LUISS

Dipartimento
di Impresa e Management

Cattedra di Asset Pricing

Dynamic Portfolio Optimization: a simulation and regression approach applied to a multi-asset portfolio choice problem

Prof. Paolo Porchia

RELATORE

Prof. Alfio Torrisi

CORRELATORE

Luigi Pretti - Matr. 711381

CANDIDATO

Anno Accademico 2019/2020

Contents

Chapter 1

Introduction

1.1 Motivations

There has always been a tension in economics between the endeavour to describe the optimal decisions of perfectly rational individuals ("positive economics") and the attempt to use different models to optimize people's imperfect decisions ("normative economics").

This thesis takes its origin in the work of Michael W. Brandt, Pedro Santa-Clara and Jonathan R. Stroud, that presented a simulation-based approach to solving dynamic portfolio allocation issues.

Brandt et al. introduced a straightforward method applied in discrete-time, "involving non-standard preferences, a large number of assets with arbitrary return distribution, and, most importantly, a large number of state variables with potentially path-dependent or non-stationary dynamics."[1]

The most important feature of their study is that this method could accommodate a large number of assets, with arbitrary return distribution determined by a large number of state variables with potential path-dependency

[1] Brandt, Michael W., et al. "A simulation approach to dynamic portfolio choice with an application to learning about return predictability." The Review of Financial Studies 18.3 (2005): 831-873.

and non-stationary dynamics.

In other words, their article introduced a simulation and regression method that is based on the predictability of the dynamic evolution of returns by the presence of one or more state variables.

The simulation-based method implemented in this thesis is a flexible, fast and dynamic application of the method of Brandt et al. about portfolio choice problems, applied to multi-asset investment opportunities that could accommodate both portfolio constraint and non-standard preferences.

1.2 Aims

The optimal portfolio choices depend on the characteristics of the environment: the availability of financial securities, their historical returns, their expected returns, their risks and the investor's preferences.

In this thesis, the methodology is applied to build an algorithm that could include a realistic investor's environment and solve portfolio choice problems for long-term horizons, where these details are particularly relevant.

Long-term investors are not only interested in the short-term expected return and risks, but also in how they may change over time.

This attention in the dynamic portfolio choice follows both the renewed interest in the study of non fully rational investor's preferences and the recent empirical evidence of return predictability.

Financial planners have traditionally emphasized the need for each investor to build a diversified portfolio that could reflect his unique personal preferences and situations.

To construct a balanced, flexible and fast dynamic method to determine the investor's optimal portfolio allocation strategy it is necessary to argue that the traditional academic analysis of portfolio problems should be modified to accommodate the long-term investment horizons in its peculiarities and details.

The main purpose of this thesis is to implement such a method and, as people face continuously financial decisions, it is interesting to ask whether an investor with a long-term horizon allocates his wealth differently form the optimal short-term allocation.

1.3 Outline

The rest of this document is organized into the following chapters:

Chapter 2 - Myopic portfolio Choice This chapter contains an introduction to the general set-up and frameworks in the Modern Portfolio Theory and how these concepts can be used to improve traditional academic analysis.

Chapter 3 - Utility Functions This chapter provides an introduction to the utility functions, which can capture the risk averseness of investors and thus enable ranking between possible portfolios.

Chapter 4 - Dynamic Portfolio Choice This chapter explains the structure and contents of the method implemented.

Chapter 5 - Methodology Here, the methodology is extensively discussed and have an in-depth study of the features and assumptions used in this thesis.

Chapter 6 - Implementation This chapter presents the method applied to a setting with multiple assets with simulated returns, describing the code structure implemented for the model.

Chapter 7 - Results This chapter presents all the test results and some preliminary conclusions.

Chapter 8 - Conclusions Conclusions and hints for future improvements and research.

Chapter 2

Myopic Portfolio Choice

This chapter is a review analysis that aims to introduce in detail some of the concepts and tools contained and used in the follow-up of the thesis. The definition of Excess returns, Performance and Risk are recalled, while the Mean-Variance optimization is discussed in detail. Then a single-period, static portfolio choice is introduced and extended with a risk-free investment.

2.1 Introduction

In the financial field, the problem of portfolio choice plays a major role, as "one of the most important decisions many people face is the choice of a portfolio of assets for retirement savings".[1]

Institutional investors face complex investment decisions, and some of them are similar to individuals in that they seek to finance a long term stream of discretionary spending.

The problem to be addressed concerns the determination of the optimal portfolio allocation.

This selection is configured as the decision making between different al-

[1] Campbell, John Y., and Luis M. Viceira. "Strategic asset allocation." Book Manuscript, Harvard University, November (2000).

ternatives to invest in the portfolio that minimizes risks and maximizes its value.

Modern financial theory about portfolio allocation probably began with the theoretical model developed in the article “Portfolio Selection”, published in 1952 by Harry Markowitz. In his paper, he outlined a framework for static optimal portfolio allocation based on the Mean-Variance analysis.

Despite the validity of the first model of portfolio selection, it is developed purely on the theoretical level and is based on some strict assumptions that diminish its value.

The main assumption of the static model is the invariance of the expected returns and volatilities, based on the idea that the characteristics of the assets and their composition should not change over time.

Therefore, the underlying investment strategy suggested by this method is the determination of the optimal asset allocation, based on short-term historical data, and then holding the portfolio for the entire investment horizon.

Despite the model does not consider the possibility of changes in the conditions and could be useful only for short-term, myopic, investors, it is considered the starting point of the Modern Portfolio Theory.

Empirically portfolio choices depend on a great number of factors such as investor preferences, availability of securities in the market and expected returns of assets and risk. All these factors become more relevant for investors with a long-term horizon.

Investors with long-term goals are not only interested in the expected return and risk, but above all in how returns vary over time. Optimal portfolio allocations for investors with a long-term horizon must have a different composition than those for short-term investors.

In 1994 Siegel, with his widespread work “Stocks for the long run”, “sealed the conventional wisdom that most of us should be in the stock market” [2], recommending investing more in shares than in fixed income over the long

[2] Siegel, Jeremy J. "Stocks for the Long Run McGraw-Hill." New York (1998).

term.

The construction of an efficient portfolio has been the subject of several research and theories about the ideal combination of investments that allows investors to minimize risk and maximize the overall return.

2.2 Defining a Financial Asset Return

Most financial studies about portfolio selection do not consider prices but returns, which represent the link between the final wealth and the initial investment.

A generic rational investor is not attracted by the absolute gain, i.e. the difference between the purchase and the sale price of a security, as by the percentage gain, which does not depend on the amount initially invested.

Returns represent a complete, non-dimensional summary of the investment opportunities, and their historical data are easier to deal with from a statistical point of view.

Asset's Excess Returns are defined as the difference between the actual return on a security, or a portfolio, R_t and the return on a risk-free asset R^f.

Excess Returns are shown in literature statistically more attractive properties than the price series, that evolve following a stochastic Random Walk process with drift.

The random walk process describes a path where the current value of a variable consists of random steps composed by the past value of the variable plus an error term defined as white noise (a random variable that is normally distributed with mean zero and variance σ^2).

This model implies that the best prediction of the variable for the next period is the current value, as the process does not allow to predict future values.

The price of financial securities follows a random walk model with drift, where the drift acts like a trend.

Defining P_t the price of a financial security at time t, the process is not stationary and has the following form:

$$log(P_t) = \mu + log(P_{t-1}) + u_t \tag{2.1}$$

Where μ is the drift parameter and u_t is white noise with mean 0 and variance σ_u. The random shocks are also called innovations since they indicate the new information that arrives at time t.

Before start introducing the definition of financial return, it is important to define the concept of stationarity.

Despite there are different definitions of stationarity, known as weaker and stronger, for the purposes of this introduction, it is sufficient to consider the definition of weak stationarity.

A generic time series $\{y_t\}$ is said to be weakly stationary if it satisfies the following three conditions:

$$\mathbb{E}(y_t) = \mu \tag{2.2}$$

$$var(y_t) = \mathbb{E}(y_t - \mu)^2 = \sigma^2 \tag{2.3}$$

and

$$cov(y_t, y_{t-s}) = \mathbb{E}(y_t - \mu)(y_{t-s} - \mu) = \gamma_s \tag{2.4}$$

The first condition states that the mean of the series is μ in every time t. The second condition states that also the variance of the series is the same in every time t, while the third condition states that the covariance between two observations y_t and y_{t-s} depends only on the lag between them, s, and is the same in every time period.[3]

The price of a generic financial security that follows a random walk process with drift is a trended time series with a non-constant mean, that violates the fist condition for weak stationarity.

The use of non-stationary time series data in financial models produces unreliable results and could lead to an incorrect forecast.

For this reason, it is useful to introduce the concept of financial returns that, as introduced above, shown in literature statistically more attractive properties than the price series.

There are different definition of Return, but the most common definitions are:

- the linear Gross Rate of Return between times t and $t-1$, given by:

$$1 + R_t = \frac{P_t}{P_{t-1}} \tag{2.5}$$

- the Continuously Compounded Return between times t and $t-1$, given by:

$$r_t = log(1 + R_t) = log\left(\frac{P_t}{P_{t-1}}\right) \tag{2.6}$$

Using the Continuously Compounded Return the yield is given by the natural logarithm of the simple gross return and represents the compounded growth rate in prices between $t-1$ and t.

The latter return is also called log return and, as stated in the literature, is more compatible with the hypothesis of normality than the gross rate of return.

[3] Pierse, Richard G. "Economic Forecasting Lecture 2: Forecasting the Trend."

The assumption of the normality of the log-returns of financial securities is one of the most important assumptions in econometrics.

It is a fundamental assumption because, assuming that r_t is normally distributed with mean μ and standard deviation σ, the simple return has a log-normal distribution and the multiperiod log-return is just the sum of the single-period log-returns. Therefore, the assumption of normality remains valid for the multiperiod returns.

The two ways of defining returns lead to very similar results, especially for short-term periods.

This could be verified by developing a second-order Taylor expansion of P_t as a series of P_{t-1}. It can be noted that $r_t < R_t$ and that the difference between the two definitions is usually very small.

The choice of which type of return to use is strictly dependent on the type of analysis to be conducted.

The linear return works easily with single investment period calculation, but involve more complex calculations for the multiperiod horizons.

When time-series of returns are considered, the log-returns are used because of several advantages as the additive property, the prevention of negative prices and the symmetry between returns i.e. returns of equal magnitude cancel each other, while linear returns do not.

As the model implemented in this thesis deals with time-series, continuously compounded returns are preferred because it would be easier to aggregate consecutive returns.

2.3 Portfolio Returns and The Effect of Diversification

A portfolio is a set of financial assets in which the wealth is allocated.

It is quite common to find that individuals tend to diversify their investment in portfolios made up of various securities, rather than concentrate their wealth in a single security.

Defining $R_{i,t}$ the gross rate of return of the security i with weight w_i in a portfolio of n assets, the gross portfolio return is:

$$R_{p,t} = \sum_{i=1}^{n} w_i R_{i,t} \tag{2.7}$$

with the weights summing up to 1.

The estimation of the portfolio risk measure is a complex problem because it can not be calculated as a weighted average of the risk measures associated with the securities in the portfolio. Such a measure would neglect the effect of diversification given by the imperfect correlation of the returns on the securities in the portfolio.

The estimation of the riskiness of a portfolio requires the statistical indicator known as the Linear Correlation Coefficient ρ, which measures the direction and the strength of the linear association between two variables.

Recalling that the linear correlation coefficient between two security x and y is :

$$\rho_{x,y} = \frac{Cov_{x,y}}{\sigma_x \sigma_y} \tag{2.8}$$

The use of ρ allows evaluating how the return of a financial security change when the return of another security varies.

Diversification aims to mitigate unsystematic risk events in a portfolio of risky assets. In a diversified portfolio, if the securities respond in different ways to market movements, the negative performance of some assets are reduced by the positive performance of others.

Thus, the diversification of the portfolio investments leads to manage risk and reduce the volatility of the portfolio returns compared to the risk associated with the various asset's volatility.

To take account of this effect, the portfolio variance is calculated as:

$$\sigma_{p,t}^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij} = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j Cov(R_{i,t}, R_{j,t}) \tag{2.9}$$

Obviously, for any value of $\rho < 0$ the variance of the portfolio returns is lower than the weighted average of the variances of the individual securities return.

However, it should be noted that the benefits in terms of portfolio variance reduction generally occur for any value of $\rho < 1$. In some special cases, it is possible to obtain a portfolio variance lower than the variance of each of the securities in the portfolio.

In practice, the benefits of diversification in terms of risk reduction justify the existence of financial institutions such as mutual funds and financial securities such as the exchange-traded funds (ETF).

Both of them allow an investor to directly purchase a highly diversified portfolio without incurring high transaction and information gathering costs that would involve investing in a range of individual financial securities.

2.4 The Markowitz Approach

As stated above, the Modern Portfolio Choice Theory was originally developed by Harry Markowitz in 1952 through the publication of his article "Portfolio Selection", which outlined the problem of portfolio allocation.

The main outcome of this paper is the theoretical mean-variance model for the solution of short-term asset allocation problems.

This theory is made by two large conceptual blocks: an optimization phase, related to the identification of the efficient portfolios among those considered, and a maximizing phase, where the goal is to maximize the satisfaction of the decision-maker.

In the latter step, the introduction of an individual's preferences allows comparing optimal portfolios with others.

The Portfolio Theory of Markowitz is based on the Perfect Market Assumptions, under which the investor environment is an economy entirely efficient in terms of both equal access to information and rational economic actors.

The most important assumptions are the same as Merton[4]:

- All assets have limited liability
- Each security can be sold or purchased in any amount at the market price at any time.
- The bid-ask spread is zero, which means that the selling price of each asset is the same as the purchasing price.
- The investors are price-takers, as they can't influence the market prices.
- There are no transaction costs or taxes involved.

These are the standard assumptions of a perfect market, but transaction costs and indivisibles do exist.

Assuming that in the market exist n risky assets and a generic investor chooses his portfolio at time 0, intending to hold it for one period.

After a period, at time 1 the true value of the portfolio is determined. Denoting by $P_{i,0}$ the price of asset i at time 0, analogously at time 1 the price of the asset i is $P_{i,1}$.

As stated above, the relative change in price during one period is the gross return on the asset $R_{i,t}$.

In this model, the gross return of a security is assumed to be a random variable with conditional mean μ_i and conditional variance σ_i^2, where the conditional mean and the conditional variance are the mean and variance conditional on the investor's information at time t.

[4]Merton, Robert C. "An intertemporal capital asset pricing model." Econometrica: Journal of the Econometric Society (1973): 867-887.

The model assumes that the investor allocates his wealth entirely over the n risky assets, with the weights adding up to one and the possibility to impose constraints about the borrowing and the shorting assets, which impose to invest only in long positions ($w_i \geqslant 0$).

Once that the allocation is determined, it is possible to compute the conditional expected mean μ_p and variance σ_p of the portfolio return.

By denoting as w and μ respectively the column vectors composed by the portfolio weights w_i and the means μ_i, and denoting by Σ the covariance matrix which contains the covariances between the asset returns, the mean and the variance of the portfolio returns are easily calculated as:

$$\mu_p = \sum_{i=1}^{n} w_i \mu_i = \mu' w \tag{2.10}$$

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} = w' \Sigma w \tag{2.11}$$

imposing that Σ is invertible.

Moreover, being Σ a covariance matrix, it is positive semi-definite and symmetric.

In his paper, Markowitz uses the expected variance of the portfolio returns σ_p as the only measure of risk, without taking account of the other conditional moments of the distribution.

A useful concept in this analysis is the Sharpe Ratio S_t, defined as the mean of the excess return over a risk-free asset return to the standard deviation:

$$S_t = \frac{\mathbb{E}_t[R_{t+1}] - R_{t+1}^f}{\sigma_t} \tag{2.12}$$

The Sharpe Ratio is an important tool as it is used from investors to understand the expected return of a specific investment compared to its risk.

As stated before, the main reason to invest in a portfolio instead of investing in a single asset the whole wealth is the possibility to find combinations of assets that are more mean-variance efficient than a single asset, due to the benefits of diversification.

"Diversification allows the investor to take advantage of the potential risk reduction by combining assets, which are not perfectly correlated and which do not react to movements in the market in the same way respectively".[5]

2.5 Efficient Frontier

Simulating a great number of portfolios, infinitely different allocation could be constructed, but what matters to a generic investor are the best performing portfolios, in terms of both return and risk.

The so-called Efficient Portfolios are the asset combinations characterized by the minimum expected risk associated with a given expected return, or symmetrically, the maximum expected return associated with a given degree of risk.

By determining all the efficient portfolio achieved with different combinations of expected return and risk, given a set of n risky assets, it is possible to construct the Efficient Frontier.

The Efficient Frontier is a theoretical concept that includes all the efficient portfolios, based on the available underlying single securities historical data.

The results of the Markowian portfolio analysis are shown in the mean-variance standard diagram (Figure 2.1), where the vertical axis contains the expected return and the horizontal axis show the risk as measured by the standard deviation.

The Efficient Frontier is a curved line that expresses a set of means and standard deviations that could be achieved by combining the n risky asset

[5] Levy, Haim, and Marshall Sarnat. "International diversification of investment portfolios." The American Economic Review 60.4 (1970): 668- 675.

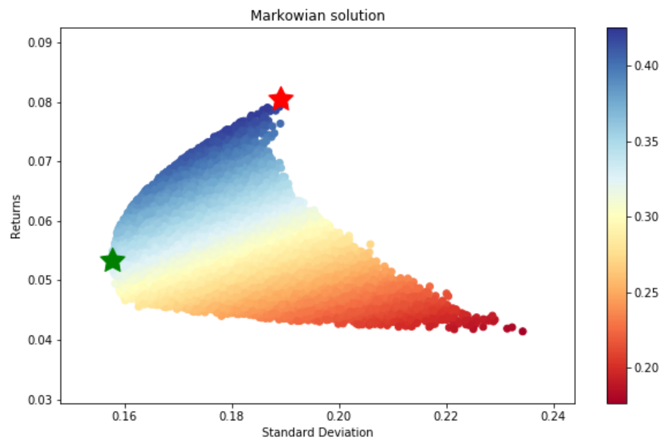

Figure 2.1

in a risky portfolio.

Stocks are shown to offer a high mean return and a high standard deviation, while Bonds are shown to offer a lower mean return and lower standard deviation.

All the portfolios lying below the efficient frontier are defined inefficient because it is possible to enhance the expected return with the same risk by changing the portfolio allocation. While it is not possible that a portfolio lies above the efficient frontier.

The portfolio characterised by the lowest degree of risk is called the minimum variation portfolio, defined as the collection of securities which, taken as a whole, minimise the volatility of the overall portfolio.

2.6 Including a Risk-Free Asset

Usually, portfolio theory, next to the n risky assets, assumes the existence of a risk-free asset, characterized by a gross rate of return R_f and the absence of risk i.e. $\sigma_f = 0$.

The risk-free security used in practice is identified in financial securities characterized by a low risk of default, as a bank deposit or a short-term government bond.

The returns of these securities, deemed as to be risk-free, may fluctuate and appear risky, as their sample variance is different from zero.

However, their sample variance is always minimal compared to that of risky securities and could be approximated to zero.

A generic investor may choose to invest a part of his wealth in the risk-free asset as well as in the n risky securities.

Denoting with w_0 the portion of wealth invested in the risk-free asset with certain return R_f, being r the column vector composed by the rate of return $R_{i,t}$ of each risky asset i in the portfolio, the rate of return of the portfolio is given by:

$$R_p = w_0 R_f + w'r = r_f + w'(r - r_f \iota) \tag{2.13}$$

with $w_0 + \iota' w = 1$, where ι is the unit vector, and the vector $r - r_f \iota$ represents the excess return of the n risky assets over the risk-free asset.

By including the risk-free asset in the portfolio calculation, the set of possible means and standard deviations that can be achieved is a straight line, called the Capital Market Line, on the diagram in Figure 2.2.

2.7 Capital Market Line

The Capital Market Line is a theoretical representation defined in terms of the total risk/return of all the efficient portfolios that could include the risk-free asset.

As with the Efficient Frontier, any portfolio that lies below the Capital Market Line doesn't represent the best allocation for the investor because, with the same overall risk and different weights, the portfolio could deliver higher returns.

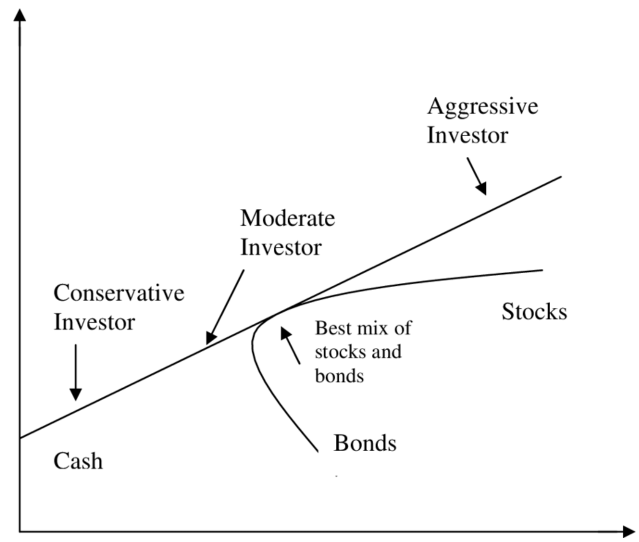

Figure 2.2: Campbell and Viceira, 2002

The Capital Market Line, which starts at the risk-free rate R_f and is tangent to the Efficient Frontier described above, represents the combinations of the highest mean return for any given standard deviation.

All the portfolios with the best risk/return ratio lie on this line, while the tangent point is called the Market Portfolio and is the “Best Mix of Stocks and Bonds”.

The striking conclusion of this analysis is that, under the assumption of homogeneous expectations between the market participants, all the investors will hold the same portfolio of risky assets, the Market Portfolio.

Thus, the “tangency portfolio represents the best allocation of risky assets and no investor should alter the relative proportions of the weights in the overall portfolio, but only choose how much to invest in the market portfolio and in the risk-free asset”[6], this result is the mutual fund theorem of James Tobin (1958).

Tobin’s Mutual Fund Separation Theorem underlines the relevance of di-

[6] Tobin, James. "Liquidity preference as behavior towards risk." The review of economic studies 25.2 (1958): 65-86.

versification in the asset allocation process and suggests to build portfolios using mutual funds as their presence could easily increase the portfolio diversification and reduce transaction costs associated with the individual assets purchasing.

