LUISS

Department
of Business and Management

Course of Asset Pricing

Copula models and the Financial crisis: how to price synthetic CDOs' tranches

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ABSTRACT

In this thesis I would like to analyze and emphasize the importance that some derivative contracts have had during the 2007-2009 financial turmoil.

Different aspects of the uncertainty embedded in the markets during those years will be considered along with the possible triggers of the resulting domino effect.

The discussion will be introduced by a quick overview of the financial markets during the period and of the main events that have leaded to the market crash.

After that, the principal contracts at the core of the conversation will be introduced, as well as their primary characteristics.

The mortgage market and more generally, the housing market bubble will be identified as the major cause of the ABSs default and of the resulting bankruptcy for the main financial institutions in the market.

In chapter 2 the conversation will vert toward the credit risk world and the main models used to identify and monitor it.

After the delineation of the main variables involved and of its role in the crisis, we will focus on the mechanics exploited for the modelling of the default risk, and further highlights the mathematical tools used to implement them.

The heart of the discussion will be the introduction of a new market practice for the pricing of credit derivatives: the copula functions. More specifically the presentation of the correlation between defaultable assets will allow us to create a bridge between, single asset default and portfolios of assets' default via the use of copula functions.

With the aim of investigating the market incongruencies arisen during the crisis, we will furnish a practical application of the one factor Gaussian Copula to tranches of CDO in order to conduct a sensitivity analysis that will help us coming up with a motivated conclusion.

CHAPTER 1
AN INTRODUCTION TO THE CRISIS

As we already know, the 2007-2009 have been characterized by a severe market break down, caused by several factors; the sub-prime mortgage market have been in the spotlight, supported by a market bubble in the real estate industry.

Let's start the analysis by first describing the situation immediately before the turmoil.

The market was characterized by a series of different products that allowed investors to diversify their portfolio's risk, either by choosing 0 correlation items or by hedging their positions through derivatives contracts. Those contracts are products whose payoff and price are directly correlated to the behavior of another item, the underlying, and that allow to cover (to hedge) the position in another investment, reducing (or even fully eliminating) the credit risk associated with the latter.

When talking about investors we should remember that banks, as active counterparties to a series of transactions, should fall under this category.

As main players in the lending field, banks also bear sources of risk. The main one, of course is the counterparty or credit risk. Those financial institutions can be seen as firms that still have to register each transaction on their balance sheet where the elements of risk are calculated and managed. For example, when they issue a loan, they are funding such cash out with money deposited by other investors in an account, meaning that if the latter would withdraw their amount back, the bank should be liable for the refunding of money through and adequate management of cash ins and cash outs.

Following this reasoning securitization have becoming more and more widespread over years.

With securitization it is meant the requirement by a counterparty (in this case the bank) to securitize a promised payment (which in the case of the loan is composed by periodical principal and interests payments) furnishing a back-up asset that could be marketed in the case of default so to create liquidity.

To mitigate the need of collaterals held by financial institutions, Asset Backed securities were integrated in the market in 1970.

We will see in the following chapter that one of the main roles played by ABSs is to allow banks to share or completely transfer the credit risk associated to their clients, to a third party.

By now we will just stress out the fact that, although not fully being derivatives, as in this case the contract provides a partial ownership of the underlying, they are still linked to a pool of assets from which their value is extrapolated.

Usually the underlying pool was composed by consumer and business loans, even though they could have been backed by any cash-generating asset.

This characteristic is fundamental as it will help us explaining the origins of the financial collapse by further introducing Mortgage Backed Securities. The latter are distinguished from the broader concept of ABSs because instead of exploiting commercial loans, they are constructed on mortgage loans.

A whole paragraph will be dedicated to the description of the housing market and of those instruments, and on the factors that have leaded to some market incongruencies giving birth to the domino effect presented in the markets during 2007-2009.

Let's first start from the simplest themes.

1.1 SECURITIZATION

The rise in funds demand and in borrowers' defaults, together with the introduction of new market regulations, have pushed banks to ask for a back- up asset when lending money to investors. Bearing in mind that the principal source of lending comes from other investors' deposits, it is easy to understand the fact that banks needed to be sure to be able to refund depositors whenever they wanted, managing at the same time the cash outs related to loans.

Holding large portfolios of loans of different kinds, banks were able to diversify idiosyncratic risks associated with each contract, exploiting the independence between obligors. That being said, the systematic market risks affecting the portfolio was still under their management.

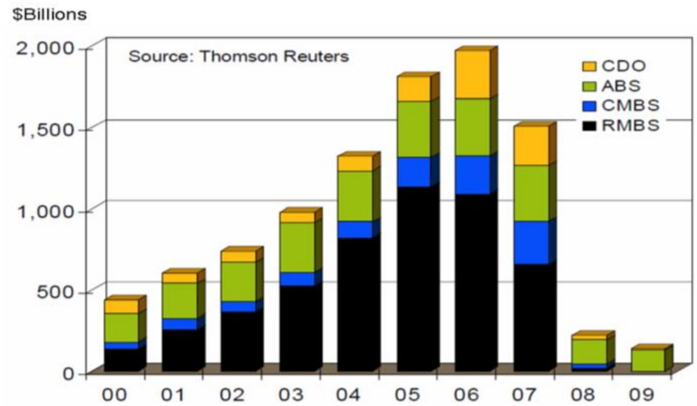
The increasing trend of collateralization, by the way, has originated new issues for the institutions holding them. The primary one was that in order to satisfy the need of liquidity in case of depositors' withdrawal, banks had to manage huge amounts of collateral assets and the buy and hold fund-strategy was difficult. Moreover, their speed in creating new funding was affected as well.

The creation of ABS has become a way for them to foster the growth in lending more than the one in deposits so to decrease the dependence between the two services and allow the transfer of risk in an easy and quick fashion.

Capital requirements for banks liquidity triggered, as well, the growing issuance of those securities.

Specifically, the need to maintain an adequate level of cash (liquid assets) to satisfy their assets side, pushed them to particularly appreciate the appearance of those financial instruments.

Figure 1.1: Securitization Market activity



Source: By Farcaster (talk) 04:36-10 October 2010 (UTC)

1.2 ABSs

In order to alleviate the burden held by banks and to make lending easier and faster ABSs were introduced in the market. They basically allow a financial institution to package income generating assets from their balance sheets, such as loans, and to sell them under the form of securities, so to immediately receive cash flows that can be then used to create other loans. More specifically an originator entity pools the asset together and transfer (sell) them to another party which is said to be bankruptcy-remote. The latter can be either a subsidiary or an affiliate institution or a Special purpose vehicle, as more popular during the 2000s, appositely created for the accomplishment of the transactions.

The term bankruptcy-remote refers to the fact that the entity to whom the receivables are transferred is characterized by an asset/liabilities composition that allows it to keep its obligations safe in the case of default of the obligors. The latter will then be accountable for the distribution of the tranches to external investors. This is to say that the intermediary should be able to fulfill the promised payments to the tranches' holders even under the circumstance of default by the original obligors. The separation between the originator and the SPV, further assures a higher credit rating to the issued tranche, as the financial position of the intermediary is not troubled by any other contract, besides the ABS. Special purpose vehicles, can further engage in two separate kinds of deals; they can either act as a pass-through, in which they just transfer payments between the originator and the investors, or they can behave as pay-through. In the latter case, the intermediary not only buys assets, but also can engage in reinvestment strategies in new assets.

Government-sponsored enterprises, such as Fannie Mae and Freddie Mac, have been instituted as well to interpret the role of the third party in those securities creations.

The fundamental function of those instruments, besides the increment in the lending ability, is that they allow the transfer of risk.

We can see them as a bond backed by loans or other assets, which generates cash ins for the issuer proportional to the cash flows of the underlying pool of assets.

For example, let's suppose that a financial institution holds a number of commercial loans. The latter can be bundled together, and the cash flows associated with them can be extrapolated and repackaged into new securities subsequently sold to a third party. By doing so the bank will no longer have the loans on its balance sheet, but at the same time will have a stream of cash ins equal to the one promised by the borrowers, with the difference that now the associated default risk is mitigated.

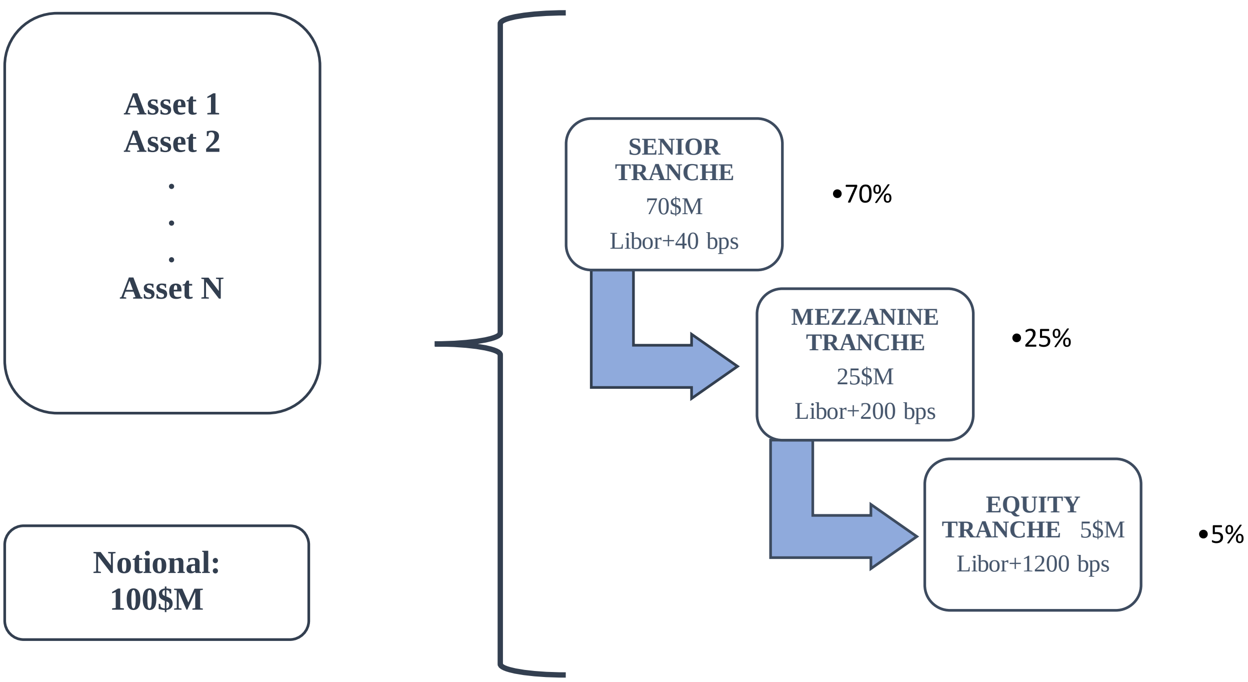
A key feature of asset backed securities is that the cash flows are afterwards split into tranches.

The resulting products are then categorized in different levels of claims, with different credit ratings. Usually three categories are created. Those are the senior tranche (rated AAA) which has the highest priority over the others; the mezzanine tranche (rated AAA-/BBB) positioned in the middle in terms of priority; the Equity tranche (not rated) which categorizes the residual claims over the pool of assets. In regard of the profit and loss allocation with respect to the underlying cash flows, the used technique is the waterfall cascade. The latter describes a hierarchy according to which profits will be distributed with a top down structure (that is why the last tranche is characterized by residual claims); on the other hand, losses will be shared following a bottom up argument.

Another property is that each tranche represents a percentage of the cumulative notional represented by the underlying's pool (Of course, the sum of all the percentages should give 100%). This implies that also profits and loss are allocated accordingly. For example, following the cascade scheme the first tranche that will earn profits (taking into account the percentage of the pool's nominal represented by the latter) will be the senior tranche. Whenever both the senior and the mezzanine tranches have been satisfied, the equity tranche will be entitled to collect the remaining profits realized by the pool. This means that according to the magnitude of profits the last tranche could be either not repaid, fully or partially repaid.

To make a practical example we can consider a portfolio of underlying assets worth 100$M. suppose the three simplified tranches are described as follows:

Figure 1.2: Example of ABS



The above scheme demonstrates that the entity issuing the ABS is practically issuing bonds (claims) backed by the asset's pool.

Being focused on the profit allocation, the cascade scheme assures that the senior tranche will be entitled to the first 70% of gains created by the pool. Whenever this percentage is satisfied profits will then be allocated to the mezzanine tranche up to the 95% of the cumulative collateral pool's profits. Moving further, whenever both the senior and the mezzanine tranches are paid, the equity tranche will earn the remaining 5%.

What happens, on the other hand, when a loss occurs is that the first tranche to suffer is the equity one. It will absorb up to the 5% of total losses of the pool, while keeping the other two tranches profitable. When the loss will exceed the 5% the mezzanine tranche will absorb up to the 25% of the total loss. This means that in order for the senior tranche to be touched the underlying pool must reach a loss of 30% and more.

We can here introduce the concept of attachment and detachment points of a tranche. By attachment point it is meant the percentage up to which the tranche does not incur in losses. The detachment point is the percentage above which the entire tranche is wiped out.

In our example the mezzanine tranche has an attachment point of 5% and a detachment point of 30%. So, we can say that it is a 5%-30% tranche.

Let's further suppose that our underlying pool is composed by 100 loans, each having a notional of 1$M. This implies that there are 100 obligors that could default.

If 1 obligor out of the 100 total defaults, the underlying pool is affected by a default of 1$M.

In the event of default, under the assumption of zero recovery rate (which is the value of the security after default has occurred expressed in FV percentage.) the loss should be borne by the equity tranche. So, what happens is that the investor holding the tranche will have a reduction in its notional of 1$M, so that the new principal will now be 4$M rather than 5$M. As a consequence, now on, he will earn its interest rate (LIBOR+1200 bps) on the new notional and will receive 4$M at the expiration date.

We can now consider the event in which 7 obligors default. The underlying pool will exhibit a total loss of 7%. Again, we expect a zero-recovery rate.

As we already know the first 5% of losses will be absorbed by the equity tranche, meaning that its holder, from the moment of default of the 5th obligor, will no longer receive neither the interest nor the principal payments.

Moving further up, the mezzanine tranche will now be entitled to deal with the remaining percentage of loss. Having already wiped out 5M, the remaining amount to be assimilated is of 2$M.

This means that the mezzanine tranche will now be calculated on a notional of 23$M.

As said above, the holder of the tranche will, continue to earn interests but on a lower amount, and will receive at the end of the period 23$M.

Of course, we could continue further with the example and show that the senior tranche will start to bear the burden of losses, when the number of defaults will reach the 30%. (attachment point of the tranche).

1.3 MBSs AND THE HOUSING BUBBLE

Now that the mechanism behind Asset Backed Securities have been examined, it will be straightforward to understand how mortgage backed securities function and how they have contributed to the housing market bubble.

Having in their assets' pool mortgage loans written by banks, MBSs present exactly the same apparatus as ABSs. They usually exploit SPE for the distribution of tranches and the allocation of profits, through which the cascade mechanism is accomplished.

The aim is to conduct a step by step description of the circumstances, together with its components, starting with an introduction about the housing market prior to crisis and the role plaid by mortgage loans, up to the creation of the housing bubble that has subsequently generated the infection of the whole financial market and of the general one.

Before the introduction in the market of ABS and MBS as they are known today, securitization was carried out through simpler contracts; specifically, pass-through contracts which involved the transfer of receivables started to be popular in the market.

In 1970 the Government National Mortgage Association first issued government guaranteed pass-through securities that allowed the passage of principal and interest payments from borrower to investors who were purchasing bonds written on the Federal Housing Administration and Veterans Administration 30-year-single-family mortgages. Subsequently, other institutions began issuing securities with those features, allowing investors to purchase credit-risk free instruments and being able to enhance their own liquidity for new loan' s issuances.

Although really close to the MBSs' mechanism, they were still lacking of the ability to please different kind of investors with different risk appetites.

The next step toward the creation of mortgage backed securities was made in 1977 when Bank of America issued the first privately-managed collateralized mortgage obligation.

Together with the risk associated to default, the mortgage market bears the prepayment risk. The latter is related to the probability of occurrence of a refinancing of debt, required by the borrower due to more favorable market conditions (dramatically lower interest rates) or by a significative change in their credit position. In both cases the consequences on the security contracts are that the borrower will repay its debt faster due to a reduction in interest paid, and the securities' investors will have to reinvest the principal earned at lower markets rates.

Through the introduction of different tranches described by different maturities and ratings, CMOs started to recognize and to deal with this source of risk.

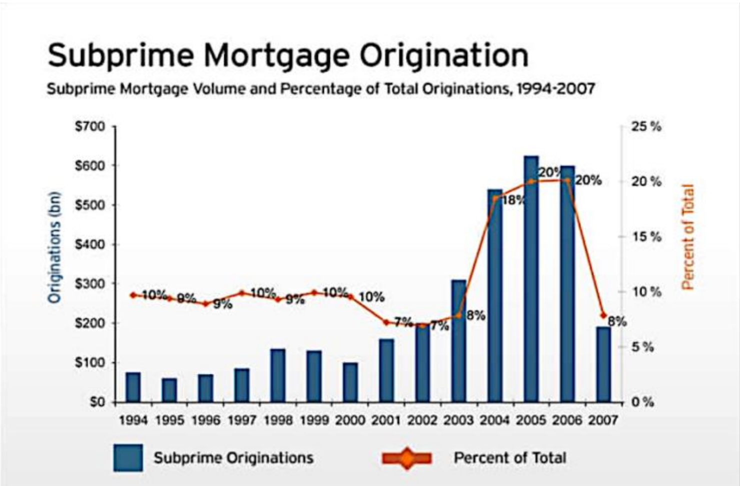
The aforementioned securities were soon extended to include many other types of assets, such as business loans, credit card loans. In the late 1990s they experienced a substantial growth in market share, that subsequently led the ABS market to its boom in 2008, with $2,671.8 billion in value of outstanding securities.

The us Government, since the 1990s, has shown a remarkable interest in boosting the housing market development by encouraging banks and other financial institutions to sustain buyers through adequate and feasible lending, expanding their services to low and middle-income households. Concurrently with that from 2002 on, the US market have been marked by low interest rates, as the Government was trying to stimulate the economy after the terrorist attack in 11 September 2001.

The combination of these two events, lead to flourishing conditions for a house buyer; the cost of funds was at its minimum since the previous depression and banks were able and willing to provide favorable terms in mortgage loans.

Subprime mortgage lending became one of the major trends in the market by 2005. Even though banks were already familiar with this category of products, at this time they started to have their own interests in issuing them. Due to the higher demand, house prices rose significantly, allowing banks to have a safer position in mortgage contracts. In fact, the latter were in a certain way ensured by the fact that in the event of default by the borrower, the foreclosure of the house would have covered any incurred loss.
In addition to that, banks were highly interested in the MBSs market, at the point that they seek to issue as many mortgages as possible in order to sell them afterwards.
This led to a relevant relaxation of lending standards, increasing the possibility for borrowers with weak credit rates to join the market.
Many households, being aware of the substandard control exercised by the banks, have even been able to lie on their soundness. This have been caused by the fact that the tranches' investors were requesting only few details about the creditworthiness of the initial borrower. In particular, the FICO rate and the loan to value became enough to evaluate the riskiness of the investment, without taking into account any other borrowing ability or history of the obligor.
In the following figure the increase in subprime mortgage loans over the years is shown as a percentage of the total lending. It is clearly visible that in the years preceding the market crash, their volume increased substantially.

Figure 1.3



Source: T2 Partners, LLC, from CNBC-june2009

Agency costs plaid likewise a role in the disruption of the market. The players involved in the construction of mortgage backed securities were the originators of the mortgages (bank), the rating agency and the institution responsible for the issuance of the tranches (SPV or other financial institutions). Of course, the initial borrower and the final investor were implicated as well. It is clear that none of them shared the same incentives. Banks were interested in issuing loans satisfying the required conditions for the creation of MBSs. The parties involved in the assessment of the appropriateness of the houses for the mortgages, were engaged in coming up with high evaluations, so to please the lenders and continue making businesses with them. The tranches' issuer, on the other hand was committed in receiving the highest rates. The rating agencies, in turns were recompensated by the issuer, so they were pressed to satisfy their willings.

