The Idiosyncratic Volatility Puzzle

Abstract

Monthly sorting stocks into quintiles based on idiosyncratic volatility levels we find a negative relationship between idiosyncratic volatility and returns, providing additional evidence of the Idiosyncratic Volatility Puzzle after the crisis in the American equity market. We test several holding periods (1, 3, 6 and 12 months) finding that the relation between idiosyncratic volatility and returns is holding-period dependent, because it presents an inverted U-shape trend. A wavelet multi-resolution analysis performed on our data shows the contribution of different frequencies to the Puzzle, reporting the relevance of heterogeneity of investors' investment horizon hypothesis. Several performance evaluation measures are then computed for two trading strategies exploiting the Puzzle.

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1 Introduction

We examine the relationship between idiosyncratic volatility and cross-section of returns, aiming to verify if the negative relationship found by Ang et al. (2006)[1] is still present in the post-crisis period (2010-2018). Hence we measure the performance difference between a portfolio made of stocks with highest level of idiosyncratic risk and a portfolio based on stocks with the lowest level of idiosyncratic risk. Afterwards a trading strategy exploiting the patterns we find in the data is tested with several evaluation performance measures. We then analyse if the relationship between idiosyncratic volatility and returns is holding-period dependent, in order to test if heterogeneity of investors' investment horizon hypothesis is verified by our findings. To test the hypothesis, we apply the wavelet transform to study the contribution of each frequency in our data to the Puzzle.

Our novel contribution is to use Ang et al. (2006)'s approach to examine the idiosyncratic volatility-return relationship in a different sample period[2] and to evaluate the performance of two trading strategies exploiting the Puzzle. In addition, we analyse if the relationship is holding-period dependent using a strategy with increasing holding periods (1, 3, 6 and 12 months) and to relate the inverted U-shaped relation we observe to the heterogeneity of investors' investment horizons hypothesis. Finally, we perform a time-frequency analysis on the American equity market post-crisis to shed a light on the reasons behind the Puzzle.

In literature, the theme of this thesis is known as "the Volatility Puzzle". We focus on the idiosyncratic side of the volatility, following other papers' example from now on we refer to it as the "IVOL Puzzle" (Idiosyncratic VOLatility Puzzle). Studying the IVOL Puzzle can be helpful both from a factor investing point of view (as a trading strategy) and as a stronger theoretical framework for all the investors which fail to diversify. About the latter, even if the idiosyncratic risk can be diversified away, there is evidence that it is still present in investments nowadays (Evans and Archer (1968); Statman (1987); Campbell et al. (2001)). By theory facing under-diversification should bring higher returns as a compensation for the higher risk borne (Merton (1987)).

Is the negative relationship between lagged IVOL and returns still there after the crisis? To answer, we firstly apply the L/M/N trading strategy with setting 1/0/1 as in Ang et al. (2006). We then test the trading strategy for increasing holding periods (1, 3, 6 and 12 months), analysing if the relationship between IVOL and returns is holding-period dependent. Setting 1/0/1 means we sort stock returns in 5 portfolios based on the level of 1 month (L=1) lagged idiosyncratic volatilities, we wait 0 month (M=0) and we hold them for 1 month (N=1). This strategy has monthly rebalancing period and it's done over the whole sample-period[3], aiming to compare the returns of portfolio 1 and 5.

[1] Their sample period is 1963-2000.
[2] In the American equity market, as for Ang et al. (2006).
[3] Our sample period is 2010-2018.

We use a dataset from Wharton Research Data Service, composed by the daily returns of the stocks belonging to the same exchanges Ang et al. (2006) used, for the post-crisis period. We monthly construct the 5 portfolios formed on IVOLs using the trading strategy 1/0/1. We find evidence of a negative relationship between IVOL and returns because the portfolio 1 (formed on low volatility stocks) outperforms the portfolio 5 (based on high volatility stocks). The fact the alphas are substantial (-0.72% relative to CAPM and -0.59% relative from Fama-French three-factor model) and statistically significant brings additional evidence of the existence of the IVOL Puzzle after the crisis.
Given our findings, we compute performance evaluation measures of four well-known strategies based on: Market, Size, Value and Momentum factors. We compare these strategies to the strategy of going long on low idiosyncratic volatility stocks and short on high idiosyncratic volatility stocks. We find our strategy and the momentum strategy performing well relative to the market index. Afterwards we quantify the cost for a mean-variance optimizer investor, with an indexed position, of ignoring low idiosyncratic volatility stocks. By tilting its position towards the low idiosyncratic volatility stocks it increases substantially its utility function.
The trading strategy 1/0/3, 1/0/6 and 1/0/12 bring other relevant discoveries. The trading strategy 1/0/3 shows the same patterns of the trading strategy 1/0/1, where portfolio 5 underperforms the portfolio 1. The alphas are relevant and statically significant, meaning that the CAPM and FF3 models still fail to price the portfolios. The trading strategy 1/0/6 shows the absence of a difference in performance both in returns and alphas between portfolio 1 and 5. The trading strategy 1/0/12 displays again the Puzzle's existence with the same patterns of the strategy 1/0/1 and 1/0/3.
Malagon et al. (2015); Yin et al. (2019) use the Wavelet Multi-Resolution Analysis to separate investor classes and decompose a time series into different time horizons. In our case instead of decomposing the time series, playing with the setting of the L/M/N strategy we test different holding periods, separating the investors from active (frequent rebalancing) to more passive investors (yearly rebalance). Anyway, time scales determine the overall return we capture, therefore different compensations required by investors with different time horizons affects the compensation of the overall holding periods. The sum of the compensations required for all the time scales inside an holding period makes the final compensation we observe, hence a change in compensation for increasing holding periods implies a compensation for bearing idiosyncratic risk that is investment-horizon dependent.
Finally, we test the hetereogeneity of investors' investment horizons hypothesis. Following the framework of Malagon et al. (2015) we apply the wavelet multi-resolution analysis, decomposing our data into 7 different frequencies representing the behaviour of different kind of investors. The analysis reports a negative relationship between firm-specific risk and returns for short term investors (investment horizon from 2 to 32 days), a positive one for medium term investors (32 to 128 days) and a negative one for longer investment horizons (> 128 days).

The remainder of the thesis is organized as follows. In Section 2, we examine the past literature about the Puzzle. In Section 3, we describe the theoretical background behind risk and the models used to extract idiosyncratic volatilities. Section 4 illustrates the main features of time-series volatility and in Section 5 we do the same for its idiosyncratic part. In section 6 we describe the trading strategy used to address the Puzzle. Section 7 displays the results for each trading strategy. Section 8 computes several performance evaluation measures on strategies exploiting our findings and the cost of ignoring the Puzzle for a hypothetical investor. Section 9 we describe the theoretical framework behind Fourier and Wavelet methods in Finance. In Section 10 we apply the wavelet trasform to test the heterogeneity of investors' investment horizons hypothesis. Finally, Section 11 concludes.

2 Literature review

2.1 Preamble

By theory, there should be a premium to compensate investors for holding assets that are not diversified. Diversification smooths out the firm specific risk by holding an enough large number of assets. The consequence of diversification is a lower risk faced, hence a better return for unit of risk in our portfolio. The reason why facing less risk means a better mean-variance optimization, is that under certain general assumptions the idiosyncratic risk is not priced (compensated) as the systematic risk. If investors were rational individuals, they should not face idiosyncratic risk and it should not even be priced. Empirically several researches prove that under-diversification is alive in financial markets (Evans and Archer (1968); Statman (1987); Campbell et al. (2001)).Goetzmann and Kumar (2004) show that, based on a sample of more than 62,000 household investors in the period of 1991 to 1996, more than 25% of the investor portfolios contain only one stock, over a half of the investor portfolios contain no more than three stocks and less than 10% of the investor portfolios contain more than ten stocks" (Fu (2009)). Goetzmann and Kumar (2004) find also a relationship between under-diversification and over-confidence preference and trend-following behaviour. Furthermore, a small share of these investors earn from under-diversification because of superior information

Given the empirical evidence, investors fail to fully diversify their investments, therefore an investigation on how idiosyncratic risk affects portfolio's performance was needed both for theoretical and investment purposes. Several papers over the years, investigating how idiosyncratic risk is priced by the market, have found mixed evidence. The main topic of this thesis is known as a Puzzle, therefore we start the analysis describing how idiosyncratic volatility (IVOL) became a Puzzle. We go through the debate about the Puzzle following the time-line of the publications, in order to show how the literature developed over the theme. In academic literature the evidence about IVOL Puzzle is mixed: there are researchers that found a significant positive relationship between idiosyncratic volatility and average returns (as Fu (2009)), there are others which failed to find a significant relationship between these two variables (as Bali and Cakici (2008)) and finally there is also evidence of a negative relationship as Ang et al.

(2006). I'm going to use Ang et al. (2006) as starting point.

2.2 How a topic became a Puzzle

Ang et al. (2006) brought up the debate with their controversial results, which stimulated an academic dispute in the following years. Sorting monthly stocks into 5 portfolios based on IVOL levels, they found out that "stocks with high idiosyncratic volatility relative to the Fama and French (1993) model have abysmally low average returns" Ang et al. (2006). Applying a trading strategy which is based on L months of estimation period, M months of waiting period and N months of holding period (L/M/N), they shed the first controversial light over idiosyncratic volatility. The difference in monthly returns between the portfolio 5 (the one with highest IVOL level) and the portfolio 1 (the one with lowest IVOL level) is statistically significant and has negative sign (-1.06%). Moreover, the Jensen's monthly alphas difference between portfolio 5 and 1 computed relative to CAPM and Fama-French three-factor models are strongly statistically significant and relevant in size (respectively -1.38% and -1.31%). The characteristics of these alphas show how even controlling for additional source of risks, portfolio 1 and 5 are mispriced by both the models. They tested their results controlling for several source of risks as: size, book-to-market, leverage, liquidity, volume, turnover, bid-ask spread, coskewness and dispersion in analysts' forecasts. They controlled even for the new factor they studied, the aggregate volatility, but accounts slightly to the low returns of stocks with high IVOL. They ended the paper stating that this unexplained returns' dynamic is robust to several exposures and therefore "the cross-sectional expected return patterns found by sorting on idiosyncratic volatility present something of a puzzle" (Ang et al. (2006)).

This result was controversial because, as stated by Malagon et al. (2015), is against the modern portfolio theory and under-diversification models (Merton (1987)). The finding has been considered provocative as the portfolios with the highest level of IVOL, associated to the lowest returns, were the ones with the smaller firms (on average their market capitalization relative to the total was 1.9%). The literature reaction to Ang et al. (2006) has been defined by Malagon et al. (2015) as reactionary. Several researches started arguing against those results and the main critics were about their robustness. Bali and Cakici (2008), pointed out that IVOL Puzzle was affected by the specification of data frequency, the weighting scheme of calculating the average portfolio return and the breakpoints to sort portfolios' quintiles. They, using the same trading strategy used by Ang et al. (2006), carried out a huge amount of tests to examine the cross-sectional relationship between IVOL and returns. They stated that using a daily measure of IVOL (as Ang et al. (2006)) they found a negative relationship IVOL-returns only when the value-weighted portfolio are constructed using CRSP breakpoints. When the same analysis is done using NYSE breakpoint, the 20% market-share or any other different weighting scheme they fail to find any significant relationship. This is an important contribution since some of the robustness tests they performed, were already used by Ang et al. (2006). Therefore, different results using the same trading strategy (even if they used 4 years longer sample period) shouldn't be found. We can interpret this saying that if from one hand the trading strategy technique is conceptually easy, on the other hand small setting differences can affect strongly the results. Bali and Cakici (2008) tested the results using a monthly IVOL instead of a daily one, they found no significance with the new measure. Studying the accuracy of both volatility measures, they stated that the monthly IVOL provides better forecast of expected volatility. For all the above reasons, they "safely" concluded "that the negative trade-off between risk and return does not exist" (Bali and Cakici (2008)). Fu (2009), criticized Ang et al. (2006)'s results digging into the idiosyncratic volatility's time-varying nature, deepening the point made by Bali and Cakici (2008) about the ability of lagged IVOL to forecasts expected IVOL. Using the one month lagged IVOL as a proxy for expected IVOL implies that process followed by volatility is a unit root, which can be a random walk process with or without drift:

$$IVOL_t = IVOL_(t-1) + u_t \tag{1}$$

Where u_t is IID with zero mean and fixed variance. Fu (2009) computed the average first order autocorrelation (0.33) and used the Dickey-Fuller test to prove the time-varying nature of IVOL. He concluded that using lagged IVOL as a proxy for expected IVOL is misleading. Therefore, his approach to the Puzzle was different compared to Ang et al. (2006). Instead of sorting based on a measurable proxy's level, he estimated expected idiosyncratic volatility. To capture the time-varying features of IVOLs, Fu (2009) applied the Exponential Generalized Auto-Regressive Conditional Heteroskedasticity (EGARCH) on the train data and used out-of-sample data to estimate the expected volatilities. Simpler model as ARCH and GARCH catch volatility features as leptokurtosis and volatility clustering. EGARCH model improves the previous models since is able to capture the leverage effect too. The model EGARCH(p, q) has the following explicit function form:

$$R_{i,t} - r_t = \alpha_i + \beta_i(R_{M,t} - r_t) + s_i SMB_t + h_i HML_t + \epsilon_{i,t} \quad where \quad \epsilon_{i,t} \sim \mathbb{N}(0, \sigma^2_{i,t}) \tag{2}$$

$$\ln \sigma^2_{i,t} = \alpha_i + \sum_{l=1}^{p} b_{i,l} \ln \sigma^2_{i,t-l} + \sum_{k=1}^{q} c_{i,k} \left\{ \theta \left(\frac{\epsilon_{i,t-k}}{\sigma_{i,t-k}} \right) + \gamma \left[\left| \frac{\epsilon_{i,t-k}}{\sigma_{i,t-k}} \right| - (2/\pi)^{1/2} \right] \right\} \quad where \quad 1 \le p \le 3, 1 \le q \le 3 \tag{3}$$

Leverage effect is referring to the fact volatility rises more following a decrease in price than an increase in price. Fu (2009) found a positive relationship between estimate IVOLs and returns. He judged Ang et al. (2006)'s results not reliable as idiosyncratic volatility has been treated wrongly as a persistent process. Fu (2009) tested this assumption reporting that was not verified, undermining the robustness of their findings. To prove the robustness of their results, Ang et al. (2009) tested them not only for USA data but for all others G7 countries data. The paper brought three main contribution to the Puzzle. Firstly, the IVOL Puzzle exists not just in the American data but is present in all the G7 equity markets (Canada, France, Germany, Italy, Japan, United States and The United Kingdom). Increasing the datasets, they verified the

IVOL Puzzle was not a data snooping effect, increasing the chance there is an economical reason for it. Data snooping refers to the wrong statistical inference which a research can state after looking to the data. To avoid data snooping, application of several tests over the same dataset and/or the use of different datasets can solve the issue[4]. Secondly, the difference between the returns of portfolios made by high IVOL stocks with portfolios composed by low IVOL stocks in international markets strongly comove with the same difference in American market. Finally, Ang et al. (2009) ruled out possible reasons behind the puzzle in the American market as: market frictions, information dissemination and option pricing. Interesting to read is how they answered to Fu (2009)'s critique. "The idiosyncratic volatility effect that we document in both U.S. and international markets is not necessarily a relation that involves expected volatility, which is unobservable and must be estimated. In contrast, past idiosyncratic volatility is an observable easily calculated stock characteristic" (Ang et al. (2009)). If till 2009 literature was trying to weaken Ang et al. (2006) results, after their second publication the Puzzle gained consensus meaning that after that more researchers started to find possible explanations for it. Between them, Brandt et al. (2009) made a difference between retail and institutional investors. They found the Puzzle was driven by the preference for high IVOL stocks of the retail investors which behave as "noise traders". The new point of view, the heterogeneity of investors in the financial markets as a driver for the Puzzle, was further developed by Malagon et al. (2015). The heterogeneity of investors' investment horizon assumption implies that to understand all the possible relationships behind the formation of a price, we need to decompose it by different investment time horizons. With this goal in mind, Malagon et al. (2015) were the first to use the Wavelet Multi-Resolution Analysis to shed a light on the Puzzle. This technique allows to disentangle the influence of several kind of investors (short-term, medium-term and long-term) to the price formation aiming to test heterogeneity of investors' investment horizons hypothesis. This statistical tool, provides a decomposition that instead of decomposing a time series trend into a seasonal and a cyclic component, the time series is seen as the resulting of a sum of time scales (our investment horizons) accounting for local changes Malagon et al. (2015). Mandelbrot (1972) observed the dependence on market returns series are non-stationary and that to asses financial risk more than two moments of the distribution are needed. Fourier analysis works well when the time series is stationary and doesn't have sudden changes Malagon et al. (2015), moreover it loses the time-dependence information. Wavelets are instead localized by both time and frequency so this transform doesn't lose time information. Wavelet behave well with sudden change in signals. To apply the technique, a time series X_0 is decomposed into a blurred approximation S_i which represents the long-run horizons. The short-time horizons are represented by by the details D_i. The decomposition has the following functional form:

$$x_t = s_{f,t} + d_{f,t} \tag{4}$$

[4]https://web.ma.utexas.edu/users/mks/statmistakes/datasnooping.html

where f represent the level of the decomposition so that $s_{f,t}$ is the level f smooth and $d_{f,t}$ is the level f smooth. The levels behave with the following functional form:

$$s_{1,t} = s_{2,t} + d_{2,t} \tag{5}$$

Therefore we can write the original time series, if Multi-Resolution Analysis is conducted to level J, as:

$$x_t = s_{j,t} + \sum_{i=1}^{J} d_{j,t} \tag{6}$$

Applying the above decomposition Malagon et al. (2015) have found a negative relationship between IVOL and returns for the short-term investors (which for them is the time scale from 2 to 4 days) while for long-term investors (more than 16 days) the Puzzle disappears. The results are robust to different way to compute idiosyncratic risk and to different definition of what a "short-term" investor means.

Herskovic et al. (2016) used the unsupervised learning technique called Principal Component Analysis to create a proxy for IVOL, they called it CIV (common idiosyncratic volatility) and used it as a state variable. To inspect the IVOL Puzzle, they regressed returns over CIV and a proxy for the market variance to see the exposure to idiosyncratic risk for each stock. With the Ang et al. (2006) trading strategy setting 1/0/1, they constructed the portfolios sorting monthly based on the level of the beta related to CIV. Comparing the returns of portfolio 5 (high beta CIV) to portfolio 1 (low beta CIV) their results confirm the Ang et al. (2006, 2009) findings. The IVOL puzzle hence turned in part into the CIV puzzle.

A recent research from Yin, Shu and Su (2018) used Wavelet Multi-Resolution Analysis and CIV to examine the Puzzle, discovering that for a short-term investment horizons (less than 4 months) the relationship is negative, then is positive for an intermediate investment horizons (between 4-16 months) and finally is negative again for a long-term investment horizons (more than 16 months).

2.3 Hypotheses

The literature's debate over the Idiosyncratic Volatility Puzzle had two main periods. From 2006 till 2009, the main effort has been to weaken the robustness of Ang et al. (2006)'s findings. Even if some critiques may have been economically reasonable and empirically proved, see Fu (2009) about the time-varying volatility's nature, statistically significant evidence of the Puzzle was still unexplained. Ang et al. (2009) have brought huge consensus about the robustness of their results, in fact after 2009 more possible explanations to the Puzzle came up. The most famous one are: heterogeneity of investors' investment horizons, lottery preference (behavioural explanation), market frictions, average variance beta (Chen and Petkova (2012)) and IVOL as an information content (Jiang et al. (2009)). Hou and Loh (2016), compares one versus one all the previous hypotheses to quantify the success of each explanation (but the heterogeneity of investors' investment horizon). They find that most explanations explain less than 10% of the Puzzle. Anyway they point, as best explanations, lottery preference and market frictions. The residual part of the Puzzle not explained by the best candidates chosen by Hou and Loh (2016) is statistically significant. Between all the hypotheses which have been used as reason behind the Puzzle, heterogeneity of investors' investment horizons captures our attention because it can potentially explain both the significant negative relationship between IVOL and returns and the mixed evidence found in literature. This hypothesis states that the compensation investors demand for bearing idiosyncratic risk could be horizon dependent, implying that a different sign for IVOL-returns relationship could arise applying different framework or different setting inside an equal framework to the Puzzle's studies. Malagon et al. (2015), applying Wavelet Multi-Resolution Analysis to disentangle the different time horizons, find a negative relationship between IVOL and returns for the short term investors while the relationship gets positive for long-term investors. Yin et al. (2019) find the Puzzle for short-term investors, a positive relationship for middle-term investors and a negative relationship again for long term investors.

Aim of the thesis is to search for evidence about IVOL Puzzle, considering the post-crisis sample period (2010-2018). Given the evidence found in the literature, applying the same framework of Ang et al. (2006, 2009) we expect to find supporting evidence to the Puzzle, since we fail to notice reasons why the compensation demanded by investors exposed to idiosyncratic risk should have been changed[5].
About the heterogeneity of investors' investment horizons hypothesis, we are going to test different holding period (different values for N) of the trading strategy L/M/N, to check if there is a change in the compensation required by investors bearing the idiosycnratic risk for a holding period of 1,3,6 and 12 months. Given the Malagon et al. (2015); Yin et al. (2019)'s results, we expect to observe supporting evidence to the heterogeneity of investors' investment horizons hypothesis.

