LUISS

Department
of Economics and Finance

Chair of "*IO and Competition Theory*"

Position Auctions in the Online Ad Market: Auction Theory and the Society

ID number 219961

Prof. Gian Luigi Albano | *Lorenzo Santoro*
Supervisor | Candidate

Academic year 2019-2020

Table of contents

1. Introduction

The objective of this thesis is to describe the Position Auction mechanism, focusing on its application in the online ad market. This auction is the framework above which the market for sponsored search is built. All the main search engines have as core business the advertising activity. Their stream of revenues consists mainly in the sale of advertisement slots in their web pages. These ad slots are assigned to advertisers according to the output of a particular auction, called Position Auction. The implementation of this mechanism has carried huge benefits, firstly to search engines that use it. Among all, Google is the most prominent example. These positive effects reach not only web's giants, but also all the users that utilize the search engine.

In this sense, in chapter 2 we start to describe the effect of the Position Auction on the society. The huge amount of money that circulate in the advertising industry, the majority of which are collected by Google, makes us aware of the prominent role played by this auction mechanism. The importance that search engines have gained in our everyday life and the efficiency provided by Google Ads make Position Auctions an interesting topic to study.

In order to understand fully how this model works, first we must introduce the fundamental features of Auction Theory. In chapter 3 we provide a general characterization of this branch of Economic Theory. We describe how the application of auction mechanisms in the online ad market has evolved during the years, until the theorization of the Position Auction. Before moving to the mathematical formalization of this auction mechanism, we analyze some theories that can be considered ancestors of our model.

In chapter 4 we formalize the Position Auction framework, following the theorization provided by Google's Chief Economist Hal Varian. We start talking about the design of this kind of auction, defining the main variables. Then, we provide two definitions of Nash equilibrium and we argument the fact that equilibrium prices can be defined explicitly via a simple recursive formula. Other characteristics, such as bounds, revenues and values, are analyzed in detail. At the end, the model is refined to include some characteristics employed by Google Ads.

Varian's work is not the only one dealing with this auction model. The literature in this field is very rich. In chapter 5 we provide a depiction of the most relevant works. Here we analyze the different ways in which economists have extended the analysis on Position Auctions. Some of these contributed to the auction design, while others focused the attention on other issues. In particular we introduce the concepts of reserve prices and click-weights.

We complete our study of the Position Auction, and our discussion of its effects on the society, in chapter 6. Here we analyze how the implementation of this auction mechanism has affected the other side of the market: the consumer's side. Including consumers in our framework, we can provide a description of how total wealth is affected by the implementation of our auction model and how surplus is distributed among the players. In this way we can conclude our work on Position Auctions, giving a complete depiction of their implementation in the online ad market.

2. Auction Theory and the society: the source of Google's revenues

Google is the leader in the search engine industry. In October 2015, the American multinational conglomerate Alphabet Inc. was created and became the parent company of Google and several former Google subsidiaries. Alphabet's market capitalization in September 2019 was 825 billion U.S. dollars[1]. According to Fortune Global 500[2], it is the world's fourth-largest company by revenues in the technological sector and it lays in the top 40 most valuable firms in the world. In October 2019, Google's market share among the leading search engines was almost 88%.

Beside these facts that characterize how valuable is this company, we want to focus our analysis on a more interesting aspect: the source of such a value. Our objective is to investigate the foundation of this wealth. Analyzing the Google's business model, we discover that the core business of Google is advertisement. Through the platform Google Ads, everyone can advertise its website, taking advantage of the Google's network. Ads will be shown above search results in the browser's web page, in other Google platforms, such as Google Maps, and on the Google Display Network, that is a collection of websites that display ads provided by Google. The sponsored search channel is the most important one and it is the topic of our work.

In 2019, 134.8 of the company's 160.7 billion U.S. dollars revenues came from advertising. For this reason, in order to analyze properly its market share with respect to its direct competitors, we must examine the advertising industry. The spending on advertising by companies worldwide is increasing steadily and reached 543.71 billion U.S. dollars in 2018. In that year, TV advertisement amounted for 182 billion U.S. dollars and it is expected to decrease in the following years. Advertising on print media shares the same destiny and, in 2018, expenditure in such a sector amounted for 47 billion U.S. dollars. The driving force in this industry is the digital advertising market. It is growing at an impressive rate and alone is expected to surpass 517 billion U.S. dollars in 2023[3]. Google is not the only internet company that operates in the digital advertising sector. Other important players in this sense are social media companies, such as Facebook and Twitter. Focusing on search engines,

[1] Data regarding Google are taken from (Clement, Google - Statistics & Facts, 2020).
[2] Every year Fortune ranks companies according to their total revenues.
[3] Data regarding the advertising market are taken from (Guttmann, 2019).

all the biggest ones, such as Yahoo![4] and Bing[5], follow the same business model of Google and, also in their case, the biggest part of their revenues come from advertisement.

From this market analysis we have understood the huge quantity of wealth that the advertising industry moves, and we have seen that the biggest part comes from the digital advertisement sector. In particular, Google is the driving force of this market. It has almost the 90% of the market share in the search engine industry[6] and it accounts for biggest part of the digital advertisement market revenues. Chart 1 summarizes these data and gives us a complete overview of the topic. Data regarding 2019 were estimated at the time all the data were collected.

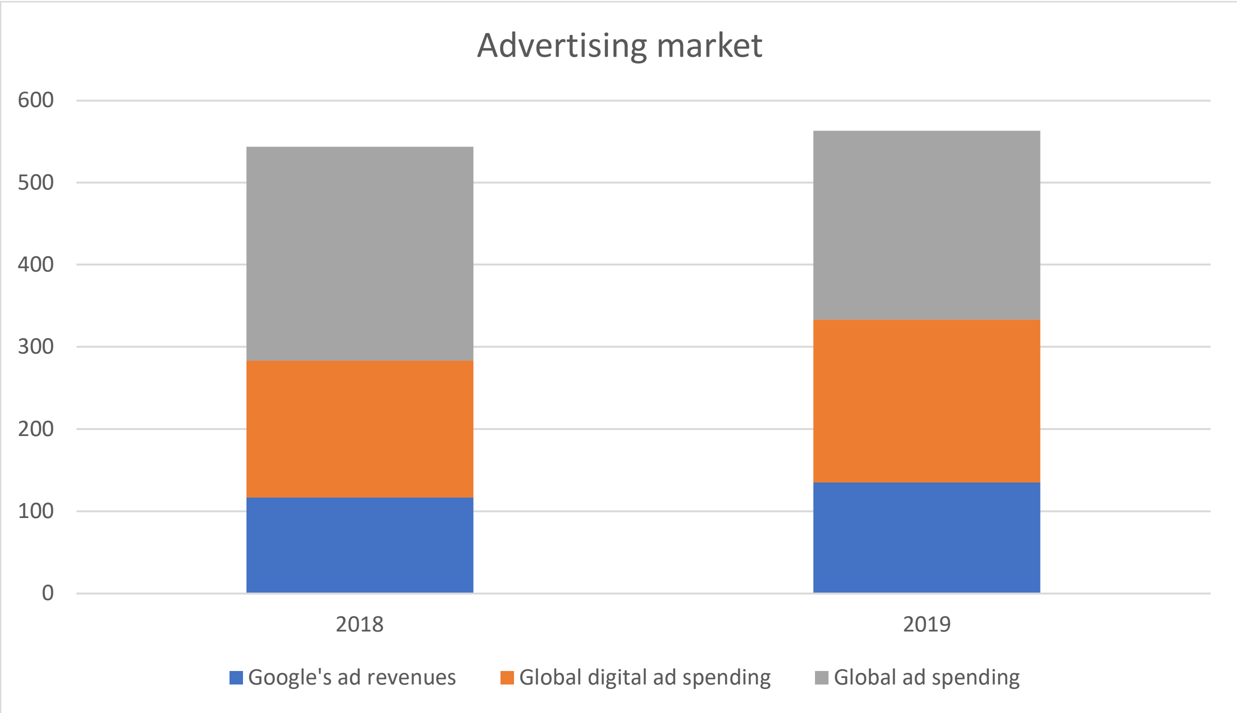

Chart 1

The market analysis explains why Google is so relevant for our study and let us conscious of the importance that an auction mechanism, that we will describe deeply in the next chapters, has in the economy. The advertising activity performed by search engines is, indeed, based on an auction framework known as "Position Auction". This can be considered the source of Google's revenues and wealth.