According to Tobin's Theorem, the problem of definition of the market portfolio and the best possible asset allocation are two different problems.

The ratio behind this separation between the asset allocation problems relies on the risk-aversion of the investor.

Despite the concept of risk-aversion is described in the third chapter of this thesis, it might be useful to mention that the trade-off between expected portfolio return and variance can be modeled by the risk aversion parameter γ. Depending on this parameter value, each investor chooses how to solving this trade-off between the certainty of the risk-free asset and the Market Portfolio.

Risk-averse investors would select portfolio close to the risk-free asset, preferring portfolios with a low variance to portfolios with a high expected return, while investors which preferences are characterized by low risk-averseness would higher up to the Capital Market Line, increasing both expected portfolio returns and risk.

2.8 Limits of the Myopic Portfolio Choice

In recent years, the attention of economists has been focused on models for the portfolio allocation that could reflect the investor's situation and characteristics.

Financial planners usually suggest to conservative investors holding more bonds in their portfolio than aggressive investors, in contradiction with the Markowian solution about the portfolio allocation which suggests a fixed bond-shares ration for any type of investor.

The static model introduced my Markowitz assumes that investor's preferences are described by a mean-variance utility function, which is a very restrictive assumption.

Another consideration about this model concerns the mono-periodical nature of the portfolio allocation problem, as it is based on the assumption that investors are interested only in the distribution of their wealth in the following period, while empirical investors are more interested in maintaining a certain standard of living through long-term investments.

To summarize, the main theoretical weakness of the model developed by Markowitz are:

- The limited one-period investment horizon, that appears unrealistic and does not include the possibility to rebalance the portfolio allocation and hedge risks.

- The dependency of the model on the first two conditional moments of the return distribution, leaving aside other possible statistical indicators that could include useful information about performance and risk of the assets. The standard deviation of returns appears a simplistic risk measure, that appears inadequate to express the riskiness of the security.

- Data used to carry out the optimization process could bring to mistakes in results.

- The efficient portfolio identified through the mean-variance model is unstable to slight variation in the expected return or variance of securities.

All these limitations, together, make the mean-variance analysis unrealistic and it should be argued if the traditional analysis of portfolio choice needs to be modified to handle with long investment horizons

Empirically, the single securities returns and their combinations are time-varying.

Therefore, for a generic investor, it is more appropriate to manage frequently and regularly his investment decisions, but this process requires deep market knowledge, it is expensive and consuming both in terms of time and effort.

Most individuals lack the experience or training to make a wise saving, investment, and withdrawal decisions.

Due to the empirical inadequacy of this static model, it is interesting to ask whether and how it is possible to modify the assumption and the methodology to construct a model that could better reflect the differences between investors, in terms of risk aversion and investment horizons.

Chapter 3

Utility Functions

So far it was assumed that investors care only about the mean and the variance of portfolio returns.

This chapter deals with the problem of defying the most appropriate Utility Function for the model implemented in this thesis, an extension to the myopic portfolio theory that concerns the use of the loss aversion function, developed by Kahneman and Tversky[1] in the Prospect Theory, to describe the preferences of a generic investor in a condition of uncertainty.

3.1 Definition of Expected Utility

Investors are assumed to make intertemporal choices in uncertainty as the future payoffs are not deterministic but stochastic, i.e. given a generic investment at time t, its value in $t+1$ will be known only after its realization, and will be different based on the state of nature that will be realized.

A model that introduces the behaviors and choices of rational investors is the Expected Utility Theory developed by Von Neumann and Morgenstern

[1] Kahneman, Daniel, and Amos Tversky. "Prospect theory: An analysis of decision under risk." Handbook of the fundamentals of financial decision making: Part I. 2013. 99-127.

in 1947.[2]

Their analysis is based on the assumption that an agent in a condition of uncertainty can determine the expected utility by calculating the average utility of each possible state, weighted by the estimated probabilities.

The authors have shown how it is possible to determine numerical values representing the investor's subjective values in such a way that actions with probabilistic consequences are preferred if the expected utility of the same is greater than the expected utility of the other options not chosen.

The idea is that rational individuals should evaluate the financial payoffs not directly on their amount, but rather on the level of financial "satisfaction" that subjectively attribute to their possession.

Therefore, the goal of a generic investor is not to maximize his expected terminal wealth but it is to maximize his expected utility of terminal wealth.

The Utility Function is defined as a function of the wealth achieved W or consumption C.

It is strictly concave, strictly increasing, and continuous, where the curvature of the function expresses the investor's risk aversion.

The risk aversion is the degree to which an investor prefers a lower and more certain gain than a possible higher return with a lower certainty about results.

The individual preferences, represented by the Utility Function, are characterized by two properties: must reflect the subject preferences and must be increasing, i.e. must have marginal utility positive in wealth because it is natural to associate greater payoffs with greater utility.

The Expected Utility Theory can be applied in a condition of uncertainty, i.e. when the individual has to make decisions without knowing with certainty what state of nature will occur but knows the possible events and the probability of realizations of such events.

[2]Von Neumann, John, and Oskar Morgenstern. "Theory of games and economic behavior, 2nd rev." (1947).

Thus, investors that have to choose between different alternatives characterized by uncertain outcomes, whose probability of occurrence is known, associate an expected utility value to each alternative.

Then the risky alternatives are ordered by their expected utility, expressed as a function of the possible results and the probability that these results will occur.

Given a function $u(x)$, where x is the value of wealth in $t+1$ and assuming $u'(x_{t+1}) > 0$, the expected utility of wealth is:

$$\mathbb{E}[u(x)] = \sum_{i=1}^{S} p_i u(x_i) \tag{3.1}$$

Where S is the number of states of natures, p_i is the probability of occurrence of each possible result x_i and $\sum_{i=1}^{S} p_i = 1$.

As various risky combinations exist, investor's decisions are based on his expected utility values.

The rational investor will choose the alternative that provides him the greatest utility, referring not to the expected value of the consequences, but the expected value of the corresponding utility, i.e. the combination associated with the highest Expected Utility.

Considering a generic investor that has to choose between two investment opportunities, one risk-free with a certain return R_f and one risky asset, with expected return R, the choice is made comparing both $u\left(\mathbb{E}([x])\right)$ and $\mathbb{E}[u(x)]$.

A decision-maker is defined as to be risk-averse if, for the same return, he prefers the certain return, i.e. $u\left(\mathbb{E}([x])\right) > \mathbb{E}[u(x)]$. In this case, the investor is willing to give up part of the possible gain to get a certain outcome, as uncertainty is considered a negative element.

3.2 Quadric Utility, Exponential Utility and Power Utility

The Utility Functions can take several forms:

- Concave when describing the preferences of a risk-averse individual.
- Convex when describing the preferences of a risk-taking individual, i.e. when $u\left(\mathbb{E}([x])\right) < \mathbb{E}[u(x)]$.
- Linear for a risk-neutral individual, i.e. when the investor is indifferent between the two opportunities.

In this thesis, the risk-aversion hypothesis is considered in order to include the behavior of investors who, when have to cope with uncertainty, attempt to reduce the risks as they prefer more predictable outcomes to greater but riskier payoffs.

To measure the intensity of risk aversion the second derivative of the utility function $u''(x)$ is considered but, of course, it is not invariant to the form of $u(x)$.

The introduction of the Arrow-Pratt's risk aversion indicators is useful to solve this problem.

Arrow-Pratt's Coefficient of Absolute Risk Aversion and the Coefficient of Relative Risk Aversion are defined, respectively, as:

$$ARA(x) = \frac{u''(x)}{u'(x)} \tag{3.2}$$

and

$$RRA(x) = x\frac{u''(x)}{u'(x)} \tag{3.3}$$

both positive due to the concavity of the utility function.

The reciprocals of these measures are known as Absolute Risk Tolerance and Relative Risk Tolerance.

As in the classical results contained in the works of Pratt[3] and Arrow[4] referred to small gambles, the ARA determines the absolute dollar amount that an investor is willing to pay to avoid a gamble of a given absolute size and it is commonly assumed that it decreases with wealth.

The RRA determines the fraction of wealth that an investor will pay to avoid a gamble of a given size relative to wealth.

The starting point of the models of portfolio choice, and more generally of every utility calculation, require assumptions about the form of the utility function.

Three alternative sets of assumptions are consistent with the risk adverseness analysis: Quadric Utility, Exponential Utility and Power Utility.

The Quadric Utility Function is defined as $u(x) = x - bx^2$, implying that ARA and RRA are increasing in wealth and that the maximization process is equivalent to the maximization of a linear combination of mean and variance of the asset returns.

The Exponential Utility Function is defined as $u(x) = 1 - e^{-\theta x}$, where asset returns are normally distributed, implying that ARA is always equal to θ and that RRA increases in wealth.

The Power Utility Function is defined as $u(x) = \frac{x^{\gamma-1}-1}{1-\gamma}$, where asset returns are log-normally distributed, implying that ARA is declining in wealth and RRA is constant.

The latter function, which takes also the name of Constant Relative Risk Aversion Function (CRRA), appears the most suitable for the study of long-term portfolio problems for two main reasons: the assumption about the

[3]Pratt, John W. "Risk aversion in the small and in the large." Uncertainty in economics. Academic Press, 1978. 59-79.

[4]Arrow, K. J. "The Theory of Risk Aversion in Aspects of the theory of risk-bearing. Helsinki: yrjo Jahnssonin Saatio." Reprint in Arrow, KJ (1971). Essays in the Theory of Risk-Bearing. Chicago: Markham Publishing (1965): 28-44.

asset returns log distribution and the consistency of the assumptions about the Arrow-Pratt's risk aversion indicators.

The assumption of log-normal random variables holds for every time horizon since, as stated in the previous chapter, the product of log-normal random variables is themselves distributed log normally.

Although this reasoning, the empirical results about the type of risk aversion and the empirical values of the parameters are mixed. For example, Schooley and Worden[5], report that an individual's RRA is constant or decreasing in wealth, depending on the samples and wealth measurement. Friend and Blume[6], found that in the context of myopic allocation households typically have constant RRA with γ being at least 1, and more likely to exceed 2.

Institutional investors probably have a higher γ than households.

Bali[7], has used several time-series to conclude that γ is between 1 and 5.

[5] Schooley, Diane K., and Debra Drecnik Worden. "Risk aversion measures: Comparing attitudes and asset allocation." Financial services review 5.2 (1996): 87-99.

[6] Friend, Irwin, and Marshall E. Blume. "The demand for risky assets." The American Economic Review (1975): 900-922.

[7] Bali, Turan G. "The intertemporal relation between expected returns and risk." Journal of Financial Economics 87.1 (2008): 101-131.

Chapter 4

Dynamic Portfolio Choice

So far it was assumed that investors have short investment horizons and care only about the distribution of wealth at the end of the next period.

The framework developed by Markowitz is a useful construction that could easily be implemented and extended with a risk-free asset, but the obvious shortcoming of the application of this portfolio theory is the assumption of constant mean and constant covariance between the assets returns during the investment horizon.

Hence, the assumption that long-horizon investor cannot rebalance his portfolio appears superficial, because it makes the long-term investment problem formally analogous to the short-horizon problem.

The static portfolio choice, introduced in the second chapter of this thesis, is stated to as myopic portfolio choice because it is not capable of adapting to possible changes in the financial markets, while the dynamic asset allocation overcomes this shortcoming enabling the adaption of the model to changes in market conditions.

As it is explained in this chapter, the long-term investor's optimal portfolio depends not only on his objective but also on what the decision-maker is allowed to do in each period, particularly on whether the investor is allowed to rebalance his portfolio each period.

The dynamic portfolio choice appears more realistic than the myopic Markowitz model because it allows the portfolio rebalancing, but it involves several complexities in the implementation.

As stated by Jhon. H. Cochrane: "Classical mean-variance brilliantly declares victory and goes home just before the hard part begins". [1]

4.1 Merton's approach to Portfolio Choice in Continuous Time

The first authors to contribute to the modern literature about Dynamic Portfolio Choice was Merton[2] and Samuelson.[3]

Both their theories are based on the idea that means and variances of asset returns are time-varying and do not remain fixed over time, as they change in response to economic conditions.

The Dynamic Portfolio baseline is that the construction of an efficient investment strategy must include protection against the fluctuations of the first and second moments of asset returns, including an "intertemporal hedging component to provide insurance against shocks in returns moments".[4]

Merton considered a dynamic portfolio problem in continuous time with intertemporal consumption, designing the solution in order to protect the investment against fluctuations and, eventually, taking advantage of these fluctuations.

[1] Cochrane, John H. "A mean-variance benchmark for intertemporal portfolio theory." Manuscript, University of Chicago (2008).

[2] Merton, Robert C. "Lifetime portfolio selection under uncertainty: The continuous-time case." The review of Economics and Statistics (1969): 247-257.

[3] Samuelson, Paul A. "Lifetime portfolio selection by dynamic stochastic programming." The review of economics and statistics (1969): 239-246.

[4] Detemple, Jérôme, René Garcia, and Marcel Rindisbacher. "Intertemporal asset allocation: A comparison of methods." Journal of Banking & Finance 29.11 (2005): 2821-2848.

Then Merton[5] proves a mutual fund theorem very similar to Tobin[6], stating that: “Given n assets with prices P_t whose changes are log-normally distributed, then there exists a unique pair of mutual funds constructed from linear combinations of these assets such that, independent of preferences (i.e. the form of the utility function), wealth distribution, or time horizon, individuals will be indifferent between choosing from a linear combination of these two funds or a linear combination of the original n assets”.[7]

Merton’s theory starting point is the distribution of returns, that is assumed to be log-normally distributed and follow a Geometric Brownian Motion.

The latter assumption considered by the author is a generic supposition about the economic prices stochastic process.

Thus, Merton showed that the price of the assets follows the stochastic differential equation:

$$dP_t = \mu P_t dt + \sigma S_t dB_t \tag{4.1}$$

where B_t is a single Werner process, a stochastic process with three important characteristics summarized by Shreve[8]:

- $B_0 = 0$
- It is continuous
- B_t has independent increments with distribution $B_t - B_s \sim N(0, t - s)$ for $0 \leq s \leq t$.

[5] Merton, Robert C. "Optimum consumption and portfolio rules in a continuous-time model." Stochastic Optimization Models in Finance. Academic Press, 1975. 621-661.

[6] Tobin, James. "Liquidity preference as behavior towards risk." The review of economic studies 25.2 (1958): 65-86.

[7] Cass, David, and Joseph E. Stiglitz. The structure of investor preferences and asset returns, and separability in portfolio allocation: A contribution to the pure theory of mutual funds. 1970.

[8] Shreve, Steven E. Stochastic calculus for nance II: Continuous-time models. Vol. 11. Springer Science & Business Media, 2004.

Following this Dynamic Allocation Strategy introduced by Merton, a generic investor with an investment horizon T should allocate his wealth at each period on the optimal portfolio weights x_t and determining the optimal intertemporal amount to consume C_t.

The dynamic of the investor's wealth W_t is dependent on the previous asset allocation and its return, net of the consumption referred to the specific period.

The wealth dynamic could be expressed as:

$$\begin{aligned} dW_t &= W_t x_t \frac{dP_t}{P_t} + W_t(1 - x_t)R_f dt - C_t dt \\ &= ((x_t(\mu - R_f) + R_f)W_t - C_t)dt + x_t \sigma W_t dB_t \end{aligned} \tag{4.2}$$

The solution to this continuous dynamic problem is constrained by some strict assumptions about the return distribution and dynamics.

However, the solution to this method is not discussed as it is not the object of this work.

By the increasing of the number of assets, calculate analytically the closed-form solution with the Merton's model could become very complicated.

For this reason, the dynamic portfolio choice problem is often discretized to be solved numerically in discrete time.

4.2 Dynamic Portfolio Choice in Discrete Time

Even if the discretization of the problem appears as a theoretical assumption that could limit the prediction power of the model, it is more realistic than the continuous-time approach, as it overcomes the limitations of the continuous-time formulation.

Despite the continuous-time represent an efficient analytical approximation of the portfolio choice problem, an investor will not continuously trade during his investment horizon, but it is more empirically correct to consider the portfolio rebalancing over discrete timestamps.

The first step of the dynamic methodology implemented in this thesis is the simulation of a large number of sample paths of asset return and state variables, through their known or estimated joint dynamics.

As stated by Brandt et al. (2005), the most relevant idea of these simulations is that: “the joint dynamics of the asset returns and state variables can be high-dimensional, arbitrary complicated, path-dependent, and even non-stationary”.[9]

The problem of portfolio selection is then addressed recursively in standard dynamic programming.

Starting from $T-1$, for each simulated path, the optimal portfolio allocation is computed as the weights that maximize a Tylor expansion of the investor’s value function.

This problem has a straightforward semi-closed form solution that involves conditional moments of the value function, its derivatives and asset returns.

These conditional expectations are calculated through ordinary least square regression of the realized utility, its conditional moments and asset returns at the following period based on functions of the realized state variables at $T-1$ across the simulated paths.

Then the model proceeds backward until time zero, in order to find the portfolio allocation that “maximizes the conditional expectation of the investor’s utility, given the optimal portfolios for all future periods until the end of the horizon”.[10]

---

[9] Brandt, Michael W., et al. "A simulation approach to dynamic portfolio choice with an application to learning about return predictability." The Review of Financial Studies 18.3 (2005): 831-873.

[10] See (9)

To summarize, this method allows evaluating the closed-form solution of the approximate optimal portfolio allocation by simulating the asset returns and the state variables paths and then computing a set of across-paths regression for each period.

This approach is inspired by a method for pricing American-style options introduced in the paper "Valuing American Option by Simulation: a Simple Least-Squares Approach" published by Longstaff and Schwartz in 2001.

These authors used across-paths regression on simulated sample paths in order to estimate the conditional expectations about the payoff to the option holder. The model is based on the calculation of the expectation of the "continuation value of the option and compares these conditional expectations to the immediate exercise value at all future dates along each simulated path".[11]

This approach is "readily applicable in path-dependent and multifactor situations where traditional finite difference techniques cannot be used".[12]

This method is adopted to use conditional expectations as input in the portfolio optimization process.

4.3 Description of the Problem

The investor problem could be summarized as the maximization of a given utility function.

At each time, the investor aims to "maximize the expected utility of wealth at some terminal date T, by by trading n risky assets and a risk-free asset at times $t, t+1, ..., T-1$".[13]

[11]Brandt, Michael W., et al. "A simulation approach to dynamic portfolio choice with an application to learning about return predictability." The Review of Financial Studies 18.3 (2005): 831-873.

[12]Longstaff, Francis A., and Eduardo S. Schwartz. "Valuing American options by simulation: a simple least-squares approach." The review of financial studies 14.1 (2001): 113-147.

[13]van Binsbergen, Jules H., and Michael W. Brandt. "Solving dynamic portfolio choice

As stated before, the investor wealth at the end of each period is directly related to his previous allocation, so that the investor's problem could be expressed by a value function $J(.)$:

$$\begin{aligned} J_t(W_t, Z_t) &= \max_{\{x_s\}_{s=t}^{T-1}} \mathbb{E}_t[u(W_T) \mid W_t, Z_t] \\ s.t.: \; W_{s+1} &= W_s(x_s' R_{s+1}^e + R_f) \; \forall s \geq t \end{aligned} \tag{4.3}$$

Where x_s represents the vector of weights of the portfolio over the risky asset chosen at time s and held until the next period.

The gross return on the risk-free asset is denominated as R_f and R_{s+1}^e represents the vector of the risky assets excess return over the risk-free return.

W_t is the investor wealth at time t and it is an endogenous variable as it is influenced only by the previous decisions about the portfolio allocation.

Z_t represents the collection of the state variables at time t, exogenous variables that include economic factors as interest rates, inflation rate, or stock index:

$$Z_t = \begin{bmatrix} z_{1,t} \\ z_{2,t} \\ \vdots \\ z_{k,t} \end{bmatrix} \tag{4.4}$$

The main concept of this dynamic approach is that future decisions about the optimal allocation of the investor's wealth depend on the allocations made earlier. i.e. an investor that holds a high realized portfolio return might behave differently from an investor with a low realized past portfolio return.

problems by recursing on optimized portfolio weights or on the value function?." Computational Economics 29.3-4 (2007): 355-367.

This concept is described by Richard Bellman in the Principle of Optimality: “An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy concerning the state resulting from the first decision”.[14]

In his article ‘Dynamic Programming and Stochastic Control Processes’ Richard E. Bellman discussed the analytic translation of the previous statement, rewriting the investor problem as a recursive problem.

“A Bellman equation, named after Richard E. Bellman, is a necessary condition for optimality associated with the mathematical optimization method known as dynamic programming”.[15]

“This breaks a dynamic optimization problem into a sequence of simpler sub-problems, as Bellman’s Principle of Optimality prescribes”.[16]

Therefore, in each period, the optimal portfolio allocation has to be made taking into account that the future optimal portfolios will depend on his previous decisions.

The Value function expressed above could be written as:

$$
\begin{aligned}
J_t(W_t, Z_t) &= \max_{\{x_s\}_{s=t}^{T-1}} \mathbb{E}_t[J_{t+1}(W_{t+1}, Z_{t+1})] \\
s.t.:\ W_{s+1} &= W_s(x_s' R_{s+1}^e + R_f)\ \forall s \geq t
\end{aligned} \tag{4.5}
$$

Where, for notational convenience, the conditional expectation is written with a subscript:

$$\mathbb{E}[X] = \mathbb{E}[X \mid W_t, Z_t]$$

[14]Bellman, Richard. "Dynamic programming and stochastic control processes." Information and control 1.3 (1958): 228-239.

[15]Dixit, Avinash K., and John JF Sherrerd. Optimization in economic theory. Oxford University Press on Demand, 1990.

[16]Kirk, Donald E. "optimal control theory: an introduction. Printice- Hall." Englewood Clis, NJ 228 (1970).

The fists step of this problem is straightforward and follows the law of iterated expectations:

$$J_t(W_t, Z_t) = \max_{\{x_s\}_{s=t}^{T-1}} \mathbb{E}_t[u(W_T)] = \max_{x_t} \max_{\{x_s\}_{s=t}^{T-1}} \mathbb{E}_t[\mathbb{E}_{t+1}[u(W_T)]] \tag{4.6}$$

While the second step is to include the interchange of expectation:

$$\max_{x_t} \max_{\{x_s\}_{s=t}^{T-1}} \mathbb{E}_t[\mathbb{E}_{t+1}[u(W_T)]] = \max_{x_t} \mathbb{E}_t[\max_{\{x_s\}_{s=t}^{T-1}} \mathbb{E}_{t+1}[u(W_T)]] \tag{4.7}$$

The formula that relates the current wealth and the wealth one step before is called the budget constraint.

