Department of Finance *Chair: Mathematical Finance*

Pricing interest rate derivatives: Pre and Post crisis comparison

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ACADEMIC YEAR
2017/2018

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1 INTRODUCTION: PROBLEM STATEMENT

After the 2007-2009 credit crunch, which has strongly impacted the financial markets causing severe liquidity and credit problems, some abnormalities have appeared in the fixed income market. Before the crisis interest rates showed a consistency that allowed to construct a single spot curve from which extrapolate the corresponding forward rates; as the crisis began those rates started to diverge substantially from each other, requiring different curves for different maturities.

The main cause of these changes has been the increase in the impact of liquidity and credit risks due to the requirement of different premiums on different tenors. This has resulted in fundamental variations in the pricing of interest rate derivatives, which have been since now the main issue to overcome. In order to embody different premiums, it is required to model a new framework in which different curves describe the dynamics for each tenor; the single yield curve model is no longer consistent with the characteristics of the market.

The LIBOR-OIS spread and the LIBOR-OIS swap spread are an example of the mutations brought by the settlement of the crisis; as opposed to the pre-crisis environment in which LIBOR and overnight indexed swap rate for the same maturity had negligible divergences, now it can be clearly observed how they started to move in different directions widening the spread between the two. This is one of the consequences of the shut downs of several banks such as Goldman Sacks, which has induced investors to revaluate the soundness of those banks, causing the risk associated with them to rise and pushing, so the interbank lending rate, such as EURIBOR and LIBOR, up.

Another peculiarity of the crisis and post-crisis scenarios has been an impressive increase in the derivatives transactions' collateralization, also due to the high interdependence that financial institutions showed during those years, which has contributed to the market distortions. This led to an increase in the default risk associated with banks even in the interbank lending, so that the rates that they charged to each other were no longer seen as the best proxy for risk-free rates and the need for a new discounting curve arises.

A highly segmented market characterized this period, in which derivatives started to be classified in different groups tenor-dependent, causing, among the other consequences, the basis-spread to increase. Due to this, a new condition became necessary in pricing a derivative, which is the requirement of *homogeneity*, according to which the derivatives selected for the construction of the forward curve should share the same tenor as the one of the derivatives which has to be priced. This gave birth to the multiple-curve model, as in derivatives' pricing a single curve is no longer adequate for the calculation of the fair rate, but rather as many curves as the underlying interest rates are required, so that each curve applies to a single tenor.

As mentioned above the interbank lending rates increased rapidly, substantially diverging from the overnight rates from which they were really close until the crisis; this was due to the default and liquidity risks, so practitioners decided to move on another set of rates that could have possibly better fitted for the case. They found out that OIS rates could have been better proxies, also for their short tenor which allowed to have a lower risk, and so the OIS discounting technique replaced the standard one based on LIBOR rates.

After all the changes that have revolutionized the market, some of the main relationships have been challenged, as the relation between the FRA rate and the forward rate, which has been really important in the derivatives pricing since it permitted to embody in a unique curve both the forward and the discount curves. As it will be shown later, this has been no longer possible.

Many solutions and different frameworks have been suggested since now, and all of them imply the modelling of different curves in order to give a correct pricing method for interest rate derivatives which still represents in the market an essential risk-management tool.

2 FIXED INCOME MARKETS, INTEREST RATE DERIVATIVES

The fixed income market is a segment of the overall financial markets in which the trade of interest rate-sensitive instruments takes place. FRAs, bonds, swaps, caps and floors fall under this category.

Discount bonds (or zero- coupon bonds) are the simplest fixed income product, also definable as vanilla instruments.

Now the main characteristics and components of the fixed income market will be discussed, as to provide, afterwards, a clear overview of the pre and post-crisis scenarios.

2.1 LIBOR, EURIBOR, EONIA AND OIS RATES.

The majority of the fixed income instruments involves the use of market rates as underlying rates. The most widespread and used market rates are the LIBOR rates or ICE LIBOR which represents a benchmark that banks charge each other when it comes to price loans. LIBOR rates are regulated by the Financial Conduct Authority (FCA) and are generated each business day for five different currencies (Euro, US Dollar, Japanese Yen, Swiss Franc) and for maturities of *1 day, 1 week, 1,2,3,6 and 12 months*. LIBOR stands for “London interbank offered rate” and it is defined as a *polled rate* as it is the result of the combination of rates submitted from a panel of representative banks. For each currency a different panel of banks is selected. Each business day selected banks are required to submit a rate that satisfies the ICE LIBOR question: "*At what rate would you borrow funds, were you to do so by asking for and then accepting inter-bank offers in a reasonable market size just prior to 11 a.m.";* afterwards a trimmed average is made of each rate submitted, as the highest and lowest 25% proposals are excluded. They reflect the short-term funding costs that London's banks face.

The European equivalent of the LIBOR rates are the EURIBOR rates (Euro interbank offer rates). Regulated by the European Money Markets Institute (EMMI), they are computed as trimmed average for spot values (T+2), for maturities *of 1 and 2 weeks, 1,2,3,6,9,12 months*. Panel banks are represented not only by European banks, but also by large international banks which participate in eurozone financial operations. Rates are submitted coherently to represent the rate at which *Euro interbank term deposits are offered by one prime bank to another prime bank within the EMU area*. As the LIBOR, they are calculated at 11 a.m. so to allow the daily use of these rates.

Strictly linked to the EURIBOR is the EONIA, Euro Overnight Index Average; calculated by the European central bank it represents the reference rate for the shortest maturity of *1 day* in the Eurozone.

It is calculated as a weighted average of inter-banks overnight unsecured lending transactions which occur within the 28 panel banks. The average is weighted according to the volume of transactions that each bank records and it is announced on an act/360day market convention.

The corresponding overnight rate calculated in US is the Federal Funds effective rate, which is calculated as the weighted average of all unsecured transactions that occur overnight within depository institutions that hold balances at the federal reserve (Federal Funds).

Remaining in the overnight rates field, it is necessary to highlight the OIS rate, which represent a market swap rate of an overnight indexed swap (OIS). It represents the rate on which the floating payments are constructed in order to be exchanged with fixed rate payments at each tenor in the overnight swap, as it is the best benchmark for the interbank credit market. It is developed through the compound of overnight rates over corresponding intervals of time within two succeeding maturity dates and it is a discretely compounded rate.

2.2 FRAs

A FRA (Forward Rate Agreement) is a linear interest rate derivative which allows the holder to enter into an agreement at time t, $0 \leq t \leq T$, with another party, where T is the inception date, in order to fix the rate R, on a notional N, at which the payment will be made at maturity S (with $S \geq T$). At time S the holder (payer leg) will pay a fixed amount and receive a floating payment, which is usually written on LIBOR and EURIBOR rates. No notional payments are made, just interest payments are exchanged between the two parties. It is seen as a zero-sum game since the two payments made by the payer leg and the receiver leg offset each other so that the present value of the overall investment (two cashflows) at S is null. Although the cash flows are exchanged at maturity, the contract is settled at time T, so it is necessary to discount the cash flow from time S to time T. In order to understand the payoff of the payer leg (the party which pays the fixed rate) it is first of all required to introduce the interest that he pays and the one that he receives; the amount of interest that outflows to the counterparty is defined as I = NRΔT , while the amount of cash that inflows from the counterparty is $I = NL\_T\ (S)\Delta T$, so that his payoff at the inception date T is equal to

$$N(L\_T\ (S) - R)\Delta T/(1 + L\_T\ (S)\Delta T). \tag{2.1}$$

Here the discount factor is calculated with the underlying market floating rate that applies in the time interval between T and S. In order for the FRA to be fair, it must have an NPV equal to zero, so to find out the appropriate fixed rate it is enough to set the NPV equal to 0 and to solve it for R.

It is possible to rewrite the simple compounded spot rate $L\_T\ (S)$ by considering it as a combination of zero-coupon bonds prices. First of all, let's consider that, under the constant simple spot rate at time T, the bond will yield a value of 1 at maturity S, so according to *Brigo and Mercurio* we can consider that

$$P(T,S)(1 + L(T,S)\delta(T,S)) = 1 \tag{2.2}$$

so that

$$L(T,S) = (1 - P(T,S))/(\delta(T,S)P(T,S)) \tag{2.3}$$

Assuming that the no-arbitrage condition holds according to *Bianchetti*(2009), the value of a cash flow must be the same regardless of whether it is discounted directly from, period S to period t or it is discounted previously from period S to period T and then, again from T to t, (formally : $P(t,S) = P(t,T)P(T,S)$), it is possible to rewrite the payoff of the payer leg at S, $N(L(T,S) - R)\delta(T,S)$, as

$$N\delta(T,S)((1 - P(T.S))/P(T,S)\delta(T,S)\ -\ R) \tag{2.4}$$

From here by discounting it back at period t, can be derived the present value at t of the payoff at S which is

$$NP(t,S)(1/P(T,S)\ - 1 - R\delta(T,S)) \tag{2.5}$$

$$= N(P(t,T) - P(t,S) - RP(t,S)\delta(T,S))$$

As it has been stated above a FRA is fair if and only if its NPV is equal to 0, so it is possible to find its adequate fixed rate by setting equation 1.5 equal to 0 and solving for R_{FRA}

$$\begin{aligned} R_{FRA} &= (P(t,T) - P(t,S))/(P(t,S)\delta(T,S)) \\ &= \frac{1}{\delta(T,S)}\left(\frac{P(t,T)}{P(t,S)} - 1\right) \\ &= F_s(t,T,S) \end{aligned} \tag{2.6}$$

As a result, it is possible to say that a FRA is fair if and only if its fixed rate coincides with the implicit simple compounded forward rate.

2.3 FIXED AND FLOATING RATE BONDS

Fixed rate bonds, as opposed to discount bonds (or zero- coupon bonds), are financial instruments that promise to their holder a fixed stream of cash flows, called coupons, calculated as a fixed percentage paid on the notional N, which is repaid at maturity with the last coupon. Coupons are paid according to a predetermined time schedule which can be monthly, semi-annual, annual and so on and so forth. Due to their intrinsic characteristics they can be replicated by considering a portfolio of zero-coupon bonds with notional equal to the coupon paid. Let's consider the construction of the price at time t, which is the date in which the bond is emitted; denoting the coupon rate as c_i, the maturity date as T_N, the equally spaced coupon dates as $T_1, T_2, \dots, T_N$ and the discount factor as $p(t,T_i)$, with $i=1,\dots,N$ the price at t is defined as

$$P(t,T_N) = \sum_{i=1}^{N} \left[c_i N p(t,T_i)\delta_i + N p(t,T_N) \right] \tag{2.7}$$

Floating rate bonds or floating rate notes (FRN) are coupon bonds where the coupon payments are calculated on the base of a floating rate which is observed at any coupon date T_i. Usually the floating rate is a market rate such as LIBOR and EURIBOR. The rate used for the computation of coupon i is known at time $i-1$, but will occur at time i. The price of such bonds at time t, assuming the LIBOR as underlying floating rate, is calculated as

$$P(t,T_N) = \sum_{i=1}^{N} L(T_{i-1},T_i) N p(t,T_i)\delta_i + N p(t,T_N) \tag{2.8}$$

where $L(T_{i-1},T_i)$ is the LIBOR observed at T_{i-1} for the time interval between T_{i-1} and T_i.

