S(t)-S_0}{\sigma^2}}N(-\frac{S(t)-S_0+\mu t}{\sigma\sqrt{t}}$$

Down and out contracts Now we start demonstrating how to price a down and out contract, then using the put-call parity condition we can expand it to the down and in case. Later on we will discuss briefly also the cases of Up-contracts.
As explained before we develop our model under BSM construction and so fixing a lower barrier $H<S_0$ we know that if the stock price S is above the level H for the entire contract period then an amount Υ (here we are considering the general situation so we are not taking into consideration if the said contract is a put or a call) otherwise if the stock price breaches the barrier during the contract life the payoff Υ will be zero.
Consider the general contract Z without the barrier, itpays without any condition the payoff $\Upsilon(S(t))$,then we define Z_{DO} as the same contract but with the conditional payoff over the lower barrier H
Z_{DO} is defined as:

$$\begin{cases} \Upsilon(S(T)), if S(t) > H forall t \in [0,T] \\ 0 if S(t) \leq H forsome t \in [0,T] \end{cases}$$

Consider a contract with maturity T which will pay according to the function $\Upsilon(S(T))$, then the pricing function, called Z_{DO} of the corresponding down and out contract will be given by:

$$F_{DO}(t, S_0, \Upsilon) = F(t, S_0, \Upsilon_{DO}) - (\frac{H}{S})^{\frac{2\mu}{\sigma^2}} F(t, \frac{H^2}{S_0}, \Upsilon_{DO})$$

where $\mu = r - 1/2\sigma^2$. To proove this we have to remember that with S_H we are considering the process with absorption at H. Using risk neutral probability Q to measure our expectation we have that the price of the contract at time 0 is:

$$F_{DO}(0, S; \Upsilon) = e^{-rT} E^Q[Z_{DO}]$$

since the value of Z_{DO} is the same of the contract Z which pays $\Upsilon(S(T))$ without any condition, we can rewrite the payoff of Z_{DO} as $\Upsilon(S(T)) * I[inf S(t) > H]$ for $0 \leq t \leq T$ where I can be 0 or 1 depending on the condition inside the brackets. Hence:

$$F_{DO} = e^{-rT} E^Q[\Upsilon(S(T)) * I[inf S(t) > H]$$

$$F_{DO} = e^{-rT} E^Q[\Upsilon_H(S_H(T)) * I[inf S(t) > H]$$

given that the indicator $I = 1$ if we consider the absorpted process $S_H(T)$ we get:

$$F_{DO} = e^{-rT} E^Q[\Upsilon_H(S_H(T))]$$

dealing with a down and out we know that the valid region is above H, thus this last expectation can be expressed as the integral:

$$\int_H^\infty \Upsilon_H(x) h(x) d(x)$$

where $h(x)$ is the density function of the variable $S_H(T)$. Remember that the stock price S moves as a geometric Brownian motion, therefore:

$$S(T) = S_0 e^{\mu T + \sigma\sqrt{T} W}$$

considering $\mu T + \sigma\sqrt{T}W = X$ we may write the absorption process $S_H(t)$ as:

$$S_H(t) = e^{X_{lnH(t)}}$$

and the above expectation becomes

$$E^Q[\Upsilon_H(S_H(T))] = \int_{lnH}^\infty \Upsilon_H(e^{X_{lnH(t)}} f(X_{lnH(t)}) dX_{lnH(t)}$$

in which, defining $X_{lnH(t)} = x$, the density function $f(x)$ is, as we pointed before

$$f(x) = \varphi(x; \mu T + lnS_0, \sigma\sqrt{T}) - e^{-\frac{2\mu(lnS_0 - lnH}{\sigma^2}}\varphi(x; \mu T - lnS_0 + 2lnH, \sigma\sqrt{T})$$

$$f(x) = \varphi(x; \mu T + lnS_0, \sigma\sqrt{T}) - (\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}}\varphi(x; \mu T + ln(\frac{H^2}{S_0}), \sigma\sqrt{T})$$

therefore we have

$$E^Q[\Upsilon_H(S_H(T))] = \int_{lnH}^{\infty} \Upsilon_H(e^x)f(x)dx$$

$$= \int_{lnH}^{\infty} \Upsilon_H(e^x)\varphi(x; \mu T + lnS_0, \sigma\sqrt{T})dx - (\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}} \int_{lnH}^{\infty} \Upsilon_H(e^x)\varphi(x; \mu T + ln(\frac{H^2}{S_0}), \sigma\sqrt{T})dx$$

and finally we obtain the result

$$E^Q_{0,S_0}[\Upsilon_H(S(T))] - (\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}} E^Q_{0,\frac{H^2}{S_0}}[\Upsilon_H(S(T))]$$

Down and out options For what we said before the price of a down and out call is priced as follows, remember that

$$F_{DO}(t, S_0, \Upsilon) = F(t, S_0, \Upsilon_{DO}) - (\frac{H}{S})^{\frac{2\mu}{\sigma^2}} F(t, \frac{H^2}{S_0}, \Upsilon_{DO})$$

and changing the general payoff Υ with the payoff of a normal call option we get that the formula for a down and out option when $H > K$ is:

$$C_{DO} = C_{BSC} - (\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}} C_{BSC}(\frac{H^2}{S_0})$$

where BSC is the standard formula for a call. Remember that

$$C = S_0 N(x) - e^{-rT} K N(y)$$

where

$$\frac{ln(S/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = x$$

and

$$\frac{ln\frac{S}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = y$$

replacing this we finally get the formula for a down and out call option when $H < K$

$$C_{DO} = S_0N(x) - e^{-rT}KN(y) - (\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}}C_{BSC}(\frac{H^2}{S_0})$$

$$C_{DO} = S_0N(x) - e^{-rT}KN(y) - S_0(\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}}N(j) - Ke^{-rT}(\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}-2}N(j - \sigma\sqrt{T})$$

for

$$j = \frac{ln[H^2/(S_0K)]}{\sigma\sqrt{T}} + \frac{\mu}{\sigma^2}\sigma\sqrt{T}$$

Things become a bit more complicated when the barrier is higher or equal to the value of the strike, therefore if we have $H \geq K$ the current value of a down and out call option is:

$$C_{DO} = S_0N(x_1) - Ke^{-rT}N(x_1 - \sigma\sqrt{T} - S_0(\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}}N(y_1) + Ke^{-rT}(\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}-2}N(y_1 - \sigma\sqrt{T})$$

where

$$x_1 = \frac{ln(S_0/H)}{\sigma\sqrt{T}} + \frac{2\mu}{\sigma^2}\sigma\sqrt{T}$$

$$y_1 = \frac{ln(H/S_0)}{\sigma\sqrt{T}} + \frac{2\mu}{\sigma^2}\sigma\sqrt{T}$$

Down and in Options These are general features used to price a contract which is null if the underlyng touches the barrier. Now considering the more specific case of a call option.

