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Analysis of a complex investment instrument: The CPDO

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Contents

1. Introduction

The thesis will focus on presenting and analyzing a complex investment instrument such as the Constant Proportion Debt Obligation. This will be carried out by considering and studying the processes and results that Cont and Jessen presented in their paper: " *Constant Proportion Debt Obligations (CPDO): Modeling and Risk Analysis*" [6]. In order to reproduce some of the calculations in [6] and in order to implement some mathematical analysis of the instrument, for simplicity, in this thesis will be found and considered a different default intensity model based on a constant intensity of default. Moreover, the results arising from these calculations and from [6] will be briefly compared.

1.1 Outline of the thesis

The thesis is structured as follows. After an initial description of the structure of Constant Proportion Debt Obligations (CPDO) in subsection 1.2, it will be analyzed the case study of a CPDO in Australia and the process against the rating agencies in subsection 1.3. In section 2 and its subsections, it will be explained more in detail and described the functioning and the strategy of an elaborate investment instrument such as CPDO, including the use of the leverage, the cash flow structure and the risk factors.
Section 3 with its subsections will be the mathematical core of the thesis. It will be provided a simplified model in order to assess default risk and intensity with respect to the one provided by Cont and Jessen in their paper [6] in order to try to reproduce some of their findings. Moreover, it will follow an estimation of the loss given default of the CPDO as well as the cumulative discount losses. In subsection 3.5, an expected shortfall definition will be provided and an analysis of the results on the tests conducted by Cont and Jessen will be presented.
Finally, section 4 will be a summary of the results found through the script and will conclude the discussion of the thesis.

1.2 The CPDO structure

A CPDO or Constant Proportion Debt Obligation is an incredibly complex credit investment strategy that generates high coupon payment through dynamically leveraging a position in an underlying portfolio of index default swaps[1]. A CPDO is composed of two positions, one in the money market, which consists of short term investments, and one that consists of default swaps on indexes such as ITRAXX and DJ CDX which are made of corporate names. The latter comprises Credit Default Swaps (CDSs) on the names in the above-mentioned indexes, where the seller of the protection, the CPDO and its manager, are bound to compensate the buyer of such an instrument when a default in the protected names occurs before the maturity of the contract. The CPDO earns a specific fee for this protection which the buyer needs to pay, usually quarterly, and, in the case a default occurs, the accrued fee is also paid.

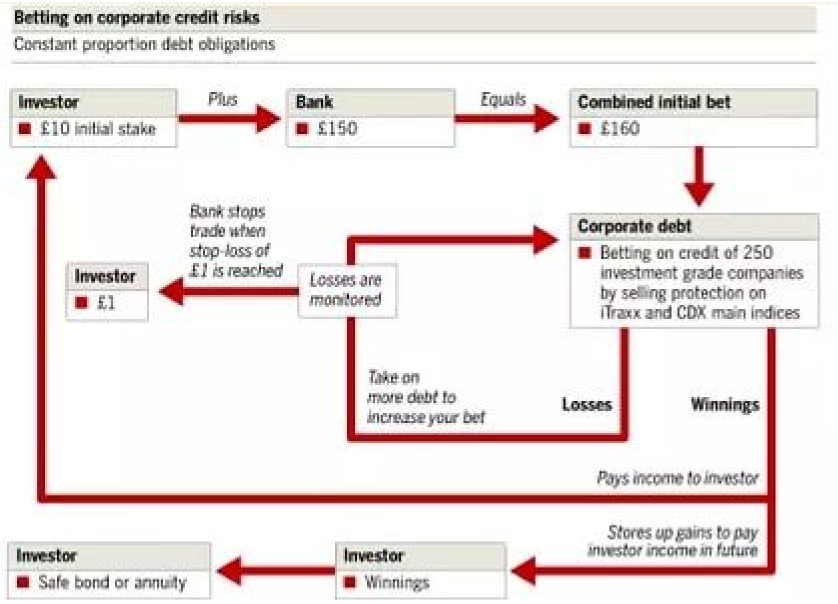
Constant Proportion Debt Obligations dynamically adjust the leverage structure and its risk exposure to ensure that revenues arising from the two positions will balance the promised customer coupon payments and all the expenditure and losses that the CPDO will face.

It is possible to idealize the structure of a CPDO (Figure1) as a basket, that does not contain actual bonds but credit default swap against bonds[2]. Unlike CDOs, in which the underlying names are fixed, a Constant Proportion Debt Obligation changes the composition of the names in the underlying portfolio at every roll date, which is usually every six months. At a roll date, the CPDO manager buys derivatives and protection on the old names he wants to be protected against and sells new swap contracts on new names of the ITRAXX or DJ CDX. This process of buying and selling and continuous customization of the underlying portfolio make it possible for the manager to personalize both the leverage strategy and the exposure at risk of the portfolio.

The rating of these products has been a very controversial issue in the past years, as it will be explored more in depth later on.

1 Cont, R., & Jessen, C. (2009) [6]
2 Chen, J. (2019, April 12) [5]

Figure 1: CPDO mechanism



Source: Financial Times

On the 29th of August 2006 the rating agency Standard and Poor's gave the first SURF CPDO arranged by ABN Amro a credit rating of AAA, the highest and safest rating in investment grade bonds. The verdict was then confirmed by Moody’s that again granted the CPDO the rating of AAA. This rating has been questioned since the first time it appeared. Jobst et al. (2007) [14] recognized that although the rating of AAA was consistent with the model used by S&P the sensitivity of the same models on its assumptions can lead to very different results going as low as BBB[3] . Later on, Moody’s, as CPDOs were proven to be riskier than they were supposed to be and his reputation became more fragile, affirmed that in the rating process of these obligations a problem with the algorithm caused the rating to be biased.

In line with other researchers, Cont and Jessen in [6] have argued that the complex framework that rating agencies have used in order to assess the rating was not necessary, and, moreover, it may have occulted the main risk factors influencing the CPDO Strategy[4], which will be analyzed in section 2, together

[3] Jobst, N., Gilkes, K., Sandstorm, N., Xuan, Y., & Zarya, S. (2007) [14]
[4] Cont, R., & Jessen, C. (2009) [6]

with its functioning. They also propose a diverse analysis based on a top-down approach of the main risks and performance drivers.

1.3 The CPDO "*Rembrandt*" case

In April 2006 the first CPDO or Constant Proportioning Debt obligation was developed by the Dutch credit institute ABN Amro. These obligations were incredibly complex and used leverage to provide a return for the investors above the average 90 days bank bill swap rate[5]. During that same year the Dutch bank asked rating agencies to rate the risk of these financial products, both S&P and Moody's responded to the call. For those two agencies, the rating of the obligation was AAA assigning the highest possible investment grade, the risk involved with investing in this product was the lowest possible. In other words, the probability of default of the CPDO was less than 0.278% and the possibility to meet payment obligations was "*extremely strong*".

In Australia, ABN Amro's CPDO also known as "*Rembrandt*" was founded very appealing by the Local Government Financial Service (LGFS), an agency that worked as financial advisor of some New South Wales local councils. This LGFS looking at an opportunity for high yield with an excellent credit rating, as the Rembrandt CPDO assured yields similar to junk bonds but with the solidity and security of a AAA bond, bought on behold of 12 local councils about 18.5 Million of Rembrandt CPDO. Running-up to the 2008 world crisis as the spread of the underlying CDS indices continued to widen, the obligations started to cash out. The cash-out process consists of unwinding all the risky exposures, ending coupon payments and returning the remaining funds to the investor when the invested principal becomes smaller than a target ratio calculated on the initially invested principal. For the Rembrandt CPDO this cash out ratio was 10%[6], so at the end of 2017 as this financial product cashed out, 90% of the principal invested by the New South Wales 12 councils was lost, and on an investment of 18.5 Million[7], the losses amounted to about 16.6 million, leaving

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5 Alderman, P. (2013, February 26) [1]
6 Alderman, P. (2013, February 26) [1]
7 Bowman, L. (2012, November 6) [4]

the councils with a dangerous budget deficit. For this reason, the 12 councils sued LGFS, S&P and ABN Amro for negligence and misleading and deceptive conduct. In particular, they contested S&P for the assignment of the rating of AAA to the "*Rembrandt*" and ABN Amro for its role in the assignment and diffusion of the rating[8]. In November 2012 the Federal Court of Australia and, in particular, judge Jagot, issued a landmark judgment in favor of the 12 councils declaring that S&P and ABN Amro baited the councils, it was said that no rating agency reasonably competent could have assigned this solid rating to such a volatile financial instrument[9]. Judge Jagot added: "*S&P's rating of AAA of the Rembrandt 2006-2 and 2006-3 CPDO notes was misleading and deceptive and involved the publication of information or statements false in material particulars and otherwise involved negligent misrepresentations to the class of potential investors in Australi*a"(Wardell, J., & Willis, K.)[10] , she also stated:*" ABN Amro was knowingly concerned in S&P's contraventions of the various statutory provisions proscribing such a misleading and deceptive conduct*" (Wardell, J., & Willis, K.)[11] . Following this sentence, the agency and the bank were forced to pay 30 Million dollars to the councils in order to cover the losses associated with their acts[12]. Regarding LGFS the judge ruled that it was negligent and guilty of misleading and deceptive conduct in failing to fully and accurately disclose all of the material risks to the councils[13]. With this first of a kind sentence for the first time a rating agency was considered accountable for the care of their ratings, and they can no longer "hide" behind the fact that their rating is only an opinion on which a customer's investment decision should not be made. The erroneous models presented by both Moody's and S&P were based on complex models for the joint transition of ratings and spreads for all names in the underlying portfolio[14] . This approach led to hundreds of state variables and it was not accessible to entities other than rating agencies due to

[8] Alderman, P. (2013, February 26) [1]
[9] Romano, L. (2012, November 5) [19]
[10] Wardell, J., & Willis, K. (2012, November 05) [25]
[11] Wardell, J., & Willis, K. (2012, November 05) [25]
[12] Gulmanelli, S. (2012, November 06) [10]
[13] Wardell, J., & Willis, K. (2012, November 05) [25]
[14] Cont, R., & Jessen, C. (2009) [6]

lack of historical data on ratings. Moreover, they have been argued to be too sensitive to parameter changes, in fact, even a small change in parameter could cause the change of rating from AAA to BBB, and its complexity may have covered the main risk factors of CPDOs.