3 The asset pricing framework

Markowitz portfolio theory developed over the years, giving a definition for risk which can be divided into systematic and unsystematic (also known as specific or idiosyncratic). One of the biggest initial improvement in investing has been the consensus that to make a optimum portfolio you cannot just take investments that are individually good, you have to study the relationships between them instead. Speaking about risk, before 1952 investors were discussing about risk but without a clear way to measure it. The first model giving a formula to quantify risk is due to Markowitz (1952), who derives the formula to obtain the expected return and especially the expected risk for a given portfolio of assets. He shows how, under a certain set of assumptions, the standard deviation of the return is a good measure of risk. The formula to

[5]Sample period of Ang et al. (2006, 2009) is 1963-2000.

asses risk is at the same time pointing the crucial role of diversifying to reduce the total risk of a portfolio and the way to do it. Markowitz (1952) shows that the expected return of a portfolio is the weighted average of the expected return of each portfolio, where the weights are the percentage of value (market capitalization) relative to the total portfolio. He also shows that to compute risk the approach is not the same, that's why we emphasize the relevance of the general formula for the standard deviation of a portfolio:

$$\sum_{i=1}^{n} w_i^2 \sigma i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov_{ij} \tag{7}$$

From this formula can be showed that adding stocks into the portfolio the total standard deviation is reduced because the sum of weighted covariances is decreased by the additional asset. Next step has been the Capital Market Theory, it extends previous theory by developing a model to price an aggregation of risky assets. The main change relative to the Markowitz Portfolio Theory is the introduction of a risk-free asset, which is defined as an asset with zero variance. Other features of it are the zero correlation with the risky assets and the risk-free rate of return it provides. Sharpe (1964) is generally recognized as the father of this capital asset pricing model and he received a Nobel for it. Must be said however that Lintner (1965) and Mossin (1966) reached similar theories autonomously. Computing the expected return and risk of a portfolio given this new asset, they created the following formula which combines the expected return of the market, its standard deviation and the risk-free asset to price a group of risky assets:

$$\mathbb{E}(R_{port}) = R_f + \sigma_{port} \left[\frac{\mathbb{E}[R_M] - R_f}{\sigma_M} \right] \tag{8}$$

This is also known as the Capital Market Line. Its interpretation is that investors should just hold the risk-free asset and the market portfolio M (with weights according to the risk aversion of the investor) which is a completely diversified portfolio since contains all the risky assets. The limitation of this model is that it prices a portfolio which in the end is a combination of all the assets in the market. Therefore, it's not able to price individual risky assets because it can't explain the role of idiosyncratic risk in asset pricing. The Capital Asset Pricing Market theory (CAPM) marks a major step forward allowing to price not just portfolios but even individual assets. The measure of risk moves from total volatility to just the undiversifiable share of it which is known as systematic risk. The "beta" is the coefficient in charge to quantify the amount of systematic risk faced by a given security. From equation (8), splitting σ_{port} into $\sigma_i \sigma_{iM}$ representing the volatility of a given firm's returns times the correlation coefficient between returns of the firm and the market portfolio. The formula became:

$$\mathbb{E}(R_i) = R_f + \beta_i [\mathbb{E}[R_M] - R_f] \quad where \quad \beta_i = \frac{\sigma_i r_{iM}}{\sigma_M} \tag{9}$$

Interpretation is that the market risk premium is the same for everyone, what makes securities' returns different is their exposure to it which is scaled case by case with the beta. A lot of critiques have been made about CAPM, some questioning the validity of the assumptions behind the model and others (as Roll (1977)) claiming the CAPM is not testable and finally others empirically testing the ability of the model to price security. Ross (1976), with the Arbitrage Pricing Theory tried to come up with an alternative model to face the mixed evidence regarding the CAPM. If in the previous model the only risk factor priced is the market risk, in APT we can have multiple risk factors, therefore the final return for a given security depends by its exposure to different source of risks.

$$\mathbb{E}(R_i) = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_k b_{ik} \tag{10}$$

Key features of APT are the low number of assumptions required (especially compared to the previous models) and the unspecified nature of the possible risk factors. The main difference between CAPM and APT is the multi-factor nature of risk compared to just the market risk of CAPM. In practice several multifactor models have been applied, the one that will be used in the thesis is the Fama-French three-factor model which is formed at a microeconomic by considering the relevant characteristics of a firm. Fama and French (1993) considered the main source of risk in the market: the market itself, the Size of the firm and the Book-to-Market factor. The size factor is a portfolio of small capitalization firms' return minus a portfolio of large capitalization firms' return. The Book-to-Market factors instead is the return of a portfolio containing stocks with high book-to-market ratio minus the return of a portfolio with low book-to-market ratio stocks. The former should capture the risk linked to the small firms, the latter should capture the risk of firms which for several reasons have their assets under-priced by the market.

In our thesis, we use CAPM and Fama-French three-factor models to test the trading strategy based on idiosyncratic volatility. Since IVOL is not a factor, instead of applying the framework used by Fama and French (1993) which sort based on betas we sort based on the 1 and 2 months lagged idiosyncratic volatility.

4 The Volatility

Volatility in Finance has been used as a risk measure since Markowitz (1952). Volatility is a measure of change of a quantity over time, in Finance it quantifies the tendency of a stock's returns to go up and down. The higher the volatility, the riskier the stock. Since volatility measure upward and downward movements, it opens positive as much negative chances to investors. From a computational point of view, *historical volatility* is computed as the square root of variance of the returns over the horizon considered. The frequency of returns used when estimating historical volatility can be: daily, weekly and monthly. Should be mentioned that even high-

frequency estimations can be done by having a frequency of 1,5 or 10 minutes[6]. A high volatility means the data is widely spread around the mean, while a low volatility means the data are clustered around the mean.
Since in Finance we refer to volatility over stocks' returns, we present some stylized facts about them. Returns have a distribution that empirically exhibit excess peakedness at the mean and fat tails compared to a normal distribution, as shown in Figure 1[7]:

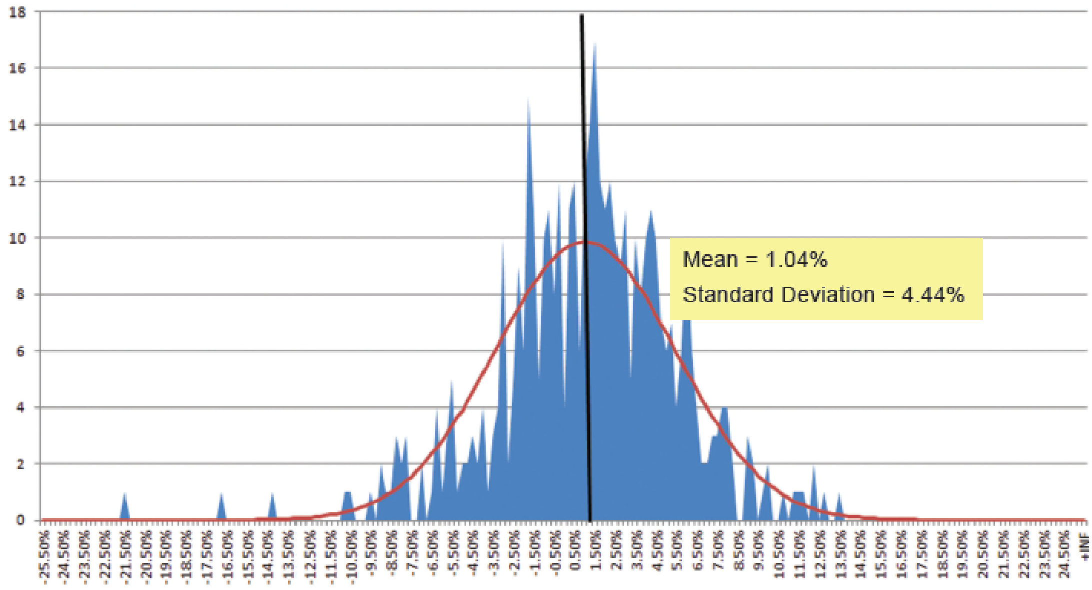


Figure 1: Empirical distribution of monthly returns for the S&P 500 vs Theoretical normal distribution given the parameters

Figure 1 represents the empirical distribution against a normal distribution with parameters estimated by the data. Monthly returns for the S&P 500 shows a *leptokurtosis distribution.* Fat tails are empirical evidence that "worst case scenarios" are more likely compared to the theoretical normal distribution.
Volatility clustering is the tendency of large changes in prices of financial assets to cluster together. Large returns are expected to follows large returns while small returns are expected to follow small returns[8], which cause the tendency for volatility to appear in bunches as in Figure 2[9]

[6] https://www.r-bloggers.com/what-is-volatility/
[7] https://www.evestment.com/resources/investment-statistics-guide/using-statistics-to-understand-return-characteristics/
[8] https://www.thoughtco.com/volatility-clustering-in-economics-1147328
[9] Schwert (2016), available at http://schwert.simon.rochester.edu/spvol.pdf.

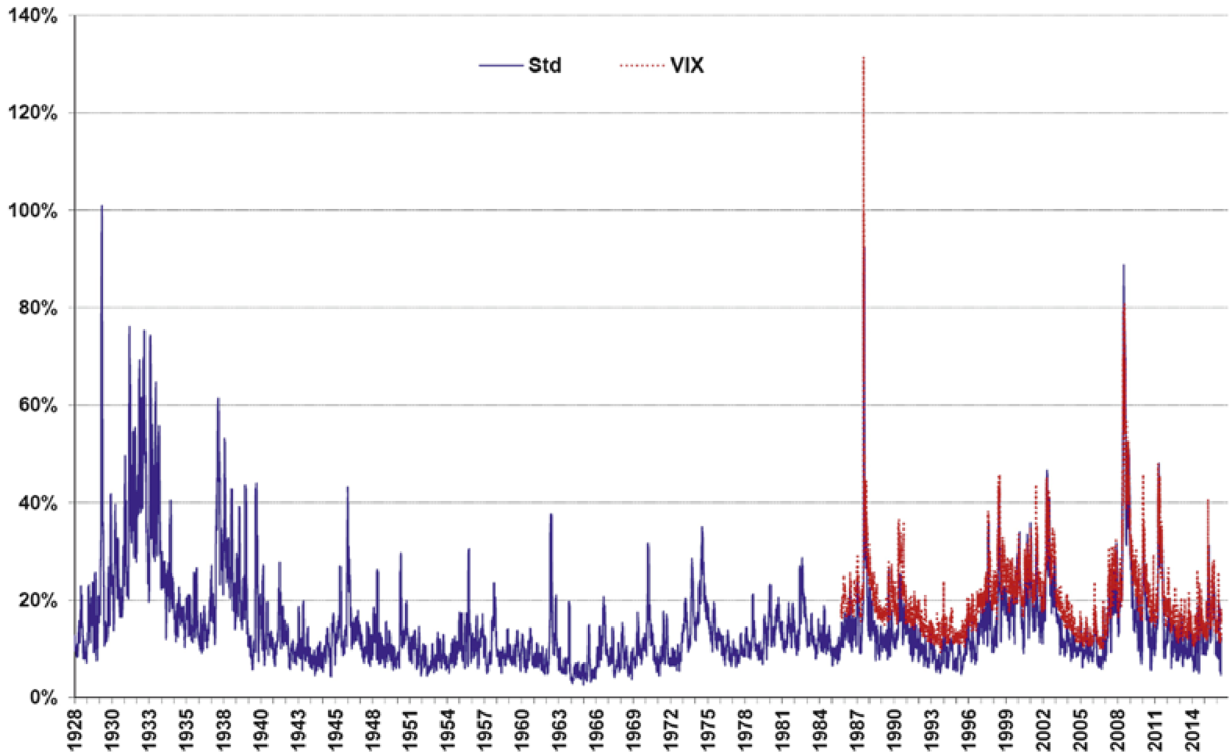


Figure 2: Rolling annualized standard deviation of S&P 500 daily returns vs VIX, 1928-2016

Volatility clustering shows how volatility shocks in financial markets prefigure periods with higher volatility. Because of that, volatility is commonly defined a persistent process because its levels tend to persist day by day till the end of the cycle.
Volatility is *cyclical*, showing a pattern of increasing trend followed by decreasing trend. The predictability of its cyclical nature is exploited by trading strategies aiming to predict the trend reversion.
Volatility is *mean-reverting*, meaning that in a long enough period it will come back to the mean value after a shock.
Volatility is known to be a *long-memory process*, which refers to the level of the statistical dependence between two points in time. As we increase the time gap between them, the rate of decay of their dependence implies if the process has long-memory. To be classified as a long-memory process the statistical dependence between the two points decay slower than an exponential decay[10]. Mean reversion and persistency together capture the nature of volatility to keep a certain level day by day but to reverse to the mean in a long enough time period. This characteristic can be easily seen in Figure 2, where volatility can have high values a year but reverse toward the mean.

We just described some empirical evidence of volatility. We said in the previous section that volatility started to be widely used as a risk measure in Finance since Markowitz (1952). We also stated that the main limitation of the Capital Market Line framework is to not be able to price individual assets, because there was not a way to quantify the individual exposure (β) to the

[10]https://prateekvjoshi.com/2016/08/27/what-is-long-memory-in-time-series-analysis/

market risk. CAPM solved the issue, moving the measure of risk from total volatility to just the undiversifiable share of it which is known as systematic risk. The CAPM's framework has given a way to make a distinction between total and idiosyncratic risk showing the big conceptual difference between them. We defined volatility of an asset as the standard deviation of returns with a given frequency, therefore it can be easily measured. On the other hand idiosyncratic volatility can only be estimated from the model's residuals, therefore is model dependent. If idiosyncratic volatility is model dependent then its accuracy is model dependent too, hence the better the model the better the idiosyncratic volatility we estimate.

5 The Idiosyncratic Volatility

Risk, intended as the standard deviation of returns over time, can be divided into two main components. When a risk is faced by all the securities in the market (can't be diversified because related to macroeconomic factors), is classified as systematic risk. As systematic are considered: the interest rate risk, the market risk, the purchasing power risk, the exchange rate risk and the political risk[11]. On the other hand, the idiosyncratic risk is an industry/firm/stock specific risk which can be diversified away just increasing the number of stocks inside the portfolio. The number required of assets to hold in order to construct a well-diversified portfolio was estimated as 10 in order to erase 70% of the unsystematic risk Evans and Archer (1968). Later, Statman (1987) increased this number to 30/40. Eventually Campbell et al. (2001) stated that "the number of randomly selected stocks needed to achieve relatively complete portfolio diversification" is about 50. Figure 3 shows that to hold a fully diversified portfolio it's sufficient to increase the number of assets inside the portfolio. The intuition is that usually assets are not perfectly correlated, therefore an additional asset will decrease the portfolio's idiosyncratic risk as shown in formula (7) from Markowitz (1952).

[11]https://efinancemanagement.com/investment-decisions/systematic-risk

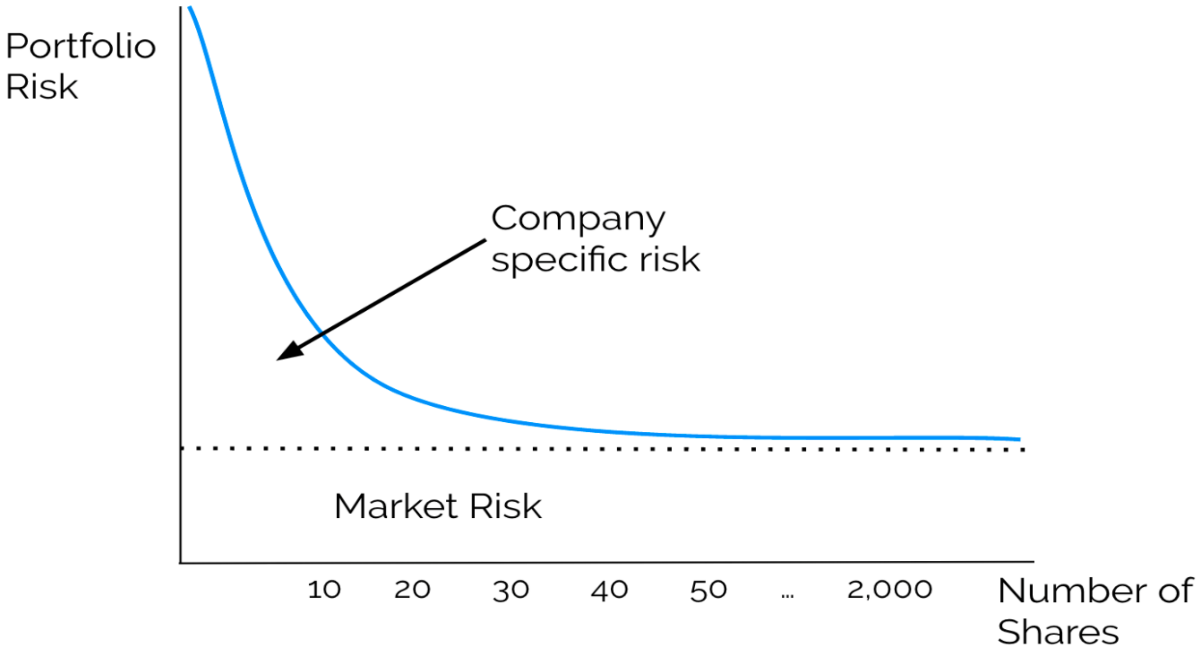


Figure 3: Diversification as function of number of assets

Using the Ang et al. (2006)'s methodology, the IVOL computation derives from the squared root of the residuals' variance $\sqrt{\mathbb{V}ar(\epsilon_{i,t})}$ from the Fama-French 3 factors model (OLS multivariate regression):

$$r_{it} = \alpha_i + \beta_{i,mkt} MKT_t + \beta_{i,SMB} SMB_t + \beta_{i,HML} HML_t + \epsilon_{i,t} \tag{11}$$

Therefore, from now on when we talk about IVOL we refer to idiosyncratic volatility relative to the Fama-French three-factor model.

5.1 Time-varying nature of idiosyncratic volatility

One of the main critiques to Ang et al. (2006) has been made by Fu (2009). He states that if IVOL risk is priced, then there should be an empirical relation between expected returns and expected IVOLs. The usual approach to test this relation is using realized returns as a dependent variable in cross-sectional regressions while as regressors we have expected IVOL and other control variables:

$$R_{i,t} = \alpha_t + \beta_{0,t} \mathbb{E}_{t-1}\left[IVOL_{i,t}\right] + \sum_{k=1}^{K} \beta_{k,t} X_{k,i,t} + \epsilon_{i,t} \quad where \quad i = 1, 2, \ldots, N_t; \quad t = 1, 2, \ldots, T \tag{12}$$

On the left-hand side, we got the realized returns for stock i over time. On the right-hand side, there is the expectation conditional to the available information at time $t-1$ of IVOL, plus the other control variables of our model. N_t is the total number of stocks available at time t while T is the total number of periods. If there were no relationship between expected return and expected IVOL then $\beta_{0,t}$ would be equal to zero or not statistically significant. Given the study over under-diversification of Merton (1987) he finds $\beta_{0,t} > 0$ while given Ang et al. (2006) should find $\beta_{0,t} < 0$. Fu (2009) states "It is crucial to have a quality estimate of $\mathbb{E}_{t-1}[IVOL_t]$, the expected idiosyncratic volatility". His intuition is that $IVOL_{i,t-1}$ to be a quality proxy of $\mathbb{E}_{t-1}[IVOL_t]$, IVOL must be a highly persistent process as the random walk. Since the idiosyncratic risk reflects industry/firm specific information, which are volatile over time, Fu (2009) test if IVOL is a highly persistent process. He computes some statistics over the IVOLs calculated for each company. The time series mean reported is 16.87% and the mean standard deviation is 9.94%. The ratio of the standard deviation over the mean, known as mean coefficient of variation, is 0.55 therefore he suggests the idiosyncratic volatilities vary substantially over time (the standard deviation covers 55% of the mean). We calculate the same statistics on IVOL computed as the monthly standard deviations on daily returns for our sample. We find a monthly time series mean of 2.0% while a standard deviation of the monthly idiosyncratic volatility of 2.47%. Our coefficient of variation is slightly bigger than one meaning that IVOLs are not clustered close around the mean in the data and that IVOLs vary a lot over time. We plot the average monthly IVOLs over time in Figure 4.

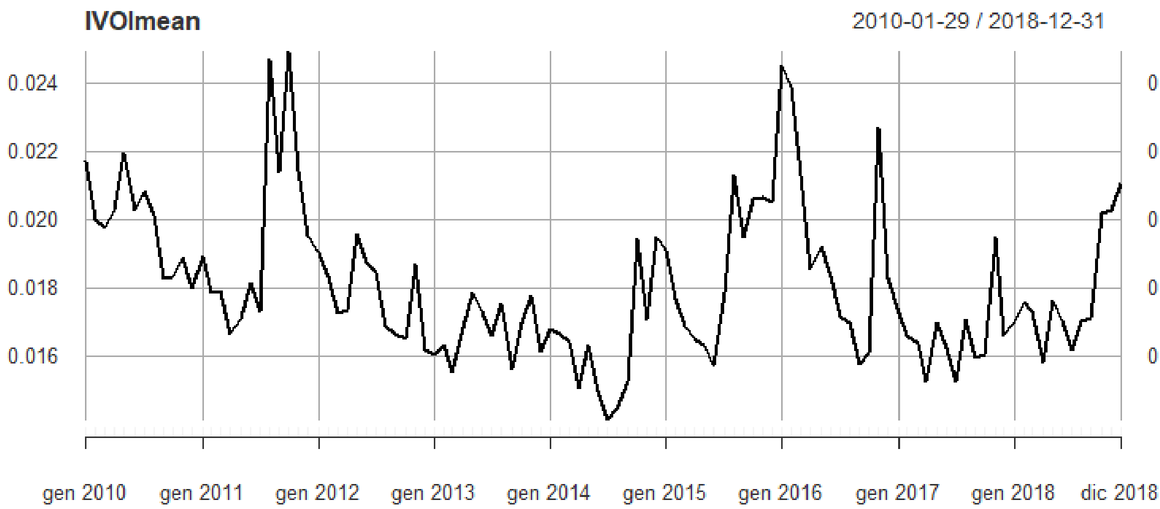


Figure 4: Plot over time of the average monthly idiosyncratic volatilities between firms in our dataset

From Figure 4 we can spot 4 peaks: one at the end of 2011, one at the start of 2016, one at the end of 2016 and the last one at the end of the sample period. We can also spot some of the features we described about time series volatility. Idiosyncratic volatility in Figure 4 is *cyclical*, as downward periods follows upward trends. Volatility *clustering* tendency is present as well the *mean-reverting* nature, we cautiously state that even if idiosyncratic volatility is a conceptually different kind of risk measure, it shares some common features with time series volatility. Fu (2009) then studies the IVOL autocorrelation function, finding a mean autocorrelation at first lag of 0.33 that decays slowly. We do the same and find out a first lag of 0.647 that decays slowly too. Figure 5 shows the autocorrelation function of average monthly IVOLs.