Rating shopping have been a widespread practice by which banks were able to designate the rating agency entitled to evaluate the MBS tranches. This allowed them to put as many pressures as they wanted to end up with high credit rates, and to enable a single entity to review all the tranches.

Rating agencies, on their side, continued to assign high ratings under the belief that the housing price rise would have last forever.

In the meanwhile, the loan to value ratio (ratio of the amount borrowed over the market value of the house) kept rising, while the use of adjustable rates mortgages strengthened. The latter allows the borrower to pay a lower fixed rate, called teaser, during the first years of the contract, and to then switch to a higher one, usually calculated as a spread over the LIBOR.

Even though teasers of different percentages have been used, varying for example from the 6% to the 2%-1%, in 2007 many obligors started to face their inability to repay their debts at the new variable rates. (after the teaser period ended).

This forced banks to foreclose the collateral furnished, injecting in the market large quantities of houses, that subsequently led to a decline in house's prices. In this market environment, banks had no longer incentives in lending as the riskiness of the deal substantially rose, existing mortgage borrowers defaulted, sometimes finding themselves with negative equity in their portfolios.

It is necessary to specify that not all of the obligors were using mortgages to physically own the house, part of them were speculating on it. These gambles were sustained by the fact that US mortgages were non-recursive, meaning that in case of default the only asset that could be attacked was the collateral; no personal assets of the obligors would have incurred in risk. From that it followed that the mortgage contract could be seen as a put option owned by the borrower. He could sell the house for a strike price equal to the nominal of the contract at any time, by defaulting in payments. Many took advantage of that, making their obligation liquid, in a moment in which house prices had already suffered from a severe decline.

Banks had no choice than to try to safeguard their liquidity, but the number of defaults and the reluctancy in investing, triggered by fear, brought them to collapse.

When an obligor defaults, the bank sells the house and tries to recover as much principal as possible, while the investors in the tranche suffer from the decrease in their principal, or of the complete termination of the investment.

Many off-balance sheets transactions had to be brought backward, making them incur in capital charges, while still striving for additional funds.

This critical situation briefly reached also other institutions strictly connected to the ABSs and MBSs' sales, such as insurance companies that were providing insurances against defaults of the latter.

The alarm had a wide breadth, expanding to the whole market. As lenders started to fill for bankruptcy, households found themselves deprived of any savings deposited, the whole economy started to experience a break down, stimulated by the uncertainty about the future and the suspiciousness regarding the integrity shown by banks.

Rating methods applied to MBSs tranches, have been strongly challenged. The tendency to regard classes of tranches as similarly rated bonds also confused investors, making them not fully aware of the risk they were facing. In this regard, in fact, the rate of loss associated with tranches was very high with respect to the one of bonds, because of the cascade structure, which made the disruption of the whole principal very likely.

1.4 CDOs, CDSs and ABS CDOs

The establishment in the market of ABSs and MBSs, was accompanied by the introduction of other products such as CDOs and CDSs.

The latter are credit derivatives, whose existence was triggered by some market imperfections, creating arbitrage opportunities for investors.

CDOs are a particular and sophisticated class of ABS. They are characterized by a collateral pool of traded bonds. This includes ABS's tranches, CDO's bonds, and other types of obligations, already traded in the market. For that reason, CDOs do not contribute to the achievement of market completeness as ABSs do; they rather allow investors to exploit arbitrage opportunities. It can be proved that whenever the underlying bonds (ABS, CDO's tranches…) are underpriced, or the resulting CDO's bonds issued are overpriced, then the security will carry a positive value. If what just said is not true, it will yield a negative value and will have no reason to exist.

This can also be seen as a proof of the fact that at that time, rating agencies were not carefully carrying out their charge. This rating arbitrage primarily rise due to the payment structure of rating agencies and because of the short-term incentive horizons for financial institution's managers.

Rather than being paid all at once, rating agencies faced periodical payments, that tended to foster a long-term relationship with their customers. On the other hand, managers' compensation mainly comprised end-of-the-year bonuses, making them weakly concerned about the future performance and consequences of their portfolios.

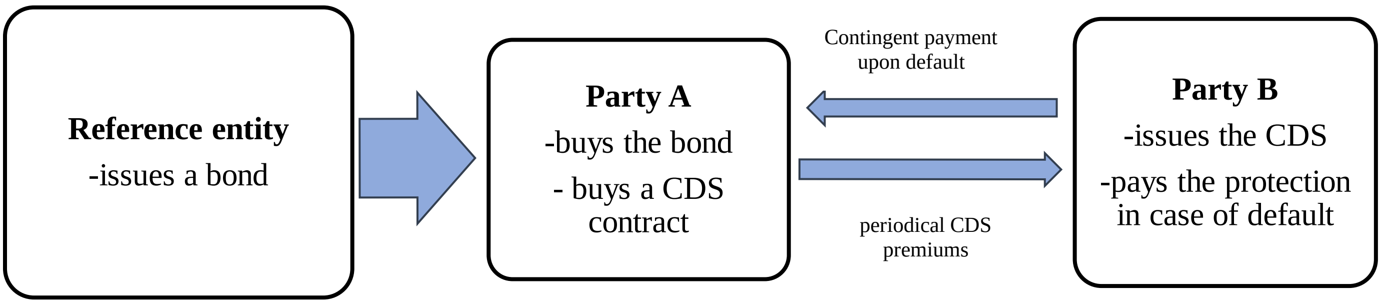
As we have seen, in the previous paragraph, this was one of the main sources of collapse, due to the agency costs and the misinformation between parties.

As opposed to ABSs' pool of assets, collaterals linked to the credit default obligations, were products already trading over the counter, and the distribution of the tranches was reserved to special purpose vehicles, specifically founded.

CDOs can take on different names depending on the pool's composition. If for example bonds are used as collaterals we are formerly dealing with a collateralized bond obligation, if loans are used instead, we will enter into a collateralized loan obligation.

Although mainly constructed with loans, bonds and other types of credit derivatives, credit default obligations can be theoretically created from pools of any defaultable title. This could include for examples companies and nations.
Two different classes of CDOs exist; cash flow and synthetic CDOs. The first category refers to the physical ownership of the bonds by the issuer. This involves a direct sensitivity to the default risk associated with the asset. As opposed to the latter, synthetic CDO's do not embed any purchase of bonds and are written on pools of credit default swaps, which creates the cash flows between the two parties.
Together with CDOs, CDS have been extremely popular during the 2000s. They basically are swap contracts in which the buyer of the security is seeking for hedge against the default of a reference entity from which he has bought an obligation (bond). The seller will promise a fixed payment equal to a preestablished nominal, in the event of default of the entity; the hedger in exchange of the sure payments, commits himself to pay a periodic premium to the other party, expressed as a percentage of the notional.

Figure 1.4: CDS contract



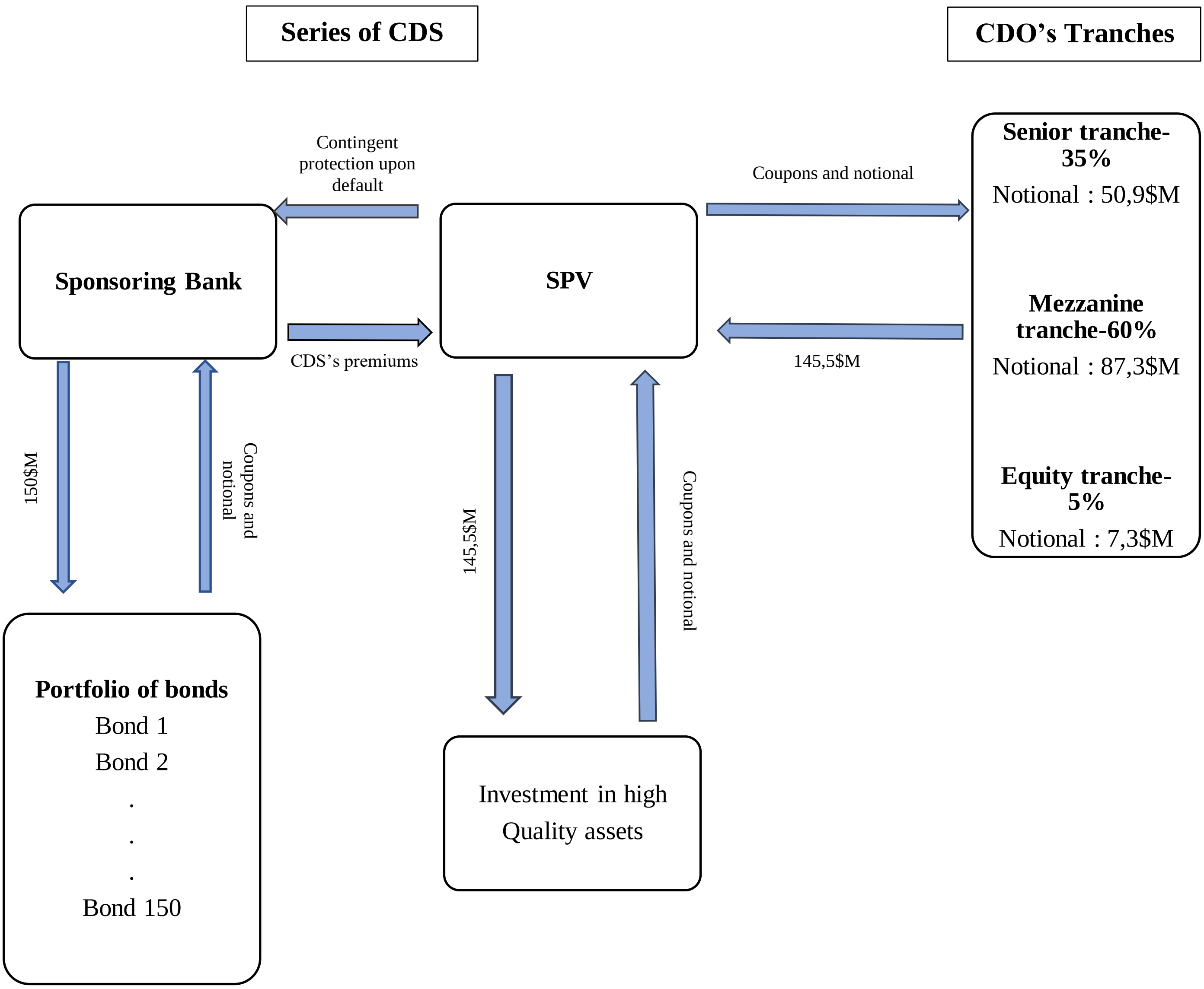
Whenever entering a credit default swaps two states of the world may occur. Either the reference entity does not default, and, in that case, the buyer does not receive any compensation for the recurrent premiums; or default does occur, so that on the date of the event the buyer will receive a payment equal to the face value and will stop the periodical fees paid up to then. To be more precise, what happens in the event of exercise of the contract, is that it can be either settled physically or in cash. In the first case the buyer is at all effects entitled to sell the bond to the counterparty at a price equal to the nominal. In contrast, when the delivery is in cash, a regulated auction is organized to sell the bond. The seller will refund the difference between the face value of the bond and the selling price.
At injection the value of the derivative is zero, as the present value of the two legs will offset each other by an arbitrage argument.
CDS have been in the spotlight during the mortgage market break down, as they have been a valid instrument to short credit risk associated with specific debt issues. Being traded over-the-counter, those securities were not exposed to strong regulations. The key feature that made them discussed is the fact that, in contrast with bonds, credit default swaps are highly leveraged. This means that a financial institution could have beneficed

from periodical floating payments bearing the credit risk of a bond without making any initial payment (when a bond is bought the Face value have to be paid at the moment of the issuance).
CDSs provided an easy and cheap way to short debt, in a market (secondary debt market) with an intrinsic low liquidity, where the portion of collateral that have to be posted by the issuer (seller) was insufficient (or even inexistent) to provide an adequate protection for the buyer against its inability to effectively receive the promised payments. Insurance companies with high ratings were not required to post any guarantee to assure their liquidity suitability. This, in turn, led to the default of a lot of contracts, as the market crashed down and the possibility to collect extra funds weakened.
With a rough understanding of the apparatus behind CDS we can go back to the concept of synthetic CDOs.

1.4.1 Example of a synthetic CDO

As early discussed, this kind of securities do not involve any physical purchase by the Special purpose Vehicle of the underlying obligations. The SPV, rather issues credit default swaps to a sponsoring bank who owns the bonds in its balance sheet.
It is usually common that the cash flows collected from the series of CDSs are not enough to satisfy the promised payments to be transferred to investors. This is why SPV tend to invest the amount of principal collected to overcome this misalignment of funds. It usually reinvests the principal collected in highly rates assets and reuse the proceeds to meet its obligations toward both the CDO's investors and the sponsoring bank.
Let us consider an example. Suppose a bank is currently holding a portfolio of bonds worth 150$M made up by 150 bonds each with a nominal of 1$M.
The SPV decides to share the credit risk with the latter in exchange of periodical payments.
150 swap contracts are issued by the purpose vehicle to provide protection for each bond holding.
CDSs are subsequently pooled and tranched. Without loss of generalities we will consider three tranches, even though usually many more are created.
Senior, mezzanine and equity tranche are then sold to investors.
We know that the CDSs contracts will be constructed so that the coverage in case of default of the reference entity may not be full.
Suppose the Sponsoring banks agrees to be recompensated whenever the loss will exceed the 3% of the total pool of bonds' nominal. That is as to say that the notional of the CDS will be the 97% of the 150$M notional, 145,5$M.
Tranches will be divided as follows: the senior tranche will represent the 35% of the pool's notional, the mezzanine the 60% and the equity tranche the 5%.

Figure 1.5: example CDO's synthetic tranches



Let us clarify that the choice of assigning the highest percentage to the mezzanine tranche, have been dictated by the fact, that it is common to have different level of mezzanine securities, such as junior mezzanine, senior mezzanine and so on and so forth.

The attachment and detachment points will then be as follows:

Table 1.1

Type of Tranche	*Attachment Point*	*Detachment Point*
Senior tranche	65%	100%
Mezzanine tranche	5%	65%
Equity tranche		5%

Following the structure described above, whenever the portfolio loss is below 3%, no changes to the tranche's notional will take place.
Conversely assume that 5 out of 150 bond issuers default, with 0 recovery rate. As we have just highlighted the first 3% of loss will not be counted; the other 2% has to be accounted through the cascade structure.
The equity tranche will absorb the loss and will suffer from a reduction in its notional of 2.9$M. Now on investors of this tranche will earn interest on a notional of 4.4$M and will receive this new amount at the end of the investment period. The reduction keeps moving up until the detachment point of the tranche, when the entire notional will be wiped out. (5%) For losses above this percentage (exceeding the 8% in aggregate) will be borne by the mezzanine tranche up to the 65% where the senior will be hurt as well.
In the end, the bank will pass through the credit risk associated with its investment while keeping the bonds in its balance sheets. The SPV will hedge the risk borne via the CDS contracts, through the issuance of the CDOs tranches. Investors preferences and needs will be satisfied through the purchase of tranches, with the possibility to eventually exploit any misprice caused by inefficiencies.
It has to be highlighted that in case of the default the tranche's investors will assimilate the percentage of loss. In that case the loss they will be accountable for is represented by the 97% of the bonds' pool.

CDOs were highly traded during the years previous the turmoil because of their capacity to remodel junk bonds and lowly rated debt in general.
They could repackage a pool of junk bonds, for example, into different tranches of debt, where a senior tranche exists. This one will be rated AAA-, for example, and still yield a higher return than a similarly rated bond in the market. The latter result is supported by the structure of the security; in fact, the cumulative return of the underlying must coincide with the weighted average of the tranches' returns (weighted by the percentage of notional each tranche describes).
This feature helped overcoming the otherwise low demand for low grade debts, making equity tranches created from those instruments appreciated from investors. This is why ABS CDOs (which fall under the category of synthetic CDOs) became very popular. The latter are CDOs tranches backed by a pool of ABS or ABS CDS contracts.
In the second case the collateral assets constituting the CDO are credit default swaps created on ABS's tranches. ABS CDSs, as opposed to standard CDSs, offer protection contingent to the default of the tranche. They additionally embed the prepayment risk introduced by the underlying tranches, due to their cashflows replication ability.
CDO^2 increased the possibility to upgrade the credit ratings beyond, by the origination of new tranches backed by CDOs' tranches, usually the equity ones.
A vanguard in the market have further been the invention of single-tranche trading; key point of those products was that no CDS were effectively created. Tranches of CDOs were constructed on a reference basket of CDS contracts, usually CDSs indexes, in order to emulate their cash flows.

It can be discussed that synthetic CDOs involve lower cost of transactions and more reliable prices than the cash ones. This is due to the fact that cash credit default obligations do involve the purchase of the underlying bonds, while in the latter case no ownership is involved.

Furthermore, being cash flow CDOs still strongly linked to the underlying bonds, they still made investors facing the illiquidity spread in the secondary market. On the other hand, synthetic products were backed by products traded over-the counter (CDSs, ABSs) allowing for a more accurate pricing of the CDO.

In the next chapters we will see how the pricing of those derivatives can be carried out, having a focus on copula models.

1.5 THE COLLAPSE

As just said in the previous paragraphs, the three main characters in the market have been ABSs, CDOs and CDSs.

Each one of those having its own benefit, they all plaid a key role in the market turmoil.

Just to summarize what have been said so far, ABS have created new accessible and liquid investment opportunities for subordinated investors, who would have alternatively had no possibilities to borrow and to afford a house. This led to an incremented market completeness, in which not only low-grade investors were able to take advantages.

CDS furnished a way to short-sell easily debt by transferring credit risk, in a world in which the short-term trade of debt securities was highly unlikely.

CDOs allowed investors to take advantage of the market inefficiencies leading to asset mispricing by rating agencies, that have been in the eye of the tornado for years.

We have already discussed the role plaid by MBS, and the behavior of the housing market. What need to be added, at this point is how credit derivatives and asset backed securities in conjunction have pressed financial institutions towards the collapse.

The starting point of the analysis is the popularity of ABS and the advantages for its originators. We know that banks tended to oversupply mortgages with the only aim or reselling them. Investors of different types were committing their selves in MBSs' tranches, mainly attracted by the highest ones (senior). For lower grade debts, CDOs provided a solution to boost the market appetite and exploit the excessively high ratings assigned by rating agencies, who were too busy to delight their clients.

CDS started to be the easiest way to construct CDOs on tranches of mortgage backed securities, allowing the intermediate special purpose vehicle to mitigate their cost of transactions in creating such derivatives.

Despite incited by diverse desires, all market participant have experienced the effects of the turmoil.

MBS's defaulted because of the inability of the mortgage obligors to repay their debts and the subsequent failure by banks to recover the loss via foreclosures.

CDS have been condemned for the general disorientation about financial institutions' riskiness.

We know that either private investors, hedge funds, or other financial institutions such as banks could invest in those securities.

What has not been said so far, is that CDS in particular involve off-balance sheet transactions. The fallout of this feature is that they are not considered when evaluating the riskiness of a party.

When the market commenced its rapid drop, it was not clear who was entitled to bear the loss resulting from mortgage securities and the uncertainty promptly rose.

As we have discussed in the previous paragraphs, credit derivatives and ABS are traded over the counter.

This provide a quicker method to trade but it is subject to limited regulations. That said, investors in this market are not allowed to assess the risk associated to their specific counterparty on the basis of other transactions they are dealing with.

The absence of transparency together with the correlation between CDS issuers, also hurt the inter-banking lending market, locking the chance for banks to borrow from each other.