It can be observed that there is the possibility to replicate the cash flows of the floating rate bond through a self-financing bond strategy which has initial value equal to $p(t, T_{i-1})$. Considering N equal to 1 and exploiting the definition of the LIBOR it can be proven that the value of the coupon payment at t is equal to $c_i = (T_i - T_{i-1})\frac{1-p(T_{i-1},T_i)}{(T_i-T_{i-1})p(T_{i-1},T_i)} = \frac{1}{p(T_{i-1},T_i)} - 1$. Knowing that the value of 1 at t is $p(t, T_i)$, by constructing the self-financing strategy it is possible to evaluate the present value of the remaining part of the equation above: an investor can buy at t equal to 0 a bond with notional equal to 1 and maturity equal to T_{i-1} , at time T_{i-1} he gets 1$ and reinvest it to buy another bond with notional equal to 1 and maturity T_i. At $T_i$he will receive $\frac{1}{p(T_{i-1},T_i)}$. So the investment required to receive $\frac{1}{p(T_{i-1},T_i)}$ at time T_i is of $-p(0, T_{i-1})$, implying that the value of the coupon payment at time 0 is equal to $p(0, T_{i-1}) - p(0, T_i)$. Putting all the components together

$$p(0) = p(0, T_N) + \sum_{i=1}^{N}[p(0, T_{i-1}) - p(0, T_i)] \tag{2.9}$$

2.4 INTEREST RATE SWAPS

An interest rate swap is a financial contract in which two parties agree to exchange interest payments periodically. There are several types of IRS such as the *Basis IRS, Currency SWAP, Amortizing IRS and the fixed for floating IRS or plain vanilla interest rate swap*. The one considered and described in this section is the plain vanilla one which is the simplest contract among the others. As stated above two parties exchange payments calculated on a predefined notional *N*, and in this case the two interest rates differ from each other, as one is fixed until the expiration date, while the other is floating and is rearranged at each period as is usually represented by market rates such as LIBOR or OIS rates. The two parties are referred to as the *payer leg*, which is the one paying the fixed rate and receiving the floating one, and the *receiver leg* which pays the floating and receives the fixed. Interest rate swaps allow to exploit the comparative advantages that companies may have in the market allowing them to earn a gain through these contracts.

The actual payment that is made at each predefined date consists of the difference of the two cash flows exchanged between the parties, and no notional payment ever occur. The payoff of the payer leg at T_i can thus be defined as

$$N\delta_i[L(T_{i-1}, T_i) - R] \tag{2.10}$$

where payments dates are considered equally spaced, formally $\delta_i = T_i - T_{i-1}$, and the cashflows exchanged in arrears. An IRS can be priced in terms of *bonds* or can be seen as a sequence of *forward rate agreements* and be so priced accordingly. In the latter case it is sufficient to consider the present value of the fixed payments separated from the present value of all the floating payment; considering $p(t, T_i)$the discount factor that applies between period t and T_i, with $t < T_i$ the PV of the fixed payments is

$$PV_{IRS,FIXED} = RN \sum_{1=1}^{N} p(t, T_i)\delta_i(T_{i-1}, T_i). \tag{2.11}$$

By the same reasoning, exploiting the fact that the fair FRA rate is $R_{FRA} = F_S(t; T, S)$ (which can be rewritten as $F_S(t; T_{i-1}, T_i)$), the PV of the floating payments is

$$PV_{IRS,FLOATING} = N \sum_{i=1}^{N} F_S(t; T_{i-1}, T_i)p(t, T_i)\delta_i(T_{i-1}, T_i). \tag{2.12}$$

The fixed interest rate is chosen in such a way that the PV of the fixed payments is equal to the PV of the floating ones, meaning that the value of the contract at *t=0* must be zero. Considering that there is no upfront payment, the fair IRS rate is found by equating the two PVs above

$$RN \sum_{1=1}^{N} p(t, T_i)\delta_i(T_{i-1}, T_i) = N \sum_{i=1}^{N} F_S(t; T_{i-1}, T_i)p(t, T_i)\delta_i(T_{i-1}, T_i) \tag{2.13}$$

$$R_{FAIR} = \frac{\sum_{i=1}^{N} F_S(t; T_{i-1}, T_i)p(t, T_i)\delta(T_{i-1}, T_i)}{\sum_{1=1}^{N} p(t, T_i)\delta(T_{i-1}, T_i)} \tag{2.14}$$

It has to be notice, however, that usually in reality the fixed and the floating payments do not necessarily occur with the same time schedule; typically, fixed payments are made annually while floating payments semi-annually so that the δ in the above formulas differs from the fixed payments case to the floating payments case.

2.5 CAPS AND FLOORS

Caps and Floors are two of the most used and important derivatives contracts and function as an insurance for investors who decide to hold them. A cap is a financial contract that provide to an investor the right to fix a predetermined maximum interest rate, *cap rate*, that he will consider for the repayment of a loan, even if the underlying rate of the loan is a floating one. A floor is a financial contract that gives its owner the right to set a predetermined fixed minimum rate, *floor rate*, that he will claim as interest rate on an amount of money invested. Although a similarity with FRAs contracts may be pointed out, there is a huge difference between them, since while the latter just give protection against future's uncertainty, floors and caps also allow to exploit eventual favourable outcomes to generate profits. Caps and Floors are sums of shortest and simplest contracts which are called respectively caplets and floorlets. Let us consider the definition of cap and floor and let's define them analytically. A caplet provide protection in the case in which the floating rate underlying a loan raises, so that the investor who enters into a loan will not pay more than the cap rate, but he will still gain profits in the case in which the rate decreases so that he will get the maximum between the difference of the floating and the cap rate and 0. Considering the case in which one knows for sure that will borrow an amount of money equal to N at time T_{i-1} for the maturity T_i and wants to protect against raising rates holding a caplet, so $\delta = (T_i - T_{i-1})$ is the time interval over which it is exercised, and $LIBOR(T_{i-1}, T_i)$ the spot floating rate underlying the caplet, the value of a caplet at time T_i is defined as

$$CAPLET_i = N\delta\, max[L(T_{i-1}, T_i) - R, 0] \tag{2.15}$$

On the other hand, if one knows for sure that will invest an amount of money equal to N at $T_{i-1}$for maturity T_i, and decides to hold a floorlet to have protection against falling rates, exploiting the symmetry between caplets and floorlets, the value of a floorlet at T_i is defined as

$$FLOORLET_i = N\delta \max[R - L(T_{i-1}, T_i), 0] \tag{2.16}$$

It can be proven that a cap can be transformed into a put option on an underlying bond with exercise price $\frac{1}{R^*}$ with exercise date T_{i-1}. Recalling that the LIBOR can be rewritten as $LIBOR(T_{i-1}, T_i) = \frac{1-p(T_{i-1},T_i)}{\delta(T_{i-1},T_i)p(T_{i-1},T_i)}$, the value of the caplet can be rewritten as

$$\begin{aligned} CAPLET_i &= \delta \max[\frac{1-p(T_{i-1},T_i)}{\delta p(T_{i-1},T_i)} - R, 0] \qquad (2.17) \\ &= \max[\frac{1}{p(T_{i-1}, T_i)} - (1 + \delta R), 0] \\ &= \max[\frac{1}{p(T_{i-1},T_i)} - R^*, 0] \\ &= \frac{R^*}{p(T_{i-1},T_i)} \max[\frac{1}{R^*} - p(T_{i-1}, T_i), 0] \end{aligned}$$

The result shows that a cap can be seen as a portfolio of put options.

2.6 OIS: OVERNIGHT INDEXED SWAPS

An overnight indexed swap is an interest rate swap in which the floating leg is linked to an overnight rate, such as EONIA rates. It concerns the exchange of interest payments, where one party (payer leg) exchange interest calculated with a fixed rate on a common notional N, for interests paid on a floating rate, which in this case is an overnight rate. Just as all the interest rate swaps it does not provide the exchange of any notional payments. Denoting with $\delta(t, T_0, T_N)$ the tenor in which the swap is exercised, where $t < T_0 < T_N$, the value of the fixed leg payments at time *t* is equal to

$$P^{OIS}(t, T_0, T_N)_{FIXED} = NR \sum_{i=1}^{N} \delta_i \, p(t, T_i). \tag{2.18}$$

Moving to the floating payments, first of all, it is required to compute the adequate rate for the interval T_{i-1}, T_i, which is done by compounding the overnight rate in T_{i-1} with the one in T_i.
Considering $T_{i-1} = t_0^i < t_1^i < \cdots < t_{N_i}^i = T_i$ the subdivisions of the time space, in which the overnight rates are observed and $\delta_{t_{j-1}^i, t_j^i} = t_j^i - t_{j-1}^i$ it follows that the compound between the two rates is structured as

$$R^{ON}(T_{i-1}, T_i) = \frac{1}{\delta_i} (\prod_{j=1}^{N_i} \left[1 + \delta_{t_{j-1}^i, t_j^i} R^{ON}\left(t_{j-1}^i, t_j^i\right)\right] - 1) \tag{2.19}$$

with the payment delivered at time T_i and with $R^{ON}\left(t_{j-1}^i, t_j^i\right)$ being the overnight rate corresponding to the time interval t_{j-1}^i, t_j^i. The value at time t of the floating payments will thus be

$$P^{OIS}(t, T_0, T_N)_{FLOATING} = N \sum_{i=1}^{N} \delta_i \, p(t, T_i) R^{ON}(t; T_{i-1}, T_i)$$

where, exploiting the link between overnight rates and OIS bond prices

$$R^{ON}(t; T_{i-1}, T_i) = \frac{1}{\delta_i} (\frac{p(t, T_{i-1})}{p(t, T_i)} - 1 \tag{2.20}$$

2.7 BASIS SWAPS

A basis swap is a financial contract in which two parties agree to exchange interest payments periodically. In these particular interest rate swaps the two legs both pay a floating amount, the difference is that the two underlying rates may have different tenors or, may be two different markets rates. For example, the payer leg can pay a floating amount linked to the 6 months EURIBOR, while receiving a floating amount linked to the 3 months EURIBOR. The scope of such an instrument is to hedge risk when an investor's portfolio is made of assets or liabilities, tied to different market rates, or to different tenor structures, and the risk associated with the mutations of the normal relationship between those floating rates is defined as *basis risk.* As it will be described more in details later, this type of derivatives has been highly discussed during the crisis since, due to the market transformations the basis risk has increased notably causing pre-crisis pricing method to be no longer useful and adequate. Before the crisis the environment was characterized by a market free of arbitrages and risk, so that the basis swap spread was considered to have mean equal to 0. In order to simplify the calculations for the price of a basis swap it can be decomposed considering two plain vanilla swaps in which the fixed legs are identical, while each floating leg assumes the tenor of one of the two market rates considered in the original contract. Here it will be simply considered a floating-for-floating swap linked to, let's say 6months EURIBOR and the 3months EURIBOR, where the latter presents a basis swap spread. The value of the contract for the payer leg, assuming that his payments are linked to the 3months EURIBOR, at time $T_i = T_j$ where T_i belongs to the term structure of the EUR(0,3months), while T_i to the EUR(0,6months) term structure, is

$$N[(EUR(0{,}6months)\delta_j) - (EUR(0{,}3months)\delta_i + R_{BS})] \tag{2.21}$$

From here it is easy to derive the present value of such a contract, bearing in mind that as usual its initial value must be equal to 0.