We starting considering the simpler case in which the barrier H is also below the strike price K. The price of a Call down and in is very similar to the price of a standard call, with the exception that the BSM formula for the barrier option consider a differ interval for the validity of the option, therefore we just need to change the Merton formula with $\frac{H^2}{S_0}$ replacing S and add $(\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}}$. Applying what we just have said we get:

$$C_{DI} = (\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}}C_{BSC}(\frac{H^2}{S_0})$$

replacing $C_{BSC}(\frac{H^2}{S_0})$ with the above explicit formula we get the final result for a down and in barrier option which is:

$$C_{DI} = S_0(\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}} N(j) - Ke^{-rT}(\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}-2} N(j - \sigma\sqrt{T})$$

and again:

$$j = \frac{ln[H^2/(S_0K)]}{\sigma\sqrt{T}} + \frac{\mu}{\sigma^2}\sigma\sqrt{T}$$

where $\mu = r - 1/2\sigma^2$. We can notice that this formula can easily be obtained from the relation $C_{DI} = C - C_{DO}$ and viceversa for the down and out option, infact a call option is exactly equal to the sum of the respective down in and down out options, because the two region considered in the expected value offset each other.
Infact the price of a down and in call when $H > K$ is obtained using this relation and replacing it with the above formula for a C_{DO}.

Up barrier options Before we demostrated how knock-down barrier options are priced starting from the Black-Scholes formula, now we will describe the formula for barrier options which have an upper barrier, both higher and lower than the strike price.
The story is more or less the same, infact consider again a normal up-type contract which pays $\Upsilon(S(T))$ and fix an upper barrier H higher than the initial stock price S_0. We define a general up and out contract Z^{UO} which has payoff Υ^H equal to:

$$\Upsilon^H(S) = \begin{cases} \Upsilon(S), ifS < H \\ 0, ifS \geq H \end{cases}$$

using the indicator I we can also say that $\Upsilon^H(S) = \Upsilon(S) * I[S < H]$. This is the same situation of the down and out contract, with the exception that we are considering a different standard option. The up and out contract price is:

$$F^{UO}(t, S_0, \Upsilon) = F(t, S_0, \Upsilon^H) - (\frac{H}{S})^{\frac{2\mu}{\sigma^2}} F(t, \frac{H^2}{S_0}, \Upsilon_H)$$

and we can derive the formula for a general Up and Out Put option when the barrier is higher then the strike price:

$$P^{UO} = e^{-rT}KN(-y) - S_0N(-x) - [S_0(\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}} N(-j) + Ke^{-rT}\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}-2} N(-j - \sigma\sqrt{T})]$$

the price of an Up-in Put option, similarly to what we said before for a Down-in is

$$S_0(\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}}N(-j) + Ke^{-rT}\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}-2}N(-j-\sigma\sqrt{T})$$

and it's easy to see how the two formulas are related, infact it holds that $P^{UI} = P - P^{UO}$. When instead we have $H \leq K$ the formula for a Up and Out Put option becomes:

$P^{UO} = -S(\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}}N(-x_1)+Ke^{-rT}N(-x_1+\sigma\sqrt{T})+S_0(H/S_0)^{\frac{2\mu}{\sigma^2}}N(-y_1)-Ke^{-rT}(H/S_0)^{\frac{2\mu}{\sigma^2}-2}N(-y_1+\sigma\sqrt{T})$

and the respective Up and In Put is obtained through the relation $P^{UI} = P - P^{UO}$.

In this section I have analyzed call options for downward options and put options for upward options. The introductory mathematical background can be extended to all types of barrier options, we just need to adjust the formula depending on the interval on which the expected value is determined. Remember that we are dealing with expectations of stochastic variables, so changing the interval of validity of the option means also change the lower/upper bound of the respective integral.

2.2 Monte Carlo simulation for continuous monitored options

For our purpose we will take in examination the general case of a knock-out call option, that is equivalent to a normal standard call which will have value only if the underlying asset price does not hit the barrier during the option life, otherwise the option payoff will be equal to zero.
The general formula for pricing this type of call is given by Black-Sholes-Merton equation:

$$C(0) = e^{-rT} E_{0,S_0}[max(S_T - K), 0) 1_{\tau_H \geq T}]$$

where K is the strike price and r is the spot risk-free interest. At the end of the formula we multiply the 'Call value' by 1 if the event τ_H is higher than out maturity T: it means that during our monitoring time the underlying price of the stock did not cross the barrier, hence our call is valuable (otherwise it would be equal to zero). We are analyzing the case for a one-constant barrier (ex. $U = 4$) but the event τ_H may include more than one barrier and they may not be constant. For the sake of simplicity in our model we will use the one-single-constant barrier case.
There are several ways that can be used to price an option, for the barrier options case the most used one is **Monte Carlo simulation**.
Anyway in the case of barrier options this simulation may take to some biased estimation, mostly in the case of continuous monitored barrier options, because to simulate them we have to discretize the monitoring of the underlying asset price and it may happen that the time in which the stock price crosses the barrier is in between two monitoring istants. Then not taking it into consideration may bring our simulation to an error and specifically to an overestimation of the price of the option (consider for example the case of a knock/down-out barrier option). Moreover, apart from the distinction of discrete and continuous monitoring, we have to study different cases depending on which type of barrier characterized the option.
In the case of one constant barrier we are estimating a probability that corresponds exactly with the exit probability of the asset price from the boundary, hence we will have un unbiased estimator of the option price.
We have to underlyne that in the case of multidimensional barrier options (as Double and Time-dependent barriers) the computed probability do not correspond exactly to the exit probability of the options, however, thanks to the use of Large Sharp Deviation formulas, we will obtain a good approximation of this probabilities.

