LUISS

Libera Università Internazionale degli Studi Sociali

FACOLTÀ DI ECONOMIA E FINANZA
Corso di Laurea in Finance

Tesi di Laurea

An Advanced Application of Black-Litterman

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Anno Accademico 2018-2019

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October 6, 2019

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1 Introduction

The investment process has changed dramatically over time, along with the development of new tools in Finance. We have observed a gradual switch from the so-called naïve investing to fully automated quantitative strategies. Nowadays, most types of strategies that one can think of can be coded using computer software and turned to reality. The strategies that involve investing are many and differ based on time horizons, risk profile of the investor, market conditions, wealth available.

The starting point of modern portfolio construction is the basic mean-variance framework (or Markowitz model), dating back to the 1950's. It was a major breakthrough in Finance and introduced a new way of rethinking investments, but still contained many limitations. Research was sparked by this innovation and this clever yet rudimental model has been enhanced, modified and fine-tuned. The products of these changes have in turn been subject to similar transformations, leading to a great multitude of models with different characteristics.

One of such models is called Black-Litterman Model and, when it was developed in the 1990's, constituted another major breakthrough in the field. In particular, it solved major problems of the Markowitz model and it produced a quantitative way for an investor to express his own personal beliefs over future outcomes. This model came along with the mathematics and theory backing it up, making, on the one hand, huge progress in the investment scene but also making, on the other hand, the portfolio construction process more complex. However, this didn't constitute a major applicability problem, because the model provides a closed-form solution. This means that it can be applied by anyone, without necessarily understanding the mathematics or economics behind it.

As every other novelty, the Black-Litterman Model was studied in-depth and modified in multiple ways to fit the specific needs of investors and overcome some limitations. The ways it has been modified are multiple, but they give up the advantage of having a closed-form solution. This means that the various steps taken in the actual construction of new models, often highly demanding in terms of knowledge of maths or coding, have to be made by the individual investor without the help of some universal formula. For this reason, most of these new models could never be accessible to some investors.

Among the assumptions overcome by the modified versions of Black-Litterman, there is normality of returns. A complex model produced by Attilio Meucci overcomes this assumption and some others, but causes non-professional investors to suffer from its extreme technicality.

Today we propose a model which overcomes the same assumptions in a more heuristic and accessible way. This model will be constructed step-by-step and compared to a benchmark in order to measure its performance and discover its strengths and weaknesses.

2 Background

2.1 Portfolio Construction Process

As a first thing, we try to answer the following question: What is an investment? A simplistic way to define it is a way to move resources across time so that we can have more tomorrow by giving up something today (or viceversa). The ways in which we can invest are infinite and the first problem imposed on us is how to allocate our wealth available for investment. This process is called Asset Allocation and concerns the choice of the asset classes to invest in. An asset class can be defined as a group of securities with a set of characteristics that are considered to be similar (e.g. all US equities, Hedge Funds, Italian Corporate Bonds). We notice that in asset allocation we are also deciding where we are investing in a geographical sense (in Italy, in the US, Russia, and so on...). Moreover, notice that an asset class is broadly defined and it can set its "boundaries" at different levels, leaving room to interpretation. An investor should make considerations over the outlook of the market and the individual asset classes before constructing its investment portfolio.

In reality, some strategies don't require any qualitative analysis and can be applied in "mathematical" way. These strategies usually belong to what is called **technical analysis** and select the individual securities, regardless of their asset class, based on some indicator. Technical analysis is beyond the scope of this discussion but, to mention some common examples, includes mean-reverting strategies or price momentum.

Another way in which we can allocate wealth is defined as **sector rotation strategy**. This strategy tries to look at the business cycle and predict its next moves. This allows an investor to choose a sector whose performance is positively correlated to the particular phase of the cycle. For example, at a market peak we could invest in IT and industrials whereas in a market trough we look more at consumer staples.

These dynamic strategies can be considered short term as they change over time to adapt to market changes. In our discussion though we will focus on more long-term strategies.

For an investor with a long-term horizon, it is the study of the overall market risk-return framework which leads to a first allocation of resources among asset classes, and this process is called **Strategic Asset Allocation (SAA)**. In this part of the process, portfolio managers decide the weights assigned to each asset class and then proceed to selecting the best securities within each one of them (in accordance with the asset class weights). The SAA is usually reviewed annually but, in normal times, it is not affected by recent market changes and has an horizon of more or less 5 years. Due to its long-term horizon, the portfolio is usually not adjusted to contemporaneous news and is held until the predefined horizon. Whenever the investor (or management) decides to take short- or medium-term bets, an alternative solution is given by **Tactical Asset Allocation (TAA)** in which the initial investment weights can be changed more frequently to adapt to temporary market changes. An SAA and TAA can coexist in the following way. The initial strategic allocation is tilted in favor or against the assets and/or asset classes on which we wish to bet, creating a tactical bet. This weights are reverted back to the SAA (or to a new TAA) as soon as the bet is realized. Which of the strategies we use depends on the conditions of the market. In a variable market it may be more logical to use a TAA whereas in a trending and predictable market it can be more efficient to rely on an SAA. The mix of the two can also be employed whenver it suits at best the needs of the investor.

A portfolio can also be constructed in a **naïve** way, that is, just through common sense diversification and intuition over the future. For instance, an investor can decide *a priori* that he wants to be invested in:

- 40% US equity
- 30% Global Bonds
- 30% Real Estate

These numbers are purely discretionary, but this can actually constitute a real-life example. We can reach a similar result, with percentages of allocation across asset classes in more advanced ways. The strategies we will consider involve quantitative optimization.

2.2 Markowitz and the Mean-Variance framework

The models used nowadays are many, but not so long ago, the **naïve** way was the only valid solution. It is thanks to the **mean-variance** optimization process, first proposed by Markowitz (1952), that the world of optimization developed so widely.

As we know, the driver of return is risk. Meaning that there is "no free lunch" and that we cannot have the former without the latter. This first optimization created an unpreceded mathematical way to deal with the trade-off between the risk and return by formalizing their measures for all individual assets:

- Risk is now measured by standard deviation (for asset i, σ_i)
- Returns are proxied by their expectations (for asset i, μ_i)

The optimization accounts for the multitude of assets in the market and requires these two measures for all of them. Moreover, the way in which the assets behave in relation with one another is measured by their covariance (for assets i and j, σ_{ij}).

With the use of this information, Markowitz provides the formulas for the expected return and risk of a portfolio given the vector of weights (w_i) invested in each of the n assets at disposal:

$$E(R_p) = \sum_{i=1}^{n} w_i \cdot \mu_i$$

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j$$

The formula for portfolio variance relies on the statistical knowledge that the covariance of two assets is the product of their standard deviations times their correlation:

$$\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$$

In the summation, when $i = j$, ρ_{ij} becomes $\rho_{ii} = 1$ (an asset has perfect correlation with itself). Therefore, $\sigma_{ii} = \sigma_i^2$ (the covariance of an asset with itself is its variance).

Imagine an investing universe made up of n assets; we wish to assign to each one of them an expected return and quantify its risk. Moreover, we can estimate the correlation among the n assets to understand how they perform related to each other. The concept of **diversification** in finance concerns the fact that assets don't covary perfectly and that there is an optimal way to take advantage of this. Notice from the formula for the variance of the portfolio that a value of correlation ρ_{ij} equal to 0 eliminates the whole contribution of the two assets to the risk of the portfolio. A negative correlation does even more, and reduces the overall risk of the portfolio. The secret of diversification, as we can see, lies in the correlation among the assets.

Markowitz provided, for the first time, a mathematical way to consider the findings above and produce **efficient** portfolios. Before we look at the way it actually works, let us consider more in-depth the concepts we are dealing with: risk and expected returns.

An expected return is the forecast of the realized return of an asset which we cannot see today but we will observe tomorrow. We need an *a priori* measure to quantify the possibility of the realized return to be different from our expectation. This possibility can be interpreted as "risk" and many ways to measure it are available. The most simply understood one, and also widely used, is standard deviation (also known as volatility). Standard deviation, as we have seen, is represented by σ, and it is the positive square root of the variance of the returns time series of an asset; it can be easily computed through historical data and it can proxy the standard deviation of future returns.

Among the issues with this measunre there is the fact that standard deviation is symmetric. This means that risk concerns downside events as well as upside events (in practical words, an asset that performs much better than expected would still be considered risky).

Extending the concept of standard deviation to a multivariate framework, it is convenient to introduce vector notation. If we are dealing with n assets, the covariance matrix (notation: Σ) is the matrix in which the diagonal represents the variances of the n assets (σ_i^2), and the off-diagonal elements their covariances (σ_{ij}).

The structure of a covariance matrix Σ is the following:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{21} & \dots & \sigma_{n1} \\ \sigma_{12} & \sigma_2^2 & \dots & \vdots \\ \vdots & \dots & \ddots & \vdots \\ \sigma_{1n} & \dots & d & \sigma_n^2 \end{bmatrix}$$

An alternative way to measure risk that we will use later on in the pages is EWMA.

EWMA stands for "Exponentially Weighted Moving Average" and is a common and straightforward way to measure volatility. The box below provides its description.

EWMA volatility

The main advantage of this method is that it enables us to differentiate between recent market events and events of a very distant past. The formula of this method is the following:

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1-\lambda) r_{t-1}^2$$

What the formula tells us is that the variance today (σ_t^2) is a function of yesterday's variance (σ_{t-1}^2) and yesterday's squared return (r_{t-1}^2). If we look at the formula in recursive way, at any time t, σ_t^2 incorporates all the squared returns of the series up to that date, excluding the contemporaneous one. We understand now that the key parameter is the "memory" coefficient, λ, which weights the last return versus all the previous ones.
For this reason, we will have a more or less "reactive" volatility based on the weights assigned to the last squared return $(1-\lambda)$. Meaning that for λ close to 0 volatility is very responsive and for λ close to 1 volatility it is more "sticky". For our purposes we will refer to RiskMetrics, which calibrates $\lambda = 0.94$.

Switching to the other main ingredient of Markowitz's model, expected returns, we have a very basic approach to quantify them, using the average return of the different assets. This average can serve as proxy for future returns. Of course, there are ways to generate expected returns with more sophisticated procedures but the arithmetic (or, alternatively, geometric) mean represents an instantaneous and simple approach.

Note that for both risk and expected returns we are using an **historical** approach. The quantities used today stem out of the past observations. The underlying assumption is that the past is representative of the future. This is not usually a desired result in Finance, and can bring to misleading decisions. To use a typical financial expression, it is like driving a car just by looking at the rear mirror.

The risk-return trade-off in creating a portfolio marks the birth of what is called **"Modern Portfolio Theory"** or simply **MPT**. More returns require more risk and, viceversa, if we don't want to face risk we cannot expect high returns.

Going back to the framework with n assets available, we can invest our budget across them in many possible ways (assigning all possible weights to all assets). Every portfolio we can construct is defined as **achievable**. If we plot all the achievable portfolios based on their risk and return, we obtain a graph similar to the one below. In the graph the blue shaded area represents all achievable portfolios but only the ones on the so-called "Efficient Frontier" are **efficient**. The envelope that contains all possible portfolios is considered efficient because the portfolios in this set are the ones that have, for their levels of risk, the highest expected return.

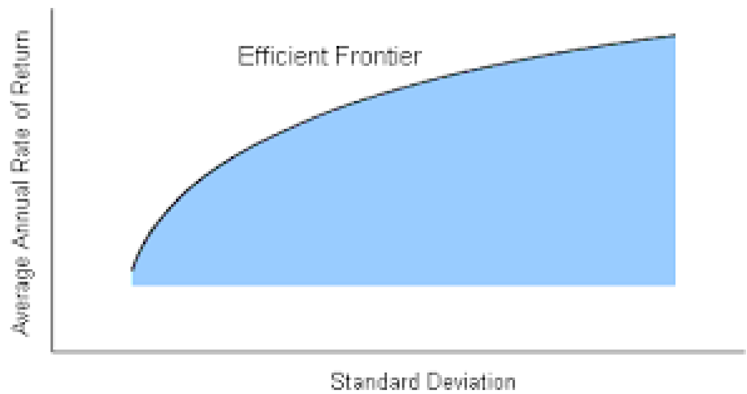

Figure 1: An example of the Efficient Frontier

Efficient, in Markowitz terms, means that there is no portfolio that has higher expected returns for that level of risk (standard deviation) or, the other way around, there is no portfolio that achieves the same expected return but faces less risk. As a consequence of this, all rational investors are going to construct one of the portfolios that lie on the Efficient Frontier.

Note that this breakthrough concept introduced by Markowitz can find applications elsewhere. As a matter of fact, later on, we will use the same concept while looking at other models; that is, the Efficient Frontier will be constructed given new ways to specify risk and returns, but producing similar results.

Each point on the frontier is a portfolio in which all assets have a weight. For these portfolios we can compute as before the expected return and variance. Using matrix notation:

$$\mu_p = \alpha_1 \cdot \mu_1 + \cdots + \alpha_i \cdot \mu_i + \cdots + \alpha_n \cdot \mu_n = \alpha' \mu$$

$$\sigma_p^2 = \alpha' \Sigma \alpha$$

This notation uses the following elements:

- α is the ordered vector of assets' weights (α_i).
- μ is the ordered vector of assets' expected returns (μ_i).
- Σ is the covariance matrix of the assets (containing σ_i^2's and σ_{ij}).

The Efficient Frontier is a continuous function in the graph provided, but if we are only allowed to invest in discrete weights (for example, we can change weights by 1 basis point each time) it becomes a step function. Later on, we will construct our Efficient Frontiers using 25 points (25 portfolios).

2.3 Utility functions

But how do we choose the best point to be at on the frontier? To answer this question we need some additional information, specifically, an utility function. Utility can be seen as the personal satisfaction of an individual. Because it is not easily quantifiable, utility functions have an ambitious goal of trying to assign a value to this degree of satisfaction. What we know is that satisfaction, in this framework, is a fucntion of two inputs, risk and return. Inevitably, more returns imply more satisfaction but, at the same time, more risk. Bearing risk, on the other hand, lowers our level of satisfaction, leading us once more to the risk-return trade-off. What a utility function does in practice is:

1. Provide for a rule for the trade-off between risk and return;
2. Take the efficient frontier portfolios and, through their expected returns and variances, compute the utility of each of them using such rule;
3. Return the portfolio with the highest utility;

In financial litterature there are different utility function families which differ significantly in their implications. They are not cited in this paper because we will use, in accordance with common practice in Finance, an **exponential utility** function, which is defined as follows:

$$U(W_{t+1}) = -\exp(-\lambda W_{t+1})$$

where W_{t+1} is next period's wealth and $\lambda \geqslant 0$ (different from the λ we have encountered in EWMA) is the absolute risk aversion coefficient.

The aim of the optimization is now to maximize *a priori* the expected utility of tomorrow's wealth:

$$E[U(W_{t+1}]$$

To do this we must first quantify tomorrow's wealth. W_{t+1} depends on the outcome of our investments today. We have, as before n assets to invest in, plus, we introduce now a risk-free asset. The assumption that there is a risk-free asset means that we can invest as much as we want (where negative investment is also allowed and means borrowing) and obtain a return r_f which has, by definition, zero risk (in the case in which we borrow we simply return after one period the amount borrowed times $1 + r_f$).

Based on how much we allocate across the risk-free and risky assets today we generate a wealth tomorrow equal to:

$$W_{t+1} = (1 + r_f) + \alpha'(1 + r_{t+1} - (1 + r_f)e)$$

We now look for optimal weights assigned to risk-free and all other assets (the vector α). The maximization problem, subject to α can be written as:

$$\max_{\alpha} E[U(W_{t+1}]$$

It can be shown that, using exponential utility, the optimal solution to this problem is equivalent to the optimal solution of the following simpler problem:

$$\max_{\alpha} \mu_p - \frac{\lambda}{2}\sigma_p^2$$

As we know, we can rewrite μ_p and σ_p^2 as:

$$\mu_p = \alpha' \mu$$

$$\sigma_p^2 = \alpha' \Sigma \alpha$$

In this way we can express returns and variance as a function of α. The coefficient λ is, therefore, the only unknown that prevents us from using first order conditions and solving for α. This parameter λ is strictly specific to every different investor and tries to quantify how much the investor is averse to taking risk. Assigning a value to this coefficient can be very hard and is a topic which resides under the scope of behavioral finance. Let us say that the value for this coefficient ranges between 1 and 8 and that, on average, it is approximately 2.5/3. A higher value of λ implies more risk aversion and, as consequence less risky portfolios. Lower λ means higher risk tolerance and riskier portfolios.

The solution, α^*,to the problem above is given by the following closed-form solution:

$$\alpha^* = \frac{1}{\lambda}\Sigma^{-1}\mu$$

Substituting for the individual risk aversion coefficient of an individual provides the optimal weights for such investor.

Given that the average coefficient λ in the overall market is approximately 2.5, one can plug-in $\lambda = 2.5$ in order to find the weights α which constitute current market capitalization. This procedure will find an application later on in our discussion.

2.4 Criticism of Markowitz and newer findings

Although Markowitz remains a milestone in Finance, it has some limitations which have been studied and, in some cases, surpassed by more modern research.