This case and this unique verdict allowed to underline the deceiving rating of CPDO and its complicated strategy; for this reason, in the next section the latter will be treated more in detail.

2. The CPDO strategy

Constant Proportion Debt Obligations (CPDOs) are leveraged credit investment strategies that appeared in the low credit spread environment of 2006 intending to generate high coupons while investing in investment grade credit. As stated above, the asset side of the CPDO contains two positions: a money market account and a leveraged credit exposure via index default swaps on indices of corporate names, typically the ITRAXX and DJ CDX (the two leading credit default swap indices). The dynamically adjusted risky exposure is chosen such as to ensure that the CPDO generates enough income to meet its promised liabilities and also to cover for eventual fees, expenses and credit losses due to defaults in the reference portfolio and mark-to-market losses linked to the fair value of the index default swap contract[15].

2.1 Description

The functioning of CPDOs is mainly based on two agents, an investor, and a CPDO manager. The investor initially injects capital, which from now on will be considered 1 for simplicity, in order to receive coupon payments made periodically until the end of the contract, at time T, of an agreed spread on the LIBOR rate. The manager utilizes these funds and invests them in leverage by selling protections on credit indexes such as ITRAXX and DJ CDX through default swaps. For this reason, the CPDO portfolio is composed of two open

[15] Cont, R., & Jessen, C. (2009) [6]

positions: a short term investment denoted by $(A_t)_{0\leq t\leq T}$ and a position in a T^I-year index default swap where T^I represents the maturity of the swap contract which can be greater or smaller than the maturity of the CPDO. The sum of the money market account and the swap contract is indicated by $(V_t)_{t\in[0,T]}$. At first, the money coming from the investor is placed in the money market account which earns interest at LIBOR rate (denoting L(t,s) as the spot LIBOR rate at t for maturity s, where s > t). The investor will receive, as stated before, coupon from the CPDO, which are paid as a spread δ over LIBOR, at dates called CD or coupon dates: $CD = \{tl \leq T \mid l = 1, 2, \ldots\}$.

$$c_{tl} = \Delta(t_l)[L(t_{l-1}, t_l) + \delta]$$

where:

c_{tl} = coupon paid to the investor.

$\Delta(t_l)$ = the time passed since the last coupon that has been paid.

$L(t_{l-1}, t_l)$ = the spot Libor rate at time t_{l-1} for time t_l.

δ = contracted spread that invest has to receive above LIBOR.

Referring to the present value of these coupons as the target value, it could be expressed as follows:

$$TV_t = B(t,T) + \sum_{tl\in CD\bigcap[t,T]} EQ\,[C_{tl}\, e^{-\int_t^{tl} rs\, ds}\, given\ F_t]$$

where:

$B(t,T)$ = spot discount factor at time t with maturity T.

$EQ\,[c_{tl}\, e^{-\int_t^{tl} rs\, ds}\, given\ F_t] = B(t,u)$ = discount factor associated at some short term process r and market information available at time t (F_t).

It is now possible to delineate a first and simple scenario, that is, when $V_t \geq TV_t$. In this case, the CPDO manager has the power to meet the promised obligation by investing the received funds, or part of them, only in the money market.

If this is not the case, the CPDO manager will utilize part of the funds, leveraging them by a factor *m*, in order to sell protections on credit indexes (that are previously cited) through Credit Default Swaps. In this way, the CPDO will earn a periodic spread called *S(t, T)* observed at time t for a swap that will expire at T. Furthermore, the present value of this spread payment will be indicated as P_t and $DT = \{\tau_1 \leq \tau_2 \leq \ldots \leq \tau_{N^I}\}$ will be defined as the set of default times in the index, τ_i is the date of the i-th default, N^I is the number of names in the index underlying and N_t represents the number of default in the index until time t :

$$N_t = \sum_{i=1}^{N^I} 1_{\{\tau_i \leq t\}}$$

In any case, the CPDO portfolio can either *cash in* or *cash out*. The event of cashing in is favorable for the investor and for the CPDO itself. It represents the situation in which the portfolio reaches a value that is sufficient to meet all future liabilities $V_t \geq TV_t$. In this case, the funds will be invested only in short term investments in the money market and all swap contracts will be liquidated. If instead, the CPDO cashes out, the value of the portfolio have plunged below a certain threshold *k* (of the investors' placement) and the CPDO lays out all the risky exposure, in order to return all the remaining to the investors, without further payment of coupons.

Until the expiration date of the CPDO contract or until a cash in /out event, the CPDO manager will dynamically adjust the leverage *m* of the position in order to comply with the contractual obligations (present value of the future stream of coupon payments) and with the trend of the markets. This mechanism will be analyzed in depth in the following chapter.

2.2 Leverage

The CPDO is a structured credit product where the proceeds received by investors are utilized as collateral for a long position in a CDS portfolio. The notional of this position in CDS does not match with the amount collected but it is geared up by a factor *m* that is the leverage factor of the CPDO. This leverage is adjusted dynamically during the lifetime of the CPDO, every six months at Roll Dates (RD) and whenever there is a default in the underlying index name, The set of dates where the leverage is rebalanced will be referred to in this thesis as Rebalancing Dates (RBD)**.** At initiation, the CPDO manager calculates the shortfall between the net asset value V_t of assets and the present value of future liabilities, also known as target value TV_t. The manager will realize that $TV_0 \geq V_0$ and thus he will calculate target leverage as an increasing function of the shortfall. This formula might be diverse for every CPDO but, in general, it is possible to define the target leverage m_t *as:*

$$m_t = \beta \frac{TV_t - V_t}{P_t}$$

In which β is a factor that controls the strategy and its aggressivity and P_t is the income generated by the swap.

It is possible to rewrite this formula in order to express the notation included in it:

$$m_t = multiplier * \frac{PV\ (future\ liabilities) - NAV}{index\ spread * remaining\ maturity}$$

After setting the target leverage During the CPDO life, the actual leverage is adjusted every six months at RD as the composition of the underlying index changes and therefore it is necessary to adjust the actual leverage factor $m^{i(t)}$ in order to set it equal to the targeted factor m_t:

$$m^{i(t)} = m_t \text{ when } t \in RD = [T_j \mid T_j = \tfrac{j}{2}, j = 1, \ldots, 2T]$$

Moreover, the leverage factor needs to be adjusted every time it differs more than $\varepsilon(\pm 25\%)$ form target leverage m_t:

$$m^{i(t)} = m_t \text{ if } m^{i(t)-1} \notin [(1-\varepsilon)m_t, (1+\varepsilon)m_t]$$

Another occasion in which the leverage factor is changed automatically is when there is a default in the name of the underlying index:

$$m^{i(t)} = \frac{Ni - N_t}{Ni - N_{t-}} m^{i(t)-1}, for\ t \in DT$$

The leverage factor has an upper bound, that is, it is capped generally at 15 (M =15) so that it will reduce the overall loss.

The name of this special obligation specifically arises from this leverage structure as it is piecewise constant and therefore the name "*Constant Proportion*" Debt Obligation or CPDO. Moreover, it is feasible to state that CPDOs have a strategy that will lead to an increase in the leverage in case of loss and a decrease in case of gains. This is a " *Buy low and sell high strategy*" (Cont and Jessen)[16]**.**

2.3 Structure of cash flows

In the previous sections, it has been delineated how a CPDO contract behaves at inception, that is: the CPDO receives an initial amount of money from the investors, this amount minus a fee (usually 1%) is placed in a short term investment, usually in the money market, earning risk-free interests. The manager of this obligation sells swap contracts on the European and American indices ITRAXX and DJ CDX with a notional equal to the notional received by investors time a leverage factor m_0. After its inception, during the lifetime of the CPDO, there are many subsequent cash flows that can be decomposed and analyze part by part. On any date before maturity the CPDO will:

- Receive interest payments from the money market account on the notional invested:

$$t \in [0, T] \ : \ A_{t-\Delta} * L(t - \Delta, t) * \Delta$$

[16] Cont, R., & Jessen, C. (2009) [6]

where Δ is the time between payments dates t and $A_{t-\Delta}$ is the notional deposited in short-term invested at time t- Δ

- Pay coupon corresponding to the Libor plus spread negotiated with investors at inception:

$$t \in CD \quad -c_{tl} = -\Delta(t_l)[L(t_{l-1}, t_l) + \delta]$$

- Pay default losses to the owner of the swap contracts arising from the default of one or more names in the ITRAXX or DJ CDX indices:

$$\tau \in DT \quad -m^{i(t_l)} \frac{(1-R)}{N^I}$$

where R is the rate of recovery for one single event.

- Receive income arising from the spread:

$$t_1 \in CD \quad m^{i(t_l)} S^{i(t_l)} \Delta(t_l)$$

- liquidate swap contract with the following cash flow:

$$m^{i(t)} * (S^{i(t)} - S(t, T_{j(t)} + T^I)) * D_t^{swap} 1_{RD}(t)$$
$$+ (m^{i(t)-1} - m_t) * (S^{i(t)} - S(t, T_{j(t)} + T^I)) * D_t^{swap} 1_{(RBD \cap \{m_t < m^{i(t)-1}\})}(t)$$

where $T_{j(t)}$ indicates the last roll date before time t and D^{swap}_t indicates the duration of the swap contract.

Now the components of the structure of the CPDO's cash flows will be further analyzed. The income generated from the spread on swap contract is calculated as the average spread on the contracts, which, every time a CPDO enters new contracts, changes. At time 0, the observed spread and contracted spread are equal therefore $S^0 = S(0,T^I)$. When the composition of the swap contract changes, at index roll dates *RD*, some contract are liquidated and some new ones enter in the composition, at this point the spread will be:

$$S^{i(t)} = S(t, t + T^I) \quad For\ t \in RD$$

When the leverage is adjusted, at rebalancing dates *RBD*, the new contract entered in the portfolio of the CPDO will influence the contracted spread:

$$S^{i(t)} = S^{i(t)-1} \quad for\ m^{i(t)} < m^{i(t)-1}$$

$$S^{i(t)} = wS^{i(t)-1} + (1 - w)\, S(t, T_{j(t)} + T^I) \quad for\ m^{i(T)} > m^{i(t)-1}$$

Where $t \in RBD$ and $w = \frac{m^{i(t)-1}}{m^{i(t)}}$ is the weight of old contracts in the portfolio.