The importance of this auction mechanism leads us to an equally important conclusion: the active character of the Economic Theory. Generally, we tend to think about economics as a description of our society, an ex-post theorization of the complex dynamics of human interactions.

[4] Data regarding Yahoo! are taken from (Yahoo Statista Dossier, 2017).
[5] Data regarding Bing are taken from (Bing Statista Dossier, 2019).
[6] Data regarding search engines industry are taken from (Clement, Global market share of search engines 2010-2020, 2020).

The case of the online ad market is a notable example of how Economic Theory, and in particular Game Theory, can be implemented to reach a higher level of efficiency and wealth when we play and interact in our society. The importance of an active approach to economics can be seen in practice in the constant growing number of economists hired in private corporations. In the recent years, tech companies started to employ teams of PhD economists leaded by a Chief Economist. Amazon, Facebook, but also Airbnb and Uber are some notable example in this sense. However, the sector in which we have the biggest gain from these new employments is the search engine industry. The fundamental role that an auction mechanism plays in this kind of companies makes the activity of an economic consultant really important. Michael Otrovsky and Michael Schwartz helped Yahoo! to increase profit by millions of dollars when they noticed that reserve prices were lower than theory would suggest; the ideas of Susan Athey helped Microsoft to improve the quality of Bing's advertisements[7]; Hal Varian developed in Google the Position Auction's model over which the online ad market is based. These eminent economists have contributed in a fundamental way in the development of Position Auctions and their work will be analyzed deeply in the following chapters.

Hal Varian has been one of the first Chief Economist in a private company. The Google's big sphere of influence and its huge presence in the online world have created an economic system that rotates around its gravity center. This has led to the creation of a word that describes such an environment: Googleconomics. Varian has been the Google's Chief Economist since May 2002. When he arrived in the company, an auction mechanism was already implemented to sell part of the ad's slots, but it is with his contribution that we have the creation of the model founded on Position Auctions.

The success of this framework has convinced Google to implement auction in several different situations. It begun to use auctions in internal operations, such as the allocation of servers among its various units. The most ambitious move was to set a variation of a Dutch auction, a first-price oral auction that we will define in the next chapter, for the company's IPO in 2004. Google believed that in this way the playing field between small investors and powerful brokerage firms could be leveled. Moreover, they wanted to minimize the typical underpricing that affects the biggest part of the IPOs, earning in this way a fair price on their stocks. Despite the skepticism of the financial world and the high level of uncertainty related to such an innovation in an activity traditionally committed to a powerful institution, such as an investment bank, the IPO did not fail at all. Google went public at 85$ a share, price included in the company's expectations[8]. However, the high level of uncertainty

[7] An article written by Athey and Luca analyzes the growing trend among tech companies to hire many economists (Athey & Luca, Why Tech Companies Hire So Many Economists , 2019).

[8] The innovative way in which Google has conducted its IPO has been described in (Dutch Auction, s.d.).

related to IPOs and auction results, the low familiarity that financial players have with this innovative process and the certainty provided by investment banks have not let every corporation be confident to implement this mechanism in their IPOs[9]. On the other hand, auction models are widely used in the sales of Treasury Bills and Bonds.

The successful implementations of auctions in the online ad market and in the financial market are just some examples of how Game Theory can be used to design efficient interactions between players in the society. The basic principles of economics have different applications in several fields of the society. The examples that we have provided underline the active role that Economic Theory has in our world.

3. Auction Theory behind the online ad market

In this section, we define the basis upon which we will construct our auction mechanism. First of all, the fundamental characteristics of the Auction Theory must be outlined. Once the building blocks of this economic theory are characterized, we can provide a depiction of how its application in the online ad market has evolved. Finally, we must make a last step before we can start to describe our model. We will provide an analysis of the theories that can be considered as ancestors of our Position Auction.

3.1. An introduction to Auction Theory

To explain properly the model above which the online ad market is built, we must introduce the main features of the Auction Theory, which, in turn, is a branch of Game Theory.

Generally, an auction is a process of buying or selling goods or services. An auctioneer collects bids from bidders and assigns the object to the highest offeree, or to the lower one in case of a buying auction. Broadly speaking, two main kind of auctions there exist: oral auctions and sealed bid auctions. In the first case, bidders meet in a physical or digital place and submit their bids. Oral auctions can follow an ascending or descending mechanism. In the former, price rises until only one bidder remains and pays his final bid. In the descending one, the price is lowered until someone stops it; he will pay that amount of money. On the other hand, sealed-bid auctions consist in private bids submitted to the auctioneer by bidders. The highest bid wins. However, the amount paid depends on the auction mechanism that has been chosen by the auctioneer. Auction design has been implemented to develop several different mechanisms that, however, can be grouped in two families: first-price

[9] Levy in his article (Levy, 2014) describes the difficulties that an IPO conducted via a Dutch Auction could present to a company.

auctions and second-price auctions. In the former the bidder submitting the highest bid wins and pays his own bid. In the latter the amount paid is the second highest bid. In the standard case in which bidders compete for just one object, this last kind of auction is also known as “Vickrey Auctions”, because the output respects the one defined in the Vickrey-Clarke-Groves Mechanism. This paper will focus on sealed-bid auctions and a special kind of second-price auction will be analyzed.

Before starting to formalize mathematically this theory, two other fundamental issues arise: whenever there is complete or incomplete information and whenever bids are submitted simultaneously or sequentially. This study will focus only on simultaneous move games, for this reason we do not describe auctions with bids submitted sequentially. Moreover, complete information will be considered a reasonable assumption in this model.

3.2. A model for the online ad market

The main features and the specific characteristics of the market for Internet advertising must be analyzed to describe properly the mechanism above which Google and other search engines have built their sponsored search auctions. This mechanism is called “Position Auction” and will be studied deeply in the next section. Here, beside a description of the online ad environment, an historical overview of the development of the mechanism used by search engines is given. This description will follow the one provided by Edelman, Ostrovsky and Schwarz (Edelman, Ostrovsky , & Schwarz, 2007).

In 1994 the business for online advertisement stared to develop. Internet advertisements were largely sold following a per-impression principle. Search engines used to charge a flat fee to advertisers in order to show their ads a fixed number of times. This process was slow, contracts were negotiated on a case-by-case basis and each contract involved a large amount of money finalized to the purchase of a certain number of impressions. In 1997 there was a turning point. Overture, that now is part of Yahoo!, introduced a radically new model based on an auction mechanism. Each advertiser had to submit a bid that correspond to his willingness to pay on a per-click basis for a particular keyword. Two are the important innovative characteristics of this new mechanism: the targeting aspect and the per-click principle. Targeting some specific keywords of interest, advertisers could focus only on potential consumers interested in their product and, moreover, they could quantify how much each keyword was worth to them. Also the payment became much more flexible and smarter. Advertising was no longer sold on a basis of thousand showings. Every time a sponsored link was clicked by a consumer, the advertiser had to pay an amount equal to the advertiser’s last bid. The highest bid was shown in the highest position, following a descending order of bids. These characteristics quickly led to the success of the Overture’s auction mechanism. However, soon, some imperfections arose. The fact that bids could be changed very frequently made the auction unstable. If advertisers respond each other using a best response function that maximizes their utility, given the other players' bid, there is no pure strategy equilibrium in the one-shot version of the game. This model can be called "Generalized First-Price Auction".

In February 2002, Google introduced its own pay-per-click mechanism, AdWords Select. Google's auction was designed following a second-price principle. Considering a number of slot equal to N, where 1 is the top position and N is the last slot, and a number of bidders equal to A, an advertiser in position i had to pay a price-per-click equal to the bid of the advertiser in position $i+1$ plus a minimum increment ε, typically equal to \$0.01. The imposition of a price-per-click equal to the next highest advertiser's bid has caused the coinage of the appellative "Generalized Second-Price Auction". More generally, we can define the price payed by each agent as the minimum amount necessary to retain his position. This auction mechanism has some similarities to the VCG mechanism: both set each agent's payments only on the allocation and bids of the other players, their own bid does not directly affect their own payment. However, the VCG mechanism would have charged on each bidder a cost equal to the externality that he imposes on others by taking one slot away from them: the total payment of the advertiser in i is equal to the difference between the total value of clicks that all other advertisers would have received if that player was not present in the market, and the total value of clicks that the other advertisers receive if he is present. This Nobel Prize winning theory respects two important characteristics: there is an equilibrium in dominant strategies and this equilibrium is achieved bidding according to the player's willingness to pay, giving a socially optimal solution. On the other hand, GSP is not characterized by truth-telling and by a Nesh Equilibrium in dominant strategies.