A major critique of this model is given by Michaud (1989) in which the author claims that mean-variance optimization is actually an "estimation-error maximizer". Meaning that the inputs estimated historically through Markowitz generate portfolios which tend to overweight assets with high estimation error in returns and low estimation error in risk (and viceversa). Michaud claims that the optimization proposed by Markowitz is really optimal only when the true population parameters are know. As we have seen though, in Markowitz, these inputs are produced using an historical method. The consequences of this can be disastrous for an investor, as there are high chances that the returns from the market will be at least less advantageous than the expected ones for the portfolio.

Moreover, especially when short selling is allowed, we face a new problem called "corner solution". The issue is that, given some inputs for returns and volatility, we may obtain a set of optimal weights which is very unstable. For unstable we mean that the model heavily relies on the preciseness of the inputs and, if the latter are slightly changed, great changes happen to the portfolio composition. This translates into high model risk, on top of "normal" market risk. Not only we face the risk of market fluctuations when we use our investment model, we also face the risk that the model itself does not produce results that match our intents.

Because the main issue is that in Markowitz the inputs are taken as 100% certain there two possibilities available:

1. Heuristic approach, in which we take not just one set of inputs, rather, we take many differnt inputs and average them out.

2. Bayesian approach, in which we take the inputs as given on one hand and then we make use of Bayesian statistics to mix them with our own views.

The first method is primarily represented by the Resampled Frontier™ proposed by Michaud in 1998 and takes a bootstrapping approach. Michaud, through his licensed methodology, resamples thousands of times the datasets and produces thousands of Efficient Frontiers which are then averaged. This produces a final Efficient Frontier which does not present the problems of mean-variance described above.

The second option, which makes use of Bayesian statistics, is covered by the famous Black-Litterman (BL) model, developed in 1990. Because we are going to deal with this latter approach, let us, before we introduce Black-Litterman formally, introduce Bayesian statistics.

2.5 Bayesian Statistics

Under the mathematical perspective, it is worthwhile briefly discussing the maths underlying the construction of the BL model. In the proposed model for portfolio construction we make use of the theory provided by Bayesian statistics. In general Bayesian probability theory is based on the principle that observed data should somehow help in estimating probabilities of future events. Using this approach, we will update the information given to us by markets with our private information.

From probability theory we know that the joint probability of two events can be decomposed as:

$$p(x, y) = p(y|x)p(x)$$

or alternatively:

$$p(x, y) = p(x|y)p(y)$$

By equalizing the two expressions above we obtain the most important result in this field of statistics, **Bayes's Theorem**. The latter is formulated as:

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)} \propto p(y|x)p(x)$$

The $\propto$ symbol means "proportional to" and allows us to disregard the denominator of the expression.
In Bayesian statistics, usually, x is an event and y is some observed data, the probabilities acquire the following interpretations:

- $p(x)$ is the **prior** probability of event x.
- $p(y|x)$ is the **likelihood** function. The probability of the "evidence" y given that x is true.
- $p(x|y)$ is the **posterior** probability of x. The new probability we assign to event x given that we observed y.

The posterior $p(x|y)$, is proportional to the likelihood, $p(y|x)$ times the prior, $p(x)$.
To reconnect with the criticisms of Markowitz, this methodology can be used to **mix** the inputs we have with our views to construct more reliable results.

3 Black-Litterman

The Bayesian approach can be used to incorporate beliefs about many market outcomes into our estimation. In particular, we place ourselves in the shoes of an asset manager who wants to base his asset allocation on his beliefs. We could start by using the market portfolio as a beginning point and then tilting the weights in each asset class according to our views, overweighting whenever we have positive views and underweighting in the opposite situation. This approach could in principle be used for investing across all securities, expressing views on each, but could lead to dimensionality issues whenever we are dealing with many assets.

The Black-Litterman model was developed in 1990 by Fisher Black and Robert Litterman at Goldman Sachs and it can be easily applied by means of a closed-form solution. The result of the model, which we will present later on, is a vector of expected returns, produced by the views expressed.

The reason why the mix between views and sample information will lead to better results than plain Mean-Variance is related to the concept of **shrinkage**. We describe this concept by taking a small step behind and presenting a fundamental fact:

Stein's Paradox (1956)
Consider the estimation of a mean of n multivariate normal random variables $X_i \sim N(\mu, 1)$. The paradox tells us that:

- If $n = 1$ and we obtain a mean of X_1, the best estimator for μ is $\hat{\mu} = X_1$;
- If $n = 2$ and we obtain a mean of (X_1, X_2), the best estimator for μ is $\hat{\mu} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$;
- If $n = 3$ and we obtain a mean of (X_1, X_2, X_3), the best estimator for μ **IS NOT** $\hat{\mu} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$;

Meaning that estimating the expected returns μ_i separately is not appropriate for $n > 2$. A better estimator has instead the following form:

$$\hat{\mu}_i^{shrinkage} = \delta \cdot \mu_0 + (1 - \delta) \cdot \hat{\mu}_i$$

The intuition is that simple averages, μ_i, are inefficient estimates, therefore, they are "shrunk" towards a **non-sample** target values, μ_0, which can be determined in different ways. The paradox also states that any shrinkage target leads to better estimation of means. Commonly used targets, μ_0, are:

- zero
- cross-section mean
- theoretical values

The same concept is applied when estimating a covariance matrix. Imagine we wish to estimate a covariance matrix, we can take these two following paths:

1. Use a single factor model covariance, call it F (the CAPM could be an example of single-factor model);

2. Use sample covariance matrix, Σ;

The first one is a highly structured estimator, which assumes a single factor explains all there is to know and has no estimation error as long as the model is true, and the second one has no structure but is subject to a lot of estimation error. A shrinkage estimator tries to mix the two by taking the structure imposed by the model and combining it with the sample estimate to reduce its estimation error. The two estimators are combined by means of a third ingredient, the shrinkage constant, δ. This constant has the purpose of minimizing the distance between the true covariance matrix and the shrinkage estimator:

$$\hat{\Sigma}_{shrinkage} = \delta F + (1 - \delta)\Sigma$$

Because we cannot observe the true covariance matrix, we cannot compute the distance between the estimated and the true values. For this reason also the shrinkage constant, δ, must be estimated. There are many possible ways in which this can happen, in our framework, $\hat{\delta}$ is computed based on uncertainty of estimates (more uncertainty reduces the weight assigned to the shrinkage component).

To summarize, for the Black-Litterman model, shrinkage is achieved for both the vector of returns and the covariance matrix. The interesting thing of the model is the choice of the shrinkage elements. The shrinkage target is derived from an equilibrium condition (which, we will see, are market neutral weights) and the shrinkage constant is linked to our confidence over the outcomes.

In the attempt to mitigate the estimation error and model risk of the traditional mean-variance optimization, BL provides a new vector of expected returns which will be a linear combination of the priors (market information) and the views, with the confidence in the latter variables (measured by their standard deviation) as a scaling factor. As we mentioned, the model leads to a closed-form solution. The latter is:

$$\mathrm{E}[\mu|v] = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\mu_{eq} + P'\Omega^{-1}v]$$

The variables which enter the formula are the following. Some may be familiar already, other will be discussed soon:

- μ: the final expected returns over the available assets;
- v: vector of views over portfolios of assets;
- τ: interpreted as confidence over the estimated covariance matrix;
- Σ: covariance matrix;

- P: selection matrix, used to describe our views;
- μ_{eq}: market neutral expected returns;
- Ω: confidence over view;

In practice the model combines the implied expected returns of the market and the personal views of the investor. We obtain a final result which can be used for an efficient mean-variance optimization, the same as Markowitz but with different "ingredients".

We will consider the information derived from the market as our prior, we will then mix with our subjective views to obtain our posterior. In particular, we extract from the current market capitalization the vector of implied returns (μ_{eq}) for a given risk aversion and considering the historical covariance matrix.

The reason why this model is considered a break-through in Finance is that it combines a versatile yet rational way to apply what is known as shrinkage. We will now address the description of the missing elements which we have mentioned above: v, μ_{eq}, τ, P and Ω.

3.1 Neutral Expected Returns

Let us start off from μ_{eq}. Black and Litterman don't take as input the vector of average returns, they take instead the equilibrium vector of expected returns (μ_{eq}). They do this to avoid the noise generated by sample means. Moreover, using a sample mean would favor past winners over past losers, disregarding that we are interested instead in the future. The way they obtain a vector, μ_{eq}, of equilibrium returns is by **reverse optimization**. We know from Markowitz that the equilibrium weights are computed using the formula:

$$w = \frac{1}{\lambda}\Sigma^{-1}\mu$$

From the above we reverse engineer the vector of market equilibrium returns:

$$\implies \mu_{eq} = \lambda_{mkt} w_{mkt} \Sigma$$

Because we observe the weights (market capitalization) we can simply reverse compute the equilibrium vector of expected returns. The only thing we are missing is the value of λ_{mkt} we need to use.

To calibrate the parameter we can take advantage of some formula manipulation: take the result we found above for μ_{eq} and premultiply by w'_{mkt}. This leads to:

$$w'_{mkt}\mu_{eq} = \lambda w'_{mkt}\Sigma w_{mkt} \implies \lambda = \frac{w'_{mkt}\mu_{eq}}{w'_{mkt}\Sigma w_{mkt}} = \frac{\mu_{mkt}}{\sigma^2_{mkt}}$$

Using this trick to calibrate λ, we reach the conclusion that the historical Sharpe ratio (defined as average returns over volatility) of the market can be used as proxy for γ. In practice, this value is close to the value of 2.5. This value is a proxy of the average risk aversion in the market.

As our prior we have that returns are distributed with mean equal to the implied vector of returns. As variance of the prior we take the scaled covariance matrix of returns. The way we scale the covariance matrix is based on the confidence that we have over the accuracy of market expectations, and this is where τ comes in. Normally, a value given to the scaling parameter τ, is $1/T$, one over the number of observations, and this is going to be also our approach. Because Black-Litterman also assumes normality of returns, we end up with the following result for the market prior:

$$\mu \sim \mathrm{MVN}(\mu_{eq}, \tau\Sigma)$$

To briefly comment on the above, we use a reverse engineering technique to obtain a prior input. This prior corresponds to the equilibrium condition extracted from current market conditions.

3.2 Subjective Views

Once the investor has the market neutral vector of expected returns at disposal, he can point out some differences between what the market believes and what he believes as an individual. More specifically, with n assets at disposal, the investor can express relative or absolute views over a subset of $k \leqslant n$ linear combinations of the returns of the assets. His views are expressed as follows:

$$v = \mathrm{P}\mu$$

The selection matrix P, is used to choose the assets subject over our views, v defines the view itself and is the difference between the prior expected return contained in μ and the investor's personal view. The way in which the various components interact is best described through the use of an example.

Example

There are 3 assets each with its own expected return. The investor has an absolute view on the first two which states that the return of the first will be 15% and the return of the second will be 10%. This can be summarized in the following way:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad v = \begin{bmatrix} 15\% \\ 10\% \end{bmatrix}$$

Imagine now that the investor has a relative view that the difference in the returns of the first and third asset will be 7%, the information will be summarized as follows:

$$P = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad v = \begin{bmatrix} 7\% \end{bmatrix}$$

Because the views are not precise we also need a precision matrix, Ω, which determines the level of uncertainty in our views. For example, we provide two examples of such a precision matrix, the left one stands for high confidence in our views and the one the right stands for lower confidence:

$$\Omega = \begin{bmatrix} (1\%)^2 & 0 \\ 0 & (1\%)^2 \end{bmatrix} \qquad \text{and} \qquad \Omega = \begin{bmatrix} (10\%)^2 & 0 \\ 0 & (10\%)^2 \end{bmatrix}$$

The diagonal elements tell us, respectively, the uncertainty we have on the views expressed. The off-diagonal elements represent the covariance between forecast uncertainty and are usually set to 0, meaning that a view's certainty is not correlated with another view's. To set the confidence of the investor over the view is a hard job and this cannot be achieved in a fully scientifical way, the numbers are usually discretionary.

Now that we have our prior (market neutral condition) and our views defined, by use of Bayesian methods we can mix the two and obtain the closed-form solution shown before. To summarize, the Black-Litterman (BL) model is constructed as follows:

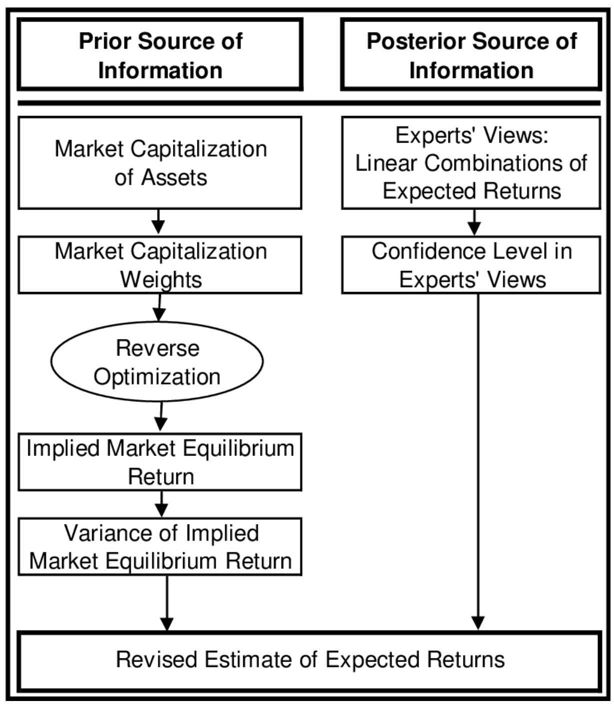

Figure 2: BL Procedure

From a methodological point of view, BL mitigates some of the problems that come from the plain mean-variance optimization. As we have seen, the investor uses market equilibrium weights as a starting point and then applies his views, if any. As a consequence, portfolio concentration is avoided automatically because we start off with a prior which is already diversified. The investor has, on top of this, the opportunity to incorporate his views in the investment strategy, making the method versatile and practical. Still, it seems like there is margin for improvement.

Two of the issues with Black-Litterman that we will discuss in our model are:

1. The assumption of normality of returns which is in conflict with some basic results in Finance.

2. The construction of the confidence matrix Ω which is completely arbitrary and does not account for interdependence of the confidence over different views.

There has been a wide research over these two issues and some authors have proposed complex models overcoming them. To cite the most relevant, see the following approaches:

- Entropy Pooling approach from Meucci (2008)

- COP from Meucci (2006)

4 An Advanced Application of Black-Litterman

Before we present the new model proposed by this thesis, we briefly describe the advances happening regarding Black-Litterman and Bayesian Asset Allocation. One very prolific researcher in the field is Attilio Meucci, who has produced different types of models incorporating macro factors, non-normality of the market and entropy pooling (see Meucci (2009), Meucci (2006),Meucci (2008)). The problem with some of these models is that they are difficult to be implemented and used in practice, due to the complexity of the theory and the computations required for their implementation. Moreover, they cannot be applied directly because they do not present any closed-form solution.

Some of the things that have been considered by Meucci are the properties of the marginal distributions of returns (the individual assets) and their dependence structure. In Finance, it is common knowledge that asset returns have negative skewness and excess kurtosis. This implies that a time series should be not be assumed to be unconditionally normal, rather, to have another type of distribution.

Starting from this point we try to formulate a simpler approach to tackle the same issues as the ones considered by Meucci. In this paper we will assume a distribution of the marginal which is the Skewed t-Distribution, the reasons for this will be provided. We will model the individual assets after clearing any form of predictability, by imposing a model for both mean and volatility. This also allows us to consider the fact that volatility is not constant in time, rather, as we would intuitively observe empirically, time varying. To model means and volatilities we will introduce, respectively, ARMA and GARCH dynamics in the return series.

Because of this previous consideration we will obtain marginal distributions for assets which are going to be non-normal (Skewed t-Distribution). Moreover, the dependence structure which links the returns of these assets is not going to be a multivariate normal as assumed by BL, it will be instead derived from a copula. A copula allows to generate an *ad hoc* joint distribution between assets, so to reflect observed data. We will choose a Student t copula, which allows for high dependence in the tails and reflects market behavior. (We tend to see cocrashes and cobooms in the market).

In the following sections we will proceed by steps, initially we will consider the time series of the assets independently and then we will consider the dependence structure that links them. This process, we will show, allows us to infer an empirical correlation matrix which we will be key in the advanced model we are proposing. We start by imposing an ARMA(1,1) to the individual time series so to remove any serial correlation. Then, we proceed by applying a GARCH(1,1) to remove additional dependence in the residuals. The choice of these processes is in line with common practice in the industry. The residuals obtained at the end of the process will then be subject of the distribution fitting, in particular the Skewed t-Distribution.

We present a compact description of the general model which describes link between the dynamics involved. In the notation r_t is the process for returns (ARMA) and σ_t is the process for volatility (GARCH):

$$r_t = \mu_t(\theta) + \varepsilon_t$$
$$\varepsilon_t = (\theta)z_t$$
$$z_t \sim g(z_t|\eta)$$

where $z_t = (r_t - \mu_t(\theta))/\sigma_t(\theta)$ represents the residuals after imposing the mean and volatility models.

In this framework, θ is a vector containing all the parameters of the mean and variance processes. Moreover, the innovations have zero mean and unit variance and are distributed as a conditional distribution $g(\cdot)$ with shape parameters η. As previously mentioned, we take the mean process to be an ARMA and the variance process to be a GARCH. The distribution of the innovations will be the Skewed t-Distribution.