Defining the two term structures as $T_i = T_1, T_2, \dots T_N$ and $T_j = T_1, T_2, \dots T_M$, the present value is equal to

$$PV^{BS} = N(\sum_{j=1}^{M} P(t,T_J)EUR(t,T_J)\delta(t,T_J) - \sum_{i=1}^{N} P(t,T_i)[EUR(t,T_i) + R_{BS}]\delta(t,T_i)) \tag{2.22}$$

Setting the above equation equal to 0, it is possible to define the basis swap spread as

$$R_{BS} = \frac{\sum_{j=1}^{M} P(t,T_J)EUR(t,T_j)\delta(t,T_J) - \sum_{i=1}^{N} P(t,T_i)EUR(t,T_i)\delta(t,T_i)}{\sum_{i=1}^{N} P(t,T_i)\delta(t,T_i)} \tag{2.23}$$

It can be observed that if the payments dates coincide so that i=j and M=N then

$$R_{BS} = EUR(t,T_j) - EUR(t,T_i) \tag{2.24}$$

which in this case is

$$R_{BS} = EUR(0{,}6months) - EUR(0{,}3months) \tag{2.25}$$

Before the financial crisis it was roughly equal to 0, afterwards it has starts increasing dramatically.

2.8 SWAPTIONS

A swaption (swap option) is a contract which provides to its holder the right but not the duty to purchase a swap at a predetermined rate R, called the strike rate, and at a predefined date. It is so a call option which provides to its purchaser the possibility to enter into a payer swap contract in which he will pay fixed interests payments and will receive floating interests payments as in a traditional payer-interest rate swap; on the other hand, his counterparty will agree to enter into a put option which ensures him to have the possibility to pay floating interest amounts receiving fixed ones. Of course, here it is considered a European option which, as opposed to American ones, cannot be exercised before the expiration date, but presents a fixed date in which it can be operated rather than an interval of time. As just said, the swap starts at expiring date of the option and it has a specific maturity attached, for example the swaption may be exercised within 12 months, and the swap may last for 6 years. At time equal to 1 year (=12 months is the expiration date) the holder can face two different scenarios: it may be that the market swap rate is higher than the strike one, and in this case the investor will choose to exercise his option, or it may be that the market swap rate is lower than the strike rate, in this case the investor will better pay the market rate without exercising his option. The payoff of the swaption at $t < T$, where the latter is the maturity date of the swaption and corresponds to T_0, which represents the date in which the holder signs the swap contract for exchanging payments in the future dates $t_1 < t_2 < \cdots < t_i < \cdots < t_N$ is equal to

$$P^{swn}(t; T_0, T_n, R) = p(t, T_0) E^{Q^{T_0}}\{(P^{sw}(T_0; T_n, R)^+ | F_t\} \tag{2.26}$$

where R is the market swap rate of the underlying swap.

The payoff of the counterparty is symmetric so will be the opposite. In order to price swaption contracts a modified version of the Black-Scholes model is also used.

3 PRE-CRISIS SCENARIO. SINGLE CURVE FRAMEWORK.

In this section an overview of the pre-crisis, traditional single curve model will be exposed, in order to better understand, later on, how the relationships among derivatives' fundamental variables have been modified and intensively disrupted by the changes brought by the financial crisis. The focus will be mainly on the discount, forward and spot curves, as they represented the main tools on which all the traditional standard market practice was based. The aim is to emphasize how, before the crisis, a single spot curve was enough for pricing derivatives, since from it, it was easy to extrapolate forward rates and discount factors, as all the linkages between derivatives were as simpler as possible.

3.1 DISCOUNT CURVE

The main problem to address when it comes to price derivatives or other financial assets, is to apply the adequate technique which allows to consider the time value of money; the discount curve has always served this scope without any controversial results, but it will be later, pointed out that when rates tend to diverge within different maturities, such a framework can no longer be considered. The key reasoning is that in order to have a market which does not present any arbitrage opportunity, the price of any financial instrument must be equal to the present value of its expected future cash flows, discounted with the appropriate risk measure. As it has been said in chapter 1, zero coupon bonds or discount bonds, are the simplest financial derivatives and, thanks to their payment schedule which is composed by only two dates, t which is today, and maturity T, it is easy to see that their price today must be equal to the amount that they promise at time T discounted back at time 0(today). If one considers that the amount delivered at T is 1, the current price of a zero-coupon bond is

$$P(t,T) = \frac{1}{(1+y)^N} \tag{3.1}$$

where N is equal to the maturity of the bond, so for example, if it is a 5 years bond, N will be 5, as the amount has to be discounted from T=5 back to t=0.

Brigo and Mercurio defined the discount factor between t and T as "the amount at time t equivalent to one unit of currency payable at time T" as the reasoning underlying the calculations is that the two amounts must have the same value when compared in the same date.

Putting all the zero-coupon bonds prices in the (P, T) space, it is possible to derive the discount curve, which provides the adequate discount factor for each maturity, and allows to extrapolate all corresponding interest rates.

3.2 THE SPOT CURVE

From the relationship pointed out in the previous paragraph, between the discount bond price and its rate of return, one can derive the spot rates for different maturities, which are technically seen as the interest rate that an investor will gain if he invests an amount equal to $P(t,T)$ today, (t) for maturity T.

A main distinction must be made between simply compounding interest rates and continuously compounding interest rates, as the former compounds m times a year, while the later compound continuously. An investment of 1 today, using a simply compounding rate, will be equal to an amount of $1(1+\frac{r_{0,T}}{m})^{\,m}$ at time T, where m is the number of compounding periods in a year. Similarly, the current price of an amount equal to 1 promised at T is equal to

$$P(t,T) = \frac{1}{\left(1+\frac{r_{0,T}}{m}\right)^{m}} \tag{3.2}$$

It can be observed that as $m \to \infty$, so as the compounding frequency increases, the simply compounded rate converges to the continuously compounded one, so that

$$\lim_{m\to\infty}\left(1+\frac{r_{0,T}}{m}\right)^{m} = \ e^{r^{C}_{0,T}\delta(0,T)} \tag{3.3}$$

The factor $\delta(0,T)$ depends on the day counting convention, which is usually, Actual/360.
It is now possible to the derive the spot curve, exploiting the relationship that exists between zero coupon bond prices and the spot rates.
It has already been said what the price of an amount equal to 1 at T should be, considering a simply compounding rate, so following this reasoning we know that investing an amount equal to P(t,T) today should yield an amount equal to 1 at T. Knowing that the investment will grow by the factor $\left(1+\frac{r_{0,T}}{m}\right)^{m}$between t and T, this is equivalent as saying that

$$P(t,T)\left(1+\frac{r_{0,T}}{m}\right)^{m} = 1 \tag{3.4}$$

Using the notation of *Brigo and Mercurio*, always considering that “the spot rate at time t is the constant rate for which the zero- coupon bonds yields 1 at maturity T”, one gets that

$$P(t,T)\big(1+L(t,T)\delta(t,T)\big) = 1 \tag{3.5}$$

So, the spot rate for maturity T is derived as

$$L(t,T) = \frac{1-P(t,T)}{P(t,T)\delta(t,T)} \tag{3.6}$$

The same applies when a continuously compounding interest rate is used, as from

$$P(t,T)e^{R(t,T)\delta(t,T)} = 1 \quad (3.7)$$

one can find

$$R(t,T) = -\frac{\ln P(t,T)}{\delta(t,T)}. \quad (3.8)$$

Plotting all the spot rates together one can find the spot curve, where each rate corresponds to a specific maturity.

3.3 FORWARD CURVE

Forward rates are rates which are set today (time t), but which do not apply immediately but rather in a interval of time between time T_1 and T_2 where $T_2 > T_1 > t$. If t=T_1 then it becomes a spot rate.

In order to define forward rates, it will be useful to exploit again, the no arbitrage condition which *Bianchetti* has stressed in his paper (2009). Recalling that he stated that, given rates that are known today, any cash flow that is discounted from T_2 to t must always have the same value whether it has been directly discounted from T_2 to t, or previously discounted from T_2 to T_1, and then again from T_1 to t, it follows the below relationship between discount factors

$$P(t,T_2) = P(t,T_1)P(T_1,T_2) \quad (3.9)$$

Here the discount factor $P(T_1,T_2)$ represents the forward discount factor between T_1 and T_2, and can also be seen as the price at T_1 of a zero- coupon bond with maturity T_2. Following this reasoning it can rewritten as the present value at time t of a unit of currency invested at time T_1 until maturity T_2, that is

$$P(t,T_1,T_2) = \frac{1}{1+F(t,T_1,T_2)\delta(T_1,T_2)} \quad (3.10)$$

From here, by solving the equation for $F(t,T_1,T_2)$, the simply compounded forward rate is founded

$$F(t,T_1,T_2) = \frac{1}{\delta(T_1,T_2)P(t,T_1,T_2)} - 1 \quad (3.11)$$
$$= \frac{1}{\delta(T_1,T_2)}[\frac{1}{P(t,T_1,T_2)} - 1]$$

From equation “3.9” it follows that $P(t,T_1,T_2) = \frac{P(t,T_2)}{P(t,T_1)}$. Substituting in equation “3.10” gets

$$F(t, T_1, T_2) = \frac{P(t,T_1) - P(t,T_2)}{\delta(T_1,T_2)P(t,T_2)} \tag{3.12}$$