This Generalized Second-Price Auction is commonly known as "Position Auction". In the next section we will go through the mathematical formalization of the game, providing a detailed analysis of the structure of this kind of auctions.

3.3. A step backward in the previous theory

In order to understand fully the theory behind Position Auctions, we must take a step backward. As we will see at the beginning of the next chapter, when we will start to formalize our model, this kind of auction mechanism can be considered as an assignment problem. For this reason, before going into the details of this auction theory, we must introduce the main features that characterize an assignment game.

This kind of game was firstly theorized by Shapley[10] and Shubik (Shapley & Shubik, 1971) and constitutes one of the building blocks of Game Theory. These two economists built a model for a two-sided market in which a product that comes in large, indivisible units, such as houses or cars, is exchanged for money, and in which participant either supplies or demands exactly one unit. The assignment problem asks for the assignment of objects to agents that maximizes value. This problem can be solved by linear programming or by other specialized algorithms.

We can see that several similarities arise between this general assignment game and our auction model. They share the same scope: S objects, or slots, must be assigned to A agents.

An optimal assignment is characterized by a set of numbers p_s such that each agent a would weakly prefer the object assigned to him over any other object. Defining u_{as} the utility that agent a gains from object s, mathematically we have

$u_{as}-p_s \geq u_{at}-p_t$ for all s and t

As we will see, the position auction game is simply a competitive equilibrium of an assignment game that has a special structure for the utility.

At the same time, a similarity from the point of view of the auction design can be detected with the multi-good auction studied by Demange, Gale[11] and Sotomayor (Demange, Gale, & Sotomayor, 1986), that can be seen as an ancestor of our auction mechanism. They developed a generalization of the single item second-price auction first described by Vickery. However, a part the multi-item characteristic, the framework of this model is quite different from our Position Auction.

4. Position Auction

The paper "Position Auctions" published by Hal Varian (Varian, 2007) will be taken as guide in the following discussion of the auction model employed by search engines in the online advertisement market. In his work, Varian has analyzed the equilibria of the game based on the ad auction employed by Google and has provided some empirical evidence that the Nash equilibria of the Position Auction describe the basic properties of the prices observed in Google's ad auction.

[10] Nobel-Prize winning economist, Shapley is considered one of the most important contribution to the development of Game Theory.
[11] Demange and Gale published the previous year (1985) another paper regarding two-sided matching markets.

4.1. Position Auction design

This auction mechanism can be modelized as an assignment problem. Agents $a = 1,\ldots, A$ must be assigned to slots $s = 1,\ldots, S$ and agent a's valuation for slot s is given by $u_{as} = v_a x_s$. Slots are numbered so that $x_1 > x_2 > \ldots > x_S$ and $x_s = 0$ for all $s > S$. The number of agents is assumed to be greater than the number of slots.

In our environment, agents are advertisers and the slots are positions on a web page. Higher positions receive more clicks, so x_s can be interpreted as the click-through-rate for the slot s. The value $v_a > 0$ can be interpreted as the expected profit per click. So, $u_{as} = v_a x_s$ indicates the expected profit to advertiser a from appearing in slot s.

This problem is resolved via an auction. Each advertiser bids an amount b_a. the slot in the highest position is assigned to the advertiser with the highest bid, the second-best slot to the advertiser with the second highest bid, and so on. The price that agent in position s faces is the bid of the agent immediately below him, so $p_s = b_{s+1}$. Hence, the net profit that agent a makes, if he acquires slot s, is $(v_a - p_s)x_s = (v_a - b_{s+1})x_s$.

Formally, we model the Position Auction as a simultaneous move game with complete information. Each agent a simultaneously chooses a bid b_a.. We build an auction model assuming that the environment in which it is implemented is characterized by the complete information property. The legitimacy of this assumption could be questioned. However, situations in which complete information can be considered as a credible approximation are very easy to experiment in real-world ad auctions. Google reports click and impression data on an hour-by-hour basis. In a short period of time, an agent could estimate the click-through-rates of each position. Moreover, Google offers a "Traffic Estimator" that provides an estimation of the number of clicks per day and the cost per day associated with the advertiser's choice of keywords. Finally, there exists also third-party companies known as "Search Engine Managers (SEMs)" that offer different services related to bids management. The presence of several tools and services, along with the ease of experimentation, suggest that the full-information assumption is a reasonable first approximation.

The characteristics of our auction are summarized in Table 1. It depicts the positions, click-through-rates, values, bids, prices, profits and net profits associated with a position auction consisting of $S = 4$ available slots and $A = 5$ agents with their respective valuations. We know that $x_s > x_{s+1}$ by assumption and that $b_s > b_{s+1}$ by the rules of the auction.

Position	*CTR*	*Value*	*Bid*	*Price*	*Profits*	*Net Profits*
1	x_1	v_1	b_1	$p_1=b_2$	v_1x_1	$(v_1-b_2)x_1$
2	x_2	v_2	b_2	$p_2=b_3$	v_2x_2	$(v_2-b_3)x_2$
3	x_3	v_3	b_3	$p_3=b_4$	v_3x_3	$(v_3-b_4)x_3$
4	x_4	v_4	b_4	$p_4=b_5$	v_4x_4	$(v_4-b_5)x_4$
5	x_5	v_5	b_5	*0*	*0*	*0*

Table 1

4.2. Nash equilibrium of Position Auction

In equilibrium, each agent should prefer his current slot to any other slot. Taking in consideration Table 1, we see that if agent 3 wanted to move up by one position, it would have to bid at least b_2, the bid of agent 2. But if agent 2 wanted to move down by one position it would only have bid at least b_4, the bid of agent 4. To move to a higher slot you have to beat the bid that the agent who currently occupies that slot is making; to move to a lower slot you only have to beat the price that the agent who currently occupies that slot is paying. We can know define a Nash equilibrium set of prices for the Position Auction.

<u>Definition 1</u>: A Nash equilibrium set of prices (NE) satisfies

$(v_s-p_s)x_s \geq (v_s-p_t)x_t$ for $t > s$

$(v_s-p_s)x_s \geq (v_s-p_{t-1})x_t$ for $t < s$

where $p_t = b_{t+1}$. •

In general, there is a set of bids and prices that satisfy these inequalities. Agents can vary their bids over a range without either changing their positions or affecting their payments. These inequalities are linear in the prices. Hence, given (v_s) and (x_s) we can use a simple linear program to solve for the maximum and minimum equilibrium revenue attainable by the auction. Furthermore, we can simplify our analysis of the Position Auction by examining a particular subset of Nash equilibria.

<u>Definition 2</u>: A symmetric Nash equilibrium set of prices (SNE) satisfies

$(v_s-p_s)x_s \geq (v_s-p_t)x_t$ for all t and s.

Equivalently,

$v_s(x_s-x_t) \geq p_sx_s-p_tx_t$ for all t and s. •

Identifying $u_{as} = v_a x_s$ and $p_s = b_{s+1} x_s$, we have

u_{as}-$p_s \geq u_{at}$-p_t for all s and t

This demonstrates the correspondence between Position Auctions and an assignment game, characteristic that we have discussed in the previous chapter. As we have already seen, this symmetric Nash equilibrium can be considered a competitive equilibrium of an assignment game.

In general, these prices can only be calculated using a linear program or related algorithm. However, a series of short arguments can be used to show that, in this special case, a simple recursive formula can be employed to compute explicitly the prices. We are going to develop our reasoning, proving a series of arguments, in order to be able to provide an explicit characterization of equilibrium prices and bids.

Argument 1: Non-negative surplus.

In a SNE, $v_s \geq p_s$. □

Proof: Using the inequalities that define a SNE, $(v_s-p_s)x_s \geq (v_{S+1}-p_{S+1})x_{S+1} = 0$ since $x_{S+1} = 0$. Solving algebraically the inequality and reminding the non-negativity of x_s, we get that $v_s \geq p_s$.