The formula that expresses the recursive relation between the value function at time t and the conditional expectation of the value function one step ahead is known as the Bellman Equation and it is the basis of the dynamic discretized portfolio problem.

4.4 Numerical Approaches

In recent years both the computing power and the numerical methods have seen huge advances, to the point that multivariate analysis and multiperiod portfolio selection problems could be solved numerically.

During this period, several authors have published different articles about numerical methods to solve the problem of portfolio allocation and they tried to incorporate realistic features into the problem. For example, Brennan, Schwartz and Lagnado[17] solve numerically the partial differential equation characterizing the solution to the dynamic optimization. Campbell and

[17] Brennan, Michael J., Eduardo S. Schwartz, and Ronald Lagnado. "Strategic asset allocation." Journal of Economic Dynamics and Control 21.8- 9 (1997): 1377-1403.

Viceira[18] log-linearize the first-order conditions in order to obtain closed-form solutions. While Das and Sundaram[19] and Kogan and Uppal[20] solved analytically the problem trough different expansions of the value function.

The most popular approaches involve the discretization of the state space, which is done by Barberis[21], Balduzzi and Lynch[22], Dammon, Spatt, and Zhang[23] and Brandt[24], among many others.

"Once the state space is discretized, the value function can be evaluated by a choice of quadrature integration (Balduzzi and Lynch), simulations (Barberis), binomial discretizations (Dammon, Spatt, and Zhang), or non-parametric regressions (Brandt), and then the dynamic optimization can be solved by backward recursion".[25]

All these methods rely on Constant Relative Risk Aversion preferences in order to eliminate the dependence of the portfolio policies on wealth and thereby make the portfolio allocation problem path-independent.

Despite a large number of possible approaches, unfortunately, almost all of them assume unrealistically simple return distribution and cannot include

[18] Campbell, John Y., and Luis M. Viceira. "Consumption and portfolio decisions when expected returns are time varying." The Quarterly Journal of Economics 114.2 (1999): 433-495.

[19] Das, Sanjiv Ranjan, and Rangarajan K. Sundaram. "Of smiles and smirks: A term structure perspective." Journal of financial and quantitative analysis (1999): 211-239.

[20] Kogan, Leonid, and Raman Uppal. Risk aversion and optimal portfolio policies in partial and general equilibrium economies. No. w8609. National Bureau of Economic Research, 2001.

[21] Barberis, Nicholas. "Investing for the long run when returns are predictable." The Journal of Finance 55.1 (2000): 225-264.

[22] Balduzzi, Pierluigi, and Anthony W. Lynch. "Transaction costs and predictability: Some utility cost calculations." Journal of Financial Economics 52.1 (1999): 47-78.

[23] Dammon, Robert M., Chester S. Spatt, and Harold H. Zhang. "Optimal consumption and investment with capital gains taxes." The Review of Financial Studies 14.3 (2001): 583-616.

[24] Brandt, Michael W. "Estimating portfolio and consumption choice: A conditional Euler equations approach." The Journal of Finance 54.5 (1999): 1609-1645.

[25] Brandt, Michael W., et al. "A simulation approach to dynamic portfolio choice with an application to learning about return predictability." The Review of Financial Studies 18.3 (2005): 831-873.

constraints on the portfolio weights.

In 2005 Brandt, Goyal, Santa Clara and Stroud published an article that includes an innovative approach to solve the problem of portfolio choice, called the BGSS method, trough a simulation-based method that uses a Tylor expansion of the value function and regressions on all simulated samples paths.

The idea behind the Brand's model came from an innovative and powerful approach of Longstaff and Schwartz for the approximation of the expected payoffs of American options.

The BGSS method overcomes the limitations of the other methods cited above, enabling the simulation of a large number of hypothetical sample paths of asset returns and state variables to form the known, estimated, or bootstrapped joint dynamics of the returns and state variables. "The key feature of this simulation is that the joint dynamics of the asset returns and the state variables can be high-dimensional, arbitrarily complicated, path-dependent and even non-stationary".[26]

Given the set of simulated paths, the optimal portfolio policies are solved recursively in a standard dynamic programming fashion deeply explained in the following chapter.

This method is promising for its speed and flexibility to accommodate both asset returns and state variable returns complex dynamics.

As observed by Detemple[27], and Garlappi and Skoulakis[28], the addition of constraints on the portfolio weights may be necessary to constrain the error of this method.

[26] Brandt, Michael W., et al. "A simulation approach to dynamic portfolio choice with an application to learning about return predictability." The Review of Financial Studies 18.3 (2005): 831-873.

[27] Detemple, Jérôme, René Garcia, and Marcel Rindisbacher. "Intertemporal asset allocation: A comparison of methods." Journal of Banking & Finance 29.11 (2005): 2821-2848.

[28] Garlappi, Lorenzo, and Georgios Skoulakis. "Numerical solutions to dynamic portfolio problems: The case for value function iteration using Taylor approximation." Computational Economics 33.2 (2009): 193-207.

In this thesis, the BGSS method is applied to a simple environment of a multiple asset case.

Chapter 5

Methodology

This chapter aims to describe the methodology as published by Brandt et al. in 2005[1]. It involves the expansion of the value function, the backward recursion by approximating terminal wealth and the determinants of a generic investor's preferences characterized by constant relative risk aversion.

The conditional expectations are calculated using several across-paths regressions.

The optimal portfolio allocation strategy is discussed for higher orders of Taylor series and the constraints used in the model.

5.1 Expanding the value function

The starting point of the methodology of this thesis is the Bellman equation and the budget constraints introduced in the previous chapter:

[1] Brandt, Michael W., et al. "A simulation approach to dynamic portfolio choice with an application to learning about return predictability." The Review of Financial Studies 18.3 (2005): 831-873.

$$
\begin{aligned}
J_t(W_t, Z_t) &= \max_{\{x_s\}_{s=t}^{T-1}} \mathbb{E}_t[J_{t+1}(W_{t+1}, Z_{t+1})] \\
s.t.:\ W_{s+1} &= W_s(x_s' R_{s+1}^e + R_f)\ \forall s \geq t
\end{aligned}
\tag{5.1}
$$

To implement the BGSS Method the budget constraint is substituted in the one step ahead value function:

$$
J_{t+1}(W_{t+1}, Z_{t+1}) = J_{t+1}(W_t(x_t' R_{t+1}^e + R_f), Z_{t+1})
$$

The next step is the employment of a Taylor series of the Value function around $W_t R_f$, which lead to an explicit solution for the portfolio weights x_t:

$$
\begin{aligned}
J_{t+1}(W_t(x_t' R_{t+1}^e + R_f), Z_{t+1}) \approx J_{t+1}(W_t R_f, Z_{t+1}) \\
+ \partial_1 J_{t+1}(W_t R_f, Z_{t+1})(W_t x_t' R_{t+1}^e) \\
+ \frac{1}{2}\partial_1^2 J_{t+1}(W_t R_f, Z_{t+1})(W_t x_t' R_{t+1}^e)^2
\end{aligned}
\tag{5.2}
$$

Where ∂_1 denotes the partial derivative concerning the first variable of the value function.

Then the value function is substituted in the Bellman equation to obtain the approximation of the value function in t:

$$
\begin{aligned}
\bar{J}_t(W_t, Z_t) = \max_{x_t} \mathbb{E}_t[J_{t+1}(W_t R_f, Z_{t+1}) \\
+ \partial_1 J_{t+1}(W_t R_f, Z_{t+1})(W_t x_t' R_{t+1}^e) \\
+ \frac{1}{2}\partial_1^2 J_{t+1}(W_t R_f, Z_{t+1})(W_t x_t' R_{t+1}^e)^2]
\end{aligned}
\tag{5.3}
$$

The gradient towards x_t is taken and imposed equal to zero, in order to find the weights that maximizes the right-hand side of the value function approximation at each time t:

$$\begin{aligned}\overrightarrow{0} = \nabla\{\mathbb{E}_t[J_{t+1}(W_t R_f, Z_{t+1}) \\ + \partial_1 J_{t+1}(W_t R_f, Z_{t+1})(W_t x_t' R^e_{t+1}) \\ + \frac{1}{2}\partial_1^2 J_{t+1}(W_t R_f, Z_{t+1})(W_t x_t' R^e_{t+1})^2]\}\end{aligned} \quad (5.4)$$

If it is assumed that the Lebesgue's dominated convergence theorem is allowed, which "provides sufficient conditions under which almost everywhere convergence of a sequence of functions implies convergence in the L norm"[2] then the interchange of expectation and derivative is allowed.

It is possible to write:

$$\begin{aligned}\overrightarrow{0} = W_t \mathbb{E}_t[\partial_1 J_{t+1}(W_t R_f, Z_{t+1}) R^e_{t+1}] \\ + W_t^2 \mathbb{E}_t[\partial_1^2 J_{t+1}(W_t R_f, Z_{t+1})(x_t' R^e_{t+1}) R^e_{t+1}]\end{aligned} \quad (5.5)$$

Because the commutativity of the inner product and the associativity of matrix operations allows to rewrite $(x_t' R^e_{t+1}) R^e_{t+1} = (R^e_{t+1} R^{e'}_{t+1}) x_t$.

This leads to an explicit expression for x_t, which depends on the conditional expectations.

It is denoted by $\bar{x}_t$ because it is an approximation of the true value of x_t. The explicit expression for the approximated value of the weights is:

[2]Bartle, R.G. (1995). The Elements of Integration and Lebesgue Measure. Wiley Interscience.

$$\bar{x}_t = -\{W_t \mathbb{E}_t[\partial_1^2 J_{t+1}(W_t R^f, Z_{t+1})(R^e_{t+1} R^{e'}_{t+1})]\}^{-1} \\ \times \mathbb{E}_t[\partial_1 J_{t+1}(W_t R^f, Z_{t+1})(R^e_{t+1})] \quad (5.6)$$

This reformulation relies on the assumption of non-singularity of the matrix $\mathbb{E}_t[\partial_1^2 J_{t+1}(W_t R^f, Z_{t+1})(R^e_{t+1} R^{e'}_{t+1})]$, that thus is invertible.

Analytically it is not necessary true, as for a single asset, neglecting $\partial_1^2 J_{t+1}(W_t R^f, Z_{t+1})$, the matrix is not invertible for $\mathbb{E}_t[(R^e_{1,1+t})^2] = 0$.

For two assets, the matrix is not invertible if $Cov(R^e_{1,t+1}, R^e_{2,t+1}) = 0$. But the conditional expectations are calculated separately for each element of the matrix, based on the simulation and regression analysis and, as the number of simulation grows, the probability that the approximated matrix is not invertible becomes very small.

Defining:

$$A_{t+1} := \partial_1 J_{t+1}(W_t R^f, Z_{t+1}) R^e_{t+1} \quad (5.7)$$

$$B_{t+1} := \partial_1^2 J_{t+1}(W_t R^f, Z_{t+1}) R^e_{t+1} R^{e'}_{t+1} \quad (5.8)$$

It is possible to rewrite the explicit expression for the approximation of x_t as:

$$\bar{x}_t = -\{W_t \mathbb{E}_t[B_{t+1}]\}^{-1} \times \mathbb{E}_t[A_{t+1}] \quad (5.9)$$

“The two conditional expectations in the formula above are the second-moment matrix of returns scaled by the second derivative of the value function and the risk premia of the assets scaled by the first derivative of the value function”.[3]

[3]Brandt, Michael W., et al. "A simulation approach to dynamic portfolio choice with

Taking a closer look at the derivatives used in the previous formula it is necessary to recall the initial maximization problem:

$$J_{t+1}(W_{t+1}, Z_{t+1}) = \max_{\{x_s\}_{s=t+1}^{T-1}} \mathbb{E}_{t+1}[u(W_T)] \tag{5.10}$$

Using the budget constraint formula, it is possible to rewrite the terminal wealth in terms of current wealth:

$$W_T = W_t \prod_{s=t}^{T-1} (x_s' R_{s+1}^e + R^f) \tag{5.11}$$

Then, assuming that all the optimal portfolio allocations for every timestamp in the investor horizon are already determined and denoted by $\hat{x}_t$, the terminal wealth W_T, in terms of W_{t+1}, can be written as:

$$W_T = W_{t+1} \prod_{s=t}^{T-1} (\hat{x}_s' R_{s+1}^e + R^f) \tag{5.12}$$

For notational convenience, used also by Brandt et al. (2005), the portfolio returns from $t+1$ to the end of the investment horizon T, under the optimal portfolio allocation strategy, are defined by:

$$\psi_{t+1} = \prod_{s=t}^{T-1} (\hat{x}_s' R_{s+1}^e + R^f) \tag{5.13}$$

So that the terminal wealth W_T can be written as $W_T = W_{t+1}\psi_{t+1}$.

As stated above, assuming that all the future portfolio weights are already known, the maximizer disappears.

By substituting the above equation in the objective equation, the value function turns out to be:

an application to learning about return predictability." The Review of Financial Studies 18.3 (2005): 831-873.

$$J_{t+1}(W_{t+1}, Z_{t+1}) = \mathbb{E}_{t+1}[u(W_{t+1}\psi_{t+1})] \tag{5.14}$$

Computing the partial derivative of the previous expectations towards the first variable, which is W_{t+1} in the above, it can be stated that:

$$\partial_1 J_{t+1}(W_t, Z_{t+1}) = \mathbb{E}_{t+1}[\partial u(W_{t+1}\psi_{t+1})\psi_{t+1}] \tag{5.15}$$

Because u is a function of just one variable, it is possible to write ∂ instead of ∂_1.

Analogously for the second derivative of J_{t+1}:

$$\partial_1^2 J_{t+1}(W_t, Z_{t+1}) = \mathbb{E}_{t+1}[\partial^2 u(W_{t+1}\psi_{t+1})\psi_{t+1}^2] \tag{5.16}$$

5.2 Simulating sample paths

One of the most important steps of this methodology consists in the data generating model.

Monte Carlo simulation is used to generate a specific number M of independent sample paths of the vector $\{Y_s\}_{s=1}^T = \{R_s^e, Z_s)\}_{s=1}^T$:

$$Y_{t+1} = f(Y_t, Y_{t-1}, ...; \epsilon_{t+1}) \tag{5.17}$$

where ϵ_{t+1} is a random innovation.

Each sample-path generated in the Monte Carlo simulation describes a hypothetical realized evolution of the asset returns and state variable from the beginning until the end of the investment horizon.

The most common way to model the economy in econometrics is the VAR(p)-model, which is also known as a vector autoregressive model of order p.

It is an extension of the univariate autoregression model, as often provides superior forecast to those from univariate time series models, and represent a flexible and straightforward model for the analysis of multivariate time series.

It describes the evolution of a generic vector Y over the given sample horizon period from t to T as a linear function of its past evolution.

The VAR(p) model has proven to be very useful in describing the dynamics of economic and financial time series.

5.3 Backward recursion by approximating terminal wealth

Assuming that the optimal portfolio weights have already been determined and denoted by $\hat{x}_s$, it is possible to solve recursively the optimal portfolio problem backward for each date t and sample-path m.

Denoting $W_t R^f$ the current wealth growing at the risk-free rate of return and ψ_t the portfolio return generated by the optimal portfolio weights $\hat{x}_s$, the terminal wealth approximation is known and equal to:

$$\hat{W}_T = W_t R^f \prod_{s=t+1}^{T-1} (\hat{x}_s R^e_{s+1} + R^f) = W_t R^f \psi_{t+1} \tag{5.18}$$

Using the above approximation it is possible to rewrite the equation of the value function as:

$$J_{t+1}(W_t R^f, Z_{t+1}) = \mathbb{E}_{t+1}[u(\hat{W}_T)] \tag{5.19}$$

Where the maximizer disappears because the optimal future portfolio allocation has already been determined.

In order to compute the value function approximation through the Tylor expansion, it is necessary to evaluate the expressions previously defined $\mathbb{E}_t[A_{t+1}]$ and $\mathbb{E}_t[B_{t+1}]$.

The first expression is computed by substituting W_{t+1} with $W_t R^f$:

$$\mathbb{E}_t[A_{t+1}] = \mathbb{E}_t[\mathbb{E}_{t+1}[\partial u(W_t R^f \psi_{t+1})\psi_{t+1}]R^e_{t+1}] \tag{5.20}$$

That is different from the original article because $R^f \psi_{t+1}$ is used as a chain factor.

Applying the law of iterated expectations gives:

$$\mathbb{E}_t[A_{t+1}] = \mathbb{E}_t[\partial u(W_t R^f \psi_{t+1})\psi_{t+1} R^e_{t+1}] \tag{5.21}$$

Analogously for $\mathbb{E}_t[B_{t+1}]$:

$$\mathbb{E}_t[B_{t+1}] = \mathbb{E}_t[\partial^2 u(W_t R^f \psi_{t+1})\psi^2_{t+1} R^e_{t+1} R^{e'}_{t+1}] \tag{5.22}$$

Where both these equations depend on the determination of the utility function u that better describes the investor's preferences.

As the majority of literature, the utility function is assumed to be a power function, with the generic investor that is characterized by a constant relative risk aversion.

As stated in chapter 3, this is consistent with the power utility function described by $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, for $\gamma \neq 0$.

For this type of function it is straightforward to compute that $\partial u(W) = W^{-\gamma}$ and $\partial^2 u(W) = -\gamma W^{-\gamma-1}$.

Now that the utility function and his derivatives are explicated it is possible to use them to rewrite the conditional expectations of A_{t+1} and B_{t+1} as:

$$\mathbb{E}_t[A_{t+1}] = (W_t R^f)^{-\gamma} \mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^e] \tag{5.23}$$

$$\mathbb{E}_t[B_{t+1}] = -\gamma (W_t R^f)^{-\gamma-1} \mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^e R_{t+1}^{e'}] \tag{5.24}$$

Assuming $W_t \neq 0$.

By using these simplifications in the explicit expression for the approximated value of the optimal weights, it becomes:

$$\bar{x}_t = \frac{R^f}{\gamma} \{\mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^e R_{t+1}^{e'}]\}^{-1} \mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^e] \tag{5.25}$$

From the latter expression, it is possible to conclude that the optimal portfolio allocation for a generic investor with constant relative risk aversion is independent of the current or initial wealth W_t.

Moreover, assuming $\gamma = 1$, the optimal portfolio allocation is also independent of the future portfolio returns ψ_{t+1}.

The above expression is a generic step to be applied in the backward recursion, in order to determine the optimal portfolio allocation for all the timestamps in the investment horizon.

The first step of this recursion is the determination of the optimal portfolio weight at time $T-1$, defined as $\bar{x}_{T-1}$.

It is necessary to notice that at time $T-1$ the product of ψ_t is empty, so $\psi_T = 1$.

This leads to:

$$\bar{x}_{T-1} = \frac{R^f}{\gamma} \{\mathbb{E}_{T-1}[R_T^e R_T^{e'}]\}^{-1} \mathbb{E}_{T-1}[R_T^e] \tag{5.26}$$

The latter formula slightly differs from the formula found by Brandt et al. in their calculations because they lose the factor Rf.

The linear combination between the portfolio allocation in risky assets and the Risk-free gross return is counter-intuitive: if Rf increases the allocation to the risky asset will be higher, while, following the CRRA assumptions, if the Rf is higher and the Excess return remains the same, the relative volatility of the asset return compared to the mean return decreases, therefore, for an investor with CRRA this implies a higher allocation to the risky asset.

5.4 Compute expectation through regression

“Regression analysis is almost certainly the most important tool at the econometrician’s disposal”.[4]

As mentioned before, the BGSS model relies on the approximation of the two conditional expectation explicated above, to determine the optimal portfolio allocation.

The approximation process of these conditional expectations is computed by an across-path regression as the idea formulated by Longstaff and Schwartz[5] in their paper “Valuing American Option by Simulation: A Simple Least-Squares Approach”.

Their article was based on a simple approach that solves the most important problem in option pricing theory, the valuation of the optimal exercise of American-style options.

This method is a powerful alternative to the traditional approaches, since the key intuition is that the conditional expectation can be estimated from the cross-sectional information in the simulation using the least squares.

[4] Brooks, Chris. Introductory econometrics for nance. Cambridge university press, 2019.

[5] Longstaff, Francis A., and Eduardo S. Schwartz. "Valuing American options by simulation: a simple least-squares approach." The review of financial studies 14.1 (2001): 113-147.

As stated in their paper: “Specifically, we regress the ex-post realized payoffs from continuation on functions of the values of the state variables. The fitted value from this regression provides a direct estimate of the conditional expectation function”.[6]

They refer to this method as the Least Squares Monte Carlo (LSM) approach. It is straightforward to implement as nothing more than the simple least square method is required.

In the BGSS model, this approach is applied in order to approximate the conditional expectations in the portfolio optimization problem trough an across-paths regression.

This regression is employed at each timestamp on the M sample paths of the N asset returns generated in the Monte Carlo simulation, fitted in a linear model with the observations at time $t+1$ for each generic element of A_{t+1} and B_{t+1}.

Let X_t denote the matrix with the M realization of the state variables at time t at all simulation path.

For simplicity, it is assumed that in X_t there is only one state variable Z_t and a quadric polynomial is used as a basis. When more state variables are involved, as in the model that will be implemented in the following chapter, X_t grows larger as the asset returns and the cross-sections of the state variables should be involved.

With only one state variable, X_t is:

$$X_t = \begin{pmatrix} 1 & Z_{1,t} & Z_{1,t}^2 \\ 1 & Z_{2,t} & Z_{2,t}^2 \\ \vdots & & \\ 1 & Z_{M,t} & Z_{M,t}^2 \end{pmatrix} \tag{5.27}$$

[6] Longstaff, Francis A., and Eduardo S. Schwartz. "Valuing American options by simulation: a simple least-squares approach." The review of financial studies 14.1 (2001): 113-147.

Assuming that for each path m there exist a linear combination between the element in the m^{th} row of the matrix X_t and the realizations $y_{m,t+1}$:

$$y_{m,t+1} = X_{m,t}\beta + \epsilon_m \tag{5.28}$$

where ϵ_m are residuals assumed to be an i.i.d. sequence.

The aim of this regression is the determination of an approximation $\hat{\beta}$ of β.

The fitted model is:

$$y_{m,t+1} = X_{m,t}\hat{\beta} + e_m \tag{5.29}$$

where e_m are the residuals of the fitted regression.

In this methodology n regressions are performed for $\mathbb{E}_t[A_{t+1}]$ and n^2 regressions are performed for $\mathbb{E}_t[B_{t+1}]$, following the procedure explained above.

