Department of Economics and Business

Chair Mathematical Finance

Pricing techniques for complete and incomplete markets

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Abstract

The current study was conducted in order to investigate different approaches to price European Options in Complete and Incomplete Market Models.

After the required Probability Fundamentals, the study introduces Stochastic processes and defines their main characteristics and, by the end of the paragraph it gives a first statement of Martingale processes. The following focus on Brownian motion is driven by the great importance they have with the models then discussed, in particular when facing the Black Scholes model.

From here the discussion moves towards the analysis of market models for the pricing of Options, firstly in Discrete-time, considering the Binomial tree model. Once the structure of the model is given, the concept of Arbitrage is presented. To state the completeness of the model two different approaches are used: Replicating strategy and Martingale measure.

The study goes on introducing the Trinomial tree model, which in contrast with the Binomial tree model features incompleteness, still addressed by both the previous approaches.

From Discrete-time models, then the study considers Continuous-time models and particularly deals with the famous Black-Scholes Model. The Black-Scholes Partial Differential Equation is derived through the Replicating strategy approach and considering the boundaries for an European Call Option a demonstration for the closed formulas is proposed. Furthermore, for an European Options without dividends, it shows a convergence of the Binomial tree model to Black-Scholes closed formulas.

In the end the study proposes a numerical example of a Monte Carlo Simulation, which is then tested with the Black-Scholes model.

1 Introduction

Options are a financial contract offering the buyer the right to buy or sell the underlying asset at a specified price, called strike price, on a specified date. An underlying options security can be a stock, an index, a currency or even another derivative and the value of the derivative, in our case the option, is based on the underlying, meaning that price movements of the underlying necessarily affects all the derivatives written on it.

The option which gives the investor the ability to buy the underlying asset at a certain price is called Call Option, while the one which instead gives the investor the possibility to sell the underlying asset at a certain price is called Put Option.

For example, a trader which expects the price of a stock to increase in the next future can hedge against this raise by entering in a Long Call Option with strike price equal to the spot market price. If the investor's expectation turns out to be right and the stock's price increases over the strike price agreed in the contract, then he would exercise his option and would receive a positive cash flow, exactly equal to the difference in prices. Furthermore, if the option contract is of an American-style the investor would be entitled to exercise his option at any time until the expiration date, while if it is of an European-style then it would only be possible for the investor to exercise his option on the exact expiration date.

Differently from a Forward contract, which is binding on the investor and has initial cost equal to 0, an Option contract gives the investor the right, not the obligation of exercise and in this sense, options always have an initial non-negative price.

The determination of the 'fair' initial price, also called premium, is exactly the aim of the models proposed in this study.

The main idea to avoid mispricing between the option's price and the underlying is that options cannot be priced arbitrarily, i.e. by computing the expected value of the discounted stochastic future payoff, but rather their values need to be determined in a way that is consistent with the market prices of the underlying.

2 An Introduction to the Stochastic Processes in Continuous time

2.1 Fundamentals

2.1.1 Probability Space

A Probability Space is defined as the set of $\{\Omega, \mathcal{F}, \mathbb{P}\}$, where:

- Ω is the set of all possible outcomes
- $\mathcal{F}$ the sigma-algebra on Ω
- $\mathbb{P}$ is the probability attached to each event.

Being the Probability Space the base of our discussion it is important to well define all its characteristics.

- The set of all possible outcomes Ω , in considering our case is the path space of the possible future prices.
- $\mathcal{F}$, the σ-algebra of the set Ω, is a collection of subsets having the following properties:

  1- Ø, the empty set, and Ω are contained in $\mathcal{F}$
  2- For a subset A in $\mathcal{F}$, then its complementary A^c is in $\mathcal{F}$
  3- If A_1, A_2, A_3 are in $\mathcal{F}$, then their union is in $\mathcal{F}$ (i.e. $A_1 \cap A_2$)

The Borel Set is the smallest possible σ-algebra of a set X, that is $\sigma(A_1)$

$$\sigma(A_1) = \{ \emptyset, X, A_1, A^c{}_1\}$$

If $\mathcal{F}$ is a σ-algebra of X, then $(X, \mathcal{F})$ is a measurable space.

- The probability $\mathbb{P}$ is a non-negative normalized measure which possess certain properties:

1- $\mathbb{P}$ of $\mathcal{F}$ = [0,1]
2- $\mathbb{P}(\Omega) = 1$ (normalization property)
3- $\mathbb{P}$ is countably additive: the probability of disjoints event in sequence is equal to the sum of each probability's event:

$$\mathbb{P}(\sqcup_n A_n) = \sum_n \mathbb{P}(A_n)$$

From the properties described:

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

where A^c denotes the complement $\Omega \setminus A$ of the event A, and that $\mathbb{P}(\emptyset) = 0$.

As the space $(X, \mathcal{F})$ is measurable, so the triplet $(\Omega, \mathcal{F}, \mathbb{P})$.

2.1.2 Random Variables

A Random Variable X on $(\Omega, \mathcal{F})$ is a function on Ω, which can take values in $\mathbb{R}$.

$$X : \Omega \rightarrow \mathbb{R}$$

and is $\mathcal{F}$-measurable, meaning that the counter-image of any half line $(-\infty, x]$ is an event :

$$\{X \leq x\} \in \mathcal{F}$$

for all $x \in \mathbb{R}$.

If $\mathcal{F}$ is too big, it is also possible to define a sub-sigma-algebra G of $\mathcal{F}$, a sigma algebra in its turn. Still if X verifies:

$$\{X \leq x\} \in G, \text{ for all } x \in \mathbb{R}$$

then X is G-measurable.

Also, the sigma-algebra generated by a random variable X is:

$$\sigma(X) := (\{X \leq x\} | x \in \mathbb{R})$$

which is a sub-sigma-algebra of $\mathcal{F}$, in its turn closed for countable unions of events and complement.

All the events which can be expressed in terms of X,

for example:

$$\{a \leq X \leq b\}$$

belong to $\sigma(X)$ and is the smallest space where X is measurable on Ω.

All the elements present in the starting set are mapped to an element of the final set.

2.1.3 Filtration

Denoted precisely as $\mathcal{F}_t^X$, it means the information generated by X in the interval $[0, t]$.
A Filtration on a set Ω is an increasing collection of sub-σ-algebras, $\mathcal{F}_t$ (where t ≥ 0) such that:

$$F_{t1} \subseteq F_{t2}, \qquad \text{for any } t_1 < t_2$$

t is a time parameter either discrete or continuous and for every couple $(t, A_{\in\mathcal{F}})$ it is possible to evaluate $\mathbb{P}(A)$.
Also, if Y is a stochastic process such that:

$$Y(t) = \mathcal{F}_t^X, \quad \text{for all } t \geq 0$$

then the process is adapted to the Filtration $\{\mathcal{F}_t^X\}_{t\geq 0}$ and it is possible to define a filtered probability space, the triplet $\{\Omega, \mathcal{F}_{(t\leq[0,T])}, \mathbb{P}\}$.