The same result could have been reached through the computation of the fair rate of a forward rate agreement, as it has been done in chapter 1. Introducing expectations and the forward measure Q^T, in the computation of a Fra fair value, one ends up with the no arbitrage condition that the present value of the FRA, under a Q risk neutral, musts be equal to its current price. Considering a FRA which at time T_2 offers a payoff of $L(T_2; T_{1,} T_2) = N\delta(T_1, T_2)[L(T_1, T_2) - R]$,with a current price of $P(t, T_1)$, one gets

$$P(t, T_1) = N\delta(T_1, T_2)\, E^Q \left\{[L(T_1, T_2) - R]e^{-\int_0^{T_2} r(s)ds}\right\} \tag{3.13}$$

$$P(t, T_1) = N\delta(T_1, T_2) \frac{E^Q\left\{[L(T_1,T_2) - R]e^{-\int_0^{T_2} r(s)ds}\right\}}{P(t,T_2)}\, P(t, T_2)$$

$$P(t, T_1) = N\delta(T_1, T_2)\left\{E^{Q^T}[L(T_1, T_2) - R]\right\}P(t, T_2)$$

3.4 BOOTSTRAPPING AND INTERPOLATION

There is not a unique way to reconstruct the entire spot curve for maturities ranging from overnight to months, years lengths of time. Here the bootstrapping and the interpolation methods will be introduced, as to show how before the credit crunch through a single curve, the spot one, it was possible to combine spot rates for all maturities, and it was sufficient to extrapolate forwards rates and discount factors as well. The derivation of the curve exploits, as said before, zero coupon bonds' prices and their relationships with their internal rates. The curve can be, for simplicity, divided in three areas according to the length of maturity to which it refers, and over all its path can be described using different financial instruments which can be described in terms of discount bonds. The main identity used is the one linking spot rates to derivatives prices, which is $P(0, T) = \frac{1}{1 + L(o,T)\delta(0,T)}$, and it will be now showed how these techniques allows to derive all spot rates from the discount factor. The key assumption is that prices are equal to the sum of all the future cash flows discounted with the zero- coupon rates for the corresponding maturity.
Let's assume that prices for 5 coupon bonds are given, as well as their maturities and coupon rates, so for example, what follows applies

T coupon rate Price

1	0	98
2	0	94.25
3	5%	102.96
4	7%	107.72

and they all have a face value equal to 100. From here we already expect to have a rate greater than their coupon one for the first two bonds, since they both sell below par.

Let's compute the simple compounded spot rate for maturity 1 year, using the fact that the price is equal to the present value of the unique cash flow that the bond promises, which is its face value 100 after 1 year:

$$98 = \frac{100}{1+r_{0,1}} \qquad r_{0,1} = \frac{100}{98} - 1 = 0.02 = 2\% \tag{3.14}$$

The same applies for the two years simple compounded spot rate, so that

$$94.25 = \frac{100}{(1+r_{0,2})^2} \qquad r_{0,2} = \sqrt[2]{\frac{100}{94.25}} - 1 = 0.03 = 3\% \tag{3.15}$$

It can now be observed that, for the construction of the other two spot rates, one can consider the two coupon bonds as the sum of zero coupon bonds with different maturities and face values, So for example the 5% three years coupon bond can be seen as the sum of one zero coupon bond with maturity 1 year and face value of 5, one zero coupon bond with maturity 2 years and face value 5 again, and in the end one zero coupon bond with maturity 3 years and a face value of 105. These are nothing else than the single cash flows that the bond promises, and they can be calculated with the spot rates already found:

$$102.96 = \frac{5}{1+r_{0,1}} + \frac{5}{(1+r_{0,2})^2} + \frac{105}{(1+r_{0,3})^3} \qquad r_{0,3} = 0.04 = 4\% \tag{3.16}$$

Repeating the same reasoning for the fourth bond we get that

$$107.72 = \frac{7}{1+r_{0,1}} + \frac{7}{(1+r_{0,2})^2} + \frac{7}{\left(1+r_{0,3}\right)^3} + \frac{107}{\left(1+r_{0,4}\right)^4} \quad r_{0,4} = 0.05 = 5\%. \tag{3.17}$$

Interpolation is a statistical method that exploits related known factors to find unknown ones, where these unknowns may be securities' prices, interest rates.

After having bootstrapped interest rates, different type of interpolation can be used to find out the value of a security between two known periods, such as the polynomial one, piecewise constant interpolation and the linear one.

The linear interpolation is the simplest one and exploits linear regressions to create curves representing price variations over time. Although it is widely used, interpolation has several pitfalls, such as its tendency to create kinks whenever the slope of the yield curve isn't constant.

4 EFFECTS OF THE CRISIS

Now the main changes and distortions brought by the crisis will be explained, as to briefly understand the main features attached to the credit crunch and the reason of the departures from traditional pricing and discounting methods.

4.1 SPREADS: CREDIT AND LIQUIDITY RISKS

As just pointed out earlier in the discussion, one of the main features of the credit crunch of 2007 has been the overtime increasing spreads between markets rates, which have changed significantly with respect to their traditional (before crisis) levels. The focus here is on trying to understand the link that exists between those spreads and the different risks characterizing the market during the latest years and to study their behaviours over time.

Data show that before 2007 the Euribor-Eonia swap spread of maturities from 1 to 12 months was constant, with mean roughly around zero, while as the crisis started to settle in, they responded with a huge and immediate increase which caused several peaks, with the greatest of more than 200 basis points.

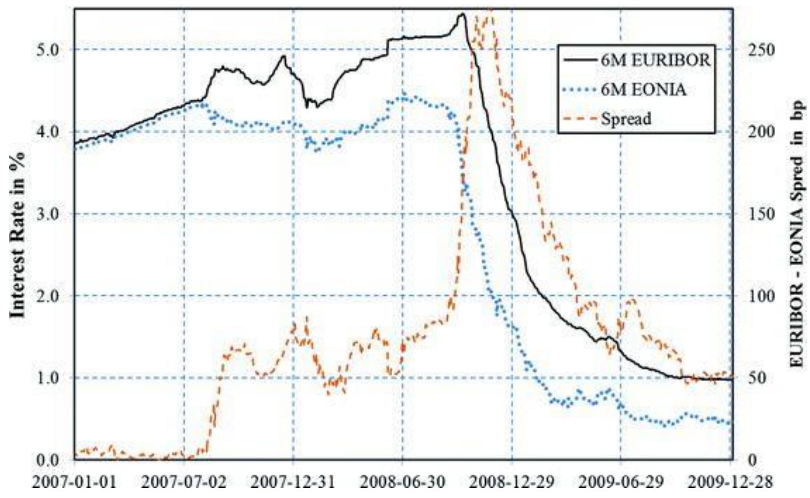

Figure 1 Source: Analytical Finance: Volume II, Jan R. M. Röman

Libor and OIS spreads exhibited a similar path causing the so- called *Libor-OIS spread* and the *Libor-OIS swap spread,* where the latter refers to the spread between the OIS rates and the Libor indexed IRS' swap rate.

This scenario has created fundamental mutations that have not been overcome even after 2012, creating an environment in which there could be evidence of arbitrage. Although this might seem the case, arbitrage opportunities seem not to have been exploited, since as said before there has not been any adjustment in the market, which should have been the consequence of the arbitrage actions eventually made; this is because even if there has been the possibility to benefice from different spreads over time, this profitable landscape has been offset by the presence of different risks, mainly tied to interbank risks.

As described in the previous chapters EONIA,OIS rates, LIBOR and EURIBOR rates are defined by groups of panel banks; those banks might cheat by uploading a rate different from the one that they think could fit to the official definition of, for example, LIBOR, that is "The rate at which an individual contributor Panel bank could borrow funds, were it to do so by asking for and then accepting interbank offers in reasonable market size, just prior to 11 London time" as a consequence of the worsening of the credit quality. This created evidence of default risk attached to panel banks, which translated in to mismatching Libor rates for different maturities. Another contributor to this result has also been the liquidity risk, referring to the day by day decrease in liquid assets ownings by banks, where with liquidity one is measuring the marketability of a contract or financial instrument, meaning the ability to hedge risk by selling or buying it quickly on the market.

Many studies and sample analysis have been made by economists such as Mercurio, Henrard, Morini and Filipović and Trolle to demonstrate how the increasing interbank risk has caused these distortions in the market. The latter has collected data for period between 2007 and 2011 showing how the credit risk has been the major cause of interbank risk as a whole and how the liquidity one has mainly impacted short term contracts.

In short what happened in contrast with the pre-crisis models, has been that the volatility of market rates has caused investors to use more than one yield curve in the pricing of derivatives, since as the rates started to diverge over different tenor, discount practices tied to the single zero coupon or yield curve has no longer served as they did.

4.2 LIBOR-OIS SPREAD

As just said in the last paragraph, spreads played a crucial role in the market mutations. LIBOR-OIS spread and LIBOR-OIS swap spread, have been much discussed due to the characteristics of the rates that represent them.

Recalling the importance of the Libor as a benchmark rate and as an indicator of the credit market conditions, it is straight forward the interpretation of its spread with respect to the OIS rate. Let's focus first of all, on the different natures of the two, in that the Libor rate, as decided by banks, is a floating rate and for that it embodies a credit risk in it, while the OIS rate is a fixed one and differs in every country; for these reasons the spread between the two is interpreted as a measure of the credit risk in the market, and for that, as it increased the market stability has decreased with it. Its growth has been so interpreted by banks as an increasing risk of default associated with interbank lending, as it actually was the situation. The increasing growth of the Libor-OIS spread has reached its peak at the end of 2008, as *Figure 4* shows and has been the determinant for the creation of new model of discounting as it will be seen in the next chapter.

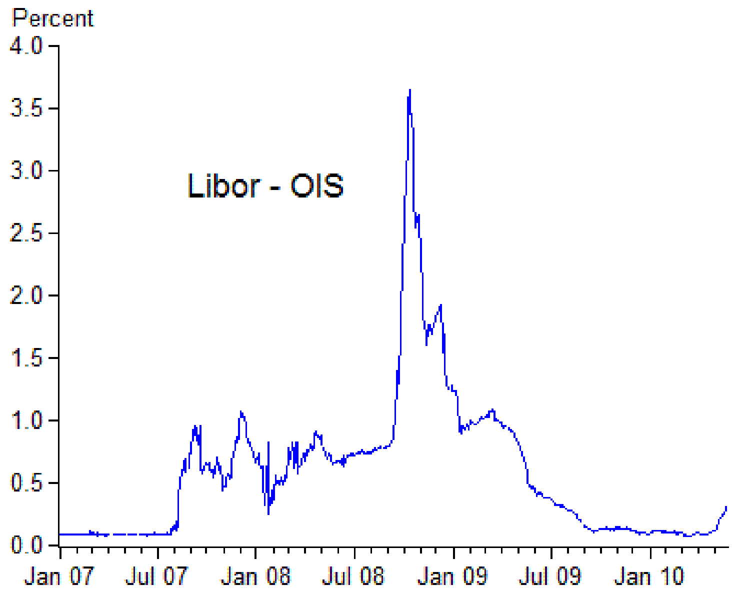

Figure 2 Source: The Fed's swap loans and Libor-OIS spread, John B. Taylor

4.3 COLLATERALIZATION

Collateralization is a way to overcome counterparty(default) risk. It technically involves that the borrowing party posts to the lender party assets or cash in order to cover fully or partially its debt in case of default, meaning in the case in which he will not be able to repay its debt. The collateral serves as a cushion and so in case of fulfilment of the reparation of his debt the borrower will eventually receive it back with the earned interests. Collateralization is a matter only for over-the-counter derivatives (OTC), since exchange-traded ones are protected from risk through the clearing from the exchange.