2.2.1 Simulation of the option

We know that the stock price behaves as a diffusion process which evolves as a geometric Brownian motion in the time interval $[0, T]$, hence the stock price path will be described by the equation:

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

where μ and σ represents respectively the drift and the 'variation' of the process and B is a standard Brownian motion.
As defined by the Stochastic Differential Equation described by Black-Scholes we know that this process at time $t+1$ will be equal to:

$$S_{t+1} = S_{t_i} e^{(r-\frac{\sigma^2}{2})\Delta t + \sigma(B_{t_{i+1}} - B_{t_i})}$$

in which r is equal to the sport risk free rate observed in the market.

2.2.2 Single constant barrier

Considering the general case of a single constant barrier option, we want to know the conditional probability that S_{t_i} hits the barrier during the interval we are not taking into consideration (t_i, t_{i+1}).
Imagine for example that we have a costant upper barrier $H(t) = H$ (where H is a fixed number) the exit probability will be given by:

$$p_H^{\Delta t}(T_0, \zeta, \gamma) = e^{-\frac{2}{\sigma^2 \Delta t}(H-\zeta)(H-\gamma)}(1 + O(\Delta t))$$

where ζ and γ are the two observations of the asset price at time t and $t+1$, more specifically $\zeta = logS_{t_i}$, $\gamma = logS_{t_{i+1}}$ and $H = logH$, $\Delta t = t_{t+1} - t$ represents the time interval $(t, t+1)$ and $O(\varepsilon)$ is the error of our conditional probability.
For the purpose of our simulation we will calculate the exit probability at each step of the process and studying it with the assumption that $O(\varepsilon) \equiv 0$ we get the simplified formula:

$$p_H^{\Delta t}(T_0, \zeta, \gamma) = exp\{-\frac{2}{\sigma^2 \Delta t}(H-\zeta)(H-\gamma)\}$$

We will calculate this exit probability at each step of the diffusion process, for example if we are monitoring the option 1000 times we will get 1000 of differente probabilities, one

for each step.
After we estimated it, we will calculate a random variable between 0 and 1 and if the exit probability will be higher than the randomized value our option will have payoff equal to zero (in the case of a knock-out barrier option) meaning that during the step we have taken into account the price of the asset breached the barrier. Otherwise we will continue our simulation with the next ε. This exit probability is calculated throughout the use of the Sharp Large Deviations developed from Fleming and James (1992) and Baldi (1995) which we are going to discuss briefly later on.
Naturally the closer the asset price gets to the barrier the more the exit probability will be equal to 1. In the end if the stock price do not breach the barrier in each of the steps of the path we had simulated, the payoff of the option will be equal to:

$$\Pi_C = e^{-rT} max[S^t - K, 0]^+$$

which is the payoff of a standard call option.

HINT: the event Σ did not happened ($\Sigma \geq T$, hence outside our monitoring period) hence we multiplied the normal payoff by 1

Consider also the symmetric case in which we have a put option with lower barrier, the exit probability will be calculated as:

$$p_L^{\Delta t}(T_0, \zeta, \gamma) = e^{-\frac{2}{\sigma^2 \Delta t}(\zeta - D)(\gamma - D)}$$

where $D = logD$ where $D =$ *fixed number* and represents the lower barrier.
Now we are going to analyze the exit probabilities for other type of barriers, but first we have to underlyne the fact that these probabilities are the best possible approximations, so we in their computation we always have to take into account some source of error $O(\Delta t)$.

2.2.3 Time dependent barrier

Earlier on we have studied the case in which our option has just one constant barrier, in that case the computational probability is very easily calculated and our Monte Carlo simulation tends to give us an unbiased price. Now we are going to analyze in details an option which barrier is time variable: so for example consider an upside barrier $H(t) = H_1 + H_2 T_0$. Remember that we are taking the logarithm of the barrier, therefore in this situation we have an exponential single barrier which may change over time.
The exit probability for the upside barrier option will be:

$$p_H^{\Delta t}(T_0,\zeta,\gamma) = exp\{-\frac{2}{\sigma^2}(H_1 + H_2T_0 - \zeta)[\frac{(H_1 + H_2T_0 - \gamma)}{\Delta t} + H_2]\}$$

and symmetrically the time dependent lower barrier in a put option will be:

$$p_D^{\Delta t}(T_0,\zeta,\gamma) = exp\{-\frac{2}{\sigma^2}(\zeta - D_1 - D_2T_0)[\frac{(\gamma - D_1 - D_2T_0)}{\Delta t} - D_2\}$$

where $D(t) = D_1 + D_2T_0$.

2.2.4 Double constant barriers

Consider now two barrier, an upper one $H(t) = H$ and a lower one $D(t) = D$, the probability that the asset price breaches the corollary is:

- $if \quad \zeta + \gamma > H + D$,

$$p_{H,D}^{\Delta t}(T_0,\zeta,\gamma) = exp\{-\frac{2}{\sigma^2\Delta t}(H-\zeta)(H-\gamma)\}(1 + o((\Delta t)^k))$$

- $if \quad \zeta + \gamma < H + D$,

$$p_{H,D}^{\Delta t}(T_0,\zeta,\gamma) = exp\{-\frac{2}{\sigma^2\Delta t}(\zeta - D)(\gamma - D)\}(1 + o((\Delta t)^k))$$

where $k \in N$. Note that these exit probabilities are obtained simplifing the infinite series:

$$p_{H,D}^{\Delta t}(T_0,\zeta,\gamma) = \sum_{k=-\infty}^{\infty}\{exp(-\frac{1}{2\Delta t}[(x+2k(H-D))^2 - x^2]) - exp(-\frac{1}{2\Delta t}[(x-2H+2k(H-D))^2 - x^2])\}$$

in which we set $\sigma = 1$ for simplicity and $x = \gamma - \zeta$ and setting in the series $k = 0$ and $k = 1$ we get our exit probability.