Using this regression method it is possible to approximate the value of the conditional expectations for each realization of the state variables.

Therefore, for each path m the approximate optimal portfolio allocation at each time t is given by:

$$\hat{x}_{m,t} = \frac{R^f}{\gamma}(\hat{b}_{m,t+1|t})^{-1}\hat{a}_{m,t+1|t} \tag{5.30}$$

5.5 Increase the order of the Tylor expansion

So far the BGSS was introduced using a second-order Taylor series of the value function. To adapt the model for the effect of the non-zero third and fourth moment of excess return, it is possible to increase the order of the Tylor expansion of the value function:

$$
\begin{aligned}
J_t(W_t, Z_t) = \max_{x_t} \mathbb{E}_t[&J_{t+1}(W_t R^f, Z_{t+1}) \\
&+ \partial_1 J_{t+1}(W_t R^f, Z_{t+1})(W_t x_t' R^e_{t+1}) \\
&+ \frac{1}{2}\partial_1^2 J_{t+1}(W_t R^f, Z_{t+1})(W_t x_t' R^e_{t+1})^2 \\
&+ \frac{1}{6}\partial_1^3 J_{t+1}(W_t R^f, Z_{t+1})(W_t x_t' R^e_{t+1})^3 \\
&+ \frac{1}{24}\partial_1^4 J_{t+1}(W_t R^f, Z_{t+1})(W_t x_t' R^e_{t+1})^4] \qquad (5.31)
\end{aligned}
$$

To obtain the maximum, it is necessary to take the gradient of the foregoing equation with respect to x_t and set it equal to zero:

$$
\begin{aligned}
\overrightarrow{0} = W_t \mathbb{E}_t[&\partial_1 J_{t+1}(W_t R^f, Z_{t+1}) R^e_{t+1} \\
&+ W_t^2 \mathbb{E}_t[\partial_1^2 J_{t+1}(W_t R^f, Z_{t+1})(x_t' R^e_{t+1}) R^e_{t+1}] \\
&+ \frac{1}{2} W_t^3 \mathbb{E}_t[\partial_1^3 J_{t+1}(W_t R^f, Z_{t+1})(x_t' R^e_{t+1})^2 R^e_{t+1}] \\
&+ \frac{1}{6} W_t^4 \mathbb{E}_t[\partial_1^4 J_{t+1}(W_t R^f, Z_{t+1})(x_t' R^e_{t+1})^3 R^e_{t+1}] \qquad (5.32)
\end{aligned}
$$

Because of the high orders of x_t it is possible to compute the above equation only implicitly.

Therefore, the solution of x_t is written in terms of the other orders:

$$\bar{x}_t = -\{W_t^2 \mathbb{E}_t[\partial_1^2 J_{t+1}(W_t R^f, Z_{t+1}) R_{t+1}^e R_{t+1}^{e\prime}]\}^{-1}$$
$$\times \{W_t \mathbb{E}_t[\partial_1 J_{t+1}(W_t R^f, Z_{t+1}) R_{t+1}^e]$$
$$+ \frac{1}{2} W_t^3 \mathbb{E}_t[\partial_1^3 J_{t+1}(W_t R^f, Z_{t+1})(x_t' R_{t+1}^e)^2 R_{t+1}^e]$$
$$+ \frac{1}{6} W_t^4 \mathbb{E}_t[\partial_1^4 J_{t+1}(W_t R^f, Z_{t+1})(x_t' R_{t+1}^e)^3 R_{t+1}^e]\} \quad (5.33)$$

The result of the above equation could be written as:

$$\bar{x}_t = -\{W_t \mathbb{E}_t[B_{t+1}]\}^{-1} \times \{\mathbb{E}_t[A_{t+1}] + \frac{W_t^2}{2} \mathbb{E}_t[C_{t+1}(x_t)] + \frac{W_t^3}{6} \mathbb{E}_t[D_{t+1}(x_t)]\} \quad (5.34)$$

Where $C_{t+1}(x_t)$ and $D_{t+1}(x_t)$ are:

$$C_{t+1}(x_t) := \partial_1^3 J_{t+1}(W_t R^f, Z_{t+1})(x_t' R_{t+1}^e)^2 R_{t+1}^e \quad (5.35)$$

$$D_{t+1}(x_t) := \partial_1^4 J_{t+1}(W_t R^f, Z_{t+1})(x_t' R_{t+1}^e)^3 R_{t+1}^e \quad (5.36)$$

For a generic investor with constant relative risk aversion preferences, it is possible to approximate the terminal wealth equation and write the above as:

$$\mathbb{E}_t[C_{t+1}(x_t)] = -\gamma(-\gamma - 1)(W_t R^f)^{-\gamma-2} \mathbb{E}_t[\psi_{t+1}^{1-\gamma}(x_t' R_{t+1}^e)^2 R_{t+1}^e] \quad (5.37)$$

$$\mathbb{E}_t[D_{t+1}(x_t)] = -\gamma(-\gamma - 1)(-\gamma - 2)(W_t R^f)^{-\gamma-3} \mathbb{E}_t[\psi_{t+1}^{1-\gamma}(x_t' R_{t+1}^e)^3 R_{t+1}^e] \quad (5.38)$$

Substituting these expression in the optimal portfolio weight formula it gives:

$$\begin{aligned}\bar{x}_t = \frac{R^f}{\gamma}\{\mathbb{E}_t[\psi_{t+1}^{1-\gamma}R_{t+1}^e R_{t+1}^{e'}]\}^{-1}\mathbb{E}_t[\psi_{t+1}^{1-\gamma}R_{t+1}^e] - \{\mathbb{E}_t[\psi_{t+1}^{1-\gamma}R_{t+1}^e R_{t+1}^{e'}]\}^{-1} \\ \times\{-\frac{1}{2}\frac{(\gamma+1)}{R^f}\mathbb{E}_t[\psi_{t+1}^{1-\gamma}(x_t' R_{t+1}^e)^2 R_{t+1}^e] \\ + \frac{1}{6}\frac{(\gamma+1)(\gamma+2)}{(R^f)^2}\mathbb{E}_t[\psi_{t+1}^{1-\gamma}(x_t' R_{t+1}^e)^3 R_{t+1}^e]\}\end{aligned} \quad (5.39)$$

Simplifying:

$$\begin{aligned}\bar{x}_t = \frac{R^f}{\gamma}\{\mathbb{E}_t[b_{t+1}]\}^{-1}\mathbb{E}_t[a_{t+1}] - \{\mathbb{E}_t[b_{t+1}]\}^{-1} \\ \times\{-\frac{1}{2}\frac{(\gamma+1)}{R^f}\mathbb{E}_t[c_{t+1}(x_t)] + \frac{1}{6}\frac{(\gamma+1)(\gamma+2)}{(R^f)^2}\mathbb{E}_t[d_{t+1}(x_t)]\}\end{aligned} \quad (5.40)$$

Calculating the conditional expectations of $C_{t+1}(x_t)$ and $D_{t+1}(x_t)$ with the same procedure used for A_{t+1} and B_{t+1}.

The above expression is an implicit expression, where the first part at the right-hand side equals the solution obtained for x_t by second-order expansion, which is written as $x_{t,0}$.

It is possible to approximate the solution above by an iteration process, known as the Newton method, that follows:

$$\begin{aligned}\tilde{x}_{t,i} = \tilde{x}_{t,0} - \{\mathbb{E}_t[b_{t+1}]\}^{-1}\times\{-\frac{1}{2}\frac{(\gamma+1)}{R^f}\mathbb{E}_t[c_{t+1}(\tilde{x}_{t,i})] \\ + \frac{1}{6}\frac{(\gamma+1)(\gamma+2)}{(R^f)^2}\mathbb{E}_t[d_{t+1}(\tilde{x}_{t,i})]\}\end{aligned} \quad (5.41)$$

Then it is possible either to choose to end the iteration process after a fixed number of iterations or to continue the iteration process until the difference between the two subsequent values is less than a specified tolerance.

5.6 Imposing constraint on the portfolio weights

Empirically, almost all investors face constraints on the portfolio allocation process. If the investor does not have the opportunity to borrow money or go short on the securities, this means that $0 \leq x_t \leq 1$.

Once that constraints are applied, “the portfolio weights is bounded and, therefore, the error is bounded as well”.[7]

Imposing constraints with only one risky asset is straightforward and does not include any particular problem in the implementation.

At each timestamp of the recursive process the weights of the tied portfolio are:

$$x_t^{constr} = max(0, min(x_t^{unconstr}, 1)) \tag{5.42}$$

In the multiple risky asset case, the $x_{i,t}^{constr}$ calculation involves more complex calculations: the constrained optimal portfolio allocation problem does not follow the same equation as the single risky asset equation. It is instead necessary to use an optimization algorithm for each timestamp t and sample-path m, which increase significantly the computation time.

The mathematical optimization deals with the problem of finding numerically minimum or maximums of an objective function.

Within this thesis the SciPy.optimize package in Python is used with the constrained algorithm for multivariate scalar functions.

With a second order expansion the objective function is :

[7] Van Binsbergen, Jules H., and Michael W. Brandt. "Solving dynamic portfolio choice problems by recursing on optimized portfolio weights or on the value function?." Computational Economics 29.3-4 (2007): 355-367.

$$\max_{x_t} x_t'\mathbb{E}_t[A_{t+1}] - \frac{1}{2}\frac{\gamma}{R^f}x_t'\mathbb{E}_t[B_{t+1}]x_t$$
$$s.t.: 0 \leq x_t \leq 1 \tag{5.43}$$

Higher order Tylor expansions cannot be computed directly with this routine, because x_t is inside the conditional expectations of C_{t+1} and D_{t+1}.

Therefore, the second-order Tyler series is used to find an initial solution that is used as the guess solution in the iteration procedure afterwards.

Due to the difficulties to apply the short-sale and borrowing constraint to the fourth-order Tyler series, in the multi-asset environment, the solution obtained by the second-order series is used as the optimal constrained allocation.

Chapter 6

Implementation

This chapter aims to illustrate the analysis conducted in this thesis, where the BGSS model is implemented using monthly data on prices and yields of certain variables.

The examined yields were calculated assuming continuous capitalization from monthly historical price series.

The U.S. Bond yields for different maturities, the U.S. 10-years break-even inflation and the S&P 500 stock market index time-series have been downloaded from the "Quandl" website.

The Treasury Bond yields refer to the monthly quotations of the U.S. Treasury Bills with a 3-month constant maturity and the monthly quotations of the 10-Year U.S Treasury Bond Rate with constant maturity.

Subsequently, the European government Bond benchmark with a duration of 10 years, the MSCI World index and the other assets considered have been derived from "Thomson Reuters Eikon".

The analysed sample consists of 108 monthly data, covering the period from January 2009 to January 2018. It was chosen to examine this 9-year interval to assess the dynamics of equity returns over a fairly long period, without incurring in the extreme global financial markets stress caused by the global financial crisis in 2007.

January 2018 was chosen as the endpoint of this study to backtest the model's outcome in the following two years, even though the Covid-19 financial crisis of 2020 leads to underestimating the results of this model.

6.1 VAR-Model

As stated before, the most common method to model the economy in econometrics is the VAR(p)-Model, that is known as a vector autoregressive model and describe the evolution of a vector y_t over the same sample period $t = 1, ..., T$, as the linear function written as:

$$y_t = c + A_1 y_{t-1} + A_2 y_{t-2} + ... + A_k y_{t-p} + \epsilon_t \tag{6.1}$$

With c that represents the intercept, A_k are the coefficient assigned to each past vector realization y_{t-k} and ϵ_t that represents a vector of errors terms $\epsilon_{i,t}$ satisfying:

- $\mathbb{E}(\epsilon_t) = 0$
- $\mathbb{E}(\epsilon_t \epsilon_t') = \Sigma$
- $\mathbb{E}(\epsilon_t \epsilon_{t-k}) = 0 \; \forall k \neq 0$

In other words, all the error terms $\epsilon_{i,t}$ have mean equal to 0, its contemporaneous matrix Σ is constant in time and there is no correlation between the errors therms across time.[1]

In this thesis, a VAR(1) is used to model the economy with log excess return of the risky assets and state variables.

[1] Brockwell, Peter J., Richard A. Davis, and Matthew V. Calder. Introduction to time series and forecasting. Vol. 2. New York: springer, 2002.

The VAR(1)-model implemented by Brandt et al. (2005)[2] with a risk-free asset, only one risky asset and one state variable, that in this example is the log dividend yield, is given by:

$$\begin{bmatrix} r^e_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} 0.227 \\ -0.155 \end{bmatrix} + \begin{bmatrix} 0 & 0.060 \\ 0 & 0.958 \end{bmatrix} \begin{bmatrix} r^e_t \\ z_t \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \end{bmatrix} \tag{6.2}$$

where the innovation are bi-normally distributed:

$$\begin{bmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \end{bmatrix} \sim N(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.0060 & -0.0051 \\ -0.0051 & 0.0049 \end{bmatrix} \tag{6.3}$$

These coefficients are equal to those used by Brand et al. (2005) in their model with a quarterly basis on the value-weighted CRSP Index. This example implies an almost perfect correlation $\rho = -0.95$ between the excess asset return and the state variable considered.

In this implementation the gross asset excess returns R^e_t are used, while in the VAR-model the log excess returns r^e_t are used.

To calculate R^e_t from the log return the following transformation is applied:

$$R^e_t = R^f(e^{r^e_t} - 1) \tag{6.4}$$

It is necessary to specify that the evolution of the state variables and the assets return considered in the multi-asset model are calculated through a VAR(1)-Model with the same specifications described above, involving multiple risky assets and a larger number of state variables.

[2]Brandt, Michael W., et al. "A simulation approach to dynamic portfolio choice with an application to learning about return predictability." The Review of Financial Studies 18.3 (2005): 831-873.

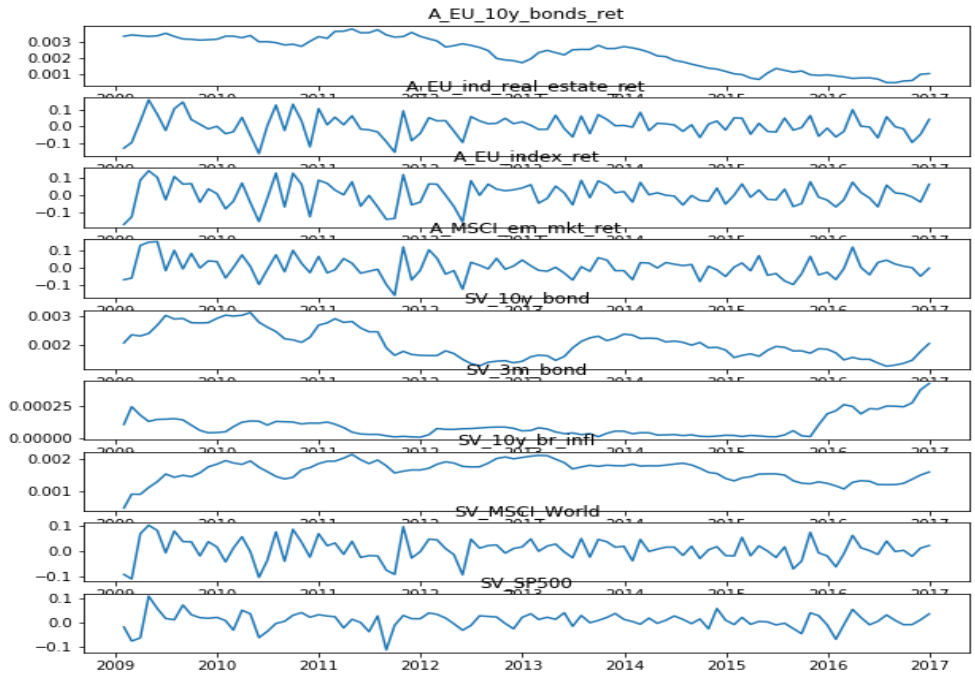

Figure 6.1: Multiasset VAR(1) input

Figure 6.1 illustrates all the time series used as inputs for the VAR(1)-Model in the multi-asset case, where the first four time-series are the asset returns historical values, while the latter five series are the state variables returns historical values.

As stated above, the VAR model is useful when the goal is to predict multiple time series variables using a single model. This model is an extension of the univariate autoregressive model to k time series regressions, where the lagged vectors of all the k series appear as regressors.

Hence, in the VAR model, each variable is modeled as a linear combination of the past values of itself and the past values of the other variables in the system.

The VAR model implemented in this thesis assumes that the passed time series of the assets and state variables returns are stationary. It is assumed that the non-stationary time series can often be transformed to be stationary by first-differencing or some other methods, because the VAR model is not appropriate for direct analysis of non-stationary time series.

In statistics, time series are stationary if they are not dependent on time. That is, the mean or the variance of the observations are consistent over time.

The Dickey-Fuller Test is a statistical test that can help determine whether a process is stationary or not. It tests the null hypothesis that a unit root is present in an autoregressive model. The alternative hypothesis states that the unit root is not present and stationarity exists.

To implement this test in Python the statsmodels.tsa.stattools module is used with the adfuller function, which performs the Augmented Dickey-Fuller Test Statistical Test, returning the p-value and the value of the test statistic.

The test statistic value should be negative and the p-value should be beneath a threshold value to reject the null hypothesis.

Using a 5% confidence level, If the p-value is less than 0.05 the null hypothesis is rejected and the time series is considered to be stationary, while if the p-value is greater than 0.05, it is not possible to reject the null hypothesis because the data has a unit root and is non-stationary.

Table 6.1 presents the results of the test on the assets and state variables returns considered in this thesis.

This table shows that all the p-values are less than the threshold value, except for the US 10 years break-even inflation that is slightly more than 0.06. Hence, it is possible to state that the time series considered are stationary.

	ADF	p-value
EU 10y Bonds	-8.54725	$9.42983e^{-14}$
EU ind RE	-9.57857	$2.18949e^{-16}$
EU index	-8.48925	$1.32718e^{-13}$
MSCI EMM	-5.03002	$1.93495e^{-05}$
US 10y Bonds	-6.96449	$8.98789e^{-10}$
US 3m Bills	-7.44679	$5.80987e^{-11}$
US 10y infl	-2.74097	0.0672131
MSCI World	-9.75107	$8.01345e^{-17}$
S&P 500	-8.20327	$7.13549e^{-13}$

Table 6.1: Augmented Dickey-Fuller Test results

After the confirmation of the stationarity of the time series, it is straightforward to estimate the VAR model.

In Python, a feasible approach is to use statsmodel.tsa.vector_ar.var_model to estimate individual equations. It contains methods that are useful for simultaneously modeling and analysing multiple time series.

For this implementation the VAR(1) model is processed and each variable is modeled as a linear combination of the past values of all the variables considered. The output is a system of 9 equations, one for each variable.

Table 6.2 shows the coefficients of the estimation for each variable. The sign of the coefficients indicates the direction of the relationship between term and response, while the size of the coefficients indicate the significance of the effect that a term has on the response variable.

These coefficient have the same interpretation as the coefficients in any regression.

The standard errors of the coefficients measure the precision of the estimation. The smaller the standard error, the more precise the estimate.

Due to the difficulties to interpret the large number of coefficient, the interest is not in these coefficients, but rather on the dynamic properties of the whole model.

In order to estimate the dynamic properties of the whole model it is important to focus on the Impulse Response Functions (IRFs) of the VAR model, which provide the cumulative total derivatives of the endogenous variables with respect to an exogenous shock to one of the variables i.e. tracks the impact of any variable on the others in the system. Technically it is equivalent to invert the VAR representation to obtain the vector moving average representation and describes the reaction of the system as a function of time.

In Python the irf function of the statsmodel.tsa.vector_ar.var_model could be used to obtain forecast error impulse responses.

The IRFs are plotted in Figure 6.2 as a matrix 9×9 of graphs, with the impulse variable (the shock) on one dimension and the response variable on the other and with a time lag of 10 periods.

The aim is to study the interdependence between the variables through the decomposition of the dynamic system in its outputs when a brief input signal, called impulse, is presented.

If the VAR is stable, then the Impulse Response Functions should converge to zero, as the time from the impulse get larger and one-time shock should not have permanent effects.

It could be noticed as almost all the IRFs of the VAR model implemented decay to zero as the time horizon increases, underlying the stability of the model.

Only few impulse response functions decay to zero slowly, hence the shock to the variable tend to change its value for many periods.

The diagonal panels in Figure 6.2 show the effects of shocks of a variable on the same variable, while the off-diagonal panels show the effects of the shocks of a variable on all the other variables.