Argument 2: Monotone values.

In a SNE, $v_{s-1} \geq v_s$, for all s. □

Proof: By definition of SNE, we have $v_s(x_s-x_t) \geq p_s x_s - p_t x_t$ and $v_t(x_t-x_s) \geq p_t x_t - p_s x_s$. Adding these two inequalities gives us $(v_t-v_s)(x_t-x_s) \geq 0$. This shows that v_s and x_s must be ordered in the same way. Since x_s is monotone by construction, v_s is a monotone sequence of values.

This argument shows that an SNE is an efficient allocation, since agents with higher values are assigned to better slots.

Argument 3: Monotone prices.

In a SNE, $p_{s-1}x_{s-1} > p_s x_s$ and $p_{s-1} > p_s$ for all s. If $v_s > p_s$, then $p_{s-1} > p_s$. □

Proof: By definition of SNE we have $(v_s-p_s)x_s \geq (v_s-p_{s-1})x_{s-1}$ which can be rearranged in $p_{s-1}x_{s-1} \geq p_s x_s + v_s(x_{s-1}-x_s)$. The right-hand side of the inequality is strictly bigger than $p_s x_s$. Hence $p_{s-1}x_{s-1} > p_s x_s$. This proves the first part of the argument. To prove the second part, we recall the proved characteristic of nonnegative surplus that states $v_s \geq p_s$. Then we write $p_{s-1}x_{s-1} \geq p_s x_s + v_s(x_{s-1}-x_s) \geq p_s x_s + p_s(x_{s-1}-x_s) = p_s x_{s-1}$. Canceling x_{s-1}, we see that $p_{s-1} \geq p_s$. If $v_s > p_s$, the second inequality is strict, which proves the last part of the argument.

Argument 4: NE ⊃ SNE.

If a set of prices is a SNE it is a NE. □

Proof: Since $p_{t-1} \geq p_t$, $(v_s-p_s)x_s \geq (v_s-p_t)x_t \geq (v_s-p_{t-1})x_t$ for all s and t.

The reason why the set of symmetric Nash equilibria is interesting is that it is only necessary to verify the inequalities for one step up or down to verify that the entire set of inequalities is satisfied.

Argument 5: One step solution.

If a set of bids satisfies the symmetric Nash equilibria for $s+1$ and $s-1$, then it satisfies these inequalities for all s. □

Proof: Suppose that the SNE relations hold for slots 1 and 2 and for slots 2 and 3; we need to show it holds for 1 and 3. Writing out the condition and using the fact that $v_1 \geq v_2$, we have

$$v_1(x_1-x_2) \geq p_1x_1-p_2x_2 \longrightarrow v_1(x_1-x_2) \geq p_1x_1-p_2x_2$$

$$v_2(x_2-x_3) \geq p_2x_2-p_3x_3 \longrightarrow v_1(x_2-x_3) \geq p_2x_2-p_3x_3$$

Adding up the left and right columns, we get $v_1(x_1-x_3) \geq p_1x_1-p_3x_3$. We have proved our argument. This process can be generalized for any slot.

These arguments allow us to provide an explicit characterization of equilibrium prices and bids. Since the agent in position s does not want to move down one slot:

$$(v_s-p_s)x_s \geq (v_s-p_{s+1})x_{s+1}$$

Rearranging this inequality, we get

$$v_s(x_s-x_{s+1})+p_{s+1}x_{s+1} \geq p_sx_s.$$

Since the agent in position $s+1$ does not want to move up one slot:

$$(v_{s+1}-p_{s+1})x_{s+1} \geq (v_{s+1}-p_s)x_s$$

Rearranging this inequality, we get

$$p_sx_s \geq v_{s+1}(x_s-x_{s+1})+p_{s+1}x_{s+1}.$$

Putting them together we see:

$$v_s(x_s-x_{s+1})+p_{s+1}x_{s+1} \geq p_sx_s \geq v_{s+1}(x_s-x_{s+1})+p_{s+1}x_{s+1}$$

Recalling that $p_s = b_{s+1}$, we can also write these inequalities as:

$v_{s-1}(x_{s-1}-x_s)+b_{s+1}x_s \geq b_s x_{s-1} \geq v_s(x_{s-1}-x_s)+b_{s+1}x_s$

Dividing everything for x_{s-1} and defining $\alpha_s = x_s / x_{s-1} < 1$, we have yet another way to write the inequalities:

$v_{s-1}(1-\alpha_s)+b_{s+1}\alpha_s \geq b_s \geq v_s(1-\alpha_s)+b_{s+1}\alpha_s$

The pure strategy Nash equilibria can be found simply by recursively choosing a sequence of bids that satisfy these inequalities.

4.3. An analysis of bounds, revenues and values

The Nash equilibrium condition shows that, in equilibrium, each agent's bid is bounded above and below by a convex combination of the bid of the agent below him and a value that is, in one case, his own value and, in the other case, the value of the agent immediately above him. We can examine the boundary cases, defining an upper bound and a lower bound. The two solutions are

$b_s^U x_{s-1} = v_{s-1}(x_{s-1}-x_s)+b_{s+1}x_s$

$b_s^L x_{s-1} = v_s(x_{s-1}-x_s)+b_{s+1}x_s$

The starting values for the recursions follow from the fact that there are only S positions, so that $x_s = 0$ for $s > S$. Writing out the upper and the lower bounds on the bid for $s = S+1$, we have

$b_{S+1}^U x_S = v_S(x_S-x_{S+1}) = v_S x_S$

$b_{S+1}^L x_S = v_{S+1}(x_S-x_{S+1}) = v_{S+1} x_S$

It is optimal for the first excluded bidder to bid his value. This result has the same argument as in the usual Vickrey auction. If you are excluded, then bidding lower than your value is pointless, but if you do happen to be shown (e.g., because one of the higher bidders drops out) you will make a profit. From this starting point we can obtain the solution of the recursions, that are

$b_s^U x_{s-1} = \sum_{t\geq s} v_{t-1}(x_{t-1}-x_t)$

$b_s^L x_{s-1} = \sum_{t\geq s} v_{t-1}(x_{t-1}-x_t)$

Any bid in the range described by the equations studied above is an SNE and, hence, a NE bid. However, there could be reasons why bidding at one end of the upper or lower bounds might be particularly attractive to the bidder. Indeed the lower bound recursion can be interpreted as the highest bid I can set so that if I happen to exceed the bid of the agent above me and I move up by one slot, I am sure to make at least as much profit as I make now. In this case I just beat the advertiser above me by a tiny amount and end up paying my bid minus a tiny amount. On the other hand, the upper bound recursion can be interpreted as the highest bid that I can set if I want to squeeze the profit of the player ahead of me, without inducing the agent above me to move down.

Summing the solutions of the recursions of the upper and lower bounds over $s = 1, \dots S$, we can define the upper and lower bounds on total revenue in a SNE. For example, if the number of sots is $S = 3$, the lower and upper bounds are

$$R^L = v_2(x_1\text{-}x_2)+2v_3(x_2\text{-}x_3)+3v_4x_3$$

$$R^U = v_1(x_1\text{-}x_2)+2v_2(x_2\text{-}x_3)+3v_3x_3.$$

Compering total revenue of SNE and NE, it turns out that the upper bound for the SNE revenue is the same as the maximum revenue for the NE, while the lower bound on revenue from the NE is generally less than the revenue bound for the SNE.

The claim that the two upper bounds coincide can be proved defining p_s^N as the prices associated with the maximum revenue Nash equilibrium and p_s^U as the prices that solve the upper recursion for the SNE. Since NE ⊃ SNE, the revenue associated with p_s^N must be at least as large as the revenue associated with p_s^U. From the definition of a NE, we have

$$p_s^N x_s \le p_{s+1}^N x_{s+1}+v_s(x_s\text{-}x_{s+1}).$$

From the definition of the upper bound recursion, we have

$$p_s^U x_s = p_{s+1}^U x_{s+1}+v_s(x_s\text{-}x_{s+1}).$$

The recursion starts at $s = S$. Since $x_{S+1} = 0$, we have

$$p_S^N \le v_S = p_S^U.$$

Hence $p_S^U \ge p_S^N$. From the initial condition it follows $p_S^U = p_S^N$.