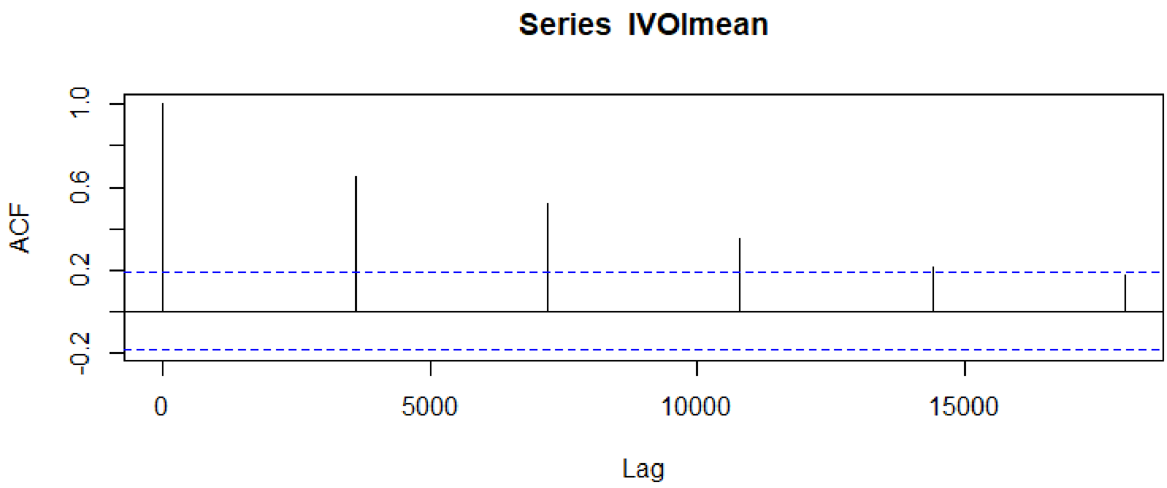


Figure 5: autocorrelation function of the average monthly volatilities with 5 lags

The point Fu (2009) proved, is that idiosyncratic volatility process doesn't follow a random walk process therefore the expected IVOL can't be predicted using the 1 month lagged IVOL. Even if theoretically Fu (2009) his critique is right, by reading Ang et al. (2006) the feeling is that they were not trying to state something about the IVOL process or constructing a conditional asset pricing model. Moreover, one of the best traits about Ang et al. (2006) findings is that they are not built on hard theoretical frameworks, but just on a trading strategy which is conceptually easy for researchers to replicate. Their paper is not about forecasting volatility but regards the fact that sorting stocks based on idiosyncratic risk (that should theoretically not even yield a compensation), on average the low IVOL stocks outperform the high IVOL stocks.

6 Trading strategy

Following Ang et al. (2006), we define our framework as the trading strategy L/M/N. At a point in time t we sort the daily stocks returns based on the L-months lagged IVOLs into 5 quantiles, then we wait M-months and eventually we hold these portfolios (the 5 quantiles) for N-months[12]. The IVOLs are constructed monthly over daily returns. We examine if going short on P5 and long on P1 is profitable. We analyse the following trading strategy's settings: 2/0/1 with monthly rebalancing, 1/0/1 with monthly rebalancing, 1/0/3 with quarterly rebalancing, 1/0/6 with semesterly rebalancing and 1/0/12 with annual rebalancing.
The difference in an investor who does monthly rebalancing based on a factor (IVOL in our case) compared to a yearly rebalancing is the different level of activeness used to manage his portfolio. Investors whom decide to rebalance every year are closer to a passive investing management while monthly rebalancing investors are more active in their portfolio management.

[12]The portfolios returns at the end of the M period are value-weighted

6.1 Trading strategy 2/0/1

The analysis starts over a dataset downloaded from the Kenneth R. French Data Library [13]. The full time period is from July 1963 to April 2019. About the portfolio's construction, they are formed monthly on the variance of the residuals from Fama-French three-factor model. The trading strategy used to form the portfolios is 2/0/1 with monthly rebalancing period. The stocks are the ones listed from NYSE, AMEX and NASDAQ as for Ang et al. (2006). The dataset contains monthly data (divided in quintiles based on IVOLs) about: equally-weighted returns, value-weighted returns and firms' size. Computed some statistics over them, the results (with sample period 1963-2019) are in Table 1. The findings show the same patterns of Ang et al.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	0.94	3.67	6980.31	0.15***	0.13***
				[2.87]	[2.99]
P2	0.96	4.46	3563.43	0.06	0.01
				[1.23]	[0.22]
P3	1.09	5.12	1943.01	0.11	0.07
				[1.59]	[1.10]
P4	1.08	6.07	968.92	0.01	-0.03
				[0.08]	[-0.46]
P5	0.66	. 7.72	266.70	-0.54***	-0.58***
				[-3.42]	[-5.49]
P5-P1	-0.28			-0.68***	-0.71***
	[-1.19]			[-3.37]	[-5.15]

Table 1: Dataset comes from Kenneth R. French Library. It contains already constructed monthly portfolio returns formed based on idiosyncratic volatility levels. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. Size is the average market capitalization of the portfolio. P5-P1 refers to the difference in monthly returns between portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 1963-2019, trading strategy is 2/0/1

(2006) findings. What is different is the magnitude of the spread return between portfolio 5 and 1 which is lower (their was 1.06%) and not statistically significant. Aside that the returns are decreasing from P1 to P5 and the spread of the alphas between portfolio 5 and 1 is sizeable and statistically significant. Hence, we find evidence of the IVOL Puzzle in this dataset with 2/0/1 trading strategy. What is interesting to see is that going from P1 to P5 the volatility of the returns increase, while the average size monotonically decreases. Another relevant fact is that the alphas, which can be interpreted as the ability from the model used to price the portfolio, are significant just for P1 and P5. It means that just the portfolios made of stocks with the highest

[13]mba.tuck.dartmouth.edu site

and the lowest IVOLs are mispriced.
Since the main goal of this thesis is to find out if evidence of IVOL Puzzle survived the crisis, we repeated the same analysis over the same dataset but with a time period that goes from 2010 to 2018. The results are in Table 2:

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	1.06	3.28	22566.64	0.14	0.10
				[1.55]	[1.29]
P2	1.15	3.97	9841.94	0.02	0.02
				[0.26]	[0.34]
P3	1.15	4.53	5405.86	-0.13	-0.10
				[-1.26]	[-1.00]
P4	1.05	5.18	2616.87	-0.38**	-0.31**
				[-2.36]	[-2.47]
P5	0.85	5.85	723.14	-0.64**	-0.49**
				[-2.33]	[-2.37]
P5-P1	-0.21			-0.78**	-0.60**
	[-0.60]			[-2.14]	[-2.30]

Table 2: Dataset comes from Kenneth R. French Library. It contains already constructed monthly portfolio returns formed based on idiosyncratic volatility levels. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. Size is the average market capitalization of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) *t*-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 2/0/1

Considering just the post-crisis period the patterns are the same of Table 1. About the alphas t-statistics, this time the only 2 portfolios that are significant are portfolio 4 and 5. Hence CAPM and Fama-French three-factor models misprice just the portfolios with high idiosyncratic risk levels. To study if there are difference in results that are caused by the different trading strategy's setting used, we run the same analysis in Table 3 for the Ang et al. (2006) sample period[14].

[14] 1963-2000.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	1.09	3.80	1908.85	0.15***	0.09*
				[1.55]	[1.29]
P2	1.15	3.97	9841.94	0.02	0.02
				[0.26]	[0.34]
P3	1.15	4.53	5405.86	-0.13	-0.10
				[-1.26]	[-1.00]
P4	1.05	5.18	2616.87	-0.38**	-0.31**
				[-2.36]	[-2.47]
P5	0.85	5.85	723.14	-0.64**	-0.49**
				[-2.33]	[-2.37]
P5-P1	-0.21			-0.78**	-0.60**
	[-1.1]			[-2.70]	[-4.42]

Table 3: Dataset comes from Kenneth R. French Library. It contains already constructed monthly portfolio returns formed based on idiosyncratic volaility levels. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. Size is the average market capitalization of the portfolio. P5-P1 refers to the difference in monthly returns between portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 1963-2000, trading strategy is 2/0/1

Compared to Ang et al. (2006) findings, the patterns are the same but there are in magnitude and robustness of some measures. The P5-P1 return is -0.21% while it should be -1.06%. Additionally, the CAPM and FF3 alphas are respectively -0.78% and -0.60% while they should be -1.38% and -1.31%.

Why are there these discrepancies? The stocks considered should be the same and the sample period considered is the same. The only difference relies in the trading strategy's setting which in our case is 2/0/1 compared to Ang et al. (2006) that is 1/0/1, meaning they sort at time t based on 1 month lagged IVOL while in this dataset stocks are sorted based on 2 month lagged IVOL. Other possible explanations can be different breaking points used, different ways to compute the weights to obtain the value-weighted returns or general computational differences of this kind.

6.1.1 Dataset from CRSP

The analysis now shifts over a dataset from Wharton Research Data Service[15]. It contains the daily returns of stocks on primary listings for NYSE, NYSE MKT (previously known as AMEX),

[15]http://www.crsp.com/products/research-products/crsp-us-stock-databases

NASDAQ and ARCA exchanges. The time period considered is the post-crisis one, therefore 2010-2018. Over the dataset will be tested the following trading strategies: 1/0/1, 1/0/3, 1/0/6 and 1/0/12. The columns/variables it contains are: daily returns, price per share and number of share outstanding. The last two variables are needed to compute the value-weighted returns of each portfolio. Multiplying them, we obtain the market capitalization which will be used to weight the returns inside the portfolios.

6.2 Trading strategy 1/0/1

On the new dataset, we construct monthly portfolios of stock returns based on five levels of the 1 month lagged IVOLs. The results are reported in a table which is in the layout similar to Ang et al. (2006)'s table for comparison purposes. This means the statistics computed for the 5 portfolios are: mean, standard deviation, market share (intended as average market capitalization of the portfolio over the sum of the 5 portfolios' market capitalizations) and alphas from CAPM and Fama-French three-factor models. In order to be as close as possible to the real application of a trading strategy, to compute the value-weighted returns we used as weights the market capitalization of the first day of the month considered. Results are in Table 4. Our

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	0.94	2.88	0.21	0.41***	0.38***
				[3.72]	[3.70]
P2	0.99	3.66	0.40	0.29***	-0.26***
				[2.63]	[2.89]
P3	0.99	4.30	0.25	0.18*	0.18*
				[1.81]	[1.68]
P4	0.81	4.82	0.11	-0.06	-0.01
				[-0.40]	[-0.06]
P5	0.63	5.66	0.04	-0.31	-0.22
				[-1.22]	[-1.00]
P5-P1	-0.32			-0.72**	-0.59**
	[-0.80]			[-2.53]	[-2.50]

Table 4: Forming value-weighted quintile portfolios every month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1

findings have the same patterns of Ang et al. (2006) but are in magnitude closer to the ones we have found on the previous dataset (Table 1, Table 2 and Table 3). In Table 4 P5-P1 return is -0.32% but it's not statistically significant. The CAPM and Fama-French three-factor models alphas are in magnitude smaller than Ang et al. (2006), but are statistically significant. Overall, we observe additional evidence of the IVOL Puzzle, since CAPM and Fama-French three-factor model misprice the P5-P1 portfolio's alphas yielding statistically significant monthly alphas of -0.72% and -0.59% on the long P5 short P1 strategy. The alphas that are statistically significant are from P1 and P2 meaning the two models used fail to price assets with low levels of idiosyncratic risk. The decreasing pattern in market share from P1 to P5 is decreasing starting from P3 as in Table 1, Table 2 and Table 3. Plotting the monthly returns of Portfolio 5 against Portfolio 1 (Figure 6) we observe a few characteristics which were shown in Table 4 too. Portfolio 5 (blue line) which contains the stocks with greatest exposure to idiosyncratic risk vary much more than the Portfolio 1 (black line). Deserve a mention also the strong *cyclical* nature of the monthly returns for both portfolios.

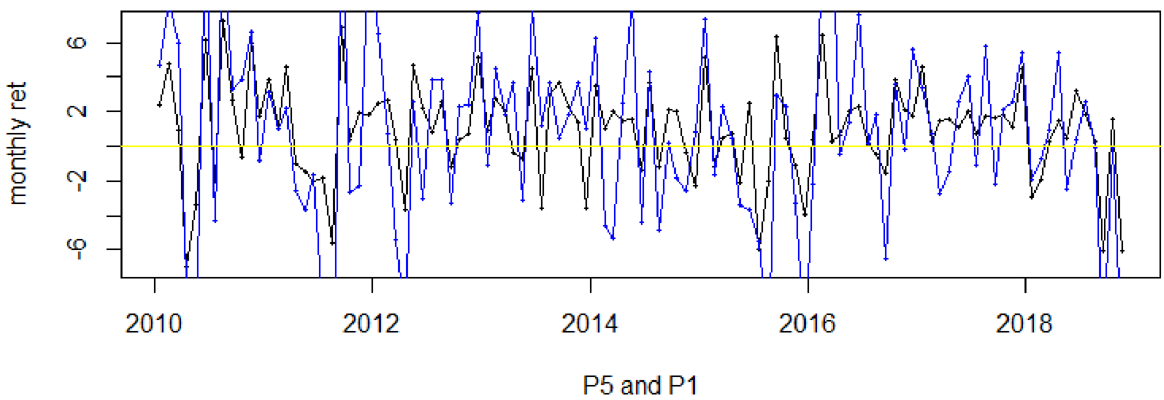


Figure 6: Plot of the monthly returns for Portfolio 1 (stocks with lowest idiosyncratic volatility levels) and Portofolio 5 (stocks with highest volatility levels). P1 is the black line, P5 is the blue line.

Figure 7 shows the monthly returns of the P5-P1 which represents the trading strategy of going short on the portfolio stocks with low idiosyncratic risk exposure and long on the portfolio formed by stocks with high idiosyncratic volatility levels. Besides two peaks of large positive returns (end of 2011 and start of 2016) and the *cyclical* nature of the monthly returns, we know by Table 4 that the average monthly performance of P5-P1 has been -0.32%.

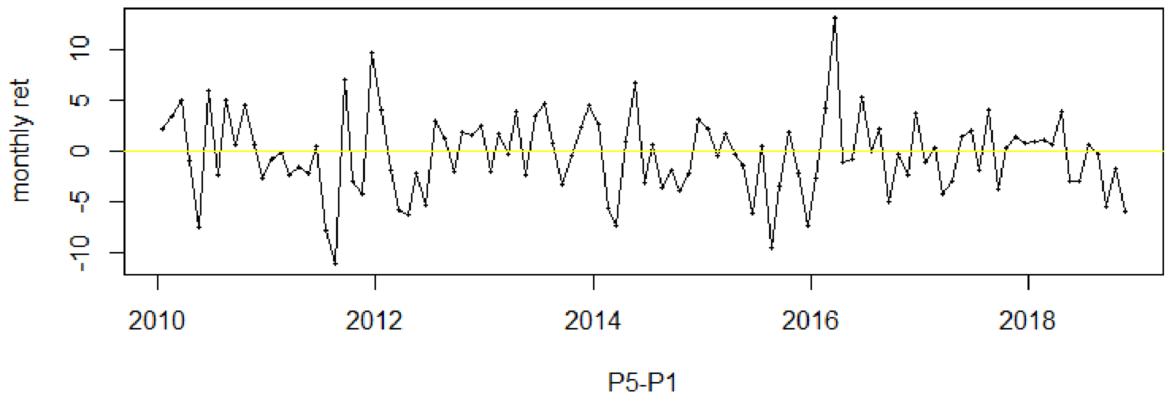


Figure 7: Plot of the monthly returns of trading strategy P5-P1 (long on Portfolio 5 and short on Portfolio 1).

6.3 Trading strategy 1/0/3

As in the previous strategy, we sort based on 1 month lagged IVOL, although this time we hold the portfolios for 3 months. The rebalancing therefore is made quarterly. Results are in Table 5. The findings have the same patterns of the previous strategies. This time the magnitude of the values is -1.30% approximately three times bigger than 1/0/1. Hence negative compensation for holding idiosyncratic risk holds keeping the portfolios for 3 months compared to one. Compared to Ang et al. (2006), here P5-P1 return is not statistically significant. Alphas relative to CAPM and Fama-French three-factor models for P5-P1 are -2.67% and -1.95% about three times bigger than strategy 1/0/1 and are strongly significant. Mean and market share are decreasing going from P1 to P5 showing the same pattern of the previous strategies.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	2.77	5.18	0.24	1.28***	1.00**
				[2.72]	[2.19]
P2	2.89	6.80	0.40	0.90***	0.66***
				[3.17]	[2.85]
P3	2.48	7.76	0.23	0.17	0.23
				[1.26]	[0.88]
P4	1.86	9.23	0.10	-0.84	-0.65
				[-1.58]	[-1.05]
P5	1.46	10.35	0.04	-1.38*	-0.94
				[-1.76]	[-1.22]
P5-P1	-1.30			-2.67***	-1.95**
	[-0.97]			[-3.04]	[-2.09]

Table 5: Forming value-weighted quintile portfolios every three months we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in quarterly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in quarterly returns between portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/3

6.4 Trading strategy 1/0/6

This time the holding period of the portfolios sorted based of 1 month lagged IVOL is 6 months. The rebalancing is semesterly. Results are in Table 6. Compared to 1/0/3, the gap in return between P5 and P1 substantially shrinks (-0.21% for 1/0/6 compared to -1.30% for 1/0/3). Since the holding period is twice the size, assuming the compensation was still negative from month 3 to 6 and given the same pattern we found with previous strategies, we were expecting a bigger gap. About the other statistics, standard deviations are just slightly increasing from P1 to P4 and decreasing from P4 to P5. The evidence of small size firms in P5 compared to the other portfolios completely disappeared. Furthermore, the alphas shrink too (-0.95% and -0.28%) compared to 1/0/3 strategy and they are not statistically significant.

The fact IVOL Puzzle vanished with a holding period of 6 months could be caused by several reasons. We state that testing different holding periods is a way to bring new evidence to the heterogeneity of investors' investment horizon hypothesis. Since we are not decomposing how the different time scales are affecting the price, holding a portfolio for a given period means to quantify the sum of the effects each time scale has. We interpret each time scale has the

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	5.63	7.22	0.25	1.83***	0.68
				[4.27]	[1.46]
P2	7.10	8.00	0.20	2.89***	2.74***
				[15.57]	[4.42]
P3	5.64	8.86	0.18	1.09	0.88
				[1.44]	[0.86]
P4	5.55	9.39	0.18	0.72	0.13
				[0.68]	[0.17]
P5	5.42	9.00	0.19	0.88	0.96
				[0.90]	[0.52]
P5-P1	-0.21			-0.95	0.28
	[-0.40]			[-1.12]	[0.25]

Table 6: Forming value-weighted quintile portfolios every six months we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in semesterly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in semesterly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/6

compensation required by the investors operating with a time horizon equal to that time scale. Therefore, the fact the IVOL Puzzle (a lower compensation for high IVOL stocks compared to low IVOL stocks) is reduced till to disappeared, can be explained by the presence of a positive compensation for bearing idiosyncratic risk approximately from the 3^{rd} to the 6^{th} month. We don't know if the sign changed exactly at the start of the 4^{th} month, as we don't know when and if the positive compensation changes sign again later.

6.5 Trading strategy 1/0/12

Holding period is set to 12 months for portfolios based on 1 month lagged IVOL. Compared to Ang et al. (2006), that monthly rebalance 1/12 of the allocation to assure an always high exposure to IVOL, our trading strategy as all the previous one has the rebalancing equal to the holding period. Results are in Table 7:

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	10.60	9.53	0.34	5.32	5.26**
				[0.61]	[1.99]
P2	10.19	13.26	0.35	2.75***	2.01***
				[4.40]	[6.96]
P3	9.78	15.31	0.20	1.41	1.04
				[1.14]	[1.14]
P4	6.63	15.91	0.09	-1.99**	-2.54***
				[-2.29]	[-7.24]
P5	6.17	22.57	0.03	-5.34***	-3.75***
				[-5.44]	[-5.34]
P5-P1	-4.44			-10.66***	-9.00***
	[-1.34]			[-4.63]	[-4.17]

Table 7: Forming value-weighted quintile portfolios every twelve months we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in annual percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in annual returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/12

Increasing additionally the holding period from 6 to 12 months, the IVOL Puzzle appears again. Standard Deviations increase from P1 to P5 while Market Share decrease from P1 to P5. The usual patterns from P1 to P5 for each statistics are back. The alphas' t-statistics are more robust than the other strategies. The 1/0/12 strategy brings again evidence of the IVOL Puzzle.

7 Trading strategy's results

7.0.1 Idiosyncratic Volatility Puzzle's post-crisis evidence

Main theme of the thesis is to search for evidence about the IVOL Puzzle in the post-crisis period. We apply several trading strategies (2/0/1, 1/0/1, 1/0/3, 1/0/6 and 1/0/12) based on IVOL, to see if the negative relationship between IVOL and returns is still present after the crisis. Across these strategies, even if the spread in returns between P5 and P1 was usually not statistically significant, we always have found relevant and strongly significant spread alphas between P5 and P1 relative to CAPM and Fama-French 3-factor models (but for trading strategy 1/0/6). What has been interesting to see is that all the patterns which made Ang et al. (2006, 2009)'s results provocative are still present in the financial markets. We refer to the low performance of high IVOL stocks (that are usually small size stocks), which made the cross-sectional relationship between idiosyncratic risk and returns the Puzzle we still face nowadays. We find for trading strategy 1/0/1 an average monthly return from going long on high IVOL stocks and short on low IVOL stocks of -0.32%, although the value is not statistically significant. However we find always relevant and statically significant alphas (but for trading strategy 1/0/6), meaning the strategies are able to "beat the market". This expression is used when active managers form portfolios capable of gaining actual returns that exceed risk-adjusted expected returns. The total actual return minus the risk-adjusted expected return equals the "alpha" gained and it measures the value the active managers bring into the investment process. We replicate the framework of Ang et al. (2006) for the post-crisis sample period (2010-2018), therefore we don't know if the missing significance for the average P5-P1 strategy return is caused by the different sample period or by a different approach to the Puzzle in the codes written on R program. Would be interesting to apply the same analysis to the Ang et al. (2006)'s sample period aiming to capture their same results. Given the sensitivity of the findings to changes in the coding side of the analysis, especially for the computation of the value-weighted returns, an equivalent result for the same sample period would confirm the validity of our model ruling out eventual doubts about the findings. The missing statistical significance of the P5-P1 average return is the only difference with Ang et al. (2006)'s findings. Our outcomes show evidence of the IVOL Puzzle in the post-crisis period[16].