CDSs contracts written on MBS and more generally ABSs' tranches suffered the damage through a domino effect hurting CDOs securities written on them and their subsequent investors.

As any other CDS' issuers, special purpose vehicles were underestimating the likelihood of default triggering their payments. This led them to find themselves with no enough liquidity, causing their incapacity in fulfilling tranches' payments.

The existence of these securities built up a massive net between financial institutions that were issuing and buying to each other. Whenever one comes to default, it affected all the others.

This is why the government decided to intervene for the rescue of some institutions such as Bear Stearns, attempting to unsuccessfully avoid the spread of the infection.

On November 2008 the FED decided to intervene providing assistance to ABS's holders, in order to avoid the complete disruption of the demand.

The Term asset backed securities loan facility (TALF) have been the major source of capital injection.

It involved the allocation of funds up to $200 billions by the Federal reserve of New York, to AAA- rated ABS tranches. The latter were backed by newly issued loans.

These NY FED was retaining an amount equal to the difference between the market value of the ABS less a haircut, which was used as a cushion in case of default.

Funds were lent to banks through different sectors-specific loans, backed by ABS, with a maturity of 1 year. (later on, some changes have been made to extend the maturity to 5 years, based on the current market necessities). Through this mechanism a boost in the banks' liquidity have allowed them to start loaning again to consumers and small businesses, even though the market circumstances created by the 2007-2008 crisis, persevered until the end of 2009 and still have left some residues today.

CHAPTER 2
CREDIT RISK

The two pillar concepts of the investment world to be considered are risk and return.

We know that broadly speaking, whenever a rational investor is concerned in evaluating an investing opportunity, he or she will seek to obtain the highest achievable return while keeping risk at the minimum possible degree.

Nevertheless, when using the word "risk", we should be aware of the fact that many different classes do exist. For example, market risk is the exposure that investors experience due to market movements in either prices or interest rates, caused by exogenous factors falling out of their domain of control.

Liquidity risk is described by the probability that, due to changed market conditions, the availability of funds is reduced or even destroyed, causing the management of financial positions to be complicated.

Finally, Credit risk is defined as the unpredictability regarding the ability of a counterparty to fulfill promised payments due to a change in its credit quality. It is commonly also referred to as Default risk, as the trigger of the loss in value is the deficiency shown by one of the two party in repaying its debts.

Credit risk does not only concern the securities world, but rather interest any contractual claim in which an investor is subject to a counterparty.

In general, we can express the payoff of any defaultable security as

$$G(S,T)[1 - 1_{DEF}(T)LGD]$$

Suppose for example the security considered is a call option on a stock. Taking default risk outside of our calculus, the payoff of the derivative at expiration will depend on the time T and underlying price S.

As we add the risk of default, we have to account for it. LGD stays for loss given default and represents the recovered amount in case of unfulfillment payments by the obligor, that in this case will be the option issuer. 1_{DEF} is an indicator function, that will take on value 1 whenever the default arrives, value 0 otherwise.

We have introduced credit risk in the first chapter, indicating it as one of the main characters of the financial turmoil.

ABS and credit derivates have been introduced to describe how the market was responding to this increasing uncertainty, mainly dictated by the expansion of over the counter transactions.

In that regard, due to the absence of a margin account in the deal, the exposure to counterparty risk became the first concern when investing in that market.

The growth in popularity of ABS and of subprime lending, together with the increasing confusion brought by rating agencies have pushed analysts to model this source of risk as to account for it in the most accurate way when pricing derivatives.

In the following chapter we will first analyze the different components characterizing the credit risk. We will then discuss the main features and indicators, subsequently used to model it by way of different techniques.

2.1 DEFINITION AND MAIN COMPONENTS OF CREDIT RISK

As already introduced so far credit risk embeds the possibility of not receiving the promised cash flows from our counterparty.

This concept is not ultimately complete.

Credit risk is composed by two ingredients: default risk and spread risk. The latter is the fear that a change in credit rating by the counterparty will lower the value of the position on hold. We have seen that during the crisis refinancing risk has constituted a huge slice of the cake. Even though refinancing happens when usually the rating of the obligor is more favorable, it can be still used as an example of how new credit conditions (in general) may have a negative impact on a deal.

Default, on the other hand, could materialize in different ways.

It can either cause a total loss of the face value underlying the transaction or a partial reduction in worth.

Two aspects have to be pointed out when talking about default and those are the default timing, "the arrival risk", and the default intensity, "magnitude risk".

The default date is identified as the date of announcement of impossibility to meet the agreed obligations.

The intensity by which event will impact the financial position of the counterparty is determined by the amount effectively lost when the event occurs.

To be more precise, besides the arrival and the magnitude characterizing the default events, other components have to be analyzed when modeling credit risk.

We can summarize them in exposure at default, transition probabilities, default probabilities and recovery rates.

Starting from the first, *exposure at default* is indicated by a random variable which takes into account the obligors outstanding and promised payments before the default occurrence. It indicates the amount that is at risk in case of default.

Transition probability refers to the likelihood that a credit migration will take place. It is an indicator of the spread risk the investor is subject to due to a change in credit rating of its counterpart.

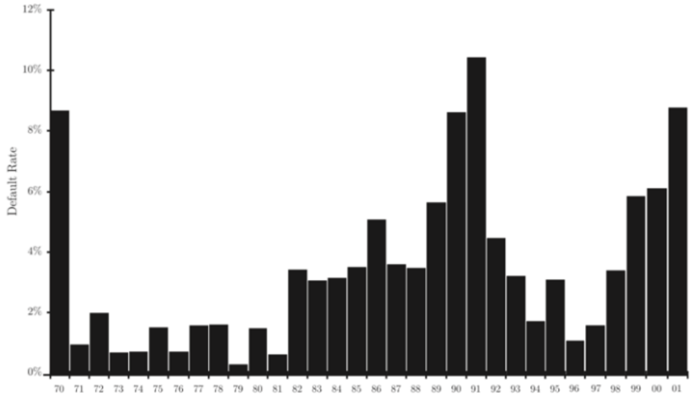
Keeping going in our list, *default probability* have already been defined as the expectation about default, and in the end, *Recovery rate* describes the proportion of payments still delivered in the case of default occurrence. It is calculated as 1- loss. The latter is better defined as loss given default and it is an index of the magnitude of default.

Credit risk management have different scopes and is carried out by diverse parties. With the common focus of identifying it and mitigate it, it has been actively regarded by financial institutions such as rating agencies, with the intent to categorize single securities or entire entities, such as firms to provide investors with a guidance. On the other hand, regulatory institutions exploit this concept in conceiving capital requirements and other arrangements, aimed at maintaining certain quality and liquidity standards.

Evidences of correlation between the circumstantial market conditions and default have been founded in different studies. According to the historical patterns followed during particular market scenarios, default rate tends to be higher during period of downturns. Of course, it has to be said that the degree of correlation might change significantly depending on other aspects, but the key result is that a negative interrelationship realizes. It has been noticed, comparing the 1981-1982 US recession and the 1990's depression, that although the first one has brought more severe repercussions, it has been characterized by lower default rates.

In the below graph the 1-year annual rates of speculative grade defaults from 1970 to 2001 is shown from Moody's data.

Figure 2.1



Source: Moody's Data

Different approaches are used in order to determine the level of credit risk outstanding. it can be observed as the market value of loss, calculated at the moment of occurrence or, for example as the potential loss foreseen at the moment of issuance of the contract.

This theme can be further extended when taking into consideration a portfolio rather than a single security. In that case, the concept of correlation needs to be introduced together with the related distributions of default and transition probabilities.

Rating agencies, for instance, adopt the average frequency with which entities of the same rating have defaulted as a measure of default probability.

We will address this broader concept later on, for now let us continue the discussion focusing on the modelling of a single product.

2.2 MODELLING TRANSITION PROBABILITIES AND DEFAULT PROBABILITIES.

As mentioned so far, credit risk brings about various aspects to study.

Transition and default probabilities represent the major issues to be addressed when modeling the two sources of credit risk, that are identifies again in spread risk and default risk.

Since years different approaches are presented to fulfill the collection of information related to it and they all involve different assumptions and point of views.

We can summarize them in three major sections.

The historical approach has been highly used at the beginning of the credit derivative spread. The latter involves the use of historical default rates and the computation of the sample average, used as indicator of the riskiness of a certain obligor.

Although based on individual's specific data, it does not allow for the inclusion in the estimates of market conditions or in general, of other exogenous variables.

We will later on discuss the Asset value approach and the Intensity based approach, where not only environmental characteristics are taken into account, but also, the default triggers are identified in different ways.

Let us first introduce some basic concepts required to set up the models.

2.2.1 FORWARD DEFAULT PROBABILITY

Let us define *p(t)* as the probability of surviving until year *t*, assessed today at date *t=0*. Indicating with τ the event of default, the latter probability will indicate $P\ (\tau \geq t)$.

Following a logical argument, we can say that the probability of default can be expressed as one minus the probability of surviving, that is *1-p(t).*

Let us further consider year *s*, with $s \geq t$. the probability of surviving between these two years will be *p(s)-p(t)*.

On the other hand, the probability of default between the two years ca be expressed as *p(t) -p(s)* (it would technically be *(1- p(s)) – (1- p(t))*).

The probability just described is called the forward unconditional default probability, as it indicates as measured today the probability of default between two future dates.

It is possible to extend this concept through the application of the Bayes's theorem, considering the probability of surviving until year *s* given that it has survived for t years:

$$p(s|t) = \frac{p(s)}{p(t)} = \frac{p\ (t|s) \cdot p(s)}{p(t)}$$

1- p (s|t) will then refers to the forward default probability, which defines the probability of default between t and s, given it has survived until years t.

Let us allow the surviving probability p(t) to be strictly positive and always differentiable, we can then see that

$$f(t) = -\frac{p'(t)}{p(t)}$$

$$p(t) = -\frac{p'(t)}{f(t)} = e^{-\int_0^t f(u)d(u)}$$

$$p(s|t) = e^{-\int_s^t f(u)d(u)}$$

F(t) is called the forward default rate and it behaves exactly as any forward interest rate. Because of this similarity it can be efficiently used to reconstruct the default rate term structure through the eventual use of bootstraps.

If *f (t)* is a continuous variable *f(t)Δ* can be approximated to the default probability between time t and *t + Δt* conditional on the fact that it has survived until *t*. We will later see that it is the same as the hazard rate.

2.2.2 HAZARD RATE

In a continuous time setting, the hazard rate is defined as a measure of probability of occurrence of an event (in that case default) at a point in time *t*, calculated as the limit, as *Δt* approaches 0, of the probability that the event will occur between *t* and *t + Δt*.

Let us define the hazard rate at t as *λ(t);* the latter will be defined so that *λ(t)Δt* represents the probability of default between time *t* and *t + Δt*, given the no earlier default until *t* (that is, conditional on survival until year *t*).

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P[t < \tau \leq t + \Delta t |\, \tau > t)]}{\Delta t}$$

Let us express the concept in terms of a firm experiencing default. The total value of the firm today, described as *V(t)*, will represent the value of its assets, that we know equalizes the sum of its debts and equity.

We define *P(t)* as the cumulative probability that the company has not yet defaulted until time *t*.

Thus, the probability of default between time t and *t + Δt*, conditional on survivance until *t* can be illustrated as follows

$$\frac{[P(t) - P(t + \Delta t)]}{P(t)} = \lambda\Delta(t)$$

We can rearrange the above equation, expressing it in terms of surviving probabilities

$$P(t + \Delta t) - P(t) = -\lambda(t)\Delta(t)P(t)$$

$$P(\Delta t) = -\lambda(t)\Delta(t)P(t)$$

$$\frac{dP(t)}{dt} = -\lambda(t)\mathrm{P}(t)$$

$$P(t) = e^{-\int_0^t \lambda(\tau)d\tau}$$

Remembering that the above indicates the probability of surviving, the default probability is defined as $Q(t) = 1 - P(t) = 1 - e^{-\int_0^t \lambda(\tau)d\tau}$.

The above formula has been derived by Lando (1998) and Duffie and Singleton (1999) and shows that the value of a risky product can be find discounting its promised payoff with a default-adjusted rate. If we define the return on Treasury bonds as r, the default adjusted rate will be $r + \lambda$, where the letter is approximated by the hazard rate.

To be precise the above described, is the *instantaneous probability of default* as it is described in an infinitesimal length of time.

We can argue that the hazard rate and the credit spread join a positive correlation since, the latter will be higher as the default intensity increases.

2.2.3 RECOVERY RATE

One of the essential ingredients to credit risk modelling is the calculation and forecast of the recovery rate.

It represents the portion of payments still delivered in case of default, expressed as a percentage of the its face value.

Using bonds to describe the concept, it is defined as the *bond market value few days after default occurrence over the bond nominal value.*

We can calculate it from the loss given default as 1- LGD, since it represents the remaining fraction, after losses have been deducted from the principal.

Generally speaking, it exhibits a positive correlation with the rating of the bonds, as the higher the credit quality of the obligor, the higher the probability of having a nonnegative recovery rate given that default has occurred.

Reconnecting the discussion to the mortgage market, it can be analyzed, on the contrary that the interaction displayed between recovery rates and default rates has on average been negative. We can explain this result, focusing on the default of a mortgage loan. We have acknowledged that in this case banks foreclose the house gave as cushion and sell it in the market trying to recover the utmost.

When the default rate started to increase significantly, the price of houses declined due to the excess supply and, as a consequence, the amount rescued by banks was decreasingly low with respect to the amount loaned to the obligor.

It can beyond be said that, usually high default rates characterize bear market conditions entailing low recovery rates. This imply that the same relationship holds for other products, such as bonds.

2.2.4 REAL WORLD AND RISK-NEUTRAL WORLD

When it comes to price financial derivatives or products in general, many assumptions are usually required in order to ease the modelling settings.

One of the fundamental suppositions is that investors show a risk neutral behavior. This imply that they will not seek to minimize the returns' uncertainty but, they will rather be focused solely on maximizing their profits. As a consequence of that, when pricing, let's say a bond, cash flows will be discounted using the risk-free rate, as investors do not require any premium over the latter for holding the risk associated with the product.

Risk neutrality is likewise entailed in the computation of probabilities.

Implied probabilities based on market yields do belong to the risk-neutral world. This is the truth as, those rates exploited are based on the calculations of expected cash flows, built on the usage of the risk-free rate, and do not involve the premiums for risk.

Historical market data, where realized returns are rather taken into account, on the other hand, reflect the real-world dynamics in which real investors do care about risk and require a compensation for the unpredictability of gains.

This distinction is required in order to introduce new models which actually depart from the basic assumptions involved with the classical Black Scholes and Merton Model, and require the use of real-world estimates.

Let us define the relationship between risk-neutral and real-world probabilities as follows

$$Q = P - \lambda \cdot \sqrt{P \cdot (1 - P)}$$

where P represents the risk-neutral probability, while Q indicates the real-world measure and λ represents the market price of risk.

2.2.4.1 GIRSANOV THEOREM

The Girsanov theorem provides means by which we can move from risk-neutrality to risk aversion easily.

Let us first start from the introduction of the process as described by Black and Scholes followed by a stock price to subsequently explain, based on this, the technique used to move onto the real world.

In a binomial world with multiple periods, the price of a stock can experiences two states of world at its expiration date. Either it enjoys an appreciation, or it suffers from a depreciation. Under the binomial trees technique for the pricing of stock options, those two cases are expressed through the use of two multipliers, *up* and *down*.

In continuous time setting, the stock price follows a diffusive Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dz(t)$$

where $\mu S(t)$ represents the mean change in price per unit of time and is called the process *drift rate*; diversely, the *variance rate* for the stochastic process is showed by σ S(t), and it displays the variance per unit of time while z(t) is a Brownian motion (with drift 0 and variance equal to 1).

The above process is defined over the filtered space $\{\Omega, \tilde{S}_t, P\}$, in which omega is the sample space, $\tilde{S}_t$ represents the collection of all the sigma-algebras and P is the probability measure under which the process is described.

Generally, P symbolize the objective probability measure, at the same time Q exemplify the risk-neutral probability of occurrence of the event.

The Girsanov theorem states that the P-measure and the Q-measure can be seen as equivalent as they share the sample space to define the equal zero probability event.

The volatility of price under the two probabilities is the same, what changes is the expected growth rate in the variable's change (the drift rate), in fact we will see how changing the drift rate will lead to a change in probabilities' space.

The core of the postulate is that, whenever we have a Wiener process *z(t)* described under the probability space $\{\Omega, \tilde{S}_t, P\}$, from the latter we can construct another process $\tilde{z}(t)$, that will be a Wiener process under the new probability space $\{\Omega, \tilde{S}_t, Q\}$.

The new process $\tilde{z}(t)$, will be a Brownian Motion as the one we considered so fare, but it will show a non-negative drift under the old probability space. Under P the process will be

$$\tilde{z}(t) = \mathrm{dz(t)} + \gamma\ \mathrm{dt}$$

Let us now consider that an investor, decides to pursue its investment goals entering into a position in a highly liquid and safe instrument. The latter will yield a constant and safe return over time expressed as the instantaneous risk-free rate *r*. Imagine this safe investment is a government BTP, with payoff at t B(t). The price of the investment will then follow the process

$$dB(t) = rB(t)$$

Notice the process is characterized by zero volatility as it is assumed to be a risk-free asset.

Combining the no-arbitrage argument and the objective measure it results that the expected change in the stock price is predicted to be equal to μdt, that is the *drift rate.* In the real world, where a bonus over the risk-free rate is required, the drift should correspond to $(r + \lambda\sigma)$, where λ denotes the market value of risk. Remember that the latter can be considered the expected value of the process, that is the rate at which the investment is expected to grow.

Going back, once more, to the process for the stock price, we can plug in the new drift rate. Under the objective measure P we will have

$$dS(t) = (r + \lambda\sigma)S(t)dt + \sigma\, S(t)dz(t)$$
$$= St[rdt + \sigma(\lambda dt + dzt)]$$
$$= St(rdt + \ \sigma d\tilde{z}t)$$

Notice that the substitution into the formula of $\tilde{z}t$ holds as we are under the old probability space, where the process is still not a Brownian motion and shows a non-0 drift rate.

Moving into the risk-neutral world, the process can be arranged as

$$dSt = rS(t)dt + \sigma S(t)d\tilde{z}t$$

The new stock price process has a drift rate equal to *rS(t)* and a variance rate of σS(t).

Notice that the new expected growth rate in stock price coincides with the expected growth rate in an investment in the BTP bond, that is with the risk-free rate.

The same rationale can be employed by means of it$\hat{o}$'s lemma, to a derivative written on the same stock.

The aforesaid contract has a payoff at T equal to *G(S(T), T)*.

Via it$\hat{o}$'s formula the resulting process followed by the latter should be

$$dg(t) = \ \mu_g g(t)dt + \frac{dg}{dS}\ \sigma_g d\tilde{z}(t)$$

The above is obtained through the same mechanism as the one used for the stock price.

We first consider the process under the P-measure, that is

$$dg(t) = \left[\frac{dg}{dS}\mu S(t) + \frac{dg}{dt} + \frac{1}{2}\frac{d^2g}{dS^2}\sigma^2 S(t)^2\right]dt + \frac{dg}{dS}\sigma S(t)dz(t)$$

We know that under the objective probability measure, the drift of the stock price would be $(r + \lambda\sigma)$, substituting we get

$$dg(t) = \left[\frac{dg}{dS}(r + \lambda\sigma)S(t) + \frac{dg}{dt} + \frac{1}{2}\frac{d^2g}{dS^2}\sigma^2 S(t)^2\right]dt + \frac{dg}{dS}\sigma S(t)dz(t)$$

Exploiting again, the no arbitrage argument, the result is that the drift of the derivative should be equal to $rg + \lambda\frac{dg}{dS}\sigma$

$$\frac{dg}{dS}(r + \lambda\sigma)S(t) + \frac{dg}{dt} + \frac{1}{2}\frac{d^2g}{dS^2}\sigma^2 S(t)^2 = rg + \lambda\frac{dg}{dS}\sigma$$

$$\frac{dg}{dt} + \frac{dg}{dS} rS(t) + \frac{1}{2}\frac{d^2 g}{dS^2}\sigma^2 S(t)^2 = rg$$

Under the new risk-neutral probability the process will then be

$$dg(t) = rg(t)dt + \frac{dg}{dS}\sigma d\tilde{z}(t)$$

Where $\tilde{z}(t)$ is a Brownian motion.