2.2.5 Double time dependent barriers

The last situation we are taking into exam is when we have an option with two barriers both changing over time, so considering an upper barrier $H(t) = H_1 + H_2t$ and a lower barrier $D(t) = D_1 + D_2t$ we will obtain again two different probabilities depending on the path followed by both the stock price and the barriers; for every $k \in N$:

- $if \quad \zeta + \gamma > H + D,$

$$p^{\Delta t}_{H,D}(T_0, \zeta, \gamma) = exp\{-\frac{2}{\sigma^2}(H_1 + H_2T_0 - \zeta)[\frac{(H_1 + H_2T_0 - \gamma)}{\Delta t}]\}(1 + o((\Delta t)^k))$$

- $if \quad \zeta + \gamma < H + D$

$$p^{\Delta t}_{H,D}(T_0, \zeta, \gamma) = exp\{-\frac{2}{\sigma^2}(\zeta - D_1 - D_2T_0)[\frac{(\gamma - D_1 - D_2T_0)}{\Delta t} - D_2]\}(1 + o((\Delta t)^k))$$

always remembering that here we are working with exponential barriers (ex: $H_1 = Log(H_1)$ which are function of time.

In our model we will divide the time interval $[0, T]$ by n number of subinterval, and each step Δt is equal to $\frac{T}{n}$. Naturally since we want to continuously monitor the path of the option we will try to work with an $\Delta t \to 0$ in order to have the best possible approximation of the option price.
Anyway trying to reduce the step size will not give us unbiased Monte Carlo simulation, because a very small step size will produce other kinds of numerical errors as demonstrated in *Baldi, Caramellino (1999)*.

2.3 Trinomial model

Another method used to value barrier options has been introduced by Ritchken and it is a trinomial lattice. Trinomial trees are better than the binomial ones but the convergence to the correct value it's very slow in the case of barrier options because the barrier made by the tree is different from the real one.
So one solution has been found by Ricthken that made a model in which the tree is specifically designed in order to have a node (or two in case of two barrier) coinciding with the barrier (or barriers)
As said before we know that the underlying asset follows a geometric Wiener process which has drift $\mu = r - \sigma_2/2$ and variance σ^2. We can represent the path followed by the asset S(t) in the following way:

$$ln(S_{t+\Delta t}) = ln(S_0) + Z(t)$$

where $Z(t) \sim (\mu\Delta t, \sigma^2\Delta t)$.

Now let $Z^a(t)$ be the the approximating distributionf for $Z(t)$ over the interval $[t, t+\Delta t]$. We can say that

$$Z^a(t) = \begin{cases} \lambda\sigma\sqrt{\Delta t} with probability p_u \\ 0 with probability p_m \\ -\lambda\sigma\sqrt{\Delta t} with probability p_d \end{cases}$$

where $\lambda \geq 1$ and we can define p_u, p_m and p_d are respectively the probabilities that the stock will go up, middle or down in the interval Δt. Then λ is the stretch parameter which controls the difference between layers of price on the lattice.

$$p_u = \frac{1}{2\lambda^2} + \frac{\mu\sqrt{\Delta t}}{2\lambda\sigma}$$

$$p_m = 1 - \frac{1}{\lambda^2}$$

$$p_d = \frac{1}{2\lambda^2 - \frac{\mu\sqrt{\Delta t}}{2\lambda\sigma}}$$

these probabilities are obtained solving a system in which we consider the first two non-central moments of the approximating distribution $Z^a(t)$ which are the same of the distribution of $Z(t)$.

$$\begin{cases} \lambda\sigma\sqrt{\Delta t}(p_u - p_d) = \mu\Delta t \\ (\lambda\sigma\sqrt{\Delta t})^2(p_u + p_d) = \sigma^2\Delta t \end{cases}$$

$$\begin{cases} p_u = \frac{\mu\Delta t}{\lambda\sigma\sqrt{\Delta t}} + p_d \\ (\lambda\sigma\sqrt{\Delta t})^2(\frac{\mu\Delta t}{\lambda\sigma\sqrt{\Delta t}} + 2p_d) = \sigma^2\Delta t \end{cases}$$

so with some algebra we get that $p_d = \frac{1}{2\lambda^2 - \frac{\mu\sqrt{\Delta t}}{2\lambda\sigma}}$ and $p_u = \frac{1}{2\lambda^2} + \frac{\mu\sqrt{\Delta t}}{2\lambda\sigma}$ as said before, then imposing $p_m = 1 - p_u - p_d$ we get $p_m = 1 - \frac{1}{\lambda^2}$.

In the extreme case where $\lambda = 1$, we have $p_m = 0$ and the trinomial lattice becomes equal to the binomial one. The stretch parameter is fundamental in the case of barrier options because it allows us to improve the model and the time of convergence to the real value of the option.

This because for any Δt we can determine a λ such that a node coincides exactly with the barrier. Considering a down-out option, with η we indicate the number of consecutive down moves such that the stock price arrive to the lowest layer of nodes above the barrier H; then we define η_0 as the largest integer smaller than η .

For $\lambda = 1$ and $\eta = \frac{ln(S_0/H)}{\sigma\sqrt{\Delta t}}$ we say that if η is an integer number we letft $\lambda = 1$, otherwise we correct it such that

$$\eta = \eta_0 \lambda$$

Using this correction our trinomial model will have a layer of nodes in which one of these is exactly on the barrier H. Later on we will see and discuss results about this model and compare this to the closed formula developed by Merton.

2.4 Discrete monitored barrier options

In the chapter before we analyzed some cases in which the barrier option was monitored in continuous time (the interval $[0, T]$ so the problem was to find the smaller step size possible for our simulation ($\Delta t \to 0$) to reduce the bias. Our correction with the exit probability works exactly in the case of one constant barrier option because the error of the probability is zero, in other situations it may lead to bigger error because the estimated probability is an approximation.
Thus we introduced the analytical formula (always used by Broadie, Glasserman and Kou 1997 but introduced by Merton 1973) to describe another useful way to price barrier options.
Now, dealing with discrete monitored barrier options, we have to discuss the different methods and their effectiveness:

- The Monte Carlo simulation is easy to implement in this case because the stock price has to be monitored only at defined istants and does not need the correction with the exit probability, anyway, as demonstrated by B.G.K(1996), it gives a 95 confidence interval with a range approximately of 0.005 cent using around one million simulation.

  If we want to have a 0.1 cent confidence interval we should compute 4.2 billion simulation trials which approximately would require 10 days of computing time on an Intel Pentium 133 MHz processor.