7.0.2 Heterogeneity of investors' investment horizons hypothesis

Second goal of the thesis is to test the heterogeneity of investors' investment horizon hypothesis, that started to be developed from Brandt et al. (2009). Key intuition behind is the presence of several kind of investors, which implies heterogeneity of their needs and consequently of their investment horizons in financial markets. Malagon et al. (2015), is the first one to use Wavelet Multi-Resolution Analysis for IVOL Puzzle in order to properly disentangle the time scale that compose the final price. They find a negative relation between IVOL and returns for short-term investors while a positive one for long-term investors. In our study, we play with the holding period (and the rebalancing frequency) to see how the relation behaves for a holding period of 1, 3, 6 and 12 months. This approach is different from the Malagon et al. (2015). For example, when we test the relationship for an holding period of one month, the result (that is the final return we find for that strategy) should have inside all the smaller investment horizons effects that converge in our framework in a unique number. Malagon et al. (2015) find a negative relation for a time scale from day 2 to day 4, while a positive one for time horizons bigger than 16 days. Therefore, when we apply strategy 1/0/1, the final relationship has inside both the short/long-term effects found by Malagon et al. (2015). Crucial is the setting of the Wavelet Multi-Resolution Analysis, which allows to capture several different time scales. This technique

[16] 2010-2018.

divides the process into a ”smooth” and one or more ”detail” effects. Malagon et al. (2015) set Wavelet Multi-Resolution Analysis with one, two and three level. Hence they test respectively from day 2 to day 4 (D1) and more than 4 days (S1), D1 and from day 4 to day 8 (D2) and more than 8 days (S1) and D1 and D2 and from day 8 to day 16 (D3) and more than 16 days (S1). Since the technique is recursive, the detail effects will be the same for all the levels. Different is the setting for Yin et al. (2019), discovering that for a short-term investment horizons (less than 4 months) the relationship is negative, then is positive for an intermediate investment horizons (between 4-16 months) and finally is negative again for a long-term investment horizons (more than 16 months). Our analysis is not considering the single time scales effects, it evaluates instead the performance of different degrees of activeness in the portfolio management. However, the two approaches are not completely separated. In fact, if increasing the holding period the IVOL Puzzle weakens, this could mean that investors with a bigger time scale are demanding a premium for bearing idiosyncratic risk. What we observe in our results is that there is a negative premium for holding high IVOL stocks for 1 month and for a 3 months holding period. With trading strategy 1/0/6, the compensation changes. If we hold the high IVOL stocks for 6 months the IVOL Puzzle disappears, since there is no significance difference between P5 and P1 in returns and in the Jensen's alphas relative to CAPM and Fama-French 3-factor models. A possible explanation is that there is an inversion of the relationship's sign between IVOL and returns from investors with an investment horizon of 4, 5 and 6 months. Since our results for trading strategy 1/0/1 and 1/0/3 still show a negative relationship, the change of sign should be between month 4 and 6. Overall the result on trading strategy 1/0/6 is that there is no significant relationship between IVOL and returns. Moving to the next strategy's results, holding the high IVOL stocks for 12 months the Puzzle shows off again.

We are going to try to interpret the overall result about the investigation of the relevance of the heterogeneity of investors' investment horizon hypothesis, given the studies provided by Malagon et al. (2015) and Yin et al. (2019). Even if both the studies disentangle the effect of different time horizons, they consider different time scales. Malagon et al. (2015) works with days and their maximum horizon is ”more than 16 days” while Yin et al. (2019) works with months and their maximum horizon is more than 16 months. We state the performance of trading strategy 1/0/6 could be explained by a change in sign of the relationship IVOL-returns somewhere between 4^{th} and 6^{th} months. The following months after the 6^{th} cannot be interpreted easily because we just know the overall relationship IVOL-returns after 12 months (which is negative and significant). This doesn't exclude a possible positive relationship IVOL-returns for investors with an investment horizon lasting slightly more than six months . What we observe is that on average at the end of the 12^{th} month the relationship is negative again. This change of sign somewhere around the middle of the year can fit with results of Malagon et al. (2015) and Yin et al. (2019). In particular Yin et al. (2019) find that before the 4th month the relationship IVOL-returns is negative which is in line with our results (both 1/0/1 and 1/0/3 have evidence of IVOL Puzzle). They found a positive relationship between 4^{th} and 16^{th} month while we found

evidence of a possible positive relationship between approximately 4^{th} and $6^{th}/7^{th}$ month. Eventually their sign changes again for investment horizons bigger than 16 months while our results show a negative relationship at least till 12^{th} month.
Since a given holding period return should be the results of all the investment horizons that compose the period, the fact we spot a different compensation for a different holding period is an evidence supporting the heterogeneity of investors' investment horizon hypothesis. Generally speaking, our results show that the sign compensation for bearing idiosyncratic risk is holding-period dependent following an inverted U-shape trend.

8 Practical applications of Idiosyncratic Volatility Puzzle

8.1 Implications for several kind of investors

Aiming to construct some applications from the theoretical effect we discover about the Idiosyncratic Volatility Puzzle, we now describe how the main investors in the markets could exploit our findings.
Speculators buy a firm's stock based on the chance the price will go up or down. Their reasoning is purely based on the amount of price change instead that on the fundamental values of the firm. Speculators aim to find past trend in prices to earn abnormal profits, they usually exploit short-term strategies aiming to perform better than long-term passive investors. Therefore, monthly rebalancing portfolio 1 (made by low IVOL stocks) and portfolio 5 (made by high IVOL stocks) they can exploit trading strategy P1-P5 with setting 1/0/1 to beat the market because it's an alpha generating strategy.
Speculators enjoy volatility because of potential large returns they can extract, hedgers instead aim to reduce risk toward zero. The goal is usually accomplished by taking an opposite position on a derivative having as underlying the security that needs to be hedged. Another way is to take an opposite position on a security highly correlated with the original security or a same sign position on a security with negative correlation with the original one. We find a correlation of -0.56 between P1-P5 strategy and the market, hence hedgers with a main position on a market index can benefit from the Puzzle. Hedging can be referred to source of a risk, meaning that we want to hedge our portfolio from a particular source of risk. For example, inside a portfolio made of global equities, an investor should consider the currency risk which is the risk derived by movement in currency prices. With our study we reveal that idiosyncratic risk is present and negatively priced by the market (not always as we find the relationship is holding period-dependent). Therefore, it is a source of risk that should be considered by a hedger.
Other players that can exploit the Puzzle are funds. They are generally divided into active or passive funds. An actively managed investment fund has a manager or a management team which decide how to invest the money [17]. Their aim is to "beat a specific benchmark", creating

[17] https://www.thebalance.com/actively-vs-passively-managed-funds-453773

value by actively investing or disinvesting following their personal strategy. By contrast, passive funds simply hold a market index. The idea of holding the market index seems unattractive compared to putting a manager skills into practice to create value by actively managing the funds, since the passive funds can't yield huge returns unless the market itself has a strong upward trend. Despite this, there is a strong evidence from literature showing that the *average* active manager doesn't capture alpha net of fees and expenses Jensen (1968). Besides this, there is a statistically significant evidence showing that a small group of them (called Superior Active Managers) have persistent skills compared to the Inferior Active Managers as pointed out by Kosowski et al. (2007).

We find above average performance of stocks with the lowest exposure to idiosyncratic risk, meaning that investors are still not fully exploiting this "inefficiency" of the market. The presence of informed traders that gathers and process information imply they earn an above average excess returns otherwise they would have no incentive to reflect the new information into prices. This intuition has been proposed by Grossman and Stiglitz (1980) where they stated that markets need to be "mostly but not completely efficient" otherwise investors would not make efforts to check whether the prices are fair or not. Therefore, active managers would exploit the Puzzle adding to their allocation exposure to the strategy of going long (short) on low (high) idiosyncratic volatility stocks. Including the P1-P5 strategy in their allocation would increase the alpha of their portfolio, which is a measure of the value added by the active manager.

The previous reasoning applies to a mean-variance optimizer investor too. Our findings shows how large and statically significant alphas can be generated with the P1-P5 strategy. Therefore, assuming that short-selling is not considered by an investor with an indexed position, going long on low idiosyncratic volatility stocks should increase the reward for unit of risk of the overall portfolio for a mean-variance optimizer investor.

8.2 Performance evaluation of P1-P5 and P1 strategies

We evaluate the performance of the P1-P5 and P1 strategies to generate risk-adjusted returns and alphas relative to: holding the market, size, value and momentum strategies. We compute risk-adjusted measures to quantify the reward for different unit of risk instead of just the holding period return for each strategy. These performance evaluation methods using mean-variance criteria came out simultaneously after CAPM has been developed. The mean of Size and Value is slightly negative while for market, momentum, P1-P5 and P1 strategies is positive (respectively 0.73%, 4.1%, 0.32% and 0.94%). Sharpe Ratios, which represent the reward for unit of total risk, are negative for Size and Value strategy. A negative Sharpe Ratio in this case is given by the slightly negative average of monthly return for both the strategies, this could represent negative expectation of returns for both the strategies but usually a negative SR doesn't bring any useful information. Pure MKT (market), MOM (momentum), P1-P5 and P1 strategies report respectively 0.19%, 0.14%, 0.08% and 0.31%.

Strategy	Mean	α	β	SR	TM	IR	σ	σ_e
MKT	0.73	0.00	1	0.19	0.73	0.00	3.81	0.00
Size	-0.059	-0.074	0.020	-0.045	-3.00	-0.056	1.33	1.32
Value	-0.012	-0.16	0.051	-0.072	-2.4	-0.094	1.70	3.70
MOM	0.41	0.52	-0.15	0.14	-2.7	0.18	2.90	11.20
P1	0.94	0.39	0.71	0.31	1.28	0.35	2.91	1.10
P1-P5	0.32	0.72	-0.56	0.08	-0.56	0.25	3.97	3.36

Table 8: Every performance evaluation measure is in percentage. Every measure is computed over the whole sample period (2010-2018) from data with monthly frequency. σ and σ_e represent respectively the volatility and the volatility of the residuals. Jensen's alphas and β are computed relative to CAPM model. SR, TM, and IR stand respectvely for Sharpe Ratio, Treynor measure and Information ratio.

Treynor measure as the Sharpe Ratio gives the reward for unit of risk, in which the risk is just the systematic one. Because of the sign of the mean of the monthly returns (for Size and Value) and of the sign of the correlation with the market (for MOM and P1-P5) the only positive Treynor measure belongs to the pure MKT and P1 strategies. When the beta β is negative the Treynor measure has not useful meaning (MOM and P1-P5 strategies).
Information Ratio measure represent the reward for unit of idiosyncratic risk. Size and Value Information Ratios are negative and don't bring useful interpretations. For MOM, P1 and P1-P5 strategies it is respectively 0.18%, 0.35% and 0.25%. The result support the P1-P5 strategy because it shows that exposure to idiosyncratic risk is rewarded better than MOM strategy which is overall performing well in most of the performance evaluation measures. Looking to the level of total and idiosyncratic risk for each strategy, we observe that the positive average return for MOM strategy is mainly because of the large idiosyncratic risk the strategy faces. P1 strategy outperforms every other strategy in each measures but for alpha α and its exposure to market risk $\beta = 0.71$ which is quite high, additionally it presents the lowest idiosyncratic risk level.

Overall the performance measures show how, besides holding the market, MOM, P1 and P1-P5 strategies perform better than Size and Value. The relevant and positive average monthly return, the Jensen's alpha, the reward for unit of total and idiosyncratic risk point out how MOM, P1 and P1-P5 are attractive and remunerative compared to Size and Value strategies.
Between MOM and P1-P5, MOM displays a mildly higher reward for unit of total risk while P1-P5 has a moderately higher reward for idiosyncratic risk. Hence the higher average mean of MOM relative to P1-P5 is because of a large exposure to idiosyncratic risk as can be seen from the idiosyncratic risk level in Table 8. P1-P5 is a long-short equity approach, given its negative beta β of -0.56 must be considered for its hedging quality too. P1 strategy instead is a long strategy as the others, given its large correlation to the market combined with its large and positive reward-risk ratios can generate sizeable returns and alphas.

8.2.1 Risk evaluation of P1-P5 and P1 strategies

We compute now two risk measures to capture the tail risks of the P1-P5 and P1 strategies compared to pure market (MKT), size, value and momentum (MOM) strategies.

Strategy	Mean	σ	σ_e	VaR	ES
MKT	0.73	3.81	0	-5.51	-7.10
Size	-0.059	1.33	-0.02	-2.24	-2.79
Value	-0.012	1.70	3.70	-2.91	-3.61
MOM	0.41	2.90	11.20	-4.36	-5.57
P1	0.94	2.91	1.10	-3.86	-5.07
P1-P5	0.32	3.97	3.36	-6.19	-7.85

Table 9: Tail-risks, values are in percentage

Tail risk must be considered during performance evaluation of a strategy. The normal distribution is commonly attributed to returns, implying that usually returns are clustered around the mean drawing the known "bell shape" along their distribution. There is evidence (Figure 1) that the empirical distribution has fatter tails compared to the theoretical normal distribution. This fact means that "tail-events" (event at both the tips of the distribution) are more likely to happen compared to theoretical distribution. Investors must always consider the likely and magnitude of left-tail events, because they can easily compromise the whole portfolio performance.

Value at Risk (VaR) measures the maximum loss a strategy/portfolio can yield with a certain level of confidence and over a given time period. Our measures (both VaR and ES) have 95% interval of confidence and are monthly tail-risk measures. Expected Shortfall computes instead the average loss in the worst 5% of scenarios. Table 9 shows that P1-P5 is the riskiest strategy compared to the others. Given the "bad" reward-risk ratios we computed in Table 9 for Size and Value strategy we compare just MKT, MOM, P1 and P1-P5 strategies. P1-P5 has the highest monthly VaR and ES, but MKT is really close in both measures. Monthly VaR for P1-P5, P1, MKT and MOM is respectively -6.19%, -3.86%, -5.51% and -4.36%. Therefore, P1-P5 stands out as the riskiest but when compared to the other two remunerative strategies (MKT and MOM) the VaR/ES difference is not huge. P1 strategy between all the remunerative strategies (MKT, MOM and P1-P5) has the lowest Value at Risk and Expected Shortfall.

8.2.2 Cumulative Returns

Plotting the cumulative returns of Table 8's strategies (Figure 8), we examine how much investing in each strategy since January 2010 yields for each month over time. We observe that P1 strategy outperforms every other strategies in terms of mean return, Sharpe Ratio and cumulative return. P5-P1 strategy slightly under-performs the momentum strategy in terms of cumulative returns. P5-P1 is a long-short equity strategy, this technique is often used by hedge funds to gain both from the increase and decrease of prices of different securities in the market. A portfolio with this setting protects itself from losses during market downturns, because of this when the strategy has a close to zero correlation relative to the market it's called a "market-neutral" strategy. The beta β of this strategy is -0.56 (Table 8) therefore the strategy, besides performing on average better than Size and Value strategies, can be used as a hedging instrument against market risk. Size and Value strategies as expected from Table 8 perform poorly post-crisis.

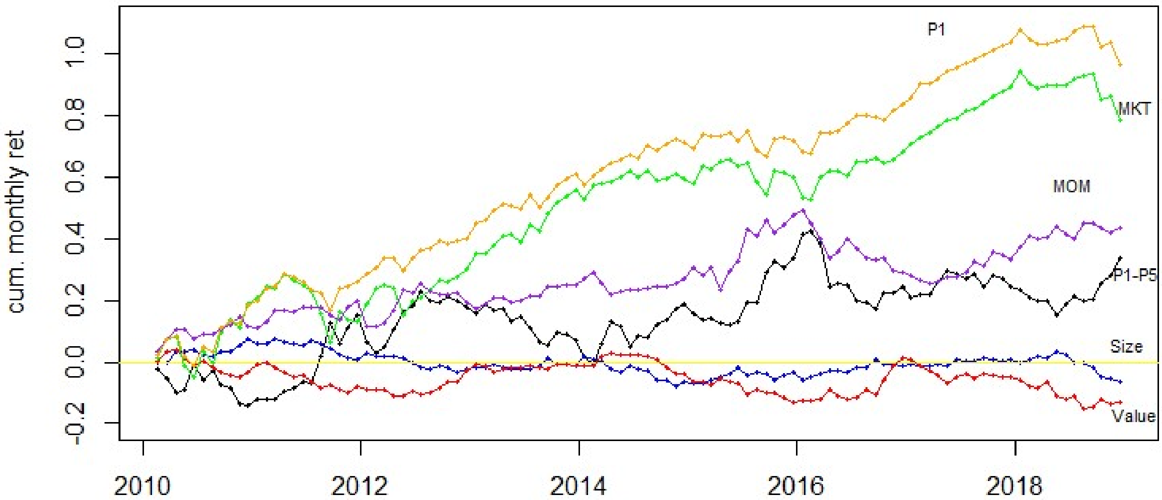


Figure 8: Cumulative returns for each strategy: P1(yellow), Market(green), Momentum(purple), P1-P5(black), Size(blue) and Value(red). Sample period: January 2010-December 2018.

8.3 Cost of ignoring low idiosyncratic volatility stocks

Having a positive alpha is a *necessary*, but not *sufficient* condition for a portfolio to outperform the index (MKT). A portfolio can yield a postive alpha but still not outperform the index because the standard deviation is high enough to decrease its Sharpe Ratio Bodie et al. (2014). This is what happens with P1 and P5 in our study, both yield positive alphas but given the higher standard deviation of P5 its Sharpe Ratio (0.11%) is smaller than MKT's one (0.19%). Since P1 Sharpe Ratio is 0.33% and P5 Sharpe Ratio is 0.11%, we consider P1 in the following tests.
We found evidence of Idiosyncratic Volatility Puzzle in Section 9 then we tested the P1-P5 strategy compared to 4 alternative strategies. In the following lines we try to quantify the cost of not considering the low idiosyncratic volatility stocks (P1).

8.3.1 The M^2 measure

The M^2 measure leads to an easy to interpret risk adjusted differential return relative to the benchmark (MKT strategy in our case). The intuition behind it is to use a risk-free asset to construct a portfolio containing the P1 and the risk-free asset. The weights are determined in order to make the overall volatility of this portfolio equal to the MKT's standard deviation, we are not considering the covariance since the risk free asset is not correlated with P1. MKT's standard deviation is 3.81% while P1's standard deviation is 2.88. Hence $3.81/2.88 = 1.32$ is the weight of P1 and $1-1.32 = -0.32$ is the weight of the risk-free asset. The average risk-free return in our data is 0.022 hence the return of our new portfolio is: $1.32 * 0.94 - 0.32 * 0.022 = 1.23$. The M^2 measure given our portfolio 1 and our index MKT is $M^2 = 1.23 - 0.73 = 0.50$. 0.50% monthly higher reward for the same amount of risk express the good performance of the P1 portfolio compared to the market.

8.3.2 Mixing an indexed allocation with active portfolio P1

For an investor which holds the market index MKT, the optimal risky portfolio is a combination of the index portfolio MKT and the active portfolio (in our case P1) Bodie et al. (2014). Our goal is to maximize the overall Sharpe Ratio of the portfolio made by MKT and P1. To find the optimal weights we use:

$$w_A^* = \frac{w_A^0}{1 + (1 - \beta_A)w_A^0} = \frac{\frac{\frac{\alpha_A}{\sigma_A^2}}{\frac{\mathbb{E}(R_{MKT})}{\sigma_M^2}}}{1 + (1 - \beta_A)\frac{\frac{\alpha_A}{\sigma_A^2}}{\frac{\mathbb{E}(R_{MKT})}{\sigma_M^2}}} = \frac{\frac{\frac{0.39}{1.21}}{\frac{0.73}{14.51}}}{1 + (1 - 0.70)\frac{\frac{0.39}{1.21}}{\frac{0.73}{14.51}}} = 2.19 \tag{13}$$

Weight for P1 is 2.19 while for MKT is -1.19. Given these weights the increased overall Sharpe Ratio is[18]:

$$S_P^2 = S_M^2 + \left[\frac{\alpha_A}{\sigma(e_A)}\right]^2 = 0.036 + 0.122 = 0.158 \tag{14}$$

The squared Sharpe Ratio increases exactly of the amount of the squared Information Ratio. The previous formula is useful when an investor wants to add an active portfolio to an indexed position. The Information Ratio is a reward to risk ratio in which the reward is the return not produced by exposure to systematic risk and the risk is the amount of idiosyncratic volatility. Mixing the indexed position with the active portfolio means to tilt the indexed portfolio toward risk could have beeen diversified. The trade-off between exposure to idiosyncratic risk and the alpha generated it's represented by the Information Ratio.

A weight of 2.19 is an extreme position towards the active portfolio P1, to avoid corner solutions, huge exposure to idiosyncratic risk and to obtain more reasonable results we put the constraint $w_A^* \leq 0.2$. Therefore we are going to set it equal to 0.2 which is nevertheless a large tilt from the initial allocation toward the P1 strategy.