Any changes in the index default swap spread will affect the MtM value (mark-to-market) that represents a measure of the fair value of accounts that can change over time[17], in this case, it is the value of entering an offsetting swap contract with same expiry and coupon dates:

$$MtM_t = (S^{i(t)} - S(t, T_{j(t)} + T^I)) * D_t^{swap}$$

where:

$$D_t^{swap} := E^Q * [\sum_{t_l \in CD \cap [t, T^I]} e^{-\int_t^{t_l} r_s ds} \Delta(t_l)\, (1 - \frac{N_{t_l}}{N^I}) | F_t]$$

At this point, it is possible to define the value of the CPDO contract as the sum of the money market account and the mark-to-market account:

$$V_t = A_t + MtM_t.$$

Another factor that will be analyzed more in detail is the liquidation of swap contracts that will either result in a loss or a gain. On roll dates *RD* all the positions held in swap contract are paid off, resulting in a profit or loss:

$$\text{Profit/Loss} = m^{i(t)} * (S^{i(t)} - S(t, T_{j(t)} + T^I)) * D_t^{swap} \quad for\ t \in RD$$

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17 Kenton, W. (2019, March 12) [16]

Moreover at rebalancing dates *RBD*, when the leverage factor is decreased hence $m_t < m^{i(t)-1}$, a liquidation of certain swap contract take place leading to a profit or loss of:

$$\text{Profit/Loss} = (m^{i(t)-1} - m_t)\ (\ S^{i(t)} - S(t, T_{j(t)} + T^I)) * D_t^{swap}\ .$$

The sum of these two cash flows will delineate the profit or loss arising from the liquidation of swap contracts.

The value of money market account and the CPDO portfolio is known up to but not at time t[18] . It is possible to calculate A_t and the total value of the CPDO V_t as it follows:

$$A_t = A_{t-\Delta} * L(t-\Delta, t) * \Delta + (m^{i(t_l)} S^{i(t_l)} \Delta(t_l) - c_{tl})\ 1_{\mathrm{CD}}(\mathrm{t})$$
$$- m^{i(t_l)} \tfrac{(1-R)}{N^I}\ 1_{\mathrm{RD}}(\mathrm{t}) + m^{i(t)} * (S^{i(t)} - S(t, T_{j(t)} + T^I)) * D_t^{swap} 1_{RD}(t)$$
$$+ (m^{i(t)-1} - m_t) * (S^{i(t)} - S(t, T_{j(t)} + T^I)) * D_t^{swap} 1_{(RBD \cap \{m_t < m^{i(t)-1}\})}(t)$$

And so

$$V_t = A_t + MtM_t$$
$$V_t = A_{t-\Delta} * L(t-\Delta, t) * \Delta + (m^{i(t_l)} S^{i(t_l)} \Delta(t_l) - c_{tl})\ 1_{\mathrm{CD}}(\mathrm{t})$$
$$- m^{i(t_l)} \tfrac{(1-R)}{N^I}\ 1_{\mathrm{DT}}(\mathrm{t}) + m^{i(t)} * (S^{i(t)} - S(t, T_{j(t)} + T^I)) * D_t^{swap} 1_{RD}(t)$$
$$+ (m^{i(t)-1} - m_t) * (S^{i(t)} - S(t, T_{j(t)} + T^I)) * D_t^{swap} 1_{(RBD \cap \{m_t < m^{i(t)-1}\})}(t)$$
$$+ (S^{i(t)} - S(t, T_{j(t)} + T^I)) * D_t^{swap}$$

2.4 Factors of risk

Having analyzed the functioning, composition, and complexity of cash flows and the structure of CPDOs it is now important to analyze the factor of risk of these financial products. It is difficult to assess all the factors of risk that, on determinate occasions, would spark a cash out situation or simply the

[18] Cont, R., & Jessen, C. (2009) [6]

impossibility to return par at maturity of the contract. A set of rules of thumbs are delineated by Isla et al. (2007)[19]:

- CPDOs are less likely to default when CDS spread volatility is low and when the rate of mean reversion is high (since in [12] Isla et al. employed a mean reverting model as Cont and Jessen did in [6]).
- The probability of default depends on the leverage rule utilized, hence the lower the coupon to be paid and the lower are all the arrangements fees, the lower is the leverage to be used and therefore the lower the probability of default.
- In general, a situation in which the spread is widening is not good news for CPDO notes, but this is not always the case. When spread initially widens but will be constant in the future, the higher carry on the future position will outweigh the initial losses of NAV. However, if the opposite is realized, that is, the spread widens later on during the life of the CPDO the possibility of recovery after the loss is very low and the probability of defaulting or not repaying par at maturity will increase substantially.
- When a market-shaking event occurs and triggers a severe spike in the spread, the probability of a cash-out event is very consistent.
- The occurrence of default events in the underlying portfolio names can generate a considerable loss in NAV, but because CPDOs, relying on the concept of index roll, limits its exposure to defaults these losses could be mitigated.

In other words, it is possible to reassume all the factors in 4 macro categories, according to Cont and Jessen[20] those categories are:

- Spread risk:

Since the index default swap spread is the main component of the cash flow of CPDOs and its evolution in time has effects on the swap income and on the profit or loss arising from roll and rebalancing dates. A sudden change in the spread will generate two main effects: on roll dates, a change in spread will generate a change in a single cash flow, but it will also have an effect on long term spread income. It

[19] Isla, L., Willemann, S., Soulier, A. (2017, April 20) [11]
[20] Cont, R., & Jessen, C. (2009) [6]

is still not clear and in-depth analysis is needed in order to assess which effect will dominate.

- Default risk:

The number of defaults in the names of the underlying portfolio is expressed through the default rate. A higher rate is not detrimental to the CPDO as it leads to higher expected credit losses.

- Interest rate risk:

The term structure of interest rates influences the cash flows of the CPDO through LIBOR, as the higher the LIBOR rate the higher the coupon payments that this product will promise to investors, and through the discount factor used in the calculation of the present value.

- Liquidity risk:

Since the cash flow of CPDOs are also affected by the bid/ask spread of the index, an important factor of risk is the liquidity of the index default swap.

3. Mathematical analysis

After the presentation of the Constant Proportion Debt Obligation that is provided in the preceding sections, now the thesis will focus on a mathematical analysis of the default intensity. A simplified approach with respect to the one proposed by Cont and Jessen will be suggested in order to try to replicate some of their calculations employing a constant expected default intensity. The first part will highlight the idea and reasoning behind this change and will delineate the new model of default risk. Furthermore, in the following sections, the new model will be utilized in order to understand the cash flow losses in the occurrence of defaults and, specifically, in order to analyze the losses given default and the cumulative discount losses. Moreover, it will be taken in consideration the value at risk (VaR) and the expected shortfall of CPDOs and a study on the results derived from Cont and Jessen's paper[21] on this subject will be conducted.

[21] Cont, R., & Jessen, C. (2009) [6]

3.1 Default risk and intensity an alternative approach

In their paper Cont and Jessen model the default events through the default intensity λ_t, which they define as the F_t-intensity of the default process N_t where F_t represents the market history up to time t. The result is a default intensity that is conditional on the probability of the next default given past market history. This model leads to a process that is self-affecting, one default may trigger a cluster of defaults through spillover effects[22]:

$$\lambda_t = \lim_{\Delta t \to o} \tfrac{1}{\Delta t} P(N_{t+\Delta t} = N_{t-} + 1 | F_t)$$

This model comprises an advanced mathematical background and its implementation is rather complex. Hereby the model by Cont and Jessen will be briefly presented. The risk-neutral intensity λ_t^Q is modeled as a CIR process with jumps at defaults event, the default intensity jumps up by a magnitude proportional to the loss at defaults and follows a diffusive process between default times. It is possible to express the intensity process as[23]:

$$d\lambda_t^Q = k(\theta - \lambda_t^Q)dt + \sigma\sqrt{\lambda_t^Q}dW_t + \eta dL_t$$

Where L represents the loss process, $k \geq 0$ is the rate at which the intensity reverts back to its long term level θ. Given that the default intensity follows a CIR-process is required that $2k\theta \geq \sigma^2$ in order to ensure $\lambda_t^Q > 0$. This process is part of the class of affine processes, where the number of defaults is given by:

$$For\ B(t) = (B_1(t), B_2(t))',\ B_2(t) = 1 \text{ where:}$$

$$B_1 = -\frac{1}{k+\eta(\frac{1-R}{N^I})} * (e^{-(k+\eta(\frac{1-R}{N^I}))(T^I - t)} - 1)$$

[22] Cont, R., & Jessen, C. (2009) [6]

[23] Cont, R., & Jessen, C. (2009) [6]

$$A(t) = \frac{k\theta}{(k+\eta(\frac{1-R}{N^I}))^2} * (e^{-(k+\eta(\frac{1-R}{N^I}))(T^I-t)} - 1) + \frac{k\theta}{(k+\eta(\frac{1-R}{N^I}))} * (T^I - t)$$

Having $E^Q[N_{T^I}|F_t] = A(t) + B_1(t)\lambda_t^Q + N_t$ which gives an analytic expression for the $T^i - year$ swap spread[24].

For simplicity, this thesis will employ a constant default intensity lambda (λ). Through a review of the data, from 1981 to 2016, an expected default per year (λ) will be estimated applying a Poisson Process and the results will be employed in order to estimate the losses and the expected shortfall of the CPDO.

3.2 The Poisson Distribution

A Poisson distribution is the probability of the number of events that occur in a given interval when the expected number of events is known and the events occur independently of one another. Defining λ as the mean for the interval of interest and it is possible to write a Poisson distribution:

$$X \sim P(\lambda)$$

where X is a random variable that represents the number of events that occurred in a specific time period, and that has a Poisson probability distribution with mean λ.