- The formula for continuous barrier options found by Merton can be used also for discrete options, but in this case it is needed a shift of the barrier to correct for discrete monitoring.

2.5 Correction in the Merton formula

As said before the price of a barrier option is given by the formula:

$$C(0) = e^{rT} E_{0,S_0}[max(S_T - K), 0) 1_{\tau_H \leq T}]$$

but now we are supposing discrete time, so the barrier is monitored only at time $i\Delta t, i = 0, 1, ..., m$ where $\Delta T/m$. Let us write $\overline{S}$ for $S_{i\Delta t}$, so that $\overline{S}, i = 0, 1, ...$ is the price of the asset at the monitoring istants. Since we are dealing with a discrete knock-in call option we will define

$$\overline{\tau}_H = inf\{n > 0 : \overline{S}_n > H\}$$

if $S_0 < H$;

$$inf\{n > 0 : \overline{S}_n < H\}$$

if $S_0 > H$.
so the price of a knock-in call is:

$$C(0) = e^{-rT} E_{0,S_0}[max(S_T - K), 0) 1_{\overline{\tau_H} \leq m}$$

Consider now the price a knock-out call continuosly monitored $C(H)$ the price of the respective discrete monitored option, with m istants is:

$$C_m(H) = C(He^{\pm \beta\sigma\sqrt{T/m}} + o(\frac{1}{\sqrt{m}})$$

where $+$ applies if $H > S_0$, $-$ applies if $H < S_0$ and $\beta = -\zeta(\frac{1}{2})/\sqrt{2\pi} \approx 0.5826$, with ζ the Riemann zeta function.
Shifting the barrier by a factor of $exp(\beta\sigma\sqrt{T})$ allows us to use the continuous barrier pricing formula as an approximation for discrete barriers.
The asset price S follows the process described by Ito's lemma $dlogS = (r - 1/2\sigma^2)dt + \sigma dz$ and for simplicity we suppose $r - 1/2\sigma^2 = 0$, so the $logS_t$ has zero drift. Then we define b and c respectively as $H = S_0 exp(b\sigma\sqrt{T})$ and $K = S_0 exp(c\sigma\sqrt{T})$.
When computing the continuous price, the value of the option is the probability

$$\begin{aligned} P(S_T < K, \tau_H \leqslant T) &= P(logS_T < c\sigma\sqrt{T}, \max_{0 \leqslant t \leqslant T} logS_t \geqslant b\sigma\sqrt{T}) \\ &= P(logS_T \geqslant (2b - c)\sigma\sqrt{T}) \\ &= 1 - \Phi(2b - c) \end{aligned}$$

where Φ is the standard normal cumulative distribution function. In this case we simplify the probability using the reflection principle.
Consider now that the option is monitored at some istants in time $\{0, \Delta t, 2\Delta t, ..., m\Delta t\}$ where m is the number of monitoring moments and $\Delta t = T/m$ is the interval between two of them. Since we are assuming zero drift, the asset price at each monitoring istant is

$$\overline{S}_n = S_0 e^{\sigma\sqrt{\Delta t}\widehat{W}_n} \text{ , with } \widehat{W}_n = \textstyle\sum_{i=1}^n Z_i$$

where Z_i are indipendent standard normal random variables. Monitoring the options in discrete istants the above price will become:

$$e^{-rT} P(\overline{S} < K, \max_{0 \leqslant n \leqslant m} \overline{S} \geq H) = e^{-rT} P(\overline{W}_n < c\sqrt{m}, \tau \leq m)$$

where $\overline{\tau}$ is the first time $\overline{W}$ exceeds $b\sqrt{m}$. The reflection principle yields so the increments of the random walk $\overline{W}$ are symmetrically distributed and

$$P(\overline{W}_m < c\sqrt{m}, \overline{\tau} \leq m) = P(\overline{W} > 2(b\sqrt{m} + R_m) - c\sqrt{m})$$

where $R_m = \overline{W}_{\overline{\tau}} - b\sqrt{m}$ is the overshoot above level $b\sqrt{m}$. Then considering $\overline{W}_m$ and R_m as independent variables and given that $P(\overline{W}_m > x\sqrt{m}) = 1 - \Phi(x)$ we can compute the above probability and get

$$P(\overline{W} > 2(b\sqrt{m} + R_m) - c\sqrt{m}) \approx E[1 - \Phi(2(b + \frac{R_m}{\sqrt{m}}) - c)]$$

Considering the standard normal density φ and using the fact that $E[R_m] \to \beta$, we get

$$P(\overline{W} > 2(b\sqrt{m} + R_m) - c\sqrt{m}) \approx E[1 - \Phi(2b - c) - \frac{2R_m}{\sqrt{m}}\varphi(2b - c) + o(\frac{1}{\sqrt{m}})]$$

$$\approx 1 - \Phi(2b - c) - \frac{2}{\sqrt{m}}E[R_m]\varphi(2b - c) + o(\frac{1}{\sqrt{m}})$$

$$1 - \Phi(2b - c) - \frac{2\beta}{\sqrt{m}}\varphi(2b - c)o(\frac{1}{\sqrt{m}})$$

$$1 - \Phi(2(b + \frac{\beta}{\sqrt{m}} - c) + o(\frac{1}{\sqrt{m}})$$

Hence in the case of discrete monitoring istants we can write the value of the option as:

$$P(S_T < K, \tau_{Hexp(\beta\sigma\sqrt{\Delta t}} \leq T) + o(\frac{1}{\sqrt{m}})$$

which is the same formula for a continuously monitored barrier option with the barrier shifted by $e^{\beta\sigma\sqrt{\Delta t}}$ plus the error term $o(\frac{1}{\sqrt{m}})$ that decreases as the number of monitoring istants increases.

3 Numerical results

3.1 Continuous monitored options

In this chapter i'm going to analyze and compare numerical results regarding methods explained before, which, depending on the value of the barrier, will have different precision and convergence to the true price.
The analysis is based on call options, since them are those better analyzed in this work; naturally the procedure is the same for put options, we just need to change the payoff formula from $max(S_T - K, 0)$ to $max(K - S_T, 0)$ when dealing with Monte Carlo simulations and trinomial model, and using a different closed formula.