To summarize:

- Thanks to the Skewed t-Distribution we eliminate the assumption of normality in the individual returns series.
- Using a copula produces a dependence structure of a Student t, eliminating the assumption of normal interdependence of the assets.
- The correlation found through the copula will be used to model the interdependence in the uncertainty over the views (the off-diagonal elements of Ω).

4.1 ABL

The combination of the elements we have seen above brings us to the creation of a new model which we call Advanced Black-Litterman approach (ABL). This model tries to solve the issues of BL and accounts for non-normality of returns by combining the market prior of BL with a modified version of the views. In particular, we perform the following steps:

1. Model the mean process for the asset data with an ARMA(1,1);
2. Model the volatility process for the asset data with an GARCH(1,1);
3. Fit a Skew-t distribution on the residuals;
4. Create a dependence structure among the assets which has a Student t distribution;
5. Extract a new correlation matrix which enters the original BL framework.

The model hereby proposed has been developed in this thesis for the first time and can be seen as a modified version of BL. It still keeps the advantage of having a closed-form solution but introduces the non-normal feature of the market and allows for interaction among views expressed (in terms of their confidence).

4.2 ARMA

The ARMA(1,1) is the first step into "cleaning" the data. Cleaning refers to the fact that we wish to arrive to a point in which deviations from expectations are purely random and unpredictable. An ARMA process allows us to link the expected return of an asset to its value in the previous observation and to the previous error term. An ARMA is a combination of two simpler process, namely, an AR and MA. The first, an AutoRegressive process, assumes that returns depend on their past values, up to a certain number of past observations, p. An AR(p) is expressed as:

$$r_t = \phi_0 + \phi_1 r_{t-1} + \cdots + \phi_p r_{t-p} + \varepsilon_t$$

By contrast, a Moving Average imposes that returns depends on the q previous errors. An MA(q) is therefore written as:

$$r_t = \theta_0 + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

An ARMA(p,q) combines the two processes, therefore returns depend both from the previous p returns and the previous q errors terms. In practice though, an ARMA(1,1) is considered to be enough to model mean returns and has the following shape:

$$r_t = c + \phi r_{t-1} + \varepsilon_t - \theta \varepsilon_{t-1}$$

The purpose of the ARMA is to remove any form of serial correlation in the returns. In a normal distribution environment, dependence is fully captured by correlation. In a non-normal framework, on the other hand, this is not enough and one should consider dependence also in functions of the error terms. It is common practice to consider also correlation in the square of the residuals. This consideration brings us to the next step.

4.3 GARCH

A GARCH is the commonly used instrument to consider the correlation in squared residuals we mentioned above. The implication of considering the square function is that it allows to account for a well-known phenomenon in time series, that is **volatility clustering**. Volatility clustering can be explained as the tendency of errors to be similar in magnitude to the following errors. In other words, big shocks (either negative or positive) are followed by big shocks and the same for small shocks. This phenomenon is particularly obvious for financial returns as in times of business expansions we observe low volatility and trending markets, whereas during market crashes we observe very high volatility and big market fluctuations driven by excitations of traders. The first attempt to model this behavior is the AutoRegressive Conditional Heteroskedasticity process, or ARCH. The model predicts squared errors by autoregressing over a number of previous squared errors. The main issue with this process was the number of lags required. For this reason a more efficient approach has been introduced, the Generalized ARCH, GARCH in Engle (1982). A GARCH(p,q) models variances for assets based on lags of the previous values of variance and of the square of shocks. The expression for a GARCH(p,q) is:

$$\sigma_t^2 = \omega + \alpha_1 \sigma_{t-1}^2 + \cdots + \alpha_p \sigma_{t-p}^2 + \beta_1 \varepsilon_{t-1}^2 + \cdots + \beta_q \varepsilon_{t-q}^2$$

In practice the common choice is to use a GARCH(1,1) to account for conditional volatility dynamics. Therefore one lag of the variance and one lagged squared return is included in the expression. In this specific case, the unconditional variance for the GARCH dynamics is:

$$\sigma^2 = \frac{\omega}{1 - \alpha - \beta}$$

Arrived at this point we can construct the standardized innovations which we introduced in the beginning of this section, namely z_t. They will be the input for the estimation of our distribution.

4.4 Skewed t-Distribution

To take a decision over how to model the individual return time series we need to account for a few different characteristics of financial returns. We know that a return series normally presents negative skewness and excess kurtosis, leading to extreme events being more frequent than the normal distribution (due to higher kurtosis) and are more likely to be negative than positive (due to negative skewness). To provide an example we present a table of descriptive statistics for a broad equity index, MSCI World, using both daily and monthly returns available from 1980 until 8^{th} July, 2019.

	Mean	St.Dev	Skewness	Kurtosis	Min	Max
Daily	0.0272	0.0087	-0.5328	14.1028	-10.3633	9.0967
Monthly	0.5704	0.0429	-0.8606	5.3382	-21.1279	10.9473

Table 1: MSCI World Descriptive Statistics

We focus on skewness and kurtosis (excess kurtosis to be precise) and notice that, both in the case of monthly and daily data frequency, we observe negative skewness and excess kurtosis. Furthermore, considering minima and maxima confirms that the worst returns are greater in absolute value than the best returns. To further confirm the non-normality of the time series we perform a standard Jarque-Bera test for normality proving non-normality at the 1% significance level.

This becomes even more evident if we plot the empirical distribution of returns versus a normal distribution.

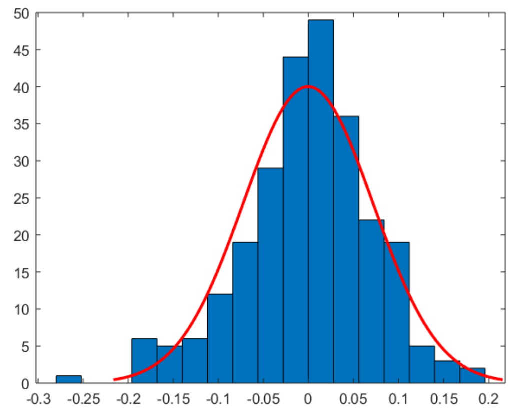

Figure 3: Empirical distribution, monthly returns

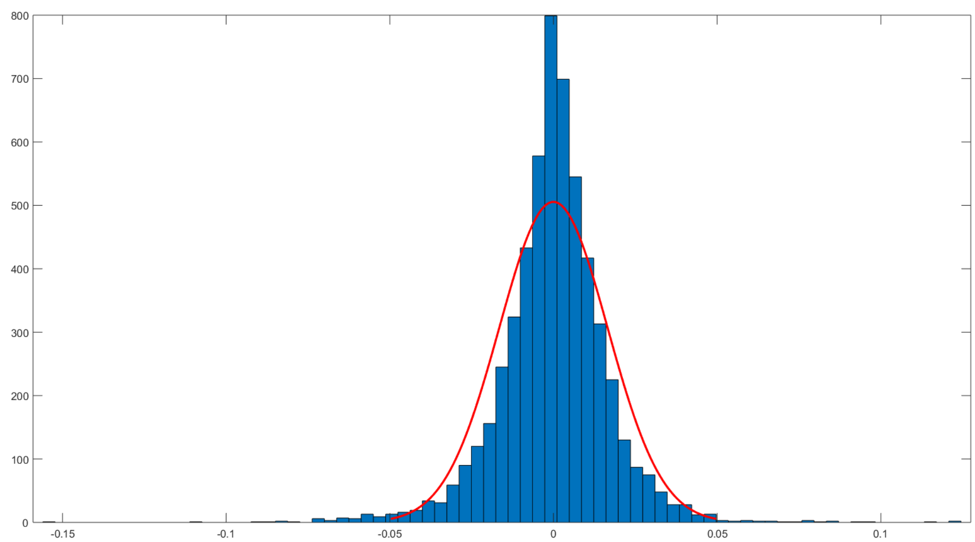

Figure 4: Empirical distribution, daily returns

Now that we have proven that non-normality and in particular, negative skewness and excess kurtosis, are standard characteristics of financial series, we ask ourself the following question: what distribution is most appropriate? If we are to focus on fat tails, the most obvious solution is the use of a Student t distribution which, for low degrees of freedom provides more common extreme events and produces higher kurtosis. The issue with this distribution is that it is symmetric (has skewness of 0). To solve the issue, the Skewed t-Distribution has been introduced by Hansen (1994) and is defined as:

$$g(z_t|\eta) = b\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi(\nu-2)}\Gamma(\frac{\nu}{2})}\left(1+\frac{\zeta_t^2}{\nu-2}\right)^{-\frac{\nu+1}{2}}$$

$$\text{where: } \zeta_t = \begin{cases} (bz_t+a)/(1-\lambda), & \text{if } z_t \leq -a/b \\ (bz_t+a)/(1+\lambda), & \text{if } z_t \geqslant -a/b \end{cases}$$

$$a = 4\lambda c\frac{\nu-2}{\nu-1} \qquad b^2 = 1+3\lambda^2-a^2 \qquad c = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi(\nu-2)}\Gamma(\frac{\nu}{2})}$$

The mathematical definition of the distribution is complicated but the principle behind it is easy to understand. The distribution treats the upside and downside shocks differently and the shape parameters, λ and ν, regulate by how much. λ is the asymmetry parameter ($-1 \leqslant \lambda \leqslant 1$) and ν represents the degrees of freedom of the distribution ($2 \leqslant \nu \leqslant \infty$). The shape parameters of the distribution can be readily estimated by Maximum Likelihood given the innovations z_t we previously found. MLE is performed to all the different assets and each one will be fitted with the appropriate parameters η. Once the marginals have been estimated for all assets we proceed to aggregate the individual results and consider how they cobehave. For this purpose, we introduce a copula.

4.5 Copula

When switching from univariate to multivariate series an important aspect is how the behavior of each marginal influences and is influenced by the others. To account for this, one needs a structure that considers multiple individual times series simultaneously. As we know, observations of frequent crashes and booms in the market suggests that assuming a multivariate normal distribution would be wrong. This assumption would generate too few extreme events and, as a consequence, the correlation among the so-called tail events (very high or very low returns) would very low.

To account for this, one could think that a multivariate Student t distribution, which allows for thick tails, is enough. As a matter of fact, we simulate a bivariate normal series imposing a fixed correlation and then simulate a bivariate student imposing the same restriction we obtain very encouraging results.

To illustrate the discussion above, we provide the results of simulation of both the bivariate distribution simulated with correlation imposed to 0.5. For the normal, we compute the overall correlation between the two series and then we compute correlation **only** in the lower tail (taken to be the portion under the -2%). We repeat the same experiment for a bivariate Student t. The output produced is the following:

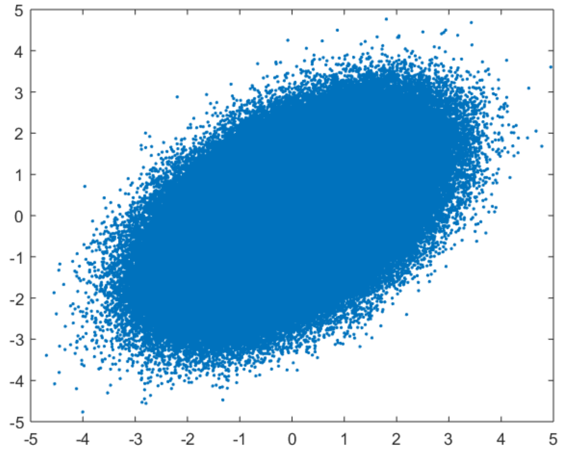

Figure 5: Bivariate Normal dependence

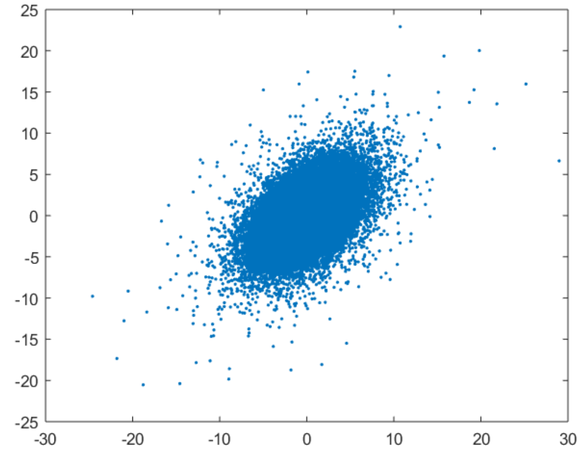

Figure 6: Bivariate Student t dependence

For the bivariate normal, we see that extreme events for the both variables are extremely rare and small in magnitude (all events are in the axis range of $\pm 5\%$). The overall correlation for the randomly generated bivariate series is 0.5003 but it is 0.1398 in the tails.

For the Student t bivariate series instead we have many more simultaneous extreme events, and when we compute the correlation in the tails it is 0.4454, much closer to the imposed correlation of 0.5. The overall correlation in the series was found to be 0.5009.

This feature, showing correlation of cocrashes and cobooms is what we are looking for and, for this reason we will find a way to specify our "view" over the dependence structure among the assets.

But, the results for the Student t distribution, which are encouraging on one hand, forego another important aspect. Imposing a multivariate Student t with ν degrees of freedom to obtain dependence among assets, implies that also the marginals have the same distribution: Student t with ν degrees of freedom. Therefore, we lose characteristics unique to the marginals which have looked for so intensively. To overcome this problem a widely used instrument in finance are copulas. They allow to impose any dependence structure of some assets, **without** imposing any restriction over the marginals.

A bivariate copula is a function which is able to measure how any two given marginals cobehave. We propose a theorem which guarantees their existence and defines them:

Sklar's Theorem (1959)
Let H be a joint distribution function of X and Y with marginal distributions F and G, respectively. Then:

- there exists a copula $C : [0,1] \times [0,1] \to [0,1]$ such that, for all real numbers (x, y)

$$H(x, y) = C(F(x), G(y))$$

  furthermore, if F and G are continuous, C is unique.

- conversely, if C is a copula and F and G are univariate distribution functions, then $H(x, y) = C(F(x), G(y))$ is a joint distribution function with marginals F and G.

From the above we understand that we could, in principle assign joint probabilities to any given marginal distributions. As we are dealing with Skewed t-Distribution marginals, we can impose a dependence structure belonging, for example, to a normal or Student t distribution. This produces an *ad hoc* joint distribution, which has all of the desired properties (both at univariate and at multivariate levels). Copulas can be of different types, namely, empirical, elliptical or Archimedean. Empirical copulas are, as the name suggests constructed based on the empirical probabilities. Archimedean copulas are produced by specific functions called copula generators. This family of copulas is very attractive because it can produce asymmetric dependence structures at the bivariate level, but is not very easily extendable to multiple series. Because they are the most suited for multivariate analysis, we will focus on elliptical copulas. The most important elliptical copulas are based on the Gaussian, Student t, Cauchy and Laplace distributions. We will base our selection of the copula based on a findings above and choose the one that allows better for the cocrash/coboom feature. For this reason the properties of the Student t copula suit our purposes at best.

To switch back from their mathematical description to their practical use, we focus now on the concrete purpose of this process. It is possible to fit a Student t copula by playing with the correlation and degrees of freedom parameters. Once the Student t copula has been fitted, it provides us with a new correlation matrix. This correlation matrix can then be used to switch back to a new covariance matrix by mean of the following passage:

$$\Sigma = D^{1/2} \cdot R \cdot D^{1/2}$$

where R is the correlation matrix we obtain and D is a diagonal matrix containing the variances of the assets.

Once we have obtained this new covariance matrix we can use it in the optimization process to measure expected dependence across the assets. This dependence is the one we wish to transfer in the matrix Ω of confidence over our views.

4.5.1 ABL vs. BL

To sum up, let us point out some differences between the regular model and the advanced one. ABL distinguishes from BL mainly in two aspects. We express our views just as in BL, but this time we eliminate the assumption of normality of returns and also specify a view over the dependence structure across the assets. This view is indirectly incorporated in the matrix Ω. In BL, such matrix is the "handmade" diagonal matrix of the uncertainty over the views, purely subjective. In ABL, such matrix, call it Ω_{ABL}, is found as follows:

$$\Omega_{ABL} = P \cdot D^{1/2} \cdot R_{Copula} \cdot D^{1/2} \cdot P'$$

Unlike in BL, Ω_{ABL} is not a diagonal matrix and it is scaled using the selection matrix, P.

We argue that using this new measure for confidence in views will bring more meaning to the views expressed and at the same time will provide for an easier version of the more complex models already present (see Meucci (2009), which are harder to replicate. The formula tells us, and this is a key element of the thesis, that not only confidence shouldn't be subjective, but also that the views' uncertainties are related among each other by their dependence structure (the off-diagonal elements should be different from 0).

Breaking down the formula we first see that the center part:

$$D^{1/2} \cdot R_{copula} \cdot D^{1/2}$$

is a new covariance matrix found by pre- and post-multiplying the correlation matrix R_{copula}, found through the copula, by $D^{1/2}$ which is the diagonal matrix containing the standard deviations of the individual assets.

On top of this, we pre- and post-multiply this new matrix by the selection matrix P, in order to scale up the matrix based on the views we have imposed and return an adequate Ω_{ABL}.

