Department of Economics and Finance

A Game Theoretical Approach to Brain Networks

Chair of Games and Strategies

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Part I

Introduction

This thesis will analyze how the union of two different theories, Game theory and Graph theory, can be applied to the study of brain network centrality.

More precisely, this document will focus on how a brain network can be studied as a graph, made of nodes (n_i), being the neuronal structures of a brain network, and edges E, which connect one neuronal structure to another, and then, how concepts of Game theory can be applied to study the centrality of these neuronal structures.

To be more accurate, to be able to study brain network centrality with the use of Game theory, the application of one precise branch of this theory is needed: Cooperative Game theory.

As the document will explain more in depth in the appropriate section, Cooperative Game theory was formulated to be able to study how coalitions of players behave and how, what coalitions as a total can get, can be distributed and divided among the participants in the specific coalition.

Moreover, one further concept will be introduced during the discussion to be able to explain how solutions of cooperative games can be fairly distributed among the participants: the concept of power indices. Power indices, which have the function to both fairly divide the outcome of a coalition among involved players and to attribute relative power to the players of a cooperative game, in this thesis will have the aim to answer the question: which is the most central, influent and powerful neuronal structure in a brain network? Throughout the paper it will be clearly seen that power indices are the right tools to study the centrality in brain networks.

To conclude, the discussion of how different underlying cooperative games, which will be interpreted to be centrality measures in this paper (six different centrality measures will be explained in this thesis), can lead to similar, but even completely different power indices' results, will take place. The reason why this happens will be explained in details in the appropriate section, when the discussion of how different centrality measures, which are applied as cooperative games to graphs, consider "centrality" of a graph to be a different thing will occur.

This thesis will be structured as follow. Part II introduces notions about Game theory and more precisely introduces concepts of Cooperative Game theory and solutions to it, with a focus on power indices, needed to study brain network centrality. Part III presents basic knowledge about Graph theory and introduces the concept of directed graphs, type of graphs needed to represent schematically brain networks. Part IV introduces six different centrality measures, among which one has been developed for the first time in this paper, the *weighted degree centrality measure*. Only after the presentation of the six different measures, an application of all of them and a comparison will take place on a simplified brain network (a toy game). To conclude, in Part V an application of the experimental *weighted degree centrality measure* on the brain's Walker's area will follow. Lastly, the results found in the Walker's area study conducted in this document, will be compared with the results found in the paper "Shapley Ratings in Brain Networks" ([1]).

Part II

Game theory

1 Introduction

Game theory is the science of strategy, that can be defined as the optimal decision-making of intelligent and rational actors in conflicts and cooperative settings. The focus of game theory are games, which are models of interaction among agents. In strategic games, participants are faced with choices, however each choice is faced by constraint and the outcome will not only depend on one agent strategy, but will be determined jointly by the strategies chosen by all participants. Each outcome is a combination of strategy described in a solution, that differs for different classes of games. Above all, the science of game theory identify two main classes of games: non cooperative and cooperative games. In this work there will be a focus on an application of a particular type of cooperative games: transferable utility (TU) games.

2 Cooperative games

Game theory offers the definition of two types of cooperative, or coalitional games: the transferable utility games (TU), games on which this section will have a focus on, and nontransferable utility games (NTU). The definition "transferable utility" means that the outcome of the game for each participant can be expressed by numbers and that there is a medium of exchange between the players, meaning that utility can be shared among the members of the coalition.

The idea of transferable utility games is that, given a finite set N, players can form coalitions $A \subset N$ and collect a utility $v(A)$, that is how much each coalition can get as a total. Furthermore, the collective utility is *transferable*, meaning that $v(A)$ can be split among the players in A, in any way that sums to no more then $v(A)$.

Definition 2.1 *A cooperative transferable utility game (TU game) is a function*

$$v : 2^N \to \mathbb{R},$$

such that $v(\emptyset) = 0$. We denote the family of the coalition of N by 2^N, while v is called the characteristic function of the game.

In addition to cooperative games, which have the properties to explain how much each coalition as a total can get, it is also interesting to learn how the total utility $v(A)$ is divided among participants, since it is important to remember that, even though players cooperate, they are only self-interested: players only care about what they are able to make for theirselves and not about $v(A)$. However, only the next section will dedicate its analysis to power indices, like the *Shapley Value*, that have the capacity to answer the above question, while

currently the attention will still be on the meaning of TU cooperative games.
A famous example of TU cooperative games is the buyers and seller games analyzed below.

Example 2.1 *Let $N = 1, 2, 3$ be the set of players, which can be divided in two groups, $N = 1 \cup 2, 3$. Player 1 is the seller, while players 2 and 3 are the buyers for an important, indivisible good. The seller evaluates the good a, while the buyers (players 1 and 2) evaluate the good b and c, respectively ($a < b < c$). The value of each coalition is the total utility that the participants can obtain from entering in the coalition, which will depend on how much participants value the underlying good.*
Thus, the game will look like this:

$$v(1) = a$$

$$v(2) = v(3) = v(2, 3) = 0$$

$$v(1, 2) = b$$

$$v(1, 3) = c$$

$$v(N) = c$$

Each single buyer, and the coalition made up by the two buyers, always get 0, due to the fact that, without the seller, they do not get any good. On the other hand, coalitions made up by the seller and one buyer, can get and share a total utility equal to how much each buyer evaluates the good.
From this first example it is understood that each coalition game is described by a vector formed by a number of coalition equal to $2^n - 1$, if the null coalition $v(\emptyset)$ is omitted.
We shall denote by $G(N)$ the set of all games having N as set of players, a vector space of a dimension $2^n - 1$.

Not all cooperative games are interesting, in the sense that in some coalitions participants do not gain anything extra, so they would be better off alone.
Which games are interesting and which are not? At this point a clear definition of *additive* and *superadditive* games needs to be given, since such definition will also be fundamental at the moment in which the concept of *Shapley Value* will be analyzed.

Definition 2.2 *A game is an additive game when*

$$v(A \cup B) = v(A) + v(B)$$

if $\forall A, B \subset N,\ A \cap B = \emptyset$

From the definition it is clearly understandable that there is no extra gain from staying together. Players are indifferent whether to join or not any coalition.

Definition 2.3 *A game is a superadditive game when*

$$v(A \cup B) \geq v(A) + v(B)$$

if $\forall A, B \subset N,\ A \cap B = \emptyset$

From this type of game, the results will be efficient coalitions. Players have incentives to stay together.

3 Solutions to Cooperative Games

Cooperative games' solutions are not as easy as non-cooperative games' solutions, in which are exactly known players' strategies and so what they will do.
The solution of a cooperative game with n players is a vector of utilities $(x_1, x_2, ..., x_n)$ in which x_i represents the total utility (or cost in case v would represent a cost function) assigned to player i. Furthermore, how the total value of a coalition will be shared among the players taking part to it, will depend on which type of solution of cooperative games will be used. Not all solutions will give the same results.

There are two minimal requirements for TU games solutions:

1. Imputation set of the game v is the set $I(v)$ of all solution vectors x such that:
   - $x_i \geq v(\{i\})\ \forall i \in N$, meaning that a player, to have an incentive, needs to get a utility greater or equal to the utility that he/she would get by playing alone
   - $\sum_{i \in N} x_i = v(N)$, condition which states that the sum of all the players' utilities in a cooperative game is efficient (this condition represents the main difference between cooperative and non-cooperative games, as the prisoner dilemma shows)
2. The core of the game v is the set $C(v)$ containing all solution vectors x such that:
   - $\sum_{i \in A} x_i \geq v(A)$ for all coalitions $A \subset N$; explaining that the aim of the core is to make happy all possible coalitions, not just some
   - $\sum_{i \in N} x_i = v(N)$.

Thus an allocation in the core is such that no coalition (not only single players) can object to.

4 Power indices

Power indices are one of the many possible solutions to cooperative games. Power indices have the aim to measure the importance of a player in a coalition. It will be analyzed later in this document how these indices can be applied to rank the centrality of players in different scenarios. The focus of future sections will be on

the application of these values to network centrality.
Shapley Value and Banzhaf Value are the two most important power indices. The goal of these two values is to attribute a power measure to each i player, which depend on the contribution of player i to the game.