2.2 The Stochastic process

A stochastic process is defined as the path followed by variables changing values overtime. This can either be:

1- A discrete process in which the variables are evaluated in certain instant of time
2- A continuous process in which instead variables are evaluated over intervals of time

Also, it can have:

1- Discrete-variable: the variables can only have determined values
2- Continuous-variable: the variables can take a value over a range

Considering a Filtered Probability Space $\{\Omega, \mathcal{F}_{(t\leq[0,T])}, \mathbb{P}\}$, then a Stochastic Process $S = \big(S(t)\big)_t$ is a collection of measurable Random Variables from $\{\Omega, \mathcal{F}_t\}$ to $\mathbb{R}$.

In order for the process to be random and non-anticipative and to be $\mathcal{F}$-measurable, S has to be:

1- $S(t)_t : \Omega \rightarrow \mathbb{R}$, for any fixed time t.
2- For all fixed real x, the set $\{S(t) \leq x\}$ belongs to $\mathcal{F}_t$

Furthermore, the process S is a function of time t and oucome ω, $S = (t, \omega)$, but

1- For a fixed $t \to t^*$, the result is the random variable $S(t^*) = S(t^*, \cdot)$, or t-marginals.
2- For a fixed $x \to x^*$ the result is a deterministic function of time $S(\cdot, \omega) : [0, T] \to \mathbb{R}$.

2.2.1 Distribution

The Cumulative Distribution Function (c.d.f.) of the process S(t) is:

$$\mathbb{P}(S_t \leq x)$$

But in order to reconstruct the whole distribution of the process S, also the Joint Distribution functions of $S(t_1), S(t_2)$ need. In our adapted process:

$$\mathbb{P}(S(t_1) \leq x_1, S(t_2) \leq x_2)\ as\ the\ sets\ \{S(t_1) \leq x_1\} \cap \{S(t_2) \leq x_2\}$$

for any x_i belonging to $\mathcal{F}_{t_2} \subseteq \mathcal{F}_T = \mathcal{F}$. Thus, they are events and can be measured by $\mathbb{P}$.

2.2.2 Expectation

The Expected Value is the average value of a random variable X possible outcomes, weighted by their probability.

For a discrete random variable X :

$$\mathrm{E[X]} = \sum_{\mathrm{i}} \mathrm{x_i} \cdot \mathrm{p_i}$$

For a continuous random variable, with density $\mathcal{P}x$, then $\mathbb{P}\ (\ x < X\ \leq x + dx\ ) =\ \mathcal{P}x(x)dx$, such that:

$$E[X] =\ \int x p_X(x) dx$$

Expectation is a linear operation, hence the expectation of a linear combination of random variables is the linear combination of the expectations:

$$\mathrm{E}\ [aX + bY] = a\mathrm{E}[X] + bE[Y]$$

A consequence of the linearity is it is possible to know the expectation of a linear combination without knowing their joint distribution.

Note this doesn't hold for the computation of $\mathrm{E}[XY]$.

Consider a continuous random variable, X, and a deterministic function of such variable, Y:

$$Y = g(X)$$

If, for example, X is the price of a stock at time T and g represents the payoff function of this stock, then one way to compute the price of the option at time 0 is to calculate the expected value of function Y.
But in case that the function g is of a Bernulli type, Y would not have density. Nevertheless, if the function g is invertible and differentiable with $g' \neq 0$, the density of y is given by:

$$p_Y(y) = p_X(g^{-1}(y)) \frac{1}{|g'(g^{-1}(y))|}$$

Then the expected value of Y is:

$$\mathrm{E}(Y) = \int y p_Y(y) dy$$

and when g is regular and invertible, substituting back:

$$E(Y) = \int g(x) p_X(x) dx$$

This last formula can be always used because it is based on the density of X so, also if Y is just an indicator and has discrete values, as in Bernulli type, it gives us the expected value useful to price the option.

2.2.3 Independence

Two real random variables X, Y are independent if for any couple of intervals I_1, I_2 the probability of the intersection $X \in I_1$, $Y \in I_2$ factorizes into the product of the probabilities:

$$\mathbb{P}(X \in I_1, Y \in I_2) = \mathbb{P}(X \in I_1)\mathbb{P}(Y \in I_2)$$

Meaning that their joint density is the product of their marginal densities:

$$p_{(x,y)} = p_X(x) p_Y(y)$$

If the variables are independent, then:

$$\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y],$$

and thus, they are uncorrelated

$$\mathrm{E}\big(X - \mathrm{E}(X)\big)\big(Y - \mathrm{E}(Y)\big) = 0$$

Note that the opposite relationship uncorrelation, independence does not hold.

2.2.4 Conditional Expectation

The conditional expectation (c.e.) of a random variable X, is the average of the possible outcomes, conditional to certain information know at the time of the estimation.

In our case the sigma algebra $\mathcal{F}_t$ of a Filtered space $\{\Omega, \mathcal{F}_{(t\leq[0,T])}, \mathbb{P}\}$ represents the information accessible relative to a particular event at time t < T .

So, if we have a set of information $\mathcal{F}_{t1}$ and we are interested in finding the expectation of the process Y which value is known at t_2, the conditional expectation of Y at $t_1 < t_2$ is expressed as:

$$\mathrm{E}[Y|\mathcal{F}_{t1}]$$

The Trivial sigma algebra $\mathcal{F}_0$, when we don't have relevant information about our process Y is equal to:

$$\mathcal{F}_0 = \{0, \Omega\}$$

and, given that the parameters in the brackets are both constants:

$$\mathrm{E}[Y|\mathcal{F}_0] = \mathrm{E}[Y|c] = \mathrm{E}[Y]$$

Given a Filtered space, consider $0 \leq t_1 \leq t_2 \leq T$ and let Y, Z be two random variables, then the following properties hold:

1. $\mathrm{E}\big[\mathrm{E}[Y|\mathcal{F}_{t1}]\big] = \mathrm{E}[Y]$
2. For Y known at time t_1, then $\mathrm{E}[Y|\mathcal{F}_{t1}] = Y$
3. Additivity: $\mathrm{E}[Y + Z|\mathcal{F}_{t1}] = \mathrm{E}[Y|\mathcal{F}_{t1}] + \mathrm{E}[Z|\mathcal{F}_{t1}]$
4. For any constant C known at time t_1: $\mathrm{E}[CY|\mathcal{F}_{t1}] = C\mathrm{E}[Y|\mathcal{F}_{t1}]$
5. If the random variable Y is independent from $\mathcal{F}_{t1}$, then

$$\mathrm{E}[Y|\mathcal{F}_{t1}]\ \textit{becomes just a constant} = \mathrm{E}[Y]$$

6. Tower law:

$$\mathrm{E}[Y|\mathcal{F}_{t1}] = \mathrm{E}[\mathrm{E}[Y|\mathcal{F}_{t1}]\,|\mathcal{F}_{t0}]$$

The tower law states that the best prediction of Y possible at time 0 can be made directly, the former, or passing through our best prediction in t_0. The same holds for any time interval.