The ISDA, International Swaps and Derivatives Association provides guidance and standards on which collateralization has to be based, and since the days of the crisis it has been strongly argued its importance in the market due to interbank risk that has been discussed earlier.

It can be done according to different mechanisms, for example the borrower could be required to post a collateral whenever the NPV of the contract sharply decreases, or whenever its credit rating worsen, in both cases the collateral has to be posted on the *collateral account*, where it remains until there's need of insurance. In particular in case of bilateral contracts, meaning contracts in which each party promises to perform a transaction in exchange of the other party's transaction, the current value of the contract is valued according to the market (marked-to market) in order to assess if any of the two parties' position has been devalued, if so the one with lost value should post the collateral.

The big issue connecting the practice of collateralization to the 2007 credit crunch is the fact that banks turned out to be strictly connected with one another, more than it was thought in the years before, meaning that the default of one could have caused a domino effect to the others.

Also the lack of liquidity has concurred to this increasing need of *securing* contracts, but as seen above both risks translate in the uncertainty accompanying banks.

In 2010 roughly the seventy per cent of financial contracts were collateralized, versus the twenty per cent in the prior years, reflecting how institutions' lost credibility in the eye of investors and marketers.

There exists another channel (instrument) through which one can protect himself against default risk in a bilateral contract and is the introduction of a CPP, that is a central counterparty. This counterparty acts as an intermediate institution between the two party effectively signing the contract and allows to forgive about their liquidity and default risks, leaving the importance and the need to monitor the risks associated with the intermediary. Although one may argue that risk still influence the contract even with a CPP, it is true that usually the latter bear negligible default risk thanks to their sufficient liquidity.

CPP uses different tools to mitigate risk in contracts, such as the *participation constraint*, which excludes parties with an associated risk above a given breakeven level from the contract, or as the *variation margin* which induce the counterparty which suffered of loss in position to post the variation margin to the central counterparty.

4.4 INTENSITY-BASED CREDIT MODELS

Intensity-based credit models are used in order to model the default time of an obligor, defined as τ , using the intensity of default. One can define the intensity of default as the rate at which this event is expected to occur, and it can be written as λ_t. According to Hebertsson (2011) the default time can be thought to be the "the first time the increasing process $\int_0^t \lambda (X_s) d_s$ reaches a random level E_1".

There are different models currently used to point out different situations, and all differ from each other in that the default intensity takes on different roles, according to its nature; for example, it can be a deterministic constant or a deterministic function of time or even a stochastic process.

In the following paragraph the stochastic model used is also known as the "Cox model".

The overall model is based on a given probability space (Ώ, F, Q) where F is the overall market information at time t, better denoted as F_t, which represents any possible outcome that could happen. It is now possible to introduce the information space generated by the stochastic process X_t, defined as G_t^X together with another filtration described as H_t which represents the indicator function $1_{\{\tau \leq t\}}$. This function explains the relationships between the sub-sigma algebra and the whole space and will be 1 in the case in which there has been a default prior to period t, 0 otherwise. These three filtrations can be also called sigma-algebras.

The basic assumption made is about the conditional probability of τ; considering a single obligor with default time τ, it can be proven that the conditional probability of default in the time period between t and $t + \Delta_t$, given the fact that it has not yet occurred before period T, so $\tau > T$, is

$$P[\tau \in (t, t + \Delta_t)|F_t] = \lambda_t \Delta_t \tag{4.1}$$

where F_t represents the market information available at time t.

What the above formula intends to illustrate is that, given the market information and given that the default has not yet occurred before t, the conditional probability of default in the period between t and $t + \Delta_t$ is equal to $\lambda_t \Delta_t$, which is the intensity of default times the time interval.

Hebertsson defined the conditional survival probability as the probability that the default time τ occurs only after a given date T considering the market information. Algebraically it can be written as

$$P[\tau > T|F_t] = 1_{\{\tau \geq t\}} E[e^{-\int_t^T \lambda(X_s) d_s} | G_t^x] \tag{4.2}$$

taking into account that the conditional probability of surviving until time t is equal to

$$P[\tau > t] = E[e^{-\int_0^t \lambda(X_s) d_s}] \; . \tag{4.3}$$

As Lando described in his paper (2004), one of the aims of the construction of an intensity-based model is to price cash flows, which are influenced by the default time τ, meaning that they are attached to a defaultable claim. If one assumes that $\lambda(X_s)$ is deterministic, it has been shown by Lando that a useful equality can be generated by pricing at time t. This equality is described as follow

$$E\left(\exp\left(-\int_t^T r(X_s)ds\right)f(X_t)1_{\{\tau>T\}}\middle|F_t\right) = 1_{\{\tau>t\}}E\left[\exp\left(-\int_t^T (r(X_s)+\lambda(X_s))d_s\right)f(X_t)\middle|G_t\right] \quad (4.4)$$

This will hold for every function of x in general, and so also applies in the case in which one considers the short risk-free interest rate r_t as a function of X_t, so that it will be also true that the short-rate process $r_t(\omega) = r_t(X_t(\omega))$ is stochastic.

It is now possible to exploit the above formula to compute the price of a defaultable discount bond $\tilde{P}(0,T)$, with maturity T. Because of the risk of default of the obligor, his counterparty is entitled to have a recovery whenever the default happens before maturity, that is whenever $\tau < T$.The recovery, denoted with the variable Φ, is in between 0 and 1 as it is expressed as a percentage in terms of present value of the promised cash flows. In order to express the meaning of the recovery and to see the influence of the intensity of default on the current price of the bond, we can compute the latter assuming that the possible recovery is paid only at maturity; the computation will evolve as follow

$$\tilde{P}(0,T) = E[e^{-\int_0^T r(X_s)ds}(1_{\{\tau>T\}} + \Phi 1_{\{\tau\leq T\}})] \quad (4.5)$$

The use of the two indicator functions helps to better understand the meaning of the recovery. The first one will equal 1 in the case in which $\tau > T$, that is in the case in which no default has yet occurred before T, while the second one following the same reasoning will equal 0 in that specific case, as the counterparty will not be entitled to receive any recovery, so the price of the bond today will just equal the present value of its cash flow in T. On the other hand, if and only if the default happens before time T, then a recovery of Φ will be required, and it will be directly counted as a percentage of the present value of the cash flow, so the price of the bond today will just equal the established recovery.

This tool has been very important and discussed during the crisis, as it has been used by Mercurio (2009) in order to calculated and track the abnormalities in the spreads over the post-crisis years.

Specifically, he focused on the spread between forward rate agreements (FRA) and implied forward rates. He considered a defaultable bond, with non-zero recovery in the time interval between t and T, with t=0. The price at inception, time t, of that bond can be computed as above

$$\tilde{P}(t,T) = E[e^{-\int_t^T r(s)ds}\left(1_{\{\tau>T\}} + \Phi 1_{\{\tau\leq T\}}\right)|F_t] \quad (4.6)$$

Since, as said earlier, the sum of the two indicator functions must be equal to 1, it is possible to rewrite equation 4.6 as follows

$$\tilde{P}(t,T) = E[e^{-\int_t^T r(s)ds}(\Phi + (1-\Phi)1_{\{\tau>T\}})] \tag{4.7}$$

If we further consider that the interest rate is independent from the default intensity, it is possible to disentangle the expectation in two parts

$$\tilde{P}(t,T) = E\left[e^{-\int_t^T r(s)ds}\right] E\left[(\Phi + (1-\Phi)1_{\{\tau>T\}})\right] \tag{4.8}$$

which can then be simplified in

$$\tilde{P}(t,T) = P(t,T)[\Phi + (1-\Phi)\mathrm{P}[\tau > T]] \tag{4.9}$$

where P(t,T) is the default-free price of a zero-coupon bond at time t , and $P[\tau > T]$ is the expectation of the indicator function $1_{\{\tau>T\}}$.

One of the peculiarities of the FRA, prior to the crisis, was that it was possible to replicate its cash flows by taking a long position on a deposit with the same maturity and short-selling another deposit with maturity equal to the FRA's settlement (inception) date; after the 2007 collapse, the spread between FRA rate and the implied forward rate started increasing so that, due to the fact that they no longer coincided, this technique for FRA's cash flows replication was no longer useful.

Mercurio proved the existence of the spread between these two rates though the intensity-based model.

First of all, let's consider a deposit written on the LIBOR rate, with inception at time t and maturity at time T. The price of that deposit can be equally described by equation "4.8", given the market information F_t

$$\tilde{P}(t,T) = P(t,T)(\Phi + (1-\Phi)E[1_{\{\tau>T\}}|F_t]) \tag{4.10}$$

Given that, it is now possible to rewrite the LIBOR rate for the time interval t,T as a function of the deposit price computed with the intensity-based credit model:

$$\begin{aligned} LIBOR(t,T) &= \frac{1}{\delta(t,T)}\left[\frac{1}{\tilde{P}(t,T)} - 1\right] \\ &= \frac{1}{\delta(t,T)}\left[\frac{1}{P(t,T)}\frac{1}{(\Phi+(1-\Phi)E[1_{\{\tau>T\}}|F_t])} - 1\right] \end{aligned} \tag{4.11}$$

Let's now try to replicate the FRA cash flows with the technique described above.

Suppose an investor enters into a payer FRA with maturity T, paying the fixed rate K_{FRA}, so that he receives

$$\frac{\delta(t,T)(LIBOR(t,T)-K_{FRA})}{1+\delta(t,T)(LIBOR(t,T)} \tag{4.12}$$

Assume that he furthermore go long on a deposit with maturity T, for an amount equal to $(1+\delta(t,T)F_D)$ so that he pays out $(1+\delta(t,T)F_D)\tilde{P}(0,T)=\tilde{P}(0,t)$, where F_D is the implied forward rate and $\tilde{P}(0,t)$ is the price of the deposit at time 0.

He also goes short on a deposit with maturity t so that he receives an amount of $\tilde{P}(0,t)$.