The Shapley Value The Shapley value is the most well-known and distinguished concept used to find a solution of transferable utility (TU) games. This solution concept was proposed by Shapley, which came up with this index while was trying to find a fair division of profits (or costs) gained cooperatively between players of a coalition game.
The Shapley value is defined in the following way (denoting it by σ):

$$\sigma_i(v) = \sum_{S:S\subset N\setminus\{i\}} \frac{s!(n-s-1)!}{n!}\left[v(S\cup\{i\}) - v(S)\right]$$

Consider the following properties:

1. *Efficiency*: for every $v \in G(v)$
$$\sum_{i\in N} \sigma_i(v) = v(N)$$

2. *Symmetry*: let $v \in G(v)$ be a game with the following property for players i, j: for every A not containing i, j, $v(S\cup\{i\}) = v(S\cup\{j\})$. then
$$\sigma_i(v) = \sigma_j(v)$$

3. *Dummy player*: let $v \in G(N)$ and $i \in N$ be such that $v(S) = v(S\cup\{i\})$ for all S. then
$$\sigma_i(v) = 0$$

4. *Additivity*: for every $v, w \in G(N)$
$$\sigma(v+w) = \sigma(v) + \sigma(w)$$

These properties are very important.
Thus the following theorem explains the importance of the Shapley value:

Theorem 4.1 *The Shapley value is the unique solution fulfilling the properties of efficiency, symmetry, dummy player and additivity.*

To clearly understand the Shapley Value formula and what it calculates, all parts of it need to be comprehended. The term $m_i(v,S) = v(S\cup\{i\} - v(S)$ is the marginal contribution of player i to the coalition $v(S\cup\{i\})$, thus, the Shapley value is the weighted sum of all marginal contributions of the player i, with the weight being equal to $\frac{s!(n-s-1)!}{n!}$. Let us now understand the meaning of the coefficient behind the marginal contribution. Suppose all players of a coalition game ($i = 1, 2, ..., N$) decide to meet at a certain time, equal for all players, in a room. Player i will enter in the coalition S if and only if he encounters only players in the coalition S at his arrival, we suppose also that all arrival's orders have the same probability. An easy calculation shows that the probability for i to find only S in the room is exactly:

$$\frac{s!(n-s-1)!}{n!}$$

Example 4.1 *There is a group of three flying companies in need of a new landing lane. If Company 1 would construct the landing lane alone, it would only need 1km, which cost c_1. Company 2 would need the first and the second km, available at a cost $c_2 > c_1$, while the last company, company 3, would need all the three km, at a cost equal to $c_3 > c_2$. Each possible coalition between these three companies will be attributed with the cost of $c(S) = max\{c_i : i \in S\}$.*

The costs shared by each coalition will take these vales:

$$c(1) = c_1$$

$$c(2) = c_2$$

$$c(3) = c_3$$

$$c(12) = c_2$$

$$c(13) = c(23) = c(123) = c_3$$

To efficiently share the total cost of the grand coalition $c(123)$ among the three companies, and thus find a solution of this cooperative game, the Shapley value can be used.
The Shapley value results are: $\sigma = (\frac{c_1}{3}, \frac{c_1}{3} + \frac{c_2-c_1}{2}, c_3 - \frac{c_1}{6} - \frac{c_2}{2})$.
Shapley value makes the cost of the first km, c_1, to be equally shared by all the three players. The marginal cost of the second km, $c_2 - c_1$, then is divided between players 2 and 3. What is left is all paid by the third player, the only player who uses the third km.

What lead the Shapley value to be considered as the main solution concept of cooperative game is connected to its *efficiency* property. However, solutions of cooperative games do not only have the aim of measuring the total utility that each player can get from a grand coalition or the cost that they will need to pay. There is a new class of Transferable Utility (TU) games for which the scope is to measure the importance and the power, thus the centrality, of nodes, representing the players of this new type of TU games, in a network. For this restricted type of TU games, the condition of efficiency is not necessary, since what really matters is the relative influence of a player in a network and not the total sum of players' power.
Given the increase in popularity of this new form of TU games and the need to find an alternative way to measure the power of nodes in networks, other indices of power have gained importance.

Definition 4.1 *A semivalue ϕ is defined in the following way:*

$$\phi_i(v) = \sum_{S:S\subset N\setminus\{i\}} p_s \left[v(S \cup \{i\}) - v(S)\right]$$

with conditions $p_s \geq 0$ *and*

$$\sum_{S:S\subset N\setminus\{i\}} \binom{n-1}{s} p_s = 1$$

The above conditions on p_s define a probability distribution on the coalitions not containing player i.

Semivalues are very important since they all fulfill the properties of the Shapley value, but the efficiency. Among these semivalues, the following one is widely used.

The Banzhaf Value The Banzhaf value, together with the Shapley value, is part of the class of semivalues. The Banzhaf value, before the application of network centrality to cooperative game theory, has never received as much attention as the Shapley value, due to the fact that this new value does not satisfy the *efficiency* condition, which, on the other hand, is satisfied by the previously analyzed semivalue. However, even though the *efficiency* property may appear necessary, it is not always essential.
As the Shapley value, the Banzhaf value bases its calculation on the marginal contribution that each player i bring to any coalition and it is defined as follow:

Definition 4.2 *The Banzhaf value,* β*, is a value on* $G(N)$ *defined for every* $i \in N$ *as follows:*

$$\beta_i(v) = \sum_{S\subseteq N-\{i\}} \frac{1}{p^{n-1}}[v(S \cup i) - v(S)]$$

with $p = 2$

As the Shapley value, the Banzhaf value is a weighted sum of all marginal contributions of player i, with the weight now being equal to $\frac{1}{p^{n-1}}$, which does not depend on the coalition S. The Banzhaf weight does not depend on S because Banzhaf believed that each player i is equally likely to join any coalition S.

In this section we have analyzed the basic concepts of game theory, the fundamentals behind cooperative game theory and how solutions to TU games, special type of coalition game, can be found. Due to the fact that finding a solution of a cooperative game is not as easy as studying solutions of non-cooperative game, in which all strategies of single players are known, more then one solution concept have been studied. In particular it has been a clear focus on the concept of power index and on its two main representatives: The Shapley value and the Banzhaf value.
Now another theory will be introduced, the Graph theory, since it will be needed to represent graphically brain networks.

Part III

Graph Theory

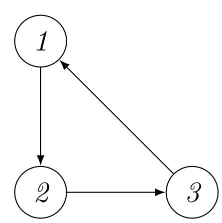
5 Introduction

The idea of graph was introduced for the first time by the mathematician L. Euler, which represented it as a set of vertices/nodes, connected by arcs, known as edges.

Definition 5.1 *Graph theory is the study of graphs, which are defined by two components:*

1. *Nodes (N), the intersection points of a graph;*
2. *Edges (E), connections between two nodes* n_i*,* n_j*.*

Example 5.1 *The graph* (N, E)*, where* $N = 3$ *is the set of nodes and* $E = 3$ *is the set of edges, can be represented as follow:*



Graph theory has many applications in real life. Examples of its application can be the use of graph theory by Google Maps, in which locations are nodes and all the roads and paths that connect these various locations are the edges. Further examples may be the application of this theory to social network, in which the nodes are the users and every connection between different users are all the edges, and to networks in general.

In the last years, graph theory, together with cooperative game theory, have earned popularity in the field of brain networks, to study the influence of neuronal structures.
We show now the basic of graph theory that will be used in the following application.

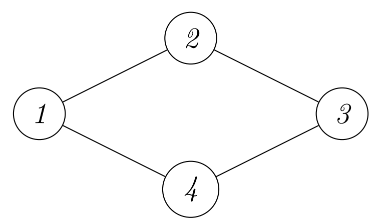
6 Types of graphs

There are a lot of different types of graphs, which can differ on number of edges, vertices and structure, but we can divide graphs mainly in two classes:

1. *Directed graphs*: graphs in which the directions of the edges are defined;

2. *Undirected graphs*: graphs in which the direction of the edges are not defined. Thus, if an edge exists between the nodes n_i and n_j, then there is a path from n_i to n_j and vice versa.

   Example 6.1 *(Undirected graph)*



However, since the focus, as stated above, will be on brain networks, which are *directed graphs*, no further characteristics of *undirected graphs* will be analyzed.

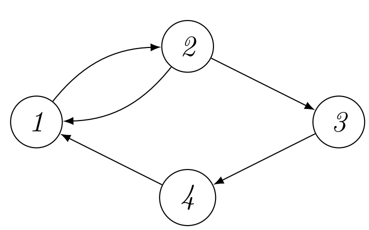
7 Directed Graphs

Directed graphs, also known as *digraphs* are formed by nodes connected by directed edges.

Definition 7.1 *A graph G is a pair (N.E), where N is a set of nodes, and E is a set of edges between the nodes* $E \subseteq \{(n_i, n_j)|n_i, n_j \in N\}$.

Principally, the main difference of *directed graphs* with *undirected graphs* is that edges E of *digraph* are ordered pairs: the edge from node n_i to node n_j is written as (n_i, n_j), while (n_j, n_i) is the opposite direction edge.

Example 7.1 *(directed graph)*



In this directed graph $E(1,2) \neq E(2,1)$.

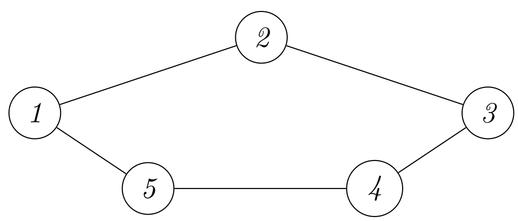
8 Matrix Representation of Graphs

In the next sections, in which an application of graph theory to brain network centrality will be presented, a matrix representation of graphs will be needed, in order to easily count how many edges E enter and exit in a node N.

Definition 8.1 *The adjacency matrix of the graph* (N, E) *is a nxn matrix* $D = (d_{ij})$*, where n is the number of nodes in the graph,* $N = \{n_1, ..., n_n\}$ *and* d_{ij} *= number of edges between* n_i *and* n_j*. If* $d_{ij} = 0$*, then there is no edge* (n_i, n_j) *in the graph.*

The *adjacency matrix* D of a graph (N, E) is symmetric, $D^T = D$ when the represented graph is *undirected*, while it is not symmetric, $D^T \neq D$, when the underlying graph is *directed*. This means that *adjacency matrices* of brain network graphs will not be symmetric.

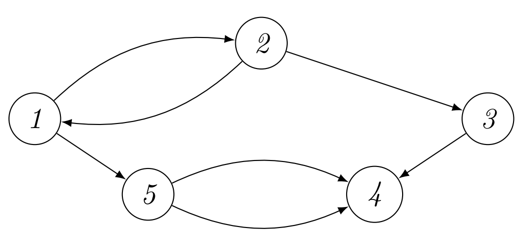
Example 8.1 *(Undirected graph)*



The adjacency matrix of such graph will look as follows:

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} = D^T$$

Example 8.2 *(Directed graph)*



The adjacency matrix of such graph will look as follows:

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \neq D^T$$

In this section we have analyzed the basic concepts of graph theory, with a focus on *directed graphs* and their representation in matrix form.
Now that all the useful theoretical concepts of game theory and graph theory, needed to study networks in general, have been explained, a practical application can be done.