Recall that the practice in Finance is to set $\mathcal{F}_0 = 0$, meaning that at time o there aren't any information available and it is only possible to see the set as a whole, Ω.

2.2.5 Martingale Processes

An adapted process M is said to be a Martingale if

$$\mathrm{E}[M(t)|\mathcal{F}_s] = M(s) \text{ , for all } 0 \leq s \leq t \leq T$$

Games which do follow this kind of process are said to be 'fair', since the entry current price is exactly equal to the conditional expectation of the future payoff. In another way martingales are not subjected to drift.

A process satisfying the following inequality:

$$\mathrm{E}[M(t)|\mathcal{F}_s] \leq M(s) \text{ , for all } 0 \leq s \leq t \leq T$$

is called Supermartingale, while if it follows:

$$\mathrm{E}[M(t)|\mathcal{F}_s] \geq M(s)\text{, for all } 0 \leq s \leq t \leq T$$

is said Submartingale.

2.2.6 The Markov processes

Also, an adapted process S is said to be a Markov process if for any deterministic function $g = g(x)\ and\ any\ \ t, \delta t$, its conditional expectation $g(S(t))$ coincides with:

$$\mathrm{E}[\, g(S(t))|\mathcal{F}_t\,] = \mathrm{E}[\, g(S(t))|S_t\,] = \bar{g}(S(t))$$

Coherent with the Weak form of Market Efficiency, in the Markov processes the stock's prices do already reflect in their values all past information, accessible to investors, so that none of them can "beat the market" by exploiting historical series. It is in fact the competition among them which assures the efficiency.

In this sense, Markov processes are memoryless, no past information can help predicting the future and the probabilistic distribution is independent from the historical path.

2.3 Different processes

2.3.1 Standard Brownian Motions

Also called Wiener process or just "Random Walk" process, Standard Brownian Motion are a particular type of Markov process with zero drift and unit variance rate.

In particular, in a filtered probability space $\{\Omega, \mathcal{F}_{(t\leq[0,T])}, \mathbb{P}\}$ where t is continuous, $W = \big(W(T)\big)_{t\leq T}$ is a Brownian Motions if:

- $W(0) = 0$.
- W is adapted to the Filtration.
- For any $s < v$, the increment $W(v) - W(s)$ is independent of F_s and its distribution is $N(0, v - s)$, respectively the mean and variance.
- The paths $W(*, \omega)$ are continuous.

Thus,

- Marginal distributions are Gaussian, since it is possible for any t to write $W(t) - W(0)$ and obtain a distribution of $N(0, t)$
- For any $u < s$, the increments $W(u), W(t) - W(s)$ are independent and have a joint normal distribution $N\left(\binom{0}{0}, \begin{pmatrix} u & 0 \\ 0 & t-s \end{pmatrix}\right)$. The zero-column vector is still the mean and, in the variance-covariance matrix, there is uncorrelation and variance equal to the interval of time.

The same reasoning applies to infinitive number of increments.

A variable z is said to follow a Brownian Motion if for a small increment of time Δt,

$$\Delta z = \varepsilon\sqrt{\Delta t}$$

where ε is a Random Variable taken from a standard normal with $\varphi(0{,}1)$ and thus Δz has null mean and variance equal to Δt.

The null mean shows exactly the idea of Martingale: the expected value of z at any time is equal to its current value.

Also, the values of Δz and Δt are independent, thus z follows a Markov process.

2.3.2 Linear Brownian Motion

Linear Brownian Motion B are a linear transformation of the Standard Brownian Motion W.

The stochastic differential is:

$$B(t) = \mu t + \sigma W(t)$$

Where still μ is the deterministic term Drift and σ > 0, the Variance rate on the Brownian Motion.

2.3.3 Geometric Brownian Motion

Or Generalized Wiener process, are an Exponential transformation of the Linear Brownian Motion.

A Generalized Wiener process x can be expressed in terms of $a, b\ and\ dz$:

$$dx = a\ dt + b\ dz$$

where a, b are constants called coefficient of the stochastic differential equation.

So that, for a small increment of time Δt,

$$\Delta x = a\ \Delta t + b\varepsilon\sqrt{\Delta t}$$

Still, ε is a Random Variable Standard Normal, thus Δx will have mean $a\Delta t$ and variance $b^2\Delta t$, say a is the drift rate and b^2 the variance rate.

Considering a mean and variance of the stock's price proportional to the stock price, the model becomes:

$$dS = \mu S dt + \sigma S dz$$

$$\frac{dS}{S} = \mu dt + \sigma dz \qquad integrating\ both\ sides \qquad \ln S = \mu t + \sigma z \quad \rightarrow \quad S = e^{\mu t + \sigma z}$$

More formally, the process $\mathcal{Y}$ can be written as:

$$\mathcal{Y} = exp(B(t)) = exp\big(\mu t + \sigma W(t)\big)$$

Being an exponential transformation of a Linear Brownian Motion, normally distributed, its marginal distributions are lognormal: the logarithm of the marginal distributions is normally distributed

The equation is the most used to model stock's prices and can be considered as the limit of the Binomial tree when the intervals of time tend to infinity, as it will be shown later on.

In a Discrete time model, the resulting version of the motion is defined:

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

2.3.4 Ito's Processes

An It's process, or a diffusion is an adapted process in which a, b, are not constants, but rather are functions of the underlying variable x, and of time t.

$$dx = a(x,t)\ dt + b(x,t)\ dz$$

Such that when $t, t + \Delta t$ also x goes to $x + \Delta x$ and the increments are defined by:

$$dx = a(x,t)dt + b(x,t)\varepsilon\sqrt{\Delta t}$$

Note that in the interval drift and variances remain constants.

2.3.5 Ito's Lemma

The Ito's Lemma states that:

For a given Ito's process of the form $dx = a(x,t)dt + b(x,t)dz$, the function $G(x,t)$ follows the process:

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}bdz$$

A non-rigorous demonstration:

For a continuous, twice-differentiable function $G(x, y)$:

$$dG \approx \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy$$

The Taylor series is:

$$dG = \frac{\delta G}{\delta x}dx + \frac{\delta G}{\delta y}dy + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}(dx)^2 + \frac{\delta^2 G}{\delta x \delta y}dxdy + \frac{1}{2}\frac{\partial^2 G}{\partial y^2}(dy)^2 + \cdots$$

for dx, dt going to 0, second order factors can be eliminated, such that:

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt$$

Considering now an Ito's process x as:

$$dx = a(x,t)dt + b(x,t)\varepsilon\sqrt{\Delta t}$$

The expansion in a Taylor series is:

$$dG = \frac{\delta G}{\delta x}dx + \frac{\delta G}{\delta t}dt + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}(dx)^2 + \frac{\delta^2 G}{\delta x \delta t}dxdt + \frac{1}{2}\frac{\partial^2 G}{\partial y^2}(dt)^2 + \ldots$$

Differently from the previous situation, here the series must comprehend the term $(dx)^2$ cannot be ignored when taking the limits, since containing a component dt.

Hence the equation becomes:

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2 dt$$

This is the Ito's Lemma and substituting back dx, dG becomes:

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}bdz$$

Note that both the processes x, G are influenced by the same source of risk dz.