It is easy to construct examples where the minimum revenue NE has less revenue than the solution to the lower recursion for the SNE; this is not surprising since the set of inequalities defining the NE strictly contains the set of inequalities defining the SNE.

Moving our attention on the values v_s, it is possible to derive useful bounds on these unobserved values of the agents by using the observed equilibrium prices. Starting from the initial definition of SNE, we have

$$v_s(x_s\text{-}x_t) \ge p_s x_s\text{-}p_t x_t$$

We want to define explicitly v_s, hence we divide by x_s-x_t. However, we must distinguish two cases: when $x_s > x_t$, implying $t > s$, and when $x_s < x_t$, implying $t < s$. In the first case x_s-x_t is positive and we have

$v_s \geq (p_s x_s - p_t x_t)/(x_s - x_t)$

In the other case x_s-x_t is negative and we have

$v_s \leq (p_s x_s - p_t x_t)/(x_s - x_t)$

Combining these two conditions together and remembering that in the first case the maximum value of the right hand side of the inequality is reached when t correspond to the position just below s, while in the second case the minimum value of the right hand side of the inequality is reached when t correspond to the position just above s, we have

$(p_s x_s - p_{s-1} x_{s-1})/(x_s - x_{s-1}) \geq v_s \geq (p_s x_s - p_{s+1} x_{s+1})/(x_s - x_{s+1})$

Multiplying by -1 numerator and denominator of the inequality on the left, we have

$(p_{s-1} x_{s-1} - p_s x_s)/(x_{s-1} - x_s) \geq v_s \geq (p_s x_s - p_{s+1} x_{s+1})/(x_s - x_{s+1})$

These inequalities can be interpreted as the incremental cost of moving up or down one position. We can recursively apply these inequalities to write

$v_1 \geq (p_1 x_1 - p_2 x_2)/(x_1 - x_2) \geq$

$v_s \geq (p_2 x_2 - p_3 x_3)/(x_2 - x_3) \geq$

…

$v_S \geq p_S$

This shows that the incremental costs must decrease as we move to lower positions. This observation has three important implications:

1. The inequalities give a necessity and sufficient condition for the existence of a pure strategy Nash equilibrium: that each of the interval must be non-empty.
2. The inequalities give some simple bidding rule for the advertisers. If their values exceed the marginal cost of moving up a position, then they have an incentive to bid higher until it is no longer true.
3. Finally, the inequalities motivate the fact that the marginal cost of a click must increase as you move to higher positions. I must be true because, if it ever decreased, there would be an advertiser who passed up cheap clicks in order to purchase expensive ones.

4.4. Position Auction in Google Ads

Until this point of the description of Position Auctions we have discussed their generic structure and features. As we have already said, the role of Google in the spread of this auction mechanism has been vital. Google play a central role in the online ad industry and, for this reason, it is interesting to investigate how Position Auctions are implemented by this search engine. Some refinements must be added to the model defined in this chapter.

Google ranks the ads by the product of a measurement of ad quality and advertiser bid, rather than just the bid alone. Advertisers are ordered according to $e_s b_s$, where e_s is the quality of the ad. This quality measurement also affects the click-through-rate that now is defined as $z_s = e_s x_s$. Each advertiser pays the minimum amount that is necessary to retain his position. Letting p_s be the amount that advertiser *a* would need to pay to be in position *s*, by construction we have

$p_s e_s = b_{s+1} e_{s+1}$.

Solving for p_s, we have

$p_s = b_{s+1} e_{s+1} / e_s$.

Nash equilibrium requires that each agent prefers his position to any other position. Recognizing that the cost and click-through-rate of the position depend on the agent's ad quality, we can define the symmetric Nash equilibrium as

$(v_s - p_s) e_s x_s \geq (v_s - p_t) e_s x_t$ for all *s* and *t*.

Substituting $p_s = b_{s+1} e_{s+1} / e_s$ into this expression and simplifying, we have

$(e_s v_s - b_{s+1} e_{s+1}) x_s \geq (e_s v_s - b_{t+1} e_{t+1}) x_t$.

Defining $q_s = b_{s+1} e_{s+1}$ and $q_t = b_{t+1} e_{t+1}$ gives us

$(e_s v_s - q_s) x_s \geq (e_s v_s - q_t) x_t$.

This is the definition of the symmetric Nash equilibrium of a Position Auction that takes into account the characteristics employed by Google.

The boundaries on the values $e_s v_s$ can by defined with the same logic used in the definition of the bounds on values developed in the paragraph 4.3. Hence, we have

$e_1v_1 \geq (q_1x_1-q_2x_2)/(x_1-x_2) \geq$

$e_2v_2 \geq (q_2x_2-q_3x_3)/(x_2-x_3) \geq$

…

$e_Sv_S \geq p_S$.

Finally, we also must mention the case of "non-fully sold pages" which are auctions where advertisers are less than the available slots, that is $A < S$. In this case, the advertisers assigned to the bottom position will be charged to pay a reserve price equal to r.

5. Literature regarding Position Auctions

Given the auction model upon which the online ad market, and in particular Google Ads, is based, we can now go through the analysis[12] of other studies that have extended the theory behind Position Auctions in several different ways. We will focus on works that have contributed to the design of the auction mechanism, we will introduce the concept of reserve prices and we will face the click-weights issue.

5.1. Auction design

The literature regarding Position Auctions has grown very rapidly at the end of the first decade of the 2000. The reason why this new auction mechanism attracted a large quantity of economists, that started to develop a solid theoretical base, is the field in which it is applied: the market for online advertisement. This is a highly remunerative sector, in which the biggest search engines compete among each other to provide the best service to a huge number of consumers.

The pivotal event in the development of these online ad auctions can be considered the employment of Hal Varian as Google's Chief Economist. His paper, published in 2007, is a landmark in auction theory applied to the online market. However, he is not alone. In the second half of the first decade of 2000, a large amount of papers, written by several eminent economists, contributed to build the literature behind this auction mechanism. Some of them have been essential in the formalization of the Generalized Second-Price Auction and laid the foundations for its design.

[12] In our analysis we will describe several works developed by several eminent economists. In order to provide a fluid narration of this theory, we will modify some notations, making them coherent with the ones used in the previous chapters.

Analyzing studies strictly related to the design aspect of auctions used to assign advertising slots on web-pages, we can mention the studies conducted by Edelman, Ostrovsky and Schwarz (Edelman, Ostrovsky , & Schwarz, 2007) and by Aggarwal, Goel, and Motwani (Aggerwal, Goel, & Motwani, 2006), beside the one repeatedly mentioned of Hal Varian (Varian, 2007).

Another work of Varian (Varian, 2009) deals with the same topic. Here the economist returns on the application of the general model of Position Auction to Google's advertisement market and analyzes a possible application of the VCG mechanism to online ad auctions, connecting this study to the one mentioned in chapter 3 conducted by Demange, Gale and Sotomayor. A further examination of the online advertisement environment is given by the analysis of bidding behaviors. Varian supposes a reasonable stable relationship between an advertiser's bid and the number of clicks that his advertisement receives during a given time period, that we express as d_a. This can be summarized as

$$b_a = B_a(d_a)$$

In the same way, a cost function can be defined as the cost that advertiser a must pay to receive d_a clicks during a given time period. This function can be written as

$$c_a = C_a(d_a)$$

Both depend on the interaction with other advertisers in the ad auction. In this way, the advertiser can maximize its surplus function $v_a d_a - C_a(d_a)$, finding his optimal number of clicks. Now, his optimal bidding behavior can be found. These relationships can be used to construct a bound on the ratio of aggregate value to aggregate cost, referred to as "surplus ratio", that is

$$\frac{value}{cost} \geq \sum_{a=1}^{A} \frac{c_a - \hat{c}_a}{d_a - \hat{d}_a} \frac{d_a}{\sum_b c_b}$$

where $\sum_b c_b$ is total cost and $\hat{d}_a$ identify a smaller number of clicks that could be achieved with another bid and for which advertiser would pay a smaller cost $\hat{c}_a$.

Another work, strictly related to the design of an auction used to price advertising slot on a web page, is the one conducted by Aggarwal, Goel and Motwani (Aggerwal, Goel, & Motwani, 2006). Their study focuses on the definition of an auction mechanism in which bidding their true valuation is a dominant strategy for advertisers. As we have already said, since truth-telling is not a characteristic of Position Auctions, optimum bids depend in a complicated dynamic fashion on externalities, such as the bids of the other agents, and it is often necessary for advertisers to hire expensive consultants or intermediaries to determine these bids. A truthful mechanism would simplify the bidding process significantly, since it would require an agent to only determine his valuation for the keyword, a quantity that is intrinsic to the agent.