Table 6.3 presents the correlation matrix of the residuals derived from the VAR model. While the correlation refers to the degree and the direction

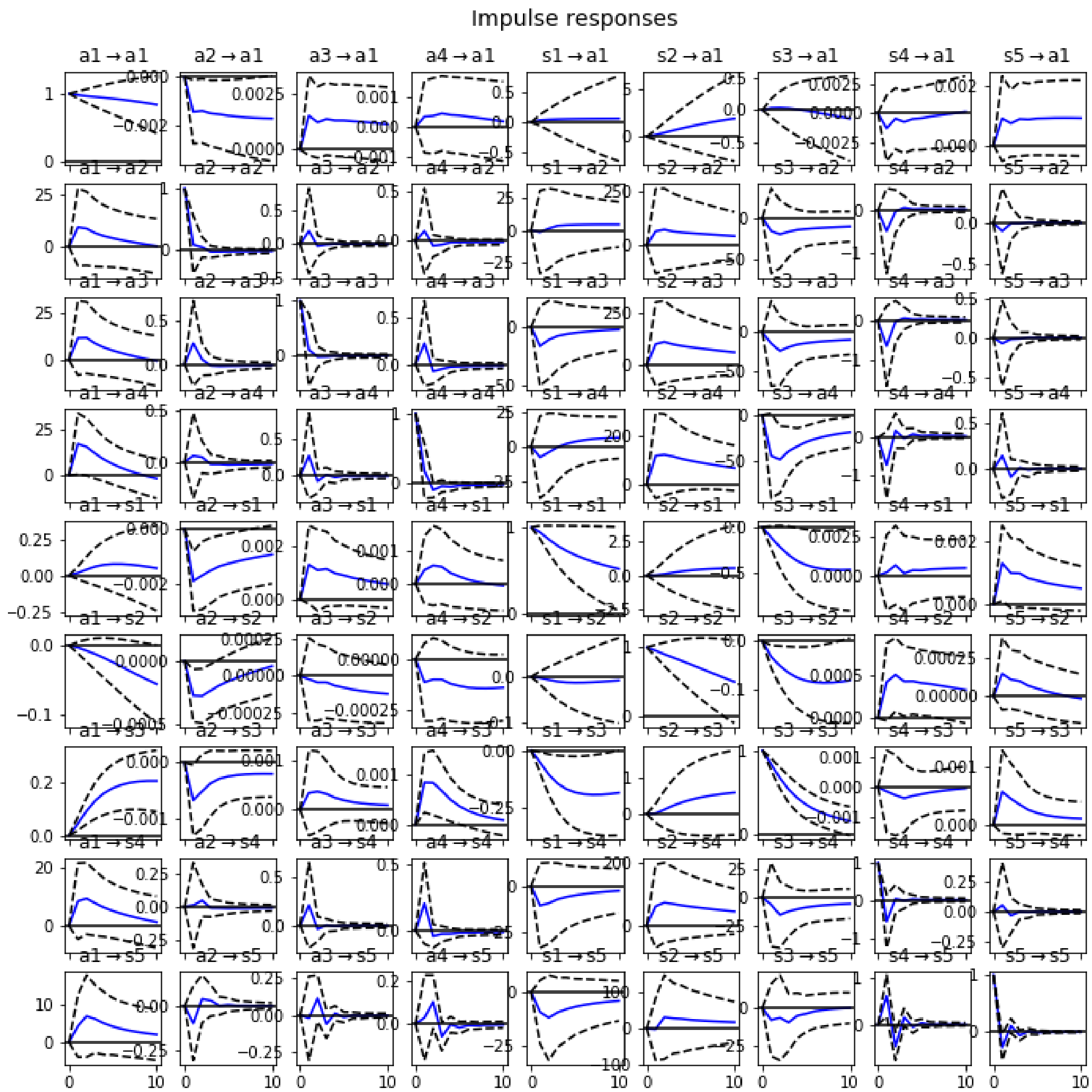

Figure 6.2: Impulse Response Analysis of the VAR model

to which a pair of variables are linearly correlated and ranges from -1.0 to $+1.0$, a correlation matrix is a square table showing the correlation coefficient between sets of variables. This matrix is symmetrical, with a line of 1 going from the top left to the bottom right.

The graphs in Figure 6.3 illustrates all the simulated paths of the scenario generation for two state variables and two risky assets.

The plotted state variables returns are derived from the US 3 month Treasury Bill monthly yields and the S&P 500 index monthly return, while the plotted assets returns are derived from the Euro area 10 year Government Benchmark Bond yields and the European Stock Index monthly return.

6.2 Set-up of the model

The BGSS model is applied to a set of four risky assets and five state variables.

The evolution of these assets and state variables is simulated with the VAR(1)-model introduced above.

The algorithm implemented assumes monthly rebalancing of the portfolio weights, which is as often as the scenario model permits.

The main constraints of the model are both the borrowing and short-sale constraints.

A polynomial of degree two is used as a basis for the regression, including the cross-terms of the state variables.

The four asset classes considered are:

	Constant	EU 10y Bonds L1	EU ind RE L1	EU index L1	MSCI EMM L1	US 10y Bonds L1	US 3m Bills L1	US 10y Inflati L1	MSCI World L1	S&P 500 L1
EU 10y Bonds	-0.00011 (0.00014)	0.97911 (0.0263)	-0.00141 (0.00066)	0.00151 (0.00090)	0.00034 (0.00062)	0.03556 (0.04632)	0.21239 (0.27456)	0.0330 (0.07399)	-0.00127 (0.00142)	0.00089 (0.00073)
EU ind RE	0.01010 (0.05046)	9.53304 (9.47510)	0.08650 (0.23785)	0.19877 (0.32421)	0.10348 (0.22256)	-1.44365 (16.62821)	65.444 (98.559)	-16.3339 (26.56121)	-0.48258 (0.51160)	-0.09178 (0.26326)
EU Index	0.02618 (0.05252)	11.5930 (9.8612)	0.23900 (0.24754)	0.10395 (0.33742)	0.22318 (0.23163)	-16.4827 (17.3058)	103.395 (102.575)	-13.7878 (27.6436)	-0.64762 (0.53244)	-0.06972 (0.27399)
MSCI EMM	0.045139 (0.04536)	17.2848 (8.5163)	0.06038 (0.21378)	0.26991 (0.29140)	0.17524 (0.20004)	-7.72891 (14.9456)	117.995 (88.5863)	-44.6072 (23.87360)	-0.73884 (0.45983)	0.14761 (0.23662)
US 10y Bonds	0.000246 (0.00011)	0.01773 (0.02219)	-0.00192 (0.00055)	0.00131 (0.00075)	0.00042 (0.00052)	0.94374 (0.03894)	0.009910 (0.23085)	-0.11086 (0.06221)	0.00020 (0.00119)	0.00131 (0.00061)
US 3m Bills	0.000068 (0.00002)	-0.00276 (0.00420)	-0.00027 (0.00010)	-0.00002 (0.00014)	-0.00011 (0.00009)	-0.00222 (0.00737)	0.95412 (0.04369)	-0.03199 (0.01177)	0.00041 (0.00022)	0.00014 (0.00011)
US 10y Infl.	0.00027 (0.00006)	0.04404 (0.01296)	-0.00067 (0.00032)	0.00034 (0.00044)	0.00084 (0.00030)	-0.05124 (0.02275)	0.01830 (0.13486)	0.83536 (0.03634)	-0.00014 (0.00070)	0.00056 (0.00036)
MSCI World	0.01605 (0.03508)	8.4345 (6.58649)	0.01083 (0.16534)	0.20721 (0.22537)	0.20546 (0.15471)	-10.4919 (11.55888)	61.2454 (68.5121)	-6.16194 (18.4636)	-0.54213 (0.35563)	0.05531 (0.18300)
S&P 500	0.02906 (0.02259)	4.3344 (4.2426)	-0.09798 (0.10650)	-0.02243 (0.14517)	0.02826 (0.09965)	-9.27636 (7.44560)	-2.26553 (44.13173)	-7.95241 (11.89329)	0.62140 (0.2290)	-0.26822 (0.11788)

Table 6.2: VAR(1) coefficients and their standard deviation

	EU 10y Bonds	EU ind RE	EU index	MSCI EMM Index	US 10y Bonds	US 3m Bills	US 10y Inflation	MSCI World Index	S&P 500 Index
EU 10y Bonds	1.000000	-0.115565	-0.076618	-0.112060	0.509575	0.046761	0.232123	-0.004408	0.094173
EU ind RE	-0.115565	1.000000	0.892328	0.728266	0.160444	-0.226293	0.186668	0.848463	0.647280
EU Index	-0.076618	0.892328	1.000000	0.794785	0.301477	-0.171631	0.254967	0.930827	0.685740
MSCI EMM	-0.112060	0.728266	0.794785	1.000000	0.125949	-0.248296	0.266498	0.850737	0.619882
US 10y Bonds	0.509575	0.160444	0.301477	0.125949	1.000000	0.107620	0.549021	0.284474	0.432451
US 3m Bills	0.046761	-0.226293	-0.171631	-0.248296	0.107620	1.000000	0.114017	-0.187047	-0.144966
US 10y Infl	0.232123	0.186668	0.254967	0.266498	0.549021	0.114017	1.000000	0.227069	0.332240
MSCI World	-0.004408	0.848463	0.930827	0.850737	0.284474	-0.187047	0.227069	1.000000	0.771646
S&P 500	0.094173	0.647280	0.685740	0.619882	0.432451	-0.144966	0.332240	0.771646	1.000000

Table 6.3: VAR(1) Correlation matrix of residuals

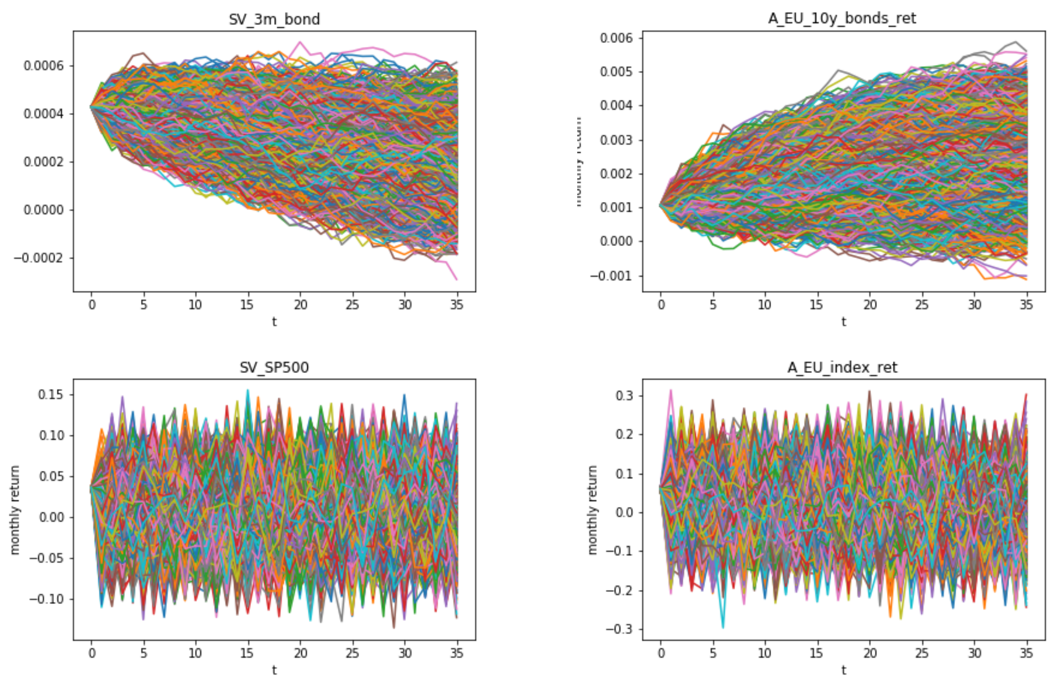

Figure 6.3: 10,000 Monte Carlo simulations of two assets and two state variables returns

	Mean R_t	Standard deviation R_t
EU 10y Bonds	0.028255	0.003455
EU ind RE	0.053131	0.215890
EU index	0.020327	0.230254
MSCI EMM	0.053788	0.202127

Table 6.4: Description of asset classes

- European government Bond benchmark with a duration of 10 years, traded monthly to keep the maturity of the portfolio constant.
- Indirect Real Estate (RE) Europe.
- Stock of European MSCI-index
- Stock of emerging markets (EMM).

All these assets are considered to be characterized by enough liquidity to be traded in each desirable amount.

The five most important drivers are selected as state variables and, even if the main assets are mostly non-American, the main drivers of the VAR-model are American:

- 3 months US nominal interest rate.
- 10 years US nominal interest rate.
- Stock of World MSCI-index.
- 10 years break-even inflation US.
- Stock of S&P500 index.

Basis statistics of these returns are in Table 6.4.

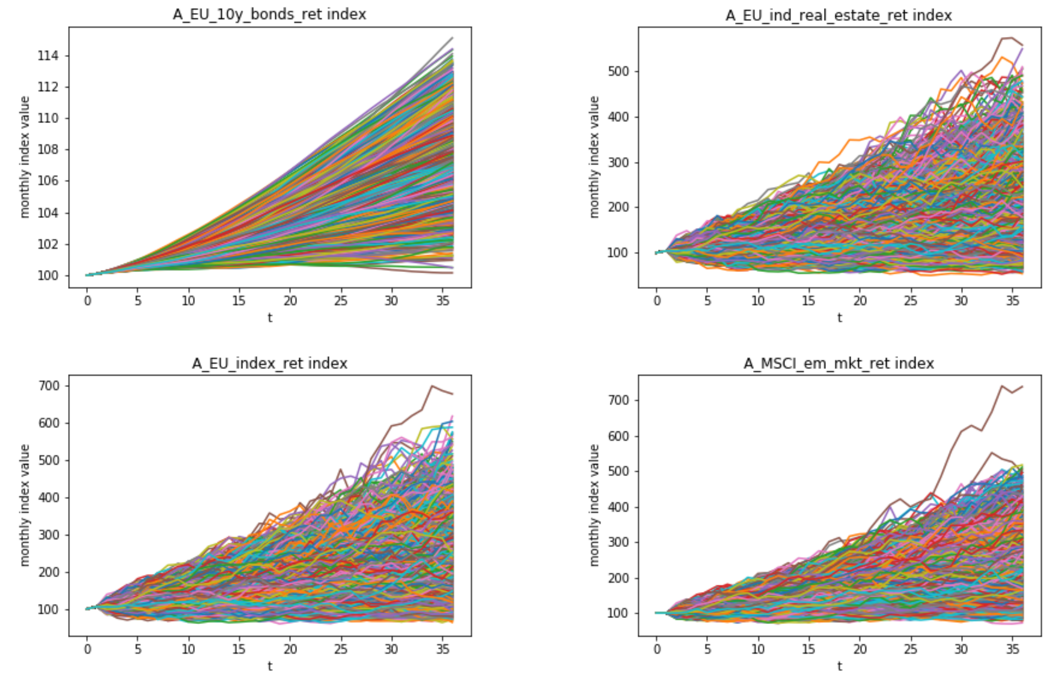

Figure 6.4: Index simulations of: European 10-years government Bond benchmark, Indirect Real Estate (RE) Europe, Stock of European MSCI-index and Stock of emerging markets (EMM).

We see that in terms of standard deviation, the bond is relatively safe, while the stocks EMM have the highest mean excess return but are also highly volatile.

In the implementation of this model, Monte Carlo simulation is used to generate a sample of 10,000 economic scenarios.

The evolution of the asset returns simulations is displayed in Figure 6.4, while the evolution of the state variables simulations are displayed in Figure 6.5. All these plots start from an initial value of 100.

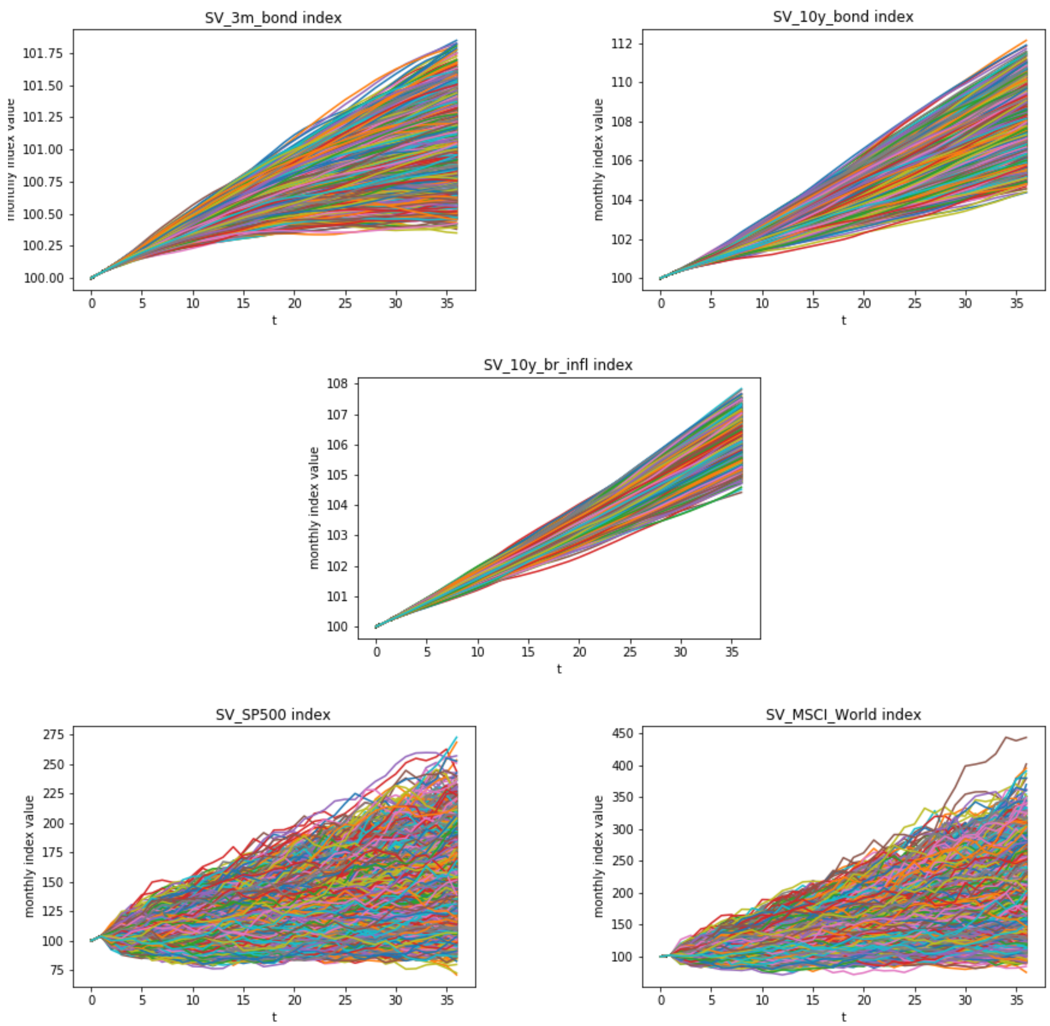

Figure 6.5: Index simulations of: 3-months US nominal interest rate, 10-years US nominal interest rate, 10 years break-even inflation US, Stock of S&P500 index and Stock of World MSCI-index.

6.3 Description of the dynamic strategy

As introduced above, the algorithm involves the generation of 10,000 simulations from the VAR-model and the application of the constrained solution method based on the second-order Tylor expansion for an investor with CRRA preferences on an investment horizon of 36 months.

While the VAR-model is used to find a model that could reflect the asset returns dynamics, the Monte Carlo simulation is involved to forecast 10,000 simulated paths for each variable.

Each path is generated from the same model, with an error term drawn randomly from a multivariate normal distribution with a vector of zeros as the mean and the covariance between the state variables and assets error terms as the covariance matrix.

All Variables are monthly generated, as in the dynamic portfolio optimization algorithm the rebalancing is assumed to be monthly.

For the regression, an ordinary least square (OLS) is used.

This method is based on the minimization of the total sum of the squares of the vertical distances from the point to the line and it is the most common method used to fit a line with the data.

A polynomial of degree two is used as a basis for the regression, where the regression matrix includes both the asset returns and the state variables cross-terms. The asset returns are considered as is assumed that the asset returns at $t + 1$ are correlated to the asset returns at t.

Under this methodology the portfolio allocation is different for each simulated path and the mean allocation is used to visualize the strategy, as well as the standard deviation.

However, behind these fluctuations, there is an upward trend in equity allocation over a long investment horizon. This trend is comparable with the results found in most literature: the longer the horizon $T - t$ is, the greater the equity allocation is.

6.4 Numerical issues

Under a second-order Tylor expansion, an horizon of 36 months, constant relative risk aversion preferences with $\gamma = 5$, OLS regression and constraints on the portfolio weights, the gains of a dynamic allocation strategy are transparent.

Unfortunately, the method does not work correctly for high values of T or γ.

An increase in the investment horizon gives peaks in the standard deviation caused by extreme portfolios allocation. The position and the magnitude of these extreme weights differ between the various simulations but their presence is persistent.

Chapter 7

Results

7.1 Mean asset allocation and performance measures

In Figure 7.1 the mean values of both the assets and the state variables simulated returns are displayed with the mean asset allocation against the remaining investment horizon.

Looking at the mean asset allocations in Figure 7.1, behind the fluctuations, it is possible to notice a trend of decreasing allocation to the risky asset, as the uncertainty about the future outcomes increases with the horizon and the optimal portfolio choice shifts to more certain allocations.

It is important to state that these figures assume a path-independent dynamic, as they are computed as the mean of all the individual dynamic strategies.

As mentioned in the previous chapter, the dynamic strategy considered in this thesis is path-dependent, therefore the optimal portfolio rebalancing at each step is dependent on each simulated path.

To underline the advantages of the dynamic strategies, it is possible to compare this strategy with the optimal mean-variance strategy and other static strategies.

To compare the performances, in Figure 7.1, with an initial wealth $W_1 = 100$, the backtested values of wealth at time t, W_t, are plotted for the dynamic strategy implemented in this thesis, for the myopic mean-variance optimal strategy and for four static strategies.

In the static allocations, the investor is considered to invest fully in the asset classes or in the optimal mean-variance portfolio, while in the dynamic strategy it is considered to monthly rebalance its investment following the mean allocations suggested by the dynamic algorithm.

The bottom-left graph of Figure 7.1 shows the backtest of the dynamic strategy, where it could be noticed that, even with the financial effects of the COVID-19 recession, the dynamic strategy terminal wealth is always higher than the wealth generated by the static strategies at every time step, slightly underperforming the strategy that invests fully in indirect European Real Estate at the end of the investment horizon.

The bottom-right graph of Figure 7.1 compares the backtested wealth values of the Dynamic Strategy with the Myopic mean-variance optimization strategy values, highlighting the gains of the Multiasset over the Myopic static strategy.

The difference between the dynamic and myopic policies is called hedging demand. It arises when, deviating from the one-period optimal portfolio choice, the investor tries to hedge against changes in the investment opportunities.

The hedging demand is defined as the "demand for financial securities that are used to diversify or reduce risk beyond normal mean-variance diversification".[1]

As stated in Chapter 2, classical Myopic portfolio selection finds portfolio weights based on first and second moments, assuming that investment opportunities are constant or returns are independently distributed over time

[1]Hedging Demand. (n.d.) Farlex Financial Dictionary. (2009).

(IID).

The myopic solution does not take into account events beyond the current period, while long-term investment problems focus on finding portfolio weights with variable investment opportunities over several periods. Thus, the multi-period investor's portfolio differs from the single-period investor due to the hedging demand.

As stated by Campbell and Viceira: “While the myopic assumption allows analytical tractability and abstracts away from dynamic hedging considerations, there is growing evidence that intertemporal hedging demands may comprise a significant part of the total risky asset demand”.[2]

As a result, investors will hold lower related assets in the current period to cover the possibility of lower expected returns in future periods.

The economic role of the hedging demands is straightforward: when the stock return is negatively related to the anticipated portfolio gains, the gains in one offset the losses in the other, leading to a lower variability of wealth.

Figure 7.2 illustrates the intertemporal mean hedging demand for stocks and bonds in the dynamic strategy implemented, as the means of the differences between the dynamic optimal portfolio allocations and the Myopic mean-variance optimal weights, shown in Table 7.1.

The main results that could be observed are the decreasing trend of the hedging demand for MSCI Emerging Markets Stocks and the increasing trend of the hedging demand for European Government Bonds.

The ratio behind these trends is that when the investment horizon increases, the hedging demand decreases for the riskier assets and increase for the safer assets due to the increased uncertainty about future returns.