A simplified example of the difference between the two is the following:

$$\Omega_{BL} = \begin{bmatrix} (5\%)^2 & 0 & 0 \\ 0 & (5\%)^2 & 0 \\ 0 & 0 & (5\%)^2 \end{bmatrix} \qquad \text{and} \qquad \Omega_{ABL} = \begin{bmatrix} 0.062 & -0.001 & 0.003 \\ -0.001 & 0.031 & -0.023 \\ 0.003 & -0.023 & 0.052 \end{bmatrix}$$

This numbers shown here are an example but already highlight the key differences. The confidence in Ω_{BL} of 5% is an arbitrary number, provided as an example, quantifying our uncertainty over the views we have expressed.

The matrix Ω in the BL framework answers to the question: "What is, respectively, the uncertainty over the views we have expressed? And, how does such uncertainty interact across views?".

In BL, the answer is discretionary. The decision maker "feels" such uncertainty for the individual views and, usually, does not specify their interaction (off-diagonal elements equal to 0). For Ω_{ABL}, the process is mathematical instead of discretionary. Notice how the off-diagonal elements of Ω_{ABL} are not equal to 0.

The positive aspect is that the introduction of considerations over normality and dependence of confidence still leaves us with a closed-form formula, ready for use:

$$\mathrm{E}[\mu|v] = [(\tau\Sigma)^{-1} + P'\Omega_{ABL}^{-1}P]^{-1}[(\tau\Sigma)^{-1}\mu_{eq} + P'\Omega_{ABL}^{-1}v]$$

The above is analogous to normal BL but contains the new matrix Ω_{ABL}.

5 Implementation

In this section we try to focus on how to implement such findings in practice. That is, how do we use this model starting from raw data over some indices of the asset classes? First of all we need the raw data. Not so long ago, data sources were very scarse and costly, this is not the case anymore and all types of data are accessible through platforms such as Bloomberg or Thomson Reuters. In this study data was taken from Datastream by Thomson Reuters. We choose six different asset classes, to provide a well-diversified SAA in which we include one alternative asset to take advantage of its potentially high returns and diversification benefits. Once we have raw data we need to manage it and make it suitable for the application of the model. When we have the data ready to be the input of the model, we apply the models described above by means of a software. We make use of MATLAB to perform the required calculations and obtain the portfolio weights as output.

5.1 Data

In this study we start off with information over prices and market values of a number of six asset classes, proxied by some indices. They are:

- US Equities. Index: S&P 500 Composite
- EU Equities. Index: MSCI Europe
- Emerging Market Equities. Index: MSCI EM
- US Bonds. Index: BoA Merrill Lynch US Total Bond Return
- EU Bonds. Index: Bloomberg Barclays Euro Aggregate
- Alternatives. Index: HFRI Fund Weighted Composite

S&P 500 is among the main US indices, it contains the 500 top firms of the country and therefore it is representative of the major stake of US equity. For Europe and Emerging Markets, the choice has fallen on two indices provided by Morgan Stanley Capital International, one of the most trusted providers of most types of index. As regards debt, we use US Total Bond index provided by Bank of America Merrill Lynch for the US and the Bloomberg Barclays Euro Aggregate index for the EU. These indices are provided by trusted institutions and can be taken as trustworthy for the purpose of our study. The bond indices are all composed by only investment grade debt securities.

The data for the first five "traditional" asset classes is easily accessible, for the last one, the "alternative" asset class, it is more complicated to find trustable data for prices and market value. As we use a hedge fund index for alternatives, the main problem is to assign a market value to assets which are traded only by sophisticated investors and therefore are not found on exchanges. For the first five indices, data is obtained by use of Datastream through a university license. Datastream is a service by Thomson Reuters which provides for the download of all types of data directly in Excel. Regarding our sixth and more troublesome index, we choose to rely on the data provided by HFR (Hedge Fund Research) for what regards prices of the HFRI Fund Weighted Composite Index. And we take the Market Values of the entire Hedge Fund industry from Preqin, another hedge fund data provider. The prices and market values will match thanks to selection of the HFRI Fund weighted index which contains all the strategies of the hedge fund world and is therefore representative of the full hedge fund market performance (hence we can consider the market value provided by Preqin).

The choice of the indices has been made in such a way that each index would be the closest available product to proxy the asset class that it matches. The indices chosen are both large enough to be considered as an asset class and also liquid enough to be considered investable and reliable.

The frequency of data used is monthly, matching the long-term horizon of the model (around five years). The data starts from July 31st 1998 and ends in June 19th 2019. The choice of the time window is very important in terms of results obtained. In our case we have a sample containing more than 20 years of data, enough to allow implementation of the model and perform additional inference and backtesting.

Looking at the events contained in the data, we must highlight the fact that our data contains two major financial crises. As we know, 2008 has marked one of the most severe financial crises of the last century, and it is a part of our data. At the same time also the sovereign debt crisis of the 2010's is included. To include a crisis in a dataset is probably a good thing and a bad thing at the same time. Including a major crisis in the time series is a way to test the functioning of the model under stress but at the same time could lead us to decisions which are biased by unique events which, in theory, could never happen again. In any case, it would be pointless to take a dataset which only contains trends and no market surprises.

5.2 Procedure

Starting from the clean data, we start applying the model in the order that we have previously mentioned. MATLAB supports financial toolboxes which allow for easier coding (mainly the MFE Toolbox provided by Kevin Sheppard). The steps taken, after importing data and making it possible to work with it, are the following:

1. Fit a model for the mean in the marginal distribution, ARMA(1,1);
2. Fit a model for the volatility of the marginals, GARCH(1,1);
3. Estimate a Skew t distribution for the residuals of the marginals;
4. Implement a Student t copula across the various marginals to find a correlation matrix;
5. Proceed with standard methodologies (historical optimization, market priors and BL) for comparison purposes;
6. Use the new correlation matrix found via copula in the new ABL framework;

In the first steps there are a few functions that can be used to simplify the procedure. In particular the Kevin Sheppard toolbox provides for *armaxfilter* and *tarch* to easily estimate ARMA and GARCH processes, the *copulafit* function allows to estimate copulas. The way things are put together is the usual way found in the Black-Litterman framework, but this time the inputs are found through preliminary steps (there is a dedicated code for construction of the selection matrix and the expression of views).

To have a form of comparison, we decide to optimize in four different ways and then analyze the results. Because the optimization produces a set of possible portfolios for many possible levels of risk, we decide to set 25 different levels of risk, producing 25 possible allocations for each methodology we use in optimizing. In other words, our Efficient Frontier is made up of 25 points, from left to right of the x-axis, representing increasingly risky portfolio compositions.

For the interested reader the step-by-step implementation of the model can be found in the section dedicated to the code.

6 Results

In this section we will show how the model described above behaves, in particular we will analyze its output and its sensitivities to inputs. We are interested in the comparison among some candidate models and check their performance through backtesting. We will, further on, present three different scenarios which will give us an understanding of how the model behaves in different circumstances.

Now that we have discussed the data used in this thesis, the model is implemented through the code (presented at the end of the writing) and provides us with the optimization outputs.

We decide to optimize in four different ways and then compare the results. The first two methods, "mean-variance" and "market prior", are independent of the views we express. The third and fourth instead depend on them heavily.

The four optimization methodologies are the following:

1. Standard mean-variance;
2. Market prior with EWMA variance (Black-Litterman with no views and modified covariance matrix);
3. Standard BL;
4. Advanced BL;

In the first case, we perform a simple mean-variance optimization to have an understanding of the starting point. We optimize using the simple mean of the assets as expected return and their historical covariance matrix for risk.

The second case is already much more reliable in terms of results. We revert the market equilibrium to find the expected returns and compute volatilities and covariances by EWMA. EWMA is a step forward we take from the historical covariance and allows us to produce a much more reliable result, yet not sophisticated enough to introduce views and more complicated processes.

For the last two methods we need to pay more attention as this is where the purpose of this paper lies. In the normal BL framework, the element Ω_{BL} is found in a discretionary way, by simply assigning some degree of confidence to our views, in the new approach, the degree of confidence, Ω_{ABL}, is computed using correlation matrix stemming out of the copula estimation.

As we provide results for 25 different levels of risk propensity, we will not obtain a single optimal portfolio, rather, a set of portfolios, each optimal for a given level of risk (for each optimization we produce Efficient Frontiers made up of 25 points each). Sometimes we will take three levels of risk propensity to proxy the methodology output as a whole.

These three levels are:

- "Minimum Volatility", the portfolio with least risk propensity (Portfolio 1, on the left side of the x-axis)

- "Middle Portfolio" (Portfolio 12, on the center of the x-axis)
- "Maximum returns", the portfolio with highest risk propensity (Portfolio 25, on the right side of the x-axis)

As they are static in the model, we first present the results of the first two optimizations. The issues with mean-variance have been previously mentioned, we expect the output to be underdiversified or possibly present corner solutions.

The graphs which depicts the product of the optimization has Risk Propensity on the x-axis and weight on the y-axis. Each asset is highlighted in different colors and has a weight going from 0 to 1. The sum of the weights will obviously add up to the total of 1 (100% of the investment). We plot the weights for 25 increasing risk levels (increasing risk propensity). The weights change as we move from left to right to produce increasingly risky portfolios.

The result for mean-variance is the following:

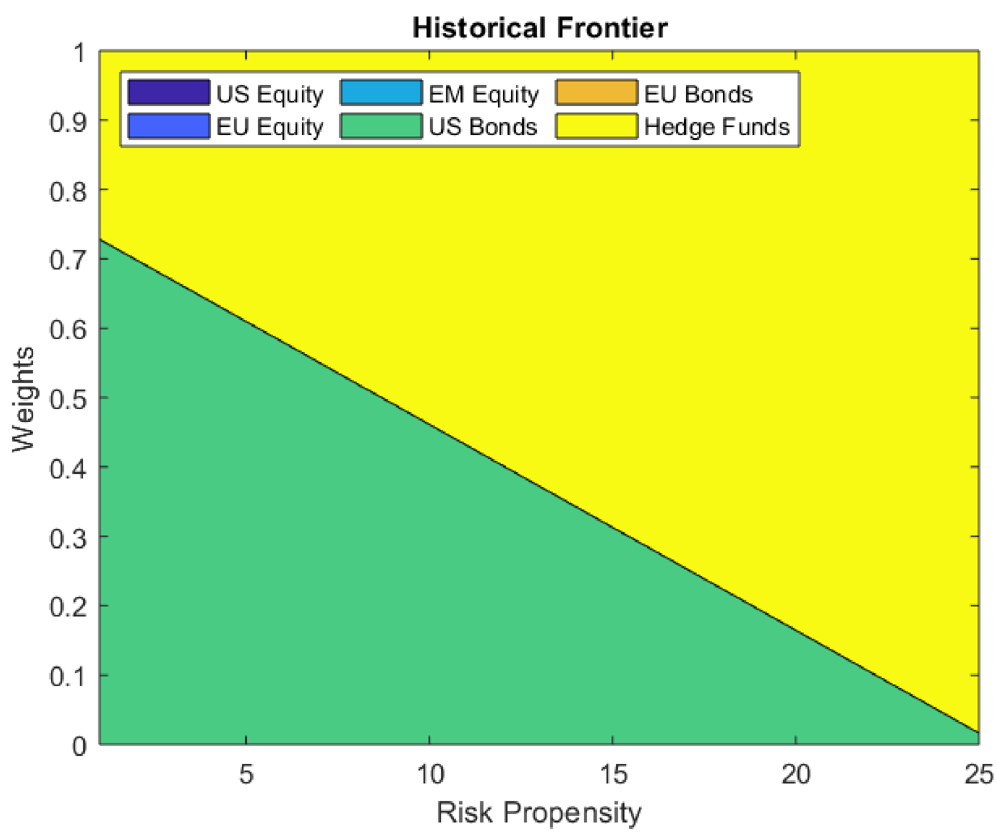

Figure 7: Mean-Variance optimized weights

The picture shows that we are fully invested in only two assets classes: US bonds and Hedge Funds. The huge weights in the individual assets imply that we are greatly exposed to shocks to individual assets and that we are not benefiting from diversification. This is not efficient in practical terms for any investor.

To highlight the disadvantages of mean-variance, let us compare the results found with the allocation (therefore, the beliefs) of the market. For this reason, we introduce neutral expected returns and EWMA volatility. We expect a great improvement in the portfolio diversification and to be invested in more assets at the same time.

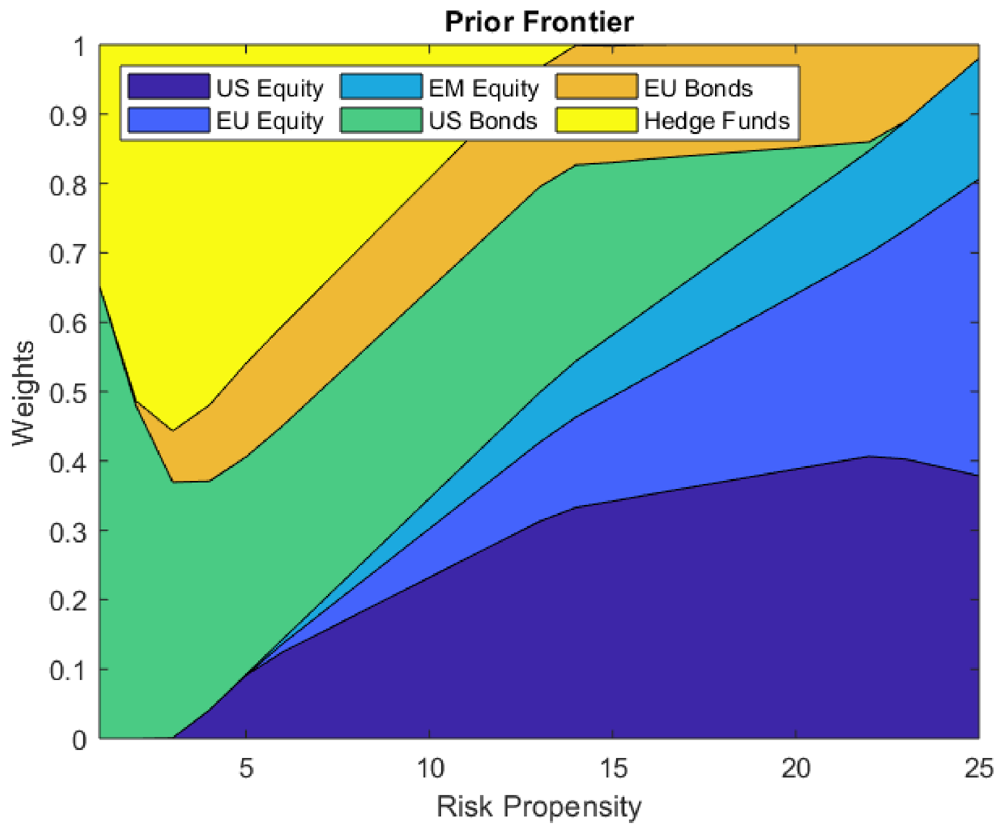

Figure 8: Market Prior optimized weights

The above in our model represents the average view over the future based on current market weights. The portfolio is diversified across the asset classes for most levels of risk propensity. The transition of the weights across the x-axis is smooth and does not present any corner solution (always keep in mind that the plot contains 25 points and is therefore discrete and not continuous). If we analyze the result, we see that we gradually shift from bonds to equity as we increase the risk of the portfolio. The least risky portfolio is almost fully invested in US bonds and hedge funds. Even though the latter asset class may sound risky to most, the fact that the two are almost uncorrelated (historical correlation of 0.2289) allows to bring down the total expected level of risk.

As expected, whenever we are more incline to risk we must switch to equities. Consistently with this, the portfolio is almost entirely invested across our three equity asset classes, US, EU and EM.

Some investors could simply stop at this point in their portfolio construction process and be happy with the benefits of this method but, in the case in which we need to express some views, we need to switch to the other two optimization methodologies.

The views we impose are arbitrary and can stem out of different considerations. We define a first scenario as follows:

Scenario 1 (Basic)

1. US equity underperforms EU equity by 4% (uncertainty of 5%).
2. US bonds overperforms EU bonds by 2% (uncertainty of 5%).
3. Hedge funds outperform equities by 3% (uncertainty of 5%).

The scenario above, which we will refer to as "basic" is simply an opinion generated by a hypothetical investor based on his personal views, these numbers could be defined as discretionary views (both the view itself and the confidence). Later on, we will consider more scenarios in order to compare the regular BL model with our proposed ABL. It is important to keep in mind that ABL is not impacted by the confidence over views, therefore confidence will be expressed **only** to allow the construction of BL allocation.

Introduction of the views should have an impact over the Market Prior allocation in order to account for the personal views of the investor. We should observe the weights tilted in the direction of our "bets". The impact of the changes in weights depends on the magnitude of the view (if it highly divergent from market neutral views) and on the confidence we express over the view itself (views for which we are certain cause big changes, views over which we have no confidence don't cause any change).

Before analyzing the output, let us remark that the confidence we express over the views leads to the following Ω_{BL} matrix:

$$\Omega_{BL} = \begin{pmatrix} 5\%^2 & 0 & 0 \\ 0 & 5\%^2 & 0 \\ 0 & 0 & 5\%^2 \end{pmatrix}$$

We notice a first impact of our decisions on the output produced. As a matter of fact, we must consider that the confidence imposed over the views has an impact on the output. As we mentioned, this impact regards the regular Black-Litterman model and not the ABL we present here. We provide evidence on this alongside with the output produced by the basic scenario.