In order to find out the probability of an event occurring under a Poisson distribution, *in primis*, there is the need to understand how this distribution works. The Poisson distribution can be obtained by taking the limit of a Bernoulli process and therefore as the limit of suitable Binomial distribution. It is possible to model the arrival of events that happen at random at a rate λ per unit time. At time $t=0$ there are no arrivals yet, so $N(0)=0$. Now the half-line $[0,\infty)$, which represents time, will be divided into tiny subintervals of length δ where the kth interval is $((k-1)\delta,k\delta]$. Assuming that in each time slot, a coin is tossed for which $P(H)=p=\lambda\delta$. If the coin lands head up, an arrival materialized in that subinterval. Otherwise, no arrival materialized in that interval. An arrival

[24] Cont, R., & Jessen, C. (2009) [6]

at time $t=k\delta$ will happen if the kth coin flip results in a head. Now, $N(t)$ is the number of arrivals (number of heads) from time 0 to time t. There are $n \approx t\delta$ time slots in the interval $(0,t]$. At this point, it is possible to conclude that $N(t) \sim Binomial(n,p)$. In this case, p=λδ and therefore:

$$\mathrm{np}=\mathrm{n}\lambda\delta \rightarrow \mathrm{np}=\tfrac{t}{\delta}\lambda\delta \rightarrow np = \lambda t$$

Thus as $\delta \rightarrow 0$ this binomial distribution converges to a Poisson distribution with rate λt.

More generally, the number of arrivals in any interval of length τ follows a Poisson(λτ) as δ→0. Mathematically this process can be explained as follows: Considering a binomial random variable B with an expected value λ, the probability of B being equal to a parameter k is as follows:

$$P(B = k) = \frac{n!}{(n-k)!k!} * \left(\frac{\lambda}{n}\right)^{k} * \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

Taking the limit as $\mathrm{n} \rightarrow \infty$

$$\lim_{n\to\infty} \frac{n!}{(n-k)!k!} * \left(\frac{\lambda}{n}\right)^{k} * \left(1 - \frac{\lambda}{n}\right)^{n-k}$$
$$= \lim_{n\to\infty} \frac{n!}{(n-k)!k!} * \left(\frac{\lambda}{n}\right)^{k} * \left(1 - \frac{\lambda}{n}\right)^{n} * \left(1 - \frac{\lambda}{n}\right)^{-k}$$
$$= \lim_{n\to\infty} \frac{n!}{(n-k)!n^{k}} * \left(\frac{\lambda^{k}}{k!}\right) * \left(1 - \frac{\lambda}{n}\right)^{n} * \left(1 - \frac{\lambda}{n}\right)^{-k}$$
$$= \frac{\lambda^{k}}{k!} * e^{-\lambda}$$

which shows that the limit distribution is a Poisson with parameter lambda. At this point, it is also reasonably easy to compute both the expected value and the variance of the Poisson distribution which are both equal to λ.

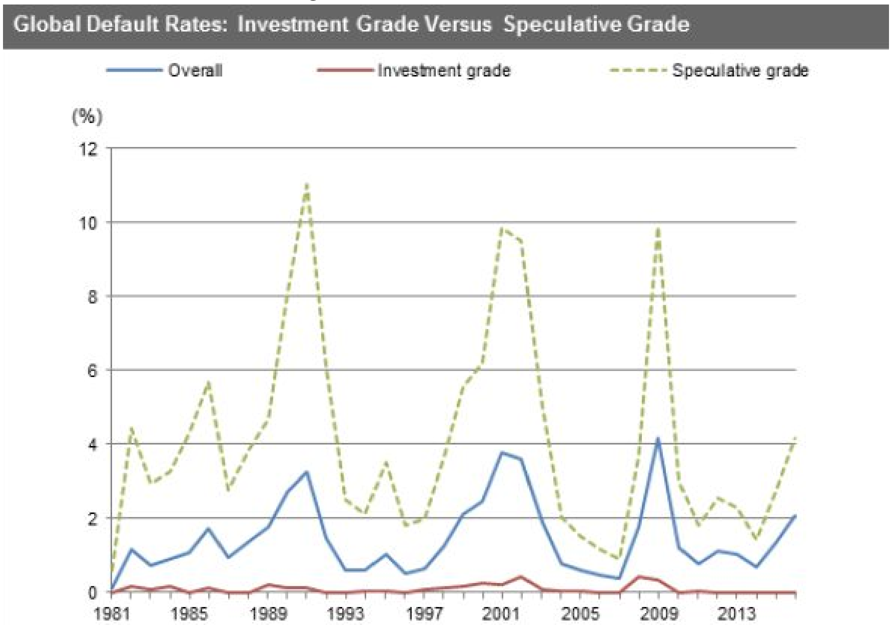
In the model that will be presented later on a Poisson distribution will be adopted, the historical data will be considered as the result of such distribution itself. It will be then possible to move backward from the results of a Poisson

Distribution to the expected outcome in order to establish a λ that represents the expected value of defaults events; this latter finding will be then employed in order to assess the expected number of defaults at time t, (N_t).

3.3 Definition of the Expected default function

Once that the concept of Poisson Process and its employment in this research has been explained it is now time to put it in practice and find the value of the expected default intensity. Firstly it is necessary to access and analyze market information about historical global corporate defaults in order to obtain the raw data needed to the Poisson Process. Figure 2 and 3 are explicative of the overall trends on corporate names defaults over time:

Figure 2: Default rates since 1981

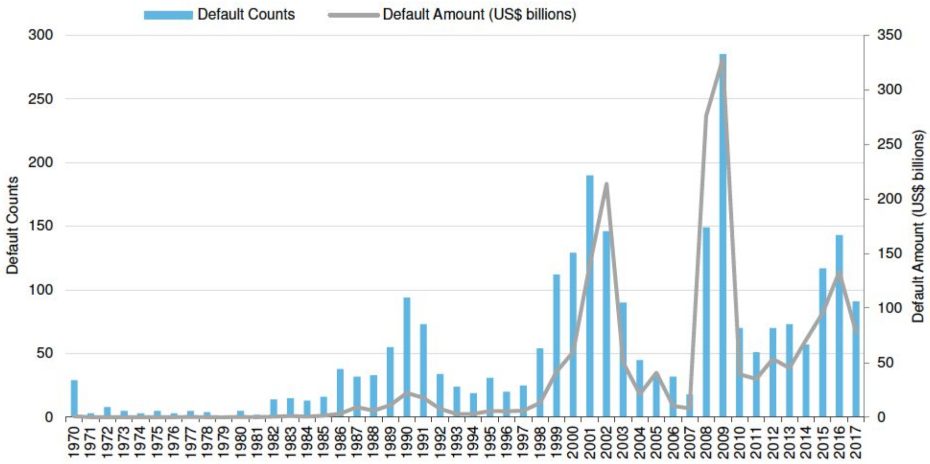


Source: S&P Global

It is clear from this analysis conducted by S&P Global[25] what the overall trends for default rates both in investment grade and speculative grade names are. Moreover, it gives a clear picture of how the market evolved over time and reacted to crisis periods.

[25] Vazza, D., Kraemer, N. W., Richhariya, N. M., Bhalla, P., Debnath, A., Gopinathan, P., & Dohadwala, A. (2017) [24]

Figure 3: Default number Vs. default amount in dollars



Source: Moody's Investor Service

Thanks to the information provided by Moody's investor service[26] it is possible to assess how default counts and default losses are correlated, as well as the amount of losses occurred during each year, given the number of default in corporate names.

Analyzing documented defaults occurrence over time it is crucial for the calculation of λ and in order to assess the number of defaults for each year, for this reason, it is presented table 1, which displays the historical evidence that will be at the base for the calculations in the model.

In order to find λ, this data set will be taught as a realization of the results of a Poisson distribution $X \sim Poisson(\lambda)$. This means that the number of defaults that occurred historically is assumed to follow such distribution and their occurrence are all independent of each other. Now recalling section 3.2 the expected value of $X \sim Poisson(\lambda)$ is equal to λ; therefore it is possible to calculate the expected value of the default intensity by:

$$\lambda = \frac{total\ \#\ of\ defaults}{\#\ of\ years} = \frac{2576}{36} = 71.56$$

[26] Ou, S., Irfan, S., Liu, Y., Jiang, J., & Kanthan, K. (2018) [18]

Table 1: Corporate Defaults Over Time

year	total defaults	Inv. Grade defaults	Spec. grade defaults	Default rate
1981	2	0	2	0.14
1982	18	2	15	0.19
1983	12	1	20	0.76
1984	14	2	12	0.91
1985	19	0	18	1.11
1986	34	2	30	1.72
1987	19	0	19	0.94
1988	32	0	29	1.38
1989	44	3	35	1.77
1990	70	2	56	2.73
1991	93	2	65	3.25
1992	39	0	32	1.49
1993	26	0	14	0.60
1994	21	1	15	0.63
1995	35	1	29	1.05
1996	20	0	16	0.51
1997	23	2	20	0.63
1998	56	4	48	1.28
1999	109	5	92	2.14
2000	136	7	109	2.48
2001	229	7	173	3.78
2002	226	13	159	3.59
2003	119	3	89	1.92
2004	56	1	38	0.78
2005	40	1	31	0.60
2006	30	0	26	0.48
2007	24	0	21	0.37
2008	127	14	89	1.80
2009	268	11	224	4.18
2010	83	0	64	1.20
2011	53	1	44	0.80
2012	83	0	66	1.14
2013	81	0	64	1.06
2014	60	0	45	0.69
2015	113	0	94	1.36
2016	162	0	143	2.06
Average	71.6	2.4	56.8	1.4

Source: S&P Global

This calculation, however, is not completely significant since CPDOs only invests in ITRAXX and DJ CDX where all underlying names are rated as investment grade, and above the data set was composed of names with different ratings. Therefore it would be more accurate to conduct the same analysis only on investment grade names defaults; the results are shown below:

$$\lambda = \frac{total\ \#\ of\ defaults}{\#\ of\ years} = \frac{85}{36} = 2.36$$

As stated before this λ represents the expected value of default occurrence per unit time, in this case, one year. Since a model that is able to quantify N_t is needed, that is the number of defaults in the CPDO's "*investment basket*" at time t, it is necessary to add a time measure to λ. This can be easily tackled by multiplying λ times the number of years passed from the inception of the CPDO. In this case, since N_t is the variable that represents the number of arrivals at time $t = \{0, 1, 2, 3, ..., T\}$, and if this follows a Poisson Process, as assumed in this thesis, the mean of the arrivals can be expressed by $\lambda * t$, hence:

$$E[N_t] = \lambda * t$$

where t represents the number of years since the inception of the CPDO

In conclusion, this section established a new model in order to estimate the expected value of defaults in the underlying index names (λ) following a Poisson Process. This will be utilized as the default intensity of investment grade names under ITRAXX and DJ CDX per year. Furthermore, it has been established a modus operandi in order to assess the number of default events occurred in a time-space and this model will be employed in order to appraise the loss given default and the cumulative discount losses in a specific CPDO case.