Basic statistics of the overall hedging demands are shown in the table.

[2]Campbell, John Y., and Luis M. Viceira. "Strategic asset allocation." Book Manuscript, Harvard University, November (2000).

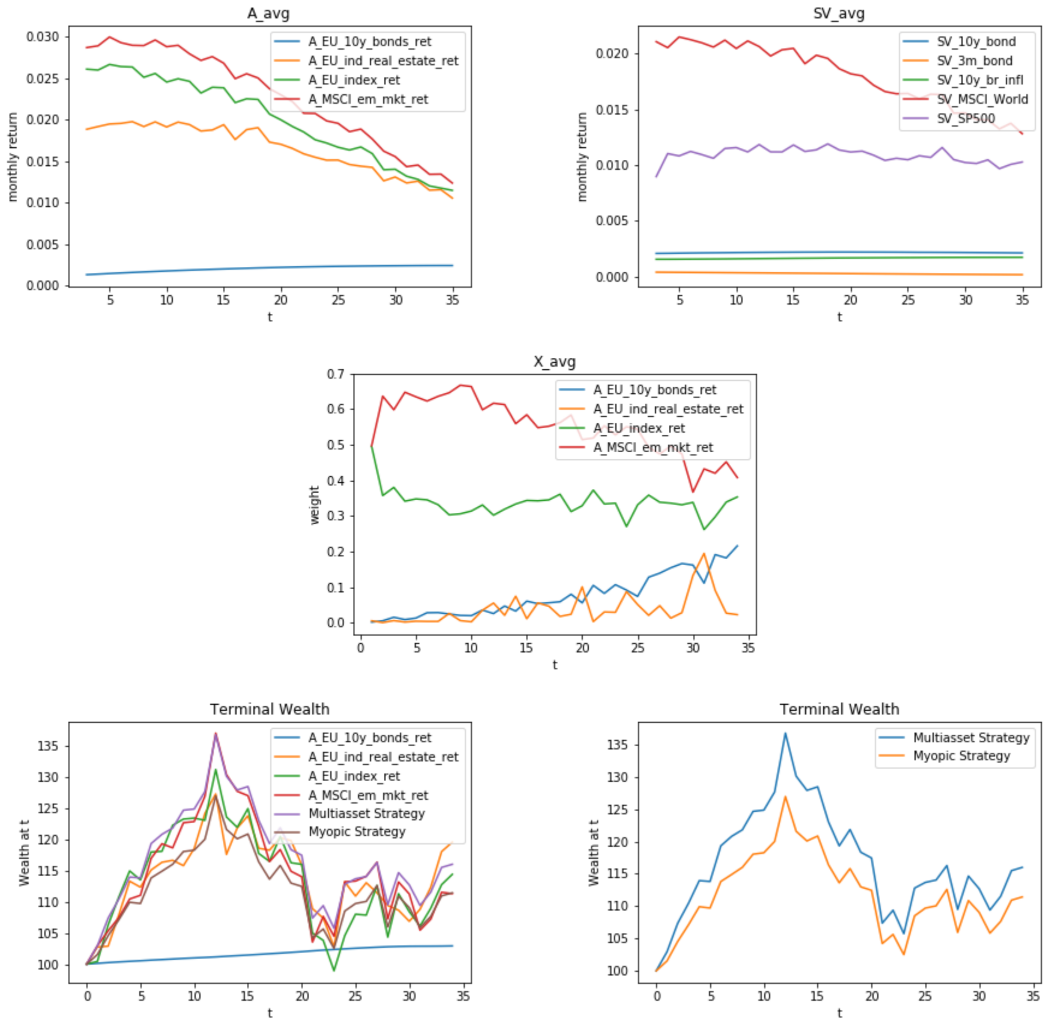

Figure 7.1: Asset's mean returns, State Variable's mean returns, mean values of the portfolio allocations and Terminal Wealth backtested values following the Dynamic Strategy, the static strategies and the myopic mean-variance strategy.

Asset Classes	EU 10y Bonds	EU ind RE	EU index	MSCI EMM
Mean	7.456%	4.028%	32.168%	56.348%
Standard Deviation	5.953%	5.226%	7.567%	11.112%

(a) Basic statistics of the dynamic strategy overall allocations

Asset Classes	EU 10y Bonds	EU ind RE	EU index	MSCI EMM
Weights	20.0%	2.8%	33.3%	43.9%

Table 7.1: Mean-Variance Myopic optimal portfolio weights

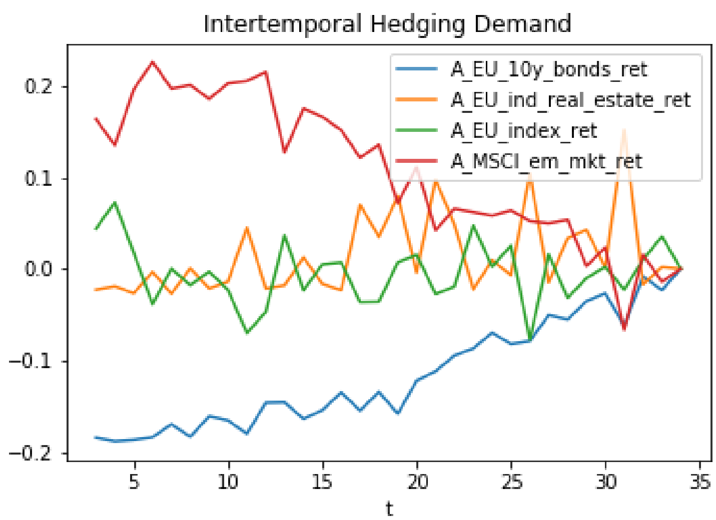

Figure 7.2: Intertemporal hedging demand

	Mean	Standard Deviation
EU 10y Bonds	-0.122813	0.0606158
EU ind RE	0.0105134	0.0439359
EU index	-0.0053567	0.0767072
MSCI EMM	0.117656	0.109402

(a) Basic statistics of the hedging demand

7.2 Gains of the dynamic strategy

In this thesis, a realistic multi-asset investment problem for an investor who has access to several asset classes is solved.

Despite the investor has to comply with borrowing and short-sale constraints, the gains of a dynamic strategy are clear.

One of the most important features of the dynamic strategy is path-dependency, not expressed it the previous figures because of the path-independent gains computed as the mean of all individual dynamic strategies.

In order to consider the path-dependency of the dynamic strategy, the histogram of the terminal values W_T, forecasted by the dynamic strategy, with an initial wealth of $W_1 = 100$, is shown in Figure 7.3 and is compared to the histograms of all the possible values of W_T, obtained by the myopic strategy and by the four static strategies of fully investing in the single risky assets.

The mean and the distribution of the dynamic strategy terminal wealth is high compared to the other possible static strategies.

The static strategies of investing fully in the risky assets, except for the bond asset class, show similar peaks on the distribution respect to the multi-asset dynamic strategy outcome.

The most important differences between the distributions can be noticed on the tails of the distribution in Figure 7.3, where it can be noticed that the static strategies present higher variances of terminal wealth and the distributions of the left tail values are more heavy-tailed.

To highlight this feature, a direct comparison between the static and the dynamic strategy is represented in Figure 7.4.

In Table 7.2 basic statistics of Terminal Wealth values of the four static strategies, the static strategy and the multi-asset strategy are displayed with the Sharpe Ratio and the yearly Value at Risk of each strategy with a 95% confidence level.

It is possible to see that, in terms of standard deviation, the Bond is

	W_T mean	W_T st.dev	Sharpe Ratio	VaR ($\alpha = 5\%$) YoY
EU 10y bond strategy	107.0713	1.8649	3.791785	-1.65922%
EU RE index strategy	174.4029	57.1305	1.302332	-31.2059%
EU Stocks index strategy	197.1392	64.4095	1.495063	-31.1243%
MSCI EMM strategy	217.2057	61.9733	1.729869	-27.1804%
Myopic Strategy	182.6368	41.5680	1.94128	-21.6817%
Dynamic Strategy	241.7225	68.4474	2.070531	-26.9751%

Table 7.2: Basic Statistics for six different strategies

relatively safe, while the Stocks of the EMM index and the Stocks of the European index have both greater mean returns but are also very volatile.

We can conclude that applying a dynamic strategy has clear gains over applying a static strategy.

By a path-dependent rebalancing strategy, it is possible to increase the mean portfolio return with significantly lower downside risk.

7.3 Alternative individual characteristics

This section involves the computation of the optimal strategies for investors with different individual characteristics.

Those characteristics involve different risk preferences, expressed by γ, and investment horizons, expressed by T.

First, the model is solved for more aggressive, $\gamma = 1$, and more conservative, $\gamma = 8$, individuals.

The results of these computations show that more aggressive individuals allocate on average more to the riskier assets than the benchmark investor with $\gamma = 5$. In contrast, more conservative investors shift earlier to long-term nominal bonds than the benchmark investor.

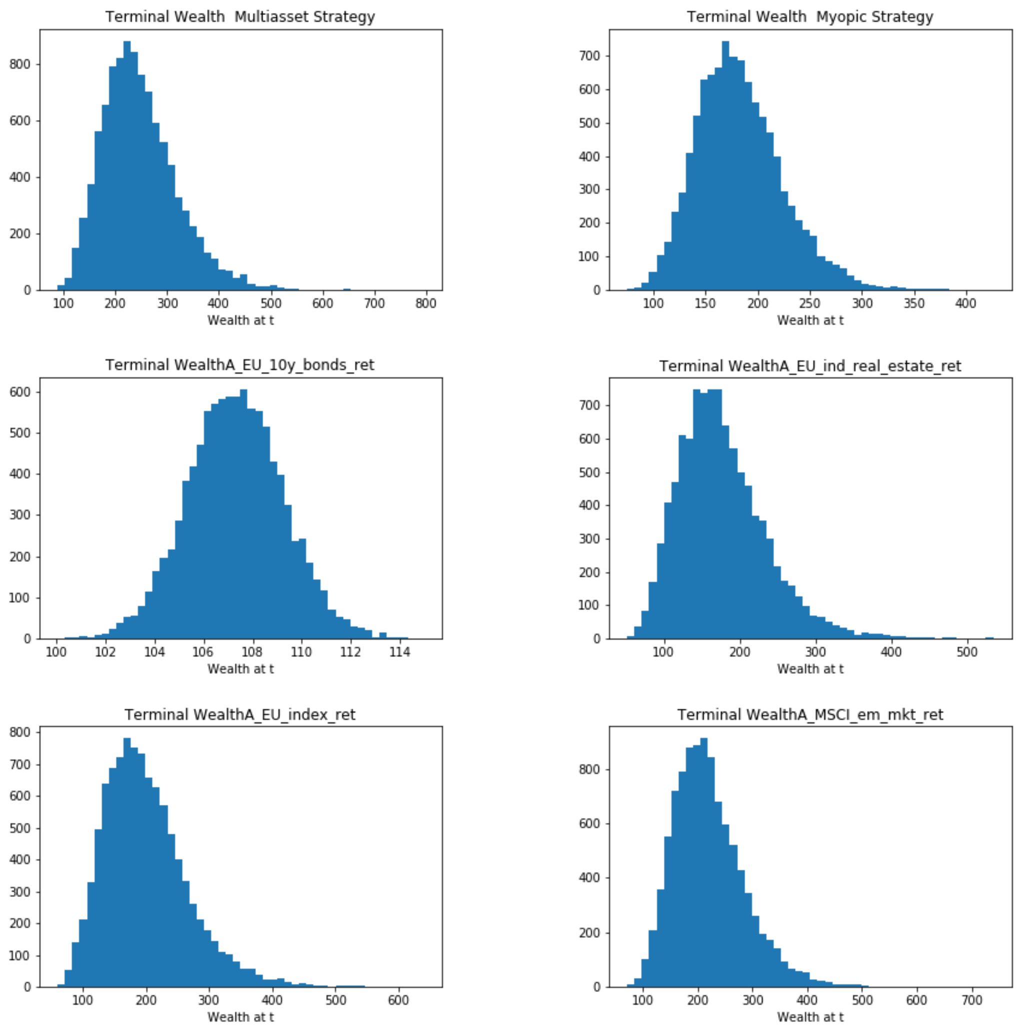

Figure 7.3: Terminal wealth values with the multi-asset dynamic strategy, Terminal wealth values with the myopic strategy and Terminal wealth values with static strategies for each of the four assets.

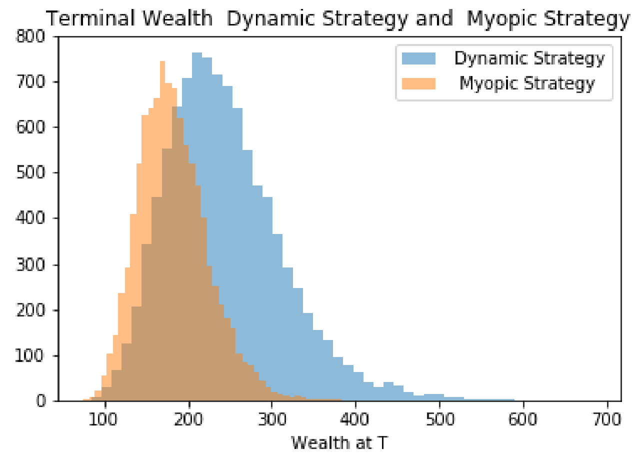

Figure 7.4: W_T for the optimal Dynamic and Myopic strategies

Comparing the asset allocation strategies for different investment horizons is more difficult than comparing different risk preferences because as the horizon increases, the method starts performing badly.

Once the combination of γ and T reaches a certain level, the regression does not work properly. The allocation to all risky assets tends to zero and the wealth is fully invested in the less risky asset, which cannot be optimal.

Investors with $\gamma = 1$ are an exception as their portfolio weights at each time does not depend on the factor $\psi_{t+1}^{1-\gamma}$.

In Figures 7.5, 7.6, 7.7 and 7.8 the distribution of terminal wealth for four values of risk aversion γ are shown for T =36, 48, 60, 72 respectively.

It is possible to notice that for lower values of γ the distribution of terminal wealth is more heavy-tailed, especially the righter tail becomes more profound, an effect which is leveraged by an increase of T.

Additionally, the tables show the values of the performance measures for the different values of risk aversion and the length of the investment horizon.

The four tables again show us the high gains of dynamic strategies.

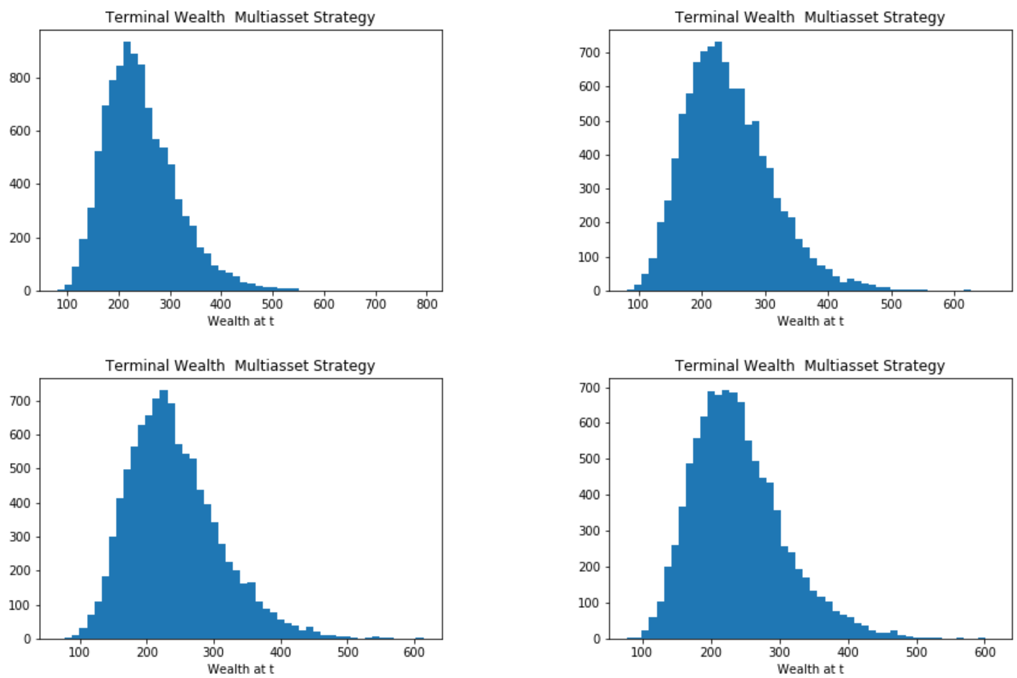

Figure 7.5: Terminal wealth distribution for $T = 36$ and γ=1,3,5,8

Risk aversion γ	W_T mean	W_T st.dev
1	243.449077	70.080151
3	242.802563	68.769423
5	241.722493	68.44743
8	240.433486	67.485338

(a) Multiasset strategy terminal wealth basic statistics for T=(36)

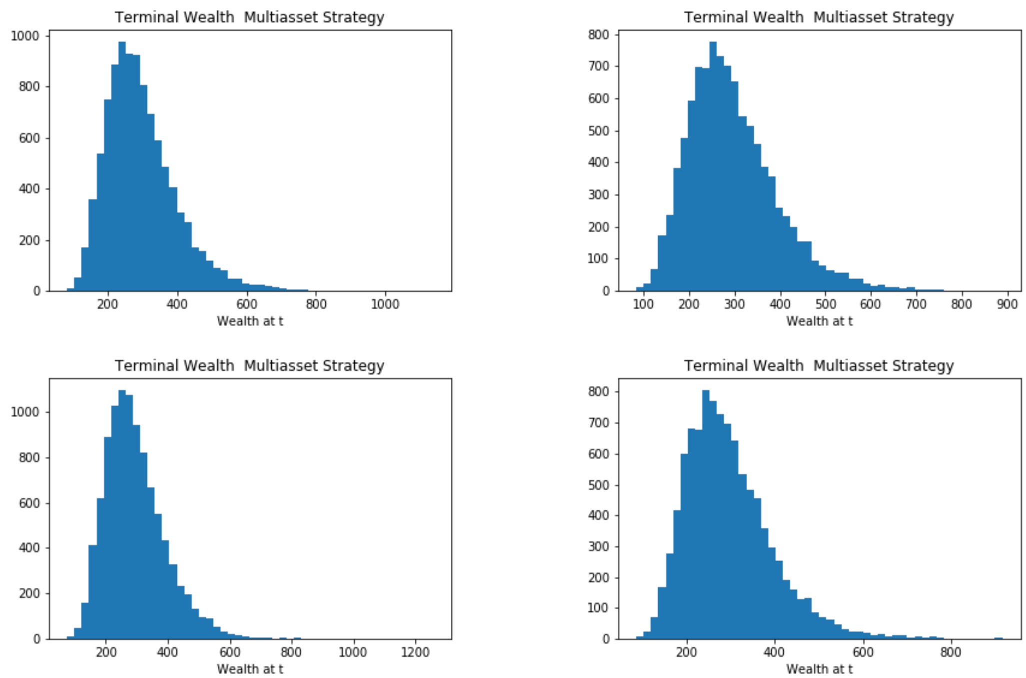

Figure 7.6: Terminal wealth distribution for $T = 48$ and γ=1,3,5,8

Risk aversion γ	W_T mean	W_T st.dev
1	296.072633	101.557279
3	295.477859	97.8137
5	294.648436	98.305707
8	295.291044	98.197249

(a) Multiasset strategy terminal wealth basic statistics for T=(48)

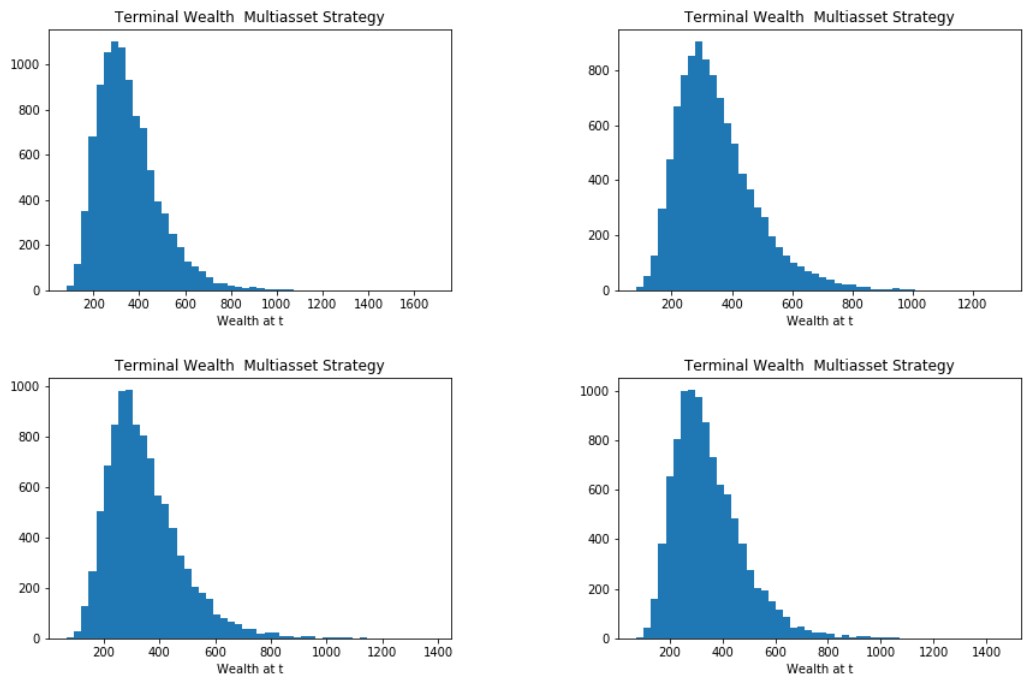

Figure 7.7: Terminal wealth distribution for $T = 60$ and γ=1,3,5,8

Risk aversion γ	W_T mean	W_T st.dev
1	353.228912	134.83212
3	350.064984	132.530982
5	347.607607	134.866769
8	348.177277	133.65403

(a) Multiasset strategy terminal wealth basic statistics for T=(60)

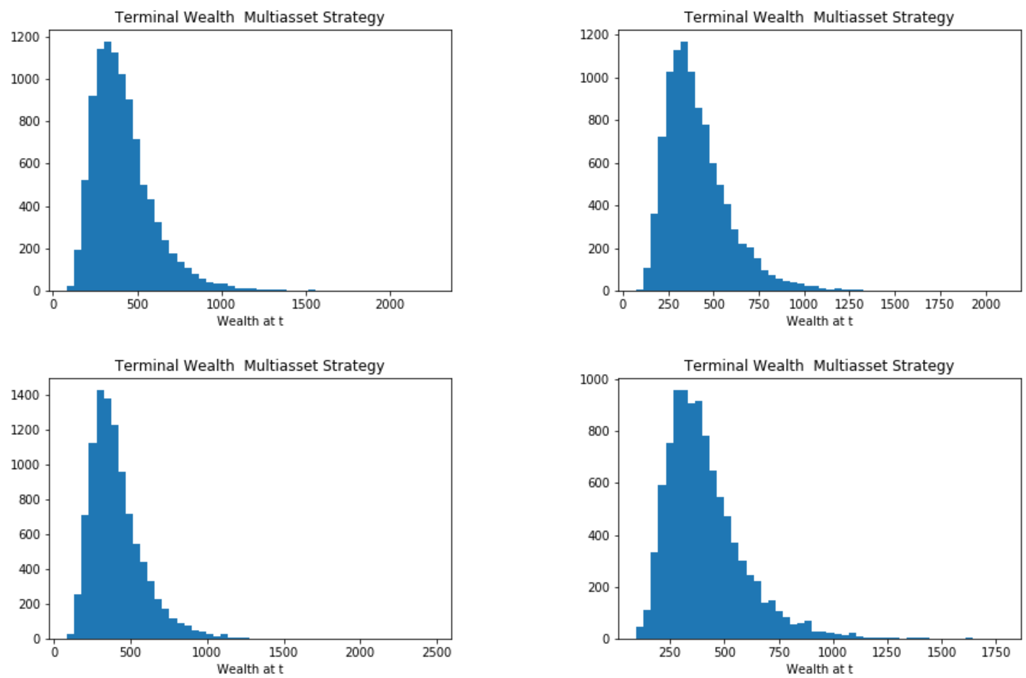

Figure 7.8: Terminal wealth distribution for $T = 72$ and γ=1,3,5,8

Risk aversion γ	W_T mean	W_T st.dev
1	416.918002	181.076248
3	413.199366	179.193595
5	412.643291	178.628045
8	412.735731	179.099352

(a) Multiasset strategy terminal wealth basic statistics for T=(72)

Chapter 8

Conclusions

In the last decades, dynamic portfolio allocation has become a popular subject of research.