The value of the strategy at time 0 must be null, so exploiting this condition it is possible to rewrite the FRA rate by discounting back to time 0 the payoff at time t

$$0=E\left[P(0,t)\frac{\delta(t,T)(LIBOR(t,T)-K_{FRA})}{1+\delta(t,T)LIBOR(t,T)}\right] \tag{4.13}$$

$$0=E\left[P(0,t)\left(1-\frac{1+\delta(t,T)K_{FRA}}{1+\delta(t,T)LIBOR(t,T)}\right)\right]$$

$$0=E[P(0,t)(1-(1+\delta(t,T)K_{FRA})P(t,T)\big(\Phi+(1-\Phi)E[1_{\{\tau>T\}}|F_t]\big]$$

$$0=P(0,t)-(1+\delta(t,T)K_{FRA})P(0,T)\big(\Phi+(1-\Phi)E[1_{\{\tau>T\}}|F_t]\big).$$

Before rewriting the above equation in terms of the FRA rate, lets recall that the implied forward rate is

$$F_D(0,t,T)=\frac{1}{\delta(t,T)}\left[\frac{P(0,t)}{P(0,T)}-1\right]. \tag{4.14}$$

From equation 4.12 it results that the FRA rate can be written as

$$\widetilde{K}_{FRA}=\frac{P(0,t)-P(0,T)(\big(\Phi+(1-\Phi)E[1_{\{\tau>T\}}|F_t]\big)}{\delta(t,T)P(0,T)(P(0,t)-P(0,T)\big(\Phi+(1-\Phi)E[1_{\{\tau>T\}}|F_t]\big)}. \tag{4.15}$$

$$\widetilde{K}_{FRA}=\frac{1}{\delta(t,T)}\left(\frac{P(0,t)}{P(0,T)}\frac{1}{(\Phi+(1-\Phi)E[1_{\{\tau>T\}}|F_t])}-1\right) \tag{4.16}$$

It has now to be noticed that since Φ is between 0 and 1, and $\left(E[1_{\{\tau>T\}}|F_t]\right)$ is equal to the probability of that event occurring , that is the probability of $\tau > T$,the whole factor $\left(\Phi + (1-\Phi)E[1_{\{\tau>T\}}|F_t]\right.$ will be greater than 0 and lower than 1 , so that it turns out that the FRA rate is greater than the implied forward rate of the two default-free deposits (bonds). It can now be seen how the two rates no longer coincide.

5 POST-CRISIS SCENARIO AND MULTIPLE CURVES MODELS

In this section some of the main post-crisis results and models will be analysed, providing a brief overview of the techniques and framework used and studied during those years. After having defined the assumptions and the reasoning underlying the new models, the focus will be shifted on the construction of the discounting and forward curves, and the price of some of the principal derivatives, already encountered in this thesis, will be computed.

5.1 BROWNIAN MOTION

It will now be better described the concept of Brownian motion, which has already been mentioned and used in the previous model but will now be explained in detail to be later used for more complex constructions. The Brownian motion is a Gaussian stochastic process defined as $W = (W(t))_{t\leq T}$, assuming that it exists on a filtered space $('\Omega, (F_t)_{t\in[0,T]}, P)$.

This particular process represents the disorganized motion of small particles suspended in a liquid, and it is someway believed that it was derived from the observation under the microscope of pollen particles floating in the water. Although Brown carried out the scientific observations and the afterwards research on it, Wiener was the one who transformed all the research in highly constructed stochastic process.

In order for $W = (W(t))_{t\leq T}$ to be a standard Brownian motion three conditions must hold: first of all that $W(0) = 0$, then that W is adapted to the filtration meaning that it is known at time t and cannot reveal the future, it is also necessary that for any s<t, the increment W(t)-W(s) is independent from the filtration F and its distribution is N(0,t-s).

5.2 HULL WHITE ONE FACTOR MODEL

The Hull-white one factor model is and interest rate term model which is consistent with the no-arbitrage conditions required in the market. It is a short-rate model and is useful in the multiple curves world as it can be used as a base for constructing two different curves. The problem that arise with the changes in the post-crunch market is that, as already said in the previous paragraphs, one discount curve can no longer suffice in the pricing of a derivative, as it can no longer be applied to rate of different tenors. For example, considering an interest rate swap with semi-annual payments exchanged, and with the floating one depending on a future 3 months floating rate, one can no longer use a single discount curve, but has the need of two different curves, one for discounting and the other to model the interest rate. In this case one can use two sets of Hull-white parameters to construct the pricing model under some accurate conditions.

The Hull-White model is characterized by the following equation, which describes the dynamics that the short rate follows

$$dr = a\left(\frac{\theta(t)}{a} - r\right)dt + \sigma dW(t) \tag{5.1}$$

where $a \; and \; \sigma$ are constant (more precisely a is called the mean reversion parameter), w(t) is a Brownian motion and $\theta(t)$ is derived from today yield curve and is a deterministic function of time.

$\theta(t)$ is defined as follow

$$\theta(t) = \frac{\delta F(0,t)}{\delta t} + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{5.2}$$

where F(0,t) is the implied forward rate at time 0 for maturity t.

Focusing on the short rate, it is assumed that it has a normal distribution, meaning that it can take on every value in R, even a negative one. From that it follows that its mean value is $E[r(t)] = e^{-at}r_0 + \frac{\theta}{a}(1 - e^{-at})$ and its variance is $Var[r(t)] = \frac{\sigma^2}{2a}(1 - e^{-2at})$; as $t \to \infty$ the mean will equal $E[r(t)] = \frac{\theta}{a}$ and the variance $Var[r(t)] = \frac{\sigma^2}{2a}$. It can be observed that as a gets bigger, the rate tends to reach "faster" its limit distribution, so the denomination of a as the mean reversion parameter is due to the fact that it describes the speed by which the short rate approaches its limit mean value.

Modelling the two curves required, the discount curve can be denoted as $D_d(T)$, while the other needed for the rate as $D_r(t)$. Given that, the short rate can be described as follows

$$dr_d(t) = [\theta_d(t) - ar_d(t)]dt + \sigma dw(t) \tag{5.3}$$

Assuming a zero-coupon bond with notional equal to 1 and maturity T, according to the Hull-White model, its price at time t will be equal to

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{5.4}$$

with

$$A(t,T) = \frac{P(0,T)}{P(0,t)}\exp[B(t,T)F(0,t) - \frac{\sigma^2}{4a}B(t,T)^2(1 - e^{-2at}) \tag{5.5}$$

and

$$B(t,T) = \frac{1}{a}[1 - e^{-a(T-t)}]. \tag{5.6}$$

The Hull-White one factor model has been also used in a dual-curve version and has been adapted to serve the market necessities arose in those past years. One of the main studies has been made by Henrard, who described a multiple-curve setting for the pricing of interest rate swaptions, based on an extension of the Hull-White one factor model with deterministic basis spreads.

5.3 MORENI & PALLAVICINI RESULTS

Many have been the contributors to the multiple curve literatures, but among them it is important to highlight Moreni and Pallavicini, which through an extension of the Heath-Jarrow-Morton framework, described in their paper of 2010 an interest rate model entirely derived from observed rates in which the dynamics of the yield-curve are based on limited number of Markov processes.

Let's, first of all, briefly introduce the H-J-M framework. As opposed to the previous interest rate models, where there was a unique explanatory variable, namely the short rate r, in this framework the entire yield curve is used for modelling.

The main assumption underlying the framework is that *the forward rate has a stochastic differential (SDE), which for every T>0 and under an objective measure P is equal to*

$$df(t,T) = a(t,T)dt + \sigma(t,T)d\bar{W}(t) \tag{5.7}$$

where $\overline{W}(t)$ is a standard Brownian motion or Wiener process, while $a(t,T)$ and $\sigma(t,T)$ are two adapted processes.

As a framework it does not provide any new model or formulas, but instead it sets up a reasoning to be followed in approaching interest rate models.

In order to better understand the basis of Moreni and Pallavicini studies, it would be useful to understand what a stochastic differential or SDE is.

The stochastic differential equation is a differential equation (ODE) which can be written in the scalar form as

$$\begin{cases} dX_t = aX_t d_t + \sigma dW_t \\ X_0 = x_0 \end{cases} \tag{5.8}$$

Treating the SDE as a normal ODE one could divide it by the factor d_t, so to arrive to a solution using ordinary calculus. After the division one would end up with

$$\frac{dX_t}{d_t} = aX_t + \sigma \frac{dW_t}{d_t} \tag{5.9}$$

so that the solution is

$$X_t = e^{at} X_0 + \sigma \int_0^t e^{a(t-s)} \frac{dW_s}{d_s} d_s \tag{5.10}$$

$$X_t = e^{at} X_0 + \sigma \int_0^t e^{a(t-s)} dW_s \tag{5.11}$$

More precisely SDE should be solved with Itô formulas, but in this case, we can approximate the result as above.

Moreni and Pallavicini's study is developed around a core problem, represented by the fact that forward rates of different tenors are no longer tied by strict constraints.

They enlarged the HJM framework to embody a multiple-curves in their setting; other economists reviewed the HJM framework, but they differed from the others in that they used only observed rates in their calculations so to avoid errors and too rough forecasting models, and in that, as said above, they used a unique family of Markov processes. Before going deep in the assumptions and development of the calculations, let's define what a Markov process is.

A Markov process is a process S, adapted to its filtration which has the characteristic of having its values in the past and in the future independent from each other. Mathematically it can be seen that for a given deterministic function $g = g(x)$ and for two given dates $s < t$, the conditional expectation of $g(S(t))$ given the sigma algebra F_t is equal to

$$E[g(S(t))|F_t] = E\left[g\left(S(t)\right)\middle|F_s\right] = \tilde{g}(S(s)) \tag{5.12}$$

which implies that to make the best bet on the expected future value of the process at t, it suffices to know the information contained in its present value at s, all the information from 0 to s are not required.

The assumptions required for the model, as stated by Moreni and Pallavicini in their paper (2009), are the following

I. the existence of a risk-free curve with forward rates $f_t(T)$;
II. the existence of Libor rates with corresponding forward rates $F_t(T, x)$;
III. no arbitrage dynamics of the two T-forward measure martingales $f_t(T)$ and $F_t(T, x)$, so that the limit case $f_t(T) = \lim_{x\to 0} F_t(T, x)$ holds;
IV. the possibility of writing both $f_t(T)$ and $F_t(T, x)$ as a function of a common family of Markov processes.

As opposed to the first two assumptions which depend on the financial quantities considered, the last two are simply imposed.

Extrapolating from market quotes $F_0(T, x)$ and $f_0(T)$, two dynamics are chosen under a T-forward measure, which are

$$df_t(T) = \sigma_t^*(T)dW_t \tag{5.13}$$

$$\frac{d(k(T,x)+F_t(T,x)}{k(T,x)+F_t(T,x)} = \Sigma_t^*(T, x)\, dW_t \tag{5.14}$$

with

$$\sigma_t(T) := \sigma_t(T; T, 0)$$

and

$$\sum_t(T, x) := \int_{T-x}^{T} du\sigma_t(u; T, x)$$

where $\sigma_t(u;T,x)$ is a row volatility vector process, $\sigma_t(T;T,0)$ is the volatility of the risk free instantaneous rate $f_t(T)$ and $k(T,x)$ is a set of shifts so that $\lim_{x\to 0} xk(T,x) = 1$ or ,equivalently, that when x approaches 0, $k(T,x)$ is equal to $\frac{1}{x}$.