3.4 Loss given default and cumulative discount losses

The default intensity model based on constant λ proposed in precedence can now be employed in order to estimate both the loss given default and the cumulative discount losses that a CPDO can potentially face. In primis, there is a need to define the two different types of losses. Loss Given Default (LGD) is defined as the amount of money a lender/investor loses when the counterparty defaults and it is not able to fulfill its obligation. In the case of CPDOs, loss given default occurs when one or more swap contracts defect. Mathematically speaking LGD is given by the following formula:

$$L_t = \frac{(1-R)}{N^I} * N_t$$

Where R is the recovery rate that it is assumed constant (0.4) across all names in the underlying index, N_t is the number of defaults until time t an N^I is the number of names in the underlying index.

While calculating LGD appears fairly simple, it is far more complicated to define and calculate the cumulative discount losses (CDL) of the CPDO. It is possible to define them as the stream of payments that the CPDO manager needs to approve in order to cover all portfolio losses as they occur during time. Calculating them is not easy, as they take into account multiple factors arising from the cash flow structure of these obligations, but through some detailed analysis it is possible to delineate an equation:

$$CDL = \frac{(1-R)}{N^I} * (B(t, T^I) * E^Q[N_{T^I}|F_t] - E[N_t] + \int_t^{T^I} R(t,s) * B(t,s) * E^Q[N_s|F_t]\, ds)$$

Where B represents the rate of a zero coupon bond with flat term structure, R is the spot yield curve and F_t serves as the market history up to time t.

Crucial in the calculation of these two types of losses is the default intensity. The model presented in section 3.3 will be integrated into this analysis and will be the determinant of N_t and N_{T^I}. Usually, a CPDO strategy is based on 250 different names in both ITRAXX and DJ CDX indices, so $N_I = 250$.

As presented above, it is possible to find the expected value of the number of defaults occurred until time t, assuming that N_t follows a Poisson Distribution by:

$$E[N_t] = \lambda * t$$

where **t** represents the number of years since the inception of the CPDO

At this stage, it is now possible to quickly solve this mathematical problem in order to find both the LGD and CDL of a CPDO at a given time t. In the following few lines, an example will be provided taking into account historical data and a CPDO with initiation in 2006 and maturity 2016, where t = 2013. The estimation of the default events having opted for the simpler model delineated in this thesis lead to the following results (Table 2, Table 3):

Table 2: Maturity of the CPDO Vs. expected number of defaults

year	Expected defaults	Total defaults
2006	2.36	2.36
2007	2.36	4.72
2008	2.36	7.08
2009	2.36	9.44
2010	2.36	11.81
2011	2.36	14.17
2012	2.36	16.53
2013	2.36	18.89
2014	2.36	21.25
2015	2.36	23.61
2016	2.36	25.97

Table 3: Average of underlying names defaults for a CPDO with maturity 10 yrs and inception in 2006

Total expected defaults during CPDO lifetime	Total expected defaults between 2006 and 2013
26	19
Percentage of defaults over CPDO lifetime	**Percentage of defaults between 2006-2013**
10%	8%

Once that $E[N_t] = 19$ has been found, and with the assumption that $E[N_{T^I}] = E[N_T] = 26$, the attention will shift towards the calculation of the loss given default of this specific obligation following the formula presented earlier (Appendix 1). Nonetheless, there is the need to make another assumption about R the recovery rate, which is considered to be 0.4 as presented in the paper by Cont and Jessen. It follows the calculation of the LGD:

$$L_{2013} = \frac{(1-R)}{N^I} * N_{2013} = \frac{(1-0.4)}{250} * 19 = 4.53\%$$

$$L_{2016} = \frac{(1-R)}{N^I} * N_{2016} = \frac{(1-0.4)}{250} * 26 = 6.23\%$$

Regarding the CDL, calculations are a little more complex and a wider set of data is needed in order to complete the task. Historical data is analyzed and utilized, and the zero coupon bond rate, as well as the spot term structures that correspond to one of the years between 2006-2016, are employed. Recalling the formula presented above in order to calculate the Cumulative Discount Losses:

$$CDL = \frac{(1-R)}{N^I} * (B(t, T^I) * E^Q[N_{T^I}|F_t] - E[N_t] + \int_t^{T^I} R(t,s) * \ B(t,s) * E^Q[N_s|F_t]\, ds)$$

now there is the need to focus on the last part of this equation; in fact, it is possible to rewrite and simplify $E^Q[N_s|F_t]$:

$$E^Q[N_s|F_t] \ = \ E[(N_s - N_t) + N_t|F_t] \ \ with\ s > t\ and\ N_t\ known\ at\ time\ t$$

Defining N as stochastic process I.I.D, that is a family of casual variables that depend upon a parameter t (time), it is possible to state that ΔN is independent of F_t (N_t included) and that it has a distribution B ($\lambda\,(s-t)$) . Reporting this on the equation:

$$E^Q[N_s|F_t] \ = \ E[(N_s - N_t) + N_t|F_t] \ = E[(N_s - N_t)|F_t] + N_t =$$
$$since\ N_t\ is\ independent\ of\ F_t$$

$$= E[(N_s - N_t)|F_t] + N_t = E[N_s - N_t] + N_t$$
$$N_t\ \ since\ \Delta N\ is\ independent\ of\ F_t$$

$$= E[N_s - N_t] + N_t = \lambda\,(s-t) + N_t\ \ with\ s > t$$

Returning now to the initial formula and substituting $E^Q[N_s|F_t]$ with $\lambda(s-t) + N_t$:

$$CDL = \frac{(1-R)}{N^I} * (B(t, T^I) * E^Q[N_{T^I}|F_t] - E[N_t] + \int_t^{T^I} R(t,s) * \ B(t,s) * (\lambda\,(s-t) + E[N_t])\, ds)$$

$$CDL_{2013} = \frac{(1-R)}{N^I} * (B(2013, 2016) * E^Q[N_{2016}|F_t] - E[N_{2013}] +$$
$$+ \int_t^{T^I} R(2013, s) * \ B(2013, s) * (\lambda\ (s - 2013) + E[N_{2013}])\ ds) =$$

$$CDL_{2013} = \frac{(1-0.4)}{250} * (B(2013, 2016) * E^Q[N_{T^I}|F_t] - 19 +$$
$$+ \int_t^{T^I} R(2013, s) * \ B(2013, s) * (2.36\ (s - 2013) + 19)\ ds) =$$

$$CDL_{2013} = \frac{(1-0.4)}{250} * (B(2013, 2016) * E^Q[N_{T^I}|F_t] - 19 +$$
$$+ \int_{2013}^{2014} R(2013, 2014) * \ B(2013, 2014) * (2.36\ (2014 - 2013) + 19)\ ds +$$
$$+ \int_{2014}^{2015} R(2013, 2015) * \ B(2013, 2015) * (2.36\ (2015 - 2013) + 19)\ ds +$$
$$+ \int_{2015}^{2016} R(2013, 2016) * \ B(2013, 2016) * (2.36\ (2016 - 2013) + 19)\ ds) =$$

Substituting the spot term structure R and the zero coupon yield B with the historical data provided respectively in appendix 3,4 and 5,6 (taking into consideration the U.S market) it is now possible to evaluate the cumulative discount losses of this specific investment instrument:

$$CDL_{2013} = \frac{(1-0.4)}{250} * (e^{-(4.468\%)} * 26 - 19 +$$
$$+ \int_{2013}^{2014} e^{-(0.137\%)} * e^{-(0.284\%)} * (2.36\ (2014 - 2013) + 19)\ ds +$$
$$+ \int_{2014}^{2015} e^{-(0.264\% \ * 2y)} * e^{-(0.284\% * 2y)} * (2.36\ (2015 - 2013) + 19)\ ds +$$
$$+ \int_{2015}^{2016} e^{-(0.389\% * 3y)} * e^{-(0.407\% * 3y)} * (2.36\ (2016 - 2013) + 19)\ ds) =$$
$$= 0.18254 \ \approx 18.25\%$$

The results are significative, in a stressed market situation corresponding to the one between 2006 and 2016 and with analysis at $t = 2013$ this investment product will have a loss given default of 4.53% and cumulative discount loss of 18.25%. There is the need to specify that this model does not take into account two factors for CPDOs. *In primis* roll dates and hence the CPDOs feature of changing the underlying index names every six months leading to a possible decrease in the number of defaulting names is not taken into account. *In secundis*, the self-affecting mechanism of defaults of the names in the indexes is not taken into account as well, in fact, in this thesis has been used a constant default mean, in this way the defaults are distributed equally during the years and are independent of each other. In order to compare the above results with the findings of Cont and Jessen Cont, R., & Jessen, C. (2009) [6][27], it will be provided an analysis of LGD and CDL also for the timeframe 2003-2013 with $t = 2010$ so that it is possible to analyze the losses arising from a CPDO also in a market not stressed at inception.