Despite this interest, closed-form and analytical solutions are available only under strict assumptions of return dynamics, and numerical approaches are needed.

Nowadays several methods have been published, but almost all of them suffer from inflexibility towards the number of assets and require a very specific structure of the asset's return dynamics.

The method published by Brand et al. (2005)[1] represents an exception as they developed a simulation-based method that involves Taylor series, backward recursion and regression analysis to predict returns. The so-called BGSS method is fast, accurate and flexible in the way asset returns dynamic are modeled.

However, in their paper, they showed results for a situation with only one risky asset and one state variable.

In this thesis, the aim was to apply this method in a realistic environment of multiple assets, without strict assumptions about the return dynamics,

[1]Brandt, Michael W., et al. "A simulation approach to dynamic portfolio choice with an application to learning about return predictability." The Review of Financial Studies 18.3 (2005): 831-873.

various state variables and portfolio constraints.

With the knowledge obtained by the Brandt et al. (2005) paper, an evaluation of a more realistic environment, that consists of four risky asset classes with different mean and volatilities, is computed.

In this model, five state variables were included and were supposed to contain sufficient predicting power.

The multivariate investment environment dynamic strategy outperforms the static strategies as described in Chapter 7.

Unfortunately for low values of γ and long investment horizon, it is still possible to observe numerical issues. These issues lead to strategies that invest fully in the less risky asset, which is suboptimal.

Of course, the methodology considered in this thesis is capable of dealing with a more realistic investor's environment.

Implementation is therefore recommended, although additional research is needed.

For general future research, it would be appropriate to investigate if other regression methods are capable of reducing the numerical issues observed.

Other researches could be appropriate about the quantification of the investor's risk aversion, to find a good proxy of the exact individual's risk aversion parameter.

If the intuitions behind the methodology of this thesis are correct, this model would be a very suitable candidate for implementation.

Bibliography

[1] Arrow, K. J. "The Theory of Risk Aversion in Aspects of the theory of risk-bearing. Helsinki: yrjo Jahnssonin Saatio." Reprint in Arrow, KJ (1971). Essays in the Theory of Risk-Bearing. Chicago: Markham Publishing (1965): 28-44.

[2] Balduzzi, Pierluigi, and Anthony W. Lynch. "Transaction costs and predictability: Some utility cost calculations." Journal of Financial Economics 52.1 (1999): 47-78.

[3] Bali, Turan G. "The intertemporal relation between expected returns and risk." Journal of Financial Economics 87.1 (2008): 101-131.

[4] Barberis, Nicholas. "Investing for the long run when returns are predictable." The Journal of Finance 55.1 (2000): 225-264.

[5] Bartle, Robert G. The elements of integration and Lebesgue measure. John Wiley & Sons, 2014.

[6] Bellman, Richard. "Dynamic programming and stochastic control processes." Information and control 1.3 (1958): 228-239.

[7] Brandt, Michael W. "Estimating portfolio and consumption choice: A

conditional Euler equations approach." The Journal of Finance 54.5 (1999): 1609-1645.

[8] Brandt, Michael W., et al. "A simulation approach to dynamic portfolio choice with an application to learning about return predictability." The Review of Financial Studies 18.3 (2005): 831-873.

[9] Brennan, Michael J., Eduardo S. Schwartz, and Ronald Lagnado. "Strategic asset allocation." Journal of Economic Dynamics and Control 21.8-9 (1997): 1377-1403.

[10] Brockwell, Peter J., Richard A. Davis, and Matthew V. Calder. Introduction to time series and forecasting. Vol. 2. New York: springer, 2002.

[11] Brooks, Chris. Introductory econometrics for finance. Cambridge university press, 2019.

[12] Campbell, John Y., and Luis M. Viceira. "Consumption and portfolio decisions when expected returns are time varying." The Quarterly Journal of Economics 114.2 (1999): 433-495.

[13] Campbell, John Y., and Luis M. Viceira. "Strategic asset allocation." Book Manuscript, Harvard University, November (2000).

[14] Cass, David, and Joseph E. Stiglitz. The structure of investor preferences and asset returns, and separability in portfolio allocation: A contribution to the pure theory of mutual funds. 1970.

[15] Cochrane, John H. "A mean-variance benchmark for intertemporal portfolio theory." Manuscript, University of Chicago (2008).

[16] Dammon, Robert M., Chester S. Spatt, and Harold H. Zhang. "Optimal consumption and investment with capital gains taxes." The Review of Financial Studies 14.3 (2001): 583-616.

[17] Das, Sanjiv Ranjan, and Rangarajan K. Sundaram. "Of smiles and smirks: A term structure perspective." Journal of financial and quantitative analysis (1999): 211-239.

[18] Detemple, Jérôme, René Garcia, and Marcel Rindisbacher. "Intertemporal asset allocation: A comparison of methods." Journal of Banking & Finance 29.11 (2005): 2821-2848.

[19] Dixit, Avinash K., and John JF Sherrerd. Optimization in economic theory. Oxford University Press on Demand, 1990.

[20] Farlex Financial Dictionary. (2009).

[21] Friend, Irwin, and Marshall E. Blume. "The demand for risky assets." The American Economic Review (1975): 900-922.

[22] Garlappi, Lorenzo, and Georgios Skoulakis. "Numerical solutions to dynamic portfolio problems: The case for value function iteration using Taylor approximation." Computational Economics 33.2 (2009): 193-207.

[23] Kahneman, Daniel, and Amos Tversky. "Prospect theory: An analysis of decision under risk." Handbook of the fundamentals of financial decision making: Part I. 2013. 99-127.

[24] Kirk, Donald E. "optimal control theory: an introduction. Printice-

Hall." Englewood Cliffs, NJ 228 (1970).

[25] Kogan, Leonid, and Raman Uppal. Risk aversion and optimal portfolio policies in partial and general equilibrium economies. No. w8609. National Bureau of Economic Research, 2001.

[26] Levy, Haim, and Marshall Sarnat. "International diversification of investment portfolios." The American Economic Review 60.4 (1970): 668-675.

[27] Longstaff, Francis A., and Eduardo S. Schwartz. "Valuing American options by simulation: a simple least-squares approach." The review of financial studies 14.1 (2001): 113-147.

[28] Merton, Robert C. "An intertemporal capital asset pricing model." Econometrica: Journal of the Econometric Society (1973): 867-887.

[29] Merton, Robert C. "Lifetime portfolio selection under uncertainty: The continuous-time case." The review of Economics and Statistics (1969): 247-257.

[30] Merton, Robert C. "Optimum consumption and portfolio rules in a continuous-time model." Stochastic Optimization Models in Finance. Academic Press, 1975. 621-661.

[31] Pierse, Richard G. "Economic Forecasting Lecture 2: Forecasting the Trend."

[32] Pratt, John W. "Risk aversion in the small and in the large." Uncertainty in economics. Academic Press, 1978. 59-79.

[33] Samuelson, Paul A. "Lifetime portfolio selection by dynamic stochastic programming." The review of economics and statistics (1969): 239-246.

[34] Schooley, Diane K., and Debra Drecnik Worden. "Risk aversion measures: Comparing attitudes and asset allocation." Financial services review 5.2 (1996): 87-99.

[35] Shreve, Steven E. Stochastic calculus for finance II: Continuous-time models. Vol. 11. Springer Science & Business Media, 2004.

[36] Siegel, Jeremy J. "Stocks for the Long Run McGraw-Hill." New York (1998).

[37] Tobin, James. "Liquidity preference as behavior towards risk." The review of economic studies 25.2 (1958): 65-86.

[38] Van Binsbergen, Jules H., and Michael W. Brandt. "Solving dynamic portfolio choice problems by recursing on optimized portfolio weights or on the value function?." Computational Economics 29.3-4 (2007): 355-367.

[39] Von Neumann, John, and Oskar Morgenstern. "Theory of games and economic behavior, 2nd rev." (1947).

Appendix

Python code:

```
import numpy as np
import statsmodels.api as sm
import pandas as pd
import quandl
import datetime
from statsmodels.tsa.stattools import adfuller
from scipy.optimize import minimize
from statsmodels.tsa.vector_ar.var_model import VAR

#---------------------------------------------------------------------------#
# FUNCTIONS
#---------------------------------------------------------------------------#

con_ON = 1    #Constraint weights 0,1 comand

#---------------------------------------------------------------------------#
# DEFINE GLOBAL PARAMETERS
#---------------------------------------------------------------------------#

M=10000             #number of paths
T=36                #Horizon (monthly data)
R_f = 1.000         #Risk free rate (monthly)
gamma = 5           #Risk aversion
start = datetime.datetime(2009, 1, 1)
end   = datetime.datetime(2018, 1, 1)

#---------------------------------------------------------------------------#
# DATA
#---------------------------------------------------------------------------#

file_loc = "path.xlsx"
```

```
df = pd.read_excel(file_loc, index_col=None, na_values=['NA'])
columns = df.columns

#------------------------------------------------------------------------------#
# ASSETS
#------------------------------------------------------------------------------#

MSCI_em_mkt = df[[columns[47],columns[48]
]].set_index(columns[47])
EU_index = df[[columns[53],columns[54] ]].set_index(columns[53])
EU_10y_bonds = quandl.get("ECB/FM_M_U2_EUR_4F_BB_U2_10Y_YLD", authtoken="
    ####")/100
EU_ind_real_estate = pd.DataFrame(quandl.get("NASDAQOMX/NQEU8600",authtoken
    = "####")['Index Value'])

#------------------------------------------------------------------------------#
# STATE VARIABLES
#------------------------------------------------------------------------------#

infl_10y = quandl.get("FRED/T10YIE", authtoken="####")/100
US_bond = quandl.get('USTREASURY/YIELD',authtoken = "####")/100
SP500_index = quandl.get("MULTPL/SP500_INFLADJ_MONTH", authtoken="####")
MSCI_World = df[[columns[29],columns[30]]].set_index([columns[29]])

#------------------------------------------------------------------------------#
# DATA MODELING
#------------------------------------------------------------------------------#

A_EU_10y_bonds_ret = (np.log(EU_10y_bonds+1)/12).dropna().resample("M").mean
    ()
A_EU_ind_real_estate_ret =(np.log(EU_ind_real_estate)-np.log(
    EU_ind_real_estate.shift(1))).dropna().resample("M").sum()
A_MSCI_em_mkt_ret =(np.log(MSCI_em_mkt)-np.log(MSCI_em_mkt.shift(1))).dropna
    ().resample("M").sum()
A_EU_index_ret=(np.log(EU_index)-np.log(EU_index.shift(1))).dropna().
    resample("M").sum()
SV_10y_bond =    (np.log(US_bond['10 YR']+1)/12).dropna().to_frame().
    resample("M").mean()
SV_3m_bond =     (np.log(US_bond['3 MO']+1)/12).dropna().to_frame().resample
    ("M").mean()
SV_10y_br_infl =  (np.log(infl_10y+1)/12).dropna().resample("M").mean()
SV_SP500_index = (np.log(SP500_index)-np.log(SP500_index.shift(1))).dropna()
    .resample("M").sum()
SV_MSCI_World =  (np.log(MSCI_World)-np.log(MSCI_World.shift(1))).resample("
    M").sum(

```

```
#-----------------------------------------------------------------------#
# DATAFRAMES
#-----------------------------------------------------------------------#

frames_a =[A_EU_10y_bonds_ret,A_EU_ind_real_estate_ret,A_EU_index_ret,
    A_MSCI_em_mkt_ret]
frames_sv=[SV_10y_bond,SV_3m_bond,SV_10y_br_infl,SV_MSCI_World,
    SV_SP500_index]

df_sv = pd.concat(frames_sv, axis = 1).dropna()[start:end]
df_a = pd.concat(frames_a, axis = 1).dropna()[start:end]

asset_list = ['A_EU_10y_bonds_ret','A_EU_ind_real_estate_ret','
    A_EU_index_ret','A_MSCI_em_mkt_ret']
sv_list= ['SV_10y_bond','SV_3m_bond','SV_10y_br_infl','SV_MSCI_World','
    SV_SP500']

df_a.columns = asset_list
df_sv.columns = sv_list

frames = [df_a,df_sv]
df = pd.concat(frames,axis=1).dropna()

#-----------------------------------------------------------------------#
# DICKY FULLER TEST
#-----------------------------------------------------------------------#

def dicky_fuller(data):

    index = data.columns
    data = data.dropna().to_numpy()
    dicky_fuller_matrix= np.zeros((len(data.T),2))

    for i in range(len(data.T)):

        adf_test = adfuller(data[:,i])
        dicky_fuller_matrix[i,0]= adf_test[0]
        dicky_fuller_matrix[i,1]= adf_test[1]
       dicky_fuller_matrix = pd.DataFrame(dicky_fuller_matrix, columns = ['
           ADF','p-value'],index=index)

       return dicky_fuller_matrix

dicky_matrix_sv = dicky_fuller(df_sv)
dicky_matrix_a  = dicky_fuller(df_a)
dicky_matrix   = dicky_fuller(df)
```

```

#------------------------------------------------------------------------------#
# VAR
#------------------------------------------------------------------------------#

def VAR_df(data):
    data = data.dropna().to_numpy()
    model = VAR(data)
    solution = model.fit()

    return solution

solution_sv = VAR_df(df_sv)
solution_a = VAR_df(df_a)
solution = VAR_df(df)

params = solution.params
resid_corr = solution.resid_corr

#------------------------------------------------------------------------------#
# MONTECARLO SIMULATION
#------------------------------------------------------------------------------#

#Monte Carlo simulations
#Simulate Paths of asset log return(r) and state variable(dp)
#simulate M hypotetical sample paths of r and dp of lenght T
#These Paths are simulated from the known estimated joint dynamics of the
    returns and state variables, given by a Vector Auto Regression

#Initialize Matrices
N_asset =(int(len(df_a.T)))
N_sv=(int(len(df_sv.T)))
N_tot =N_sv+N_asset
x = np.zeros((M,T-1,N_asset))
x_0 = np.zeros(N_asset)
r =  np.zeros((N_asset,T,M))
sv = np.zeros((N_sv,T,M))
epsilon = np.zeros((M,T,N_tot))

#Initialize epsilon with random values
E_mean = np.zeros(N_tot)

E_cov = solution.resid_acov()[0,:,:]

########################simulation
#Run all the simulation M, for each timestep T
```

```

for m in range(M) :
        for t in range(T) :
            epsilon[m,t,:] = np.random.multivariate_normal(E_mean,E_cov)

for m in range(M):

    r[:,0,m] = df_a.tail(1)
    sv[:,0,m]= df_sv.dropna().tail(1)

for t in range(T-1):
    for m in range(M):
        for j  in range(N_asset):
            r[j,t+1,m]= params[0,j] +params[1:N_asset+1,j].dot(r[:,t,m])+params[N_asset+1:,j].dot(sv[:,t,m]) + epsilon[m,t,j]

        for i in range(N_sv):
            sv[i,t+1,m]= params[0,i+N_asset]+params[1:N_asset+1,i+N_asset].dot(r[:,t,m])+params[N_asset+1:,i+N_asset].dot(sv[:,t,m])+epsilon[m,t,i+N_asset]

R_e = R_f*np.exp(r)-R_f   #excess return

#-----------------------------------------------------------------------------#
# WEIGHTS SOLVER
#-----------------------------------------------------------------------------#

t=T-1

X = np.transpose(np.array([np.ones(M), r[0,t-1,:],r[1,t-1,:],r[2,t-1,:],r[3,t-1,:],sv[0,t-1,:],sv[1,t-1,:],sv[2,t-1,:],sv[3,t-1,:],sv[4,t-1,:],sv[0,t-1,:]**2,sv[1,t-1,:]**2,sv[2,t-1,:]**2,sv[3,t-1,:]**2,sv[4,t-1,:]**2,sv[0,t-1,:]*sv[1,t-1,:],sv[0,t-1,:]*sv[2,t-1,:],sv[0,t-1,:]*sv[3,t-1,:],sv[0,t-1,:]*sv[4,t-1,:],sv[1,t-1,:]*sv[2,t-1,:],sv[1,t-1,:]*sv[3,t-1,:],sv[1,t-1,:]*sv[4,t-1,:],sv[2,t-1,:]*sv[3,t-1,:],sv[2,t-1,:]*sv[4,t-1,:],sv[3,t-1,:]*sv[4,t-1,:]]))

ahat  = np.zeros((M,N_asset))
B     = np.zeros((M,N_asset,N_asset))
bhat = np.zeros((M,N_asset,N_asset))

for i in range(N_asset):
    abeta = sm.OLS(r[i,t,:],X).fit()
    ahat[:,i] = abeta.predict(X)

for m in range(M):
```

```
        B[m,:,:] = r[:,t,m][np.newaxis].T*r[:,t,m]

for i in range(2):
    for j in range(2):
        bbeta = sm.OLS(B[:,i,j],X).fit()
        bhat[:,i,j] = bbeta.predict(X)

def objective(x) :
    y1 = x[0]
    y2 = x[1]
    y3 = x[2]
    y4 = x[3]
    y = np.array([y1,y2,y3,y4])[np.newaxis]
    output= -(y.dot(ahat[m,:][np.newaxis].T)-0.5*gamma/R_f*y.dot(bhat[m
        ,:,:]).dot(y.T))
    return output

cons = ({'type' : 'eq','fun': lambda x: np.sum(x)-1})
b1 = (0,1)
bounds = [b1]*N_asset

if con_ON == 1 :

    for m in range(M):
        sol = minimize(objective,x_0,constraints=cons, bounds=bounds, method
            = 'SLSQP')
        x[m,t-1,:] = sol.x
if con_ON == 0:

    for m in range(M):

        sol = minimize(objective,x_0,constraints=cons, method= 'SLSQP')
        x[m,t-1,:] = sol.x

#SECOND PERIOD

t=T-2

X = np.transpose(np.array([np.ones(M), r[0,t-1,:],r[1,t-1,:],r[2,t-1,:],r[3,
    t-1,:],sv[0,t-1,:],sv[1,t-1,:],sv[2,t-1,:],sv[3,t-1,:],sv[4,t-1,:],sv[0,
    t-1,:]**2,sv[1,t-1,:]**2,sv[2,t-1,:]**2,sv[3,t-1,:]**2,sv[4,t-1,:]**2,sv
    [0,t-1,:]*sv[1,t-1,:],sv[0,t-1,:]*sv[2,t-1,:],sv[0,t-1,:]*sv[3,t-1,:],sv
    [0,t-1,:]*sv[4,t-1,:],sv[1,t-1,:]*sv[2,t-1,:],sv[1,t-1,:]*sv[3,t-1,:],sv
    [1,t-1,:]*sv[4,t-1,:],sv[2,t-1,:]*sv[3,t-1,:],sv[2,t-1,:]*sv[4,t-1,:],sv
    [3,t-1,:]*sv[4,t-1,:]]))
```

```
psi = np.zeros(M)

for m in range(M):
    psi[m] = (x[m,t,:].dot(r[:,t+1,m])+R_f)**(1-gamma)

for i in range(N_asset):
    abeta = sm.OLS(psi*r[i,t,:],X).fit()
    ahat[:,i] = abeta.predict(X)

for m in range(M):
    B[m,:,:] = r[:,t,m][np.newaxis].T*r[:,t,m]

for i in range(2):
    for j in range(2):
        bbeta = sm.OLS(psi*(B[:,i,j]),X).fit()
        bhat[:,i,j] = bbeta.predict(X)

if con_ON == 1 :

    for m in range(M):

        sol = minimize(objective,x_0,constraints=cons, bounds=bounds, method
            = 'SLSQP')
        x[m,t-1,:] = sol.x

if con_ON == 0:

    for m in range(M):

        sol = minimize(objective,x_0,constraints=cons, method= 'SLSQP')
        x[m,t-1,:] = sol.x

#OTHER PERIODS

t_delta=3
while t_delta<T:
    t = T-t_delta

    X = np.transpose(np.array([np.ones(M), r[0,t-1,:],r[1,t-1,:],r[2,t-1,:],
        r[3,t-1,:],sv[0,t-1,:],sv[1,t-1,:],sv[2,t-1,:],sv[3,t-1,:],sv[4,t
        -1,:],sv[0,t-1,:]**2,sv[1,t-1,:]**2,sv[2,t-1,:]**2,sv[3,t-1,:]**2,sv
        [4,t-1,:]**2,sv[0,t-1,:]*sv[1,t-1,:],sv[0,t-1,:]*sv[2,t-1,:],sv[0,t
        -1,:]*sv[3,t-1,:],sv[0,t-1,:]*sv[4,t-1,:],sv[1,t-1,:]*sv[2,t-1,:],sv
        [1,t-1,:]*sv[3,t-1,:],sv[1,t-1,:]*sv[4,t-1,:],sv[2,t-1,:]*sv[3,t
        -1,:],sv[2,t-1,:]*sv[4,t-1,:],sv[3,t-1,:]*sv[4,t-1,:]]))

```

```
    psi = np.zeros(M)

    for m in range(M):
        psi[m] = (x[m,t ,:].dot(r[:,t+1,i])+R_f)**(1-gamma)

    for i in range(N_asset):
        abeta = sm.OLS(psi*r[i,t,:],X).fit()
        ahat[:,i] = abeta.predict(X)

    for m in range(M):
        B[m,:,:] = r[:,t,i][np.newaxis].T*r[:,t,i]

    for i in range(2):
        for j in range(2):
            bbeta = sm.OLS(psi*(B[:,i,j]),X).fit()
            bhat[:,i,j] = bbeta.predict(X)

    if con_ON == 1 :

        for m in range(M):

            sol = minimize(objective,x_0,constraints=cons, bounds=bounds,
                method= 'SLSQP')
            x[m,t-1,:] = sol.x

    if con_ON == 0:

        for m in range(M):

            sol = minimize(objective,x_0,constraints=cons, method= 'SLSQP')
            x[m,t-1,:] = sol.x

    t_delta = t_delta+1
```

Summary Report

Abstract

Title:	Dynamic Portfolio Optimization: a simulation and regression approach applied to a multi-asset portfolio choice problem
Academic year:	2019-2020
Course:	Corporate Finance - Chair in Asset Pricing
Author:	Luigi Pretti - Matr. 711381
Supervisor:	Prof. Paolo Porchia
Co-Supervisor:	Prof. Alfio Torrisi
Key words:	Asset allocation, dynamic programming, regression analysis, dynamic portfolio choice.
Purpose:	In this thesis an innovative approach to the multi-asset dynamic portfolio selection is presented.