Equation "5.13" shows a shifted forward LIBOR dynamics, which is essential in order for the third assumption to hold.

The main result of the paper concerns the dynamics of the shifted forward LIBOR under the risk neutral measure. The reasoning is that, by integrating the SDE over the time period [0,t], the dynamics of the shifted forward rate can be written as

$$\ln\left(\frac{k(T,x)+F_t(T,x)}{k(T,x)+F_0(T,x)}\right) = \int_0^T \Sigma_s^*(T,x)\left(dW_s - \frac{1}{2}\Sigma_s(T,x)\,ds + \int_S^T du\sigma_S(u;u,0)ds\right) \tag{5.15}$$

Since the requirement is to write it in terms of Markov process, an extension of the single-curve HJM model by Ritchken and Sankarasubramanian(1995) can be used by means of fixing of the following equality

$$\sigma_t(u;T,x) := h_t(q(u;T,x)g(t,u)) \tag{5.16}$$

In the above equation h_t defines as a matrix adapted process and q is a deterministic vector function, moreover the function g is described as follows

$$g(t,u) := exp\left\{-\int_t^u dv\lambda(v)\right\} \tag{5.17}$$

where λ is a deterministic array function.

Following the Ritchken and Sankarasubramanian model, it is required to also impose a condition on the deterministic vector function q, so it follows that for the HJM separability condition to hold in the case in which $x \to 0$, it is necessary that $q(u;u,0)$ is equal to 1 when $T = u$.

Considering that the dynamics for the forward LIBOR rates and for the instantaneous risk-free rates can be represented, under a risk neutral measure, as

$$\frac{d\big(k(T,x)+F_t(T,x)\big)}{k(T,x)+F_t(T,x)} = \Sigma_t^*(T,x)\left[\int_t^T du\sigma_t(u;u,0)dt + dW_t\right] \tag{5.18}$$

$$df_t(T) = \sigma_t^* \left[\int_t^T du\sigma_t(u; u, 0)dt + dW_t \right],$$

it is possible to plug the above equation in equation “5.15”, substituting in it also the expression for volatility, ending up with the following result

$$\ln\left(\frac{k(T,x)+F_t(T,x)}{k(T,x)+F_0(T,x)}\right) = G^*(t, T-x, T; T, x)\left(X_t + Y_t\left(G_0(t, t, T) - \frac{1}{2}G(t, T-x, T; T, x)\right)\right) \tag{5.19}$$

with $G_0(t, t, T)$ and $G(t, T-x, T; T, x)$ as vectorial deterministic functions.

It has to be noticed that both X_t and Y_t are two Markovian processes with dynamics correspondingly equal to

$$dX_t = (Y_t^* \cdot 1 - \lambda(t)X_t)dt + h_t^* \cdot dW_t \tag{5.20}$$

$$dY_t = \left(h_t^* \cdot ht - \left(\lambda^*(t)Y_t + Y_t\lambda(t)\right)\right). \tag{5.21}$$

5.4 MERCURIO AND THE MINIMAL BASIS VOLATILITY

As already said, the exponents of the multiple-curve phenomenon have been various. Fabio Mercurio has demonstrated since the beginning an active interest in disentangling the problem of pricing. He went through complex calculations and ended up with the idea that to solve the difficulties it could be possible to construct a model based on a minimal basis volatility.

This minimal basis volatility is related to the LIBOR-OIS spread which has showed an incoherent pattern from 2007 on. The idea develops on the relationship that exists between convexity adjustments and the correlation, which has shown a positive sign; for this reason, choosing a deterministic Libor-OIS basis could led to overestimate the correlation between the rates. Mercurio found out that by substituting the deterministic basis with another one, namely a minimal basis volatility it is possible to overcome the problem of misevaluation of the correlation between Libor and OIS rates, drawback also highlighted by Henrard in his paper (2014).

This allows to have a basis for which the correlation is lower than 0 and to have its volatility at the lowest possible level whatever the correlation between Libor and OIS rates is.

In the setup of the model he considered the OIS discounting, which will be covered in the next paragraph, and employed a multiple-curve model similar to that of Moreni and Pallavicini.

Mathematically, he just considered a different dynamic for the OIS rates movements and by fixing a correlation between the rates equal to ρ, he defined the unique OIS volatility as

$$h(t) = \frac{\rho\sigma_n(t)}{G_n}\frac{L_n(t)+a_n}{L_n+({}^{1}/_{\tau_n})} \tag{5.22}$$

where h(t) is an adapted process and $L_n(t)$ is a forward Libor dynamic.

5.5 OIS DISCOUNTING

One of the biggest issues in finance concerns the development of methods valuing financial instruments, either for the determination of fair market values of investments, either for other purposes related to financial management. The main technique still remains the discounted cash flows one, which has already been used in all the thesis; after the establishment of future cash flows value, the crucial point is determining the adequate rate for discounting. It has already been shown how in the period prior to the crisis, interbank lending rates, such as EURIBOR and LIBOR, were considered to be the risk-free rates due to the soundness and reliability of banks, bus as the financial collapse settled in, because of the increase of the risk of default of many banks, these rates were no more seen as before. For this purpose, the attention has been shifted to shorter terms rates, specifically to the overnight rates. A new curve has been consequently constructed based on overnight index swap rates such as EONIA swaps rates for the euro area. The discounting curve can be obtained by plugging the market quotes for OIS rates easily founded in the market to reconstruct the shortest maturities up to one year, as this instrument just provide a final exchange of interests, and by then focusing on OISs with greater maturities. Due to the nature of these instruments the OISs quotes in the market will not reflect the required rate for a cash flow, let' say, after three years, but would rather represent the average of previous intermediate payments. To overcome this lack of precision the bootstrapping technique is considered for the determination of the rest of the curve. The best way to do that is by considering the most liquid assets available in the market. For shorter maturities the ones who best fit for this purpose are Fed fund futures and OIS swaps, but the maximum length of time that they can cover is of 10 years; for this reason, Fed funds vs 3months LIBOR basis swaps are considered for maturities ranging up to 30 years.

Considering the nature of the derivative mentioned above, it is no longer sufficient to reconstruct a unique discount curve, as now the existence of the OIS curve and of LIBOR curve are both required to exist at the same time. The reason underlying this need is embodied in the fact that after the crisis LIBOR swaps have started to be valued considering OIS rates for discounting factors, this means that it is no longer possible to derive the forward LIBOR curve from LIBOR swaps directly, if one does not have previously derived the OIS curve.

As opposed to the pre-crisis scenario, in which the two curves could be constructed separately, following the single-curve methodology, now the necessity of having the two simultaneously leads to the development of a multiple-curve or dual-curve method.

Nowadays the time to maturities of swaps in Europe has been extended to greater maturities up to 30 years, allowing for the origination of the entire risk -free curve.

5.6 FORWARD CURVE IN THE MULTIPLE CURVE MODEL

The methodology for constructing discount and forward curves represent the main area of change during the years of the crisis. It has already been discussed how the use of OIS discounting has introduced multiple-curves models and the methodology concerning the forward curve development will now be presented.

Let's first of all remind that forward rates are considered to be the best proxies for the future cash flows coming from the floating leg (or position) of a derivative; for this reason, the construction of a forward curve is highly tied to the underlying rate of the derivative considered.

In the single-curve environment different instruments with different maturities were selected to build the entire curve, with a focus on the highly liquid assets available in the market, but now a new concept has to be introduced.

As opposed to the old framework, after the crisis a requirement for *homogeneity* has induced to select financial instruments with the same maturity as the one of the instruments to be priced in order to reconstruct the forward curve, so it is no longer adequate to combine different underlying interest rate tenors. This is due to the high market segmentation which has appeared during the crisis, where many instruments started to be classified in different segments defined by different maturities, causing so the pre-crisis no arbitrage conditions to be no longer effective.

The implications of the above reasoning are that in the case in which a portfolio is composed by derivatives having different tenors, one forward curve for each tenor needs to be considered; if one for example has a portfolio made of two instruments then two curves need to be there. This gives the name to the multiple-curve model, even if it may occur the case in which just two curves suffice.

Many approaches have been developed, as the one of Bianchetti and Ametrano (2009) which proposed the same technique used in the single-curve case with the difference that now the market quotes used to bootstrap the forward curve are consistent with the homogeneity requirement, and so all share the same underlying tenor.

Another methodology has been presented by Chibane and Sheldon (2009) which have suggested to use the OIS discounting technique also for the development of the forward curve.

After the crisis there has been a specific class of instrument which has catch the attention of investors, these instruments are the forward rate agreements (FRAs). The reason behind the continuously increasing fame of the FRAs is due to the fact, that they turned out to be the unique able to explain different tenors in the middle part of the forward curve, this is because they are quoted on the market for different maturities, but also for different starting dates, so they are able to cover a full period of time without any interruption.

Other instruments such as deposits, futures and most of all basis swaps are also really common.

5.7 PRICING INTEREST RATE DERIVATIVES IN THE MUTIPLE-CURVE MODELS

The previous paragraphs can be summarized by giving a unified methodology for pricing interest rate derivatives. Due to the increasing collateralization of assets in the period following the 2007-2009 years, it could be proper to differentiate the pricing of collateralized and of non-collateralized interest rate derivatives, although since the no-default value needs to be founded so that it would be sufficient to generalize to the full-collateralized case.

According to Hull and White (2013) it is possible to rewrite the value of a non-collateralized derivative as a function of a full collateralized one, by taking into account the risk of default coming from the two counterparties. Specifically, the value of a non-collateralized interest rate derivative can be founded by adding to the no-default value a credit and a debit valuation adjustment, namely CVA and DVA, so that it can be written as

$$V = V_{df} + CVA + DVA \,. \tag{5.23}$$

The general procedure to be followed when pricing a derivative under the multiple-curve model can be expressed in four easy steps that have already been explained in the previous paragraphs and are:

1. First of all, construct the OIS curve relating on OIS rates provided in the market, and by bootstrapping the missing ones for greater maturities;
2. Choose adequate derivatives with different tenors, which satisfy the requirement of homogeneity, to reconstruct the corresponding parts of the forward curve by bootstrapping;
3. Using the suitable forward rates generated by the forward curve, define the expected future cash flows that the derivate promises to deliver;
4. Create discount factors from the OIS curve to find out the present value of the future expected cash flows and sum them in order to find the present value of the derivative.