2.3.6 Lognormal distribution

Assuming $G(S,t) = \ln S$ and applying the Lemma:

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \quad \frac{\partial^2 G}{\partial S} = -\frac{1}{S^2}, \quad \frac{\partial G}{\partial t} = 0 \qquad for \begin{cases} a = \mu S \\ b = \sigma S \end{cases}$$

then,

$$dG = d(\ln S) = \left(\frac{1}{S}\mu S - \frac{1}{2S^2}\sigma^2 S^2\right)dt + \frac{1}{S}\sigma S dz$$

Simplifying and rearranging the process followed by G results to be:

$$dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dz$$

Being μ and σ two constants, the process G is also a Generalized Weiner process in which its variation between t_0 and T will have:

- Drift rate: $(\mu - \frac{\sigma^2}{2})T$
- Variance rate: $\sigma^2 T$

Integrating the stochastic differential equation:

$$d(\ln S) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\varepsilon\sqrt{dt}$$

Such that:

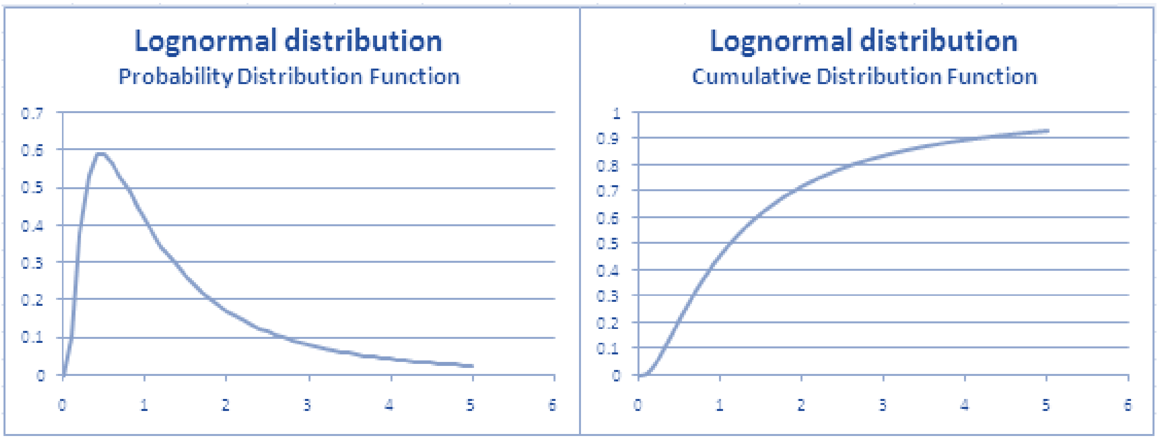
$$d\,(\ln S_T) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)T\,;\ \sigma^2 T\right)$$

$$\ln S_T - \ln S_0 \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)T\,;\ \sigma^2 T\right]$$

$$\ln S_T \sim \left[\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T \; ; \; \sigma^2 T\right)\right]$$

Meaning that, it is $\ln S_T$ that has a normal distribution, hence S_T has a lognormal distribution.

Graphically:



3 Market models in Discrete time

3.1 Binomial tree

Binomial trees are a common method for Option valuations, a similar model to the one proposed in the late 70s by Cox, Ross and Rubinstein.

3.1.1 Model

It is a Discrete time model defined over today $(t = 0)$, and tomorrow $(t = 1)$. Consider that in the market there exit only two assets: a risky Stock and a riskless Bond.

- The Bond price process is $\begin{cases} B_0 = 1 \\ B_1 = 1 + r_f \end{cases}$

- The Stock price process is $\begin{cases} S_0 = s \\ S_1 = \begin{cases} su, & with\ probability\ q_u \\ sd, & with\ probability\ q_d \end{cases} \end{cases}$ $;say\ process\ Z$

Define u as the *up movement*, and d as the *down movement*. The process Z is of a Bernoulli type.

Also, the following assumptions hold:

- Short positions and fractional holdings are allowed
- No Bid-Ask spread
- No transaction costs
- $d < u$

Consider a portfolio $h = (x, y)$ having a deterministic value at $t = 0$ and a stochastic value at $t = 1$. In particular the Value Process of the portfolio is:

$$V_t^h = xB_t + yS_t \quad for\ t = (0,1) \rightarrow \begin{cases} at\ t = 0 \ \rightarrow \quad V_0^h = x + ys \\ at\ t = 1 \ \rightarrow \quad V_1^h = x\left(1 + r_f\right) + ysZ \end{cases}$$

3.1.2 Arbitrage

A portfolio is said to be an Arbitrage portfolio if:

- $\begin{cases} V_0^h = 0 \\ V_1^h \geq 0, \quad and\ \mathbb{P}\left(V_1^h > 0\right) > 0 \end{cases}$

The Binomial tree model respects the Non-Arbitrage Condition if and only if:

$$d < 1 + r_f < u$$

which simply says that the return on the stock is not allowed to dominate the return on the bond and vice versa.

i.e. if $d < u < r_f$, then a strategy such as $h = (s, -1)$ would have:

$$V_0^h = 0 \ \ at\ t = 0, \qquad and \qquad V_1^h = s(1 + r_f) - sZ$$

which for assumption is strictly positive, thus an arbitrage opportunity.

3.1.3 Replicating Strategy (Hedging)

Exploiting the Law of One Price, insured by the NA condition, it is possible to price any Derivative written on the Stock.

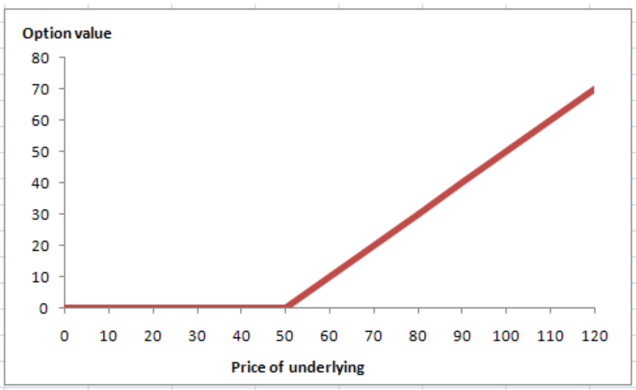
An Option contract is a unilateral contract in which the investor has the right to decide until the expiration whether to exercise it or not. In this sense an European Call Option, gives the buyer the possibility to hedge against increasing prices, giving him the option to exercise the contract and recover the losses generated by the increase in prices.

Consider now a European Call Option C written on the stock, such that:

$$C = \phi(Z), \qquad where\ Z\ is\ the\ stochastic\ price\ process\ observed\ before$$

For a Strike price K, $$C = \max(S_t - K; 0) = \begin{cases} su - K, & if\ Z = u \rightarrow \phi(u) = su - K \\ 0, & if\ Z = d \rightarrow \phi(d) = 0 \end{cases}$$

Graphically:



Here, for S_t on the x-axis and C on the y-axis, if $S_t < K$, (here K=50) the investor would not exercise the contract, such that its value would be C=0. As $S_t > K$, the investor would exercise the contract which would lead to a positive payoff C > 0.