The optimization using BL produces the following results given the market prior and expressed views:

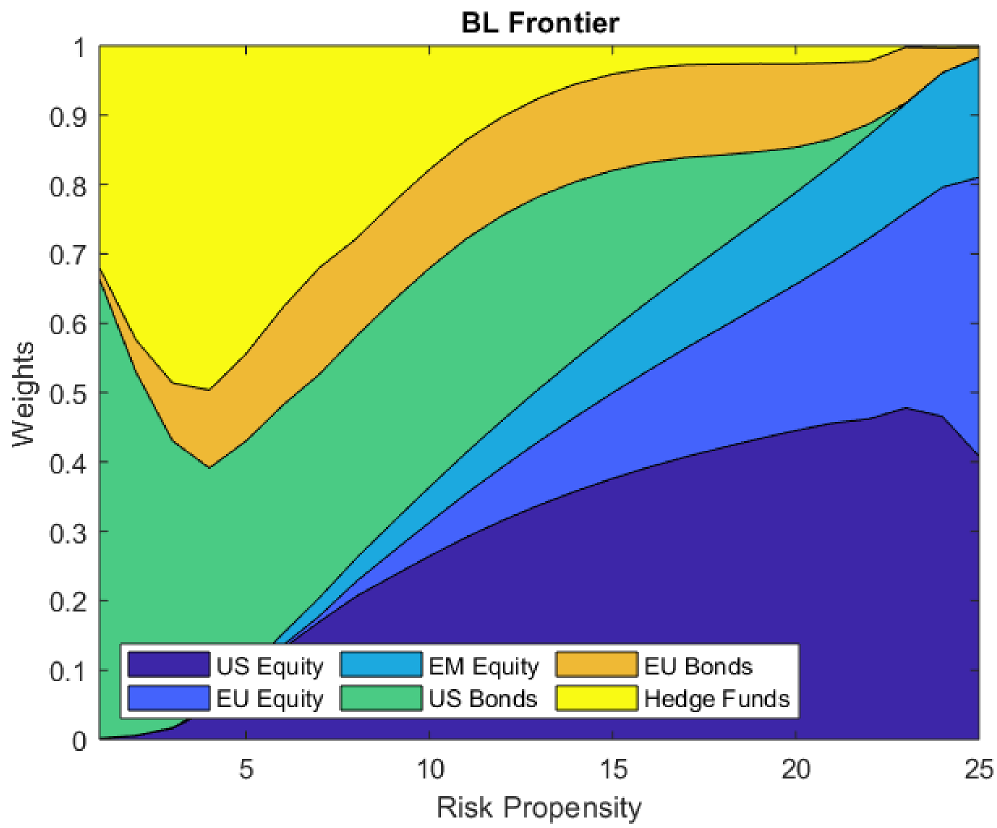

Figure 9: Black-Litterman optimized weights

The changes in the portfolio construction are visible by naked eye as we are more invested in hedge funds and US equity and less invested in EU bonds, in accordance with the views expressed. We provide the tables of the weights to highlight this fact at the end of this section.

Furthermore, we check the impact of the certainty over the views by reoptimizing using 50% and 1% as a degree of uncertainty for all the views expressed. the results are the following:

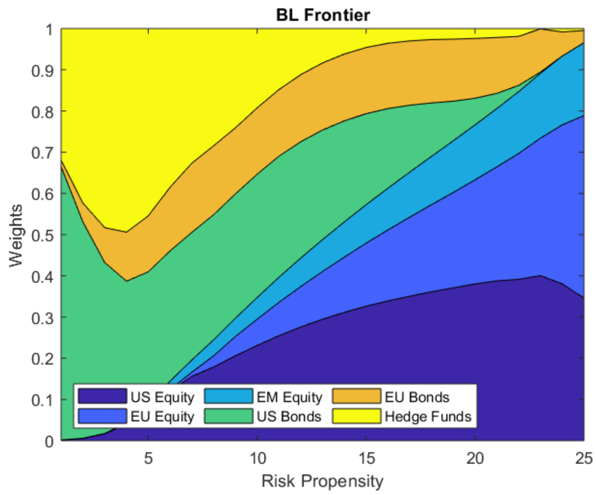

Figure 10: Uncertainty of 50%

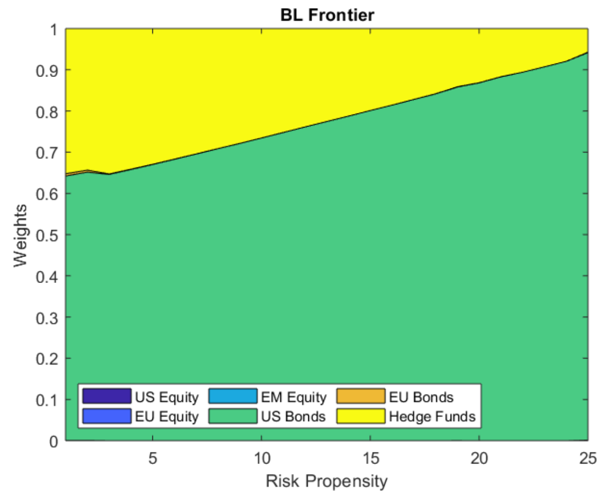

Figure 11: Uncertainty of 1%

On the left-hand side, we observe that we tilt back towards the market prior, with asset weights decreasing closer towards the market neutral values. Even if we express some views, by the fact that we are highly unconfident, the model does not make great changes to the market prior.

On the right-hand side instead, we revert to a form of output which is close to the mean-variance issue. This time the message which this picture carries is that being too confident on our views eliminates the mixture of BL and brings us back to a mean-variance scenario in which we rely completely on our inputs.

We conclude by remarking that the changes generated in the portfolio for the individual assets, even if apparently small ($\pm 5\%$) in magnitude (especially on the left hand graph) can be very significant in the overall performance of the portfolio.

Going back to the output produced by BL in the basic scenario, we can also notice that the "curves" generated by the portfolio weights are very rounded and don't present any form of spike or corner.

To sum up, the graphs above have shown that expressing views causes a shift from the prior state. On top of this, the confidence specified over our views is relevant in determining the magnitude of these changes.

Before we move to the ABL, let us briefly recall the differences between BL and ABL. Recall that the ABL model makes use of the correlation matrix found through maximization of the likelihood of a copula.

Therefore, let us first present the new correlation matrix and compare it with the common historical correlations:

	US Equity	EU Equity	EM Equity	US Bonds	EU Bonds	Hedge Funds
US Equity	1.00	0.83	0.73	0.20	0.18	0.76
EU Equity	0.83	1.00	0.79	0.23	0.45	0.80
EM Equity	0.73	0.79	1.00	0.20	0.29	0.86
US Bonds	0.20	0.23	0.20	1.00	0.26	0.16
EU Bonds	0.18	0.45	0.29	0.26	1.00	0.26
Hedge Funds	0.76	0.80	0.86	0.16	0.26	1.00

Table 2: Copula Correlation

	US Equity	EU Equity	EM Equity	US Bonds	EU Bonds	Hedge Funds
US Equity	1.00	0.86	0.80	0.26	0.22	0.77
EU Equity	0.86	1.00	0.82	0.35	0.47	0.78
EM Equity	0.80	0.82	1.00	0.29	0.30	0.88
US Bonds	0.26	0.35	0.29	1.00	0.40	0.23
EU Bonds	0.22	0.47	0.30	0.40	1.00	0.22
Hedge Funds	0.77	0.78	0.88	0.23	0.22	1.00

Table 3: Historical Correlation

The correlations among assets are slightly changed, some of them, like the correlation between US equity and bonds is exactly the same (-0.09), whereas, the correlation between other assets significantly changes (take the difference between the correlations of US equity with hedge funds, 0.47 versus 0.34).

Concluding with the final optimization, the results of the ABL method are the following:

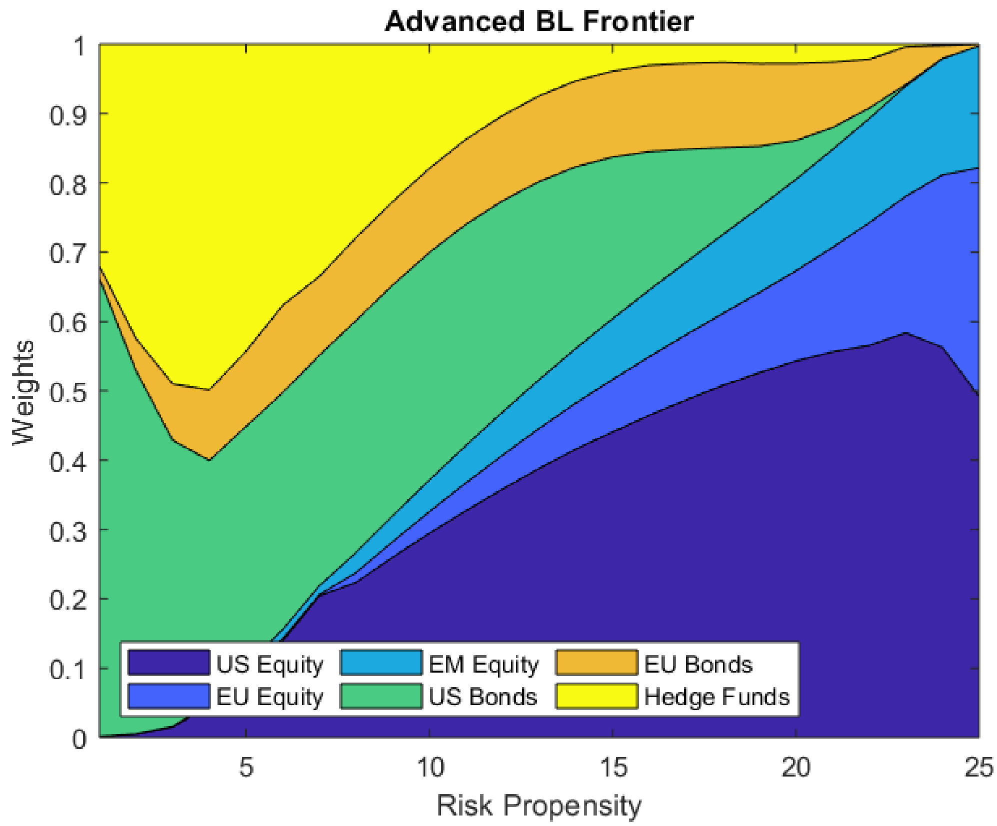

Figure 12: Advanced Black Litterman optimized weights

This time we observe much larger changes when compared to the market prior. The shifts of the weights are significant and regard especially EU bonds and US equity, significantly decreased in weight. The US bonds weight is largely increased and also EU equity is apparently more attractive. All the weights "fluctuate" across the x-axis, according to risk choices, in a smooth and curved manner, also in this case.

Again, for those who wish to see the differences in the numbers instead of using plain sight, we provide the tables for the weight of Portfolio 1, 12 and 15:

	Prior	BL	ABL
US Equity	0.00	0.00	0.00
EU Eq.	0.00	0.00	0.00
EM Eq.	0.00	0.00	0.00
US B.	0.65	0.66	0.61
EU B.	0.00	0.02	0.02
Hedge Funds	0.35	0.32	0.37

Prior	BL	ABL
0.29	0.41	0.39
0.10	0.03	0.00
0.06	0.05	0.02
0.30	0.41	0.57
0.17	0.04	0.00
0.09	0.07	0.01

Prior	BL	ABL
0.38	0.70	0.84
0.43	0.12	0.00
0.17	0.18	0.01
0.00	0.00	0.15
0.02	0.00	0.00
0.00	0.00	0.00

Table 4: Comparison BL vs. ABL for different risk propensities

6.1 ABL vs. BL (continued)

What we focus on now is the differences between the regular BL and ABL applications. We will do this mainly by backtesting over 3 possible scenarios. Backtesting will take place in the following way. We breakup our data (which goes from 31-07-98 to 28-06-19) into two parts: in-sample and out-of-sample. We will use the in-sample data to construct our model and construct a portfolio, then we will evaluate the performance of the portfolio using the out-of-sample data. We choose to match the usual timing of a strategic asset allocation and test the model over 5 years (60 observations). This means that our last in-sample observation is 30-06-14.

The views, within this framework, are expressed on the 30-06-14 as if everything following such date is unknown. We apply three sets of views which constitute our three scenarios. The confidence level has been calibrated at 5% for all scenarios to produce meaningful results. Similar levels of confidence (in the range $5-25\%$ are acceptable, whereas, more extreme (lower or higher) levels produce extreme situations.

We recall the first scenario and, for the two remaining, we will choose the views in such a way that allows to evaluate the performance of the models for correct or wrong views *a posteriori*.

1. "Basic", the scenario we have already seen above in which the views are express in a purely discretionary way. Recall that the views in this case are:
   (a) US equity overperforms EU equity by 4% (uncertainty of 5%).
   (b) US bonds overperforms EU bonds by 2% (uncertainty of 5%).
   (c) Hedge funds outperform equities by 3% (uncertainty of 5%).
2. "Advantageous", this second scenario contains the same format of views expressed above but the over-/under-performance of each view is set by using the **real** market outcome. That is, we compute the a posteriori return of the out-of-sample timeseries and use it as if it was a view. This leads to the following scenario:
   (a) US equity **over**performs EU equity by 15.41% (uncertainty of 5%).
   (b) US bonds **over**performs EU bonds by 19.89% (uncertainty of 5%).
   (c) Hedge funds **over**perform equities by 6.97% (uncertainty of 5%).
3. "Disadvantageous", this last scenario has the opposite purpose as the advantageous one. It takes as view the opposite of the realized returns. The views expressed here are:
   (a) US equity **under**performs EU equity by 15.41% (uncertainty of 5%).
   (b) US bonds **under**performs EU bonds by 19.89% (uncertainty of 5%).
   (c) Hedge funds **under**perform equities by 6.97% (uncertainty of 5%).

In each of the scenarios described above we take and compare the results of BL and ABL for the three different levels of risk propensity we specified above (abbreviated to MinVol, Middle, MaxRet):

1. Minimum Volatility (Portfolio 1);
2. Middle Portfolio (Portfolio 12);
3. Maximum Return (Portfolio 25);

At the end of this process we wish to understand how the two methodologies behave in different circumstances and for different levels of risk propensity so that an investor can choose in a rigorous way which one to apply in its portfolio construction process.

Before we proceed with the analysis, we first show the *a posteriori* results of the out-of-sample data. In this way we can get a first understanding of how the market and all the assets have actually behaved.

We present below a graph and a table of the out-of-sample performance of the individual asset classes:

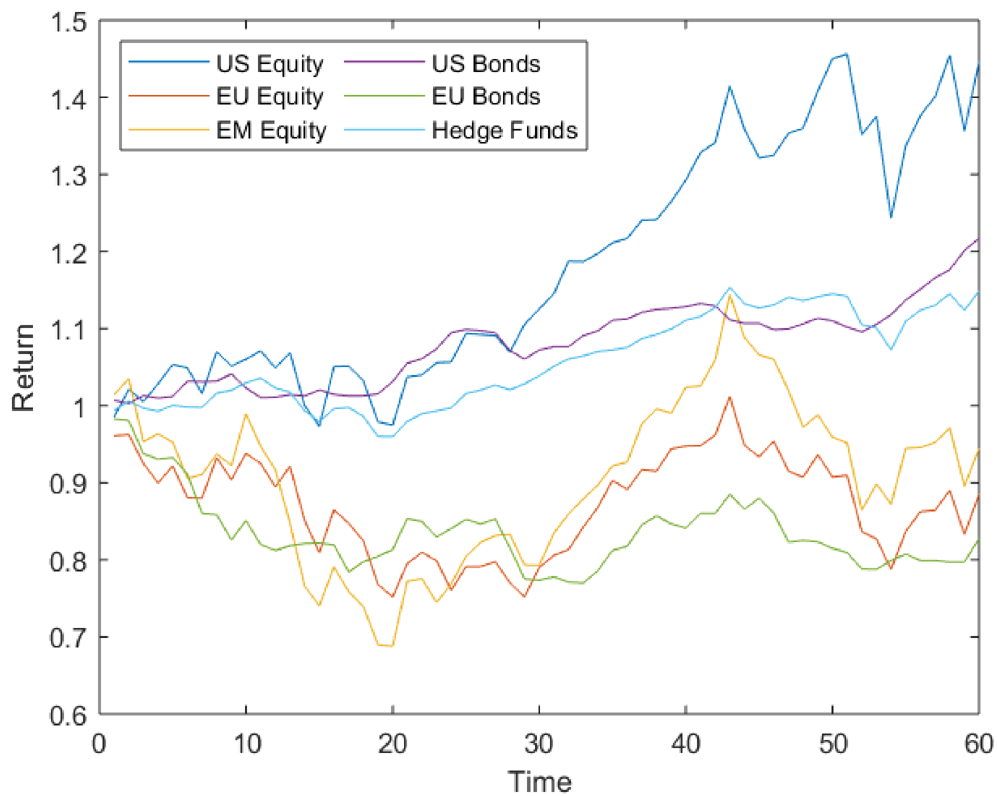

Figure 13: Asset Classes Total Returns

	US Equity	EU Equity	EM Equity	US Bonds	EU Bonds	Hedge Funds
Mean Return	0.0068	-0.0013	0.0001	0.0033	-0.0029	0.0024
Volatility	0.0347	0.0382	0.0454	0.0093	0.0226	0.0128
Total Return	1.4462	0.8853	0.9446	1.2171	0.8277	1.1487

Table 5: Asset Classes out-of-sample statistics

We observe that in the last 5 years, the top performer was US equity followed by US bonds and Hedge Funds. EU equity and bonds show negative overall performance, the same holds for EM equity. It is clear from this portrait that the winning bets are the ones on the US market and the losing bets are on the EU market. Hedge Funds perform discreetly producing 14% in 5 years.