[18] Bodie et al. (2014)

8.3.3 Utility cost of not considering low idiosyncratic volatility stocks

Our hypothetical investor holds the index (MKT) without mixing it with the risk-free asset. Being a mean-variance optimizer, he maximize its utility function:

$$\mathcal{U}_0(\mathbb{E}(R_{MKT}), \sigma^2_{MKT}) = \mathbb{E}(R_{MKT}) - \frac{1}{2}A\sigma^2_{MKT} \tag{15}$$

We don't know its $\mathcal{U}_0$ at t_0 since we ignore its risk aversion A. We estimate it knowing that he maximized:

$$\mathcal{U}(\mathbb{E}(R_p), \sigma^2_p) = \mathbb{E}(R_p) - \frac{1}{2}A\sigma^2_p = \tag{16}$$

$$= r_f + w\left[\mathbb{E}(R_{MKT}) - r_f\right] - \frac{1}{2}Aw^2\sigma^2_{MKT} \tag{17}$$

Maximizing (17) the investor finds the optimal weights between MKT and the risk-free asset. Since we assumed he just holds the market we know its weights, hence:

$$\max_w \mathcal{U}(\mathbb{E}(R_p), \sigma^2_p) = \max_w \left[r_f + w\left[\mathbb{E}(R_{MKT}) - r_f\right] - \frac{1}{2}Aw^2\sigma^2_{MKT}\right] \tag{18}$$

We calculate the First Order Condition (FOC) with respect to w and set it equal to 0:

$$FOC(w) = \frac{\partial\mathcal{U}(\mathbb{E}(R_p), \sigma^2_p)}{\partial w} = \mathbb{E}(R_{MKT}) - r_f - Aw\sigma^2_{MKT} = 0 \tag{19}$$

From (19) with a few steps we get its optimal weights:

$$w = \frac{\mathbb{E}(R_{MKT}) - r_f}{A\sigma^2_{MKT}} \tag{20}$$

$$w_{r_f} = 1 - w \tag{21}$$

We set the optimal weights equal to 1 and 0 respectively ($w = 1$ and $w_{r_f} = 0$), now we can compute its risk aversion:

$$A = \frac{\mathbb{E}(R_{MKT}) - r_f}{w\sigma^2_{MKT}} = \frac{0.0073 - 0.00022}{1 * 0.0381^2} = 4.88 \tag{22}$$

We now can compute the utility of our investor holding the indexed position:

$$\mathcal{U}_0(\mathbb{E}(R_{MKT}), \sigma^2_{MKT}) = \mathbb{E}(R_{MKT}) - \frac{1}{2}A\sigma^2_{MKT} = 0.0073 - \frac{1}{2} * 4.88 * 0.0381^2 = 0.0038 \tag{23}$$

The new utility mixing the indexed position with the active portfolio P1 using the optimal weights is:

$$\mathcal{U}_1(\mathbb{E}(R_{MKT+P1}), \sigma^2_{MKT+P1}) = \mathbb{E}(R_{MKT+P1}) - \frac{1}{2} A\sigma^2_{MKT+P1} \tag{24}$$

In which the expected return of the new portfolio MKT+P1 is:

$$\mathbb{E}(R_{MKT+P1}) = w_P 1^* \mathbb{E}(R_{P1}) + (1 - w_P 1^*)\mathbb{E}(R_{MKT}) = 0.2 * 0.94 + 0.8 * 0.73 = 0.77 \tag{25}$$

while the variance of the new portfolio MKT+A is computed using (7) from Markowitz (1952):

$$\sigma^2_{MKT+P1} = (w^*_{P1})^2 \sigma^2_{P1} + (1 - w^*_{P1})^2 \sigma^2_{MKT} + \frac{1}{2} w^*_{P1}(1 - w^*_{P1})\sigma_{MKT}\sigma_{P1}\beta_{P1} = \tag{26}$$

$$= 0.2^2 * 8.30 + 0.8^2 * 14.51 + 0.2 * 0.8 * 2.88 * 3.81 * 0.70 = 10.85 \tag{27}$$

Therefore the new utility is:

$$\mathcal{U}_1(\mathbb{E}(R_{MKT+P1}), \sigma^2_{MKT+P1}) = \mathbb{E}(R_{MKT+P1}) - \frac{1}{2} A\sigma^2_{MKT+P1} = \tag{28}$$

$$= 0.0077 - \frac{1}{2} * 4.88 * 0.033^2 = 0.0050 \tag{29}$$

The cost of ignoring low volatility for a mean-variance optimizer investor in the American financial market that has an indexed position (MKT, which represents all CRSP firms) is represented by the percentage improvement of its utility function . We find the utility function is 31% higher than the initial allocation: $(\frac{0.0050}{0.0038} - 1) * 100 = 31\%$.

9 Fourier and Wavelet methods for time series in Finance

9.1 From time domain to frequency domain

In several fields, the time domain analysis of a variable can be enhanced by the frequency domain analysis. A time domain series is a variable which is function of time, therefore is indexed in time order and plotting the variable we obtain a time amplitude representation. Studying the frequencies of a process, we can observe characteristics hidden in the frequency domain representation. The frequency is measured in Hertz which is defined in cycles/second. All the frequency components of a signal/process/series are called frequency spectrum.
The mathematical tools used to go from time to frequency domain are generally called transforms and the most popular one is the Fourier transform.

9.2 Spectral analysis and the Fourier transform

Aim of the spectral analysis is to find a new sequence $X(f)$, of the process $x(t)$ we are studying, representing the contribution of each frequency component in the original time series (Masset (2008)). The discrete version of the Fourier transform is:

$$X(f) = \sum_{t=-\infty}^{\infty} x(t)e^{-i2\pi ft} \tag{30}$$

where f represents the frequencies. Thanks to the De Moivre's theorem we can write:

$$e^{-i2\pi ft} = cos(2\pi ft) - isin(2\pi ft) \tag{31}$$

which decompose $x(t)$ into a set of sinusoidal functions representing a set of distinct frequencies component. Fourier transform is a reversible transform meaning implying that from the spectrum of a signal we can obtain the signal itself:

$$x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(f)e^{-i2\pi ft}df \tag{32}$$

One of the limitations of the Fourier transform is that frequency information is not available in the time-domain and time information is not available in frequency domain. This limitation implies we can observe which frequency component exists in the signal but not when in time the component exists. The family of series which don't require both the informations are the stationary processes, because for them all frequency components exists at all times as stated by Polikar et al. (1996).

9.3 Wavelet transform

In finance often the data doesn't satisfy the stationarity feature. We show in section 4 that volatility exhibit non trivial patterns as jumps, clustering and long memory. A recent transform which doesn't require this assumption is the wavelet transform. With wavelets we can obtain a time-frequency representation of our data.

9.3.1 Theoretical background

By wavelet we mean a wave that grows and decays in a limited time frame, different from the Fourier transform in which the sine and cosine functions have an unlimited time frame. Wavelet analysis is based on a main function called the *mother wavelet* denoted by $\psi(t)$, which must satisfy the two following conditions:

$$\int_{-\infty}^{\infty} \psi(t)dt = 0 \tag{33}$$

$$\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1 \tag{34}$$

plus the *admissibility condition* stating that if the Fourier transform of a function:

$$\Psi(f) = \int_{-\infty}^{\infty} \psi(t)e^{-i2\pi ft}dt \tag{35}$$

is such that:

$$C_\Psi = \int_{0}^{\infty} \frac{|\Psi(f)|^2}{f} df \tag{36}$$

satisfies $0 < C_\Psi < \infty$ then the wavelet function $\psi(t)$ is admissible. *Admissibility condition* allows to go from the continuous wavelet transform of a function to the function itself (Percival and Walden (2000)).

9.4 The Continuous Wavelet Transform (CWT)

Following Masset (2008), goal of CWT is to quantify the change of a function at a particular frequency and at a particular point in time. To do this, the *mother wavelet* is scaled and translated:

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right) \tag{37}$$

in which u, s are respectively the location and the scale parameters. Projecting the original signal $x(t)$ into the *mother wavelet* $\psi_{u,s}(t)$ we obtain the function $W(u, s)$ which represent the CWT:

$$W(u, s) = \int_{-\infty}^{\infty} x(t)\psi_{u,s}(t)dt \tag{38}$$

Increasing (decreasing) s we can capture the changes of the functions on a large (small) scale therefore at a low (high) frequency. As pointed by Gençay et al. (2001) there are some limitations of the CWT. First, there a computational issue trying to analyze a signal using every wavelet coefficients, making this technique more suitable for functions than for finance time-series. Second, being $W(u, s)$ a function of two parameters it has a lot of redundant information.

9.5 The Discrete Wavelet Transform (DWT)

Contrary to CWT, the Discrete Wavelet Transform has a limited amount of coefficients because the *mother wavelet* is dilated and translet a limited number of times. This is obtained setting:

$$s = 2^{-j} \quad u = k2^{-j} \tag{39}$$

where j, k are the set of discrete translation and dilatation, implying that the wavelet transform is calculated only at *dyadic* scales (2^j). Another implication is that, being N the observations of our time series, the largest number of scales is the the integer J:

$$J = [log_2(N)] = [log(N)/log(2)] \tag{40}$$

this can be an issue because if the time series is not of *dyadic* length, observations must be added or removed.

Two discrete wavelet filter are behind the DWT. One is the *mother wavelet*, denoted $h_l = (h_o, ..., h_{L-1})$. The second one is the *father wavelet*, denoted $g_l = (g_0, ..., g_{L-1})$. Properties of the *mother wavelet* are:

$$\sum_{l=0}^{L-1} h_l = 0, \quad \sum_{l=0}^{L-1} h_l^2 = 1, \quad \sum_{l=0}^{L-1} h_l h_{l+2n} = 0 \quad \forall n \in \mathbb{N}_0 \tag{41}$$

Thanks to the above properties, h_l is a difference operator, the DWT has the variance of the original data and a multiresolution analysis can be performed. The *father wavelet* is a low pass filter and captures the long scales, hence the low frequency, smooth components of the series computing the "scaling" coefficients. The *mother wavelet* is an high pass filter and captures the short scales, high frequency, details components of the series. The *father wavelet* has the following condition:

$$\sum_{l=0}^{L-1} g_l = 1 \tag{42}$$

The first level of decomposition computes the wavelet and the scaling coefficients of the first scale, respectively $w_1(t)$ and $v_1(t)$ that are obtained in the following way:

$$w_1(t) = \sum_{l=0}^{L-1} h_l x(t') \quad and \quad v_1(t) = \sum_{l=0}^{L-1} g_l x(t') \tag{43}$$

in which $t = 0, 1, ..., T/2 - 1$ and $t' = 2t + 1 - lmodT$.

Thanks to the pyramid algorithm (procedure in figure 9), we can further decompose the low frequency scaling coefficients $v_1(t)$ into other to components. Therefore the second level decomposition has $w = [w_1, w_2, v_2]$ and the J level decomposition has $w = [w_1, ..., w_J, v_J]$.

algorithm.png

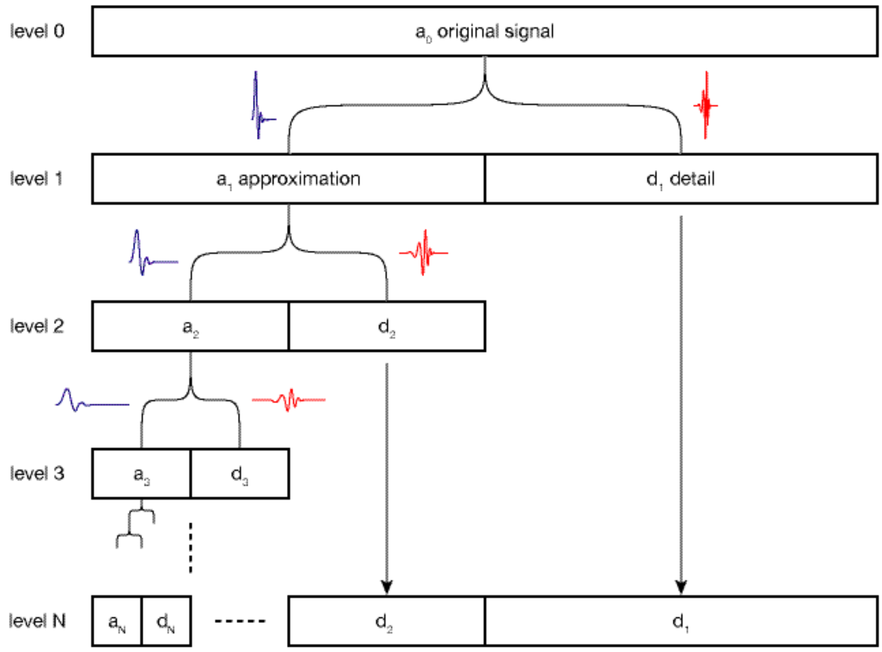


Figure 9: Pyramid algorithm doing a N level decomposition from Sundling et al. (2006)

9.6 The Maximal Overlap Discrete Wavelet Transform (MODWT)

To overcome the limitations we briefly described of DWT, we can use the MODWT. Contrary to the DWT, MODWT consider all the possible (integer) translations. Hence for every scale the wavelet coefficients, the scaling coefficients and the original series have the same length. At the first level decomposition with MODWT we have:

$$\widetilde{w}_1(t) = \sum_{l=0}^{L-1} h_l x(t') \quad and \quad \widetilde{v}_1(t) = \sum_{l=0}^{L-1} g_l x(t') \tag{44}$$

in which $t = 0, 1, \ldots, T$ and $t' = t - lmodT$. Using the pyramid algorithm, we can obtain the MODWT coefficients for further level of decomposition.

9.7 Wavelet filters

Many filters exists, the choice between them is done on a case by case basis. The most famous are Haar, Daubechies and Least-Asymetric filters. The properties they can have are: *symmetry*, *orthogonality*, *smoothness* and the number of *vanishing moments*.

Symmetric filters are appealing because they ensure there will be no displacements of the series in the time domain. One of the few filters which has this property is the Haar wavelet, *symmetry* is not an issue for MODWT because by construction all the coefficients are aligned.

The degree of smoothness of the filter must be related to the degree of smoothness of the function/series we are studying. The least the function/series is smooth, the least the filter must satisfy this property. Haar filter, being the least smooth, is particularly suitable for jump processes.

Orthogonality implies that wavelet and scaling coefficients have different information. Daubechies and Least-Asymmetric wavelets show this property.
Number of *vanishing moments* properties means that if the process is a polynomial of order p, the wavelet transform is able to capture it only if it has p *vanishing moments*. More vanishing moments means complex functions can be represented with a sparser set of wavelet coefficients.
A graphical representation of the main wavelet filter is in figure 10.

filters.png

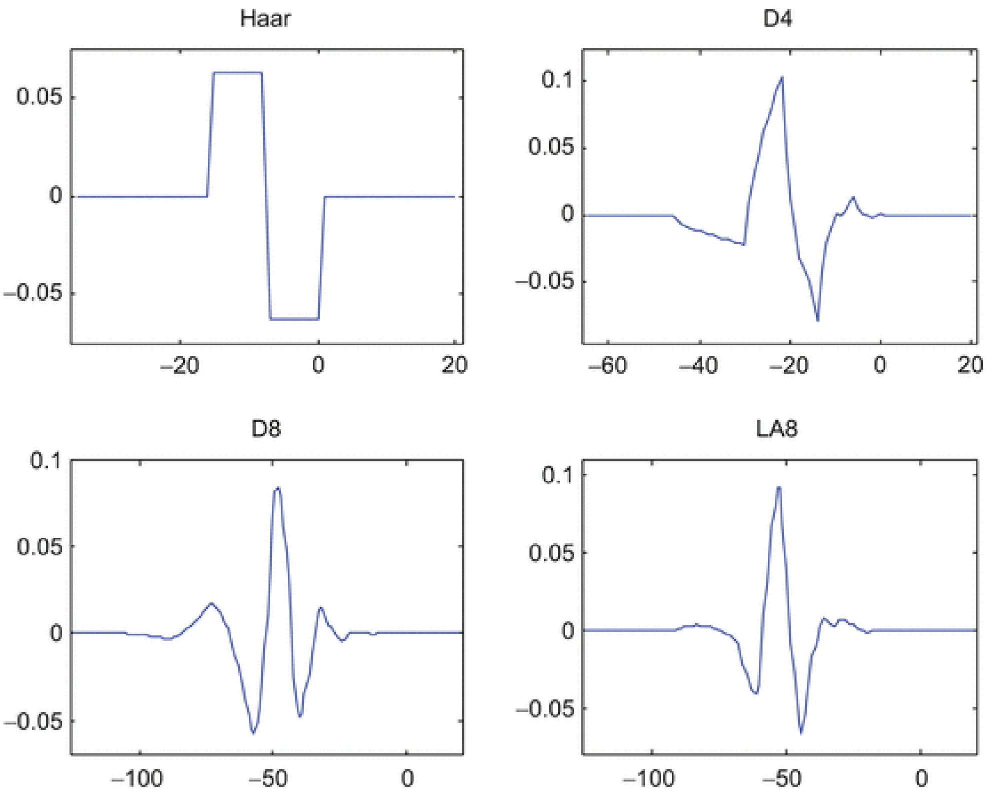


Figure 10: Most famous wavelet filters from Masset (2008)

10 Heterogeneity of investors' investment horizons hypothesis and Wavelet transform

Comparing the compensation required by investors, with different holding period, to bear idiosyncratic risk we observe for short holding period (1-3 months) a negative premium, for medium holding period (3-6 months) a zero premium and for long term holding period (6-12 months) again a negative compensation. Our hypothesis is that this result can be driven by investors with different investment horizon requiring different compensation for bearing idiosyncratic risk. Testing the trading strategy with increasing holding period can show the compensation of all the investors with investment horizon smaller or equal to the holding period but can't properly disentangle every required compensation. Therefore the return we observe at the end of the holding period of p months, is the aggregation of all the compensation required by investment horizons smaller or equal to p.

To study in detail the Idiosyncratic Volatility Puzzle, we use the Wavelet transform to study the contribution of each frequency (time scale) to the final holding period return of portfolio 5 and 1.

10.1 From frequencies to investors

We summarized in Section 9 the theoretical background, the properties and the applications of the wavelet transform. In finance, decomposing in frequencies a time series of returns, we look for an economical interpretation of the frequencies. Considering a short (long) time scale of our time series, means to capture the high (low) frequency contributions to our series. Therefore in our case, the high frequencies (short time scale) are the contribution of the short term investors to the series, while the low frequencies (long time scale) represent the contribution of the long term investors. This reasoning is reasonable as the short (long) term investors contribute to the most (least) frequent movements of the price. Different investors have different trading frequencies (Malagon et al. (2015))
The definition of short/medium/long term investors depends by the frequency of the data on which we do the wavelet transform. The technique creates frequency bands separated by multiples of 2^J. Therefore with daily data we can capture the contribution of investors with investment horizon of 2-4 days, 4-8 days, 8-16 days till the maximum admitted level of decomposition. Performing the wavelet transform, we decompose our time series S_0 into an approximation S_j (long time scale/low frequency) and details D_j (short time scale /high frequency).

10.2 Application to trading strategy 1/0/1

We apply a wavelet transform of level 6 to our framework, decomposing the daily returns and the daily factors into 6 details and one smooth component (example in figure 11). We use the la8 filter and the Maximal Overlap Discrete Wavelet Transform. The reason behind the MODWT is that we need to apply the transform to approximately 11.000 firms. To avoid issues linked to the required *dyadic* length of the data by the Discrete Wavelet Transform, we choose the MODWT. Once the data is decomposed, we test the trading strategy 1/0/1 for each details and smooth components. Aim of this framework is to capture the compensation required, for bearing idiosyncratic risk, by specific investors' investment horizon identified by each of the details and smooth components. The next sections cover just detail 1, 5 and 6 while the others are in the appendix.

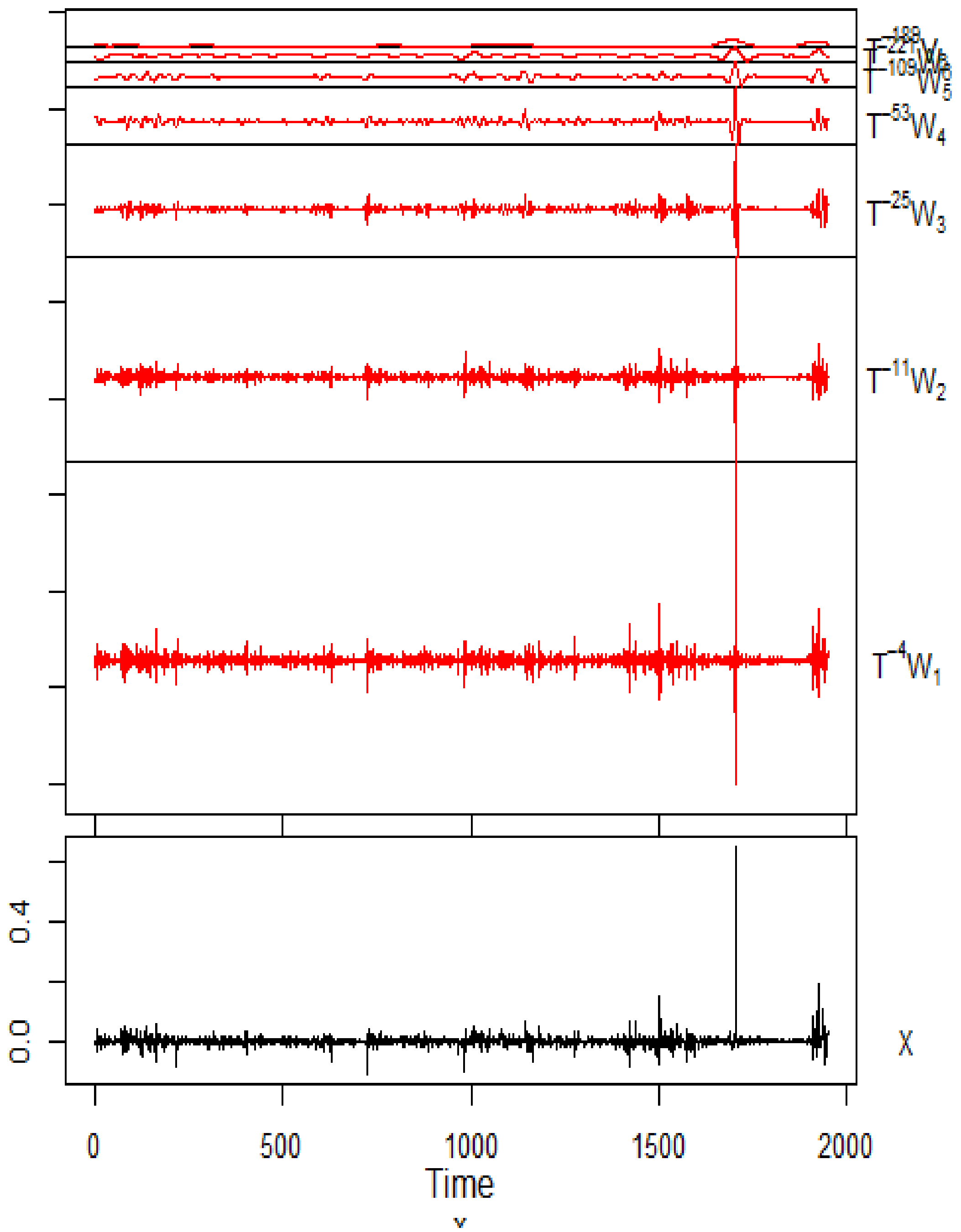


Figure 11: X is the time series of daily returns of an American firm in our dataset. Applying the wavelet transform with level 6, we obtain six details component called W and one smooth component defined V

10.3 D1 component

The D1 component, since the wavelet transform is done on daily data, represents the investment horizon from 2 to 4 days. Results are in Table 10.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	-0.06	0.40	0.23	-0.08***	-0.08***
				[-2.79]	[-3.06]
P2	-0.10	0.44	0.38	-0.12***	-0.12***
				[-4.13]	[-3.97]
P3	-0.15	0.51	0.24	-0.16***	-0.16***
				[-4.29]	[-4.51]
P4	-0.24	0.58	0.12	-0.26***	-0.26***
				[-7.45]	[-7.30]
P5	-0.42	0.70	0.04	-0.43***	-0.43***
				[-9.54]	[-9.96]
P5-P1	-0.35***			-0.34***	-0.34***
	[-12.4]			[-12.54]	[-12.67]

Table 10: D1 component, 2-4 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1

The trading strategy P5-P1, which represents the compensation for exposure to idiosyncratic risk, is negative and statistically significant. This result is in line with our previous results because we find strategy 1/0/1 (Table 4) having a negative monthly compensation. Alphas relative to CAPM and Fama-French 3-factor model are negative and statistically significant. We find the same patterns for standard deviation market share observed in Table 4.