For a given derivative C, a hedging portfolio is the one able to reach exactly: $V_1^h = C$

For this reason, the pricing principle states that the only price process for C can be:

$$V_T^h = f(T\,;\,S)\,, \qquad for\ T = (0, t)$$

since any price at today $T = 0$, different by V_0^h would lead to an arbitrage possibility.

Moreover, if all derivatives are replicable, then the market is complete.

Considering the system $\begin{cases} \phi(u) = (1 + r_f)x + suy \\ \phi(d) = (1 + r_f)x + sdy \end{cases}$

Since by assumption $d < u$, then the system has a unique solution given by:

$$x = \frac{1}{1+r_f}\left(\frac{\phi(d)u - \phi(u)d}{u-d}\right) \quad while \quad y = \frac{1}{s}\left(\frac{\phi(u) - \phi(d)}{u-d}\right)$$

Such that at $T = 0$, the price process:

$$f(0\,;S) = V_0^h = x + sy \quad \rightarrow \quad \frac{1}{1+r_f}\left\{\frac{(1+r_f)-d}{u-d}\phi(u) + \frac{u-(1+r_f)}{u-d}\phi(d)\right\}$$

Note: the number of Stock shares used in the replicating portfolio for each Bond is often called the Δ of the derivative.

3.1.4 Completeness

When any Option' payoffs can be replicated by a combination of Bond and Stock, we say that the model is complete. Nevertheless, if at any time the risky asset could assume not only two but three (or more) different values, such new market model would not any longer be complete.

More in general, assume a market composed by a Bond and *n* risky assets $S^1, S^2, \dots S^n$ and two intermediate dates $t = 0,\ t = 1$.

The Bond price is a deterministic function:

$$B_1 = B_0(1+r)$$

The Stock price is random:

$$S_1^i = S_1^i(\omega) \ , \qquad for\ i = 1, 2, \dots n$$

with $\omega\ \epsilon\ \Omega = \{\omega_1, \omega_2, \dots, \omega_m\}$ and $m\ \epsilon\ \mathbb{N}$, a finite simple space.

Then it is possible to state that a model is complete if and only if the matrix:

$$\begin{pmatrix} B_1(\omega_1) & S_1^1(\omega_1) & \cdots & S_1^n(\omega_1) \\ B_1(\omega_2) & S_1^1(\omega_2) & & S_1^n(\omega_2) \\ \vdots & & \ddots & \vdots \\ B_1(\omega_m) & S_1^1(\omega_m) & \cdots & S_1^n(\omega_m) \end{pmatrix}$$

has rank equal to the cardinality of Ω, that is m: the number of underlying assets equals the number of outcomes in the sample space.

3.1.5 Risk-Neutral approach

In order to determine the 'fair' price today of any Derivative in a Binomial tree model it is also possible to consider another point of view, using the popular Risk-Neutral approach. Based on the concept of martingale, in the one period Binomial Tree model, a probability measure $\mathbb{Q}$ is called a Martingale measure (or risk-neutral measure) if:

$$S_0 = s = \frac{1}{1+r_f} E^{\mathbb{Q}}[S_1|\mathcal{F}_0]$$

since it makes the whole discounted process a Martingale in its turn:

$$E^{\mathbb{Q}}\left[\frac{S_1}{1+r_f}|\mathcal{F}_0\right] = S_0$$

Such $\mathbb{Q}$, which is not necessarily unique, is defined as Equivalent Martingale Measure (*EMM*) for the process S.

3.1.6 Martingale Measure

Moreover, Martingale Measures have a key role in determining features of market models, such as Non-Arbitrage Condition and Completeness.

Introduce the Fundamental Theorems of Asset Pricing (FTAPs):

1. The First Fundamental Theorem of Asset Pricing (FTAP1) states that Non-Arbitrage is equivalent to the existence of at least one probability $\mathbb{Q}$ on $\mathcal{F}_{\mathrm{t}}$ with the martingale property derived above.
2. The Second Fundamental Theorem of Asset Pricing (FTAP2) moreover specifies that the market is complete only if the Equivalent Martingale Measure exists and is unique.

For a Binomial tree model, a Martingale probability measure $\mathbb{Q}(Z = u) = q_u$; $\mathbb{Q}(Z = d) = q_d$ is:

- a probability measure, hence: $\begin{cases} q_u \geq 0 \\ q_d \geq 0 \\ q_u + q_d = 1 \end{cases}$

- a Martingale measure for the process, hence:

$$\frac{1}{1+r_f} E^{\mathbb{Q}}[S_1|\mathcal{F}_0] = S_0, \quad where \begin{cases} E^{\mathbb{Q}}[S_1|F_0] = suq_u + sdq_d \\ S_0 = s \end{cases}$$

- Equivalent to $\mathbb{P}$, thus $\mathbb{Q}(A) = 0$, if and only if $\mathbb{P}(A) = 0$, hence in particular $q_i > 0$.

The system to solve has two equations and two unknows, hence a unique solution.

$$\begin{cases} \frac{1}{1+r_f}(suq_u + sdq_d) = s \\ q_d = 1 - q_u \end{cases} \rightarrow \begin{cases} \frac{1}{1+r_f}(q_u u + (1-q_u)d) = 1 \\ q_d = 1 - q_u \end{cases} \rightarrow \begin{cases} q_u u + d - q_u d = 1 + r_f \\ q_d = 1 - q_u \end{cases}$$

the unique possible values for q_u, q_d are given by:

$$\begin{cases} q_u = \dfrac{(1+r_f) - d}{u - d} \\ q_d = 1 - q_u = \dfrac{u - (1+r_f)}{u - d} \end{cases}$$

note that, in order to have $q_u \geq 0$, $q_d \geq 0$, we find out again the Non-Arbitrage condition:

$$u > 1 + r_f > d$$

- *i.e. for:* $r_f = 2,\ u = 5,\ d = 1$
  - $q_u = \frac{1+2-1}{5-1} = \frac{1}{2}, \quad q_d = \frac{1}{2}$

Also, recalling the formula for the value today of the replicating portfolio

$$f(0\,;S) = V_0^h = x + sy \quad \rightarrow \quad \frac{1}{1+r_f}\left\{\frac{(1+r_f) - d}{u-d}\phi(u) + \frac{u-(1+r_f)}{u-d}\phi(d)\right\}$$

we find exactly the Martingale probabilities q_u, q_d.

In some sense, the replicable derivatives are redundant: they can be synthetically reproduced using Stocks and Bonds, hence these concepts are more useful as pricing tools, i.e. when placing a new Derivative on the market and looking for the unique fair, non-arbitrage price.

Note, here we have introduced the model considering only one period, but the principle of evaluating the Option in a risk-neutral-world keep holding as the number of states grow. At any point the Option's price is equal to its expected value in a risk-neutral-world, actualized with the risk-free rate.

3.1.7 Volatility

Note that in order for the tree to be coherent with the volatility of the underlying stock, the parameters u, d must be chosen correctly.