Methodology: The methodology relies on a simulation-based approach to solve realistic discrete-time portfolio choice problem involving a large number of assets with arbitrary distribution and non-standard preferences.

Statistical Software: Python

Literature review: This study is based on theories of econometrics, Asset Allocation and Dynamic Programming, as well as on previous findings about the dynamic portfolio selection.

Empirical framework: The quantitative study is based on a sample of four asset classes and five state variables, while the analysed sample consists of 108 monthly data.

Findings: The study concludes that the dynamic strategy implemented has a significant impact on the portfolio overall performances. As a result, the dynamic strategy outperforms the static strategies considered.

1 Introduction

This thesis takes its origin in the work of Michael W. Brandt, Pedro Santa-Clara and Jonathan R. Stroud, that presented a simulation-based approach to solve dynamic portfolio allocation issues.

Their article introduced a simulation and regression method that is based on the predictability of the dynamic evolution of returns by the presence of one or more state variables.

The most important feature of their study is that this method could accommodate a large number of assets, with arbitrary return distribution determined by a large number of state variables with potential path-dependency and non-stationary dynamics.

The simulation-based method implemented in this thesis is a flexible, fast and dynamic application of the method of Brandt et al. about portfolio choice problems, applied to multi-asset investment opportunities, that could accommodate both portfolio constraint and non-standard preferences.

The methodology is applied to build an algorithm that could include a realistic investor's environment and solve portfolio choice problems for long-term investment horizons, that are not only interested in the short-term expected returns and risks, but also in how they may change over time.

To construct a balanced, flexible and fast dynamic method to determine the investor's optimal portfolio allocation strategy it is necessary to argue that the traditional academic analysis of portfolio problems should be modified to accommodate the long-term investment horizons in its peculiarities and details.

The main purpose of this thesis is to implement such a method and, as people continuously face financial decisions, it is interesting to ask whether an investor with a long-term horizon allocates his wealth differently form the optimal short-term allocation.

2 Theory

In the financial field, the issue of portfolio choice plays a major role.

The problem to be addressed concerns the determination of the optimal portfolio allocation.

Modern financial theory about portfolio allocation probably began with the theoretical model developed in the article Portfolio Selection, published in 1952 by Harry Markowitz. In his paper, he outlined a framework for static optimal portfolio allocation based on the Mean-Variance analysis.

Despite the validity of this model, it is developed purely on a theoretical level and is based on some strict assumptions that diminish its value.

The main assumption of the static model is the invariance of the expected returns and volatilities, based on the idea that the characteristics of the assets and their composition should not change over time.

Empirically portfolio choices depend on a great number of factors such as investor preferences, availability of securities in the market and expected returns of assets and risk. All these factors become more relevant for investors with a long-term horizon.

The Portfolio Theory of Markowitz is based on the Perfect Market Assumptions, according to which the investor's environment is an entirely efficient economy, in terms of both equal access to information and rational economic actors.

The striking conclusion of this analysis is that, under the assumption of homogeneous expectations between the market participants, all the investors will hold the same portfolio of risky assets, the Market Portfolio.

In recent years, the attention of economists has been focused on models for the portfolio allocation that could reflect the investor's situation and characteristics.

The first authors to contribute to the modern literature about Dynamic Portfolio Choice were Merton and Samuelson.

Both their theories are based on the idea that means and variances of asset returns are time-varying and do not remain fixed over time, as they change in response to economic conditions.

The Dynamic Portfolio baseline is that the construction of an efficient investment strategy must include protection against the fluctuations of the first and second moments of asset returns.

Despite the continuous-time formulation of Merton and Samuelson represent an efficient analytical approximation of the portfolio choice problem, an investor will not continuously trade during his investment horizon, but it is more empirically correct to consider the portfolio rebalancing over discrete timestamps.

The first step of the discrete dynamic methodology considered in this thesis is the simulation of a large number of sample paths of asset return and state variables, through their known or estimated joint dynamics.

The problem of portfolio selection is then addressed recursively in standard dynamic programming fashion.

Starting from $T-1$, for each simulated path, the optimal portfolio allocation is computed as the weights that maximize a Tylor expansion of the investor's value function.

This problem has a straightforward semi-closed form solution that involves conditional moments of the value function, its derivatives and asset returns. These conditional expectations are calculated through ordinary least square regression of the realized utility, its conditional moments and asset returns at the following period based on functions of the realized state variables at $T-1$ across the simulated paths.

Then the model proceeds backward until time zero.

To summarize, this method allows evaluating the closed-form solution of the approximate optimal portfolio allocation by simulating the asset returns and the state variables paths and then computing a set of across-paths regression for each period.

3 Methodology

The starting point of the methodology of this thesis is the Bellman equation and the budget constraints introduced in the previous chapter:

$$
\begin{aligned}
J_t(W_t, Z_t) &= \max_{\{x_s\}_{s=t}^{T-1}} \mathbb{E}_t[J_{t+1}(W_{t+1}, Z_{t+1})] \\
s.t.:\ W_{s+1} &= W_s(x_s' R_{s+1}^e + R_f)\ \forall s \geq t
\end{aligned}
$$

To implement the BGSS Method the budget constraint is substituted in the one step ahead value function:

$$J_{t+1}(W_{t+1}, Z_{t+1}) = J_{t+1}(W_t(x_t' R_{t+1}^e + R_f), Z_{t+1})$$

The next step is the employment of a Taylor series of the Value function around $W_t R_f$, which lead to an explicit solution for the portfolio weights x_t:

$$
\begin{aligned}
J_{t+1}(W_t(x_t' R_{t+1}^e + R_f), Z_{t+1}) \approx\ & J_{t+1}(W_t R_f, Z_{t+1}) \\
& + \partial_1 J_{t+1}(W_t R_f, Z_{t+1})(W_t x_t' R_{t+1}^e) \\
& + \frac{1}{2}\partial_1^2 J_{t+1}(W_t R_f, Z_{t+1})(W_t x_t' R_{t+1}^e)^2
\end{aligned}
$$

Where ∂_1 denotes the partial derivative concerning the first variable of the value function.

Then the value function is substituted in the Bellman equation to obtain the approximation of the value function in t:

$$
\begin{aligned}
\bar{J}_t(W_t, Z_t) = \max_{x_t} \mathbb{E}_t[J_{t+1}(W_t R_f, Z_{t+1}) \\
+ \partial_1 J_{t+1}(W_t R_f, Z_{t+1})(W_t x_t' R^e_{t+1}) \\
+ \frac{1}{2}\partial_1^2 J_{t+1}(W_t R_f, Z_{t+1})(W_t x_t' R^e_{t+1})^2]
\end{aligned}
$$

The gradient towards x_t is taken and imposed equal to zero, in order to find the weights that maximizes the right-hand side of the value function approximation at each time t:

$$
\begin{aligned}
\overrightarrow{0} = \nabla\{\mathbb{E}_t[J_{t+1}(W_t R_f, Z_{t+1}) \\
+ \partial_1 J_{t+1}(W_t R_f, Z_{t+1})(W_t x_t' R^e_{t+1}) \\
+ \frac{1}{2}\partial_1^2 J_{t+1}(W_t R_f, Z_{t+1})(W_t x_t' R^e_{t+1})^2]\}
\end{aligned}
$$

This leads to an explicit expression for x_t, which depends on the conditional expectations.

It is denoted by $\bar{x}_t$ because it is an approximation of the true value of x_t. The explicit expression for the approximated value of the weights is:

$$
\begin{aligned}
\bar{x}_t = -\{W_t \mathbb{E}_t[\partial_1^2 J_{t+1}(W_t R^f, Z_{t+1})(R^e_{t+1} R^{e'}_{t+1})]\}^{-1} \\
\times \mathbb{E}_t[\partial_1 J_{t+1}(W_t R^f, Z_{t+1})(R^e_{t+1})]
\end{aligned}
$$

Defining:

$$
A_{t+1} := \partial_1 J_{t+1}(W_t R^f, Z_{t+1}) R^e_{t+1}
$$

$$B_{t+1} := \partial_1^2 J_{t+1}(W_t R^f, Z_{t+1}) R^e_{t+1} R^{e'}_{t+1}$$

It is possible to rewrite the explicit expression for the approximation of x_t as:

$$\bar{x}_t = -\{W_t \mathbb{E}_t[B_{t+1}]\}^{-1} \times \mathbb{E}_t[A_{t+1}]$$

As mentioned before, the BGSS model relies on the approximation of the two conditional expectation explicated above, to determine the optimal portfolio allocation.

The approximation process of these conditional expectations is computed by an across-path regression as the idea formulated by Longstaff and Schwartz in their paper "Valuing American Option by Simulation: A Simple Least-Squares Approach".

Their article was based on a simple approach that solves the most important problem in option pricing theory, the valuation of the optimal exercise of American-style options.

This method is a powerful alternative to the traditional approaches, since the key intuition is that the conditional expectation can be estimated from the cross-sectional information in the simulation using the least squares.

They refer to this method as the Least Squares Monte Carlo (LSM) approach, that is straightforward to implement as nothing more than the simple least square method is required.

In the BGSS model, this approach is applied in order to approximate the conditional expectations in the portfolio optimization problem trough an across-paths regression.

This regression is employed at each timestamp on the M sample paths of the N asset returns generated in the Monte Carlo simulation, fitted in a linear model with the observations at time $t+1$ for each generic element of A_{t+1} and B_{t+1}.

4 Implementation

The examined yields were calculated assuming continuous capitalization from monthly historical price series.

The analysed sample consists of 108 monthly data, covering the period from January 2009 to January 2018. It was chosen to examine this 9-year interval to assess the dynamics of returns over a fairly long period, without incurring in the extreme global financial markets stress caused by the global financial crisis in 2007.

January 2018 was chosen as the endpoint of this study to backtest the model's outcome in the following two years, even though the COVID-19 financial crisis of 2020 leads to underestimating the results of this model.

The BGSS model is applied to a set of four risky assets and five state variables.

The evolution of these assets and state variables is simulated with the VAR(1)-model.

The algorithm implemented assumes monthly rebalancing of the portfolio weights, which is as often as the scenario model permits.

The main constraints of the model are both the borrowing and short-sale constraints.

A polynomial of degree two is used as a basis for the regression, including the cross-terms of the state variables.

The four asset classes considered are:

- European government Bond benchmark with a duration of 10 years, traded monthly to keep the maturity of the portfolio constant.
- Indirect Real Estate (RE) Europe.
- Stock of European MSCI-index
- Stock of emerging markets (EMM).

All these assets are considered to be characterized by enough liquidity to be traded in each desirable amount.

The five most important drivers are selected as state variables and, even if the main assets are mostly non-American, the main drivers of the VAR-model are American:

- 3 months US nominal interest rate.
- 10 years US nominal interest rate.
- Stock of World MSCI-index.
- 10 years break-even inflation US.
- Stock of S&P500 index.

In the implementation of this model, Monte Carlo simulation is used to generate a sample of 10,000 economic scenarios.

Hence, the algorithm involves the generation of 10,000 simulations from the VAR-model and the application of the constrained solution method based on the second-order Tylor expansion for an investor with CRRA preferences on an investment horizon of 36 months.

While the VAR-model is used to find a model that could reflect the asset returns dynamics, the Monte Carlo simulation is involved to forecast 10,000 simulated paths for each variable. Each path is generated from the same model, with an error term drawn randomly from a multivariate normal distribution with a vector of zeros as the mean and the covariance between the state variables and assets error terms as the covariance matrix.

All Variables are monthly generated, as in the dynamic portfolio optimization algorithm the rebalancing is assumed to be monthly.

For the regression, an ordinary least square (OLS) is used.

A polynomial of degree two is used as a basis for the regression, where the regression matrix includes both the asset returns and the state variables cross-terms.

The asset returns are considered as is assumed that the asset returns at $t+1$ are correlated to the asset returns at t . Under this methodology the portfolio allocation is different for each simulated path and the mean allocation is used to visualize the strategy, as well as the standard deviation.

5 Results

In this thesis, a realistic multi-asset investment problem for an investor who has access to several asset classes is solved.

Despite the investor has to comply with borrowing and short-sale constraints, the gains of a dynamic strategy are clear.

In Figure 8.1 the mean values of both the assets and the state variables simulated returns are displayed with the mean asset allocation against the remaining investment horizon.

Looking at the mean asset allocations in Figure 8.1, behind the fluctuations, it is possible to notice a trend of decreasing allocation to the risky asset, as the uncertainty about the future outcomes increases with the horizon and the optimal portfolio choice shifts to more certain allocations.

It is important to state that these figures assume a path-independent dynamic, as they are computed as the mean of all the individual dynamic strategies.

To underline the advantages of the dynamic strategies, it is possible to compare this strategy with the myopic optimal mean-variance strategy.

The difference between the dynamic and myopic policies is called hedging demand. It arises when, deviating from the one-period optimal portfolio choice, the investor tries to hedge against changes in the investment opportunities.

The myopic solution does not take into account events beyond the current period, while long-term investment problems focus on finding portfolio

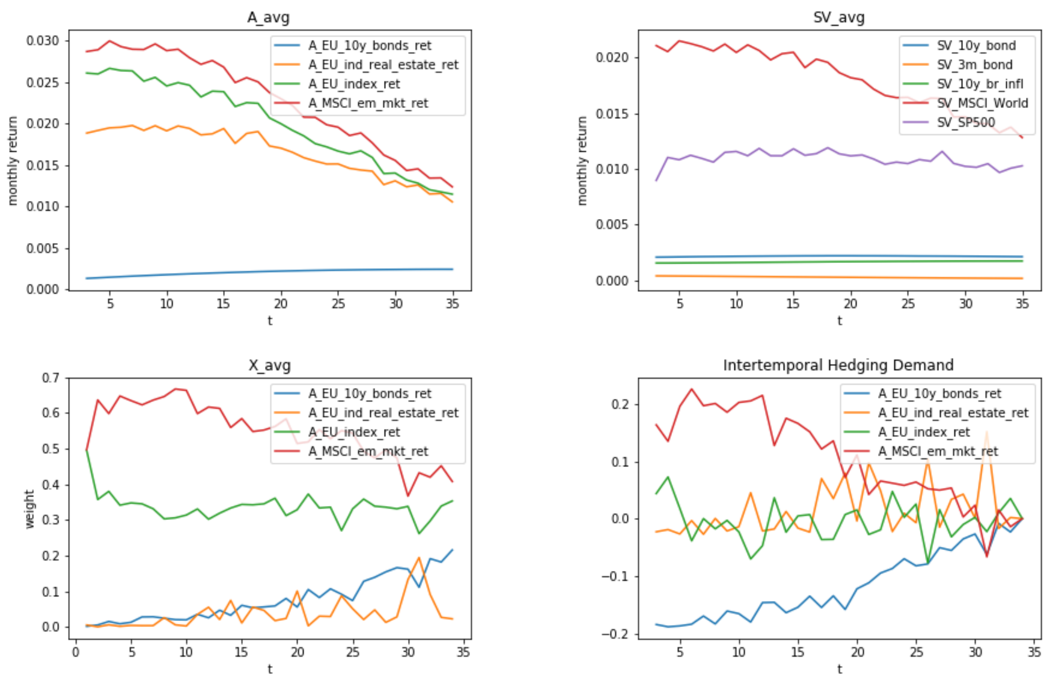

Figure 8.1: Asset's mean returns, State Variable's mean returns, mean values of the portfolio allocations and intertemporal hedging demand.

Asset Classes	EU 10y Bonds	EU ind RE	EU index	MSCI EMM
Mean	7.456%	4.028%	32.168%	56.348%
Standard Deviation	5.953%	5.226%	7.567%	11.112%

(a) Basic statistics of the dynamic strategy overall allocations

Asset Classes	EU 10y Bonds	EU ind RE	EU index	MSCI EMM
Weights	20.0%	2.8%	33.3%	43.9%

Table 8.1: Mean-Variance Myopic optimal portfolio weights

weights with variable investment opportunities over several periods. Thus, the multi-period investor's portfolio differs from the single-period investor due to the hedging demand.

As a result, investors will hold lower related assets in the current period to cover the possibility of lower expected returns in future periods.

The bottom-right graph in Figure 8.1 illustrates the intertemporal mean hedging demand for stocks and bonds in the dynamic strategy implemented, as the means of the differences between the dynamic optimal portfolio allocations and the Myopic mean-variance optimal weights, shown in Table 8.1.

The main results that could be observed are the decreasing trend of the hedging demand for MSCI Emerging Markets Stocks and the increasing trend of the hedging demand for European Government Bonds.

The ratio behind these trends is that when the investment horizon increases, the hedging demand decreases for the riskier assets and increase for the safer assets due to the increased uncertainty about future returns.

One of the most important features of the dynamic strategy is path-dependency, not expressed it the previous figures because of the path-independent gains computed as the mean of all individual dynamic strategies.

In order to consider the path-dependency of the dynamic strategy, histograms of the terminal values W_T, forecasted by the dynamic strategy for both the Dynamic and Myopic allocations, with an initial wealth of $W_1 = 100$, are shown in Figure 8.2.

In Table 8.2 basic statistics of Terminal Wealth of the five static strategies

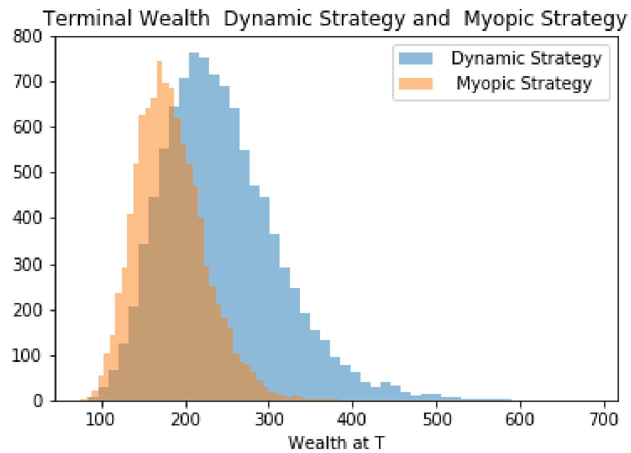

Figure 8.2: W_T for the optimal Dynamic and Myopic strategies

	W_T mean	W_T st.dev	Sharpe Ratio	VaR ($\alpha = 5\%$) YoY
EU 10y bond strategy	107.0713	1.8649	3.791785	-1.65922%
EU RE index strategy	174.4029	57.1305	1.302332	-31.2059%
EU Stocks index strategy	197.1392	64.4095	1.495063	-31.1243%
MSCI EMM strategy	217.2057	61.9733	1.729869	-27.1804%
Myopic Strategy	182.6368	41.5680	1.94128	-21.6817%
Dynamic Strategy	241.7225	68.4474	2.070531	-26.9751%

Table 8.2: Basic Statistics for six different strategies

and of the multi-asset strategy are displayed with the Sharpe Ratio and the yearly Value at Risk of each strategy with a 95% confidence level.

We can conclude that applying a dynamic strategy has clear gains over applying a static strategy.

By a path-dependent rebalancing strategy, it is possible to increase the mean portfolio return with significantly lower downside risk.

6 Conclusion

In the last decades, dynamic portfolio allocation has become a popular subject of research.

Despite this interest, closed-form and analytical solutions are available only under strict assumptions of return dynamics, and numerical approaches are needed.

Nowadays several methods have been published, but almost all of them suffer from inflexibility towards the number of assets and require a very specific structure of the asset's return dynamics.

The method published by Brand et al. (2005) represents an exception as they developed a simulation-based method that involves Taylor series, backward recursion and regression analysis to predict returns. The so-called BGSS method is fast, accurate and flexible in the way asset returns dynamic are modeled.

However, in their paper, they showed results for a situation with only one risky asset and one state variable.

In this thesis, the aim was to apply this method in a realistic environment of multiple assets, without strict assumptions about the return dynamics, with various state variables and portfolio constraints.

With the knowledge obtained by the Brandt et al. (2005) paper, an evaluation of a more realistic environment, that consists of four risky asset classes with different mean and volatilities, is computed.

In this model, five state variables were included and were supposed to contain sufficient predicting power.

Despite the investor has to comply with borrowing and short-sale constraints, the gains of the dynamic strategy are clear.

Of course, the methodology considered in this thesis is capable of dealing with a more realistic investor's environment.

Implementation is therefore recommended, although additional research is needed.

Other researches could be appropriate about the quantification of the investor's risk aversion, to find a good proxy of the exact individual's risk aversion parameter.

If the intuitions behind the methodology of this thesis are correct, this model would be a very suitable candidate for implementation.