To begin with, an application to a very simplified brain network, which we will call a toy game, will be done and only afterwords all these previously studied concepts will be tested on a real brain network.

Part IV

Network centrality of a simplified brain network

9 Introduction

A brain network is a *directed graph* (N, E) where N is the set of nodes, representing a set of neuronal structures, and E is the set of ordered edges, which represent connections between the nodes.
To study the influence and power of neuronal structures in a brain network, power indices, as the Shapley and Banzhaf values, are used. However, before being able to rate brain nodes, a clarification needs to be made: brain network centrality can be calculated taking in consideration different types of coalition games, known as centrality measures, which calculate the value of single nodes and of coalitions made of nodes under that and only that specific centrality measure.

10 Centrality measures

In literature a wide range of different centrality measures have been proposed, each one having a different point of view on how to calculate and evaluate the centrality of node in a network, leading then to situation in which the same node may have different influence under different types of games. Exactly for this reason, later on a toy game representing a very simplified version of a brain network will be presented, on which different centrality measures will be calculated and on which it will be tested how much different will be the leading results.
However, before being able to apply these different measures, the six different centrality measures, among which one, the *weighted degree centrality* measure, will be presented for the first time, applied as coalition games, need to be presented and explained.

Strongly connected component game ([2]) A *strongly connected component* (SCC) is a maximal induced subgraph which is strongly connected:

Definition 10.1 *A directed graph (N, E) is called strongly connected if for every two nodes n_i and n_j in N there is a directed path from n_i to n_j and from n_j to n_i.*

With the concept of *strongly connected component* a cooperative game, that works as a centrality measure, can be formulated.

Definition 10.2 *The coalitional game (N, w^E) corresponding to a brain network (N, E), can be defined as follow:*

$$w^E(S) = SCC(S, E[S])$$

for all $S \subseteq N$. In which the worth of the coalition is defined by the number of strongly connected components in the subgraph $(S, E(S))$.

In this game, the lower is the value of w^E, the stronger and the more central is the subgraph, thus, when calculating the power indices on this type of game, the lower will be the value obtained from a node, the higher will be the influence of it.

Ordered pairs game ([2]) With the concept of ordered pairs (n_i, n_j) a second coalitional game can be formulated.

Definition 10.3 *The brain network game* (N, v^E) *corresponding to a brain network* (N, E)*, can be defined as follows:*

$$v^E(S) = |E[S]|$$

for all $S \subseteq N$*. In which the worth of the coalition is defined by the number of ordered pairs* (n_i, n_j) *of edges in* S *for which there exist a directed path from* n_i *to* n_j *in* $(S, E[S])$*.*

This type of cooperative game, opposite to the *strongly connected component game*, is *superadditive*, meaning that it is more convenient for players, in this case for neuronal structures, to stay together, rather then to stay alone.
Furthermore, in this type of game, the higher is the value of v^E, the more influent is the subgraph, meaning that the higher is the value obtained from a power index calculation of a node, the stronger and powerful is that specific node.

Degree centrality game ([3]) The *degree centrality* classifies nodes and coalitions of nodes on the base of the direct connections they have to other nodes, outside the considered coalition, when the goal is to measure the *degree centrality* of a group.

Definition 10.4 *Degree centrality ranks nodes and group of nodes according to how many neighbours they have. Formally:*

$$degree(S) = |\{n_i : n_i \in E(S)\}|$$

with $E(S) = \bigcup_{i \in S} E(i) \backslash S$ *and* $E(i) = \{j : (i, j) \in E)\}$ *is the set of neighbours of a node* n_i*.*

In this type of centrality measure, which calculates the number of nodes with which a single node or a group of nodes have a direct connection with, the larger the neighbourhood of a node and of a group of nodes, the more central, thus powerful, it is. The direct consequence can be seen on the values obtained from the calculation of power indices: the higher is the value obtained for a node, the more fundamental it is.

Weighted Degree centrality game [1]

The *weighted degree centrality* measure is a modified version of the classic *degree centrality* measure, which takes into account only once the number of neighbours with which the coalition has a direct connection.

[1] Weighted version of the degree centrality measure, developed for the first time in this paper

Definition 10.5 *Weighted degree centrality ranks nodes and groups of nodes according to the number of edges E which begin from a node n_i inside the coalition S, but are directed toward a node n_j outside the coalition S. Formally:*

$$weighteddegree(S) = \sum_{l \in S} \sum_{i=1}^{n} a_{il} - \sum_{l_1 l_2 \in S} a_{l_1 l_2}$$

This modified version of the *degree centrality* measure, opposite to what the *degree centrality* does, does not only count the number of neighbours with which the node or the coalition of nodes has a direct connection, but counts all the edges that start inside the coalition and are directed towards nodes outside of it. Exactly for this reason the name *weighted degree* centrality has been attributed to this measure.

Betweenness centrality game ([3]) The *betweenness centrality* calculates the shortest path between any two nodes and between any group of nodes and another nodes in a network.

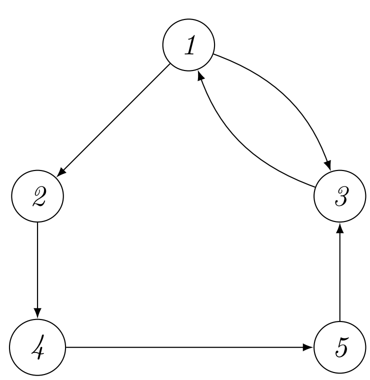
Definition 10.6 *Betweenness centrality ranks nodes and group of nodes according to how many shortest paths in the network they lie on. Formally:*

$$betweenness(S) = \sum_{n_i, n_j \notin S} \frac{\sigma_{(n_i,n_j)}(S)}{\sigma_{(n_i,n_j)}}$$

with $\sigma_{(n_i,n_j)}(S)$ be the number of shortest paths between n_i ($\notin S$) and n_j ($\notin S$) that visit at least one node in S, and $\sigma_{(n_i,n_j)} = |path(n_i, n_j)|$.

As the intuition would suggest, the higher the number of shortest paths a node and a group of nodes lies on, the more central it is.
Since the *betweenness centrality* measure is not as intuitive as the other centrality measures, an example of how this measure is calculated is provides. The other centrality measures' calculations will be clarified while solving the toy example.

Example 10.1 *(Betweenness centrality)*



S	*betweenness(S)*	*S*	*betweenness(S)*
{1}	*6*	*{2}*	*5*
{3}	*6*	*{4}*	*5*
{5}	*5*	*{12}*	*3*
{13}	*3*	*{14}*	*5*
{15}	*5*	*{23}*	*5*
{24}	*2*	*{25}*	*4*
{34}	*5*	*{35}*	*3*
{45}	*2*	*{123}*	*1*
{124}	*1*	*{125}*	*2*
{134}	*2*	*{135}*	*1*
{145}	*2*	*{234}*	*2*
{235}	*2*	*{245}*	*1*
{345}	*1*	*{1234}*	*0*
{1235}	*0*	*{1245}*	*0*
{1345}	*0*	*{2345}*	*0*
{12345}	*0*		

To understand how the values attributed to each coalition S *were found, an explanation of how the result of one coalition* S, $S = \{4, 5\}$, *can be given:*
betweenness$(\{4, 5\}) = 2$ *because nodes* n_1 *and* n_3 *do not need to pass through the coalition* $\{4, 5\}$ *to reach respectively* n_2 *and* n_3, *and* n_1 *and* n_3. *Only* n_2 *needs to pass through the coalition* S *to reach both* n_1 *and* n_3.

Closeness centrality game ([3]) The *closeness centrality* calculates the distance between nodes and between coalitions of nodes and nodes.

Definition 10.7 *Closeness centrality ranks nodes and group of nodes based on their distances to other nodes. Formally:*

$$closeness(S) = \sum_{n_i \in N} dist(S, n_i)$$

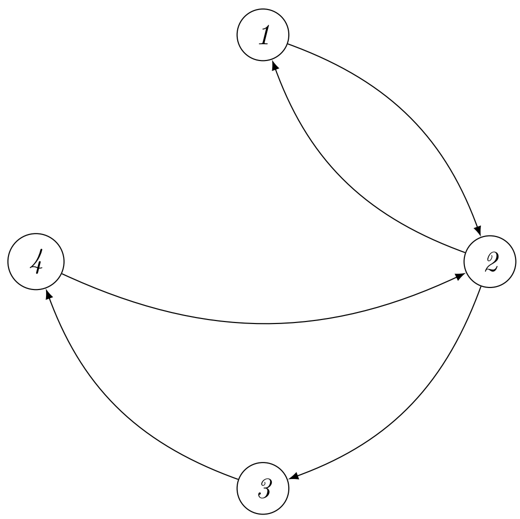
where $\sum_{n_i \in N} dist(S, n_i)$ *is the sum of the distances from* S *to any node outside of* S.

This centrality measure results in a ranking in which nodes and groups of nodes with lower values are on average closer to nodes outside the group, thus the lower is the *closeness centrality* measure, the more central and influential is the node or the coalition.

11 Application of centrality measures to the simplified brain network

Now that the basic concepts of game theory and graph theory, including centrality measures, have been presented, these concepts can been applied to a simplified brain network, to a toy game taken from the article "On Shapley Ratings in Brain Networks" ([2]), and can be used to study how different centrality measures, so how different cooperative games applied to this network, can lead to different results of nodes' influence.