Considering that the variance rate of the Stock is $\sigma^2 \Delta t$ and that the variance of a Random Variable is $E(X^2) - [E(X)^2]$, in a given interval of time Δt:

$$\sigma^2 \Delta t = q_u u^2 + q_d d^2 - [q_u u + q_d d]^2$$

Substituting q_u, q_d from previous and considering and expansion in series before the terms of order Δt^2 and superior, the solutions proposed by Cox, Ross and Rubinstein for u, d are:

$$u = e^{\sigma\sqrt{\Delta t}} \quad and \quad d = e^{-\sigma\sqrt{\Delta t}}, \qquad note: u = \frac{1}{d}$$

3.1.8 Delta

Equating the initial and the future payoffs and solving for Δ, the ratio of the rate of change of the Option over the rate of change of the Stock in an interval, we get:

$$\Delta = \frac{\phi(u) - \phi(d)}{su - sd}$$

Furthermore, the Δ is exactly the y used before as the number of Stocks to invest into for each Option shorted. Being a Linear function, the Δ of the portfolio is a linear combination of the Δ_s of each position.

Note, it is $0 < \Delta < 1$ for Call Options and $-1 < \Delta < 0$ for Put Options.

3.2 Trinomial Tree

It can be thought as an extension of the Binomial Tree and it is conceptually similar.

3.2.1 Model

Still in a Discrete one time model, the market comprises of the previous two assets, but now the Stock price is allowed for a future third middle state m.

Such that:

- The Bond price process is $\begin{cases} B_0 = 1 \\ B_1 = 1 + r_f \end{cases}$

- The Stock price process is $\begin{cases} S_0 = s \\ S_1 = \begin{cases} su, & with\ probability\ q_u \\ sm, & with\ probability\ q_m \\ sd, & with\ probabiliy\ q_d \end{cases} \end{cases}$

say process T.

Then a financial derivative as $X = \phi(T)$ would solve the system:

$$\begin{cases} (1+r)x + suy = \phi(u) \\ (1+r)x + smy = \phi(m) \\ (1+r)x + sdy = \phi(d) \end{cases}$$

Considering the completeness statement, the resulting matrix would be:

$$\begin{pmatrix} (1+r) & su & \phi(u) \\ (1+r) & sm & \phi(m) \\ (1+r) & sd & \phi(d) \end{pmatrix}$$

3.2.2 Incompleteness

It is not true that this matrix never has solution, but since the completeness condition requests that it must be always possible to replicate any Option payoffs by going short and long on a certain number of Bonds and Stocks, then the trinomial tree model is incomplete. There are two underlying assets for three possible outcomes.

3.2.3 Martingale Measure

In a Trinomial Model a Martingale probability measure $\mathbb{Q}$, being:

$\mathbb{Q}(T = u) = q_u$; $\mathbb{Q}(T = m) = q_m$; $\mathbb{Q}(T = d) = q_d$

is a probability, hence: $\begin{cases} q_i \geq 0 \\ \sum q_i = 1 \end{cases}$

and a Martingale measure for the process hence:

$$\frac{1}{1+r_f}E^{\mathbb{Q}}[S_1|\mathcal{F}_0] = S_0, \ \ where \begin{cases} E^{\mathbb{Q}}[S_1|F_0] = suq_u + smq_m + sdq_d \\ S_0 = s \end{cases}$$

The resulting system has two equations and three unknows, hence infinite solutions.

$$\begin{cases} \frac{1}{1+r_f}(suq_u + smq_m + sdq_d) = s \\ q_d = 1 - q_u - q_m \end{cases} = \begin{cases} q_u = \frac{(1+r_f) + dq_m - mq_m - d}{u-d} \\ q_d = 1 - q_u - q_m \end{cases}$$

This implies that the Trinomial tree model still respects the First Fundamental Theorem of Asset Pricing, but not the Second, thus also under this method it results incomplete.

- *i.e. for:* $r_f = 2, \ u = 5, \ d = 1, \ m = 4$
  - if $q_m = \frac{1}{2} \ \rightarrow \ \begin{cases} q_u = \frac{3+\frac{1}{2}-2-1}{4} = \frac{1}{8} \\ q_d = 1 - \frac{1}{8} - \frac{1}{2} = \frac{3}{8} \end{cases}$
  - if $q_m = \frac{1}{4} \ \rightarrow \ \begin{cases} q_u = \frac{3+\frac{1}{4}-1-1}{4} = \frac{5}{16} \\ q_d = 1 - \frac{1}{4} - \frac{5}{16} = \frac{7}{16} \end{cases}$

both the triplet are Martingale measures for the process.

4 Market models in Continuous time

4.1 The Black-Scholes Model

The model awarded with a Nobel Prize in Economic sciences in 1997 is a milestone of Finance.

4.1.1 Model

For the binomial assumptions still holding, (perfect market) consider a market composed of only two financial instruments:

- A riskless Bond B continuously paying the risk-free rate $r \geq 0$

$$price\ process\ given\ by \begin{cases} B(0) = 1 \\ dB(t) = rB(t)dt \end{cases} \ \ , for\ B(t) = e^{rt}$$

- A risky Stock S satisfying the Stochastic Differential Equation with initial condition (Cauchy Problem):

$$\begin{cases} S(0) = S_0 \\ dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \end{cases}$$

where S_0 is the initial empirical market price, and μ , σ $(\sigma > 0)$ are constants respectively called drift and volatility.

The unique solution for S to the Cauchy problem is:

$$S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$$

Where the marginals of $S(t)$, recalling the lognormal distribution for generalized Wiener processes, is:

$$\ln \frac{S(t)}{S_0} \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)t \,;\, \sigma^2 t\right)$$

Hence

$$\ln S(t) \sim N\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t \,;\, \sigma^2 t\right]$$

The logreturn of mean and variance grow linearly with time.

Differently from normal distribution, lognormal distribution is positively asymmetric: the mode is smaller than the median, in its turn smaller than the mean.

In particular μ is the exponential growth rate of the average stock price:

$$E[S(t)] = S_0 e^{\mu t}$$

and σ is the standard deviation of the annual logreturn:

$$var(S_t) = S_0^2 e^{2\mu t}(e^{\sigma 2t} - 1)$$

4.1.2 Black-Scholes differential equation

Considering the price process of the Stock

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

and an Option S with price $f(S,t)$, applying the Ito's Lemma, we know that:

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S dW$$

recall that the process dW in both the stock and the option prices, is the same, and this is the key idea on which to build the hedging portfolio.

4.1.3 Replicating portfolio (Delta-Hedging)

The value today of a portfolio made by a short position on one Option f and a long position on $\frac{\partial f}{\partial s}$ Stocks would be:

$$\Pi = -f + \frac{\partial f}{\partial S}S, \quad \textit{such that continuously becomes}: \qquad d\Pi = -df + \frac{\partial f}{\partial S}dS$$

Substituting df from previous, the portfolio becomes riskless:

$$d\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt$$

The source of risk which was in the Geometric Brownian motion gets canceled out.