2.3 STRUCTURAL MODEL FOR DEFAULT PROBABILITIES.

When addressing the pricing of a defaultable security, the first obstacle regards the modelling of the default probability associated with the cash flows of the product.

We have introduced before, the general payoff of a defaultable derivative, that is

$$G(S,T) \cdot [1 - 1_{DEF}(T)LGD]$$

With respect to the default-free world, where the uncertainty of cash flows was merely related to a market risk, addressable through binomial settings, we now have to deal with an additional unpredictability linked to the credit worthiness of our counterparty, that is the probability of loss.

It is then straightforward to understand how different approaches may lead to diverse perceptions of the risk.

Let us start the description of default risk modelling, introducing one of the main theories elaborated and adopted in the market.

Structural models find their roots back to Merton's findings (1974); exploiting the balance sheet distributions of firms the pricing of defaultable products can be easily extrapolated.

We will see how, as opposed to intensity-based model, where the default intensity is assumed, here the trigger of loss are defined via the concept of distance to default.

The discussion will be focused on the most plain vanilla item existent in the market, that is a ZCB.

We consider a defaultable ZCB and we assume that an entity embedded with default risk is issuing those products with face value $\overline{DD}$.

The firm issues today debt instruments with no coupons, meaning that the only (defaultable) payment will be transferred at the end of the holding period, that is at T.

The assets underlying the business are funded via equity and debt and displays a value equal to V(T).

The key idea behind the model is that, being shareholders residual claimants, their payoff is related to the probability of occurrence that the asset value at the end of the year exceeds the amount of debt to be repaid. In fact, whenever the latter happens, they will get the difference between the asset value and debt and receive 0 otherwise. This can be characterized with a call option on the assets value of the firm with strike price equal to the face value of debt, that is $\overline{DD}$.

The payoff for shareholders at T can then be written as

$$C(T) = \max\,(V(T) - \overline{DD}, 0)$$

Debtholders, differently, are entitled to receive at T the face value; this will be the case whenever the asset value is sufficiently high to fulfill the promised amount. By the way, it could happen that at maturity the company is not liquid enough to meet its obligations. Assuming that no limits on debtors are imposed for the takeover of the firm, they will have the possibility to recover an amount equal to the residual value of assets at that time. Their payoff at time T can then, be expressed as

$$\mathrm{DD(T)} = \min\,(\overline{DD}; V(T))$$

Recalling that the asset value must match the liability side, we have that

$$\mathrm{V(T)} = \mathrm{DD(T)} + \max\,(\mathrm{V(T)} - \overline{\mathrm{DD}}, 0)$$

$$\overline{DD} = \mathrm{V(T)} - \max\,(V(T) - \overline{DD}, 0)$$

The model goes on by recognizing that debt value can be further decomposed in two ingredients:

$$\mathrm{DD(T)} = \min\,(\mathrm{V(T)}; \overline{DD})$$

$$\mathrm{DD(T)} = \overline{DD} - \max\,(\,\overline{DD} - V(T); 0)$$

The above result suggests that the value of a defaultable debt can replicated with a portfolio including a non-defaultable bond ($\overline{DD}$) and a short position in a put option on the asset value of the firm with strike equal to $\overline{DD}$.

V(T) can then be expressed as a portfolio of securities comprising:

- a long position in a non-defaultable bond with FV= $\overline{DD}$

- long position in a call option with K= $\overline{DD}$ and underlying V(T)

-short position in a put option with K= $\overline{DD}$ and underlying V(T).

Nevertheless, it is necessary to recognize that some sharp assumptions have been made, such as the abolition of taxes, transaction costs and of bankruptcy limitations.

Turning the above into calculus and assuming the put call parity holds, we have

$$\mathrm{V(T)} = \overline{\mathrm{DD}} - \max(\,\overline{\mathrm{DD}} - \mathrm{V(T)}; 0) + \max\,(\mathrm{V(T)} - \overline{\mathrm{DD}}, 0)$$

Through the Modigliani and Miller theorem, we see that changing the structure of debt the value of the assets side is unaffected.

The asset value follows a geometric Brownian motion and under the risk-neutral assumption is defined as

$$\mathrm{dV(t) = r\,V(t)dt + \sigma V(t)d\tilde{z}(t)}$$

Since the requirements for the application of the BSM model are meet, we can exploit the option pricing techniques to recover the value of assets.

Let us first focus on the call option representing the equity value of the firm.

Its value at t can be recovered via the Black Scholes equation as follows

$$\mathrm{C(t) = V(t)\phi(d1) - \exp\left(-r(T-t)\right)\overline{DD}\phi(d2)}$$

$$d1 = \frac{\ln\left(\frac{v(T)}{\overline{DD}}\right) + (r + \frac{\sigma_V^2}{2})(T-t)}{\sigma_V\sqrt{T-t}}$$

$$d2 = d1 - \sigma_V\sqrt{T-t}$$

The current price of debt will then be

$$DD(t) = V(t) - \left[V(t)\phi(-\mathrm{d1}) - \exp\left(-\mathrm{r(T-t)}\right)\overline{\mathrm{DD}}\phi(\mathrm{d2})\right]$$

Or

$$DD(t) = \exp\,(-r(T-t))\,\overline{\mathrm{DD}} - \left[-V(t)\phi(-\mathrm{d1}) + \exp\left(-\mathrm{r(T-t)}\right)\overline{\mathrm{DD}}\phi(-\mathrm{d2})\right]$$

$$DD(t) = V(t)\phi(-\mathrm{d1}) + \exp\left(-\mathrm{r(T-t)}\right)\overline{\mathrm{DD}}\phi(\mathrm{d2})$$

We can now exploit the notion of quasi-leverage. The latter represents the fraction of debt used to finance the business assets, over the asset value. We here consider the “quasi” leverage as the book value of liabilities today is calculated on the basis of risk-neutrality, that is using the risk-free rate as discount rate.

We define the quasi leverage as d

$$d = \frac{\exp\left(-\mathrm{r(T-t)}\right)\overline{\mathrm{DD}}}{V(t)}$$

Multiplying and diving the equation by “$\exp\left(-\mathrm{r(T-t)}\right)\overline{\mathrm{DD}}$”, the result will be

$$\mathrm{DD}(t) = \overline{\mathrm{DD}}\exp\big(-r(T-t)\big) - \left[-\frac{1}{d}\phi(-d1) + \phi(-d2)\right]\overline{\mathrm{DD}}\exp\big(-r(T-t)\big)$$

$$= \overline{\mathrm{DD}}\exp\big(-r(T-t)\big)\left\{1 - \left[-\frac{1}{d}\phi(-d1) + \phi(-d2)\right]\right\}$$

Our goal is to provide an expression for the defaultable bond in terms of its default risk.

From the Black Scholes model we know that $\phi(-d2)$ is the probability that the put option will be in the money; namely it represents the likelihood by which the strike price ($\overline{\mathrm{DD}}$) will be greater than the underlying value at maturity (V(T)). In our scenario it does represent the probability of default of the bond, as in that case the firm will not be able to meet the promised payments and bondholders will be obliged to takeover the business.

Let us combine the latter concept with the pricing formula, previously provided.

We highlight the term $\phi(-d2)$ in the brackets and rearrange the equation as follows

$$DD(t) = \overline{\mathrm{DD}}\exp\big(-r(T-t)\big)\left\{1 - \phi(-d2)\left[1 - \frac{1}{d}\frac{\phi(-d1)}{\phi(-d2)}\right]\right\}$$

The next step will be to recognize that the term in squared brackets can be approximated to be the loss given default rate so that the price of the bond becomes

$$DD(t) = \overline{\mathrm{DD}}\exp\big(-r(T-t)\big)\{1 - DP \cdot LGD\}$$

The result is coherent with the solution provided at the beginning of the chapter for a general defaultable security. $\overline{\mathrm{DD}}\exp\big(-r(T-t)\big)$ represents the expected discounted payoff of the product and is then multiplies by one minus the default rate (which was represented through an indicator function) multiplies by the loss incurred in case of default.

Structural models have paved the way for practitioners to experiment different conditions under which the model can be described. Here the conceptual structure of default implies that the event will not occur until the maturity date of the contracts specified in the portfolio.

Other economists such as Black and Cox, have investigated the effect of other timings for the event.

The latter in particular in the first passage model, identified the possibility of default for the firm at any time. In fact, whenever the asset value falls underneath a specified level, a default may occur, without imposing any time constraint, due to the existence of other agency related costs.

Although the ease of application, the use of this technique has usually generated an underestimation of credit risk. As we can see from the formulas provided above, both the recovery rate and the probability of default depend on the asset value. In turns, the only source of risk associated with the replicating portfolio is represented by the options held by the company; no other factors affecting the asset value are taken into consideration.

Another pitfall concerns the choice of the process followed by the asset value. As it is considered a Brownian motion, default is treated as a predictable event.

2.4 REDUCED FORM MODEL

Modelling default probabilities and recovery rates simultaneously has shown a persistent inefficiency in recognizing a correct level of credit risk embedded in issued bonds.

One way to overcome such incongruencies is to model the default probability separately.

Reduced form models suggest methodology to do so.

The core of the theory is based on the idea that recovery rates are exogenously given, and so they do not enter in the stochastic calculations of the model. As a result, the default probability can be treated as a Poisson process indexed by an intensity rate, λ, which corresponds to its hazard rate. The latter characteristic implies another label by which those models are known, that is intensity-based models.

Reduced form models as opposed to structural ones, treat default as an unpredictable event, that occurs randomly and cannot be standardized through thresholds level of some underlying variables.

Let us remind that Brownian motion does not allow for the existence of jumps in the process, that is it only considers negligible changes in the variable in a short period of time.

Default events are more perceived as a shock, as in a small time interval the occurrence of the episode mat lead to considerable changes.

Poisson process are counting process that allow the modelling of a series of jumps (default events). Those jumps are independent from one another, and they may occur in an interval of time. N(T), will indicate the number of jumps up to time T, so N(T)-N(t), reflects the number of defaults between time t and T (assuming $t<T$).

The timing of the event is not known, but it is characterized by a rate of occurrence which enters the model (λ). Lambda represents the rate of occurrence of the jumps per unit of time, which can also be interpreted as the number of defaults expected per unit time.

Assumed that the default rate process is orthogonal to the interest rate, we can define the probability of survivance until time T (probability that default τ has not yet occurred by time T) as

$$P(\tau > T) = \exp[-\lambda(T - t)]$$

Notice that the hazard rate is assumed to be deterministic for now, implying its independence to both the interest rate and the recovery rate.

That being said, under the risk-neutral probability measure Q, the conditional probability of default, given no earlier default is given by

$$E_Q[1 - 1_{DEF}] = P(\tau > T) = \exp[-\lambda(T - t)]$$

Let us consider gain a defaultable zero coupon bond with maturity T and payoff at expiration equal to DD(T,T)=1.

We first consider the case in which the Recovery rate is equal to 0, implying a loss given default equal to 100%. The current price of the bond will be equal to its expected payoff discounted and weighted by the probability of survivance:

$$\begin{aligned} DD(t, T; RR = 0) &= D(t, T)E_Q[1 - 1_{DEF}] \\ &= D(t, T)\exp[-\lambda(T - t)] \end{aligned}$$

If, on the other hand, the recovery rate is assumed to be positive, equal to 1-Loss Given Default, the price can be break down into a hold in a default free bond and in a bond with the same default risk and recovery rate equal to 0.

Let us get back to the above pricing formula. When the recovery rate is different from 0 $E_Q[1 - 1_{DEF}]$

Is different from $\exp[-\lambda(T - t)]$, as in case of default there will still be the possibility to recoup part of the losses.

This can be expressed as follows

$$\begin{aligned} DD(t, T; RR \neq 0) &= D(t, T)E_Q[1 - 1_{DEF}LGD] \\ &= D(t, T)E_Q[1 - 1_{DEF}(1 - RR)] \\ &= D(t, T)E_Q[(1 - 1_{DEF} + 1_{DEF}RR)] \\ &= D(t, T)\{RR + (1 - RR)E_Q[(1 - 1_{DEF})]\} \\ &= D(t, T)RR + D(t, T)(1 - RR)E_Q[(1 - 1_{DEF})] \\ &= D(t, T)RR + (1 - RR)DD(t, T; RR = 0) \end{aligned}$$

What we are interested in for pricing purposes is to get the credit spread associated with the bond, that is the difference between the rate of return of the investment and the US treasury risk-free rate.

The credit spread can be extrapolated in both cases from the pricing formula. In the first case of zero recovery we can observe how the spread coincides with the hazard rate. The result arises from the shape of the credit spread curve.

Many variations to the model can be provided by letting the hazard rate follow different process.
For example, one can argue the latter is stochastic and let it follow a Cox process, that is a generalized version of the Poisson process where the indexing intensity is embedded with some sources of randomness.
Other extensions might regard the nature of the recovery rate. Once again, we have considered it to be constant and given, but of course versions which allow for stochasticity of the latter do exist.
Moreover, the computational choices regarding the recovered value may provide different evaluations.
Three possibilities do exist. The first is to consider the recovery rate in terms of market value of debt right before the default occurrence (Duffie and Singleton-1998); further it can be calculated as a fraction of notional amount as used by Hull and White (2005); last it can be considered as a percentage of the total of principal and accrued interests.
Just to sum up the discussion regarding the two default probability models, let us rewrite the two proposed pricing formulas, applying them to a general derivative contract:

STRUCTURAL MODEL

$$g(t) = D(t,T)E_Q[G(S,T)]\left[1 - \underbrace{\phi(-d2)}_{\text{DP}}\underbrace{\left[1 - \frac{1}{d}\frac{\phi(-d1)}{\phi(-d2)}\right]}_{\text{LGD}}\right]$$

INTENSITY MODEL

$$g(t) = D(t,T)E_Q[G(S,T)]\left[1 - \underbrace{\left(1 - \exp\left(-\lambda(T-t)\right)\right)}_{\text{DP}}LGD\right]$$

This is just to highlight, that despite the divergent conceptual assumptions pertinent to each model, the primary form behind both is the same.

2.5 HOW TO EXTRACT DEFAULT PROBABILITIES

Different information from disparate markets are used to model credit risk.
Structural model proposed by Merton, takes its roots from the equity markets, while the reduced form model exploits corporate bonds markets to represent the counterparty risk.
Default probabilities can be extrapolated in different ways and it can be discussed that inputs from each market are required.
In the following paragraph we will refer to implied default probabilities. Being extrapolated from different products' pricings, they reflect the assumptions made in modelling the latter. That is, since risk-neutrality is

used for the discounting of these items' cash flows, the resulting deduced default probability belongs to the risk-neutral world (under the Q measure).
One way to do so, is by considering corporate bonds annual yields. In that case hazard rates for each maturity are constructed on the basis of the fact that the average density of default over an investment period should be equal to the credit spread associated with the bond over the loss rate.(As we have, this is the case with zero recovery rates under intensity based models).
Let us better define the above said concept. Consider a T-year bond has a credit spread (defined as its annual yield in excess of the US treasury annual yield) of y(T). Since the latter is the percentage of return the product promises over the risk-free rate each year, it can also be seen as the loss rate associated with the same. This argument holds as, the spread represents the premium required to hold the counterparty risk, so it also indicates the maximum loss attainable.
The loss rate can also be expressed as average hazard rate over the T years horizons times one minus recovery rate; from that it follows that

$$\bar{\lambda}(T)(1 - RR) = y(T)$$

$$\bar{\lambda}(T) = \frac{y(T)}{(1 - RR)}$$

From the above we see that, adopting the loss given default weighted by the default intensity we can naturally approximate the effective loss suffered by the investor.
From here, it will possible to calculate the average hazard rate of a specific obligor over an exact year, given the credit spread associated with the latter. The result will be the implied default rate, as it has been extrapolated from market data.
Remaining in the corporate bond field, it is viable to exploit the reduced form models, to achieve the conditional default probability.
We know that under such models the conditional probability of survivance is $P(\tau > T) = exp[-\lambda(T - t))]$, assuming the intensity is fully deterministic.
We decompose the corporate defaultable bond with recovery rate different from 0 in a risk-free bond and a defaultable bond with the same probability of default and 0 recovery rate

$$DD(t, T; RR \neq 0) = DD(t, T)RR + DD(t, T; RR = 0)(1 - RR)$$

Recall the price for a 0 recovery rate bond equates

$$DD(t, T; RR = 0) = D(t, T)exp[-\lambda(T - t)]$$

Substituting the above value in the initial price

$$DD(t,T;RR\neq 0)=D(t,T)RR+D(t,T)P(\tau>T)(1-RR)$$

and then solving it for the survivance probability

$$P(\tau>T)=\frac{\frac{DD(t,T;RR\neq 0)}{DD(t,T)}-RR}{(1-RR)}$$

provides us with an adequate estimate of the default probability.

Notice that as the hazard rate λ represents the probability of default between time t and t+Δt, given no earlier default, whenever we consider t to coincide with time 0, the latter estimation will coincide to the default probability.

Bonds market does not represent the unique alternative in estimating credit risk probabilities.

Stocks prices present a different approach. We have introduced the possibility to assess the equity value of a firm under the Merton model via option prices.

Option quotes can be easily derived applying the standard Black Sholes model, even though, when considering the asset value of a company as underlying, its price cannot be easily found in the market, as it is not actively traded.

From a practical stand points, the unknowns of the equation will be the asset value and the volatility associated with the latter. Nevertheless, we can have access to the equity market value and proceed from here to find the missing information.

What we are interested in, in analytical terms is the probability of exercise of the call option $\phi(d2)$, from which we can subsequently, derive the probability of default $\phi(-d2)$.

Equity price is

$$C(t)=V(t)\phi(d1)-exp[-r(T-t)]\overline{DD}\phi(d2)$$

The equity approach suggests to solve the equation for the unknown asset value. What is missed here, is the level of volatility associated with the latter which can be either recovered from historical data, and then adapted to the risk-neutral measure through a change in drift (Girsanov theorem), or it can be estimated via the itos lemma considering its relationship with the call option's volatility.

Following the latter alternative, we can specify the two volatilities as

$$\sigma_c=\frac{V(t)}{C(t)}\sigma_V\phi(d1)$$

Let us now introduce the concept of distance to default.

We know that under the structural models we can adhere to different views about the timing of default.

In the last paragraph we have assumed that default could only happen at the expiration date of the bond, that is at T. Other models such as the one proposed by Black and Cox, suggest that as the event can happen at any time, we can calculate the distance to default. Technically speaking the latter indicates the number of standard deviations (used as a numeraire) by which the assets value exceeds the book value of liabilities.

Assume we can calculate the probability of default as

$$\phi(-d2) = \phi\left(-\frac{ln\frac{V(T)}{\overline{DD}} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}} \right)$$

Notice that again the above value will be specified under the risk-neutral probability measure Q, as the assumed return over the asset is the risk-free rate.

Again, we can consider the real drift rate driven by the market value of risk (premium over the risk-free return) and reformulate the above calculation substituting the value.

A well known company, dealing with default rates calculation for the majority of the firms in the market, KMV, has adopted the distance to default approach, where the latter variable is defined as

$$\phi\left(-\frac{ln\frac{V(T)}{\overline{DD}} + \left(\mu_V + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}} \right)$$

and where the process followed by the asset value is considered to be

$$dV(t) = (r + \lambda\sigma_V)V(t)dt + \sigma_V V(T)d\tilde{z}_t.$$

The distance to default X, is modelled as a Brownian motion with constant drift and constant variance.