Example 11.1 *Consider the simplified brain network, represented as a directed graph,* (N, E) *with* $N = \{1, 2, 3, 4\}$*:*



with the corresponding adjacency matrix:

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

To this simplified brain network, the six centrality measures presented above, with a corresponding calculation of the Shapley and Banzhaf values for each of the six cooperative games, can be measured and presented.

Strongly connected component game *The worths of every coalitions under the SCC game are presented below:*

S	$w^E(S) = SCC(S, E[S])$
{1}	1
{2}	1
{3}	1
{4}	1
{12}	1
{13}	2
{14}	2
{23}	2
{24}	2
{34}	2
{123}	2
{124}	2
{134}	3
{234}	1
{1234}	1

To understand how these calculations have been made, results of two different coalitions can be explained:

- $w^E(14) = SCC(\{1,4\}, E[\{1,4\}]) = 2$ *because the subgraph induced by* $\{1,4\}$ *consists of two strongly connected components: the subgraph induced by* $\{1\}$*and* $\{4\}$*;*
- $w^E(124) = SCC(\{1,2,4\}, E[\{1,2,4\}] = 2$ *because the subgraph induced by* $\{1,2,4\}$ *consists of two strongly connected components. the subgraph induced by* $\{1,2\}$ *and* $\{4\}$.

In this SCC game player 3 and player 4 are symmetric, meaning that they bring the same contribution to any coalition S, thus the Shapley value and the Banzhaf value will attribute them the same value.
Now that the SCC game have been solved and it has been recognized the symmetry of two players, the two power indices can be calculated.

1. *Shapley value results:*
$$\sigma_i = (\frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, \frac{1}{3})$$
2. *Banzhaf value results:*
$$\beta_i = (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})$$

Since under the SCC game the lower is the value of the coalition, the higher is the power and influence of it, then the lower are the values attributed to nodes by the Shapley and the Banzhaf values, the higher will be the strength of that specific node. This means that this type of game will determine a ranking:

1. *Shapley value ranking:*

$$(2, 3, 4, 1)$$

   with a tie between players 3 and 4;

2. *Banzhaf value ranking:*

$$(2, 1, 3, 4)$$

   with a tie between players 1, 3 and 4.

Ordered pairs game *The worths of every coalitions under the ordered pairs game are presented below:*

S	$v^E(S) = \|E[S]\|$
{1}	0
{2}	0
{3}	0
{4}	0
{12}	2
{13}	0
{14}	0
{23}	1
{24}	1
{34}	1
{123}	4
{124}	4
{134}	1
{234}	6
{1234}	12

To understand how these calculations have been made, results of two different coalitions can be explained:

- $v^E(124) = |E[\{1,2,4\}]| = 4$ *because* $E[\{1,2,4\}] = \{(1,2),(2,1),(4,2),(4,1)\}$*;*
- $v^E(234) = |E[\{2,3,4\}] = 6$ *because* $E[\{2,3,4\}] = \{(2,3),(2,4),(3,2),(3,4),(4,2),(4,3)\}$.

In this cooperative game player 3 and player 4 are symmetric, thus will be attributed with the same values by both the Shapley value and the Banzhaf value.
Knowing the value of each coalition under this specific coalition game, the two power indices can be calculated.

1. *Shapley value results:*

$$\sigma_i = (2\frac{1}{6}, 4\frac{1}{6}, 2\frac{5}{6}, 2\frac{5}{6})$$

2. Banzhaf value results:

$$\beta_i = (\frac{7}{4}, \frac{7}{2}, \frac{9}{4}, \frac{9}{4})$$

Opposite to the SCC game, under the ordered pairs game the higher is the value attributed to the coalition, the stronger that coalition is. This means that the higher is the value attributed to a node by the Shapley and Banzhaf values, the more central will be that node. Knowing this, a ranking of nodes under this specific game can be determined:

1. *Shapley value ranking:*

$$(2, 3, 4, 1)$$

   with a tie between players 3 and 4;

2. *Banzhaf value ranking:*

$$(2, 3, 4, 1)$$

   with a tie between players 3 and 4.

Weighted Degree Centrality game *The worths of every coalitions under the weighted degree centrality game are presented below:*

S	$weighteddegree(S) = \sum_{l \in S} \sum_{i=1}^{n} a_{il} - \sum_{l_1 l_2 \in S} a_{l_1 l_2}$
{1}	1
{2}	2
{3}	1
{4}	1
{12}	1
{13}	2
{14}	2
{23}	2
{24}	2
{34}	1
{123}	1
{124}	1
{134}	2
{234}	1
{1234}	0

To understand how these calculations have been made, results of two different coalitions can be explained:

- *weighteddegree*$(14) = 2$ *because the coalition* $\{1, 4\}$ *has two edges,* (n_1, n_2) *and* (n_4, n_2)*, directed toward nodes outside the coalition. Even though both the edges are directed toward* n_2*, the result attributed to this coalition is still* 2 *since the weighted degree centrality counts the number of outstanding edges and not the number of neighbours;*

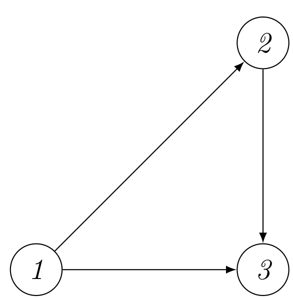
- *weighteddegree*$(134) = 2$ *because the coalition* $\{1, 3, 4\}$ *has two edges,* (n_1, n_2) *and* (n_4, n_2)*, directed toward nodes outside the coalition. The reason why the result is 2 and not 1, even though both the outstanding edges are directed toward 2, is the same as the point before.*

The value of the grand coalition in this game is 0, thus the outcomes of the Shapley and Banzhaf values could be two: some players getting a positive impact and other having a negative result or, the second possibility could be all players getting a 0 value. By calculating the two values, the results will look as follow:

1. *Shapley value results:*
$$\sigma_i = (0, 0, 0, 0)$$

2. *Banzhaf value results:*
$$\beta_i = (0, 0, 0, 0)$$

Under this game, the larger is the number of edges which starts inside the coalition S and end outside of it, hence the larger is the neighborhood of the node and of the coalition of nodes, the more influent and central is the node and the coalition. This means that, with this coalition game, the higher is the Shapley and Banzhaf values, the more powerful is that node. However, in this simplified brain network, the two power indices attribute to all nodes a null value. In my opinion, two interpretations are possible for these results. On one hand, the first possibility could just be that, under this specific brain network, when it is the weighted degree centrality the measure applied, all nodes have the same power. On the other hand, it could even be that the weighted degree centrality measure is meaningless when applied to networks.
To prove that it is not the second option the case verified, a counter example is provided:



If it is the weighted degree the centrality measure applied to this network, the results obtained for each coalition are the following:

S	$weighteddegree(S) = \sum_{l \in S} \sum_{i=1}^{n} a_{il} - \sum_{l_1 l_2 \in S} a_{l_1 l_2}$
{1}	2
{2}	1
{3}	0
{12}	2
{13}	1
{23}	0
{123}	0

When the Shapley value is calculated on the three nodes of the analyzed network, these are the results:

$$\sigma_i = (1, 0, -1)$$

From the above results, the hypothesis that the weighted degree centrality measure is meaningless when applied to networks can be excluded. This means that it can be supposed that the Shapley value and Banzhaf value results obtained for the toy example, all the 4 players attributed with a null value, just mean that all the four nodes have the same power.

Degree centrality game *The worths of every coalitions under the degree centrality game are presented below:*

S	$degree(S) = \lvert\{n_i : n_i \in E(S)\}\rvert$
{1}	1
{2}	2
{3}	1
{4}	1
{12}	1
{13}	2
{14}	1
{23}	2
{24}	2
{34}	1
{123}	1
{124}	1
{134}	1
{234}	1
{1234}	0

To understand how these calculations have been made, results of two different coalitions can be explained:

- *degree*$(14) = 1$ *because both the edges which start inside the coalition* $\{1, 4\}$, (n_1, n_2) *and* (n_4, n_2), *and are directed toward nodes outside the coalition, are directed toward node* n_2. *Hence, when it is the degree centrality the measure applied, we only need to consider the neighbour of the coalition once;*

- *degree*$(123) = 1$ *because, as the point above, both the edges which start inside the coalition* $\{1, 2, 3\}$, (n_1, n_2) *and* (n_4, n_2), *are directed toward node* n_2. *So, since the degree centrality counts the number of neighbours of a coalition, and not the number of outstanding edges, the neighbour* n_2 *needs to be count only once.*

In the degree centrality game none of the players are symmetric. This means that all nodes, under both the Shapley and Banzhaf values, could be attributed with different values.
However, by calculating the two power indices for this game, it is noticed that nodes 1 and 4 are attributed with the same value by both the indices, even though not symmetric. Below the results of the two power indices are provided:

1. *Shapley value results:*
$$\sigma_i = (-\frac{1}{6}, \frac{1}{3}, 0, -\frac{1}{6})$$

2. *Banzhaf value results:*
$$\beta_i = (-\frac{1}{4}, \frac{1}{4}, 0, -\frac{1}{4})$$

As the weighted degree centrality game, which, as specified above, is just a slightly modified version (the weighted version) of the degree centrality game, the larger is the neighborhood of each node, the more powerful is that specific node. So, as for the Shapley and Banzhaf values' calculations for the weighted degree centrality game, the higher is the value attributed by these two power indices to each node, the more central will be. Now a ranking of the four nodes in this game can be presented:

1. *Shapley value ranking:*
$$(2, 3, 1, 4)$$
with a tie between players 1 and 4;

2. *Banzhaf value ranking:*
$$(2, 3, 1, 4)$$
with a tie between players 1 and 4.