In the first case, I analyze an Up and out call option with the following characteristics:
S0 = 100, K = 105, sigma = 0.25, r = 0.025, T = 1y, dt = 1/365. The barrier will have different values in order to show the effectiveness of methods when the barrier get closer to the initial stock price.
The results are compared to the closed formula studied by Merton, so errors will be calculated respect to this method and they will be reported under the respective price as a percentage (positive or negative depending on overestimation or underestimation of the price). In the standard Monte Carlo I made one hundred simulations based on 10000 simulated-paths of the underlying asset price, the step size is 1/365 (daily monitoring); in the corrected Monte Carlo I applied the probability correction explained in Baldi, Caramellino (1999) and the results are obtained always with 100 simulations of 10000 paths-simulated asset price with step size 1/365. Then, since we are making 100 simulations, we have also to consider the standard deviation of our result,which will be below the relative error.
In the last column I used the trinomial model introduced by Ritchken, using fifty time steps, which is a reasonable number for this model: in the case of up and out option, it shows many problems relating to the fact that we are imposing a parameter lambda such that the barrier coincides exactly with one layer of the node, this creates errors in pricing the option depending on the amount of time steps we are using. Anyway, I decided to use 50 time steps because it is the more efficient in terms of time and errors. (Naturally using a high number of time steps will reduce the error but it will need much more time, and it is not always the optimal choice since the errors arising from choosing lambda increase). So focusing on this model when pricing upward barrier options is not the best solution, but we will see later that in the case of down and out options it gives us very precise results increasing the number of time steps.

Up and In Call option

BARRIER	MERTON	MC CORRECTED	MC NOT CORR	RITCHKEN
140	6,1572	6,0627	5,9587	6,1635
		-0,01558711	-0,03331264	0,00102215
		0,2207	0,1756	
130	7,6614	7,6127	7,4892	6,9646
		-0,0063972	-0,02299311	-0,10004882
		0,243	0,1726	
120	8,6226	8,5891	8,4784	8,614
		-0,00390029	-0,01700793	-0,00099837
		0,1533	0,2016	
115	8,8308	8,8357	8,8398	8,6062
		0,00055457	0,00101812	-0,02609746
		0,2258	0,1659	

In this case the method with the better convergence to the Merton formula price is the corrected Monte Carlo, which surprisingly shows a better result when the barrier is close to the initial stock price. Infact, with a barrier at 115 the relative error is an overestimation of the 0.05 per cent and in general the error is very small also with other barrier's values. The probability introduce by Baldi,Caramellino(1999) are very functional, the higher precision respect to the not corrected Monte Carlo permits a better estimation of the options lifetime and thereafter a more good approximation of the price. The Ritchken formula here demostrates all its limits: starting with a very low error (0.1 per cent of the true price) it increases when the barrier gets closer to the initial price of the stock and, when the barrier is at 130, the price is strongly biased (it shows a 10 per cent underestimation).

Up and Out Call option

BARRIER	MERTON	MC CORRECTED	MC NOT CORR	RITCHKEN
140	2,7517	2,7709	2,9601	2,7242
		0,006929	0,070403	-0,01009
		0,066	0,0768	
130	1,2476	1,269	1,4123	1,9232
		0,016864	0,116618	0,35129
		0,0395	0,03	
120	0,2863	0,2949	0,3521	0,2738
		0,029162	0,186879	-0,04565
		0,0151	0,0186	
115	0,0781	0,0841	0,1094	0,2816
		0,071344	0,286106	0,722656
		0,0067	0,101	

The results in the table above are in line with what said before, apart from the fact that here the relative percentage errors for corrected Monte Carlo is increasing while moving the barrier closer to the initial stock price. Anyway the relative errors are still lower and the simulations is again performing with a high degree of precision respect to other methods. The normal Monte Carlo has an error of 28 per cent when the barrier is at 115 and the Ritchken method fails when the barrier is at 130 and 115 with errors respectively of 35 and 75 per cent.

Down and In Call option

BARRIER	MERTON	MC CORRECTED	MC NOT CORR	RITCHKEN
80	0,2447	0,2036	0,2019	0,1976
		-0,20187	-0,21199	-0,23836
		0,0244	0,0204	
90	2,1665	1,9254	1,9235	2,1477
		-0,12522	-0,12633	-0,00875
		0,0715	0,0747	
95	4,7428	4,3015	4,2741	4,7221
		-0,10259	-0,10966	-0,00438
		0,127	0,1153	
96	5,4406	4,9616	4,925	5,4144
		-0,09654	-0,10469	-0,00484
		0,1675	0,1171	

For what concern Down and In options the correction applied to Monte Carlo simulations again improves the accuracy of pricing, but has not the same degree of precision respect to upward barrier options. The relative errors decreases when the barrier gets closer to

the initial stock price.
In this situation the model which give us the best results is the Ritchken trinomial tree which, starting with a higher underestimation of 23 per cent, improves its performance decrasing the relative error to 0.4 per cent without any biased results.

Down and Out Call option

BARRIER	MERTON	MC CORRECTED	MC NOT CORR	RITCHKEN
80	8,6642	8,6474	8,5994	8,6901
		-0,00194	-0,00754	0,00298
		0,1583	0,1701	
90	6,7424	6,7722	6,9738	6,7401
		0,0044	0,033181	-0,00034
		0,2168	0,2	
95	4,161	4,0981	4,6242	4,1657
		-0,01535	0,100169	0,001128
		0,1513	0,1251	
96	3,4683	3,4598	4,0523	3,4733
		-0,00246	0,144116	0,00144
		0,1062	0,1495	

Also looking at the table above we can sign a line and draw our general conclusions. The probability improves a lot the Monte Carlo simulation errors decreasing the errors, and the corrected simulations result to be the most realiable method because it tends to have a low relative percentage error in almost all the situaions. Anyways the Ritchken formula sometimes performs better and faster but it is not as costant as the MC Corrected because, applying the correction for lambda which rely on the barrier's value, sometimes, we get biased prices which have large errors.