10.4 D5 component

The D5 component, represents the investment horizon from 32 to 64 days. Running the analysis we find the results in Table 11.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	-0.03	1.40	0.19	-0.23***	-0.24***
				[-3.02]	[-3.03]
P2	-0.01	1.52	0.28	-0.21**	-0.23***
				[-2.21]	[-2.78]
P3	-0.01	1.59	0.26	-0.23***	-0.25***
				[-3.00]	[-3.34]
P4	-0.07	1.75	0.18	-0.32***	-0.34***
				[-3.59]	[-4.05]
P5	0.04	2.08	0.08	-0.22**	-0.24**
				[-2.15]	[-2.29]
P5-P1	0.07			0.01	0.01
	[1.22]			[0.08]	[0.04]

Table 11: D5 component, 32-64 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1

The trading strategy P5-P1, which represents the compensation for exposure to idiosyncratic risk, is positive. Alphas relative to CAPM and Fama-French 3-factor model are positive too. Table 11 shows investors with time horizon from 32 to 64 days demand a positive compensation for bearing idiosyncratic risk.

10.5 D6 component

The D6 component, represents the investment horizon from 64 to 128 days. In Table 12 there are our findings.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	-0.01	1.18	0.21	-0.15*	-0.15*
				[-1.75]	[-1.80]
P2	-0.02	1.46	0.28	-0.21***	-0.20***
				[-2.62]	[-2.66]
P3	0.02	1.56	0.26	-0.18**	-0.18*
				[-1.96]	[-1.90]
P4	0.07	1.92	0.18	-0.19*	-0.17
				[-1.70]	[-1.58]
P5	0.09	2.23	0.09	-0.19	-0.17
				[-1.53]	[-1.34]
P5-P1	0.10			-0.04	-0.02
	[1.51]			[-0.31]	[-0.15]

Table 12: D6 component, 64-128 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1

The trading strategy P5-P1, which represents the compensation for exposure to idiosyncratic risk, is positive. Alphas relative to CAPM and Fama-French 3-factor model are positive too. Table 12 shows investors with time horizon from 64 to 128 days demand a positive compensation for bearing idiosyncratic risk.

10.6 The smooth component

The smooth component, represents the time scales longer than 128 days. Results are in Table 13.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	1.26	1.70	0.14	1.07***	1.10***
				[4.20]	[4.66]
P2	1.11	1.80	0.23	0.92***	0.96***
				[4.00]	[4.00]
P3	1.05	1.59	0.27	0.88***	0.90***
				[4.11]	[4.29]
P4	1.14	1.76	0.24	0.93***	0.97***
				[3.92]	[3.70]
P5	1.19	2.31	0.12	0.94***	0.99***
				[2.75]	[3.23]
P5-P1	-0.06			-0.13	-0.11
	[-0.47]			[-1.01]	[-0.85]

Table 13: Smooth component, $>$ 128 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1

The compensation to bear idiosyncratic risk is negative again. The inverted U-shape the compensation draws for increasing time scales further explains our results in Section 7. The not constant performance of portfolio with increasing holding and rebalancing period, finds justification in the heterogeneity contribution of different frequencies to the series.

10.7 Wavelet transform's results

In Section 7 we interpret the performance of a trading strategy based on idiosyncratic volatility with increasing holding period. Referring to the new hypothesis in literature about the Puzzle being driven by the heterogeneity of investors' investment horizons hypothesis, we state that the almost null performance of trading strategy 1/0/6 could be motivated by a positive compensation demanded by investors with investment horizon between the 3rd and 6th month. This hypothesis finds support in section 9 where we apply the wavelet transform to study the different frequencies in our data. We observe a negative compensation for time scales going from 2 to 32 days, while the the investors with time scales from 32 to 128 days require a positive compensation. The smooth component, representing the long run gets negative again. Since in our data months are approximately 20 days long, we have a positive compensation in time scales from 1 month and a half to 6 months. The decomposition we perform brings additional evidence about the relevance of the heterogeneity of investors' investment horizons hypothesis. Our setting for the wavelet transform follows Malagon et al. (2015), who work with daily data and performs the Ang et al. (2006) trading strategy. Malagon et al. (2015) finds a negative compensation for short term investors and a positive one for the smooth component but fails to find the time scale where the compensation gets negative again. Yin et al. (2019) instead, working with monthly data and the Common Idiosyncratic Volatility factor (CIV, derived by principal component analysis), finds an inverted U-shape in the factor loadings of CIV. Because we observe an inverted U-shape too for the compensation for bearing idiosyncratic risk, our findings are in line with Yin et al. (2019).

11 Conclusion

Our study use the Ang et al. (2006)'s framework to examine if the Idiosyncratic Volatility Puzzle post-crisis (2010-2018) is still present. Additionally, to analyse if the relationship between idiosyncratic volatility and returns is holding-period dependent we apply the following trading strategies: 2/0/1, 1/0/1, 1/0/3, 1/0/6 and 1/0/12. The strategies are constructed applying the L/M/N framework changing the parameters to test different holding periods. Our study shows a negative relationship between idiosyncratic volatility and returns for trading strategy 1/0/1. We compute several performance evaluation measures P1-P5, P1, Market, Size, Value and Momentum strategies. We find P1 (low idiosyncratic volatility stocks) outperforms every other strategy in reward-risk terms. Moreover P1-P5 strategy can be used for hedging purposes because of its long-short equity structure and its negative correlation with the market. Moreover, we discover that the cost of ignoring the low idiosyncratic volatility cost for a mean-variance optimizer investor, holding an indexed allocation, is a sizeable increase of its utility function.
Second goal of the thesis is to test if the Puzzle is holding-period dependent. Studying the L/M/N strategy with different N (and different rebalancing periods), we bring some evidence in line with the heterogeneity of investors' investment horizons hypothesis. All the strategies but 1/0/6 present a negative compensation for bearing idiosyncratic risk. Trading strategy 1/0/6 shows no presence of IVOL Puzzle, because there is no difference in performance between portfolios made of high IVOL stocks and portfolios made of low IVOL stocks. The absence of a compensation for bearing firm-specific risk means that investors with investment horizon from 4 to 6 months (at least) start to require a positive compensation to bear idiosyncratic risk, making the final compensation required close to zero at the end of 6^{th} month.
Overall the findings of this thesis prove the presence of the IVOL Puzzle post-crisis and bring evidence in line with heterogeneity of investors' investment horizons hypothesis supported by Malagon et al. (2015) and Yin et al. (2019).
Thanks to the decomposition of our data provided by the wavelet transform, we prove as the

almost null performance of strategy 1/0/6 is driven by a positive compensation for bearing idiosyncratic risk demanded in time scales preceding the 6^{th} month. Moreover this technique furnishes additional evidence to the importance of the heterogeneity of investors' investment horizons hypothesis, increasing the accuracy of the analysis and quality of the model.
The heterogeneity of investors' investment horizon hypothesis can be one of the reasons behind the mixed literature. The several approaches to the topic, the different equity markets studied and the different time period considered have contributed to make the construction of a proper consensus over the IVOL Puzzle difficult. Nevertheless, besides being reasonable the heterogeneity of investors' investment horizon hypothesis seems to put together different results instead of generating additional stances.

12 Limitations and recommendations for future research

The Idiosyncratic Puzzle has been investigated with several approaches in literature. Interesting would be to search for evidence supporting our results trying to estimate expected IVOL as Fu (2009) instead of using the 1 month lagged IVOL as a proxy. Additional evidence in line with our finding coming from a different framework would bring robustness to the study.
A limitation regards how to handle properly the delisting firms over our time series. Some firms enter into the dataset after 2010 and some disappear before 2018. When a firm's time series of returns stops, that firm has been delisted. Delisting is usually a bad sign but it's not always the case, for example delisting can be a firm's choice or due to a merger. The limitation in our case is that delisted stocks, especially when we compute trading strategy with holding period bigger than 1 month, were simply not considered in the next period portfolio creating a loss of information. A method to handle this could improve the accuracy of the results.
A more general limitation of our study is the lack of robustness check. Ang et al. (2006) checked their results for size, book-to-market, leverage, liquidity, volume, turnover, bid-ask spread, coskewness and dispersion of analysts' forecasts. In addition Ang et al. (2009) tested their results for: market frictions, information dissemination and option pricing.

References

Ang, A., Hodrick, R. J., Xing, Y., and Zhang, X. (2006). The cross-section of volatility and expected returns. *The Journal of Finance*, 61(1):259–299.

Ang, A., Hodrick, R. J., Xing, Y., and Zhang, X. (2009). High idiosyncratic volatility and low returns: International and further us evidence. *Journal of Financial Economics*, 91(1):1–23.

Bali, T. G. and Cakici, N. (2008). Idiosyncratic volatility and the cross section of expected returns. *Journal of Financial and Quantitative Analysis*, 43(1):29–58.

Bodie, Z., Kane, A., and Marcus, A. (2014). Investments (vol. 10). *London: McGraw-Hill Education-Europe. Hentet September*, 5:2017.

Brandt, M. W., Brav, A., Graham, J. R., and Kumar, A. (2009). The idiosyncratic volatility puzzle: Time trend or speculative episodes? *The Review of Financial Studies*, 23(2):863–899.

Campbell, J. Y., Lettau, M., Malkiel, B. G., and Xu, Y. (2001). Have individual stocks become more volatile? an empirical exploration of idiosyncratic risk. *The Journal of Finance*, 56(1):1–43.

Chen, Z. and Petkova, R. (2012). Does idiosyncratic volatility proxy for risk exposure? *The Review of Financial Studies*, 25(9):2745–2787.

Evans, J. L. and Archer, S. H. (1968). Diversification and the reduction of dispersion: an empirical analysis. *The Journal of Finance*, 23(5):761–767.

Fama, E. F. and French, K. R. (1993). Common risk factors in the returns on stocks and bonds. *Journal of financial economics*, 33(1):3–56.

Fu, F. (2009). Idiosyncratic risk and the cross-section of expected stock returns. *Journal of financial Economics*, 91(1):24–37.

Gençay, R., Selçuk, F., and Whitcher, B. J. (2001). *An introduction to wavelets and other filtering methods in finance and economics.* Elsevier.

Goetzmann, W. N. and Kumar, A. (2004). Diversification decisions of individual investors and asset prices. *Yale School of Management Working Papers (Yale School of Management.).*

Grossman, S. J. and Stiglitz, J. E. (1980). On the impossibility of informationally efficient markets. *The American economic review*, 70(3):393–408.

Herskovic, B., Kelly, B., Lustig, H., and Van Nieuwerburgh, S. (2016). The common factor in idiosyncratic volatility: Quantitative asset pricing implications. *Journal of Financial Economics*, 119(2):249–283.

Hou, K. and Loh, R. K. (2016). Have we solved the idiosyncratic volatility puzzle? *Journal of Financial Economics*, 121(1):167–194.

Jensen, M. C. (1968). The performance of mutual funds in the period 1945–1964. *The Journal of finance*, 23(2):389–416.

Jiang, G. J., Xu, D., and Yao, T. (2009). The information content of idiosyncratic volatility. *Journal of Financial and Quantitative Analysis*, 44(1):1–28.

Kosowski, R., Naik, N. Y., and Teo, M. (2007). Do hedge funds deliver alpha? a bayesian and bootstrap analysis. *Journal of Financial Economics*, 84(1):229–264.

Lintner, J. (1965). Security prices, risk, and maximal gains from diversification. *The journal of finance*, 20(4):587–615.

Malagon, J., Moreno, D., and Rodríguez, R. (2015). Time horizon trading and the idiosyncratic risk puzzle. *Quantitative Finance*, 15(2):327–343.

Mandelbrot, B. (1972). Statistical methodology for nonperiodic cycles: from the covariance to r/s analysis. In *Annals of Economic and Social Measurement, Volume 1, Number 3*, pages 259–290. NBER.

Markowitz, H. (1952). Portfolio selection. *The journal of finance*, 7(1):77–91.

Masset, P. (2008). Analysis of financial time-series using fourier and wavelet methods. *Available at SSRN 1289420.*

Merton, R. C. (1987). A simple model of capital market equilibrium with incomplete information. *The journal of finance*, 42(3):483–510.

Mossin, J. (1966). Equilibrium in a capital asset market. *Econometrica: Journal of the econometric society*, pages 768–783.

Newey, W. K. and West, K. D. (1986). A simple, positive semi-definite, heteroskedasticity and autocorrelationconsistent covariance matrix.

Percival, D. B. and Walden, A. T. (2000). *Wavelet methods for time series analysis*, volume 4. Cambridge university press.

Polikar, R. et al. (1996). The wavelet tutorial.

Roll, R. (1977). A critique of the asset pricing theory's tests part i: On past and potential testability of the theory. *Journal of financial economics*, 4(2):129–176.

Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *Journal of Economic Theory*, 13:341–360.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *The journal of finance*, 19(3):425–442.

Statman, M. (1987). How many stocks make a diversified portfolio? *Journal of financial and quantitative analysis*, 22(3):353–363.

Sundling, C. M., Sukumar, N., Zhang, H., Embrechts, M., and Breneman, C. (2006). Wavelets in chemistry and cheminformatics. *Reviews in Computational Chemistry*, 22:295–329.

Yin, L., Shu, T., and Su, Z. (2019). Common idiosyncratic volatility and returns: From an investment horizon perspective. *International Journal of Finance & Economics*, 24(1):370–390.

A Trading Strategy 1/0/3

Plotting the quarterly returns of Portfolio 5 against Portfolio 1 (Figure 12) we observe a characteristic which were shown in Table 5 too. Portfolio 5 (blue line) which contains the stocks with greatest exposure to idiosyncratic risk vary much more than the Portfolio 1 (black line).

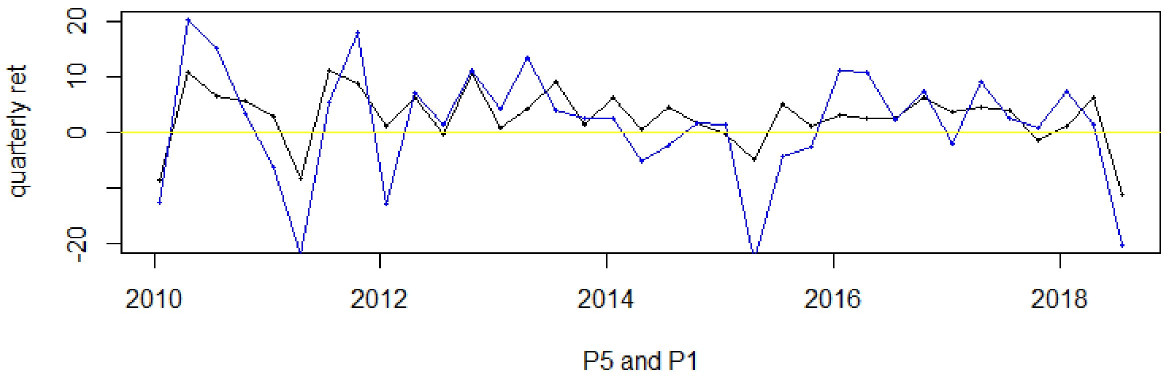


Figure 12: Plot of the quarterly returns for Portfolio 1 (stocks with lowest idiosyncratic volatility levels) and Portofolio 5 (stocks with highest volatility levels). P1 is the black line, P5 is the blue line.

Figure 13 shows the quarterly returns of the P5-P1 which represents the trading strategy of going short on the portfolio stocks with low idiosyncratic risk exposure and long on the portfolio formed by stocks with high idiosyncratic volatility levels. Besides two peaks where the strategy gained positive returns for two quarters consecutively (middle of 2010 and start of 2016) we know by Table 5 that the average quarterly performance of P5-P1 is -1.30%.

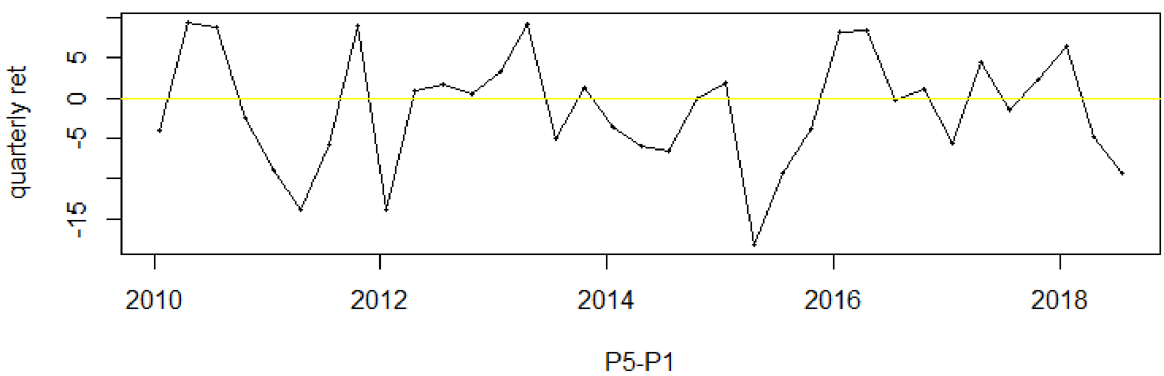


Figure 13: Plot of the quarterly returns of trading strategy P5-P1 (long on Portfolio 5 and short on Portfolio 1).

B Trading Strategy 1/0/6

Plotting the semesterly returns of Portfolio 5 against Portfolio 1 (Figure 14) we observe again that stocks facing high idiosyncratic risk usually have higher total volatility compared to stocks with low idiosyncratic volatility levels. Portfolio 5 (blue line) which contains the stocks with greatest exposure to idiosyncratic risk vary more than the Portfolio 1 (black line).

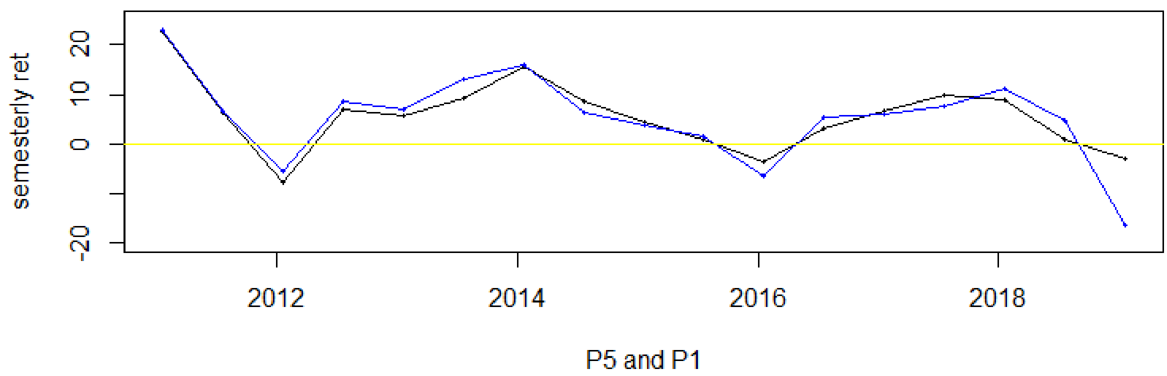


Figure 14: Plot of the semesterly returns for Portfolio 1 (stocks with lowest idiosyncratic volatility levels) and Portofolio 5 (stocks with highest volatility levels). P1 is the black line, P5 is the blue line.

Figure 15 shows the semesterly returns of the P5-P1 which represents the trading strategy of going short on the portfolio stocks with low idiosyncratic risk exposure and long on the portfolio formed by stocks with high idiosyncratic volatility levels. Besides one peak where the strategy gained abysmally negative return (end of 2018) we know by Table 6 that the average semesterly performance of P5-P1 is -0.21%.

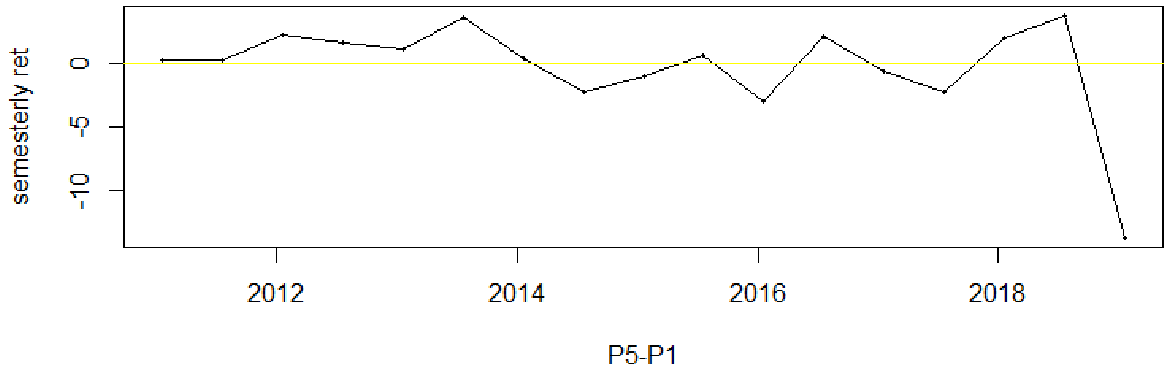


Figure 15: Plot of the semesterly returns of trading strategy P5-P1 (long on Portfolio 5 and short on Portfolio 1).