Regarding the loss given default (LGD), since the defaults model presented above utilizes a constant λ, the number of defaults until time t (which for both situation was 7 years from inception) will be the same, as will the default until maturity, assuming the same recovery rate $R = 0.4$, the results will not change (Appendix 2):

$$L_{2010} = \frac{(1-R)}{N^I} * N_{2010} = \frac{(1-0.4)}{250} * 19 = 4.53\%$$
$$L_{2013} = \frac{(1-R)}{N^I} * N_{2013} = \frac{(1-0.4)}{250} * 26 = 6.23\%$$

Regarding cumulative discount losses the result will be different as it will be based on the different rates that the market presented in between 2003-2013 (Appendix 7-10) and 2006-2016 (appendix 3-6):

$$CDL = \frac{(1-R)}{N^I} * (B(t, T^I) * E^Q[N_{T^I}|F_t] - E[N_t] + \int_t^{T^I} R(t,s) * \ B(t,s) * (\lambda\ (s-t) + E[N_t])\ ds) =$$

[27] Cont, R., & Jessen, C. (2009) [6]

$$= CDL_{2010} = \frac{(1-0.4)}{250} * (e^{-4.470\%} * 26 - 19 +$$

$$+ \int_{2010}^{2011} e^{-(0.405\%)} * e^{-(0.385\%)} * (2.36\,(2011-2010) + 19)\, ds +$$

$$+ \int_{2011}^{2012} e^{-(1.068\% \ * 2y)} * e^{-(0.697\% * 2y)} * (2.36\,(2012-2010) + 19)\, ds +$$

$$+ \int_{2012}^{2013} e^{-(1.623\% * 3y)} * e^{-(1.560\% * 3y)} * (2.36\,(2013-2010) + 19)\, ds) =$$

$$= 0.17678 \ \approx 17.68\%$$

This further calculation will allow to compare the results of the cumulative discount losses of a CPDO in two different market settings, one stressed at inception (2006-2016) and one not stressed at inception (2003-2013). Although the difference is minimal, the CDL that arose from the stressed market setting are higher than the one from the non-stressed market setting. these results are in line with the findings of [6], despite the fact that the two default models are significantly different, and their model is taking into account the clustering of defaults and self-affecting defaults while the one presented above has a constant expected value every year. In fact, in both this thesis and [6] it is possible to delineate how CPDO responds to situations in which the market at inception is not stressed, leading to lower cumulative discount losses and, according to Cont and Jessen[28], a lower expected shortfall $ES_{0.99} = 6\%$ compared to the one of a stressed market of 10.5%. The average cash-in time when CPDOs do not defaults is 5.1 years for the non-stressed market and 5 years for the stressed market with a default rate of respectively 1.8% and 1.2%.

3.5 Expected shortfall analysis

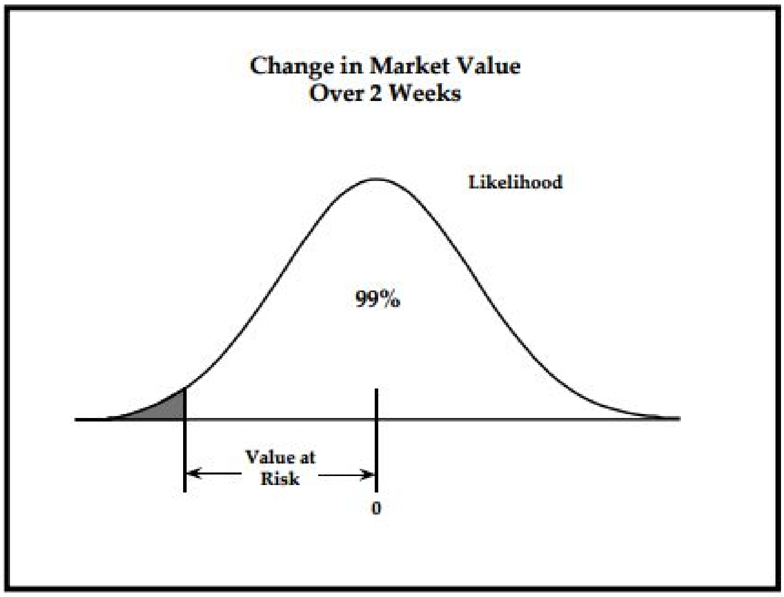
After having depicted the model of default intensity and the losses that could arise from defaults and from the stream of payments necessary to cover portfolio losses as they occur, in this thesis the concept of expected shortfall will be expressed. Furthermore, the calculations and results that arose from [6] will be employed in order to further analyze the behavior of these investments

[28] Cont, R., & Jessen, C. (2009) [6]

instruments. First of all, it is essential to define what an expected shortfall is. In order to define it, it is of prime importance to further assess the meaning and the definition of Value at Risk (VaR). The VaR measures the potential loss in value of a risky asset, or a portfolio, over a defined period of time for a given confidence interval α. Statistically speaking the VaR measure is the critical value, given a certain confidence level, of the probability distribution of changes in the market value[29] (Figure 3). Now it is possible to define the expected shortfall (ES) as the statistic used to quantify the risk of a portfolio. Given a confidence level α, this measure represents the expected loss when this is greater than the VaR calculated with the same confidence level α. In other words, it is the average value of all the values exceeding a specific threshold, the VaR (Figure 4).

This measure is used to assess what the expected losses could be in case the investment go bad[30].

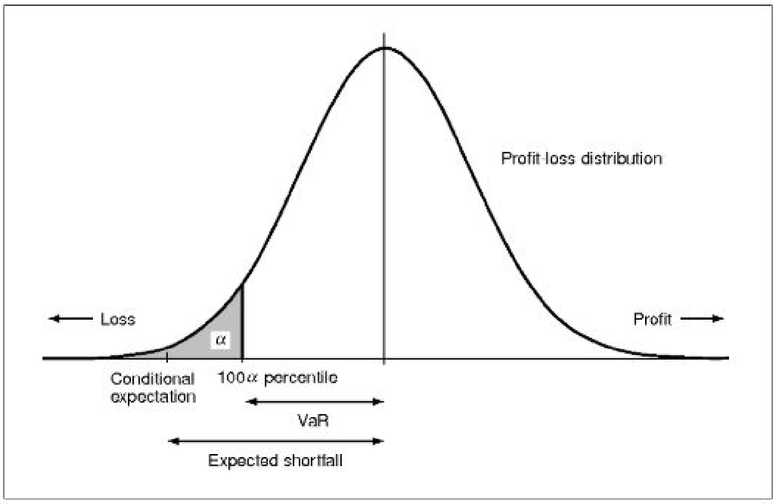
Figure 4: Graphical representation of VaR with gaussian loss



Source: An overview of Value At risk (D. Duffie and J. Pan)

[29] Duffie, D., & Pan, J. (1997, January 21) [8]
[30] Sanchez, P. (2018, September 26) [21]

Figure 5: Graphical representation of ES



Source: Semanticscholar.org

It is possible to represent this definition with a mathematical notation:

$$ES_\alpha = E[L|L > VaR_\alpha] = \int_\alpha^1 \frac{1}{1-\alpha} Var\, dp$$

$$\text{Where } VaR_\alpha = inf\ \{l|P(L > l)\ <\ (1 - \alpha)\}$$

In this case, L represents the loss process, l is an arbitrary threshold and α is the intended confidence level.

It is now feasible, after having defined these two main elements, to evaluate and explain the results of the expected shortfall analysis conducted by Cont and Jessen. Their results are based on 10 000 Monte Carlo simulations of the behavior of CPDO in order to asses default and cash out probabilities as well as loss distribution and expected shortfall. It is the results of the latter that will be analyzed in the following lines.

Through their simulations the two researchers found the probability of a CPDO defaulting when markets are not stressed at inception, to be 1.8%, the loss given default to be 3.5% of the notional and generally, the instrument is found to cash in after 5.1 years with a probability of cashing out of 0.04%. Moreover, the results of their expected shortfall analysis suggested that $ES_{0.99}$ = 6.0% of note notional, this means that investors expect to lose less than 10% in the worst 1%

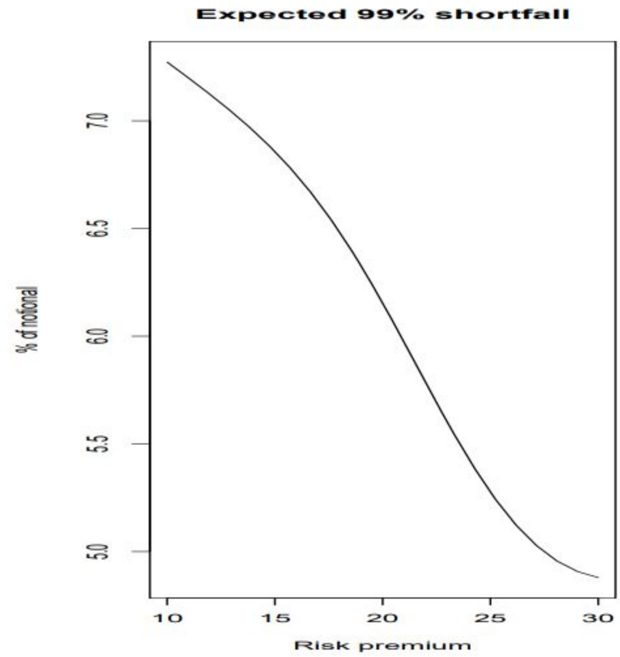
of the scenarios. It is possible now to compare these results with the results arising from the case in which markets are stressed at inception. In this case, the probability of default of a CPDO is found to be 1.2%, the loss given default 9.0% and usually, the instrument cashes in after 5.0 years with a probability of cashing out of 0.10%. The expected shortfall in this setting is $ES_{0.99}$ = 10.5% so in the worst 1% of the scenarios investors will lose a little above 10% of the note notional.

Comparing the expected shortfalls in the two different market settings, it is clear how in the case when the market is stressed at inception the worst 1% of the scenarios lead to almost double the loss of the case in which the market is not stressed. This is possibly due to the relation with risk premium ϑ since it determines the average level of spread income relative to the credit losses incurred[31]. It is possible to express the expected 99% shortfall a function of the risk premium ϑ and this will result in a downward sloping curve (Figure 5).