Since now the portfolio is void of risk, for the NA condition its return needs to be the same of a risk-free asset. That is:

$$d\Pi = r\Pi dt \qquad \rightarrow \quad \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt = r\left(f - \frac{\partial f}{\partial S}S\right)dt$$

Finally:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

This is the famous Differential equation of Black-Scholes.

note that here $\frac{\partial f}{\partial t}$ varies continuously, so that the portfolio needs to be rebalanced accordingly.

The solution of this equation can be found by considering the boundary conditions of each derivative written on the stock.

4.1.4 The Black-Scholes Closed Formulas

If we keep developing these concepts for a European Call Option, the boundary condition would be:

$$f = max(S - K, 0)$$

In order to solve the Partial Differential Equation, it is possible to adopt a so-called heuristic reasoning, or the Feynman-Kac formula. We now briefly show one other approach to derive the closed formulas for European Call Option.

Consider $g(S)$ the density of S, and $s = \sigma\sqrt{T}$,

then

$$E[max(S - K, 0)] = \int_K^\infty (S - K)g(S)d(S)$$

knowing that the Stock has lognormal distribution, it can be shown that its mean is

$$m = \ln[E(S)] - \frac{s^2}{2}$$

Introduce a new variable Q, being

$$Q = \frac{\ln(S) - m}{s}$$

Having a Normal distribution $h(Q)$, with null mean and unit standard deviation:

$$h(Q) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Q^2}$$

such that it is possible to change the variable of integration from S, to Q

$$E[max(S-K,0)] = \int_K^{\infty} (S-K)g(S)d(S) = \int_{\frac{\ln(K)-m}{s}}^{\infty} (e^{Qs+m} - K)h(Q)dQ$$

which can also be rewritten as

$$E[max(S-K,0)] = \int_{\frac{\ln(K)-m}{s}}^{\infty} e^{Qs+m} h(Q)dQ - K \quad \int_{\frac{\ln(K)-m}{s}}^{\infty} h(Q)dQ$$

Also, for

$$e^{Qs+m}h(Q) = \frac{1}{\sqrt{2\pi}} e^{\frac{(-Q^2+2Qs+2m)}{2}} = \frac{1}{\sqrt{2\pi}} e^{\frac{[-(Q-s)^2+2m+s^2]}{2}} = \frac{e^{\frac{m+s^2}{2}}}{\sqrt{2\pi}} e^{\frac{[-(Q-s)^2]}{2}}$$

hence

$$e^{m+\frac{s^2}{2}} h(Q-s)$$

The equation becomes

$$E[max(S-K,0)] = e^{m+\frac{s^2}{2}} \quad \int_{\frac{\ln(K)-m}{s}}^{\infty} h(Q-s)dQ - K \int_{\frac{\ln(K)-m}{s}}^{\infty} h(Q)dQ$$

Define $N(x)$, the probability that a normal random variable is smaller than x, the first integral results to be

$$1 - N\left\{\frac{[\ln(K)-m]}{s} - s\right\}$$

substituting m from previous

$$N(d_1) = N\left\{\frac{\ln[E(S)/K] + s^2/2}{s}\right\}$$

and similarly for $N(d_2)$, such that, once substituted back m too, the equation becomes

$$E[max(S-K,0)] = E(S)N(d_1) - KN(d_2)$$

with

$$\begin{cases} d_1 = \frac{\ln[E(S)/K]+s^2/2}{s} \\ d_2 = \frac{\ln[E(S)/K]-s^2/2}{s} \end{cases}$$

In particular for $E(S) = S_0 e^{rT}$ and, recall $s = \sigma\sqrt{T}$

$$c = S_0 N(d_1) - Ke^{-rT}N(d_2) \qquad for \begin{cases} d_1 = \dfrac{\ln[S_0/K] + \left(r + \dfrac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ d_2 = \dfrac{\ln[S_0/K] + \left(r - \dfrac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \end{cases}$$

Those are the closed formulas proposed by Black and Scholes for the evaluation of European Call Options.

4.1.5 Convergence of Binomial tree model to Black-Scholes model

It is possible to derive the Black-Scholes Partial Differential Equation also by extending the Binomial Tree.

Considering a Binomial tree market model

$$f(0;S) = \frac{1}{1+r_f}\left\{\frac{\left(1+r_f\right)-d}{u-d}\phi(u) + \frac{u-\left(1+r_f\right)}{u-d}\phi(d)\right\}$$

with changes in an interval of time, $t + dt$ being reflected in the stock price as

$$S_{t+dt} = \begin{cases} uS_t \quad with\, probability \;\; q_u = \left(\dfrac{e^{rdt} - d}{u - d}\right) \\ dS_t \quad with\, probability \; q_d = \left(\dfrac{u - e^{rdt}}{u - d}\right) \end{cases}$$

the Binomial tree model closed formula can be written:

$$f\ (0\ ;\ S) = \ e^{-rt}\{q_u\phi(u) + q_d\phi(d)\} = q_u\big(\phi(u) - \phi(d)\big) + \phi(d)$$

Consider Taylor expansions up to dt of $\phi(u), \phi(d), e^{rdt}, u\ and\ d$:

- $\phi(u) \approx f + \frac{\partial f}{\partial S}(S^u_{t+dt} - S_t) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(S^u_{t+dt} - S_t)^2 + \frac{\partial f}{\partial t}dt$

$$= f + \frac{\partial f}{\partial S}S_t(u-1) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}{S_t}^2(u-1)^2 + \frac{\partial f}{\partial t}dt$$

similarly:

- $\phi(d) \approx f + \frac{\partial f}{\partial S}S_t(d-1) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}S_t^2(d-1)^2 + \frac{\partial f}{\partial t}dt$
- $e^{rdt} \approx 1 + rdt$
- $u \approx 1 + \sigma\sqrt{dt} + \frac{1}{2}\sigma^2 dt$
- $d \approx 1 - \ \sigma\sqrt{dt} + \frac{1}{2}\sigma^2 dt$

note that $(u-1)^2 \approx (d-1)^2 \approx \sigma^2 dt$, such that:

- $q_u\big(\phi(u) - \phi(d)\big) = q_u(u-d)\frac{\partial f}{\partial S}S_t = \ \left(rdt + \sigma\sqrt{dt} - \frac{1}{2}\sigma^2 dt\right)\frac{\partial f}{\partial S}S_t$

substituting $\phi(u),\ \phi(d)$ in this latter equation

$$f(1 + rdt) = rS_t\frac{\partial f}{\partial S}dt + f + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}dt$$

which removing f from both sides and dividing by dt, becomes exactly:

$$\frac{\partial f}{\partial t} + rS_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} = rf$$

5 Monte Carlo Simulation

This latter section is dedicated to the application of Monte Carlo methods, a popular tool to evaluate and analyze portfolios, investments or in our case financial instruments.

5.1 Theory

In particular the Monte Carlo Simulation is one of the computational algorithms classified as Monte Carlo methods which relies on repeated random sampling able to obtain numerical results. Essentially it exploits randomness to solve deterministic problems.