Betweenness centrality game *The worths of every coalitions under the betweenness centrality game are presented below:*

S	$betweenness(S) = \sum_{n_i n_j \notin S} \frac{\sigma_{(n_i,n_j)}(S)}{\sigma_{(n_i,n_j)}}$
{1}	0
{2}	5
{3}	2
{4}	2
{12}	1
{13}	1
{14}	1
{23}	2
{24}	2
{34}	0
{123}	0
{124}	0
{134}	0
{234}	0
{1234}	0

To understand how these calculations have been made, results of two different coalitions can be explained:

- *$betweenness(2) = 5$ because n_1 needs to pass through n_2 to reach both n_3 and n_4, n_3 needs to pass through n_2 to reach n_1 and n_4 needs to go through n_2 to reach both n_1 and n_3;*
- *$betweenness(34) = 0$ because both n_1 and n_2 can reach the other without passing through the coalition $\{3, 4\}$.*

In this network, when the betweenness centrality game is applied, players 3 and 4 are again symmetric. So, the Shapley and the Banzhaf values will lead again to results which attribute to these two players the same values.

1. *Shapley value results:*
$$\sigma_i = (-\frac{5}{6}, \frac{7}{6}, -\frac{1}{6}, -\frac{1}{6})$$
2. *Banzhaf value results:*
$$\beta_i = (-\frac{5}{4}, \frac{1}{2}, -\frac{3}{4}, -\frac{3}{4})$$

Again in this type of game the higher is the value attributed to a node and to a coalition, the more influent will be that node and that coalition in the network. This type of game will then lead more central and influential nodes to have higher values of Shapley and Banzhaf. Once this concept is clear, a ranking of the nodes present in the network can be made:

1. *Shapley value ranking:*

$$(2, 3, 4, 1)$$

with a tie between players 3 and 4;

2. *Banzhaf value ranking:*

$$(2, 3, 4, 1)$$

with a tie between players 3 and 4.

Closeness centrality game *The worths of every coalitions under the closeness centrality game are presented below:*

S	$closenss(S) = \sum_{n_i \in N} dist(S, n_i)$
{1}	6
{2}	4
{3}	6
{4}	5
{12}	3
{13}	2
{14}	3
{23}	2
{24}	2
{34}	3
{123}	1
{124}	1
{134}	1
{234}	1
{1234}	0

To understand how these calculations have been made, results of two different coalitions can be explained:

- $closeness(2) = 4$ *because the coalition* $\{2\}$ *to reach* n_1 *needs to cross the edge* (n_2, n_1)*, to reach* n_3 *needs to cross the edge* (n_2, n_3) *and to reach* n_4 *needs to cross both the edges* (n_2, n_3) *and* (n_3, n_4)*;*

- $closeness(34) = 3$ *because the coalition* $\{3, 4\}$ *to reach* n_1 *needs to cross the edges* (n_4, n_2) *and* (n_2, n_1) *and to reach* n_2 *needs only to cross the edge* (n_4, n_2)*.*

In this specific game, the two players that in the previous cooperative games were symmetric, so players 3 and 4, are no more symmetric. This means that the Shapley and the Banzhaf values for these two players under this specific game, no more need to be the same.

1. *Shapley value results:*

$$\sigma_i = (\frac{1}{3}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{6})$$

2. *Banzhaf value results:*

$$\beta_i = (-\frac{3}{4}, -\frac{5}{4}, -1, -1)$$

Under the closeness centrality game, the lower is the value attributed to a node and to a coalition of nodes, the closer is the node and the coalition to the other nodes in the network. This means that the smaller is the value attributed by Shapley and Banzhaf to a node, the higher will be the centrality of that node.

1. *Shapley value ranking:*

$$(2, 4, 3, 1)$$

with no node having a tie;

2. *Banzhaf value ranking:*

$$(2, 3, 4, 1)$$

with a tie between players 3 and 4.

As expected, different centrality measures calculate nodes' influence in different ways, leading then power indices to rank these nodes, which represent neuronal structures, in different ways.

Furthermore, not only the Shapley value was used to measure the respective power of each node under each different centrality measure, but also the Banzhaf was applied. The decision to use even this second power index follows what was explained above about the main difference between these two indices of power: even thought the Banzhaf value does not respect the *efficiency* property, this condition is not necessary when the scope of the game is to measure the importance and the influence of nodes in a network, and not to efficiently distribute the utility or divide the cost between the players in the grand coalition.

As it can be seen from the exercise, the Shapley value and the Banzhaf value, even thought it is not efficient, lead to equal, and only in few cases similar, but not identical, ranking of nodes in this simplified brain network, under the same centrality measure game. More precisely, in the *ordered pairs game*, *degree centrality game* and *betweenness centrality game* the two power indices lead to the same ranking, meaning that, under these three specific cooperative games, the four nodes analyzed in this example have the same influence for both power indices. Even the *weighted degree centrality* game lead the four nodes to have the same ranking under both power indices. Even though it could be argued that the Shapley and Banzhaf results obtained when it is the *weighted degree centrality* the measure applied are a bit ambiguous, the counter example provided above proved that the results obtained are not the outcome of a meaningless centrality measure, but just the fact that all the four nodes can be interpreted to have the same power when it is the *weighted degree centrality* the measure applied. On the other hand, the two power indices do not lead to the same ranking in the *SCC game* and the *closeness centrality game*. Going a bit more in detail, it can be seen that the Shapley value attributes a tie only to node 3 and node 4 in the *SCC game*, while the Banzhaf value attributes a tie to all nodes, except node

2. While, in the *closeness centrality game*, the Shapley value calculates that node 4 is stronger then node 3 (no tie between the two players), whereas the Banzhaf value attributes again the same influence rank to node 3 and 4.

However, what really matters in this exercise is not the comparison between the Shapley result and the Banzhaf result within a single centrality measure game, but the comparison of nodes' ranking between different centrality measure games.

When we compare all the six cooperative games at the same time we notice that, except for the *weighted degree centrality* game which classifies all the four nodes to have the same power under both power indices, it can be noticed that all games, and both power indices, evaluate node 2 the neuronal structure with the highest influence and power in the simplified brain network. Why is this result obtained? First of all, the first reason why we have obtained this result could be because if node 2 would be removed from the simplified brain network, the directed graph representing this network would become unconnected, meaning that it would become impossible to go from node 1 to nodes 3 and 4 and vice versa. Furthermore, this result obtained could be explained by the fact that node 2 is the only node from which it starts two edges, (n_2, n_1) and (n_2, n_3), and from which it ends other two edges, (n_1, n_2) and (n_4, n_2).

However, how much more influent and central is node 2 compared to the node with the lowest ranking, thus node 1 for almost all centrality measure games, cannot be said for games which allow the Shapley and Banzhaf values to be negative, therefore it cannot be said for the *SCC game*, the *betweenness centrality game*, the *closeness centrality game* and the *degree centrality game*. This means that how much more central is node 2 compared to the lowest central node can only be said for the *ordered pairs game* in which node 2 is almost two times more central and influent compared to the node ranked at the fourth place, node 1, under both power indices.

In this part an application of centrality measures and power indices to a very simple brain network was conducted. While solving this simplified exercise, as it was a toy game, it was understood that different centrality measures can lead to different results due to the fact that every measure calculates centrality and attribute power to nodes following different schemes and ideas. These differences on how to calculate node centrality led to different node centrality ranking. However, it was discovered that application of different power indices on the same centrality measure game almost always lead to similar, if not equal, results of centrality.

Now that it is clear how node centrality and influence can be studied, applying concepts of graph theory and cooperative game theory, an application of this study can be done to a real brain network: a variation of the Walker's area, the W40-2 network.

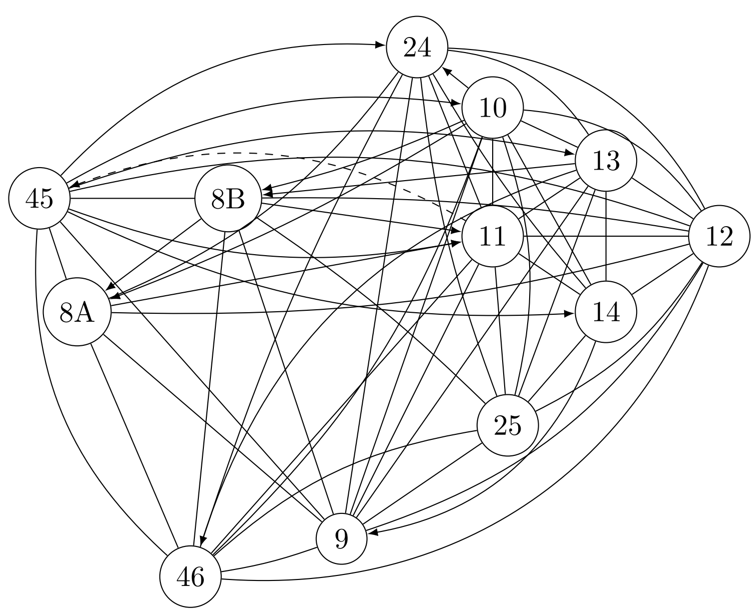
Part V

Walker's area network centrality

12 Introduction

A network centrality study, with an application of the *weighted degree centrality measure*, centrality measure formulated and defined above as a weighted version of the *degree centrality measure*, on the Walker's area, will be conducted in this section.