6.1.1 Basic Scenario

Given the views of the basic scenario we obtain the following allocations for the BL and ABL methodologies:

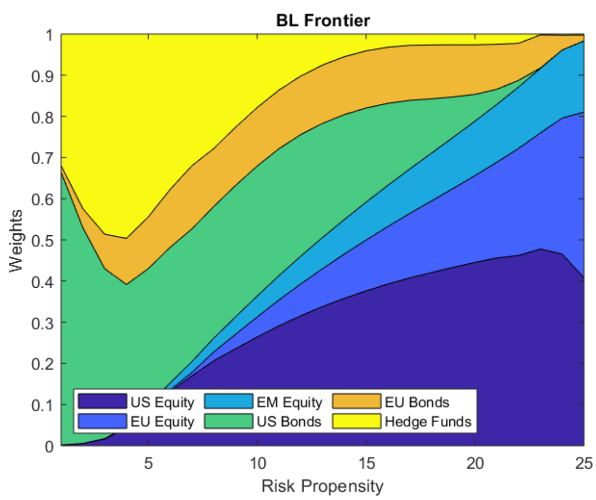

Figure 14: BL allocation

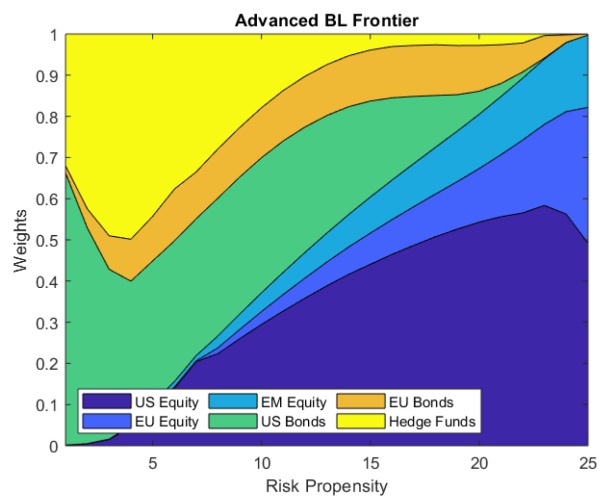

Figure 15: ABL allocation

We see that the changes from the prior setting reflects the views incorporated in the scenario in terms of investment in the asset classes. Furthermore, we see that the ABL portfolio incorporates the views in a "larger" way, creating larger tilts from the prior allocation towards our views. This can be seen through the weights assigned to US Equity and EU Bonds in particular.

To understand the differences in the results we provide a table of the weights computed for the four different methodologies for the minimum variance portfolio, the maximum expected return portfolio and the middle portfolio.

	MV	Prior	BL	ABL
US Equity	0.00	0.00	0.00	0.00
EU Equity	0.00	0.00	0.00	0.00
EM Equity	0.00	0.00	0.00	0.00
US Bonds	0.73	0.65	0.66	0.66
EU Bonds	0.00	0.00	0.02	0.02
Hedge Funds	0.27	0.35	0.32	0.32

Table 6: Minimum Volatility Portfolio

	MV	Prior	BL	ABL
US Equity	0.00	0.29	0.32	0.36
EU Equity	0.00	0.10	0.08	0.05
EM Equity	0.00	0.06	0.07	0.06
US Bonds	0.40	0.30	0.30	0.30
EU Bonds	0.00	0.17	0.14	0.12
Hedge Funds	0.60	0.09	0.10	0.10

Table 7: Middle Portfolio

	MV	Prior	BL	ABL
US Equity	0.00	0.38	0.41	0.49
EU Equity	0.00	0.43	0.40	0.33
EM Equity	0.00	0.17	0.17	0.18
US Bonds	0.02	0.00	0.00	0.00
EU Bonds	0.00	0.02	0.01	0.00
Hedge Funds	0.98	0.00	0.00	0.00

Table 8: Maximum Return Portfolio

The tables show that the difference between the weights produced for this scenario increases with risk propensity. In particular, if we compare the percentages produced by BL and ABL for minimum volatility we see no discrepancies at all. For the middle portfolio we see some discrepancies but of small magnitude and the discrepancies become more evident in the last table, for the maximum return portfolio. We observe that the magnitude of the differences increases as we move towards riskier portfolios. Notice that the tables show also the weighting of the MV and Market Prior portfolio in order to have a ready comparison. For the time being, we focus on BL vs. ABL.

To understand how the portfolios behave in a real framework we show the results of the backtesting of the three different risk propensities in the basic scenario:

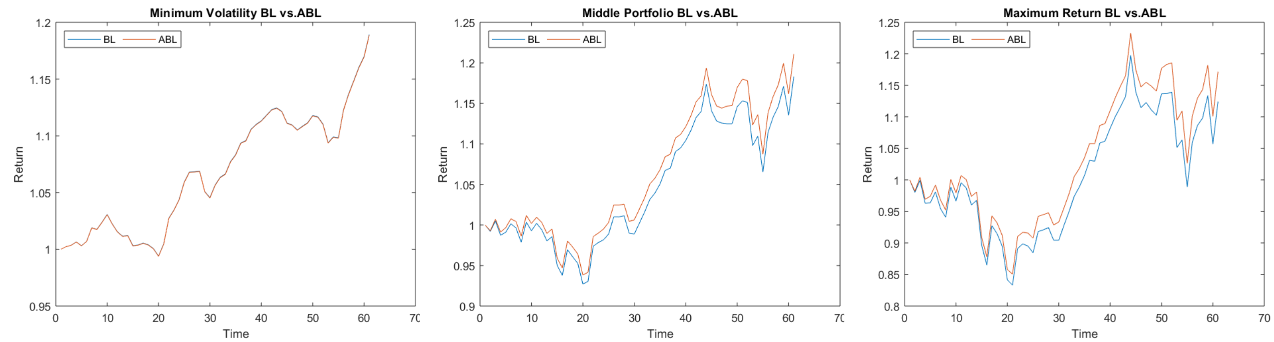

Figure 16: Comparison BL vs. ABL for different risk propensities

The graphs show the returns of the ABL versus BL portfolios *a posteriori*. That is, we invest in the portfolio optimal at the last observation date, 30-06-14 and then compute the returns for those weights using the out-of-sample returns. The plots show that ABL and BL perform almost equally for the "minimum volatility" investor (the lines basically overlap), but ABL outperforms BL for both the middle portfolio and the maximum return portfolio. The total returns for the portfolios are the following:

	Min Vol	Middle Port	Max Ret
BL	1.1891	1.1831	1.1243
ABL	1.1887	1.2110	1.1716

Table 9: Total Returns Basic Scenario

We note that BL and ABL have the same total return up to two decimal spots for the Min Vol case. For higher levels of risk propensity, we observe increasing relative performance of the ABL portfolio with excess return close to 3% for Middle Port and close to 5% for Max Ret. It would seem as if ABL is better than BL at first, but remember that this scenario is based on mild views which can be considered by some as trivial. Therefore, it is worthwhile continuing our analysis to observe the behavior of the models under "extreme" scenarios.

6.1.2 Advantageous Scenario

As a second scenario we take the advantageous one. For this scenario we use the return observed in the first year of out-of-sample and use it as view. This inevitably leads to a perfect situation in which we place a bet and get it correct in an exact way. Notice that we do not choose the full five years *a posteriori* performance as view, rather, we look just one year ahead. The reason for this is that we want to limit the scenario to what we can call a "good start scenario"; we are not interested in a perfect view as distant as 5 years ahead, we just want the portfolio to **start** in the right direction. Allowing for the portfolio to start well at first is analogous to an investor having a confident short term view but not as confident on the medium term. Because we always have the option of switching to Tactical Asset Allocation therefore it would be superflous to consider such long scenario.

The results of the optimization in Scenario 2 are the following:

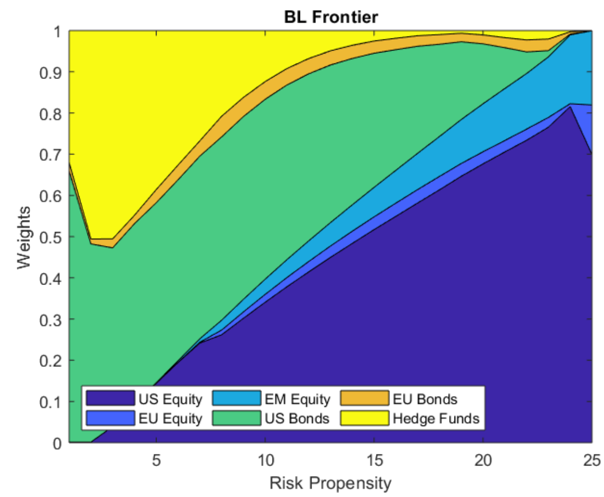

Figure 17: BL allocation

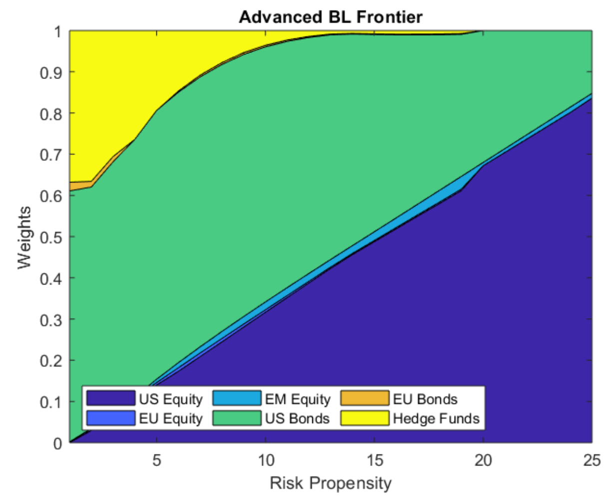

Figure 18: ABL allocation

What we notice at first sight is that for both scenarios the extreme views have reduced the diversification of the portfolio significantly, leading in the case of BL to an unexpected corner solution. The reasons behind this can be found in the magnitude of our views. Indeed, we are fairly confident over the (high) returns of some risky asset classes such as equity and hedge funds. This constitutes in some way a reversion back to the issues caused by Markowitz optimization. In this sense it is not surprising to observe a MV-like issue in BL. The ABL avoids the issue of the corner solution as it relies on more modelled inputs but still presents significant decrease in diversification, in a more exacerbated manner than BL. Here we have a first symptom of the "aggressiveness" of the new approach.

We can confirm the findings above by analyzing the tables below:

	MV	Prior	BL	ABL
US Equity	0.00	0.00	0.00	0.00
EU Equity	0.00	0.00	0.00	0.00
EM Equity	0.00	0.00	0.00	0.00
US Bonds	0.73	0.65	0.66	0.61
EU Bonds	0.00	0.00	0.02	0.02
Hedge Funds	0.27	0.35	0.32	0.37

Table 10: Minimum Volatility Portfolio

	MV	Prior	BL	ABL
US Equity	0.00	0.29	0.41	0.39
EU Equity	0.00	0.10	0.03	0.00
EM Equity	0.00	0.06	0.05	0.02
US Bonds	0.40	0.30	0.41	0.57
EU Bonds	0.00	0.17	0.04	0.00
Hedge Funds	0.60	0.09	0.07	0.01

Table 11: Middle Portfolio

	MV	Prior	BL	ABL
US Equity	0.00	0.38	0.70	0.84
EU Equity	0.00	0.43	0.12	0.00
EM Equity	0.00	0.17	0.18	0.01
US Bonds	0.02	0.00	0.00	0.15
EU Bonds	0.00	0.02	0.00	0.00
Hedge Funds	0.98	0.00	0.00	0.00

Table 12: Maximum Return Portfolio

The weights are very extreme especially for higher risk propensities (Portfolio 12 and 25) and it seems as if the "safe" choice of sticking to market prior would lead to a better performance. For comparison, one can check that diversification seems higher in the allocation of the Market Prior views found in Fig.9 above. Below instead, are the performances of BL and ABL for the present scenario.

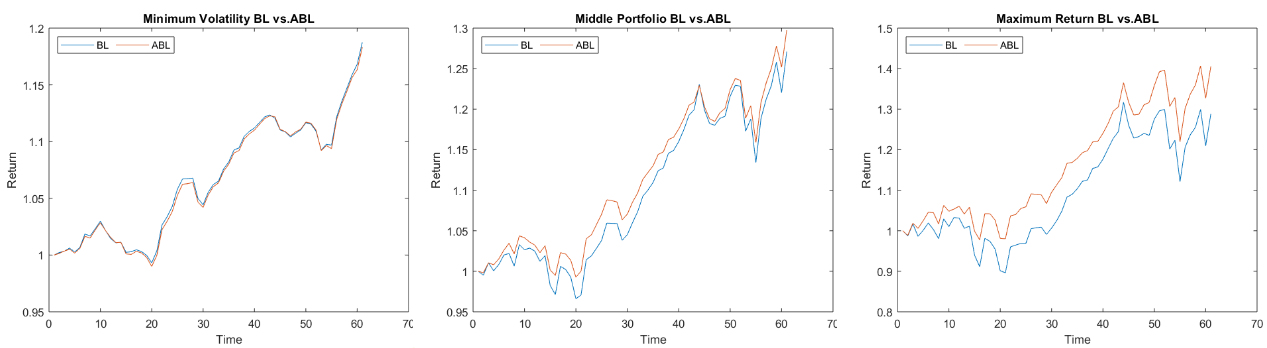

Figure 19: Comparison BL vs. ABL for different risk propensities, perfect view

If we are to compare the performance of the two methods we observe, as in the basic scenario, that MinVol produces very similar results. For the Middle Port risk level, we observe a mildly better performance of the Advanced model. As we can see from the total return table below, it is for the maximum return portfolio that we observe the highest difference, with ABL significantly outperforming BL.

	Min Vol	Middle Port	Max Ret
BL	1.1874	1.2710	1.2881
ABL	1.1831	1.2975	1.4054

Table 13: Total Returns Advantageous Scenario

As in the basic case, the MinVol stategy produces equal returns for the two strategies up to two decimal spots. ABL middle portfolio outperforms the BL competitor by slightly more than 2%. In the maximum return portfolio we observe the new model outperforming the original by almost 12%. The reasons for this can be found in the aggressiveness of ABL. It incorporates the views in a more extreme way but is still resilient to the corner solution issue. On the other hand, a problem with such aggressiveness could be experienced if we select the wrong views. The next scenario tries to analyze this latter case.

6.1.3 Disadvantageous Scenario

This last scenario represents the opposite case as the one we have just seen. It focuses on the analysis of a negative scenario in which the views we have expressed turn out to be exactly the opposite of real-life market developments. Therefore, we reoptimize on the new set of views and obtain the following results for our allocations:

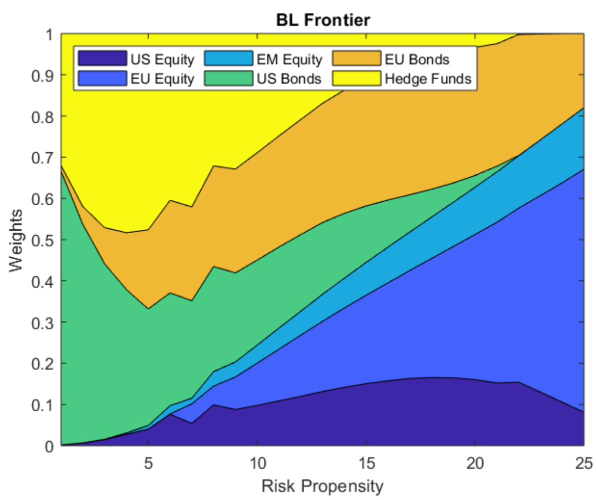

Figure 20: BL allocation

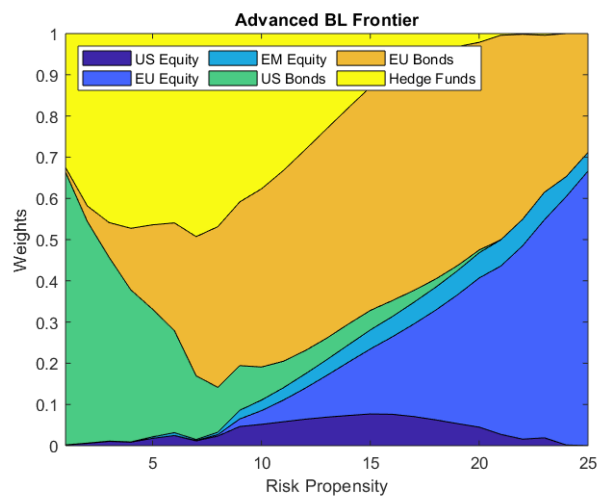

Figure 21: ABL allocation

Considering that the views expressed are quite extreme, the results are encouraging. Both BL and ABL produced fairly diversified portfolios, but, again we observe that ABL is more aggressive in incorporating views. We notice a large decrease in US equity and bonds and a large increase in EU bonds. *A posteriori* we already know that this will probably lead to even more negative returns than BL.