The last approach considered for the computation of implied probability, is to consider the credit spread implicit in credit default swaps contract. From the pricing formula of CDS, we can derive the survival probability for the underlying bond conditional on no early default. A term structure can be further recovered from the bootstraps of the missing maturities.

CHAPTER 3

COPULA MODELS AND THE PRICING OF A CDO

In the last chapter we have introduced and dealt with the concept of default. We have found some different methodologies to model the probability of occurrence of the event, developed on strong assumptions. We will later on see how the intensity-based model represent the market standard model for pricing CDOs. Recalling that CDO's are tranched portions of debt backed by a pool of underlyings, another issue arises in our modelling. The implicit construction of those contracts suggests that we now have to handle with more than one obligor exposed to the risk of default.

Considering default probabilities as single separate items will not lead us to draw the right conclusion as some interrelationships between default do exist.

We need to come up with a technique able to model different defaults probabilities jointly. Copula functions allow us to do.

In the next paragraph we will analyze the concept of default dependence between different obligors, in order to clarify our necessities and to construct a bridge between single default world (single security pricing) and multiple defaults world (pool of different securities).

3.1 THE IMPORTANCE OF DEFAULT DEPENDENCE

During the financial collapse of 2007-2009 failures in the estimates of credit spreads and so of credit risk, have triggered unconscious portfolio management by the majority of investors in the market.

Wrong perceptions about the interdependence among entities caused the market to be unprepared to a radical shock that underwent different sectors.

The capital allocation embedded in ABS products, implies the dependence of the tranche profits and loss on the different defaults of the obligors characterizing the underlying. This means, that when it comes to pricing, the credit risk of the tranche must be modelled with respect to the joint probabilities of default of the pool rather than on a single marginal probability, based on the idea that the tot loss of the pool (on which the tranche depends) cannot be considered as a mere sum of the single defaults.

One of the substantial failures of rating agencies and other evaluators have been to underestimate the correlation existent between the different obligors, that is, the consequences of a default on other obligations outstanding.

It has to be argued that such dependence is usually exacerbated by the surrounding economic conditions. Specifically, when the market is in a phase of down-turn, the probability of default of a security becomes more likely, given the default of others. This represents part of a vicious circle, in which recovery rates tend to decrease, due to weak market conditions that obstacle the liquidity of the securities and the resulting possibility to recover part of their value.

Furthermore, we have already investigated the fact that financial institutions involved in the trading of ABS, and CDOs on the latter, were strictly connecting themselves to the worthiness of other participants in the market, so intensifying the risk of having a whole market breakdown in case of bankruptcy of one of them.

In CDS particularly the correlation between the reference entity and the protection issuer caused an additional risk born by the hedger, as a default of the obligor could trigger a default in the CDS seller as well.

The underestimation of risk brought to the downgrade of many products. In turns, the correlation shown by bonds pooled in the same CDOs, decreased sharply due to the different credit levels displayed a-posteriori.

Financial institutions that were playing hedging strategies holding different positions in the tranches indexed to the same underlying suffered great losses.

In particular hedge funds that were shorting the equity tranche and longing the mezzanine ones, borne the decrease in correlation due to the increase in equity-tranche spreads and the decrease in the mezzanine-tranche one, resulting in losses in both positions.

A deterioration of the highest-grade tranches occurred concurrently due to the heightened correlation between tranches generated by the encompassing status of crisis, which reevoked the prospects to protect them via the lower grade portions.

In the standard market techniques used to model default probabilities, correlation cannot be assumed to be linear.

We have to notice that this event can be caused by several factors, which can be either idiosyncratic or dictated by market conditions. As noticed before the second group is the one playing the major role during financial turmoil.

Linear correlation, on its own, does not allow us to take care of all these inputs. Moreover, in risk management we are interested to model the probability of the event jointly simultaneously for all the items included in the portfolio; linear correlation just considers pairwise interdependencies providing the bivariate marginal distributions.

Models to provide correlation estimates between assets, are divided in three groups: *Stochastic Financial correlation models*, where, as the name suggests, the dependence between defaults times is considered to follow a stochastic process with some randomness; *Statistical correlations models* which involves parameters such as the Kendall's tau; in the end Deterministic financial correlation models.

Multi-asset Gaussian copula (Li) and the One-factor Gaussian Copula (Vasicek) fall under the latter category. More specifically they are part of the bottom-up models that, as opposed to the top-down ones, first collect single asset data, quantify it and then find a model to join them together to come up with a comprehensive correlation result.

3.2 JOINT DEFAULT PROBABILITIES, MARGINAL DEFAULT PROBABILITIES: SKLAR'S THEOREM

In order to address the asset dependence in pools of underlying, Ly (2000) first had the idea to apply the copula models to financial problems.

Let us first define the marginal default probabilities of each obligor.

Consider we have a portfolio of bonds made of 2 bonds issued by two separate entities, firm A and firm B.

The probability that a variable will be equal to a specific value, can be identified with the marginal probability density function $P(x = y) = f(x)$. As a consequence, there will exist a function, the marginal cumulative distribution function, that allows us to predict the likelihood of the variable to fall within a certain interval [0,y], that is $P(x \leq y) = F(x)$.

The latter function is used to model the default probability of the obligor. If we let x be the event of default, we can then calculate the probability of having a default before time t via the CDF.

In our example we have a pool of bonds, meaning that the event of default can occurs in both products. We have already stressed out that considering the correlation between the two to be linear would bring us to underestimate the effective dependence shown due to other exogenous conditions.

We now need to model both jointly.

The Sklar Theorem provides us with a viable solution to the latter problem.

Let us make an example considering an European style bivariate digital put option. The latter is an option written on two underlying, two stocks, and that will pay 1$ whenever both prices will flow under their respective thresholds levels at time T (expiration date of the contract)

We define K1 and K2 as the strike prices, S1 and S2 as the stock prices observed in the market.

We can observe via the cumulative density function the probability of prices being lower than the strikes

$$P(S_1 \leq K_1) = F(K_1)$$
$$P(S_2 \leq K_2) = F(K_2)$$

The payoff of the option will depend on both probability distributions, so that the joint probability of both events occurring concurrently must be modelled.

The joint probability can be written as

$$P(S_1 \leq K_1, S_2 \leq K_2) = F(K_1, K_2)$$

The Sklar theorem states that whenever we have the two marginal probability distributions, there will exist a copula function able to synthetize the joint probability of the two events.

It states that whenever we have the marginal probability distributions of two events F(K_1), F(K_2), the multivariate distribution F(K_1, K_2) can be represented through a copula function having the two marginal distributions as arguments C(F(K_1), F(K_2)).
The price of the option can then be expressed as follows

$$DP_t = 1\$ \exp\big(-r(T-t)\big)\, F(K_1, K_2)$$
$$= 1\$ \exp\big(-r(T-t)\big)\, C(\, F(K_1), F(K_2))$$

If we consider the structural model for default calibration, we know that the trigger of default will be the asset value being lower than the value of liabilities (outstanding bonds).
Consider for example a cash CDO tranche backed by a pool composed by just two bonds, the probability of default of the said tranche will be built upon the joint probability of default of the underlying. Remember value of assets at T is expressed as V(T), and book value of debt as $\overline{DD}$. The probability of bond A defaulting can be described by the probability of V(T) being lower than $\overline{DD}$ for entity A, $P(V_A(T) \le \overline{DD}_A\,) = F_A(\overline{DD})$. The same applies to company B. In order to derive the tranche's value, we express the joint cumulative probability function via the copula

$$P(V_A(T) \le \overline{DD}_A, V_B(T) \le \overline{DD}_B) = C(F(\overline{DD}_A), F(\overline{DD}_B))$$

Assuming lognormality of the assets' value, we can apply the Black Shole formula to derive the marginal probabilities of default.
Lognormality implies that the ln of the Asset value returns, $ln\frac{V_A(0)}{V_A}$ follows a normal distribution having mean $(r - \frac{\sigma_A^s}{2})T$ and variance $\sigma_A^2 T$.
Under the above assumption, the probability of default is approximated with $\phi(-d_{2A})$

$$P(V_A(T) \le \overline{DD}_A) = \phi(-d_{2A}) = \phi\left(\frac{\ln\left(\frac{V_A(0)}{\overline{DD}_A}\right) + \left(r - \frac{\sigma_A^s}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}}\right)$$

$$P(V_B(T) \le \overline{DD}_B) = \phi(-d_{2B}) = \phi\left(\frac{\ln\left(\frac{V_B(0)}{\overline{DD}_B}\right) + \left(r - \frac{\sigma_B^s}{2}\right)(T-t)}{\sigma_B\sqrt{(T-t)}}\right)$$

So that the joint cumulative probability can be defined with a copula as

$$P(V_A(T) \le \overline{DD}_A, V_B(T) \le \overline{DD}_B) = F(\overline{DD}_A, \overline{DD}_B) = C\big(F(\overline{DD}_A), F(\overline{DD}_B)\big) = C(\phi(-d_{2A}), \phi(-d_{2B}))$$

The opposite also holds true, that is whenever we have a copula function taking univariate marginal distributions as arguments, there will exist a multivariate joint distribution.

Many different copulas can be created to represent the multivariate cumulative distributions, unless the marginal ones are continuous. In the latter case the copula will be unique.

The copula defined so far is a bivariate copula as it allows for the modelling of two arguments.

Multivariate copulas can be created as well exploiting the property described by the Sklar's Theorem, assumed that four requirements are met.

First of all, the arguments must be contained in the unit interval as they are representing probabilities and as a consequence the output of the copula must fall within the same interval.

Considering the bivariate case, whenever the probability of the first event becomes 0, then the probability of them jointly occurring should be 0 as meaning that the output of the copula is 0, in general terms

$$F(x) = 0, F(y) \neq 0 \Longrightarrow C(F(x), F(y)) = 0$$

If the first event has a probability of occurrence equal to 1, then the copula function should equal the likelihood of the second event to occur. In other words, as we know for sure that one out of the two companies will default, the probability of having both defaulting will equal the expectations about the second defaulting as well.

$$F(x) = 1 , F(y) \neq 0 \Longrightarrow C(F(x), F(y)) = F(y)$$

Last, the copula function must be increasing in both arguments, so that as the likelihood of one of the two companies of defaulting increases, also the joint probability must increase.

Sklar theorem further covers the multivariate case.

If we have a set of n random variables $X = x_1, x_2, \ldots, x_n$ with marginal distributions $F_i(x_i)$, there will exists a subcopula such that

$$F(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n))$$

Furthermore, if the marginal probabilities are continuous, then the sub copula is a copula.

3.3 THE GAUSSIAN COPULA

Copula functions were first introduced in the statistical literature in the 1959 by Abe Sklar, it was David Li (2000) who originally had the idea to apply those models to the resolution of financial problems.

We have seen in the previous section that the Sklar theorem allows us to define a multivariate probability distribution in terms of copula, simplifying so the modelling of n functions via the introduction of an n-variate single function (the copula itself).
The apex of popularity has been reached during the financial crisis, mainly due to the use of the Gaussian copula for the pricing of CDO's tranches, adopted by rating agencies.
The assumptions made for the modelling of correlations among assets in the pool were at the core of the criticism related to the adoption of this technique, even though it provided a valid alternative for the adjustment of the multivariate distribution to include different levels of dependence.
Let us introduce the latter and describe the main advantages and disadvantages.
A Copula function, technically speaking, is a function able to transform an n-dimensional function into a unit-dimensional one.
Let us consider n univariate marginal distributions, defined as $Q_i(x_i) = P(X_i \leq x_i)$ with $i = 1,2, \dots, n$. In the previous paragraph we have defined the latter probability in terms of strike price and stock price and we have denote it by $F(K_1)$ and $F(K_2)$.
Each marginal probability belongs to the unit-dimension [0,1] and we know that there exists a copula function such that

$$F(X_1, X_2, \dots, X_n) = P[X_1 \leq x_1;\ X_2 \leq x_2; \dots;\ X_n \leq x_n) \qquad \in [0,1]^n$$

$$F(X_1, X_2, \dots, X_n) = C(\ F(X_1), F(X_2), \dots, F(X_n))$$

To express the above in terms of multivariate density function we change the variable $x_i \to F_i^{-1}(x_i)$

$$C(x_1, x_2, \dots, x_n) = F_n(F_1^{-1}(x_1), F_2^{-1}(x_2), \dots, F_n^{-1}(x_n))$$

Copula functions are helpful in that they allow us to consider the correlated default probabilities.
In fact, the function defining the copula is characterized by a correlation factor that allows for the interdependence of the marginal distributions.
Let us consider the case in which the latter cannot be described as a perfect correlation ($\rho = \pm 1$) , namely the specific characteristics of the variables are such that they do not own similar properties.
For corporate bonds, it could be the case; for example, if they are issued by entities belonging to different sectors, or if they have significant different ratings attached to them.
In that case, a non-parametric variable representing correlation can be introduced so that the joint cumulative distribution computed in the inverse of the arguments satisfies the requirements of the copula function

$$C(x_1, x_2, \dots, x_n) = F_n[F_1^{-1}(x_1), F_2^{-1}(x_2), \dots, F_n^{-1}(x_n); \rho_F]$$

The above implies that given the marginal distributions, there exist a function that allows us to map $F_i(x_i)$ from i=1 to i=n, through a function F_i^{-1}, and that will subsequently permit us to join them into a single univariate function $F_n[F_1^{-1}(x_1), F_2^{-1}(x_2), \dots, F_n^{-1}(x_n); \rho_F]$, described by the correlation ρ_F.
Copula functions can be divided in one parameter copulas and Archimedian family copulas.
Gaussian copulas fall under the first category. The main idea behind them is that the standard normal distribution can be exploited for the construction of the copula. Specifically, if we let F_n be the Gaussian n-variate distribution, and the marginal probabilities to follow the standard normal distribution, we end up with the Gaussian copula that allow us to define the following

$$C^{Ga}\big(F_1(x_1), F_2(x_2), \dots, F_n(x_n)\big) = M_n\big[N_1^{-1}\big(F_1(x_1)\big), N_2^{-1}\big(F_2(x_2)\big), \dots, N_n^{-1}\big(F_n(x_n)\big); \rho_M\big]$$

where M_n is the n-variate standard normal cumulative function. Further assuming that the marginal probabilities are uniform will allow us to state that $N_i^{-1}\big(F_i(x_i)\big)$ follows a standard normal distribution as well. The latter represents the x-axis values of the standard normal distribution, in that through the function N_i^{-1} we are mapping percentile to percentile the cumulative marginal probabilities to a standard normal distribution.
Moreover ρ_M is an $n \times n$ symmetric, positive-definite correlation matrix of the normal distribution.
Applying the notation provided by Li in his paper, we define the marginal probability distribution of entity i at a fixed time t as $Q_i(t)$. There exists a Gaussian copula C^{GA} such that the cumulative default probabilities can be mapped through $N^{-1}$to standard normal, and then merged together in the Gaussian function M_n.
Let us make an example in a bivariate copula case.

3.3.1 EXAMPLE OF A BIVARIATE GAUSSIAN COPULA

We have two speculative-grade (BBB-) corporate bonds (two entities), with given cumulative default probabilities over the holding period.
The cumulative default probabilities represent the likelihood of default every year, not conditional on any prior default or survivance, to rephrase it they are unconditional.
Suppose the time to maturity of the bonds is T= 5y. The table below synthetizes the cumulative probabilities over the years.

Table 3.1

TIME	$Q_A(t)$	$Q_B(t)$
1	6,3%	2,6%
2	13,4%	7,3%
3	21,3%	14,2%
4	29,8%	23,6%
5	39,8%	38,9%

Notice that the default probabilities tend to increase with time, so does the cumulative one. We are here considering two companies with no evidence of distress likelihood, that is, for the information currently available about their financial positions, they are both conducting their business with an assets' liquidity at average levels, even though some other variables may cause them to default subsequently.

Having just two companies, our copula will have two arguments, and it is defined as

$$C^{GA}\big(Q_A(t), Q_B(t)\big) = M_2\big[N^{-1}\big(Q_A(t)\big), N^{-1}\big(Q_B(t)\big); \rho_M \ \big]$$

Applying our Gaussian copula, we can map the above marginal probabilities to the standard normal through the inverse of the standard normal distribution N_2^{-1}. Notice that in our example it will be a bivariate distribution.

We do so using the excel function "normsinv($Q_i(t)$)". The results are displayed in the following table

Table 3.2

Time	$Q_A(t)$	$N^{-1}(Q_A(t))$	$Q_B(t)$	$N^{-1}(Q_B(t))$
1	6,3%	-1,5301	2,6%	-1,9431
2	13,4%	-1,1077	7,3%	-1,4538
3	21,3%	-0,7961	14,2%	-1,0714
4	29,8%	-0,5302	23,6%	-0,7192
5	39,8%	-0,2585	38,9%	-0,2819

Graphically, we are mapping percentile to percentile each cumulative distribution to a standard normal one. Once we have reconstructed the curve, we look at the abscise corresponding to each mapped point.

We are halfway through the resolution of the problem. We now have the mapped default probabilities; we can now join them in the Gaussian multivariate distribution M_n, as the formula suggests.
Since we are dealing with two entities, the correlation between them can be expressed as a single number, as opposed to the multivariate case, in which a correlation matrix is involved.
Let us assume their Gaussian correlation for the next year is 0,5, that is their defaults are subject to positive comovements, due to similar economic conditions, and so similar ratings.
The joint cumulative probability of both company A and B defaulting in year 1, given the correlation rate, can be defined as

$$Q(t_A \leq 1 \cap t_b \leq 1) = M_2(x_a \leq -1{,}5301 \cap x_b \leq -1{,}9431, \rho = 0{,}5)$$

In the above we express the joint probability of having the default before the end of year 1, by considering the joint probability of having the mapped values x_a and x_b lower or equal than the abscise corresponding to each mapped distribution.
We can solve it in excel through the bivariate distribution code, so to have

$$M_2(x_a \leq -1{,}5301 \cap x_b \leq -1{,}9431, \rho = 0{,}5) = 0{,}0088 = 0{,}88\%.$$

The copula function furthermore allows us to model the joint probability of default in two different years. If we want to know the joint probability of company A defaulting in year 5, and company B defaulting in year 2, assuming the gaussian correlation remains constant over time, we can model the problem as below

$$Q(t_A \leq 5 \cap t_b \leq 3) = M_2(x_a \leq -0{,}2585 \cap x_b \leq -1{,}0714, \rho = 0{,}5) = 10{,}28\%.$$

3.4 GAUSSIAN COPULA WITH MULTIPLE ASSETS

When dealing with multiple assets, such as in the case of CDOs, we can still rely on the use of copula functions for the modelling of the joint cumulative probability distribution.
We have shown that the Sklar theorem, applies even when we have a set of n variables, so that the requirements necessary for the transformation of the marginal probabilities are met.
Consider we have a portfolio of n-bonds. We seek to find their correlated probability of default. Assuming they are described in continuous time, we can find a unique copula for our computation.
In that case, a correlation matrix will characterize our standard normal distribution M_N.
We start by considering a sample $M(.)$ from the n-variate distribution. We. Equate the latter with the marginal probability distribution of company i $Q_i(\tau_i)$ and we solve it in terms of default time τ_i)

$$Q_i(\tau_i) = M(.)$$
$$\tau_i = {Q_i}^{-1}(M(.))$$

Mathematically this is done through the Cholesky Decomposition theorem, by which the correlation matrix can be decomposed into a matrix multiplication between a special symmetric, positive definite lower triangular matrix M, and its transpose M^T. Uncorrelated random samples are then extracted from the standard normal distribution and are multiplied to the M matrix giving the correlated random values we can equate to our marginal probability distributions.

Whenever the mapped marginal probabilities $N^{-1}(Q_i(x_i))$ are greater than the extracted samples, the company will default.