The model that they built is a general model that can be applied to every different kind of online ad auction. As we have observed in chapter 2, an Overture model and a Google model can be identified. They defined a ranking function $R=(w_1, w_2, \ldots, w_A)$ where w_a is a weight assigned a priori to agent a independently of his bid. Agent are ranked accordingly to $w_a b_a$ where, as usual, b_a is agent a's bid. In the Overture model w_a is equal to one, while in the Google model w_a is the click-through-rate attached to agent a and, more precisely, the quality measure that we have defined with e_a. The price-per-click in their truthful auction is than defined as

$$p_s = \sum_{j=s}^{S} \left(\frac{CTR_j - CTR_{j+1}}{CTR_s} \right) \frac{w_{j+1}}{w_j} \, b_{j+1}$$

or, equivalently,

$$CTR_s p_s = \sum_{j=s}^{S} \left(CTR_j - CTR_{j+1} \right) \frac{w_{j+1}}{w_j} \, b_{j+1}$$

where CTR_j is the click-through-rate in the position s. It can be detected a correspondence between this CTR_j and x_s in our Position Auction's model. Since step one requires computation of the price that the agent would pay at position $s+1$, this formula is recursive, or laddered. For this reason, this auction has been called "Laddered Auction".

The auction mechanism implemented by search engines to sell advertisement slots was theorized, beside Varian, by Edelman, Ostrovsky and Schwarz (Edelman, Ostrovsky , & Schwarz, 2007). They designed the main features of the Generalized Second-Price Auction and coined this name. As we said in chapter 2, this mechanism has some characteristics in common with the well-known VCG mechanism, but it has also two fundamental differences: truth-telling is not an equilibrium strategy and there is not a Nash equilibrium in dominant strategies. VCG would reduce incentives for strategizing and would make life easier for advertisers. The reasons why search engines do not implement this kind of auction are several. First, VCG is hard to explain to typical advertising buyers. Second, switching to VCG may entail substantial transition cost. Indeed, it can be demonstrated that, if all advertisers were to bid the same amounts under the two mechanisms, VCG revenues are lower than GSP revenues and, moreover, advertisers may be slow to stop shading their bids. Third, the revenues consequences of switching to VCG are uncertain.

They imposed some assumptions. First, they assumed that all values are common knowledge. Over time, advertisers are likely to learn all relevant information about each other's values. Second, since bids can be changed at any time, stable bids must be best responses to each other. Hence, bids form an equilibrium in a simultaneous-move, one shot game of complete information. Given this assumptions, they defined the vectors of bids that identify a Nash equilibrium in such a game as those bids that correspond to a situation in which each advertiser *a* do not want to exchange his position *s* with the advertiser in the higher position *s-1*. They called such an equilibrium a "locally envy-free" equilibrium.

Definition: An equilibrium of the simultaneous-move game induced by GSP is locally envy-free if a player cannot improve his payoff by exchanging bids with the player ranked one position above him. More formally, in a locally envy-free equilibrium, for any position $s=1 \ldots S$

$$x_s v_a - p_s \geq x_{s-1} v_s - p_{s-1} \quad \bullet$$

As in the previous section, x_s identifies the click-through-rate in position s, v_s identifies the value per click to the advertiser that occupies position s, and p_s is the price that he must pay. In particular, by construction, $p_s = b_{s+1}x_s$. This "locally envy-free" equilibrium correspond to the lower bound on bids in the symmetric Nash equilibrium found in the previous chapter.

Another interesting analysis that Edelman, Ostrovsky and Schwarz conducted in their work is the investigation of how advertisers converged in such a steady state characterized by perfect information. They built another auction mechanism, called "Generalized English Auction", characterized by imperfect information. This game can be considered to characterize the online ad market until players have acquired a complete knowledge about other players' values. This is an analogue of the standard English auction that follows, however, the framework of the GSP. In the Generalized English Auction, there is a clock showing the current price, which continuously increases over time. An advertiser can drop out at any time, and his bid is the price on the clock at the time when he drops out. The ad of the only advertiser remaining, after the next-to-last advertiser drops out, will be placed in the higher position and his payment per-click is equal to the price at which the next-to-last advertiser dropped out, and so on for the other players. Click-through-rates x_s are commonly known and advertisers' per-click valuations v_s are drawn from a continuous distribution with a continuous density function. Each advertiser knows his valuation and the distribution of other advertisers' valuations.

Defining $h= (b_{s+1}, \ldots, b_{S+1})$ as the history of prices at which previous advertisers have dropped out, we can express the strategy of an advertiser a as a function $p_a (s, h, v_a)$ where p_a is the price at which he drops out. The perfect Bayesian equilibrium of the Generalized English Auction is defined by

$p_a(s, h, v_a) = v_a - \frac{x_s}{x_{s-1}}(v_a - b_{s+1})$

This is a unique perfect Bayesian equilibrium with strategies continuous in advertisers' valuations and payoffs equal to VCG payoffs.

Given this works that are strictly related to the construction of an auction mechanism that is, or could be, employed in the online ad market, we can now describe those works that have extended the analysis in various and different directions, focusing their attention on particular aspects or assumptions of the Position Auction's mechanism.

5.2. Reserve prices

Edelman and Schwarz (Edelman & Schwarz, 2010) were the first to analyze optimal reserve prices. Following the work that they made with Ostrovsky, they started with a model that captures the dynamic aspects of the mechanism used in practice. They described a dynamic game of incomplete information and, then, provided a one-shot game of complete information as a reasonable approximation, identifying the impossibility to solve analytically such a dynamic game. The innovative aspect of their description is the presence of a reserve price. The per click payment of the advertiser in the bottom position equals the reserve price. Returning on the incomplete information issue, they defined how the Generalized English Auction, already described in their previous work, can be considered an optimal mechanism in its one-shot variation, taking into consideration the presence of a reserve price *r*. They proved the fact that, as the reserve price increases, the total payment of every advertiser increases. The reserve price directly affects the lower bidder and indirectly affects the others. This indirect effect is due to the fact that reserve prices may impact equilibrium behaviors of all players. Hence, under realistic assumption, an optimal reserve price can yield a notable increase in search engine revenues.

5.3. Click-weights

The value of past performance information in the contest of keyword advertising auctions has been investigated by Liu and Chan (Liu & Chen , 2006). Advertisers differ both in valuation-per-click and in the numbers of clicks they can generate: their performance. This characteristic can be expressed in the auction mechanism as a weight attached to each advertiser's bid. For simplicity, they divided bidders into two groups: L-type and H-type. The former group has a lower expected click-through-rate than the latter. The auctioneer assigns a weight to L-type bidders and another weight to H-type ones. Without loss of generality the H-type weight is assumed to be *1* while the other is indicated with $\gamma < 1$. The score of an advertiser is given by the product of his bid and his weight. The problem for the auctioneer is to choose the optimal γ. They provided a characterization of this optimal value according to a revenue-maximizing and an efficiency-maximizing objective. Liu, Chan and Whinston (Liu, Chen , & Whinston , 2010) conducted another study on the same click-weighting issue.

The quantity of works devoted to the study of this auction mechanism is not limited to the ones that we have presented in this section. We have selected some of them in order to provide a complete overview on every aspect of the topic, but without focusing too much on each characteristic of this environment.

6. Auction Theory and the society: the consumer's side of the market

We have already discussed the active role that Economic Theory has in our society when we have talked about the source of Google's revenues. Now we want to analyze the same topic, but from the other side of the market: the consumer's side.

All the works and studies mentioned until now deal exclusively with the side of the market in which advertisers interact with search engines in order to have their websites sponsored. The discussion has not taken into consideration the other side of the market: the consumer's side. With consumers, we want to identify everybody that, using a search engine, is looking for something in the World Wide Web. This need, identified by a keyword, can be satisfied by a website which, in turn, can advertise its capacity to meet consumers' needs with an ad on the web browser's page corresponding to the search of that particular keyword.