C Trading Strategy 1/0/12

Plotting the annual returns of Portfolio 5 against Portfolio 1 (Figure 16) we observe again the difference in volatility between P5 and P1. Portfolio 5 (blue line) which contains the stocks with greatest exposure to idiosyncratic risk vary much more than the Portfolio 1 (black line). Figure 17 shows the annual returns of the P5-P1 which represents the trading strategy of going

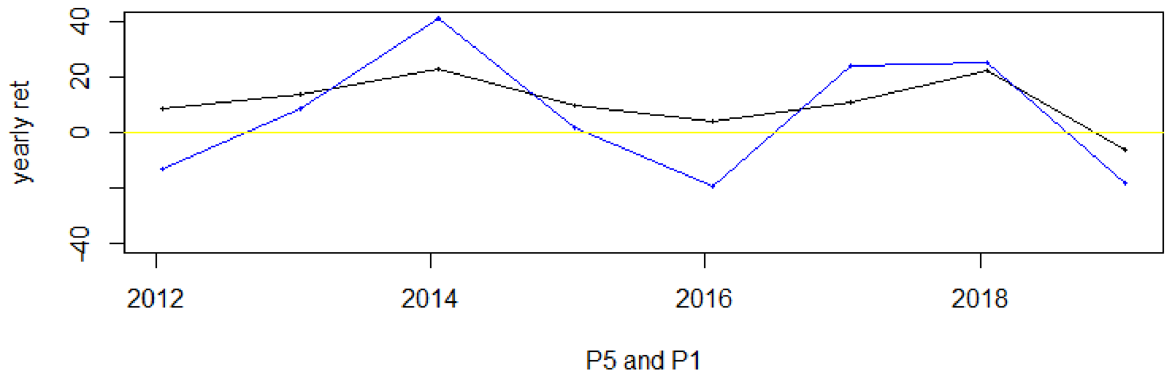


Figure 16: Plot of the annual returns for Portfolio 1 (stocks with lowest idiosyncratic volatility levels) and Portofolio 5 (stocks with highest volatility levels). P1 is the black line, P5 is the blue line.

short on the portfolio stocks with low idiosyncratic risk exposure and long on the portfolio formed by stocks with high idiosyncratic volatility levels. There aren't abnormal peaks over the eight years, we notice again a strong *cyclicality*. From Table 7 average annual performance of P5-P1 is -4.44%.

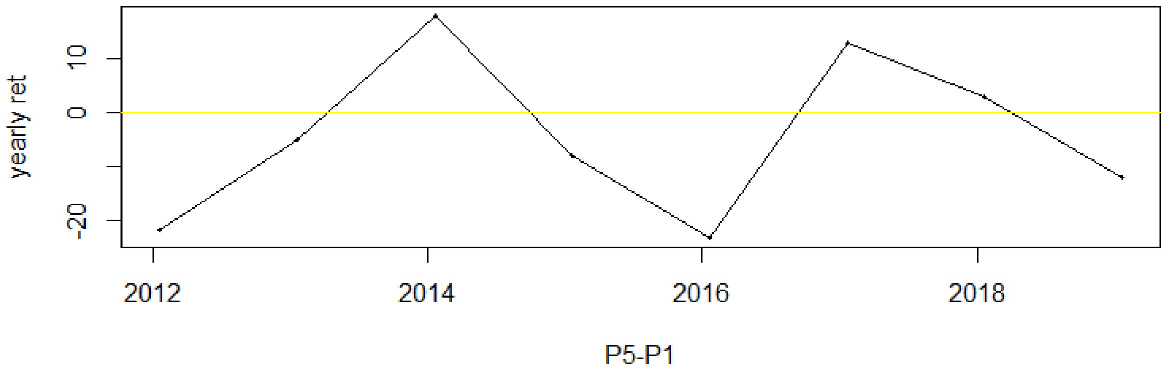


Figure 17: Plot of the annual returns of trading strategy P5-P1 (long on Portfolio 5 and short on Portfolio 1).

D D2 component

The D2 component, represents the investment horizon from 4 to 8 days. Running the analysis we obtain Table 14.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	-0.01	0.39	0.22	-0.04*	-0.05**
				[-1.81]	[-2.30]
P2	-0.03	0.47	0.37	-0.06**	-0.07***
				[-2.33]	[-2.79]
P3	-0.06	0.53	0.24	-0.10***	-0.10***
				[-3.26]	[-3.92]
P4	-0.07	0.60	0.13	-0.11***	-0.12***
				[-3.15]	[-3.95]
P5	-0.15	0.68	0.05	-0.19***	-0.20***
				[-7.20]	[-6.20]
P5-P1	-0.14***			-0.15***	-0.16***
	[-6.62]			[-7.80]	[-7.50]

Table 14: D2 component, 4 to 8 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/6

The trading strategy P5-P1, which represents the compensation for exposure to idiosyncratic risk, is almost zero. This result is in line with our results because we find strategy 1/0/1 having a negative monthly compensation. Alphas relative to CAPM and Fama-French 3-factor model are negative.

E D3 component

The D3 component, since the wavelet transform is done on daily data, represents the investment horizon from 8 to 16 days. Results in Table 15.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	-0.03	0.55	0.21	-0.09***	-0.10***
				[-3.20]	[-3.75]
P2	-0.01	0.64	0.34	-0.07***	-0.08***
				[-2.57]	[-2.80]
P3	-0.03	0.77	0.25	-0.11***	-0.12***
				[-3.30]	[-4.42]
P4	-0.03	0.85	0.14	-0.12***	-0.13***
				[-2.73]	[-3.18]
P5	-0.05	0.96	0.05	-0.14***	-0.15***
				[-2.68]	[-3.54]
P5-P1	-0.02			-0.05	-0.06
	[-0.86]			[-1.43]	[-1.40]

Table 15: D3 component, 8-16 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1

The trading strategy P5-P1, which represents the compensation for exposure to idiosyncratic risk, is almost zero. This result is in line with our results because we find strategy 1/0/1 having a negative monthly compensation. Alphas relative to CAPM and Fama-French 3-factor model are negative.

F D4 component

The D4 component, since the wavelet transform is done on daily data, represents the investment horizon from 16-32 to 4 days. Results in Table 16.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	0.96	2.78	0.20	0.46	0.40
				[2.84]	[3.12]
P2	1.00	3.47	0.37	0.36	0.32
				[2.74]	[2.80]
P3	0.87	4.37	0.26	0.05	0.05
				[0.46]	[0.40]
P4	0.77	4.99	0.13	-0.15	-0.11
				[-0.94]	[-0.73]
P5	0.51	5.64	0.05	-0.48	-0.40
				[-2.04]	[-1.86]
P5-P1	-0.45			-0.94	-0.80
	[-1.05]			[-2.86]	[-3.05]

Table 16: D4 component, 16-32 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1

The trading strategy P5-P1, which represents the compensation for exposure to idiosyncratic risk, is negative. This result is in line with our results because we find strategy 1/0/1 having a negative monthly compensation. Alphas relative to CAPM and Fama-French 3-factor model are negative.

ABSTRACT

1 Introduction

We examine the relationship between idiosyncratic volatility and cross-section of returns, aiming to verify if the negative relationship found by Ang et al. (2006)[1] is still present in the post-crisis period (2010-2018). Hence, for each firm in the American equity market, we construct from the individual stocks 5 portfolios with different levels of idiosycnratic risk. Then we measure the performance difference between the portfolio made of stocks with highest level of idiosyncratic risk and the portfolio based on stocks with the lowest level of idiosyncratic risk. Afterwards a trading strategy exploiting the patterns we find in the data is tested with several evaluation performance measures. We then analyse if the relationship between idiosyncratic volatility and returns is holding-period dependent, in order to test if heterogeneity of investors' investment horizon hypothesis is verified by our findings. To further test this recent hypothesis in literature, we apply the wavelet transform to study the contribution of each frequency in our data to the Puzzle.
Our *novel contribution* is to use Ang et al. (2006)'s approach to examine the idiosyncratic volatility-return relationship in a different sample period[2] and to evaluate the performance of two trading strategies exploiting the Puzzle. In addition, we analyse if the relationship is holding-period dependent using a strategy with increasing holding periods (1, 3, 6 and 12 months). To relate the inverted U-shaped relation we observe to the heterogeneity of investors' investment horizons hypothesis, we perform a time-frequency analysis with the wavelet transform on the American equity market post-crisis.
In literature, the theme of this thesis is known as "the Volatility Puzzle". We focus on the idiosyncratic side of the volatility, following other papers' example from now on we refer to it as the "IVOL Puzzle" (Idiosyncratic VOLatility Puzzle). Studying the IVOL Puzzle can be helpful both from a factor investing point of view (as a trading strategy) and as a stronger theoretical framework for all the investors which fail to diversify.

We use a dataset from Wharton Research Data Service, composed by the daily returns of the stocks belonging to the same exchanges Ang et al. (2006) used, for the post-crisis period. We monthly construct the 5 portfolios formed on IVOLs using the trading strategy 1/0/1. We find evidence of a negative relationship between IVOL and returns because the portfolio 1 (formed on low volatility stocks) outperforms the portfolio 5 (based on high volatility stocks). The fact the alphas are substantial (-0.72% relative to CAPM and -0.59% relative from Fama-French three-factor model) and statistically significant brings additional evidence of the existence of the IVOL Puzzle after the crisis.
Given our findings, we compute performance evaluation measures of four well-known strategies based on: Market, Size, Value and Momentum factors. We compare these strategies to the strategy of going long on low idiosyncratic volatility stocks and short on high idiosyncratic volatility stocks. We find our strategy and the momentum strategy performing well relative to the market index. Afterwards we quantify the cost for a mean-variance optimizer investor, with an indexed position, of ignoring low idiosyncratic volatility stocks. By tilting its position towards the low idiosyncratic volatility stocks it increases substantially its utility function.
The trading strategy 1/0/3, 1/0/6 and 1/0/12 bring other relevant discoveries. The trading strategy 1/0/3 shows the same patterns of the trading strategy 1/0/1, where portfolio 5 underperforms the portfolio 1. The alphas

[1] Their sample period is 1963-2000.
[2] In the American equity market, as for Ang et al. (2006).

are relevant and statically significant, meaning that the CAPM and FF3 models still fail to price the portfolios. The trading strategy 1/0/6 shows the absence of a difference in performance both in returns and alphas between portfolio 1 and 5. The trading strategy 1/0/12 displays again the Puzzle's existence with the same patterns of the strategy 1/0/1 and 1/0/3.

Malagon et al. (2015); Yin et al. (2019) use the Wavelet Multi-Resolution Analysis to separate investor classes and decompose a time series into different time horizons. In our case instead of decomposing the time series, playing with the setting of the L/M/N strategy we test different holding periods, separating the investors from active (frequent rebalancing) to more passive investors (yearly rebalance). Anyway, time scales determine the overall return we capture, therefore different compensations required by investors with different time horizons affects the compensation of the overall holding periods. The sum of the compensations required for all the time scales inside an holding period makes the final compensation we observe, hence a change in compensation for increasing holding periods implies a compensation for bearing idiosyncratic risk that is investment-horizon dependent.

Finally, we test the hetereogeneity of investors' investment horizons hypothesis. Following the framework of Malagon et al. (2015) we apply the wavelet multi-resolution analysis, decomposing our data into 7 different frequencies representing the behaviour of different kind of investors. The analysis reports a negative relationship between firm-specific risk and returns for short term investors (investment horizon from 2 to 32 days), a positive one for medium term investors (32 to 128 days) and a negative one for longer investment horizons (> 128 days).

2 Literature on the Idiosyncratic Volatility Puzzle

2.1 Preamble

By theory, there should be a premium to compensate investors for holding assets that are not diversified. Diversification smooths out the firm specific risk by holding an enough large number of assets. The consequence of diversification is a lower risk faced, hence a better return for unit of risk in our portfolio. The reason why facing less risk means a better mean-variance optimization, is that under certain general assumptions the idiosyncratic risk is not priced (compensated) as the systematic risk. If investors were rational individuals, they should not face idiosyncratic risk and it should not even be priced. Given the empirical evidence, investors fail to fully diversify their investments, therefore an investigation on how idiosyncratic risk affects portfolio's performance was needed both for theoretical and investment purposes. Several papers over the years, investigating how idiosyncratic risk is priced by the market, have found mixed evidence. In academic literature the evidence about Idiosyncratic Volatility Puzzle is mixed: there are researchers that found a significant positive relationship between idiosyncratic volatility and average returns (as Fu (2009)), there are others which failed to find a significant relationship between these two variables (as Bali and Cakici (2008)) and finally there is also evidence of a negative relationship as Ang et al. (2006).

2.2 Hypotheses

The literature's debate over the Idiosyncratic Volatility Puzzle had two main periods. From 2006 till 2009, the main effort has been to weaken the robustness of Ang et al. (2006)'s findings. Even if some critiques may have been economically reasonable and empirically proved, see Fu (2009) about the time-varying volatility's nature, statistically significant evidence of the Puzzle was still unexplained. Ang et al. (2009) have brought huge consensus about the robustness of their results, in fact after 2009 more possible explanations to the Puzzle came up. The most famous one are: heterogeneity of investors' investment horizons, lottery preference (behavioural explanation), market frictions, average variance beta (Chen and Petkova (2012)) and IVOL as an information content (Jiang et al. (2009)). Between all the hypotheses which have been used as reason behind the Puzzle, heterogeneity of investors' investment horizons captures our attention because it can potentially explain both the significant negative relationship between IVOL and returns and the mixed evidence found in literature. This hypothesis states that the compensation investors demand for bearing idiosyncratic risk could be horizon dependent. Malagon et al. (2015), applying Wavelet Multi-Resolution Analysis to disentangle the different time horizons, find a negative relationship between IVOL and returns for the short term investors while the relationship gets positive for long-term investors. Yin et al. (2019) find the Puzzle for short-term investors, a positive relationship for middle-term investors and a negative relationship again for long term investors.

Aim of the thesis is to search for evidence about IVOL Puzzle, considering the post-crisis sample period (2010-2018). Given the evidence found in the literature, applying the same framework of Ang et al. (2006, 2009) we expect to find supporting evidence to the Puzzle, since we fail to notice reasons why the compensation demanded by investors exposed to idiosyncratic risk should have been changed[3].
About the heterogeneity of investors' investment horizons hypothesis, we are going to test different holding period (different values for N) of the trading strategy L/M/N, to check if there is a change in the compensation required by investors bearing the idiosycnratic risk for a holding period of 1,3,6 and 12 months. Given the Malagon et al. (2015); Yin et al. (2019)'s results, we expect to observe supporting evidence to the heterogeneity of investors' investment horizons hypothesis.

2.3 The Idiosyncratic Volatility

We defined volatility of an asset as the standard deviation of returns with a given frequency, therefore it can be easily measured. On the other hand, idiosyncratic volatility can only be estimated from the model's residuals, therefore is model dependent. If idiosyncratic volatility is model dependent then its accuracy is model dependent too, hence the better the model the better the idiosyncratic volatility we estimate.

Risk, intended as the standard deviation of returns over time, can be divided into two main components. When a risk is faced by all the securities in the market (can't be diversified because related to macroeconomic factors), is classified as systematic risk. As systematic are considered: the interest rate risk, the market risk, the purchasing power risk, the exchange rate risk and the political risk[4]. On the other hand, the idiosyncratic risk is an industry/firm/stock specific risk which can be diversified away just increasing the number of stocks inside the portfolio. Campbell et al. (2001) stated that "the number of randomly selected stocks needed to achieve relatively complete portfolio diversification" is about 50. The intuition is that usually assets are not perfectly correlated, therefore an additional asset will decrease the portfolio's idiosyncratic risk. Using the Ang et al. (2006)'s methodology, the IVOL computation derives from the squared root of the residuals'

[3]Sample period of Ang et al. (2006, 2009) is 1963-2000.
[4]https://efinancemanagement.com/investment-decisions/systematic-risk

variance $\sqrt{\mathbb{V}ar(\epsilon_{i,t})}$ from the Fama-French 3-factors model (OLS multivariate regression):

$$r_{it} = \alpha_i + \beta_{i,mkt}MKT_t + \beta_{i,SMB}SMB_t + \beta_{i,HML}HML_t + \epsilon_{i,t} \tag{1}$$

Therefore, from now on when we talk about IVOL we refer to idiosyncratic volatility relative to the Fama-French three-factor model.

3 Trading strategy

Following Ang et al. (2006), we define our framework as the trading strategy L/M/N. At a point in time t we sort the daily stocks returns based on the L-months lagged IVOLs into 5 quantiles, then we wait M-months and eventually we hold these portfolios (the 5 quantiles) for N-months[5]. The IVOLs are constructed monthly over daily returns. We examine if going short on P5 and long on P1 is profitable. We analyse the following trading strategy's settings: 2/0/1 with monthly rebalancing, 1/0/1 with monthly rebalancing, 1/0/3 with quarterly rebalancing, 1/0/6 with semesterly rebalancing and 1/0/12 with annual rebalancing.
The difference in an investor who does monthly rebalancing based on a factor (IVOL in our case) compared to a yearly rebalancing is the different level of activeness used to manage his portfolio. Investors whom decide to rebalance every year are closer to a passive investing management while monthly rebalancing investors are more active in their portfolio management.

3.0.1 Dataset from CRSP

The analysis now shifts over a dataset from Wharton Research Data Service[6]. It contains the daily returns of stocks on primary listings for NYSE, NYSE MKT (previously known as AMEX), NASDAQ and ARCA exchanges. The time period considered is the post-crisis one, therefore 2010-2018. Over the dataset will be tested the following trading strategies: 1/0/1, 1/0/3, 1/0/6 and 1/0/12. The columns/variables it contains are: daily returns, price per share and number of share outstanding. The last two variables are needed to compute the value-weighted returns of each portfolio. Multiplying them, we obtain the market capitalization which will be used to weight the returns inside the portfolios.

3.1 Trading strategy 1/0/1

We construct monthly portfolios of stock returns based on five levels of the 1 month lagged IVOLs. The results are reported in a table which is in the layout similar to Ang et al. (2006)'s table for comparison purposes. This means the statistics computed for the 5 portfolios are: mean, standard deviation, market share (intended as average market capitalization of the portfolio over the sum of the 5 portfolios' market capitalizations) and alphas from CAPM and Fama-French three-factor models. In order to be as close as possible to the real application of a trading strategy, to compute the value-weighted returns we used as weights the market capitalization of the first day of the month considered. Results are in Table 1.

[5]The portfolios returns at the end of the M period are value-weighted
[6]http://www.crsp.com/products/research-products/crsp-us-stock-databases

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	0.94	2.88	0.21	0.41***	0.38***
				[3.72]	[3.70]
P2	0.99	3.66	0.40	0.29***	-0.26***
				[2.63]	[2.89]
P3	0.99	4.30	0.25	0.18*	0.18*
				[1.81]	[1.68]
P4	0.81	4.82	0.11	-0.06	-0.01
				[-0.40]	[-0.06]
P5	0.63	5.66	0.04	-0.31	-0.22
				[-1.22]	[-1.00]
P5-P1	-0.32			-0.72**	-0.59**
	[-0.80]			[-2.53]	[-2.50]

Table 1: Forming value-weighted quintile portfolios every month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1

Our findings have the same patterns of Ang et al. (2006). In Table 1 P5-P1 return is -0.32% but it's not statistically significant. The CAPM and Fama-French three-factor models alphas are in magnitude smaller than Ang et al. (2006), but are statistically significant. Overall, we observe additional evidence of the IVOL Puzzle, since CAPM and Fama-French three-factor model misprice the P5-P1 portfolio's alphas yielding statistically significant monthly alphas of -0.72% and -0.59% on the long P5 short P1 strategy. The decreasing pattern in market share from P1 to P5 is decreasing starting from P3.

3.2 Trading strategy 1/0/6

This time the holding period of the portfolios sorted based of 1 month lagged IVOL is 6 months. The rebalancing is semesterly. Results are in Table 2.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	5.63	7.22	0.25	1.83***	0.68
				[4.27]	[1.46]
P2	7.10	8.00	0.20	2.89***	2.74***
				[15.57]	[4.42]
P3	5.64	8.86	0.18	1.09	0.88
				[1.44]	[0.86]
P4	5.55	9.39	0.18	0.72	0.13
				[0.68]	[0.17]
P5	5.42	9.00	0.19	0.88	0.96
				[0.90]	[0.52]
P5-P1	-0.21			-0.95	0.28
	[-0.40]			[-1.12]	[0.25]

Table 2: Forming value-weighted quintile portfolios every six months we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in semesterly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in semesterly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/6

Compared to 1/0/3, the gap in return between P5 and P1 substantially shrinks (-0.21% for 1/0/6 compared to -1.30% for 1/0/3). Since the holding period is twice the size, assuming the compensation was still negative from month 3 to 6 and given the same pattern we found with previous strategies, we were expecting a bigger gap.

The fact IVOL Puzzle vanished with a holding period of 6 months could be caused by several reasons. We state that testing different holding periods is a way to bring new evidence to the heterogeneity of investors' investment horizon hypothesis. The fact the IVOL Puzzle (a lower compensation for high IVOL stocks compared to low IVOL stocks) is reduced till to disappeared, can be explained by the presence of a positive compensation for bearing idiosyncratic risk approximately from the 3rd to the 6th month.

3.3 Trading strategy 1/0/12

Holding period is set to 12 months for portfolios based on 1 month lagged IVOL. Results are in Table 3.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	10.60	9.53	0.34	5.32	5.26**
				[0.61]	[1.99]
P2	10.19	13.26	0.35	2.75***	2.01***
				[4.40]	[6.96]
P3	9.78	15.31	0.20	1.41	1.04
				[1.14]	[1.14]
P4	6.63	15.91	0.09	-1.99**	-2.54***
				[-2.29]	[-7.24]
P5	6.17	22.57	0.03	-5.34***	-3.75***
				[-5.44]	[-5.34]
P5-P1	-4.44			-10.66***	-9.00***
	[-1.34]			[-4.63]	[-4.17]

Table 3: Forming value-weighted quintile portfolios every twelve months we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in annual percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in annual returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/12

Increasing additionally the holding period from 6 to 12 months, the IVOL Puzzle appears again. Standard Deviations increase from P1 to P5 while Market Share decrease from P1 to P5. The usual patterns from P1 to P5 for each statistics are back. The alphas' t-statistics are more robust than the other strategies. The 1/0/12 strategy brings again evidence of the IVOL Puzzle.