The Walker's area is a brain network which connects regions of macaque prefrontal cortex. Three variations of this area, the W40 network, had been hand-selected from anatomical tracing studies represented in the CoCoMac database (Kotter, 2004; Stephen et al., 2001). Between these three variants, only one will be analyzed and compared with the results obtained by the authors of the paper "Shapley ratings in brain networks" ([1]): the W40-2 variant. This variant is the most recent one and is the outcome of the application of newest evolution of tracing data. This variant has been formulated following the most up to date mapping procedures. Due to the fact that new and more advanced procedures have been used to map this brain network, new direct connections between neuronal structures within the walker's network have been found and thus included in the analysis. However, only the presence of one edge, at the moment in which the new version of the Walker's area was derived, was unknown. To understand which of the edges in the W40-2 is still uncertain, a representation of the network is provided below[2].



As it can be immediately deducted from the graphical representation of the W40-2 version of the Walker's area, only the presence of one edge, the edge (n_{11}, n_{45}), is uncertain.

[2]undirected edges in the graph mean that the edge goes in both directions

Due to the presence of this uncertain edge, two different analysis will be conducted on this brain network:

1. Network centrality analysis when the edge (n_{11}, n_{45}) is present;
2. Network centrality analysis when the edge (n_{11}, n_{45}) is not present.

To conclude, two different comparisons will be made:

1. A comparison between the two results obtained from the two different analysis;
2. A comparison between the results that will be obtained in this paper and the results obtained by the paper "Shapley ratings in brain networks" ([1]).

13 Weighted Degree centrality measure applied to the W40-2 brain network

Before being able to apply the *weighted degree centrality* measure to the analyzed brain network, a matrix representation of it needs to be made, due to the fact that the chosen centrality measure (the *weighted degree centrality* measure) will be calculated exploiting the matrix representation of the underlying directed graph of the Walker's area network.
Given that two analysis will be conducted (when the edge (n_{11}, n_{45}) is present and when it is not), two matrix representations will be needed. However, the rest of the analysis will be conducted in the same way for both cases:

1. First of all the *weighted degree centrality* measure will be calculated for all the possible coalitions obtained with the 12 neuronal structures present in the W40-2 network;
2. Only after the centrality measure will be calculated for all the $2^{12} - 1$ coalitions, the chosen power index, the Shapley value, will be applied in order to calculate which neuronal structure can be considered to have an higher influence under this centrality measure.

(It has been decided to use only one power index, the Shapley value, since from the previous part of the paper it has been obtained the result that the Shapley value and the Banzhaf value lead to similar, if not equal, results).

Now that it has been explained what will be done next, the only thing that is left is to understand how the chosen centrality measure and the corresponding Shapley value can be calculated on a brain network that involves 12 neuronal structures, thus 12 players.
However, this is not a problem since a method to conduct these two analysis will be explained in the next two paragraphs.

W40-2 network with the (n_{11}, n_{45}) edge First of all, the analysis will be conducted on the case in which the edge (n_{11}, n_{45}) is present. However, before being able to explain how this analysis has been conducted, a matrix representation of the the brain network need to be made:

$$D = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

with:

$$(pl_1, pl_2, pl_3, pl_4, pl_5, pl_6, pl_7, pl_8, pl_9 pl_{10}, pl_{11}, pl_{12}) = (n_{45}, n_{24}, n_{8A}, n_{8B}, n_{10}, n_{13}, n_{12}, n_{11}, n_{14}, n_{25}, n_9, n_{46})$$

It can be understood by looking at the matrix that the uncertain edge is present in this analysis since: $(n_{11}, n_{45}) = 1$ (1 means that a directed edge between the two nodes exist, while 0 means that no edge is present between the two nodes).

Now that a matrix representation has been provided, the centrality analysis can begin.
In order to be able to calculate the *weighted degree centrality* measure for each coalition and then calculate the Shapley value for all the 12 neuronal structures, two programming softwares have been used: Julia and R.
The software Julia will be used to calculate the *weighted degree centrality* for each of the $2^{12} - 1$ coalitions. On the other hand, R will then calculate the Shapley values using the *weighted degree centrality* measures exported from Julia.
Given for what each programming software will be used, the steps needed to reach the final scope of this analysis (i.e. study the influence of each node of the Walker's area under the selected centrality measure), can be analyzed and explained. The steps conducted on Julia and R are the following:

1. First of all, on the software Julia, the *weighted degree centrality* is calculated for all the possible coalitions between the 12 neuronal structures in the W40-2 network. In order to do so, few passages need to be conducted on the chosen program. The needed steps are the following:

- Write the 12x12 matrix as follows

$$D = [0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1; ...; 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0]$$

- Perform the following operation

$$SumRows = Sum(A, dims = 2)$$

- Tell Julia to use a specific package to conduct the future operations

$$Using\ Combinatorics$$

- Define the number of players in the game

$$nplayers = 12$$

$$players = 1 : nplayers$$

- Define all the possible coalitions in the game with the function *powerset*

$$coalitions = collect(powerset(players))$$

(which in total need to be $2^{12} = 4096$ if the empty coalition is included as well)

- To check if all the 4096 coalitions have been listed by Julia, another function is used

$$v = zeros(size(coalitions))$$

(if Julia will list 4096 zeros it will mean that all the possible coalitions have been taken into account)

- At this point, a function able to calculate for all the possible coalitions the *weighted degree centrality* using the adjacency matrix D defined in the first step, need to be programmed. To do so, it is possible to start from the *weighted degree centrality* measure, which, as specified above, can be written as follow

$$weighteddegree(S) = \sum_{l \in S} \sum_{i=1}^{n} a_{il} - \sum_{l_1 l_2 \in S} a_{l_1 l_2}$$

Once it is understood that the *weighted degree centrality* measure can be written as follow when the adjacency matrix of the brain network is given, a program to calculate it for all the 4096 coalitions can be written on Julia:

$$for\ i = 2 : length(coalitions)$$

$$s = coalitions[i]$$

$$v[i] = sum(SumRows[i] - sum(A[i, j]\ for\ j\ in\ s)\ for\ i\ in\ s);$$

$$end$$

- The last step to conduct on Julia is to tell the program to calculate all the values, now that a definition of $v[i]$ has been provided

  $$[coalitions\ v]$$

  (command that will calculate, with the function defined in the previous step, all the 4096 different *weighted degree centrality* measures)

2. Secondly, all the datas calculated on Julia (i.e. the *weighted degree centrality* for all coalitions) are imported on R, programming software with which, thanks to the use of one specific package ([6]), it will be possible to calculate the Shapley values of the 12 neuronal structures in the network. To do so, the steps conducted on R need to be explained:

   - First of all the *Game Theory* package needs to be installed in R

     $$library(Game\ Theory)$$

   - Once the package is in use and thus its functions can be applied, the values found with Julia for each coalition can be imported in R using the function

     $$coalitions \leftarrow c()$$

     (in the brackets all the 4096-1, since the empty coalition does not have to be considered, will be copied and pasted from Julia)

   - Once that the values of all possible coalitions have been imported in R, the coalition game on which the Shaley Value wants to be calculated, can be defined

     $$weighteddegreecentrality \leftarrow DefineGame(12, coalitions)$$

     (this *DefineGame* function will attribute to all the 4096-1 coalitions formed with the 12 neuronal structures the values that have been imported from Julia)

   - To print the result, the command

     $$summary(weighteddegreecentrality)$$

     needs to be run

   - Before computing the Shapley values, names can be attributed to the 12 players and thus the 12 neuronal structures' names can be used

     $$names \leftarrow (45,\ 24,\ 8A,\ 8B,\ 10,\ 13,\ 12,\ 11,\ 14,\ 25,\ 9,\ 46)$$

- Only once all these operations have been conducted, the Shapley value for all the 12 players, on the *weighted degree centrality* game, can be calculated using the appropriate function in R

  $$weighteddegreecentralityshapey \leftarrow ShapleyValue(weighteddegreecentrality, names)$$