	MV	Prior	BL	ABL
US Equity	0.00	0.00	0.00	0.00
EU Equity	0.00	0.00	0.00	0.00
EM Equity	0.00	0.00	0.00	0.00
US Bonds	0.73	0.65	0.66	0.66
EU Bonds	0.00	0.00	0.01	0.01
Hedge Funds	0.27	0.35	0.32	0.33

Table 14: Minimum Volatility Portfolio

	MV	Prior	BL	ABL
US Equity	0.00	0.29	0.12	0.06
EU Equity	0.00	0.10	0.15	0.07
EM Equity	0.00	0.06	0.06	0.03
US Bonds	0.40	0.30	0.19	0.06
EU Bonds	0.00	0.17	0.28	0.49
Hedge Funds	0.60	0.09	0.21	0.28

Table 15: Middle Portfolio

	MV	Prior	BL	ABL
US Equity	0.00	0.38	0.08	0.00
EU Equity	0.00	0.43	0.59	0.67
EM Equity	0.00	0.17	0.15	0.05
US Bonds	0.02	0.00	0.00	0.00
EU Bonds	0.00	0.02	0.18	0.29
Hedge Funds	0.98	0.00	0.00	0.00

Table 16: Maximum Return Portfolio

Once again, the largest differences in the weights found can be seen in the maximum return portfolio. Also the middle portfolio shows tangible diffences between the two allocations, whereas in the minimum volatility portfolio the differences are almost inexistent. The results produced by the portfolios above in terms of performance are shown in the following graphs.

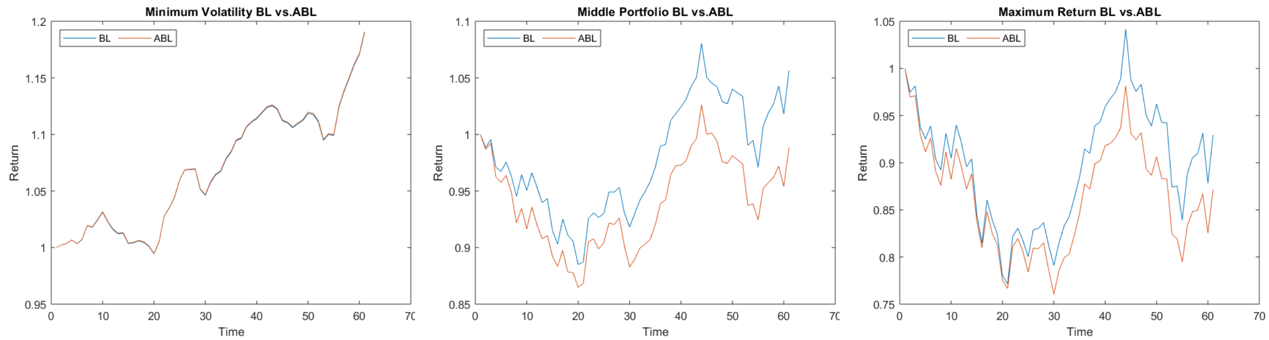

Figure 22: Comparison BL vs. ABL for different risk propensities, opposite view

The performance of the MinVol cases is inevitably almost the same as it stems out of almost identical portfolios. Opposite to the advantageous case, in this scenario it is BL outperforming ABL. The interesting thing is that the MaxRet ABL performs relatively better than the MiddlePort when compared to BL. The total return tables below provide the numbers.

	Min Vol	Middle Port	Max Ret
BL	1.1901	1.0564	0.9295
ABL	1.1909	0.9884	0.8716

Table 17: Total Returns Disadvantageous Scenario

The BL middle portfolio surpasses the ABL middle portfolio by almost 7% whereas the ABL maximum return portfolio is approximately 5% behind its competitor.

Taking the results of the three scenarios together we can form an opinion on the BL vs. ABL comparison. First of all, we state that for low risk profile investors, which will construct portfolios closer to the minimum volatility one, it is almost indifferent to choose ABL over BL or vice versa as the results produced in the end are very close to identical. For identical output, we choose the ABL to account for the theoretical results we have mentioned (negative skewness and excess kurtosis). The investor which places himself in a central position on the risk propensity axis has a more complicated decision to take. He must weigh the benefits of performing more thanks to ABL in a positive situation against the downturns of being in a negative setting and holding the ABL portfolio. From what we have seen, ABL produces better results in the basic scenario, with mild views. Nevertheless, ABL outperforms the BL portfolio much less in the advantageous case than the underperformance observed in the disadvantageous scenario (+2% vs. -7%). For this reason, this type of investor will most likely choose ABL in the a general case, but will revert to a BL optimization if his primary concern is not losing money. A risk loving investor who wants to construct a risky portfolio will probably have less undecisiveness. As we have seen ABL outperforms BL by 3% in the basic case and by 12% in the advantageous case. ABL underperforms BL by 5% in the disvantageous case. The choice for the risk loving investor will likely be an ABL portfolio.

	Min Vol	Middle Port	Max Ret
Choice	ABL	Uncertain	ABL

Table 18: Model best choice

The choice of three investors belonging to the three categories of risk propensity can be summarized as follows:

For two out of three "proxy investors", ABL is better over BL. Only for the middle investor there is uncertainty over the optimal choice. Therefore, taking the two as models competing in being used by an investor with views, we can state that ABL is, in many but not all cases, a better performing optimization tool for an investor. Because of this finding above we can now focus on comparing ABL performance with the Prior methodology to have an in-depth understanding of their differences. This is going to be the focus of the next section and will tell us whether it is better or not to express views at all, on average.

6.2 Performance

Because of the simplicity and popularity of ETFs, it is very easy to be passively invested in what can be defined to be a market portfolio. For this reason, as we try to understand the performance of the ABL portfolio, we compare with the market (in this case we do not use an ETF, instead we use market neutral weights). Obviously, beating the market is a must whenever we switch from simple replication of an index to an active strategy. As this model is being proposed for the first time, in this discussion we disregard transaction costs. This is really a big issue in any investment strategy, in our case it can be disregarded as we are not concerned yet with the actual purchase of securities (subject to transaction costs); we are interested, instead, in allocating among asset classes. The issue of security selection and, therefore, transaction costs will enter the discussion only in subsequent phases which follow. In this section we first analyze the performance of the market (taken to be the Prior allocation with market neutral weights) and then we compare key statistics to understand the strengths and weaknesses of the ABL strategy.

The last section confirmed that an investor who wants to express some views is recommended to use ABL over BL optimization. In this section we try to understand how expressing views can help our portfolio performance. It is trivial that expressing perfect views or completely wrong views (Scenarios 2 and 3) will lead, respectively, to a major over- and underperformance. For this reason what we compare to the market is our "Basic" scenario. As we mentioned, what is defined for us as "Market" portfolio, is the market prior portfolio generated by market neutral weights.

Recall the allocations of the Market Prior and its weights for the three standard risk propensities in the following graph and table:

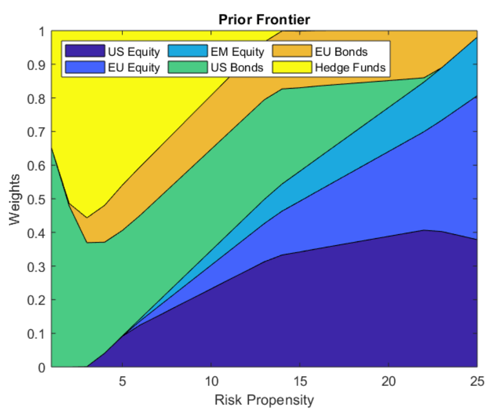

Figure 23: Prior allocation

	Min Vol	Middle Port	Max Ret
US Equity	0.00	0.29	0.38
EU Equity	0.00	0.10	0.43
EM Equity	0.00	0.06	0.17
US Bonds	0.65	0.30	0.00
EU Bonds	0.00	0.17	0.02
Hedge Funds	0.35	0.09	0.00

Table 19: Weights Market Prior

Remark The neutral weights are the ones derived using a risk aversion coeffient of 2.5. The risk aversion coefficient is equivalent in meaning for the risk propensity which we deal with. The market neutral weights are:

	US Equity	EU Equity	EM Equity	US Bonds	EU Bonds	Hedge Funds
Initial Weights	0.31	0.11	0.07	0.30	0.17	0.04

Table 20: Market Weights

Taking the latter weights as benchmark we plot its performance alongside the performance of the individual assets over our backtesting window.

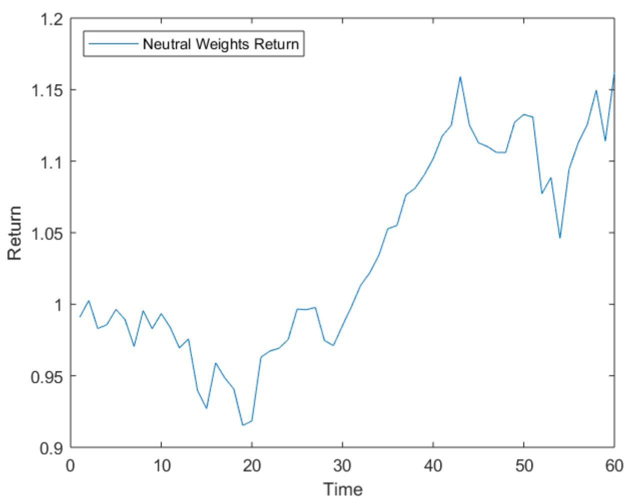

Figure 24: Market Return

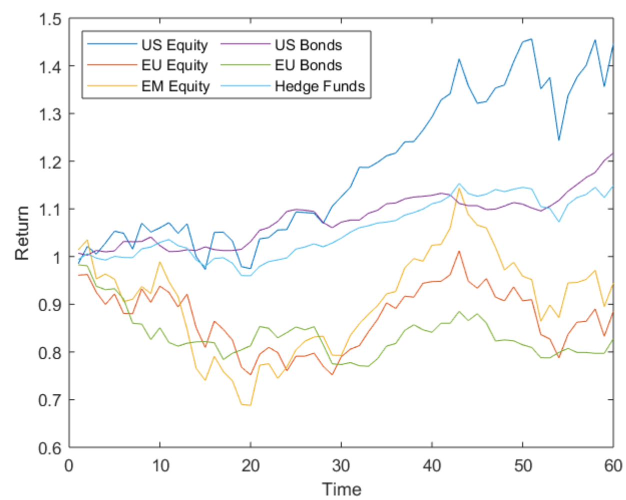

Figure 25: Returns of the individual assets

We provide some statistics for the market assets to understand which asset class performed well in the out-of-sample period.

	US Equity	EU Equity	EM Equity	US Bonds	EU Bonds	Hedge Funds
Mean Return (%)	0.68	-0.13	0.01	0.33	-0.29	0.24
St.Dev	0.16	0.06	0.10	0.05	0.05	0.06
Sharpe Ratio	1.81	-0.84	0.03	2.62	-2.54	1.66
Geometric Mean	1.17	0.87	0.90	1.08	0.83	1.06
Total Return	1.45	0.89	0.94	1.22	0.83	1.15

Table 21: Market Statistics

We now provide the performance of the three standard risk propensities applied to the ABL:

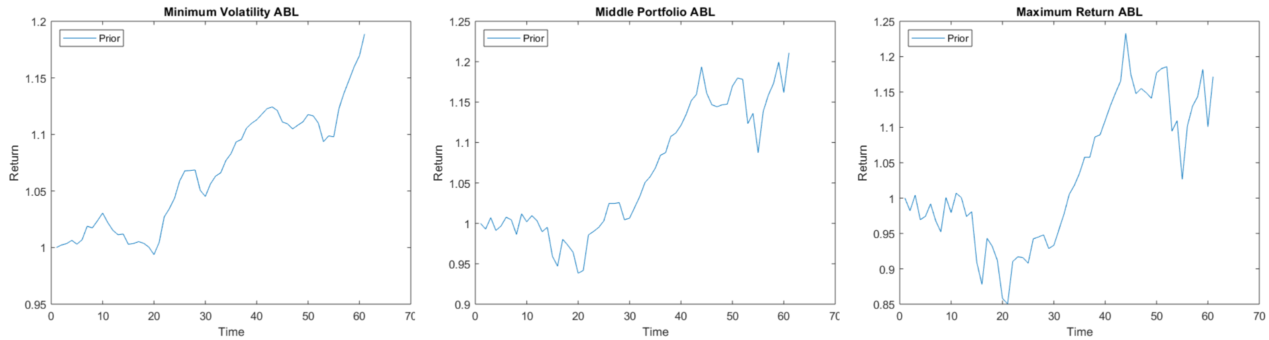

Figure 26: ABL for different risk propensities

The performance in terms of total return for the three risk propensities of the ABL portfolio plus the market neutral portfolio are the following:

	Market	Min Vol	Middle Port	Max Ret
Total Return	1.16	1.19	1.18	1.12

Table 22: Total returns ABL

By the graphs and the table above we find that the total return generated by ABL beats the market for lower risk propensities: MinVol and MiddlePort. On the contrary, the MaxRet ABL portfolio is beaten by the market by 4%.

Considering the intrinsic aggressive nature of the portfolio, it is reassuring that it performs at best for low levels of risk propensity. But we must also consider other key measures to understand fully the drivers of performance.

To start, we rely on some key statistics: the mean return, the volatility and the annualized Sharpe Ratio. Sharpe Ratio is a very popular measure of performance, it measures the amount of return per unit of risk taken.

$$\text{Sharpe Ratio} = \frac{\text{Average return}(\bar{\mu})}{\text{Volatility}(\sigma)}$$

	Market	Min Vol ABL	Middle Port ABL	Max Ret ABL
Mean Return (%)	0.25	0.29	0.33	0.29
Volatility	0.0195	0.0077	0.0188	0.0346
Sharpe Ratio	0.45	1.31	0.60	0.29

Table 23: Key Statistics

It is interesting to find the highest Sharpe Ratio for the MinVol portfolio. Another striking feature is that the market has the lowest mean return, still, it manages to achieve a Sharpe Ratio which is better than MaxRet thanks to its lower volatility. We can look at higher moments of the return series to understand more of our results.

	Market	Min Vol ABL	Middle Port ABL	Max Ret ABL
Min (%)	-4.5121	-1.6709	-4.3725	-7.8112
Max (%)	5.0798	2.2680	4.8507	7.3666
Skewness	0.0020	0.0755	-0.1062	-0.2517
Kurtosis	3.1971	3.4864	3.3969	3.2117

Table 24: Advanced Statistics

As regards these additional statistics, we observe that the MinVol portfolio is the one with the narrowest Min-Max gap, with both values which are quite low. The market and MiddlePort have similar gaps, with the market one slightly wider. The MaxRet Portfolio shows a gap going from over -7% to +7%, showing significantly higher volatility. Asymmetry, measured by skewness, shows that the most attractive portfolios for this characteristic (we like positive skewness and dislike negative) are the market and MinVol ones. Furthermore, investors normally prefer thin tails, translated in statistical terms, they like low kurtosis. The series of returns with the lowest kurtosis is the market portfolio.

We continue with even more measures of performance and risk. This time we focus on measuring the downside risks of the portfolios. We use for this purpose the downside volatility and ValueAtRisk (VaR) computed at 95%.

Downside volatility (σ_{-}) takes into account the fact that volatility is a symmetric measure. For this reason, an increase in volatility can be due to both positive and negative returns. Downside volatility only focuses on the negative returns when computing volatility and disregards the "good" volatility (due to positive returns). Value at Risk on the other hand is a measure of "how badly can things go?". It answers to the question: what is the return that I will observe in the worst 5% of possible scenarios?". To proxy this quantity we use the empirical 5% quantile of the distribution. From another perspective, the value we obtain for this measure tells us that 95% of the times we will perform better than such returns. A final measure we use is Expected Shortfall (ES), a concept which is very close to VaR. If the VaR tells us what the 5% worst scenario is, ES tells us what is the expected loss given that we are in the worst 5% of possible outcomes.

Using the downside risk measure we also provide another performance measure, the Sortino Ratio. The concept is analogous to the Sharpe Ratio but uses downside volatilities instead of normal standard deviations.

$$\text{Sortino Ratio} = \frac{\text{Average return}(\bar{\mu})}{\text{Downside Volatility}(\sigma_{-})}$$

	Market	Min Vol ABL	Middle Port ABL	Max Ret ABL
Downside Risk	0.0120	0.0044	0.0125	0.0241
VaR (95%)	-3.04	-0.89	-3.17	-6.85
ES (95%)	-3.79	-1.34	-3.85	-7.36
Sortino Ratio	0.73	2.28	0.90	0.41

Table 25: Additional Measures

We notice from the table above that MinVol ABL, due to its conservative nature has the lowest measure of downside risk, making it less subject to negative market swings. The Market and MiddlePort perform similarly and MaxRet has, inevitably, much more exposure to negative shocks. The same pattern applies to ValueAtRisk, with MinVol showing a worst 5% outcome of -0.89%, followed by -3.04% for the market portfolio and -3.17% for MiddlePort. Much higher is the VaR value for MaxRet, equal to -6.85%. We notice that the order for the ES is the same, with more negative values by construction.