4 Trading strategy's results

4.0.1 Idiosyncratic Volatility Puzzle's post-crisis evidence

Main theme of the thesis is to search for evidence about the IVOL Puzzle in the post-crisis period. We apply several trading strategies (2/0/1, 1/0/1, 1/0/3, 1/0/6 and 1/0/12) based on IVOL, to see if the negative relationship between IVOL and returns is still present after the crisis. Across these strategies, we always have found relevant and strongly significant spread alphas between P5 and P1 relative to CAPM and Fama-French 3-factor models (but for trading strategy 1/0/6). We find relevant and statically significant alphas (but for trading strategy 1/0/6), meaning the strategies are able to "beat the market". This expression is used when active managers form portfolios capable of gaining actual returns that exceed risk-adjusted expected returns. The total actual return minus the risk-adjusted expected return equals the "alpha" gained and it measures the value the active managers bring into the investment process.Our outcomes show evidence of the IVOL Puzzle in the post-crisis period[7].

[7] 2010-2018.

4.0.2 Heterogeneity of investors' investment horizons hypothesis

Second goal of the thesis is to test the heterogeneity of investors' investment horizon hypothesis, that started to be developed from Brandt et al. (2009). Key intuition behind is the presence of several kind of investors, which implies heterogeneity of their needs and consequently of their investment horizons in financial markets. Our analysis, till now, doesn't consider the single time scales contribution (as Malagon et al. (2015); Yin et al. (2019)), it evaluates instead the performance of different degrees of activeness in the portfolio management. However, the two approaches are not completely separated. In fact, if increasing the holding period the IVOL Puzzle weakens, this could mean that investors with a bigger time scale are demanding a premium for bearing idiosyncratic risk.

Since a given holding period return should be the results of all the investment horizons that compose the period, the fact we spot a different compensation for a different holding period is an evidence supporting the heterogeneity of investors' investment horizon hypothesis. Generally speaking, our results show that the sign compensation for bearing idiosyncratic risk is holding-period dependent following an inverted U-shape trend.

4.1 Cumulative Returns

Plotting the cumulative returns some famous strategies (Figure 1), we examine how much investing in each strategy since January 2010 yields for each month over time. We observe that P1 strategy outperforms every other strategies in terms of mean return, Sharpe Ratio and cumulative return. P5-P1 strategy slightly underperforms the momentum strategy in terms of cumulative returns. P5-P1 is a long-short equity strategy, this technique is often used by hedge funds to gain both from the increase and decrease of prices of different securities in the market. A portfolio with this setting protects itself from losses during market downturns, because of this when the strategy has a close to zero correlation relative to the market it's called a "market-neutral" strategy. The beta β of this strategy is -0.56 therefore the strategy, besides performing on average better than Size and Value strategies, can be used as a hedging instrument against market risk. Size and Value strategies as expected from summary statistics we computed, perform poorly post-crisis.

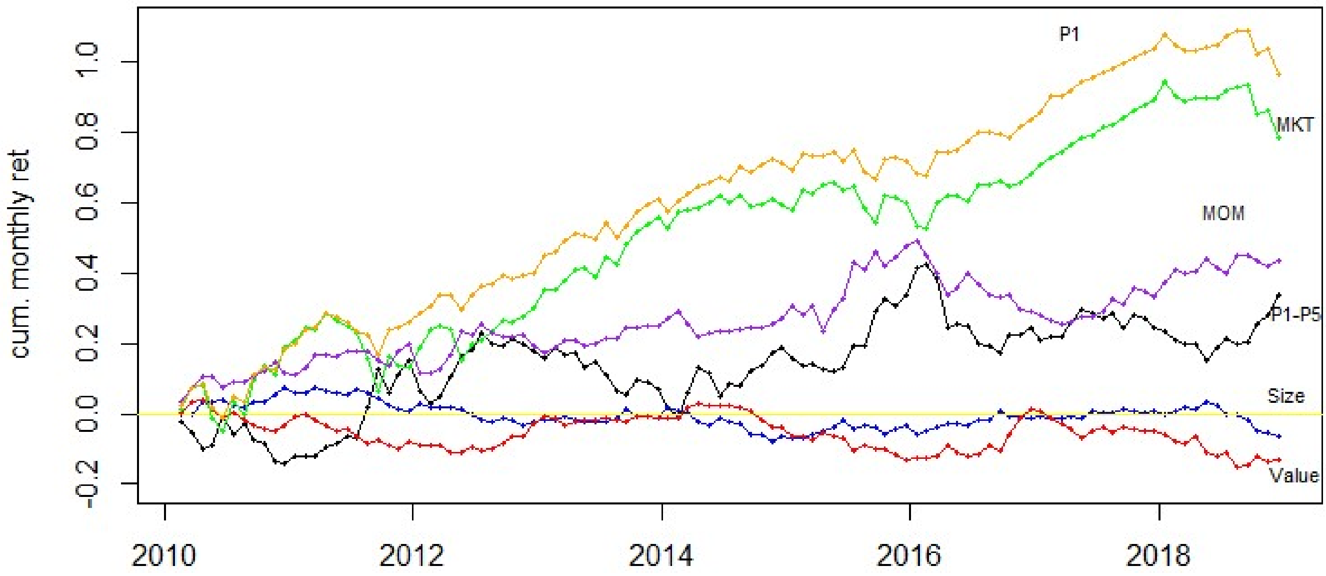


Figure 1: Cumulative returns for each strategy: P1(yellow), Market(green), Momentum(purple), P1-P5(black), Size(blue) and Value(red). Sample period: January 2010-December 2018.

5 Fourier and Wavelet methods for time series in Finance

5.1 From time domain to frequency domain

In several fields, the time domain analysis of a variable can be enhanced by the frequency domain analysis. A time domain series is a variable which is function of time, therefore is indexed in time order and plotting the variable we obtain a time amplitude representation. Studying the frequencies of a process, we can observe characteristics hidden in the frequency domain representation. All the frequency components of a signal/process/series are called frequency spectrum.
The mathematical tools used to go from time to frequency domain are generally called transforms and the most popular one is the Fourier transform.

5.2 Wavelet transform

Fourier transform requires the process to be stationary, because it goes from the time-domain to the frequency domain. In finance often the data doesn't satisfy the stationarity requirement. A recent transform which overcome this issue is the wavelet transform because with wavelets we can obtain a time-frequency representation of our data.

5.3 The Discrete Wavelet Transform (DWT)

Contrary to CWT, the Discrete Wavelet Transform has a limited amount of coefficients because the *mother wavelet* is dilated and translet a limited number of times. This is obtained setting:

$$s = 2^{-j} \quad u = k2^{-j} \tag{2}$$

where j, k are the set of discrete translation and dilatation, implying that the wavelet transform is calculated only at *dyadic* scales (2^j). Another implication is that, being N the observations of our time series, the largest number of scales is the the integer J:

$$J = [log_2(N)] = [log(N)/log(2)] \tag{3}$$

this can be an issue because if the time series is not of *dyadic* length, observations must be added or removed. Two discrete wavelet filter are behind the DWT. One is the *mother wavelet*, denoted $h_l = (h_o, ..., h_{L-1})$. The second one is the *father wavelet*, denoted $g_l = (g_0, ..., g_{L-1})$. Properties of the *mother wavelet* are:

$$\sum_{l=0}^{L-1} h_l = 0, \quad \sum_{l=0}^{L-1} h_l^2 = 1, \quad \sum_{l=0}^{L-1} h_l h_{l+2n} = 0 \quad \forall n \in \mathbb{N}_0 \tag{4}$$

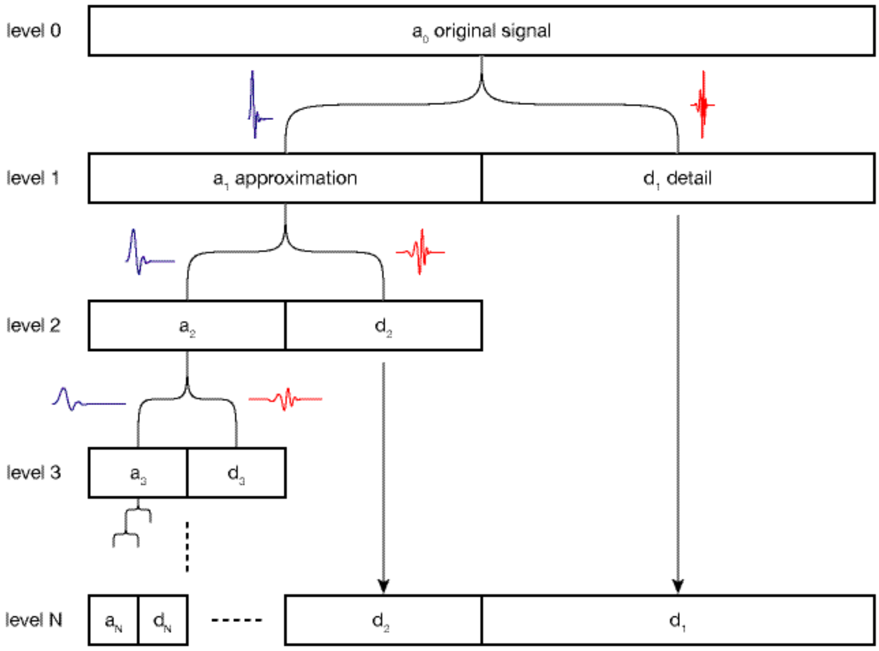
Thanks to the above properties, h_l is a difference operator, the DWT has the variance of the original data and a multiresolution analysis can be performed. The *father wavelet* is a low pass filter and captures the long scales, hence the low frequency, smooth components of the series computing the "scaling" coefficients. The *mother wavelet* is an high pass filter and captures the short scales, high frequency, details components of the series. The *father wavelet* has the following condition:

$$\sum_{l=0}^{L-1} g_l = 1 \tag{5}$$

The first level of decomposition computes the wavelet and the scaling coefficients of the first scale, respectively $w_1(t)$ and $v_1(t)$ that are obtained in the following way:

$$w_1(t) = \sum_{l=0}^{L-1} h_l x(t') \quad and \quad v_1(t) = \sum_{l=0}^{L-1} g_l x(t') \tag{6}$$

in which $t = 0, 1, ..., T/2 - 1$ and $t' = 2t + 1 - lmodT$.
Thanks to the pyramid algorithm (procedure in figure 2), we can further decompose the low frequency scaling coefficients $v_1(t)$ into other to components. Therefore the second level decomposition has $w = [w_1, w_2, v_2]$ and the J level decomposition has $w = [w_1, ..., w_J, v_J]$.



algorithm.png

Figure 2: Pyramid algorithm doing a N level decomposition from Sundling et al. (2006)

5.4 The Maximal Overlap Discrete Wavelet Transform (MODWT)

To overcome the limitations we briefly described of DWT, we can use the MODWT. Contrary to the DWT, MODWT consider all the possible (integer) translations. Hence for every scale the wavelet coefficients, the scaling coefficients and the original series have the same length. At the first level decomposition with MODWT we have:

$$\widetilde{w}_1(t) = \sum_{l=0}^{L-1} h_l x(t') \quad and \quad \widetilde{v}_1(t) = \sum_{l=0}^{L-1} g_l x(t') \tag{7}$$

in which $t = 0, 1, \ldots, T$ and $t' = t - lmodT$. Using the pyramid algorithm, we can obtain the MODWT coefficients for further level of decomposition.

6 Heterogeneity of investors' investment horizons hypothesis and Wavelet transform

Comparing the compensation required by investors, with different holding period, to bear idiosyncratic risk we observe for short holding period (1-3 months) a negative premium, for medium holding period (3-6 months) a zero premium and for long term holding period (6-12 months) again a negative compensation. Our hypothesis is that this result can be driven by investors with different investment horizon requiring different compensation for bearing idiosyncratic risk. Testing the trading strategy with increasing holding period can show the compensation of all the investors with investment horizon smaller or equal to the holding period but can't properly disentangle every required compensation. Therefore the return we observe at the end of the holding period of p months, is the aggregation of all the compensation required by investment horizons smaller or equal to p.
To study in detail the Idiosyncratic Volatility Puzzle, we use the Wavelet transform to study the contribution of each frequency (time scale) to the final holding period return of portfolio 5 and 1.

6.1 From frequencies to investors

We summarized in Section 9 the theoretical background, the properties and the applications of the wavelet transform. In finance, decomposing in frequencies a time series of returns, we look for an economical interpretation of the frequencies. Considering a short (long) time scale of our time series, means to capture the high (low) frequency contributions to our series. Therefore in our case, the high frequencies (short time scale) are the contribution of the short term investors to the series, while the low frequencies (long time scale) represent the contribution of the long term investors. This reasoning is reasonable as the short (long) term investors contribute to the most (least) frequent movements of the price. Different investors have different trading frequencies (Malagon et al. (2015))

The definition of short/medium/long term investors depends by the frequency of the data on which we do the wavelet transform. The technique creates frequency bands separated by multiples of 2^J. Therefore with daily data we can capture the contribution of investors with investment horizon of 2-4 days, 4-8 days, 8-16 days till the maximum admitted level of decomposition.

Performing the wavelet transform, we decompose our time series S_0 into an approximation S_j (long time scale/low frequency) and details D_j (short time scale /high frequency).

6.2 Application to trading strategy 1/0/1

We apply a wavelet transform of level 6 to our framework, decomposing the daily returns and the daily factors into 6 details and one smooth component (example in figure 3). We use the la8 filter and the Maximal Overlap Discrete Wavelet Transform. The reason behind the MODWT is that we need to apply the transform to approximately 11.000 firms. To avoid issues linked to the required *dyadic* length of the data by the Discrete Wavelet Transform, we choose the MODWT. Once the data is decomposed, we test the trading strategy 1/0/1 for each details and smooth components. Aim of this framework is to capture the compensation required, for bearing idiosyncratic risk, by specific investors' investment horizon identified by each of the details and smooth components. The next sections cover just detail 1, 5 and 6 while the others are in the appendix.

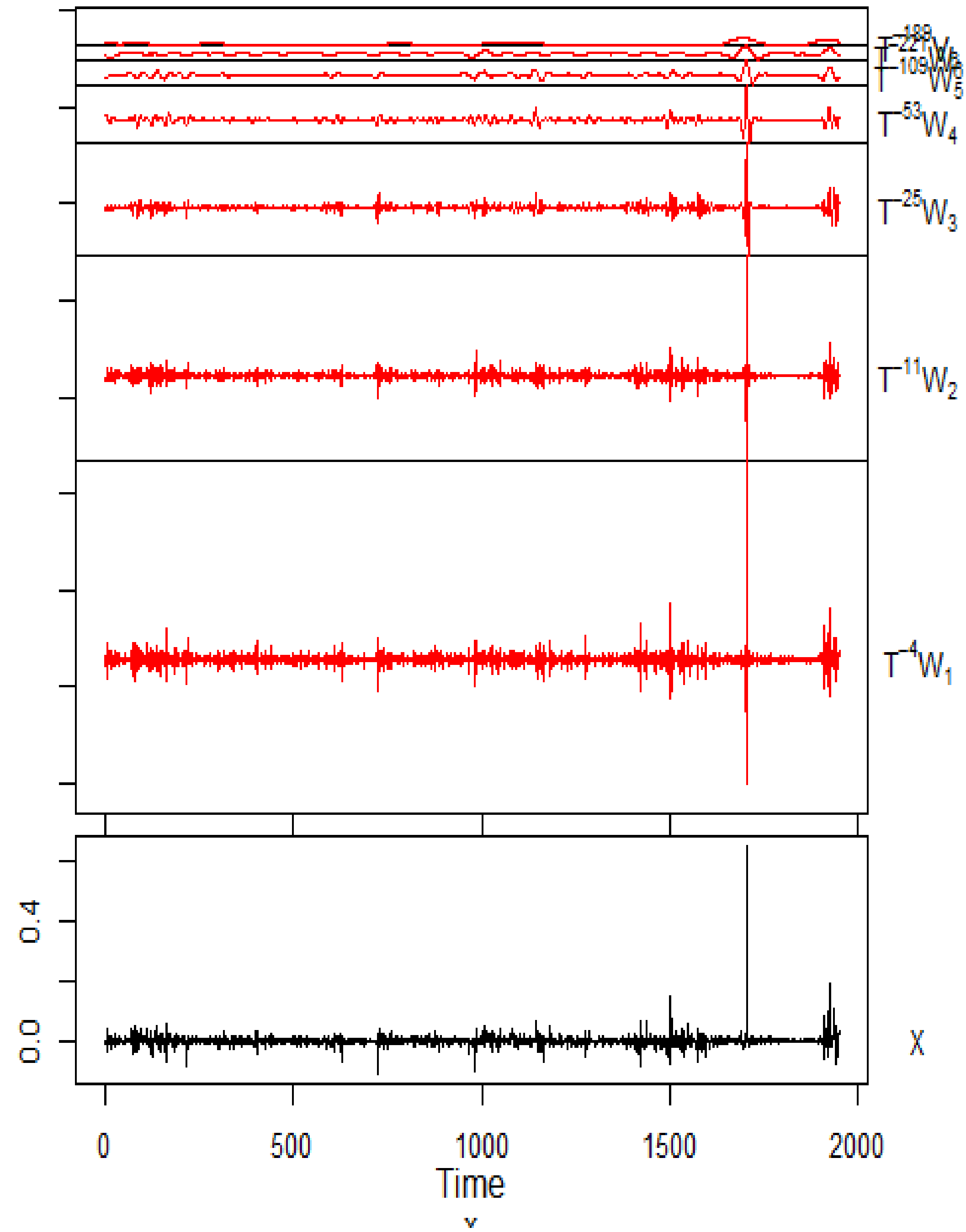


Figure 3: X is the time series of daily returns of an American firm in our dataset. Applying the wavelet transform with level 6, we obtain six details component called W and one smooth component defined V

6.3 D1, D5, D6 and S6 component

The D1 component, since the wavelet transform is done on daily data, represents the investment horizon from 2 to 4 days. Results are in Table 4.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	-0.06	0.40	0.23	-0.08***	-0.08***
				[-2.79]	[-3.06]
P2	-0.10	0.44	0.38	-0.12***	-0.12***
				[-4.13]	[-3.97]
P3	-0.15	0.51	0.24	-0.16***	-0.16***
				[-4.29]	[-4.51]
P4	-0.24	0.58	0.12	-0.26***	-0.26***
				[-7.45]	[-7.30]
P5	-0.42	0.70	0.04	-0.43***	-0.43***
				[-9.54]	[-9.96]
P5-P1	-0.35***			-0.34***	-0.34***
	[-12.4]			[-12.54]	[-12.67]

Table 4: D1 component, 2-4 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) t-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1

The trading strategy P5-P1, which represents the compensation for exposure to idiosyncratic risk, is negative and statistically significant. This result is in line with our previous results because we find strategy 1/0/1 (Table 1) having a negative monthly compensation. Alphas relative to CAPM and Fama-French 3-factor model are negative and statistically significant. We find the same patterns for standard deviation market share observed in Table 1.

Testing also the other components, we find that the compensation draws for increasing time scales an inverted U-shape which further explains our results. The non-constant performance of portfolio with increasing holding and rebalancing period, finds justification in the heterogeneity contribution of different frequencies to the series.

7 Conclusion

Our study use the Ang et al. (2006)'s framework to examine if the Idiosyncratic Volatility Puzzle post-crisis (2010-2018) is still present. Additionally, to analyse if the relationship between idiosyncratic volatility and returns is holding-period dependent we apply the following trading strategies: 2/0/1, 1/0/1, 1/0/3, 1/0/6 and 1/0/12. The strategies are constructed applying the L/M/N framework changing the parameters to test different holding periods. Our study shows a negative relationship between idiosyncratic volatility and returns for trading strategy 1/0/1. We compute several performance evaluation measures P1-P5, P1, Market, Size, Value and Momentum strategies. We find P1 (low idiosyncratic volatility stocks) outperforms every other strategy in reward-risk terms. Moreover P1-P5 strategy can be used for hedging purposes because of its long-short equity structure and its negative correlation with the market. Moreover, we discover that the cost of ignoring the low idiosyncratic volatility cost for a mean-variance optimizer investor, holding an indexed allocation, is a sizeable increase of its utility function.

Second goal of the thesis is to test if the Puzzle is holding-period dependent. Studying the L/M/N strategy with different N (and different rebalancing periods), we bring some evidence in line with the heterogeneity of investors' investment horizons hypothesis. All the strategies but 1/0/6 present a negative compensation for bearing idiosyncratic risk. Trading strategy 1/0/6 shows no presence of IVOL Puzzle, because there is no difference in performance between portfolios made of high IVOL stocks and portfolios made of low IVOL stocks. The absence of a compensation for bearing firm-specific risk means that investors with investment horizon from 4 to 6 months (at least) start to require a positive compensation to bear idiosyncratic risk, making the final compensation required close to zero at the end of 6^{th} month.

Overall the findings of this thesis prove the presence of the IVOL Puzzle post-crisis and bring evidence in line with heterogeneity of investors' investment horizons hypothesis supported by Malagon et al. (2015) and Yin et al. (2019).

Thanks to the decomposition of our data provided by the wavelet transform, we prove as the almost null performance of strategy 1/0/6 is driven by a positive compensation for bearing idiosyncratic risk demanded in time scales preceding the 6^{th} month. Moreover this technique furnishes additional evidence to the importance of the heterogeneity of investors' investment horizons hypothesis, increasing the accuracy of the analysis and quality of the model.

The heterogeneity of investors' investment horizon hypothesis can be one of the reasons behind the mixed literature. The several approaches to the topic, the different equity markets studied and the different time period considered have contributed to make the construction of a proper consensus over the IVOL Puzzle difficult. Nevertheless, besides being reasonable the heterogeneity of investors' investment horizon hypothesis seems to put together different results instead of generating additional stances.