The goal of the procedure is to find the correlated default times, solving the equation for τ_i.

Many simulations are made and the related time of default are estimated via average, so to have as much precision as possible.

Even though highly used, especially during the financial turmoil of 2007-2009, copula models do have some noticeable drawbacks.

Studies conducted by Duffie et al.(2009) and Das et al.(2007) highlight that the correlation tends to grow during periods of distress. In the course of the crisis the correlation among different companies increased, as the general market conditions were weakened by an insufficient liquidity and a heightened uncertainty. In these environmental conditions, companies are more likely to default given the default of the others.

Following the above mentioned rational, it would be necessary to use a distribution that describes a joint distribution with higher comovements in the lower tails.

Copula functions do provide a way to model this tail dependence, even though the Gaussian copula in particular does not satisfy it for any value of ρ.

Further criticisms have been moved in regard of the choice of correlation between the assets. A single correlation parameter is usually chosen in practice to fit the relationship between any two assets, even though in reality, market traders influence the choice altering the correlations to have higher resulting tranches spreads. This is true especially for the lower (equity) and the higher (senior) tranches, and it leads to a phenomenon known as the correlation smile.

It can be further argued that using Gaussian copula for defaults simulations, the latter will converge slowly, as in order to have a statistically reliable result, some of the random samples must lead us to the conclusion of non-default.

Last, copula functions are static models, in which a one-year time horizon was considered at first. Even though some subsequent extension of the methodology proposed by Ly(2000), have tried to overcome the inflexibility caused by staticity, the time to default and the default correlations still lack of a stochastic process.

3.5 ONE FACTOR GAUSSIAN COPULA

Pricing large portfolio of assets with the Gaussian copula do provide a simplicist way of modelling correlated default probabilities of each single product, but, on the other hand implies a lot of calculation and an abundant set of information.

Often market practitioners make strong assumptions in trying to simplify pricing models.

In 1987 Oldrich Vasicek introduced a model to price credit risk in homogeneous portfolios, called the one-factor copula function. In the latter two main assumptions defining the homogeneity of the asset pool are made: the first is that, when we are dealing with bonds with similar ratings or belonging to analogous sectors, the pairwise correlation among the assets may be considered to be unique. In addition to that, having comparable characteristics, their probability of default could be thought to be the same.

Three ingredients are then required for modeling. They are the *recovery rate* RR, the *default intensity* λ, and a default correlation, that in this specific case will be defined as *conditionally independent default correlation*.

We consider a portfolio of N entities issuing assets that join the homogeneous portfolio. We define a latent variable $x_i, i = 1,2, \dots, N$ that is a default indicator for the i^{th}obligor. The variable x exhibits a negative relationship with the default probability, as the higher it is the farer in years the default time

will be, and viceversa. It can also be interpreted as the company's overall health, so that the weaker it is the sooner default will occur.

Each asset is considered to be affected by a common factor M, that interprets the role of the systematic market factor, and is an indicator of the whole economy conditions.

Furthermore, an idiosyncratic component Z_i, representing each firm particular position, is included in the model. Both are distributed with o mean and unit-variance.

All the above can be summarized in the formula for the calculation of the latent variables

$$x_i = \sqrt{\rho} M + \sqrt{1 - \rho} Z_i$$

with ρ equal and constant for each pair of assets.

Here we are not considering the conditional default probabilities directly, but, instead, we are conditioning the defaults on M. For example, whenever $\rho = 1$, assets are perfectly dependent, and in fact the above formula will simplify to the common factor M. Contrarily, if $\rho = 0$, the only driver of the default indicator will be the firm specific factor.

M and Z_i are found drawing samples of random variables from standard normal distributions.

Once the x_is are calculated on the given correlation, they have to be subsequently transformed into cumulative probabilities through the cumulative standard normal distribution. For that reason, the model is called Gaussian copula model.

De facto, important divergences exist between the one-factor model and the model we described in the last paragraph.

First of all, in the latter, we have to deal with an $n \times n$ correlation matrix, while here we just use a single parameter. In that regard we do not use the matrix to correlate defaults directly and we cannot allow for different pairwise correlations. It follows that the standard normal distribution considered here is one-dimensional as opposed to the n-dimensional used before.

It can be notice that in the bivariate case the two models coincide as the correlation matrix will be made of identical entries. As a consequence, the random sampling conducted through the application of the Cholesky decomposition, will yield the same result for x_i, as the one we achieved under the one-factor Gaussian copula.

Technically speaking the process involves different steps.

a) The first one comprises the draws of random samples for our indicator variables. For each simulation we need a single sample for the common factor M and, as much samples as the number of assets for the variable Z_i.

b) Once we have our inputs, and given an exogenous correlation, suitable for each pair of assets, we calculate the x_is from $x_i = \sqrt{\rho}M + \sqrt{1-\rho}Z_i$.

c) We transform the above results into probabilities via the cumulative standard normal distribution N.

d) We calculate market thresholds by considering 1- the above probabilities.

e) Each asset survival probability is recovered from the market given hazard rates λ_i. If we have a single constant intensity value, we directly apply the formula relating it to the survival probability $s_i = (1 - \lambda_i)$. If, on the other hand we have a curve of time dependent intensities for asset i, the procedure requires to construct a cumulative survival probability curve for each asset, considering the market given default rates, and subsequently calculating the intensity for each year t.

f) We derive the correlated time to default of each entity by equating, the survival probability found in the previous step with the market threshold, provided by the model. Let us define $N(x_i) = P_i$, we then have $1 - P_i = {s_i}^t$, from which $t_i = \frac{ln(1-P_i)}{\ln(s_i)}$. In the case of dependence of the hazard rate on time, we relate the cumulative survival probabilities to the simulated thresholds. Monte Carlo is then used in both states, to derive the average of default times for each entity i.

3.6 OTHER TYPES OF COPULA FUNCTIONS

The Gaussian copula just represents one of the main instruments used for the pricing of CDOs. In reality many other copulas do exist and can be applied for the pricing of different credit products.

We considered before an example of a digital call option. Let us now define a digital call option written on two stocks. The latter will pay 1$ whenever the price of both securities will be above their respective strike prices, so that the payoff can be defined as

$$DC_t = \exp\big(-r(T-t)\big)\, F(S_1 > K_1;\ S_2 > K_2)$$

We use, again, a copula to define the joint probability

$$DC_t = \exp\big(-r(T-t)\big)\, \bar{C}(F(S_1), F(S_2))$$

$\bar{C}(F(S_1), F(S_2))$ is known as the survival copula as it is now mapping the marginal survivance probabilities, instead of the default ones and it is related to the standard copula through a no-arbitrage argument

$$\bar{C}\big(F(S_1), F(S_2)\big) = 1 - F(S_1) - F(S_2) + C\big(F(S_1), F(S_2)\big).$$

Let us go back to our example of European digital put option to define the well-known Fréchet boundaries. We have two stocks, coming from two different markets, let's say one from the Japanese market, and the other from the US one. The two products are then affected by different market comovements, assuming that the two markets are not highly correlated. We can so infer that the correlation between stocks is 0. This result will lead us to introduce the Product copula.

The latter derives from the consequent independence of the marginal probabilities, so that the copula function will yield us the product of the latter two

$$DC_t = \exp\big(-r(T-t)\big)\, C\big(F(S_1), F(S_2)\big) = \exp\big(-r(T-t)\big)\, F(S_1)F(S_2).$$

The joint probability of default can be further constrained within two boundaries level. The latter are known as the Fréchet Boundaries and are extracted from the case of perfect correlation between the underlyings. Consider our stocks are perfectly positive correlated, $\rho = +1$.

From the properties required under the Sklar theorem, we know that whenever one of the stocks' price will for sure be lower than the strike, the joint probability of default will equate the marginal probability of default of the second stock. The copula function will yield as a result, the minimum between the two marginal distributions, as when one happens with no uncertainty its marginal probability will be 1

$$DP_t = \exp\big(-r(T-t)\big)\, C\big(F(S_1), F(S_2)\big) = \exp\big(-r(T-t)\big)\ \min\,(F(S_1), F(S_2))$$

On the other hand, whenever the two, markets exhibit a persistent perfect negative correlation, the copula simplifies to

$$DP_t = \exp\big(-r(T-t)\big)\, C\big(F(S_1), F(S_2)\big) = \exp\big(-r(T-t)\big)\ \max\,(F(S_1), +F(S_2) - 1; 0)$$

The two copulas just described are known respectively, as the maximum copula and the minimum copula, and they describe the thresholds value within which the copula must be contained.

In the case of imperfect correlation on the assets (as usual), a linear combination of the above can be made so that we derive a Fréchet family copula

$$C\big(F(S_1), F(S_2)\big) = \beta \max(F(S_1), +F(S_2) - 1; 0) + (1 - \alpha - \beta)F(S_1)F(S_2) + \alpha\min\,(F(S_1), F(S_2)).$$

Regarding the theme of tail dependence, we have seen how Gaussian copula is not able to include high correlations levels at the extremes. Sometimes the t-copula is instead used to overcome this inefficiency.
The student-t copula exploits, as the name suggests, the t-distribution which is fatter and shorter with respect to the Normal one. It can be described as

$$T_C^v(x_1, x_2, \dots, x_n) = t_c^v(t_v^{-1}(x_1), t_v^{-1}(x_2), \dots, t_v^{-1}(x_n); \rho_t)$$

here the standardized n-variate t-distribution is considered to join the marginal probabilities. Notice that it is characterized by v degrees of freedom (n-1). The Cholesky decomposition is used as well to draw the random samples. Here the presence of other variables in the modelling, require the draw of new simulations.
Considering the description provided by Roncalli (2002), the Gaussian copula and the t-student copula can be expressed as

$$C^{GA}(v, z) = \int_0^v \phi\left(\frac{\phi^{-1}(z) - \rho_{x,y}\phi^{-1}(t)}{\sqrt{(1 - \rho_{x,y}{}^2)}}\right) dt$$

$$T_{\rho,v}(v, z) = \int_0^v t_{v+1}\left(\sqrt{\frac{v+1}{v + t_v^{-1}(s)^2}}\,\frac{t_v^{-1}(z) - \rho t_v^{-1}(s)}{\sqrt{1 - \rho^2}}\right) ds$$

we can intuitively see that the t-copula entails more factors to be considered. In fact, when using it to find the default time, we have to add the representation of an additional variable, that is *s*.
The latter is simulated from a chi-squared distribution with v degrees of freedom.
Even though it allows for the modelling of correlated extreme events, the t-copula is based on a symmetric distribution causing a high probability of zero losses.
Another famous class of copulas is represented by the Archimedian copulas, that as opposite to the Gaussian one, allows for the explicitation of a formula.

3.7 PRICING MODELS FOR CDOs

Diverse techniques can be applied for the computation of the fair spread in CDOs. Even though different inputs may be required, the crucial informations are the marginal default probability, the default correlation among assets, and the recovery rate in case of default.

The first peace of information required, is the marginal distribution of each single assets.
This can be either achieved through the structural models, introduced by Merton, or via the intensity-based models, in which the hazard rate is modelled. Another common practice is to use a Monte Carlo simulation, to then average the results and come up with a significant estimator of correlated defaults.
Copula models, principally the Gaussian one, have been adapted to the financial field, to create an estimate of correlated marginal default probabilities, via the n-dimensional correlation matrix, and to establish the expected time of default of each tranche, by equating the mapped probabilities with a market threshold indicated by a sample of the n-variate standard normal distribution. The same result can be achieved via an alternative procedure that exploits the reduced form models. Following the theories implemented by Lando and Duffie and Sinleton, we can use the hazard rate to discount risky claims. The probability of survivance of the defaultable entity i, will then be approximated by

$$P\{\tau_i > t\} = exp\left\{-\int_0^{\tau_i} \lambda_i(t)dt\right\}$$

Under the procedure proposed by the Gaussian copula approach, we draw a sample from the n-variate standard normal distribution, and we can now equate it to the survival probability

$$M(.) = exp\left\{-\int_0^{\tau_i} \lambda_i(t)dt\right\}$$

From which the correlated default time is extracted as

$$\tau_i = \frac{-\ln(M(.))}{\lambda_i}$$

All the above models have the aim of letting us find the correlated default times of each asset. Knowing the notional and the recovery rates of the latter, will further allow us to find the outstanding value at each time, by just subtracting the loss given default from the original principal.

$$ON = N(1 - LGD)$$

We have introduced CDOs in the first chapter, and we have already stressed out the way in which losses and profits are allocated through the different tranches.
When dealing with synthetic CDOs, that is, tranches written on pools of CDSs contracts where no actual ownership of the underlying bonds is required by the Special Purpose Vehicles, our aim is to find the fair spread that makes the deal, at least theoretically, arbitrage free.
We can define the notional of a tranche by looking at its attachment and detachment points.

We know these values provide us with the range of percentage losses of the pool, in which the tranche will be attacked. For example, if we have a pool of 100 assets with notional equal to 1$M, the equity tranche with attachment and detachment equal to 0%-4%, will be characterized a notional of 3$M. If we further consider that each asset has 1$M of notional, and that the recovery rate is 0, we can state that the considered tranche will bear losses for the first 3 entities defaulting.

The swap theory, intuitively suggests, that at $t = 0$, the value of the two legs must coincide, so that the price of the product equates 0.

We define the present value of the spread tranche, that is the PV of the payments promised to the tranche holder as

$$PV(Spread\ tranche) = Spread_i E\left(\sum_{t=1}^{N} e^{-rt} ON_i(t)\right)$$

and the present value of the payout leg, that is the payment in case of default as

$$PV(payout\ leg) = E\left(\sum_{t=1}^{N} e^{-rt}(ON_i(t-1) - ON_i(t))(1 - RR)\right)$$

Here we are assuming that the coupon payments are made in arrears and that the default of the entity can occur anyways through the middle of the year.

In regard to the recovery rates, they are usually deduced from historical rates of defaulted companies.

By the way some other considerations may be made to assign them. For example, S&P, assigns higher recovery rates to lower-graded securities. The reasoning behind it, is that higher graded companies usually only default, in harsh market conditions; when a crisis is in act recovery rates sharply decrease. Contrarily, low-grade firms, may default anyways, notwithstanding the market conditions. Consecutively, default may happen during a period of flourishing economic circumstances, which usually entail, higher recovery rates on average. Other aspects, such as the seniority and the security are taken into account.

3.8 THE GAUSSIAN COPULA AND THE CRISIS

"The formula that Felled Wall Street", this is how the financial times referenced to the Gaussian copula in 2009, when after the domino effect traversed all the financial sectors, the economy was destroyed and experts were looking for someone to blame.

One of the most severe failure have been identified in the use of the Gaussian copula for the pricing of CDO's tranches.

As we have seen, CDOs are usually constructed on huge pools of assets. When written on CDS contracts, for example, 125 underlyings with their respective default probabilities and correlations need to be model. Of course, this is not as easy and reliable as it seems.
Furthermore, the composition of synthetic CDOs exacerbated even more the chaos. The latter were usually written on assets that had never defaulted before, or on tranches of ABSs, that were relatively new in the market, so the accuracy of the estimates for recovery rates and default density was very weak. In fact, market data from normal economic conditions were considered for the pricing of those products, resulting in estimation that were not able to reflect the current crisis situation and the risk generated by it.
Correlation have been at the center of the critics. Different approaches were used when choosing the comovements describing the pools of assets, but, notwithstanding the computational difficulties, there have been no mercy in casting rating agencies. S&P first used the gaussian copula to provide tranches ratings in 2001; according to their standards correlation was considered to be the same for assets coming from the same sector. As a consequence, the correlation for MBS was considered to be roughly the same for all, as, in addition to the similarity in provenience, the default risk associated with mortgage loans was mainly described by idiosyncratic personal information, not available on the market. The latter feature allowed for the construction of a chain between mortgage loans, ABSs and CDOs. Asset backed securities were diversifying the unsystematic risk, and CDOs written on them permitted investors to ulteriorily eliminate the apparently low dependence between ABSs' tranches.
Due to their structural construction, the dependence between the CDOs and the effective original underlyings of the ABSs on which they were written, was ignored in the computation of ratings, as they were modelled separately by different entities.
As the housing market started its crash, the risk associated to the loans started to be driven by systematic market factors, influencing so the solidicity of ABSs as well. This adjustment in the nature of risk was not taken into account. Correspondingly the correlation between ABSs rose rapidly causing the given rates for CDOs to be no longer consistent with the market prices.

CHAPTER 4

APPLICATION: PRICING CDO TRANCHES WITH THE ONE FACTOR GAUSSIAN COPULA

4.1 Product Description: CDX NA IG 34 V1

In the first chapters we have discussed the different kinds of CDOs existent in the market. We have made a difference between synthetic and cash CDOs; a further distinction has to be made.

Synthetic CDOs are tranches written on a pool of CDS, those contracts, as such, are said to be "funded" in that a sale of the underlying swap contracts is provided via credit linked notes.

With the increase in trading and appreciation for such products, Credit Default Index Swaps have been created. The latter are indexes on a pool of standardized CDS contracts that trade on the OTC market, and they have become the most liquid credit derivative.

The main difference between a Tranche written on an index and the one written on a single CDS, is that in the former case when defaults occur the protection buyer continues to make spread payments, but on a reduced notional, moreover instead of embedding real CDSs contracts they just emulate its cash flows . The pricing technique of CDIS index tranches is still the same as the one for synthetic CDOs as they are economically equivalent in terms of cash flows.

Two main families of indices do exist and those are the CDX type and the iTraxx type, with the former making reference to north American markets and emerging ones, and the latter to the European and Asian markets.

Both kinds of indices share the same composition of the underlyings, that is each title in the pool is characterized by the same weight, and the total of contracts is usually 125, so that the equal weights are 0,8%. Furthermore, the standard maturities to which they make reference are 3years,5years,7 years and 10 years (where 5 and 10 are the most liquid).

We will focus on CDXs and make a further classification into high yields CDX, where the bonds underlying the CDS are assigned non-investment grades, High volatility CDX, , emerging markets CDX, crossover CDX and investment-grade CDX.

The product chosen for our application is the latter one.

CDX NA IG 34 V1 is the full name of the contract as it is traded, we have already determined what IG stands for, so we can focus on the rest of the specification.

First of all, we have to introduce the trading rules for the CDXs on the market. They are based on quarterly payments usually made in arrears and each 6 months, on the 20th of September and on the 20th of march the index is rolled over and the spread associated with it is recomputed.

Rolling the index means that the entities that have defaulted by the time, are substitute by new ones sharing same characteristics as to maintain the coherency of the entire index.

Each time the index is rolled a new version is creates, so from that we can infer that the "34" in our product stands for 34th version, with the first one being created in 2003 (34 semesters are equivalent to 17 years).

The Index is traded with a specified spread which should represent the average credit risk associated with the underlying contracts; even thought it could be tempting to compute it as a weighted average of the underlying spreads using the portfolio weights of 0,8%, it can be seen that the two do not coincide, this is due to an adjustment made in order to consider the correlation that does link the defaults and so the realization of each contract, so that the weight used is equal to the risky duration of asset i over the sum of all the risky durations in the portfolio.

It can be shown that in our case the weighted average (with reference to portion of tot Nominal) should be a CDS par spread equal to 86,96, while the index is currently trading with a par spread of 69,7.

The total notional on which the index trades is assumed to be 125.000.000.000 $, so that each contract represents a notional of 1.000.000.000 $. The CDX NA IG version 34 is currently trading with an index factor of 1 indicating that the contingent payments upon credit events occurrence will cover the tot loss perceived by the underlying.

Indexes have become really famous since the very beginning mainly because of the standardization and transparency embedding their trades.

Let us now suppose that there exist a special purpose vehicle willing to tranche this index and to transfer the credit exposure of each obligor in the pool to external investors.

In constructing tranches, we will make reference to the standard attachment and detachment point associated to the CDX investment grade. The latter are collected in the below table.