Most of the literature devoted to the study of the online ad market is mainly auction-focused and does not take into consideration the fact that the objects being auctioned are advertisements. The value of a link is due to consumers' clicks on those links and, in turn, consumer behavior is affected by the process by which links are assigned to slots.

Athey and Ellison (Athey & Ellison, 2011)[13] developed an interesting study on sponsored search auctions, focusing on the consumer's side of the market and enlarging the analysis to include these aspects in the framework. By incorporating consumers into the model, they have been able to answer questions about how the design of the advertising auction marketplace affects overall welfare, as well as the division of surplus between consumers, search engines, and advertisers. This new framework provides new insights about reserve price policies, click-through weighting, fostering

[13] Also in this case, we follow the analysis provided by Athey and Ellison in their paper "Position Auctions with consumer search" making some minor modifications to their formalization and notation, in order to develop a clear narration, coherent with our previous work.

product diversity, advertisers' incentives to write accurate ad text, and effects of different bidding mechanisms. Including consumers in the model has an impact also on the auction design. Any changes to the rules by which ads are selected affect consumers' behavior and their inferences about advertisement quality, which in turn affects the value that a position has for the advertiser.

As we have already mentioned, this new analysis starts from the fact that a number of consumers have a need. To satisfy it, they visit a search site. The search engine displays *S* sponsored links. Consumer j can click on any of these sustaining a cost equal to c_j. If their need is met, they receive a benefit of *1*. If it is not the case, they visit the next website until their need is met or until the expected benefit from an additional click falls below c_j. The environment is not assumed to satisfy the complete information property. Costs are distributed continuously over the interval [0; 1] according to a defined function *G*.

A advertisers wish to advertise on a web page with *S* slots. Firm *a* has probability q_a of meeting each consumer's need and this characteristic can be seen as the quality of the ad. Advertisers get a payoff of *1* every time they meet a need. All firms draw their q_a independently from a continuous distribution *F*, defined over [0; 1] and in common to all the players.

The base model incorporates some simplifications. Advertisers are symmetric except for their probability q of meeting a need. In order to focus on the consumer aspect, the pricing problem for the advertiser is not considered. We assume that advertisers receive no benefit when consumers see their ad but do not click on it. Moreover, we can define the expected quality of the advertisement provided by agent a that occupies position s. Considering a Bernoulli random variable z_s that is equal to *1* if agent a meets the consumer need, the expected quality is expressed as

$$\bar{q}_a = E(q_a | z_1 = \cdots = z_{s-1} = 0).$$

This correspond to the expected utility for the consumer that clicks on the links until advertisement a. At the same time, we can define a demand function that varies as the expected quality $\bar{q}_a$ of the ad varies: $G\,(\bar{q}_a)$.

The auction mechanism that we consider follows the one provided by Edelman, Ostrovsky and Schwarz (Edelman, Ostrovsky , & Schwarz, 2007) when they dealt with a situation in which there is incomplete information among players, the Generalized English Auction. The logic is the same: an agent must decide whenever to drop out and the boundary case that we consider is the one in which the agent is indifferent between maintaining his position or passing to the higher one. Differences arises from the facts that we take into consideration consumers actions and that ads are characterized only by their quality q. Hence, in the characterization of the set of bids that satisfy such an equilibrium, click-through-rates are given, in addition to the position, by the demand function of the consumers $G\ (\bar{q}_a)$. Given the fundamental similarity to the one already described, we will not go through the mathematical formalization of such an equilibrium, in order to focus our attention on those innovative aspects that are important for the purpose of our work.

Search engines auctioning sponsored links are information intermediaries. They contribute to the social welfare making consumer search more efficient. This claim can be demonstrated comparing the consumer welfare with sorted and unsorted lists. Let us firstly consider the situation in which links are sorted according to the output of a Position Auction. The agents' bids are strictly monotone in q, as can be seen formalizing the equilibrium condition, and, hence, advertisers with the highest bids are on top. This creates an incentive for consumers to follow a top-down strategy: they start at the top and continue clicking until their need is satisfied or until the expected quality of the next website is below the search cost $\bar{q}_a < c_j$. Hence, in this case, the expected consumer surplus can be expressed as

$$E[CS(c)] \ = \begin{cases} \quad 0 & if\ c \in [\bar{q}_1; 1] \\ \ \bar{q}_1 - 1 & if\ c \in [\bar{q}_2; \bar{q}_1] \\ \qquad\qquad\qquad\qquad \cdots & \\ (\bar{q}_1 - c) + (\bar{q}_2 - c)(1 - \bar{q}_1) + \cdots + (\bar{q}_a - c)\prod_{k=1}^{s-1}(1 - \bar{q}_k) & if\ c \in [\bar{q}_{a+1}; \bar{q}_a] \end{cases}$$

On the other hand, if advertisements are sorted randomly, we cannot compute the expected quality $\bar{q}$ in the usual way: we are not able, in this case, to identify the expected quality of advertiser *a*, but we can only compute the expected value of the qualities of the ads

$$\bar{q} = E[q]$$

In this case the expected consumer surplus will be

$$E[CS(c)] = \begin{cases} 0 & if\ c \in [\bar{q}; 1] \\ (\bar{q} - c)\dfrac{1 - (1 - \bar{q})^S}{\bar{q}} & if\ c \in [0; \bar{q}] \end{cases}$$

Consumers with $c > \bar{q}$ do not click on any ads, while consumers with $c < \bar{q}$ click on ads until their need is met or they run out of ads.

Compering the two expressions, we can find out that there is a welfare gain in the implementation of a Position Auction to sort advertisements. Consumers with $c \in [\bar{q}; \bar{q}_1]$ get no utility at all from an unsorted list, but positive utility from a sorted list because the higher quality of the top links makes clicking worthwhile. Consumers with lower search cost have a gain as well because sorted list enable them to find what they want more quickly. As the number *A* of advertisers increases, this gain becomes more evident.

Given that there is a welfare gain in the implementation of a Position Auction in the sponsored search market, now we must analyze more deeply how welfare is affected and how its components vary. We can define a gross consumer surplus (*GCS*) and a gross producer surplus (*GPS*). The former is equal to *GSC = Consumer Surplus + Search Costs*, the latter is *GPS = Advertisers Profit + Search Engine Profit*. Because *GCS* and *GPS* both increase by *1* if a consumer need is met and by *0* otherwise, we have that *E(GCS) = E(GPS)*. Moreover, we can define the total welfare as *W = GCS + GPS – Search Costs.* Given these definitions, we can investigate how changes in the auction design affect the welfare distribution among our players.

We focus our attention on reserve prices, topic that we have already mentioned. In the standard auction model, reserve prices increase the auctioneer's expected revenue, but they have a negative effect on social welfare, reducing it. However, we have not investigated in detail the consumer's side of the market. In this model we must make some different considerations that lead us to different results. Reserve prices can increase both profits of the auctioneer and social welfare. This occurs because consumers sustain search costs when they are looking for a website that can satisfy their need. With a reserve price, auctioneer commits not to list advertisements of low-quality reducing welfare loss for consumers and increasing the number of search that they are willing to carry out. Moreover, we can say that, supposing a uniform distribution of search costs and reminding the definitions that we have provided earlier, consumer surplus and social welfare are maximized for the same reserve price. This result is true for any bidding behavior by advertisements and can be easily generalized for any search cost distribution $G(c) = c_j$.

We have understood that consumer surplus and producer surplus are proportional and, hence, are maximized for the same reserve price. However, producer surplus is the sum of two different factors: search engine revenues and advertiser surplus. A conflict arises between these two when we investigate under which reserve price each one of these components is maximized. Search engines may prefer a reserve price much higher than the social optimum and advertisers may prefer a reserve price much lower than the social optimum. This affirmation is generally true, but some further complications may arise when we consider in more details how our players may best respond to each other in each possible scenario. We will not go into details.

Reserve prices are fixed by search engines and, hence, it is straightforward to think that they will pursue their own profit maximization target. A situation in which total welfare is achieved and auctioneers have an incentive to implement as objective function the maximization of the society surplus is when there exists perfect competition among search engines. If a search engine does not maximize total surplus, consumers will be better off using another search engine. The same logic applies to an imperfectly competitive market but that is still characterized by a dynamic competition to attract consumers. If current market shares are sensitive to the consumer surplus that a search engine provides, higher current market shares lead to higher future market shares, and then maximize consumer surplus may be a good approximation to the optimal policy for a search engine.