The technique used to value Options begins by simulating a large number of possible random price paths of the underlying Stock. From here it is possible calculate the correspondent exercise value (payoff) of the Option for each path and discounting the average value at today, we derive a consistent price for the Option.

In Excel, in order to generate random values Y with the desired gaussian distribution F it is possible to combine two different functions:

1- Function RAND(), which returns number uniformly distributed between $[0,1]$
2- Function NORM.S.INV(RANDOM()), which returns the inverse of the standard cumulative distribution

This technique exploits the following relation:

For a Random Variable X, uniformly distributed $[0,1]$, if for assumption F is invertible, then $Y := F^{-1}(X)$, has distribution exactly F, since:

$$P(Y \leq y) = P(F^{-1}(X) \leq y) = P\big(X \leq F(y)\big) = F(y)$$

Further, the price path of the Stock evolves according to a Geometric Brownian Motion in $[0, T]$, as introduced earlier.

recall the equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

yielding: $S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$

Finally, to calculate the value of the Call Option impose the condition $C = \max(S_t - K, 0)$

5.2 Numerical Example

In an Excel spreadsheet construct a table $[B2{:}B6]$ with the following values:

$$r_f = 3\%, \qquad \sigma = 15\%, \qquad S_o = 100, \qquad K = 95, \qquad t = 0{,}25$$

Then, as shown here by the firsts 10 values of a table of 1000, build the Monte Carlo table

random	ε	S(t)	Call Option
0,06324243	-1,5281115	89,5905031	0
0,46755881	-0,0814078	99,8582918	4,85829183
0,83925151	0,99138633	108,224921	13,2249207
0,34192442	-0,4072167	97,4477594	2,44775942
0,01100189	-2,2903026	84,6127522	0
0,60153733	0,2573283	102,427713	7,42771278
0,42533534	-0,1882627	99,0612138	4,06121376
0,67984772	0,46727302	104,053289	9,05328901
0,50710399	0,017808	100,604128	5,60412752

To obtain the third column with the possible values of the stock, report :

i.e. in G2: $= \$B\$4 * EXP((\$B\$2 - \$B\$3\text{^}2/2) * \$B\$6 + \$B\$3 * SQRT(\$B\$6) * F2)$

where F2 is the adjacent value of ε.

The last column with the values of the Call Option is built by:

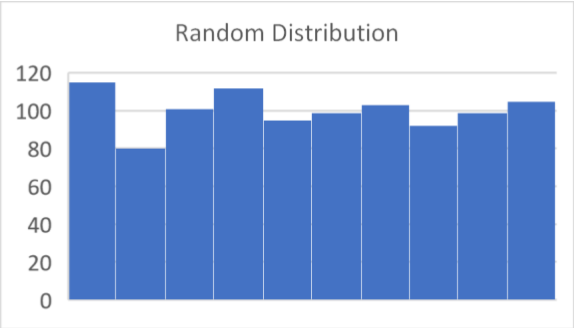
i.e. in H2: $= MAX(G2 - \$B\$5;\ 0)$

Taking the average of all possible values of C, $[H2{:}H1002]$, the price tomorrow would be 6,706166, which discounted at today by the risk-free rate:

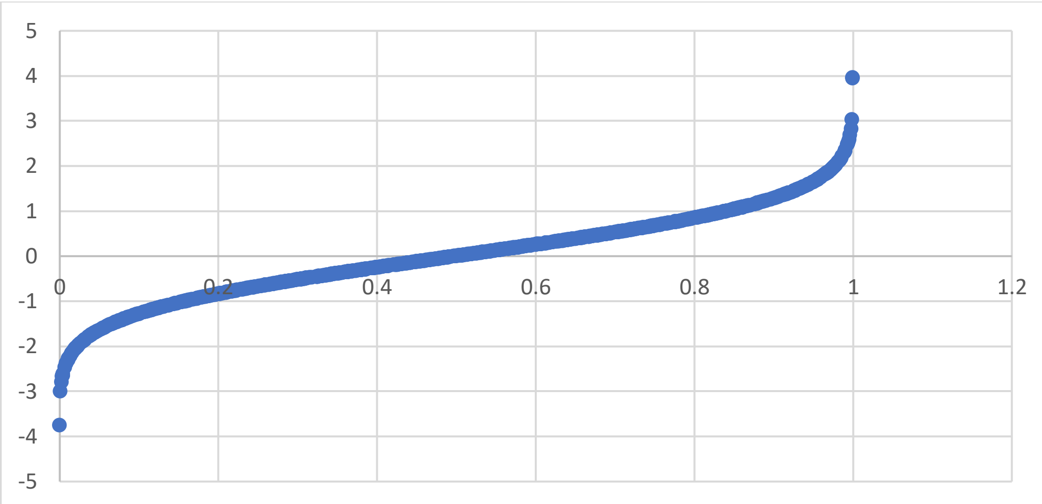
$$= 6{,}706166 * EXP(-B2 * B6)$$

results: 6,64940849.

In our example the RAND function results distributed:



so that combined with the second column ε obtain:



The figure represents the distribution function multiplier, which is the inverse of the standard normal distribution function

5.3 Comparison to Black-Scholes Model

In order to verify the strength of the simulation, it is possible to calculate for the same values what would be the price of the Option in a Black-Scholes model.
For the same assumed values, now in [Q4:Q8]:

$$\begin{cases} d1 \ = (LN(Q6/Q7) + (Q4 + Q5\text{^}2/2) * Q8)/(Q5 * SQRT(Q8)) \\ d2 \ = S2 - Q5 * SQRT(Q8) \end{cases}$$

where S2 = $d1$.

Respectively $d1 =$ 0,82141059, $d2 =$0,74641059

To derive instead $N(d1), N(d2)$, use the function NORM.S.DIS() which selecting the value and imposing True, returns the cumulative normal distribution function of the chosen value.
such that:

$$\begin{cases} N(d1) = 0{,}7942938 \\ N(d2) = 0{,}7722903 \end{cases}$$

applying the closed formula proposed by Black and Scholes for the evaluation of European Call Option

$$C = (Q6 * S6) - Q7 * EXP(-Q4 * Q8) * U6 = 6{,}60999948$$

where S6 = N(d1), U6 = N(d2).
The price today for the European Call Option given by the Monte Carlo Simulation and the closed formula by Black and Scholes are equal until the first decimal. To get a more appropriate price simulation the number of random paths needs to increase (consider 100.000 random samples to round to the second decimal).

5.4 Conclusion

The determination of an option's price is complex and requires the analysis of many variables in addition to the underlying. All methods anyway share common concepts such as risk neutrality, time value or put-call parity. In the current study three different forms of valuation are used:

1. Lattice models: Binomial tree model and Trinomial tree model
2. Closed model: Black-Scholes model
3. Monte Carlo methods

Moreover it is possible to state that Monte Carlo simulations are a very useful tool when evaluating options with multiple sources of risk or with complicated features (i.e. Asian options or lookback options), but in general, when an analytical technique for the evaluation exist, then Monte Carlo would become too slow to be competitive.

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