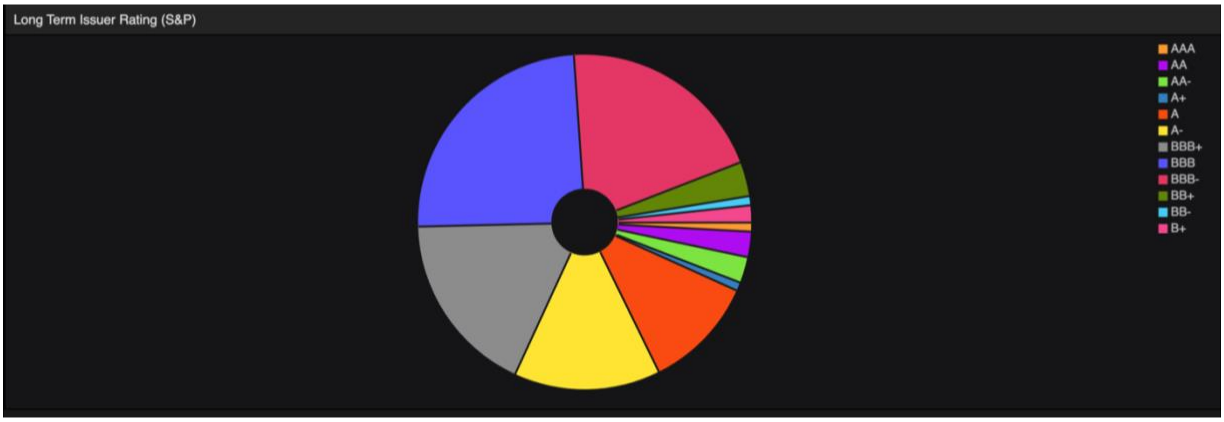
Table 4.1

TRANCHE	Attachment (%)	Detachment (%)	Attachment (M$)	Detachment (M$)	Tranche Notional (M$)
Equity	0	3%	0	37,5$	37,5$
Junior Mezzanine	3%	7%	37,5$	87,5$	50$
Senior mezzanine	7%	10%	87,5$	125$	37,5$
Senior	10%	15%	125$	187,5$	62,5$
Supersenior	15%	30%	187,5$	375$	187,5$

We can immediately recognize that the total value of the underlying index's Nominal that is tranched is the 30% and not the 100%. This is because we are dealing with investment grade assets that have a low probability of default, so that the probability that all the assets will default by the expiration, that in our case is T=5 years, is equal to 0 on average.

Below a pie chart of the composition in terms of rating is provided.

Figure 4.1: Index composition



Source: Thomson Reuteurs Data Stream

4.2 One Factor Gaussian Copula

When dealing with synthetic CDO the market practice for quoting the products is to use the one factor Gaussian copula simplification.

The same reasoning applies for indices as the simplistic assumptions can hold in a macro and theoretical view of the problem.

Let us remind that one main assumption on which the model is based is called the Large Homogeneous Portfolio Approximation (Vasicek-1987). The latter refers to the fact that since we are dealing with a portfolio of n>100 assets and each title in the pool shares the same characteristics with the others, instead of modelling the parameters for Recovery rates, hazard rates and correlation separately, we can assume they take on the same value for all components of the portfolio.

The reasoning is even more based when we deal with an index, as the market practice suggests to use the index CDS par spread as proxy for each underlying company's spread.

As a result, a unique hazard rate can be recovered and used in the model, together with a unique recovery.

Regarding the latter parameter, it can be seen from market data that on average it is always equal to RR=40% for each asset with a low variability in value. A further assumption is that it stays constant over the holding period so we can calculate the tranches price assuming the percentage as an exogenous factor.

Let us start with the description of the inputs.

The first data we have to recover is the CDX CDS par spread, which can be easily collected in the trading info of the index. In our case it is equal, again, to 69,7.

We have introduced in chapter 3 the relationship between the hazard rates and the CDS spread.

We recall that since the hazard represents the probability of default between a small time interval, given no prior default has occurred, if we multiply it by the Loss given default (1-RR), we get the expected loss. At the same time, the yield on the investment over the same period of time can be seen as another approximation of the expected loss, as it will represent the fraction of interests we will not collect due to the credit event. From that, it follows that by equating the two, the hazard can be recovered as CDS spread/(1-RR).
In our case the estimate for hazard is equal to *0,01162*. We will assume, again that the given value is constant over time, that means that the intensity of default will be the same each year.
Given that, we can easily recover the cumulative probability of survivance until T=10 by simply computing the inverse of the hazard and exponentiating it to the power of 10, so that the probability of not having any default until T is 94,33%.
We compute the inverse (1-94,33%) and we get the cumulative probability of default of the index over the time period, which turns out to be 5,68%.
It can be checked that the market given default probability is the same as the one we have just calculated (see in data stream under "Default Probability").
Let us recall the standard one-factor Gaussian copula equation

$$x_i = \sqrt{\rho}M + \sqrt{1-\rho}Z_i$$

We can now follow two paths. The first one is to recover the correlated time of default of each asset, conditioning them on the common factor M. So, we would now proceed by creating random samples for both M and Zi, and computing the simulated Probabilities of default of each asset $N(x_i)$, comparing them with the market given ones (recovered from hazard rate) and solving the equation for the *correlated time of default* τ.
The process we will follow here is slightly different.
The one-factor equation can be, in fact, also used to recover a more market oriented (realistic) loss distribution.
Basing our modelling on the homogeneity of the single assets, that we have seen implies same constant hazard rate, same recovery, same probability of default and an equally weighted portfolio (same notional for each contract), we can solve the above equation for the idiosyncratic factor Zi.

$$Zi = \left(\frac{x_i - \sqrt{\rho}M}{\sqrt{1-\rho}}\right)$$

We now map the cumulative probability of default to a standard normal cumulative distribution by computing $N^{-1}(Q_i(t))$.

Remember that we already have the cumulative probability of default, 5,68%, we now get its corresponding value in standard normal terms, that is -1,5827 (it is the x-axis value corresponding to the mapped probability in the cumulative standard normal distribution).

We set our auxiliary variable x_i equal to our $N^{-1}\big(Q_i(t)\big)$, as we know that through the LHP it will be the same for each asset.

Zi can be used to calculate the cumulative default probability of the idiosyncratic factor. And this is what we will be doing by using the Normdist function

$$N(Zi) = N\left(\frac{N^{-1}\big(Q_i(t)\big) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right)$$

The above is actually a representation of the cumulative probability of default of the i^{th} asset, conditional on M. We use the conditioning as it will allow us to define the probability of default as mutually independent.

The above formula is computed in conjunction with random drawings for the common factor. 500 values for the latter between -3 and +3 are considered, and the consequent probabilities are computed accordingly.

Let us just remark that as the defaults are independent from each other we can avoid the index "i" and consider a single value applicable to each asset.

Regarding the correlation between assets we consider the historical equity returns. Even though the latter is not the best proxy for correlation in our model, we can still use the equity markets to have an idea of the comovements. Historical daily data for Johnson & Johnson and Macy's have been collected. The choice of using the latter two have been drawn by the fact that they are elected from the two most extreme classes of ratings with respect to our index (Macy's is B+, J&J is AAA). The basic idea is that their correlation represents a sample of the minimum level of correlation we should expect among each pair of assets. Since the index is composed by assets of the same investment levels, but mainly with slightly different ratings, it can be reasonable to use the above value as a reference for the correlation among each couple of entities. Moreover, pairwise correlations between other assets in the bundle have been computed and they all turn really close to the selected one. The daily lognormal returns have been estimated and the correlation turns out to be 0,3535. We know that under the one factor copula model it will be assumed to be the same for each pair of companies.

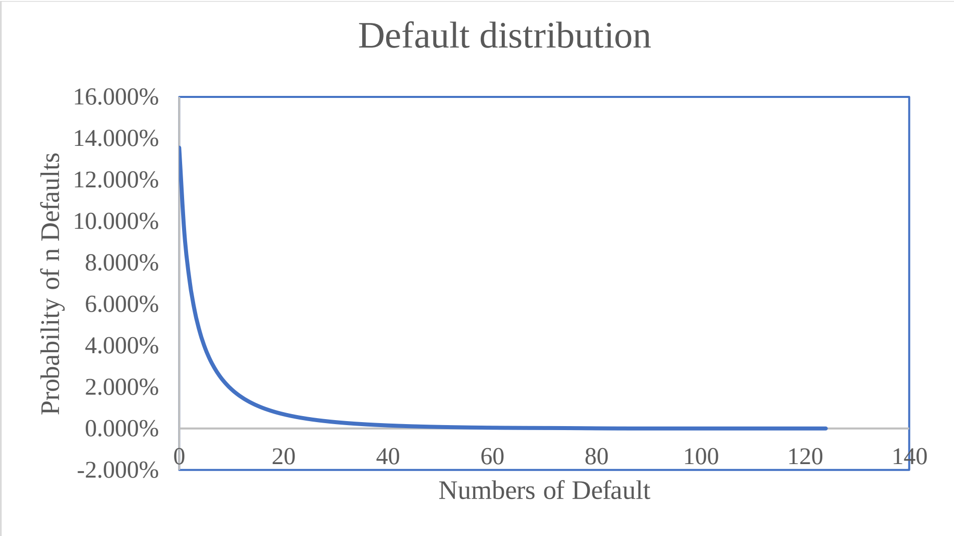
We now are left with 500 different outputs for the conditional probability. We would now consider a binomial distribution for the defaults and compute all the possible combinations, that is how many defaults could happen together over the 10 years period. We do so, using the binomial function with each trial happening with a probability equal to N(Z).

As we have said, the latter represents the cumulative probability conditional on M, so that we can write it as Q(T|M).

Conditional probabilities can now be exploited to arrive at un unconditional estimate for the probability of n defaults in 5 years. What we have to do, is to integrate over the common factor.

The results are displayed in the table below, and it can be seen that under our assumptions, as the number of defaults increase, the likelihood of occurrence decreases.

Figure 4.2



The results above are obtained by weighting the probabilities of having n defaults by time T by the piecewise integration over M, that we remind is assumed to be normally distributed with mean 0 and SD equal to 1. For the properties of its distribution the integration can be computed as the difference between the cumulative distribution for value "u" minus the cumulative distribution of value "u-1", for example the probabilities computed assigning Q(T|M) to the likelihood of occurrence of each default, with a value of M equal to -2,976, are integrated via "(N(-2,976)-N(-2,988))", where the latter is the immediate lower value chosen for the common M in our drawings.

Now that we have the cumulative probabilities of default, we can derive the expected loss on the portfolio. Recall that the expected loss can be recovered as (1-RR), that is the Loss Given Default, times the default probability. We first of all consider the percentage of loss expected in case of each combination of defaults, that is for example, for 5 defaults

$$Portfolio\ Loss = \frac{5}{125}(1 - 40\%)$$

Having the total pool loss, each single tranche expected loss as a percentage can be expressed considering the boundaries represented by the corresponding attachment and detachment points.

Each tranche under a given combination of defaults will suffer a loss equal to

$$Tranche\ Loss(T)\% = min\left[\max\left(\frac{L(T) - Attachmment}{Detachment - Attachment}; 0\right); 1\right]$$

$L(T)$ represents the portfolio percentage loss under the same combination, and the above can be shown to be true by considering different scenarios.

If the pool loss is lower than the attachment point of the tranche, the latter then, will theoretically not be involved in the allocation of losses (via the cascade); in fact, if L(T) is lower than attachment the above formula gives 0. If the tot index loss is bigger than the attachment than the result will be equal to $\frac{L(T)-Attachmment}{Detachment-Attachment}$, as the latter represents a percentage (and it is in between 0 and 1). Last whenever *L(T) > Detachment*, the tranche will be totally wiped out causing the loss to be 100% (the result of the function is 1).

We calculate the loss given the number of defaults for each tranche. We then use the probabilities of n default to weight them and to have a unique estimates of Tranche percentage expected loss over 5 years.

Table 4.2

Attach	**Detach**	**EL**
0	3%	51,77%
3%	7%	22,41%
7%	10%	11,68%
10%	15%	6,23%
15%	30%	1,65%

What we need now in order to price the tranche is to have an estimate of the percentage losses for each year. The latter can be recovered from the cumulative expected loss per tranche over different time horizons.

What can be done, is to run the whole process *T* times (in our case 5) changing the time horizon each time and so plugging the market given cumulative probability of defaults (from CDS spreads) accordingly into the initial Z estimate.

We calculate the cumulative probabilities of default to plug into our initial formula from hazard rate.

It has been already remarked that we will consider the hazard rate to be constant and equal for all assets in the index. We know the latter represents the probability of default between t and t+1. If we let t be equal to 0, then the probability of default of the first year will equate the intensity of default, as by definition the contract cannot be issued with a default already in occurrence.

Following the same path, the probability of survivance until the end of year 1, will be 1- hazard rate (= default rate).

The probabilities of default for the following years can be calculated using the definition of hazard rate.
We know that since it is a default probability conditioning on previous survivance, for year 2, for example, it will be equal to the unconditional probability of having default over the probability of survivance until the end of year 1 (that is, the survivance probability of year 1)

$$Hazard\ rate(2) = \frac{Unconditional\ Probability\ of\ default(t=2)}{Probability\ of\ survivance\ (t=1)}$$

We solve the above for the unconditional probability of default and we get 1,1482%.
We keep doing this until year 5. The cumulative probabilities are computed considering the sum of the unconditional ones.
The cumulative values for default and expected percentage loss for a time period of 5 years have already been calculated. We know the cumulative probability of default of each asset is 5,68% and we also have the expected loss over the same period, like for example the one for equity is 51,77%.
Ones we have all the cumulative probabilities we are ready to rerun the process and get the cumulative expected loss on each tranche over our entire holding period.
Let us focus on the equity tranche.

Table 4.3

Year	Cumulative expected loss
1	16,94%
2	28,78%
3	38,01%
4	45,52%
5	51,77%

From the cumulative expected loss, the marginal loss and the outstanding notional each year can be calculated.
The market practice for Tranche pricing is to consider the arbitrage-free swap methodology.
In order to avoid any costless abnormal return, theoretically, the present value of the two legs payments should be the same, so that the total present value of the contract equates 0.
The tranche involves the premium leg and the default leg.
The premium leg is the one paid by the protection buyer, paying each quarter the spread times the current outstanding notional. It has to be noticed that when we deal with an equity tranche, the premium leg also

comprises an upfront payment, quoted as a percentage of the total tranche notional. The latter is paid at inception to recompensate for the higher default risk born by the tranche. During the life of the contract the spread is set at 500 bps (on the run spread) for convention.

$$PV_{Tranche_{A,D}}(t) = UP_{A,D} + s_{A,D}\sum_{t=1}^{N} On_t D(0,t) + \sum_{t=1}^{N} EL_t\, D(0,t)$$

It is common to use the Libor/swap rate for the computation of the discount factors. As the Libor is quoted up to maturity 12 months, the rest of the discount curve is derived by interpolating the swap rates for the remaining maturities. Here the par swap rates have been used for each term.

We discount the survived outstanding notional and the marginal expected loss for each period. What we are trying to calculate for the equity tranche is the upfront payment, so we use the swap pricing methodology. We set PV equal to 0 and we solve the equation for $UP_{A,D}$, considering $s_{A,D} = 5\%$ (500bps).

The result will be an upfront spread of 20,86%.

We apply the same reasoning for the rest of the tranches, with the difference that what we will be calculating is $s_{A,D}$, as no upfront payment is planned.

Table 4.4

Equity Tranche	0-3%	20,86%
Junior Mezzanine	3%-7%	472,46 bps
Senior Mezzanine	7%-10%	213,07 bps
Senior	10%-15%	104,75bps
Supersenior	15%-30%	24,64 bps

4.3 Types of correlations

What we have used in the calculation of the tranche prices is the historical correlation, approximated to be equal for each pair of assets. In reality, mainly after the growth of standard tranches, other types of correlations have been preferred for the modelling.

As those products became more and more traded, market quotes for the standard tranches started to be easily available. From the latter the compound implied correlation of each tranche can be extrapolated. Compound correlation is defined as the constant correlation value that leads the calculated tranche spread to be equal to the market given one. By definition, each tranche will be characterized by a different value. Let us make an example, with our data to understand the meaning.

Suppose we have market quotes for each tranche written on our CDX NA V34, and that they take on the values presented below.

Table 4.5

Equity Tranche	0-3%	22,57%
Junior Mezzanine	3%-7%	326,36 bps
Senior Mezzanine	7%-10%	173,5 bps
Senior	10%-15%	86,3 bps
Supersenior	15%-30%	40,31 bps

Knowing the spread for each tranche, we can set up our model to find out which correlation will lead us to have a Tranche PV equal to 0. We can rewrite the equation for tranche present value as below

$$PV\left(Tranche_{A,D}\right) = s_{A,D} PV_{Premium}(A, D, \rho_D) + PV_{Protection}(A, D, \rho_D)$$

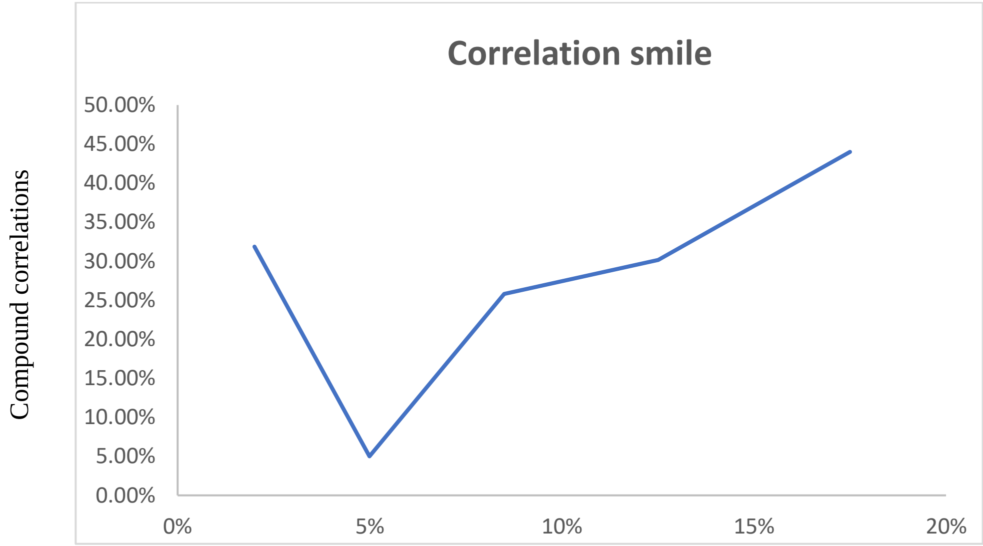
as we know that under our model the present value of both the premium and the protection leg are dependent on the correlation of the underlying assets.
We can now substitute $s_{A,D}$ with the ones given by the market and we solve it for correlation.

Table 4.6

Implied correlation (equity)	31,87%
implied correlation junior mezzanine	5,000%-78,535%
implied correlation senior mezzanine	25,81%
implied correlation senior	30,18%
implied correlation supersenior	44,00%

The first thing that falls in the spotlight is that there is no unique solution for the Junior Mezzanine tranche. We can see how the market spread can be equated either via 5% or 78,534%. This result implies that the risk profile embedded in the investment in those tranches can vary significantly. When the correlation is high mezzanine investors are said to be long on correlation, that is on credit risk, while when the latter is at low levels they are short on credit risk. This changing relationship is due to the position of the tranche in between high default-probability investments (equity tranche), and low probability ones (Senior tranches). The market practice is to consider the lower level as it is reasonably nearer to the average historical equity returns correlations.
If we plot the values against the attachment-detachment points we can see that the curve creates a smile, the famous correlation smile.