This analysis is perfectly coherent with the basic characterization of the different kinds of market, in which total welfare and deadweight loss are, respectively, directly and inversely related with the degree of competitiveness that there is in the market. Moreover, it enriches our analysis with another important consideration about the topic that we are studying: how market structure may influence auctions design.

The main idea behind the imposition of reserve prices is that consumer surplus, and hence welfare, is always improved if more information about ads' qualities are available to consumers. This concept follows another fundamental economic principle at the basis of the characterization of markets. The higher the degree of information available to players, the higher the efficiency and, hence, the higher the welfare. In an idealized environment, search engines could report the degree of quality along with each ad. A perfectly informed market cannot exist in the real world, but it is an asymptotic limit we can strive for. In practice, a more sophisticated system of reserve prices can be used, such as a decreasing lower bound limit on prices attached to each position. Another solution could be the geographical distribution of advertisements in the web page, with some positions more valuable than others. These solutions implemented by search engines constantly evolve and vary, beside the evolution of the market and the sensitivity of consumers.

We now focus our attention on another aspect of the design of Position Auctions: click-weights. We have already dealt with this aspect of the auction design in our work, the possibility to attach a weight to advertisers' bids and order ads in a better way. We introduced this possibility for the first time when we described the particular Position Auction implemented by Google, in which advertisements are ordered according to the product between their bid b and a weight e that represent the quality of the ad. In the description that we are currently developing of sponsored search, the design of a click-weighted auction presents more complications. In the standard Position Auction model, weights generally correspond to the click-through rate relative to that advertisement. In our model the quality of the ad is the discriminant upon which advertisements are sorted. Introducing a variable that reflects the click-through rates of the ads can lead to contrasting results. Advertisements are no longer sorted in a decreasing order of their qualities. This solution favors websites that can satisfy a wide range of needs, but providing a lower quality, with respect to others that are specialized in that field. Efficiency is damaged as well, except in the limit case in which search costs c are approximately 0.

Strictly related to the fact that click-weights can distort the market efficiency, the concept of obfuscation can give us more insights in this problem. Imposing click-weights on advertisements gives advertisers an incentive to pursue an obfuscation activity. Assuming that links' texts affect consumers expectations, obfuscation consists in the creation of an ad's text that misleads consumers, encouraging them to click on that advertisement also if it will be proved to not be able to satisfy their needs. When click-weights are not present, advertisers have incentives to make ad texts accurate and informative.

In chapter 2 we have discussed the relationship between an auction mechanism and the business world. We have seen how Position Auctions are the foundation of one of the biggest company's richness. Now we have analyzed how Auction Theory may enhance consumers' welfare. Consumers and companies are just different players in the same game. Every human being interacts with his equals every day in the society. This interaction can take place through the company where he works or through his activity as consumer. Economic Theory, and in particular Game Theory, may let this interaction be more efficient.

7. Conclusion

In this work we have analyzed deeply Position Auctions and their implementation in the online ad market. One of the main conclusions in our study is the recognition of the active role played by Economic Theory in our society. Starting from the benefits that an auction mechanism can carry to search engines, we have concluded our discussion analyzing the increase in welfare provided to consumers. Auction Theory influences our society. It makes more efficient the interaction between players on the world's stage. The position auction mechanism is the foundation of Google and other search engines' profits and it has carried huge benefits in term of efficiency to advertisers and users.

Google is part of our everyday life and it has contributed to the evolution of our world. It would be hard to think about the Internet without thinking about Google. The great relevance that Auction Theory has for such an innovative company makes us aware of the importance that economics has in our world. Position Auction is the result of the research for the smartest way to assign advertisement slots in the web pages. The profit maximizing purpose has led to the creation of this auction mechanism. The importance of the Position Auction in companies like Google has motivated them to rely on economic consultants. On the other hand, the powerful practical application of this model has stimulated economists to create a dynamic and growing theoretical literature on the topic. We have analyzed this literature, providing a complete characterization of the main features of this theory.

We have also drawn conclusions about a causality relationship between the market structure and the auction design. If it is true that welfare in the society is enhanced by Auction Theory, it is also true that, in a market in which competition among search engines exists, it is the seek for a big share of consumers that leads competitors to provide the most efficient model for ads allocation. The auction design, in this case, drives surplus to the total welfare maximization level.

Search engines are information intermediaries: their business is founded on the ability to provide more information as possible and in the most effective way. In this sense, Auction Theory has provided a great instrument to Google and its competitors.

Bibliography

Aggerwal, G., Goel, A., & Motwani, R. (2006). Truthful Auctions for Pricing Search Keyword. *Proceedings of the 7th ACM conference on electronic commerce*.

Athey, S., & Ellison, G. (2011). Position Auctions with Consumer Search. *The Quarterly Journal of Economics*, Vol. 126, No. 3, pp. 1213-1270.

Athey, S., & Luca, M. (2019, February 12). *Why Tech Companies Hire So Many Economists* . Retrieved from Harvard Business Review: https://hbr.org/2019/02/why-tech-companies-hire-so-many-economists

Bing Statista Dossier. (2019). Retrieved from Statista : https://www.statista.com/study/47763/bing/

Chen, Y., & He , C. (2011). Paid Placemen: Advertising and Search on the Internet . *The Economic Journal*, Vol. 121, No. 556, pp. F309-F328.

Clement, J. (2020, March 25). *Global market share of search engines 2010-2020*. Retrieved from Statista : https://www.statista.com/statistics/216573/worldwide-market-share-of-search-engines/

Clement, J. (2020, February 5). *Google - Statistics & Facts*. Retrieved from Statista : https://www.statista.com/topics/1001/google/

Demange, G., & Gale, D. (1985). The Strategy Structure of Two-Sided Matching Markets. *Econometrica*, Vol. 53, No. 4, pp. 873-888.

Demange, G., Gale, D., & Sotomayor, M. (1986). Multi-Item Auctions. *Journal of Political Economy* , Vol. 94, No. 4, pp. 863-872.

Dutch Auction. (n.d.). Retrieved from Corporate Finance Institute: https://corporatefinanceinstitute.com/resources/knowledge/finance/dutch-auction/

Edelman, B., & Schwarz, M. (2010). Optimal Auction Design and Equilibrium Selection in Sponsored Search Auctions. *The American Economic Review* , Vol. 100, No. 2, pp. 597-602.

Edelman, B., Ostrovsky , M., & Schwarz, M. (2007). Internet Advertising and the Generalized Second -Price Auction: Selling Billions of Dollars Worth of Keywords. *The American Economic Review*, Vol. 97, No. 1, pp. 242-259.

Fortune Global 500. (2019). Retrieved from Fortune.

Guttmann, A. (2019, November 12). *Global advertising market - Statistics & Facts*. Retrieved from Statista : https://www.statista.com/topics/990/global-advertising-market/

Lahaie, S. (2006). An Analysis of Alternative Slot Auction Design for Sponsored Search . *Proceedings of the 7th ACM conference on Electronic commerce*, pp. 218-227.

Levy, A. (2014, August 19). *Google shares took off, but the auction didn't*. Retrieved from CNBC: https://www.cnbc.com/2014/08/19/es-took-off-but-the-auction-didnt.html

Liu, D., & Chen , J. (2006). Designing online auctions with past performance information. *Decision Support Systems*, Vol. 42, Issue 3, pp. 1307-1320.

Liu, D., Chen , J., & Whinston , A. (2010). Ex Ante Information and the Design of Keyword Auctions. *Information System Research* , Vol. 21, No.1, pp. 133-153.

Secret of Googleconomics: Data-Fueled Recipe Brews Profitability. (2009, May 22). Retrieved from Wired : https://www.wired.com/2009/05/nep-googlenomics/

Shapley, L. S., & Shubik, M. (1971). The Assignment Game I: The Core. *International Journal of Game Theory*, Vol. 1, pp. 111-130.

Varian, H. R. (2007). Position auctions. *International Journal of Industrial Organization*, Vol. 25, Issue 6, pp. 1163-117.

Varian, H. R. (2009). Online Ad Auctions. *The American Economic Review* , Vol. 99, No. 2, pp. 430-434.

Yahoo Statista Dossier. (2017). Retrieved from Statista: https://www.statista.com/study/15692/yahoo-statista-dossier/