Moreover, another possible determinant of spread widening, and perhaps the more significant in the model presented by Cont and Jessen, is the mean reversion speed parameter κ. With a higher mean reversion speed, the index spread fluctuates more tightly around its long term mean level, which reduces the mark-to-market losses. Therefore a higher κ reduces the probability of default and leads to a lower expected shortfall[32]. It is possible to understand the relationship between expected shortfall and κ in Figure 6, where the ES is expressed as a function of the mean reversion parameter. In general to conclude this section about expected shortfall it is possible to state that the average numbers of defaults in the underlying portfolio affect the performance of the CPDO less significantly that the spread widening which leads to mark-to-market losses. Although the volatility of the default intensity does affect the volatility of the spread less significantly than other variables (such as the mean reversion speed κ) it is shown that higher volatility is harmful to the CPDO both in terms of expected shortfall and cash out probability.

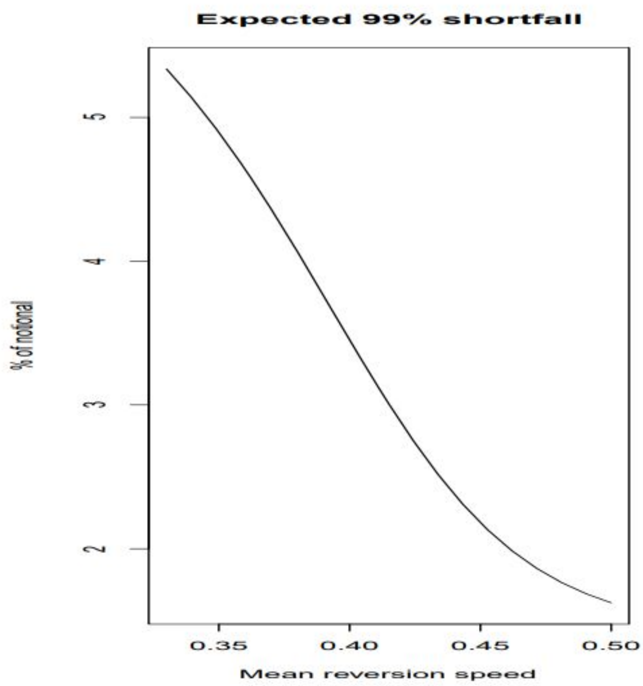
[31] Cont, R., & Jessen, C. (2009) [6]
[32] Cont, R., & Jessen, C. (2009) [6]

Figure 5: Dependence of $ES_{0.99}$ on risk premium ϑ



Source: Cont, R., & Jessen, C. (2009)

Figure 6: Dependence of CPDO $ES_{0.99}$ on mean reversion speed κ



Source: Cont, R., & Jessen, C. (2009)

4. Conclusions

This thesis analyzed CPDOs in detail, in particular, it presented an analysis of the leverage strategy, which is piecewise constant and increases when losses occur while decreases when gains occur. Moreover, a detailed description of the cash flow structure of such instruments was provided, it is found that this mainly depends on five factors that are: interest payment, coupon payments, spread income default loss given a certain recovery rate and liquidation of swap contracts. Another important element of the structure of CPDOs are the risk factor. It is possible to classify them in: spread risk as the primary determinant of the CPDO cash flow is the index default swap spread, default risk as it determines the average number of defaults during the lifetime of the investment instrument, interest risk since the term structure of interest rate affects the cash flows and liquidity risk of the index default swap.

After having portrayed the structure of CPDOs, the thesis focus shifted toward a mathematical analysis of such an instrument. An alternative model for the estimation of default risk and intensity was proposed in order to carry out some calculation of losses and expected defaults, with the aim of comparing the results with the ones that arose from the cumbersome and mathematically challenging model proposed by Cont and Jessen. This model is based on the concept of Poisson Distribution, and it has been estimated, through a careful analysis of historical data, the expected default of names in the underlying swap indexes as ITRAXX and DJ CDX to be 2.36 per year. This result is further employed in the thesis as an analysis of loss given default and cumulative discount losses of a CPDO both in case of non-stressed market condition (2003-2013) and stressed market condition (2006-2016) at inception was provided. The analysis suggested that in both cases the loss given default was 4.53% for $t = 7$ years from inception and 6.23% for the whole maturity of the CPDO. The cumulative discount losses differed between each other but only by a few figures (18.25% in the stressed case and 17.68% in the non-stressed setting). This difference is minimal as in this thesis's model the default intensity is constant over time and for this reason the only difference between the two stages of the markets arises from differences in interest rates, while default numbers stay constant. Lastly, the results on expected shortfall presented by Cont and Jessen have been explained and analyzed; it is found that expected shortfall depends on ϑ and κ which are respectively the risk premium and the mean reversion speed. The results of the ES for non-stressed market and stressed market change significantly with the latter being almost double the former (10.5% and 6%).

The thesis was designed to explain how such a sophisticated investment instrument works and to implement some calculations on the expected losses and defaults based on [6]. In order to do so, as stated before, a new model on default intensity was proposed. This model, of course, cannot be of the same accuracy of the one suggested by Cont and Jessen, but it is a reasonable estimate given the historical data in possession. The results of the analysis conducted with this model differ from the ones of the two researchers (due to the difference in the number of variables and complexity of the model) but leads to similar conclusions. A CPDO behaves differently in stressed and non-stressed markets. In the first case the probability of default is less than in the second one but the expected shortfall is almost double, the cumulative discount losses are greater in situations with stressed markets that in those with non-stressed markets and this is in line with the findings of Cont and Jessen. Moreover, a CPDO generally cashes in after 5 years in stressed market and 5.1 in non-stressed with a probability of cashing out respectively of 0.1% and 0.04%.

In conclusion, the model presented might be simple and not consider all the variables taken into account by Cont and Jessen but it is a reasonable estimation in order to conduct the mathematical analysis presented above, without prejudicing the results that, in fact, are in line with those of the two researchers.

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Appendix

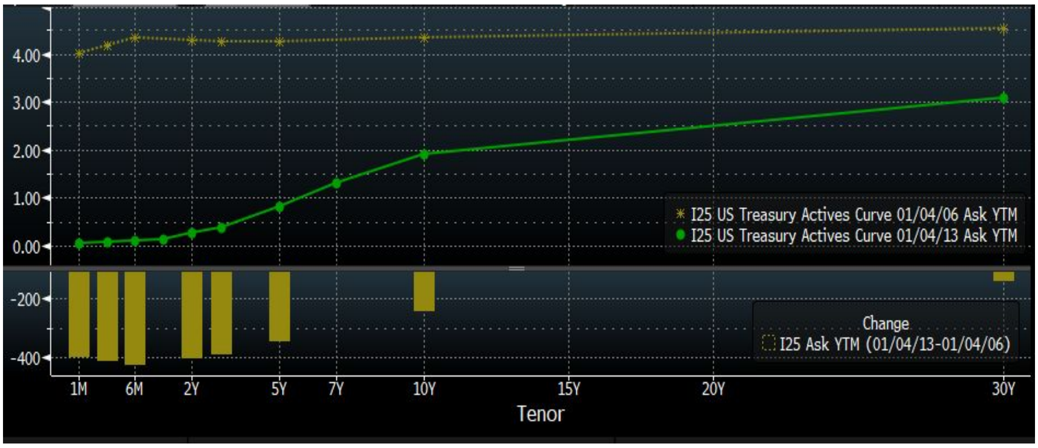
Appendix 1: LGD of a CPDO with initiation in 2006 and maturity 2016, t = 2013

	2006-2013	2006-2016
Recovery Rate	0.4	0.4
# of names	250	250
# of defaults	19	26
LGD	4.53%	6.23%

Appendix 2: LGD of a CPDO with initiation in 2003 and maturity 2013, t = 2010

	2003-2010	2003-20113
Recovery Rate	0.4	0.4
# of names	250	250
# of defaults	19	26
LGD	4.53%	6.23%

Appendix 3: Spot yield curve of U.S. Treasury securities as 04/01/2006 and 04/01/2013



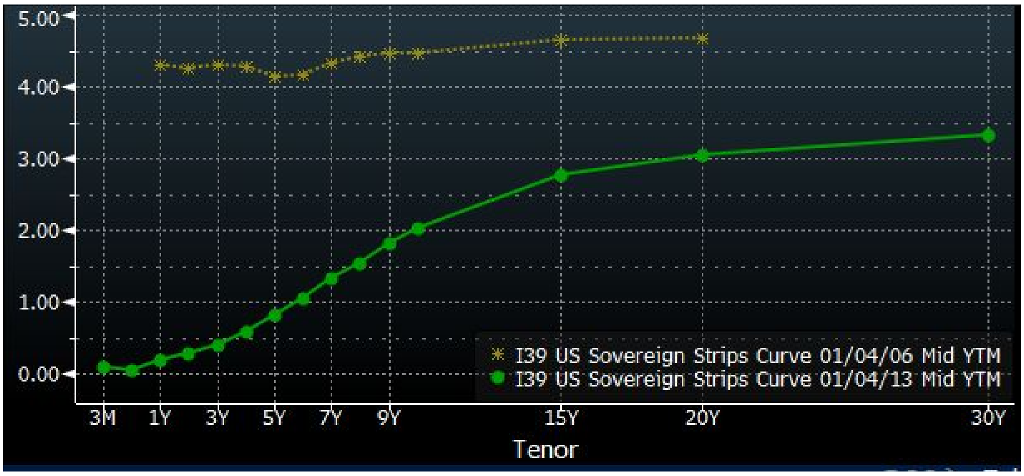
Source: Bloomberg

Appendix 4: Spot yield curve table of U.S. Treasury securities as 04/01/2006 and 04/01/2013

	Tenor	I25 Ask YTM US Treasury Actives Curve 01/04/13 Description	Yield	I25 Ask YTM US Treasury Actives Curve 01/04/06 Description	Yield	I25 Ask YTM (Change) 01/04/13-01/04/06 Yield
11)	1M	GBM Govt	0.051	Same	4.024	-397.4
12)	3M	GB3 Govt	0.066	Same	4.180	-411.4
13)	6M	GB6 Govt	0.107	Same	4.351	-424.4
14)	1Y	GB1 Govt	0.137			
15)	2Y	GT2 Govt	0.264	Same	4.300	-403.7
16)	3Y	GT3 Govt	0.389	Same	4.268	-387.8
17)	5Y	GT5 Govt	0.808	Same	4.275	-346.8
18)	7Y	GT7 Govt	1.304			
19)	10Y	GT10 Govt	1.899	Same	4.342	-244.3
20)	30Y	GT30 Govt	3.097	Same	4.537	-143.9