5.7.1 CONSTRUCTION OF THE OIS AND OF FORWARD CURVE IN THE MULTIPLE-CURVE MODEL AND PRICING OF A LIBOR SWAP CONTRACT

The theoretical tools learned so far will now be translated into practice, providing the set up and the development of all the 5 steps above in order to price an interest rate derivative.
First of all, I decided to start from a specific derivative of which the par rate will then be calculated later, so that all the material that will be developed in this paragraph will then be reused afterwards. The derivative I selected is a 2year Libor swap, which provides semi-annual payments, so that the short part of the OIS curve will suffice for the discounting issues related to the first year of our derivative.
As already said above for what concerns the OIS par rates up to 1 year, it is enough to take the market quotes and to directly use them as discount factors, since the OIS with maturities in between this tenor only provide a single payment. Having say that, it just requires to look for data and to use them. In order to price the full derivative , I will also bootstrap the OIS discount factors for greater maturities, up to 5 years and I will then interpolate them to have a continuous line in the graph.
The first step concerns the construction of the OIS discount factors; I will use here fictitious rates just to give an idea of the practice effectively used in the market.
Let’s suppose that the data relative to the OIS par rates observed in the market up to 5 years are as follows

MATURITY	OIS PAR RATES
1W	0,001640
2W	0,001620
3W	0,001690
1M	0,001640
2M	0,001670
3M	0,001630
4M	0,001640
5M	0,001610

6M	0,001630
7M	0,001640
8M	0,001580
9M	0,001610
10M	0,001560
11M	0,001600
1Y	0,001560
15M	0,001635
18M	0,001725
21M	0,001840
2Y	0,002070
3Y	0,003066
4Y	0,006041
5Y	0,008372

First of all it has to be pointed out that the structure of the payments of the OISs changes according to the tenor; we can observe how until 1 year the payments happen at the end of the contract, while from 1 to 2 years the payments appear quarterly (every 3 months) and how then on they are annual.

Having say that the rates until 1 year are to be considered as the discounting ones as, since here is supposed that the OISs are quoted at par, the value of the fixed and of the floating legs at inception coincide so that when we calculate the discount rate it turns out to be the pat rate.

The calculations for the discount rate between 1 and 2 years become a little more trivial as there is the need to use the bootstrapping method in order to find the missing ones.

Let's start by calculating the one for 15 months tenor.

We can imagine to have a 15 month OIS swap with a par rate of 0.001635, quoted at par with a face value equal to 1000. Since the two legs coincide, as said above, we can consider the fixed one as a fixed coupon bond, with quarterly payments and we can set its future cash flows as follows

$$1000 = \frac{1000(0.001635\frac{90}{360})}{1 + R_{0,3m}^{OIS}\frac{90}{360}} + \frac{1000(0.001635\frac{90}{360})}{1 + R_{0,6m}^{OIS}\frac{180}{360}} + \frac{1000(0.001635\frac{90}{360})}{1 + R_{0,9m}^{OIS}\frac{270}{360}} + \frac{1000(0.001635\frac{90}{360})}{1 + R_{0,1y}^{OIS}\frac{360}{360}} + \frac{1000 + 1000(0.001635\frac{90}{360})}{1 + R_{0,15m}^{OIS}\frac{450}{360}}$$

Since we already have the discount rate up to 1 year we can substitute them and find the 15 months discount rate by solving the above equation for it

$$1000 = \frac{1000(0.001635\frac{90}{360})}{1+0,001630\frac{90}{360}} + \frac{1000(0.001635\frac{90}{360})}{1+0.001630\frac{180}{360}} + \frac{1000(0.001635\frac{90}{360})}{1+0.001610\frac{270}{360}} + \frac{1000(0.001635\frac{90}{360})}{1+0.001560\frac{360}{360}} + \frac{1000+1000(0.001635\frac{90}{360})}{1+R_{0,15m}^{OIS}\frac{450}{360}}$$

$$1000 = 0.408583 + 0.40841714 + 0.408257 + 0.40811334 + \frac{1000.40875}{1+R_{0,15m}^{OIS}\frac{40}{360}}$$

$$1 + R_{0,15m}^{OIS}\frac{450}{360} = \frac{1000.40875}{1000-(0.408583+0.40841714+0.408257+0.40811334)}$$

$$R_{0,15m}^{OIS} = \frac{\left(\frac{1000.40875}{1000-(0.408583+0.40841714+0.408257+0.40811334)}-1\right)}{\frac{450}{360}} = 0,0016363$$

Now to find the discount factor it is sufficient to plug the above rate in the standard formula so that we find that the discount factor relative to maturity 15 months is

$$\frac{1}{1+0.0016363\frac{450}{360}} = 0.99796$$

The same can be done for the rest of the rates and will give us the result below

MATURITY	OIS DISCOUNT FACTORS
1W	0.999968
2W	0.999937
3W	0.999901
1M	0.999863
2M	0.999721
3M	0.999593
4M	0.999454
5M	0.999330
6M	0.999190
7M	0.999056
8M	0.998948
9M	0.998794
10M	0.998702
11M	0.998535

1Y	**0.99844**
15M	**0.997959**
18M	**0.997416**
21M	**0.996785**
2Y	**0.995868**
3Y	**0.989081**
4Y	**0.976081**
5Y	**0.958417**

The above calculations will not give us the final OIS discount curve, as if we plot data in a graph we will only get a series of discontinuous dots; to get the continuous line we have to use interpolation so that the curve will then be available for the pricing of any plain vanilla instrument.

Going further in the steps previously described, it is now required to have an adequate forward curve. Recalling the need for homogeneity, since the Libor swap par rate I want to find refers to a 2-year swap with tenor of 6 months, the forward curve must be a 6-months forward curve.

The instrument that better fits for this purpose is the FRA rate, so I can just use a sequence of FRA so that the whole period of 2 years is covered, hence the selected FRAs will be the following: LIBOR FRA 6×12, LIBOR FRA 12×18, LIBOR FRA 18×24.

For what concerns the time period from 0 to 6month a deposit on Libor can serve the purpose.

Let’s assume that the following rates are the one observed in the market and let’s start from the fact that the fixed and the floating leg payoff must coincide.

TIME PERIOD	**RATES**
0-6m	**0.004**
6m-12m	**0.0045**
12m-18m	**0.0051**
18m-24m	**0.0067**

The convention is to think of a portfolio made by a combination of long and short position on a fixed rate bond and on a floating rate bond so to replicate the cash flows promised by the LIBOR swap.

Assuming a notional of 1000, we can write the two legs as follows

$$\frac{1000F_{0,6m}\frac{180}{360}}{1+R_{0,6m}^{OIS}\frac{180}{360}}+\frac{1000F_{6m,12m}\frac{180}{360}}{1+R_{0,12m}^{OIS}\frac{360}{360}}+\frac{1000F_{12m,18m}\frac{180}{360}}{1+R_{0,18m}^{OIS}\frac{540}{360}}+\frac{1000+1000F_{18m,24m}\frac{180}{360}}{1+R_{0,24m}^{OIS}\frac{720}{360}}$$
$$=\frac{1000i_{SW}\frac{180}{360}}{1+R_{0,6m}^{OIS}\frac{180}{360}}+\frac{1000i_{SW}\frac{180}{360}}{1+R_{0,12m}^{OIS}\frac{360}{360}}+\frac{1000i_{SW}\frac{180}{360}}{1+R_{0,18m}^{OIS}\frac{540}{360}}+\frac{1000+1000i_{SW}\frac{180}{360}}{1+R_{0,24m}^{OIS}\frac{720}{360}}$$

where the first part refers to the floating leg and the second to the fixed one. The swap par rate is "i_{SW}"and can be found by solving the above equation for it

$$\left[1000(0.004)\frac{180}{360}\right](0.999190)+\left[1000(0.0045)\frac{180}{360}\right](0.99844)+\left[1000(0.0051)\frac{180}{360}\right](0.997416)+\left[1000+1000(0.0067)\frac{180}{360}\right](0.995868)\quad=\left(1000i_{SW}\frac{180}{360}\right)(0.999190)+\left(1000i_{SW}\frac{180}{360}\right)(0.99844)+\left(1000i_{SW}\frac{180}{360}\right)(0.997416)+\left(1000+1000i_{SW}\frac{180}{360}\right)(0.995848)$$

$$i_{swap}=\frac{(1.99838+2.24649+2.54341+999.20416)-1000(0.995848)}{\left(1000\frac{180}{360}\right)(0.999190)+\left(1000\frac{180}{360}\right)(0.99844)+\left(1000\frac{180}{360}\right)(0.997416)+\left(1000\frac{180}{360}\right)(0.995848)}$$

$$i_{swap}=0.00507924.$$

6 CONCLUSIONS

The aim of this thesis was to provide an overview of the features of the 2007-2009 financial turmoil and the consequences that it has brought in the financial markets. This has been done by presenting the main changes and abnormalities which have characterized the market in those years, and by introducing the tools and models which have concurrently appeared to overcome these singularities. A comparison with the single-curve, pre-crisis model has served the purpose to understand in an accurate way which specific variables has caused the related mutations and to better define the reasoning underlying the development of such new model, the *multiple-curve model*. It has been shown that one of the main peculiarities of the post-crisis environment has been an increasing LIBOR-OIS spread caused by the growing default risk associated with banks, which in those years were no longer able to promise risk-free investments, also due to their lack of liquidity; for the same reasoning more and more assets started to be collateralized in order to have a cushion in case of (extremely probable) unfulfillment in the repayment of debt payments. All this has shaped an environment in which the credit and liquidity risks ruled.

As it has been said along the thesis, the augmenting spread between LIBOR and OIS rates has forced investors to believe that the LIBOR rates could no longer represent the risk-free rates and as a consequence the construction of the risk-free curve should have been made with reference to other rates, the OIS rates. This has caused the first step towards the multiple-curve model, promoting a discount curve constructed upon OIS par rates to best fit the actual market conditions.

In addition to this, the market demonstrated a deep segmentation, as many new derivatives appeared to be grouped in several sub-segments classified by different tenors, this provoked a break in the no-arbitrage conditions that headed the market since then. As a consequence, it has been explained that practitioners discovered that it was no longer suitable to construct a single forward curve, and moreover to extrapolate it from the discount one; derivatives with the same tenors as the one of the product to be priced are required to construct a forward curve, and this specific curve can apply only to a specific tenor, so that for each tenor one needs to be generated.

Some of the main theories carried by the new market conditions have been presented, as the hull-white model and Moreni and Pallavicini results, but many others such as Mercurio, which implemented a model exploiting a minimal basis-volatility have contributed to the multiple-curve literature.

In the last part of the thesis I have done a practical representation of the technique for pricing a swap in order to conclude the theoretical part becoming more concrete.

In conclusion, although the various point of views and the various beliefs, all the exponents of the pricing theory share the common belief that to use different curves is still the best way to overcome the problems that arose in pricing derivatives, since it is the unique way to take into account and to respect the environmental conditions that governs the financial markets nowadays.

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