The Sortino Ratio provides a measure of the trade-off between negative outcomes and average returns. The findings show that MinVol is by far the best performer under this perspective and that MaxRet is the worst. Once again Market and MiddlePort are in the middle, with returns not far from one another.

From the discussion above we can summarize some general facts. We noticed that the performance of the minimum volatility portfolio constructed using ABL has the best performance indicators when compared to its competitors. It shows in fact, higher Sharpe Ratio and Sortino Ratio. It is also very protective, as it shows excellent downside risk measures and a narrow range of returns. Of course, it is not a valid choice for a less risk averse investor. When the desire for risk increases, the alternative are the Market and Middle portfolio. They perform similarly, with the ABL MiddlePort presenting slightly better Sharpe and Sortino Ratios, but at the same time showing negative skewness, fatter tails and higher VaR and ES. For this reason, a more simple and adequate investment would be the Market Portfolio. For the investor in search of more returns, MaxRet ABL provides for consistent positive returns at the risk of large negative shocks due its higher negative skewness. It also widens the range of high (but also low) returns showing more extreme minimum and maximum returns. The excess riskiness of the strategy, in terms of all risk measures (Standard Deviation, Downside Volatility, VaR and ES) causes the performance ratios to be low. The increase in return performance is probably not high enough to justify the higher risks taken.

7 Conclusion

We have seen that this model, although advanced in its implications, manages to incorporate the views in a new close-form solution. The model requires, as in BL, the construction of the selection matrix and specification of views. These preliminary passages lead to the introduction of a new correlation matrix which is likely to be a concept more familiar than other advanced ones.

The ABL finds its strength in its ready applicability. Still it manages to meet its purposes: to incorporate non-normality and dependence in confidence of views. From another perspective, its strength is also its weakness because sophisticated investors and market professionals will likely decide to take a further step and choose a more advanced model.

For the average investor, instead, this model can constitute a valid solution. After the comparison of our Advanced Black-Litterman model with the original BL, we noticed better performance of the ABL in most cases, with the BL never being undoubtedly better (we noticed one *ex equo* result). When compared to BL, the ABL led to enhanced performance in the case of a low-risk and high-risk investor. An investor placing himself in the middle between the two, would choose between the two models based on the type of views he expresses.

As we have seen, the ABL is a more aggressive model and the middle type of investor would choose to use it only for mild views. For more extreme views over future outcomes, the investor should rely instead on a BL model, so to limit the double impact of an aggressive model with aggressive views.

When compared to the market, we have noticed great results for the ABL constructed for low risk profiles and good performance for the middle range of risk propensity. This can be seen through the performance ratios evaluated and other key statistics. ABL for minimum volatility presents good measures of risk and return and the ABL middle risk results are very similar to the market. Still in terms of performance ratios and key statistics, it is clear that ABL is probably not suited for aggressive, highly risk seeking investors.

From this we can draw the conclusion of what it implies to incorporate non-normality (through negative skewness and excess kurtosis) and dependence across view confidence in the portfolio construction process. This model has provided a possible solution, although, with its applicability and theoretical limits. It is a challenge for future research to find a model which, incorporating the same information with the same simplicity, produces a result which is more efficient, compared to normal BL, 100% of the times.

8 Acknowledgements

At the end of this stage of my life, I'd like to thank all the ones who took part in this university experience. In Campi, in Rome, in Melbourne, in Lausanne. Family, housemates, professors and friends.

Special thanks to Prof. Massi Benedetti for his patience and professionality in accompanying me towards the conclusion of this path.

I'd like to express particular thought:

to Chiara, who taught me that we can be so different yet so alike at the same time;

to Fernando, in the hope that I can carry on with some of the things you didn't have a chance to achieve.

9 Code

```
close all
clear
clc

risk_aversion=2.5;
NumPortf=30; %meaning that optimization returns this number of portfolios
% which are then graphed

monthly.price= xlsread('Data.xlsx',1);
monthly.cap= xlsread('Data.xlsx',2);
monthly.returns=diff(log(monthly.price));
backtest=monthly.returns(end-59:end,:);
monthly.returns=monthly.returns(1:end-60,:);
N= size(monthly.returns,2);
options = optimset('Display','off');


for i=1:N

    w_init(i)=monthly.cap(end,i)/sum(monthly.cap(end,:));

end

tau= inv(size(monthly.returns,1));  %confidence in market priors

assets_names={'US Equity',
'EU Equity',
'EM Equity',
'US Bonds',
'EU Bonds',
'Hedge Funds' };

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% marginal distribution + copula
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:N

[asset(i).arma,~,asset(i).residuals]= ...
    armaxfilter(monthly.returns(:,i),1,1,1,[],[],options);
```

```
[asset(i).garch,~,asset(i).variance]= ....
    tarch(asset(i).residuals,1,0,1,[],[],[],options);
asset(i).uncvariance= ...
    asset(i).garch(1)/(1-asset(i).garch(2)-asset(i).garch(3));
asset(i).std=sqrt(asset(i).variance);
asset(i).z=asset(i).residuals./asset(i).std;
asset(i).mle= ...
    mle(asset(i).z,'pdf',@(z,a,b)SkTDens(asset(i).z,a,b),'start',[-0.1,5]);
lambda=asset(i).mle(1)+zeros(size(asset(i).z,1),1);
eta=asset(i).mle(2)+zeros(size(asset(i).z,1),1);
asset(i).u=SkTCDF(asset(i).z,lambda,eta);

end

u=asset(1).u;
for z=2:i

    u=[u asset(z).u];

end

% [rhohat] = copulafit('Gaussian',u);
[rhohat,nuhat] = copulafit('t',u);

sigma_hist=cov(monthly.returns);

% estimate historical covariance using EWMA

window= 50; % first estimation takes 50 observations
ewma_phi=0.94; %like RiskMetrics
sigma_ewma= cov(monthly.returns(1:50,:));

for i=51:size(monthly.returns,1)
sigma_ewma= ewma_phi * sigma_ewma + ...
    (1-ewma_phi)*(monthly.returns(i,:)'*monthly.returns(i,:));
end



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% historical covariance and mean returns
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
NumShown=NumPortf-5;
CovRets_Hist=sigma_hist;
ExpValRets_Hist=mean(monthly.returns)';

[E,V,Portfolios_MV]= ...
    EfficientFrontier(NumPortf, CovRets_Hist, ExpValRets_Hist);

figure;
area(Portfolios_MV,'FaceColor','flat')
ylim([0 1])
xlim([1 NumShown])
ylabel('Weights')
xlabel('Risk Propensity')
legend(assets_names,'Location','northwest','NumColumns',3);
% PlotFrontier(Portfolios)
title('Historical Frontier','fontweight','bold')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% market prior
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

CovRets_Prior=sigma_ewma;
ExpValRets_Prior=risk_aversion*CovRets_Prior*w_init';

[E,V,Portfolios_Prior]=...
    EfficientFrontier(NumPortf, CovRets_Prior, ExpValRets_Prior);

figure;
area(Portfolios_Prior,'FaceColor','flat')
ylim([0 1])
xlim([1 NumShown])
ylabel('Weights')
xlabel('Risk Propensity')
legend(assets_names,'Location','northwest','NumColumns',3);
% PlotFrontier(Portfolios)
title('Prior Frontier','fontweight','bold')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% views on the market
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Scenario 1
```

```
n_views= 3; % the number of view the investor has, sum of

P=zeros(n_views,N);
% the order of the assets is:
% 1. US Equity 2. EU Equity 3. Emerging Markets Equity
% 4. US Bonds  5. EU Bonds  6. Hedge Funds

P(1,1)=1;%here the selection matrix has to be constructed manually
P(1,2)=-1;
P(2,4)=1;
P(2,5)=-1;
P(3,1)=-1/3;
P(3,2)=-1/3;
P(3,3)=-1/3;
P(3,4)=0;
P(3,5)=-0;
P(3,6)=1;

v=[0.04  0.02  0.03]';

Omega= zeros(n_views); % uncertainty of views was measured

Omega(1,1)=0.05^2;
Omega(2,2)=0.05^2;
Omega(3,3)=0.05^2;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% Alternative Scenario
% n_views= 3; % the number of view the investor has, sum of
%
% P=zeros(n_views,N);
%% the order of the assets is:
%% 1. US Equity 2. EU Equity 3. Emerging Markets Equity
%% 4. US Bonds  5. EU Bonds  6. Hedge Funds
%
% P(1,1)=1;%here the selection matrix has to be constructed manually
% P(1,2)=-1;
% P(2,4)=1;
% P(2,5)=-1;
% P(3,1)=-1/3;
% P(3,2)=-1/3;
```

```
% P(3,3)=-1/3;
% P(3,6)=1;
%
% v=[0.1541  0.1989 0.06977]'; % correct view over 1 year
% v= -v; % wrong views
%
% Omega= zeros(n_views); % uncertainty of views was measured
%
% Omega(1,1)=0.05^2;
% Omega(2,2)=0.05^2;
% Omega(3,3)=0.05^2;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Black-Litterman
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

Mu_BL= inv(inv(tau*CovRets_Prior)+P'*inv(Omega)*P)* ...
    (inv(tau*CovRets_Prior)*ExpValRets_Prior+P'*inv(Omega)*v);
Sigma_BL=inv(inv(tau*CovRets_Prior)+P'*inv(Omega)*P);

% compute MV efficient frontier

[E,V,Portfolios_BL]=EfficientFrontier(NumPortf, Sigma_BL, Mu_BL);
figure;
area(Portfolios_BL,'FaceColor','flat')
ylim([0 1])
xlim([1 NumShown])
ylabel('Weights')
xlabel('Risk Propensity')
legend(assets_names,'Location','northwest','NumColumns',3);
% PlotFrontier(Portfolios)
title('BL Frontier','fontweight','bold')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Advanced Black-Litterman
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

D = zeros(N);
for i=1:N

    D(i,i)= CovRets_Prior(i,i);
```

```
end

OmegaABL=  P *D^(1/2) * rhohat * D^(1/2)* P';

Mu_ABL= inv(inv(tau*CovRets_Prior)+P'*inv(OmegaABL)*P)*...
    (inv(tau*CovRets_Prior)*ExpValRets_Prior+P'*inv(OmegaABL)*v);
Sigma_ABL=inv(inv(tau*CovRets_Prior)+P'*inv(OmegaABL)*P);

% compute MV efficient  frontier

[E,V,Portfolios_ABL]=EfficientFrontier(NumPortf, Sigma_ABL, Mu_ABL);
figure;
area(Portfolios_ABL,'FaceColor','flat')
ylim([0  1])
xlim([1  NumShown])
ylabel('Weights')
xlabel('Risk  Propensity')
legend(assets_names,'Location','northwest','NumColumns',3);
title('Advanced  BL  Frontier','fontweight','bold')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Analysis
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% market  performance

returns_avg = mean(backtest);

returns_market= cumprod(1+backtest);
returns_geomean=geomean(returns_market);
total_return= returns_market(end,:);

volatility_market= std(returns_market);

figure;
plot(returns_market)
ylabel('Return')
xlabel('Time')
legend(assets_names,'Location','northwest','NumColumns',2);

neutral_return= w_init*returns_market';
figure;
```

```
plot(neutral_return)
ylabel('Return')
xlabel('Time')
legend('Neutral Weights Return','Location','northwest','NumColumns',2);


% three different risk propensities ( portfolio 1, 12 and 25)

weights.minvol=[ Portfolios_MV(1,:); Portfolios_Prior(1,:); ...
    Portfolios_BL(1,:); Portfolios_ABL(1,:)   ];
weights.maxret=[ Portfolios_MV(25,:); Portfolios_Prior(25,:); ...
    Portfolios_BL(25,:); Portfolios_ABL(25,:)   ];
weights.middleport=[ Portfolios_MV(12,:); Portfolios_Prior(12,:); ...
    Portfolios_BL(12,:); Portfolios_ABL(12,:)   ];

% compute volatilities
for i=1:NumPortf

volatility_MV(i) = Portfolios_MV(i,:)*CovRets_Prior*Portfolios_MV(i,:)';

end

for i=1:NumPortf

volatility_Prior(i) = Portfolios_Prior(i,:)*CovRets_Prior...
    *Portfolios_Prior(i,:)';

end

for i=1:NumPortf

volatility_BL(i) = Portfolios_BL(i,:)*CovRets_Prior*Portfolios_BL(i,:)';

end

for i=1:NumPortf

volatility_ABL(i) = Portfolios_ABL(i,:)*CovRets_Prior*Portfolios_ABL(i,:)';

end

variances= [volatility_MV(1) volatility_MV(12) volatility_MV(25);
```

```
    volatility_Prior(1) volatility_Prior(12) volatility_Prior(25);
    volatility_BL(1) volatility_BL(12) volatility_BL(25);
    volatility_ABL(1) volatility_ABL(12) volatility_ABL(25)];


standard_devs= sqrt(variances);

for i=1:NumPortf

return_MV(i) = Portfolios_MV(i,:)*returns_avg';

end

for i=1:NumPortf

return_Prior(i) = Portfolios_Prior(i,:)*returns_avg';

end

for i=1:NumPortf

return_BL(i) = Portfolios_BL(i,:)*returns_avg';

end

for i=1:NumPortf

return_ABL(i) = Portfolios_ABL(i,:)*returns_avg';

end

returns= [return_MV(1) return_MV(12) return_MV(25);
    return_Prior(1) return_Prior(12) return_Prior(25);
    return_BL(1) return_BL(12) return_BL(25);
    return_ABL(1) return_ABL(12) return_ABL(25)];

sharpe=(returns*12)./(standard_devs*sqrt(12)); % annualized

performance.minvol= [ ones(4,1), weights.minvol * returns_market'];
performance.middleport= [ ones(4,1), weights.middleport * returns_market'];
performance.maxret= [ ones(4,1), weights.maxret * returns_market'];
```

```
% Compare ABL to Market

% order is : 1. Market  2. MinVol ABL 3. MiddlePort ABL 4. MaxRet ABL

market.series=w_init*backtest';
minvolabl.series=weights.minvol(4,:)* backtest';
middleportabl.series=weights.middleport(4,:)* backtest';
maxretabl.series=weights.maxret(4,:)* backtest';

comparison= [(w_init*backtest')' (weights.minvol(4,:)* backtest')' ...
   (weights.middleport(4,:)* backtest')' (weights.maxret(4,:)* backtest')']




j=1;
for i=1:length(comparison)

    if market.series(i) <0
       negative.market(j)= market.series(i)
    j=j+1;

    else continue
    end
end

j=1;
for i=1:length(comparison)

    if minvolabl.series(i) <0
       negative.minvolabl(j)= minvolabl.series(i)
    j=j+1;

    else continue
    end
end

j=1;
for i=1:length(comparison)
```

```
    if middleportabl.series(i) <0
       negative.middleportabl(j)= middleportabl.series(i)
    j=j+1;

    else continue
    end
end

j=1;
for i=1:length(comparison)

    if maxretabl.series(i) <0
       negative.maxretabl(j)= maxretabl.series(i)
    j=j+1;

    else continue
    end
end


downside_vol = [std(negative.market) std(negative.minvolabl) ...
    std(negative.middleportabl) std(negative.maxretabl) ];

var_95= [ quantile(market.series,.05) quantile(minvolabl.series,.05) ...
    quantile(middleportabl.series,.05) quantile(maxretabl.series,.05)];

sorted.market= sort(market.series);
sorted.minvolabl= sort(minvolabl.series) ;
sorted.middleportabl= sort(middleportabl.series) ;
sorted.maxretabl= sort(maxretabl.series) ;

es_95 = [mean(sorted.market(1:3)) mean(sorted.minvolabl(1:3)) ...
    mean(sorted.middleportabl(1:3)) mean(sorted.maxretabl(1:3))];

sortino= (12*mean(comparison))./(sqrt(12)*downside_vol);



figure;
plot(performance.minvol')
ylabel('Return')
xlabel('Time')
```

```
legend('MV','Prior','BL','ABL','Location','northwest','NumColumns',2);

figure;
plot(performance.minvol(4,:)')
ylabel('Return')
xlabel('Time')
legend('Prior','Location','northwest','NumColumns',2);
title('Minimum Volatility ABL','fontweight','bold')

figure;
plot(performance.middleport(4,:)')
ylabel('Return')
xlabel('Time')
legend('Prior','Location','northwest','NumColumns',2);
title('Middle Portfolio ABL','fontweight','bold')

figure;
plot(performance.maxret(4,:)')
ylabel('Return')
xlabel('Time')
legend('Prior','Location','northwest','NumColumns',2);
title('Maximum Return ABL','fontweight','bold')


% tables
rows= {'Mean-Variance';'Prior';'BL';'ABL'};
cols= {'Min Vol', 'Middle Port' , 'Max Ret'};
input.data= sharpe;
input.tablePositioning = 'h!';
input.tableColLabels = cols;
input.tableRowLabels = rows;
input.dataFormat = {'%.4f'};
input.tableColumnAlignment = 'c';
input.tableCaption = 'Sharpe Ratios Portfolios';
input.tableLabel = 'MLE Correlation';
latex1= latexTable(input);
```

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