Source: Bloomberg

Appendix 5: Spot yield curve of U.S. ZCB as 04/01/2006 and 04/01/2013



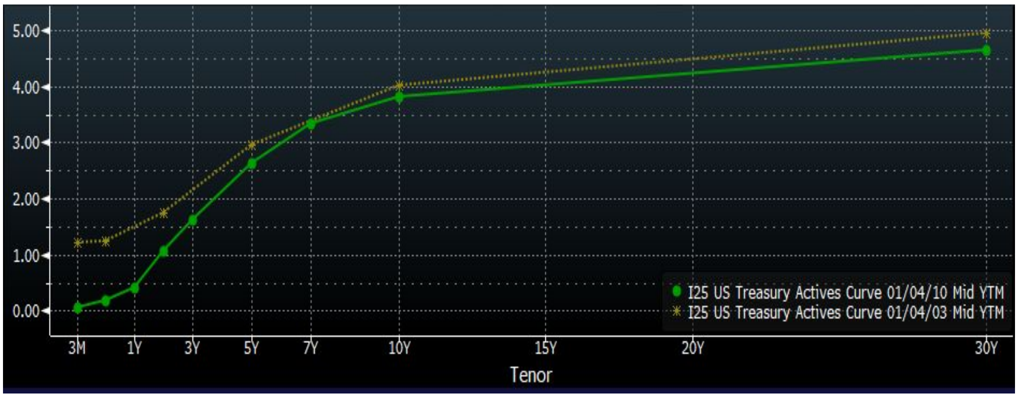
Source: Bloomberg

Appendix 6: Spot yield curve table of U.S. ZCB as 04/01/2006 and 04/01/2013

	Tenor	I39 Mid YTM US Sovereign Strips Curve 01/04/13 Description	Price	Yield	I39 Mid YTM US Sovereign Strips Curve 01/04/06 Description	Price	Yield	I39 Mid YTM (Change) 01/04/13-01/04/06 Price	Yield
11)	3M	S 0 02/15/13 Govt	99.991	0.087					
12)	6M	S 0 05/15/13 Govt	99.982	0.050					
13)	1Y	S 0 11/15/13 Govt	99.843	0.184	S 0 11/15/06 Govt	96.399	4.315	3.444	-413.1
14)	2Y	S 0 11/15/14 Govt	99.475	0.284	S 0 11/15/07 Govt	92.434	4.277	7.041	-399.3
15)	3Y	S 0 11/15/15 Govt	98.846	0.407	S 0 11/15/08 Govt	88.534	4.305	10.312	-389.8
16)	4Y	S 0 11/15/16 Govt	97.729	0.597	S 0 11/15/09 Govt	84.868	4.297	12.861	-370.0
17)	5Y	S 0 11/15/17 Govt	96.069	0.828	S 0 11/15/10 Govt	81.950	4.139	14.119	-331.1
18)	6Y	S 0 11/15/18 Govt	94.109	1.040	S 0 11/15/11 Govt	78.498	4.175	15.611	-313.5
19)	7Y	S 0 11/15/19 Govt	91.353	1.324	S 0 11/15/12 Govt	74.455	4.347	16.897	-302.3
20)	8Y	S 0 11/15/20 Govt	88.573	1.551	S 0 11/15/13 Govt	70.889	4.426	17.684	-287.5
21)	9Y	S 0 11/15/21 Govt	85.240	1.812	S 0 11/15/14 Govt	67.560	4.476	17.680	-266.4
22)	10Y	S 0 11/15/22 Govt	81.997	2.025	S 0 11/15/15 Govt	64.684	4.468	17.313	-244.3
23)	15Y	S 0 11/15/27 Govt	66.355	2.781	S 0 11/15/20 Govt	50.398	4.665	15.956	-188.5
24)	20Y	S 0 11/15/32 Govt	54.775	3.055	S 0 11/15/25 Govt	39.865	4.685	14.910	-163.0
25)	30Y	S 0 11/15/42 Govt	37.145	3.345					

Source: Bloomberg

Appendix 7: Spot yield curve of U.S. Treasury securities as 04/01/2003 and 04/01/2010



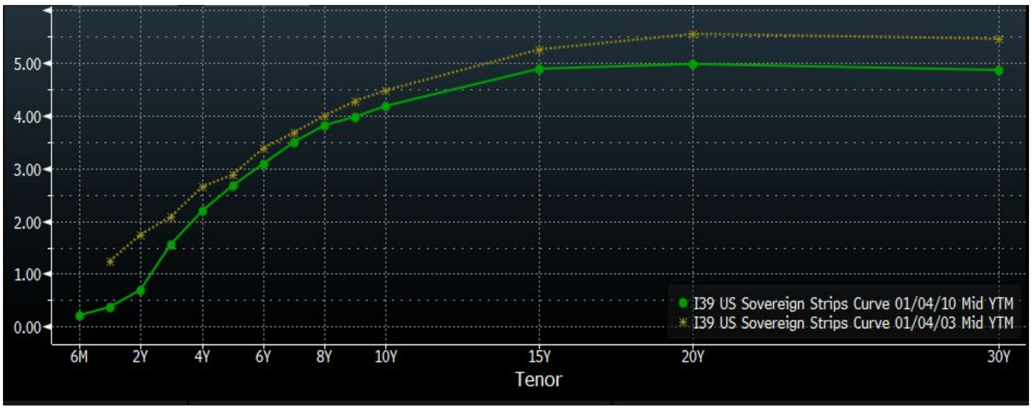
Source: Bloomberg

Appendix 8: Spot yield curve table of U.S. Treasury securities as 04/01/20033 and 04/01/2010

		I25 Mid YTM US Treasury Actives Curve 01/04/10			I25 Mid YTM US Treasury Actives Curve 01/03/03			I25 Mid YTM (Change) 01/04/10-01/03/03	
	Tenor	Description	Price	Yield	Description	Price	Yield	Price	Yield
11)	3M	B 0 04/01/10 Govt	0.059	0.060	B 0 04/03/03 Govt	1.195	1.215	-1.136	-115.5
12)	6M	B 0 07/01/10 Govt	0.172	0.175	B 0 07/03/03 Govt	1.225	1.250	-1.053	-107.5
13)	1Y	B 0 12/16/10 Govt	0.398	0.405					
14)	2Y	T 1 12/31/11 Govt	99-27¾	1.068	T 1 ¾ 12/31/04 Govt	99-31¾	1.754	-0-04	-68.6
15)	3Y	T 1 ⅛ 12/15/12 Govt	98-19¼	1.613					
16)	5Y	T 2 ⅝ 12/31/14 Govt	99-30¼	2.637	T 3 11/15/07 Govt	100-04¼	2.970	-0-06	-33.3
17)	7Y	T 3 ¼ 12/31/16 Govt	99-12¼	3.350					
18)	10Y	T 3 ⅜ 11/15/19 Govt	96-12+	3.817	T 4 11/15/12 Govt	99-27	4.019	-3-14+	-20.2
19)	30Y	T 4 ⅜ 11/15/39 Govt	95-21+	4.644	T 5 ⅜ 02/15/31 Govt	106-11	4.954	-10-21+	-31.0

Source: Bloomberg

Appendix 9: Spot yield curve of U.S. ZCB as 04/01/2003 and 04/01/2010



Source: Bloomberg

Appendix 10: Spot yield curve table of U.S. ZCB as 04/01/2003 and 04/01/2010

		I39 Mid YTM US Sovereign Strips Curve 01/04/10			I39 Mid YTM US Sovereign Strips Curve 01/03/03			I39 Mid YTM (Change) 01/04/10-01/03/03	
	Tenor	Description	Price	Yield	Description	Price	Yield	Price	Yield
11)	6M	S 0 06/30/10 Govt	99.896	0.215					
12)	1Y	S 0 11/15/10 Govt	99.670	0.385	S 0 11/15/03 Govt	98.947	1.240	0.723	-85.5
13)	2Y	S 0 11/15/11 Govt	98.715	0.697	S 0 11/15/04 Govt	96.817	1.750	1.898	-105.3
14)	3Y	S 0 11/15/12 Govt	95.654	1.560	S 0 11/15/05 Govt	94.239	2.088	1.415	-52.8
15)	4Y	S 0 11/15/13 Govt	91.889	2.204	S 0 11/15/06 Govt	90.312	2.660	1.577	-45.6
16)	5Y	S 0 11/15/14 Govt	87.904	2.671	S 0 11/15/07 Govt	87.043	2.878	0.861	-20.7
17)	6Y	S 0 11/15/15 Govt	83.565	3.088	S 0 11/15/08 Govt	82.154	3.385	1.411	-29.7
18)	7Y	S 0 11/15/16 Govt	78.832	3.498	S 0 11/15/09 Govt	77.905	3.675	0.927	-17.7
19)	8Y	S 0 11/15/17 Govt	74.367	3.804	S 0 11/15/10 Govt	73.345	3.985	1.022	-18.1
20)	9Y	S 0 11/15/18 Govt	70.591	3.970	S 0 11/15/11 Govt	68.833	4.262	1.759	-29.2
21)	10Y	S 0 11/15/19 Govt	66.558	4.172	S 0 11/15/12 Govt	64.679	4.470	1.879	-29.8
22)	15Y	S 0 11/15/24 Govt	48.857	4.879	S 0 11/15/17 Govt	46.326	5.247	2.531	-36.8
23)	20Y	S 0 11/15/29 Govt	37.700	4.973	S 0 11/15/22 Govt	33.801	5.538	3.899	-56.5
24)	30Y	S 0 11/15/39 Govt	23.820	4.863	S 0 08/15/30 Govt	22.631	5.455	1.189	-59.2

Source: Bloomberg