3.2 Discrete monitored options

Now we will focus on numerical results of methods explained in section 2.4 useful to price barrier options which are monitored at descrete istants.
Again consider the Up and out call option with the following characteristics:
S0 = 100, K = 105, sigma = 0.25, r = 0.025, T = 1y, dt = 1/50.
In the case of discrete monitored barrier options we know that Monte Carlo is a bit ineffective given that the monitoring istants are few, so also applying the correction the method will have a slower convergence to the true price.
As true price now we consider the formula developed by Merton with the correction explained in section 2.5: in order to avoid the overshoot we use the corrected barrier $H * exp(0.5826 * sigma * T/m)$ in the case of upward barrier options, and $H * exp(-0.5826 * sigma * T/m)$ in the case of downward barrier options.
The term m indicates the number of monitoring istants and for our studies we use an $m = 50$. Again, in the first column there are Merton's result, in the second the corrected Monte Carlo, in the third the Monte Carlo without correction and in the last one Ritchken trinomial tree. In this case Monte Carlo simulations spend less time but also the degree of precision is lower since the number of monitoring steps is lower.
Here below are reported some numerical results of discrete monitored options.

Up and Out Call option

BARRIER	MERTON	MC CORRECTED	MC NOT CORR	RITCHKEN
140	2,8183	2,9339	3,3588	2,7242
		0,039401	0,160921	-0,03454
		0,0697	0,0683	
130	1,2972	1,3992	1,7427	1,9232
		0,072899	0,255638	0,325499
		0,0519	0,0638	
120	0,3075	0,3586	0,5297	0,2738
		0,142499	0,419483	-0,12308
		0,013	0,017	
115	0,087	0,1177	0,1968	0,2816
		0,260833	0,557927	0,691051
		0,0058	0,0084	

Up and In Call option

BARRIER	MERTON	MC CORRECTED	MC NOT CORR	RITCHKEN
140	6,0906	5,7079	5,5182	6,1635
		-0,06705	-0,10373	0,011828
		0,1838	0,1483	
130	7,6117	7,2224	7,1467	6,9646
		-0,0539	-0,06506	-0,09291
		0,1585	0,1314	
120	8,6014	8,404	8,3164	8,4784
		-0,02349	-0,03427	-0,01451
		0,1059	0,1113	
115	8,8219	8,6776	8,5044	8,614
		-0,01663	-0,03733	-0,02414
		0,1092	0,1273	

Down and In Call option

BARRIER	MERTON	MC CORRECTED	MC NOT CORR	RITCHKEN
80	0,2298	0,1369	0,131	0,1976
		-0,6786	-0,7542	-0,16296
		0,0176	0,0091	
90	2,0693	1,4396	1,4332	2,1477
		-0,43741	-0,44383	0,036504
		0,0433	0,0528	
95	4,5614	3,3332	3,3244	4,7221
		-0,36847	-0,3721	0,034031
		0,0705	0,1134	
96	5,2392	3,939	3,8574	5,4144
		-0,33008	-0,35822	0,032358
		0,0533	0,1034	

Down and In Call option

BARRIER	MERTON	MC CORRECTED	MC NOT CORR	RITCHKEN
80	8,6791	8,648	8,7195	8,6901
		-0,0036	0,004633	0,001266
		0,1591	0,1376	
90	6,8396	6,7871	6,9323	6,7401
		-0,00774	0,013372	-0,01476
		0,1367	0,1814	
95	4,3475	4,3883	4,8845	4,1657
		0,009297	0,10994	-0,04364
		0,0883	0,1435	
96	3,6698	3,6473	4,226	3,4733
		-0,00617	0,131614	-0,05657
		0,1131	0,1152	

If the options are monitored at discrete time the situation is more or less the same, the Monte Carlo simulation strong over/underestimate the price of the option, but if we apply the Baldi,Caramellino correction the relative percentage error decreases a lot (for example in the case of Up and Out Call it goes from 16 to 3.9 per cent); but if the barrier gets closer to the initial stock price the error increases. As explained before, the Ritchken trinomial tree also gives good results but sometimes the price has a large error (again looking at the Up and Out call option when the barrier is a 130 the error is 32.5 per cent).

These results are in line with what said about barriers monitored in continuous time. The better method is the Monte Carlo with exit probability which again results to be the less unbiased and more efficient in term of time and precision.

3.3 Matlab codes

Here below are reported Matlab Codes which I've developed and used for simulations: in the first part of each one are written the characteristics of the option, then three different cycles divide it for number of Monte Carlo simulations, number of simulated-paths and number of time intervals. The probabilities are different depending on whether we are analyzing an upward or a downward option, and for knock-in options the validity is checked after the simulation since the option may always cross the barrier during its lifetime, instead for knoc-out options after the barrier is breached the simulation is interrupted since when it happens they become null.

For the reason just explained Knock in options require more computational time than knock outs.

At the end of the section I also reported the code used for Ritchken Trinomial Tree.

Up In

```
S0 = 100; % price of the stock at time 0
sig = 0.25; % volatility of the stock
K = 105; % strike price
B = 140; % barrier
r = 0.025; % risk free rate
T = 1; % maturity

nsim = 100 %number of simulation
npath = 10000; % number of simulated paths
nstep = 365; % number of monitoring steps
dt = 1/nstep; % time interval
Disc = exp(- r * T); %discount factor

for k = 1:nsim
for i = 1:npath
    S(k,i,1) = S0;
    Z(k,i,1) = 0;
    for j = 2:nstep
        Z(k,i,j) = normrnd(0,1);
        S(k,i,j) = S(k,i,j-1)*exp((r - 0.5*sig^2)*(dt) + sig * sqrt(dt) * Z(k,i,j));
        p(k,i,j) = exp(-2/(sig^2*dt)*(log(B)-log(S(k,i,j-1)))*(log(B)-log(S(k,i,j))));
        % probability introduced by Baldi and Caramellino (one single constant barrier c
        X(k,i,j) = rand;
        % random variable useful to determinate the validity of the option
    end
    if max(S(k,i,:)) >= B || p(k,i,j) > X(k,i,j) % condition to be validated
```

```
      vectpayoffs(k,i) = max(S(k,i,nstep) - K,0); %payoff in the case of call options
    else                                           %for put options we just need to chang
        vectpayoffs(k,i) = 0                       %the payoff
    end

end

Upin = sum(vectpayoffs')/npath * Disc %vector with all different simulations' results

end

Price = 1/nsim * sum(Upin); % we take the mean of simulations in order to get the
                            % final price
SD = std(Upin) %standard deviation
```