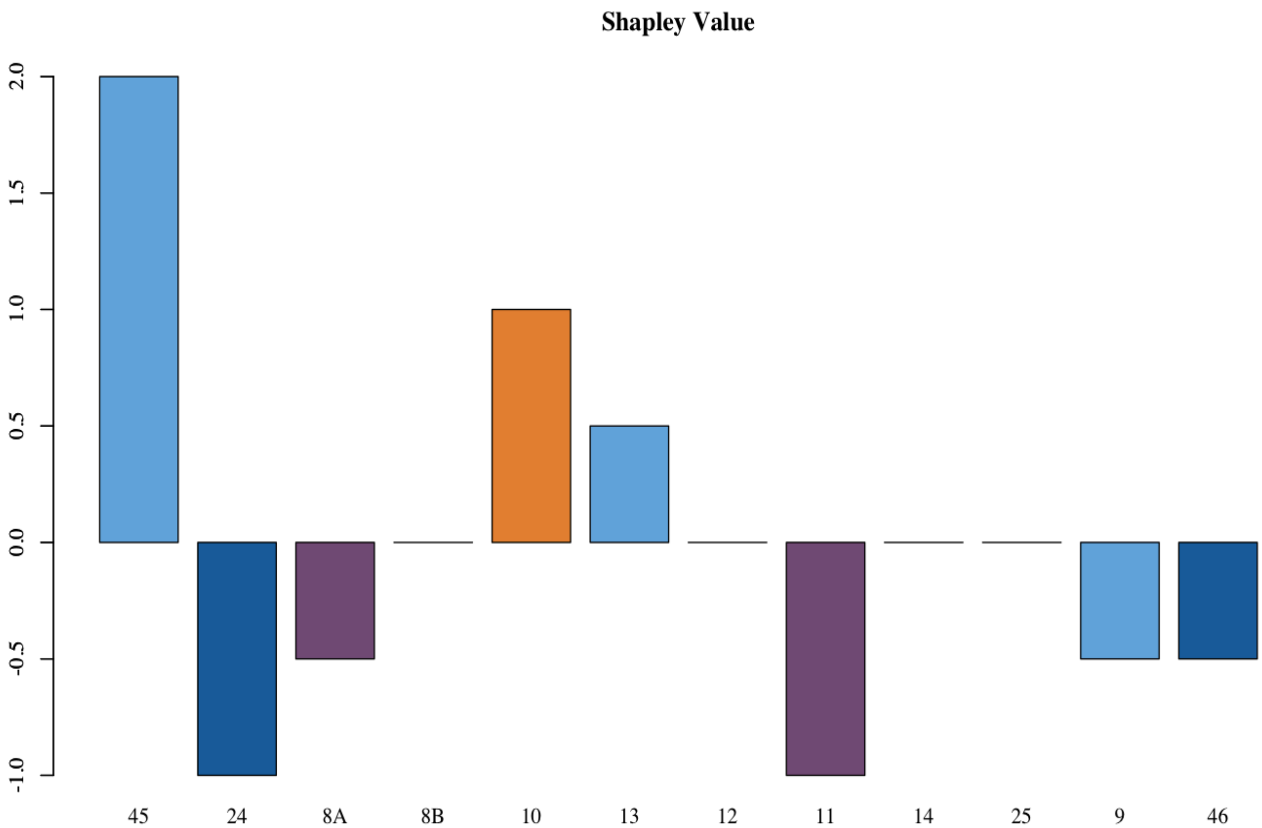
  (this function will provide a bar chart showing all the 12 results)

- Finally, to have the 12 results in numbers and not just on a bar chart, the function

  $$summary(weighteddegreecentralityshapley)$$

  can be used.

Now that all the processes needed to calculate the Shapley Value of the 12 neuronal structures in the W40-2 network, when the *weighted degree centrality* measure is applied, have been explained and analyzed, the result found can be given and analyzed.
The bar chart representing the 12 Shapley values, obtained and calculated with R, is the following:



From this bar chart it is possible to immediately notice that the neuronal structure which has the highest influence in the Walker's area, when it is the *weighted degree centrality* used as centrality measure, is the neuronal structure 45 (the reader needs to remember that when it is the *weighted degree centrality* the measure used to asses nodes' centrality, the higher the Shapley value attributed to a node, the more central and powerful will be that node in the network). However, it cannot actually be said how much more influent is the node 45 compared to the other neuronal structures. This is due to the fact that, as specified above, when the Shapley value attributes even negative powers to some nodes, comparison of values attributed to different nodes inside the game cannot be done.

Furthermore, R even calculated the exact Shapley value for each node:

i	σ_i
45	2.000000e+00
24	-1.000000e+00
8A	-5.000000e-01
8B	1.110223e-16
10	1.000000e+00
13	5.000000e-01
12	1.110223e-16
11	-1.000000e+00
14	0.000000e+00
25	0.000000e+00
9	-5.000000e+00
46	-5.000000e-01

These numbers confirm what has been detected previously from the bar chart: the neuronal structure 45 is the more influent and central node in the Walker's area when the centrality measure used is the *weighted degree centrality* and the edge (n_{11}, n_{45}) is considered to be present.

W40-2 network without the (n_{11}, n_{45}) edge Now, in this new paragraph, a second analysis will be conducted: nodes' centrality analysis of the Walker's area when the edge (n_{11}, n_{45}) is considered to not be present in the network.

As what have been done in the previous paragraph, before conduction the centrality analysis on the network, the representation of the adjacency matrix of the W40-2 network, without the (n_{11}, n_{45}) edge, needs to be provided:

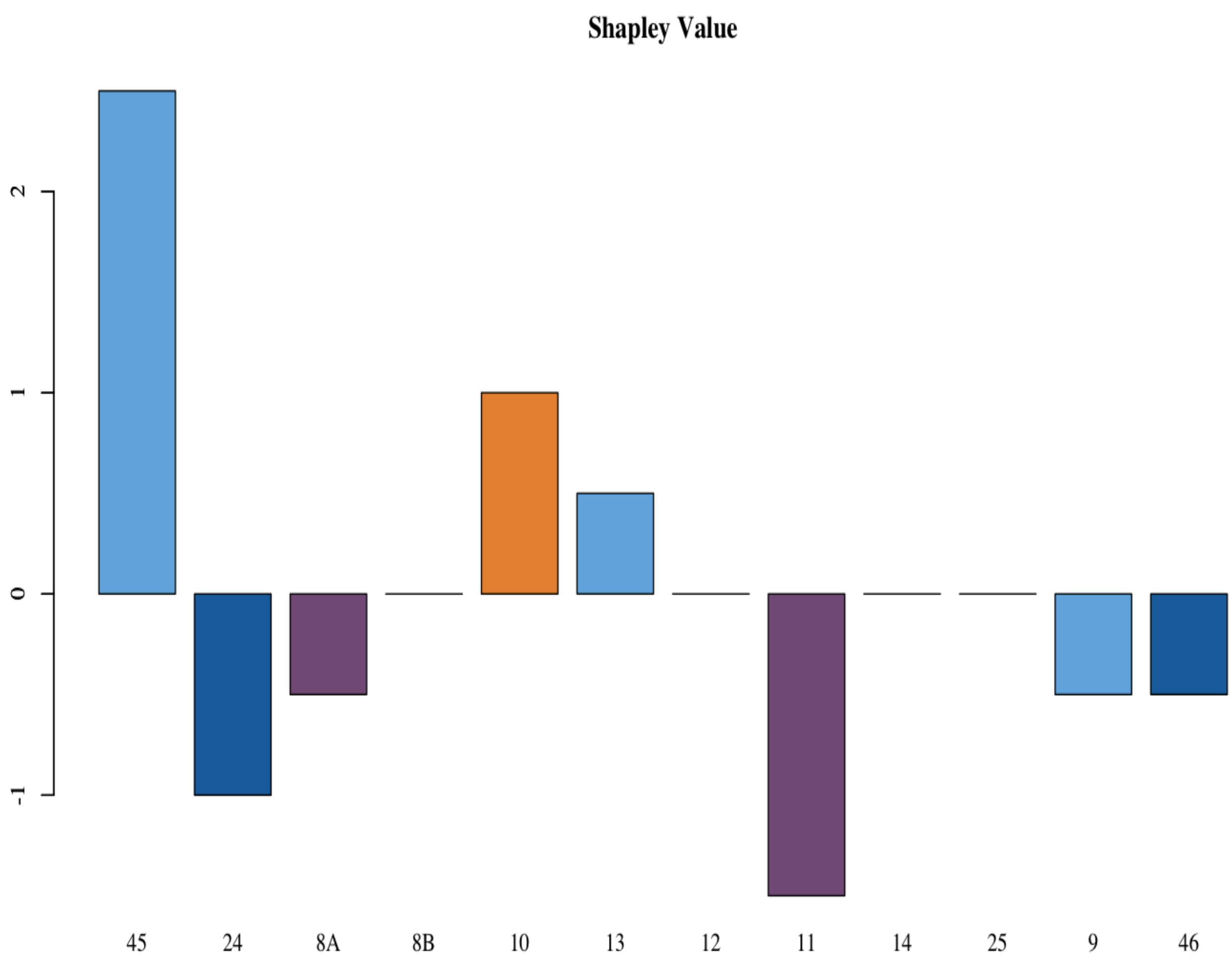
$$D = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

again with:

$$(pl_1, pl_2, pl_3, pl_4, pl_5, pl_6, pl_7, pl_8, pl_9 pl_{10}, pl_{11}, pl_{12}) = (n_{45}, n_{24}, n_{8A}, n_{8B}, n_{10}, n_{13}, n_{12}, n_{11}, n_{14}, n_{25}, n_9, n_{46})$$

By looking at the adjacency matrix it is clear that the edge (n_{11}, n_{45}) is not present in this analysis since a zero value is attributed to it.

The centrality analysis of the 12 nodes present in the network will be conducted following exactly all the steps explained in the paragraph above and used to measure the influence of nodes in the W40-2 network with the (n_{11}, n_{45}) edge being included. For this reason, there is no need to explain again all the procedures followed to calculate the Shapley value for the 12 neuronal structures when the centrality measure applied is the *weighted degree centrality*. For this reason,the results of this analysis can be immediately given.
As it has been done before, a bar chart representing the Shapley values attributed to the 12 nodes is provided:



As expected, when only the edge (n_{11}, n_{45}) is removed from the network, no big results' changes occur (i.e.

the neuronal structure 45 remains the most influent node, while 11 is calculated to be the least central node). However, to be more precise the exact Shapley values calculated by R are provided below:

i	σ_i
45	2.500000e+00
24	-1.000000e+00
8A	-5.000000e-01
8B	1.110223e-16
10	1.000000e+00
13	5.000000e-01
12	1.110223e-16
11	-1.500000e+00
14	0.000000e+00
25	0.000000e+00
9	-5.000000e-01
46	-5.000000e-01

As before, the numerical values of Shapley values confirm what the bar chart illustrated here above.

14 Comparison of results

Now that the analytical part of the analysis is done and that all the results of the two studies were provided, a comparison of the results can be made. However, as specified above, not only a comparison between the two studies conducted in this paper will be directed, but even a comparison between the "Walker's area without the (n_{11}, n_{45}) edge" study and the study conducted by the authors of the paper "Shapley ratings in brain networks" ([1]).