Up Out

```
S0 = 100; % price of the stock at time 0
sig = 0.25; % volatility of the stock
K = 105; % strike price
B = 115; % barrier
r = 0.025; % risk free rate
T = 1; % maturity

nsim = 100 %number of Monte Carlo simulation
npath = 10000; % number of paths-simulated
nstep = 365; % number of monitoring steps
dt = 1/nstep; % time interval
Disc = exp(- r * T); %discount factor

S = zeros(1,nstep)


for k = 1:nsim
for i = 1:npath
    S(k,i,1) = S0;
    Z(k,i,1) = 0;
    for j = 2:nstep
        Z(k,i,j) = normrnd(0,1);
        S(k,i,j) = S(k,i,j-1)*exp((r - 0.5*sig^2)*(dt) + sig * sqrt(dt) * Z(k,i,j));
        p(k,i,j) = exp(-2/(sig^2*dt)*(log(B)-log(S(k,i,j-1)))*(log(B)-log(S(k,i,j))));
        X(k,i,j) = rand;
        if S(k,i,j) >= B
```

```
            break
            vectpayoffs(k,i) = 0;
        else if  p(k,i,j) >= X(k,i,j)
                break
                vectpayoffs(k,i) = 0;
            else
                continue;
            end
        end

    end

    vectpayoffs(k,i) = max(S(k,i,nstep) - K,0);


end

Upout = sum(vectpayoffs')/npath * Disc

end

Price = sum(Upout)/nsim

SD = std(Upout)
```

DownOut

```
S0 = 100; % price of the stock at time 0
sig = 0.25; % volatility of the stock
K = 105; % strike price
B = 80; % barrier
r = 0.025; % risk free rate
T = 1; % maturity

nsim = 100 %number of Monte Carlo simulation
npath = 10000; % number of paths-simulated
nstep = 365; % number of monitoring steps
dt = 1/nstep; % time interval
Disc = exp(- r * T); %discount factor

S = zeros(1,nstep)
vectpayoffs = zeros(1,npath)
```

```
for k = 1:nsim
for i = 1:npath
    S(k,i,1) = S0;
    Z(k,i,1) = 0;
    for j = 2:nstep
        Z(k,i,j) = normrnd(0,1);
        S(k,i,j) = S(k,i,j-1)*exp((r - 0.5*sig^2)*(dt) + sig * sqrt(dt) * Z(k,i,j));
        p(k,i,j) = exp(-2/(sig^2*dt)*(log(S(k,i,j-1))-log(B))*(log(S(k,i,j))-log(B)));
        X(k,i,j) = rand;
        if S(k,i,j) <= B
            break
            vectpayoffs(k,i) = 0;
        else if  p(k,i,j) >= X(k,i,j)
                break
                vectpayoffs(k,i) = 0;
            else
                continue;
            end
        end

    end

  vectpayoffs(k,i) = max(S(k,i,nstep) - K,0);
end

DownOut = sum(vectpayoffs')/npath * Disc

end

Price = sum(DownOut)/nsim

SD = std(DownOut)
```

DownIn

```
S0 = 100; % price of the stock at time 0
sig = 0.25; % volatility of the stock
K = 105; % strike price
B = 80; % barrier
r = 0.025; % risk free rate
T = 1; % maturity

nsim = 100
```

```
npath = 10000; % number of simulation
nstep = 365; % number of monitoring steps
dt = 1/nstep; % time interval
Disc = exp(- r * T); %discount factor

for k = 1:nsim
for i = 1:npath
    S(k,i,1) = S0;
    Z(k,i,1) = 0;
    for j = 2:nstep
        Z(k,i,j) = normrnd(0,1);
        S(k,i,j) = S(k,i,j-1)*exp((r - 0.5*sig^2)*(dt) + sig * sqrt(dt) * Z(k,i,j));
        p(k,i,j) = exp(-2/(sig^2*dt)*(log(S(k,i,j-1))-log(B))*(log(S(k,i,j))-log(B)));
        X(k,i,j) = rand;
    end
    if min(S(k,i,:)) <= B | p(k,i,j) > X(k,i,j) % condition to be validated
      vectpayoffs(k,i) = max(S(k,i,nstep) - K,0);
    else
        vectpayoffs(k,i) = 0
    end

end

Downin = sum(vectpayoffs')/npath * Disc

end

Price = 1/nsim * sum(Downin);

SD = std(Downin)
```

Down and Out Ritchken formula

```
% Ritchken trinomial tree model for down and out calls

% inputs
T = 1; %maturity
nstep = 50 %number of time steps
r = 0.025; %risk free rate
K = 105; %strike price
S0 = 100; %initial stock price
B = 80; %barrier
sigma = 0.25; %volatility
mu = r - (sigma^2)/2;
dt = T / nstep;

eta = log(S0/B)/(sigma*sqrt(dt))  %Here I apply the Ritchken correction for
n0 = fix(eta)                     % lambda: if eta is an integer number we
if eta == n0                      % don't need to change lambda, where eta
    lambda = 1;                   % represents the needed down moves to reach
else                              % the barrier. Otherwise we have to adjust
    lambda = eta/n0;              % lambda in order to make them coincide
end

Df = exp(-r * dt);
u = exp(lambda * sigma * sqrt(dt))
d = 1/u
Pu = 1/(2 * lambda^2) + (mu * sqrt(dt))/(2 * lambda * sigma);
Pm = 1 - 1/(lambda^2);

Pd = 1/(2 * lambda^2) - (mu * sqrt(dt))/(2 * lambda * sigma);

for j = 1:nstep+1 %generating the trinomial tree
        for i = nstep-j+2:nstep+j
        S(i,j) = S0*u^(nstep-i+1)
    end
end

for i = 1:2*nstep+1 %stock and option value at the final stage
    SF(i) = S(i,nstep+1)
    if SF(i) > B
        C(i,nstep+1) = max(S(i,nstep+1) - K,0);
    else
        C(i,nstep+1) = 0
    end
```

```
end

for j = nstep %option value in the trinomial tree
    while j>0
        for i = nstep-j+2:nstep+j
            if S(i,j) > B
                C(i,j) = Df * (Pu * C(i-1,j+1) + Pm * C(i,j+1) + Pd * C(i+1,j+1));
            else
                C(i,j) = 0
            end
        end
        j=j-1
    end
end

DOWNOUT = C(nstep+1,1);
```

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