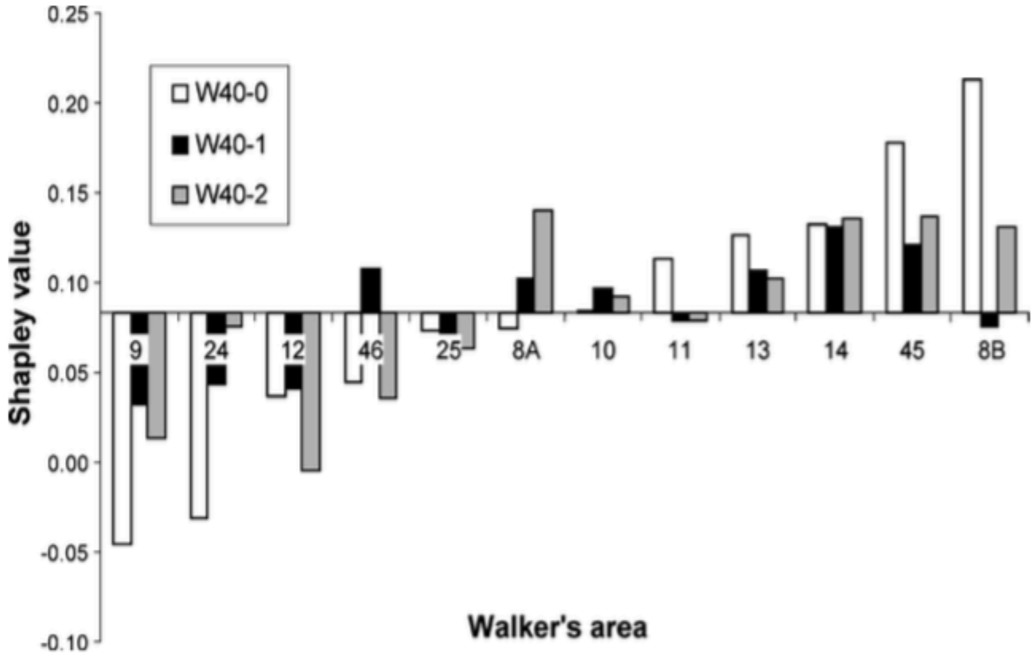
Comparison between the "W40-2 network with and without the (n_{11}, n_{45}) edge" studies When the "W40-2 network with the (n_{11}, n_{45}) edge" study is compared with the speculative study, but without the (n_{11}, n_{45}) edge, it is possible to notice that the Shapley values attributed to all the neuronal structures which are not included in the excluded edge, thus all the nodes expect n_{11} and n_{45}, are identical in both cases (i.e. the neuronal structures 24, 8A, 8B, 10, 13, 12, 14, 25, 9 and 46 have the same Shapley values in both analysis). This could mean that the centrality of all these nodes do not depend on the presence of the eliminated edge. However, what is more interesting is looking at what happens to the Shapley values of the involved nodes: n_{11} and n_{45}. By looking at the two studies' results it is noticeable that the Shapley values of these two nodes change when the edge is excluded from the analysis, thus this mean that the influence of these two neuronal structures in the network, the Walker's area, depend on the fact if the edge (n_{11}, n_{45}) is present or not. Furthermore, by looking at the two bar charts and at the numerical Shapley values, one fact comes immediately to the eyes:

when the edge (n_{11}, n_{45}) is excluded from the analysis, n_{45} experiences an increase in centrality, thus increase its influence in the network, (the Shapley value attributed to 45 increases). Oppositely, n_{11} experiences a loss, thus, even though was already attributed with a negative Shapley value, becomes even less influent and central. This decrease in centrality of node n_{11} could be explained by the fact that when the edge (n_{11}, n_{45}) is excluded from the analysis, the *weighted degree* value attributed to all coalitions S in which the node n_{11} is present, but n_{45} is not, decreases of one point, since now the outstanding edge (n_{11}, n_{45}), which starts inside the coalition S and is directed outside of it (toward n_{45}) does not have to be calculated any more. The result will then be that the node n_{11} will lose power.

Comparison between the "W40-2 network without the (n_{11}, n_{45}) edge" study and the results obtained in the "Shapley ratings in brain network" paper ([1]) Before being able to compare the results found on this paper and the results found, for the same variant of the brain network, on the paper by Kotter, Reid, Krumnack, Wanke and Sporns ([1]), a brief explanation of what the authors have done to reach their results need to be given.

The author of the paper conducted an analysis on the W40-2 variant of the Walker's area, when the edge (n_{11}, n_{45}) is non present, very similar to the one conducted on this paper. The only difference of the two analysis is that, while in this paper the centrality measure selected to calculate the centrality of nodes and coalitions has been the *weighted degree centrality* measure, the authors of the other paper decided to use the *Strongly connected component (SCC) centrality* measure as the centrality measure on which to calculate the Shapley values of the 12 neuronal structures. However, as it has been proved when the six different centrality measures have been tested on the simplified brain network previously, different centrality measures have different ways and ideas on how to measure the centrality and influence of a node in a network. For this reason, two different rankings of nodes' centrality are expected from the two different analysis.

Now, a graphical representation of the Shapley values' results found by the paper "Shapley ratings in brain network" ([1]) will be provided to then be able to compare these results with the ones found in this paper:



In this bar chart not only the Shapley values in which this comparison is interested in are provided (the Shapley values found when analyzing the W40-2 variant), but even the results found when the study was conducted on the other two variants. However, from now on in this comparison the focus will be on the light grey bars only (i.e. bars representing the 12 Shapley values found when analyzing the W40-2 variant).

For this study, the analysis conducted by the authors of the other paper, the lower is the Shaley value, the more central is the neuronal structure, due to the fact that when it is the *Strongly connected component (SCC) centrality* the measure used to calculate centrality, the lower is the value attributed to nodes and coalitions of node, the more central and influent is that node or coalition (this is exactly the opposite compared to what happened with the study conducted in this paper in which the centrality measure selected has been the *weighted degree centrality*).

Now that all the necessary clarifications have been made, the two centrality rankings, obtained by the two different studies, can be provided:

1. Centrality ranking obtained by the "W40-2 network without the (n_{11}, n_{45}) edge" study conducted on this paper:

$$(n_{45}, n_{10}, n_{13}, n_{8B}, n_{12}, n_{14} n_{25}, n_{8A}, n_{9}, n_{46}, n_{24}, n_{11})$$

   with a tie between n_{8B} and n_{12} and between n_{8A}, n_9 and n_{46};

2. Centrality ranking obtained by the study conducted on the paper "Shapley ratings in brain networks" on the W40-2 network:

$$(n_{12}, n_{9}, n_{46}, n_{25}, n_{24}, n_{11}, n_{10}, n_{13}, n_{8B}, n_{14}, n_{45}, n_{8A})$$

   with no ties between any neuronal structures.

As it can be noticed by looking at the two different rankings, the two studies lead to two completely different outcomes.
The study conducted on this paper rank n_{45} as the neuronal structure with more influence in the network, while, on the contrary, the study conducted using the *SCC centrality* measure as the centrality measure behind the Shapley value calculations, ranks n_{45} as the node with the second lowest influence in the same network. On the other hand, the *SCC centrality* measure based study identify n_{12} as the neuronal structure with major importance and influence in the Walker's area, while the analysis conducted in this paper attributes to n_{12} a Shapley value really close to 0, hence classify n_{12} as a neutral node in the brain network.
All these results found comparing these two studies can be considered additional proofs to what have been found while conducting the brain centrality study of the simplified brain network: different centrality measures can lead to different neuronal structures centrality results.

Part VI

Conclusion

In this thesis, starting from the *degree centrality* measure, analyzed in the paper "Efficient Computation of Semi-values for Game-Theoretic Network Centrality" ([3]), a new centrality measure was introduced: the *weighted degree centrality* measure. This centrality measure, as the name suggests, is a weighted version of the classic *degree centrality* measure which does not only consider once the neighbours of a node or of a coalition of nodes, as the classic centrality measure does, but counts all the edges which connect the node or the coalition of nodes towards outstanding nodes.

Thanks to the implementation of this new centrality measure, a different study, compared to the one conducted by the authors of the article "Shapley ratings in brain networks" ([1]), of the W40-2 variant of the Walker's area was conducted.

The results found in this paper, which have as underlying centrality measure the *weighted degree* measure, have then been compared to the results found in the previously cited work ([1]), which have as underlying centrality measure the *strongly connected component (SCC)* measure.

While comparing the two results, it was immediately discovered that the Shapley value calculated on the two different centrality measures, applied to the same brain network, led to two completely different outcomes: the study conducted on this thesis identify the node n_{45} to be the most central, hence powerful, node in the network, while the other study ranks node n_{12} to be the most influent node in the Walker's area and node n_{45} to be one of the least influent, opposite to what was discovered in this paper.

This comparison of results could mean two things. On one hand, it could mean that, as already anticipated, since different centrality measures identify "centrality" of a network in different ways, application of different centrality measures to the same network can lead to completely different outcomes, as what happens in this case. However, on the other hand, it could even be that the new *weighted degree centrality* measure defined in this paper is unable to properly explain brain network centrality in a coherent way.

However, we are able to exclude the second option since, thanks to the counter example provided while conducting the *weighted degree* analysis on the toy example, we eliminated the possibility of the *weighted degree centrality* measure being meaningless.

To conclude, as a result of the counter example, it is possible to say that the results obtained from the *weighted degree* analysis on the W40-2 variant of the Walker's area are just a result of differences in the way in which "centrality" is measured.

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