Figure 4.3

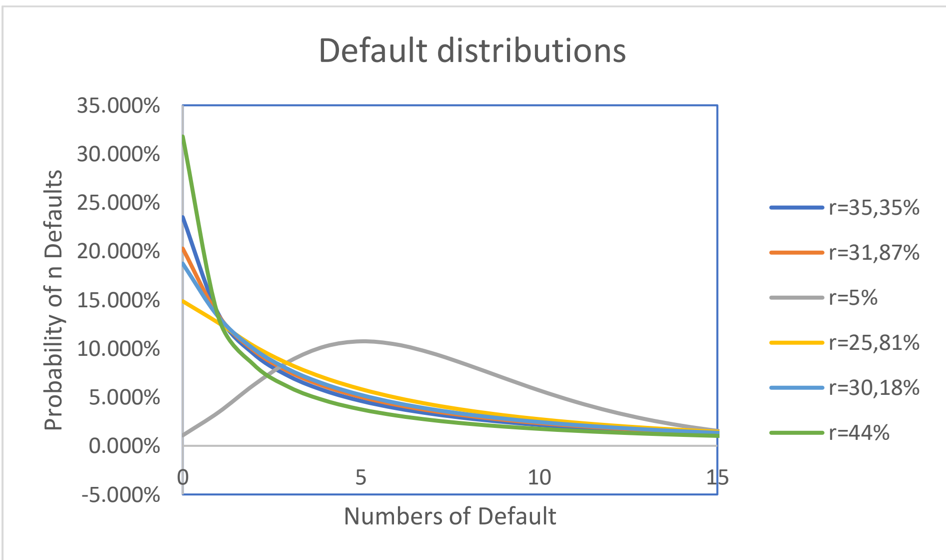


Tranches Attachment-Detachment Points

This outcome can be explained by the different relationships that each tranche has with the underlying correlation and by the inefficiency of the model to match the market perception of credit risk.

First of all, the fact that we have different correlations for each tranche suggests the incapacity of the one-factor-copula model to approximate correlation between 125 underlyings to produce market consistent prices. The way in which the model is displayed causes the underlying loss distribution of the portfolio to change as the correlation value varies, this is due to the effect caused directly on the conditional probability of default Q(T|M). As a consequence, each correlation implies a different probability mass distribution of default impacting so the portfolio loss distribution as well.

Figure 4.4



Furthermore, increasing correlation causes the equity tranche spread to decrease while induces an increase in Senior prices. This results from the fact that as correlation level presents a positive increment the likelihood of having high numbers of default or low numbers of default increases. The implications of the second scenario drive the equity tranche spread down, as the riskiness of the portfolio decreases; of course the opposite is true in that higher overall risk brings the spread up, but due to the construction of the tranche, that is capped by the detachment point, the first effect plays a major role. On the other hand, high correlation implies a greater chance for the Senior tranche to suffer from any loss.
If market quotes for the spread were higher, an upward shift in the implied correlations curve would have occurred, keeping all the other inputs fixed.
Below a sensitivity analysis of the tranche spreads have been conducted as to show how as the correlation increases the equity tranche is the only one suffering from a decrement in value, while all the others are impacted by the higher interconnections between the index's corporations and require a higher recompense for bearing the risk.

Table 4.7

Scenario Summary	Current Values:	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
correlation	35,35%	9,00%	18,00%	27,00%	45,00%	54,00%	63,00%
0%-3%	20,86%	35,86%	30,07%	25,06%	16,36%	12,42%	8,63%
3%-7%	472,46	380,76	447,00	471,11	458,01	433,58	400,86
7%-10%	213,07	57,55	127,21	179,39	237,82	249,66	252,17
10%-15%	104,75	8,12	38,54	74,19	134,06	155,03	169,74
15%-30%	24,64	0,11	3,09	12,02	42,22	59,90	77,57

Another thing to identify is that the implied correlation estimations are tied to the model used and to the maturity chosen , as it depends on the market perception of risk for the entire period so it will not be surprising to see a lower implied correlation for the CDX NA V34 10 years.
Other market factors also influence the implied correlations above such as the demand-supply relationship for each tranche. In the Mezzanine case it can be very often seen how the market quotes are lower than the one resulting from the model due to the high demand for this class of investments.
In order to solve the weaknesses caused by the compound correlation, base correlation have been considered. The main idea is that each tranche can be decomposed into two sub tranches, and its present value can be expressed in terms of base tranche, that shares the same detachment point but has an attachment of 0. Let us consider our junior mezzanine tranche with attachment and detachment point of 3% and 7% respectively. Taking a long position in this investment should be equal to take on a long position in an equity tranche with

detachment 7% and a short position in another equity tranche with detachment 3%. Following this reasoning the PV of our junior mezzanine tranche can be written as

$$\begin{aligned}\mathrm{PV}_{3\%,7\%}&\left(3\%, 7\%, \mathrm{S}_{3\%,7\%}, \rho_{3\%,7\%}\right)\\ &= \mathrm{PV}_{0\%,7\%}\left(0\%, 7\%, \mathrm{S}_{3\%,7\%}, \rho_{3\%,7\%}\right) - \mathrm{PV}_{0\%,3\%}\left(0\%, 3\%, \mathrm{S}_{3\%,7\%}, \rho_{0\%,3\%}\right)\end{aligned}$$

What we are saying in the above formula is that we can find a value for the correlation such that the difference between the two present values is equal to 0, that is such that the junior mezzanine PV is 0.

First of all, we compute the PV of the equity tranche with its implied correlation, that turns out to be also its base correlation, but we now use the market quoted spread for the 3%-7% tranche.

The resulting PV is 2,89%.

We now have to compute the PV of our base tranche 0-7%, and we do so by first adjusting the notional and expected loss for each period. For example, the cumulative expected loss for the 5 years period is expressed as

$$EL_{0-7\%} = \frac{3\%}{7\%} EL_{0\%,3\%} + \frac{(7\% - 3\%)}{3\%} EL_{3\%,7\%}$$

We compute the above for each tenor and we set up our calculation for the PV of the 0%-7% tranche.

We know that the difference between the two equity tranches PV must be equal to the PV of our Junior mezzanine, and so must give 0.

By equating the difference to 0 we solve for the correlation that allows for that value, while keeping the PV of the 0-3% equity tranche fixed, so that it doesn't change with the new correlation.

The result is a base correlation equal to 77,06%.

One of the main advantages of using base tranches, is that provides a single possible estimate for the mezzanine case, as opposed to the Compound correlation.

Repricing the junior mezzanine tranche using its base correlation we get a spread equal to 334,51 bps, which is visibly closer to the market one, meaning that it better reflects the market perceptions about the risk involved.

The procedure involves the calculation of the higher tranches in the same way so that each base correlation is explained by the one of the previous tranche and so on. For that reason, the curve is increasing as each correlation must also take into account the value already calculated for the previous base tranche. It follows that the smile encountered in the compound correlation is overcame even though, the curve now shows a skewness.

This last methodology is the preferred one as at least it allows to always obtain an estimate for each tranche, moreover it allows to interpolate correlations even for nonstandard tranches, differently from the compound one, as it is only dependent on the detachment point.

CONCLUSIONS

After the analysis of all the factors embedded in the financial collapse, there are some personal considerations I would like to draw.

First of all, it has to be recognized that a particular attention has been dedicated to the understanding of CDOs, but the latter can be expanded to include reflections about the Asset Backed Securities in general.

The use of the one factor copula has been identified by many, as the possible trigger of the whole market price mismatch and as a consequence of the whole credit risk scenario.

Let me express my opinion following a step by step explanation.

First of all it is true that the one factor copula in particular is not a precise and reliable method to price such products; first of all assuming that all the assets have the same correlation can be highly dangerous and risky, as we know that in the real market idiosyncratic components of each single entity may cause a default. This is true mostly in normal market conditions, but when a crisis is underway, we have seen how the systematic factors tend to overwhelm the firm specific ones. When the economy is in a downturn, each single default is mostly likely driven by external factors, we know that in 2007-2009 even well established companies (characterized for example by low leverage and high liquidity of assets) suffered because of the unexpected difficulties in borrowings and in the temporaneous decline in the willingness to trade. That have been said, even if an entity had enough theoretical liquidity, if there's no market, then the assets cannot be turned into cash, even if they are the highest demanded ones. As a consequence, whatever the company position was, they were all associated by the same struggle.

My opinion is that, as this was the case, considering a common pairwise correlation could turn out to be an acceptable compromise, as the dependence between each pair of companies was more driven by common factors, affecting the entirety of the portfolio.

Keeping commenting on the same topic, the fact that large pools of assets were used as collaterals for ABSs and CDOs makes it impossible and even worthless to consider different behaviors in comovements and still coming up with a consistent price estimate.

We have seen how one of the main features of CDSs contracts is that they are off balance sheet, meaning that they cannot be included in the risk evaluation of a company. We also have seen that in the years prior to the crisis and during the first period, such market was at the peak of its expansion, so it is a natural consequence that the credit ratings could not have been precise as they should, since there was a high probability of dealing with a company involved in many of such contracts, hiding the risk of such transactions, as they were not required to furnish any detail about them.

Going back to the one factor copula model, the main challenge in dealing with it is to find the right correlation level. Even though many ways have been provided by practitioners, it has been seen how each of them has its drawbacks and still doesn't allow to come up with a single value that fits all the tranches. For example, historical correlation turns out to be not a good proxy especially in changing market conditions (as they were during the credit crisis); compound correlation is intuitive and provides a value for each tranche, but creates issues when dealing with the mezzanine and shows a smile; base correlation provides a way to connect each tranche correlation to the others, but as a result needs the pricing information for all the issues, and cannot be used on single tranches issuances (which are really common in the CDS indexes world).

This is one of the hot topics, as correlation is still tranche-sensitive, meaning, that the spreads given in the market are not coherent with the base homogeneous assumptions about the pool, whatever they are.

We can see through the creation of the correlation smile, that the quotes provided by the market were overpricing the equity and the senior tranches, while they were underpricing the mezzanine one. Even though the market quotes used in the example are completely fictitious, they do provide a realistic representation of the real market quotes (especially for the one characterizing the financial crisis). Nonetheless many explanations can be provided for the arise of the smile, the more intuitive is based on the dealer-buyer dynamics. Mezzanine tranches were highly demanded and as a result the market was selling them off, while equity and senior ones were mainly used by investors to mitigate their positions in other assets.

Being considered the Black Scholes model's equivalent in the CDOs world, the One factor Gaussian copula shares the same conceptual failures in estimating the prices, remembering the volatility smile of the options world, similar to our correlation smile.

Notwithstanding the simplicity embedded in it, marketers preferred to avoid other sophisticated models, as the latter required the modelling of huge amounts of parameters with few information effectively available.

For the reasons just exposed above, my opinion is that the main issue behind the crisis cannot be blamed to be the model, nor even the parameters estimations, but rather the surrounding environment.

As all the recipes, even the one for disaster requires more than one ingredient, so even though it didn't permit fair precise estimates it is true that it had to calculate factors based on available data that were not accurately reflecting the reality. The lack of control by the government and the speculative use of asset backed securities made by institutions, together with the misalignment of interests, have for sure caused the whole economy to suffer, and to keep hiding the real risk profile behind the transactions.

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SUMMARY

In this thesis I would like to analyze and emphasize the importance that some derivative contracts have had during the 2007-2009 financial turmoil.

Different aspects of the uncertainty embedded in the markets during those years will be considered along with the possible triggers of the resulting domino effect.

The discussion will be introduced by a quick overview of the financial markets during the period and of the main events that have leaded to the market crash.

After that, the principal contracts at the core of the conversation will be introduced, as well as their primary characteristics.

The mortgage market and more generally, the housing market bubble will be identified as the major cause of the ABSs default and of the resulting bankruptcy for the main financial institutions in the market.

In chapter 2 the conversation will vert toward the credit risk world and the main models used to identify and monitor it.

After the delineation of the main variables involved and of its role in the crisis, we will focus on the mechanics exploited for the modelling of the default risk, and further highlights the mathematical tools used to implement them.

The heart of the discussion will be the introduction of a new market practice for the pricing of credit derivatives: the copula functions. More specifically the presentation of the correlation between defaultable assets will allow us to create a bridge between, single asset default and portfolios of assets' default via the use of copula functions.

With the aim of investigating the market incongruencies arisen during the crisis, we will furnish a practical application of the one factor Gaussian Copula to tranches of CDO in order to conduct a sensitivity analysis that will help us coming up with a motivated conclusion.

The three main pillars of the discussion, will then be

1. CDOs and the ABSs market.
2. Credit risk
3. One factor copula model to account for the underlyings' correlation.
4. Pricing Techniques for CDOs: Application of the one factor copula model to price tranches on the CDX NA IG.

1: CDOs and the ABSs market

As we already know, the 2007-2009 have been characterized by a severe market break down, caused by several factors; the sub-prime mortgage market have been in the spotlight, supported by a market bubble in the real estate industry.

Let's start the analysis by first describing the situation immediately before the turmoil.

The market was characterized by a series of different products that allowed investors to diversify their portfolio's risk, either by choosing 0 correlation items or by hedging their positions through derivatives contracts. Those contracts are products whose payoff and price are directly correlated to the behavior of another item, the underlying, and that allow to cover (to hedge) the position in another investment, reducing (or even fully eliminating) the credit risk associated with the latter.

When talking about investors we should remember that banks, as active counterparties to a series of transactions, should fall under this category.

As main players in the lending field, banks also bear sources of risk. The main one, of course is the counterparty or credit risk. Those financial institutions can be seen as firms that still have to register each transaction on their balance sheet where the elements of risk are calculated and managed. For example, when they issue a loan, they are funding such cash out with money deposited by other investors in an account, meaning that if the latter would withdraw their amount back, the bank should be liable for the refunding of money through and adequate management of cash ins and cash outs.

Following this reasoning securitization have becoming more and more widespread over years.

To mitigate the need of collaterals held by financial institutions, Asset Backed securities were integrated in the market in 1970.

1.6 ABSs

In order to alleviate the burden held by banks and to make lending easier and faster ABSs were introduced in the market. They basically allow a financial institution to package income generating assets from their balance sheets, such as loans, and to sell them under the form of securities, so to immediately receive cash flows that can be then used to create other loans. More specifically an originator entity pools the asset together and transfer (sell) them to another party which is said to be bankruptcy-remote. The latter can be either a subsidiary or an affiliate institution or a Special purpose vehicle, as more popular during the 2000s, appositely created for the accomplishment of the transactions.

The term bankruptcy-remote refers to the fact that the entity to whom the receivables are transferred is characterized by an asset/liabilities composition that allows it to keep its obligations safe in the case of default of the obligors. The latter will then be accountable for the distribution of the tranches to external investors.

The fundamental function of those instruments, besides the increment in the lending ability, is that they allow the transfer of risk.

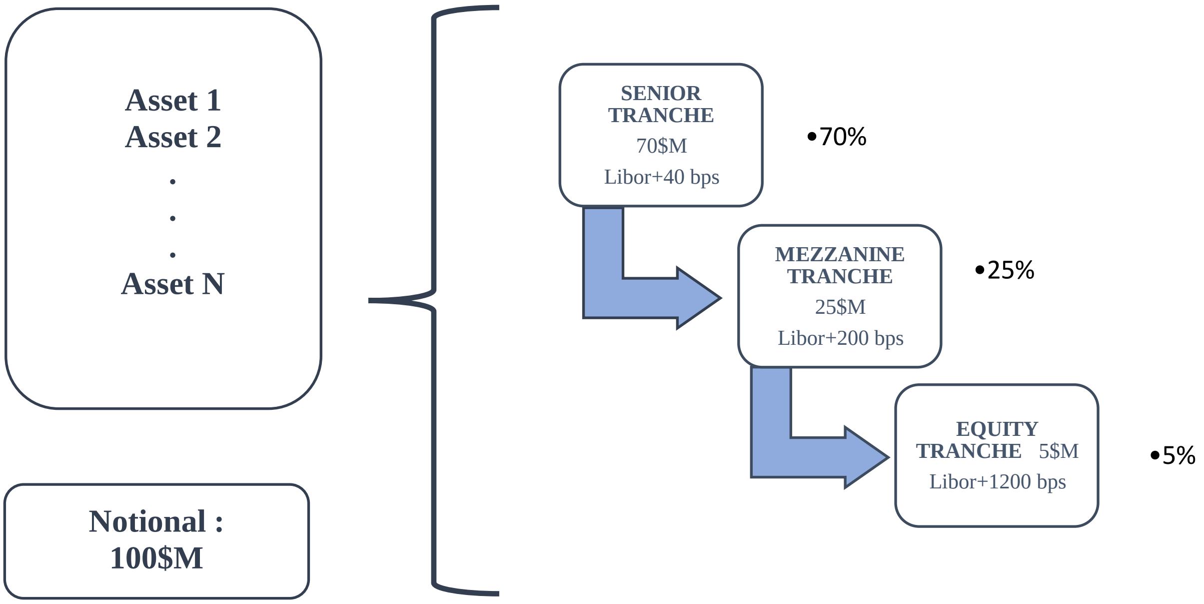
We can see them as a bond backed by loans or other assets, which generates cash ins for the issuer proportional to the cash flows of the underlying pool of assets.

A key feature of asset backed securities is that the cash flows are afterwards split into tranches.

The resulting products are then categorized in different levels of claims, with different credit ratings. Usually three categories are created. Those are the senior tranche (rated AAA) which has the highest priority over the others; the mezzanine tranche (rated AAA-/BBB) positioned in the middle in terms of priority; the Equity tranche (not rated) which categorizes the residual claims over the pool of assets. In regard of the profit and loss allocation with respect to the underlying cash flows, the used technique is the waterfall cascade. The latter describes a hierarchy according to which profits will be distributed with a top down structure (that is why the last tranche is characterized by residual claims); on the other hand, losses will be shared following a bottom up argument.

To make a practical example we can consider a portfolio of underlying assets worth 100$M. suppose the three simplified tranches are described as follows:

Figure 1.1: Example of ABS



The above scheme demonstrates that the entity issuing the ABS is practically issuing bonds (claims) backed by the asset's pool.

Being focused on the profit allocation, the cascade scheme assures that the senior tranche will be entitled to the first 70% of gains created by the pool. Whenever this percentage is satisfied profits will then be allocated to the mezzanine tranche up to the 95% of the cumulative collateral pool's profits. Moving further, whenever both the senior and the mezzanine tranches are paid, the equity tranche will earn the remaining 5%.

What happens, on the other hand, when a loss occurs is that the first tranche to suffer is the equity one. It will absorb up to the 5% of total losses of the pool, while keeping the other two tranches profitable. When the loss will exceed the 5% the mezzanine tranche will absorb up to the 25% of the total loss. This means that in order for the senior tranche to be touched the underlying pool must reach a loss of 30% and more.
Let's further suppose that our underlying pool is composed by 100 loans, each having a notional of 1$M. This implies that there are 100 obligors that could default.
If 1 obligor out of the 100 total defaults, the underlying pool is affected by a default of 1$M.
In the event of default, under the assumption of zero recovery rate (which is the value of the security after default has occurred expressed in FV percentage.) the loss should be borne by the equity tranche. So, what happens is that the investor holding the tranche will have a reduction in its notional of 1$M, so that the new principal will now be 4$M rather than 5$M. As a consequence, now on, he will earn its interest rate (LIBOR+1200 bps) on the new notional and will receive 4$M at the expiration date.

1.7 CDOs

The establishment in the market of ABSs and MBSs, was accompanied by the introduction of other products such as CDOs and CDSs.
The latter are credit derivatives, whose existence was triggered by some market imperfections, creating arbitrage opportunities for investors.
CDOs are a particular and sophisticated class of ABS. They are characterized by a collateral pool of traded bonds. This includes ABS's tranches, CDO's bonds, and other types of obligations, already traded in the market. For that reason, CDOs do not contribute to the achievement of market completeness as ABSs do; they rather allow investors to exploit arbitrage opportunities. It can be proved that whenever the underlying bonds (ABS, CDO's tranches…) are underpriced, or the resulting CDO's bonds issued are overpriced, then the security will carry a positive value. If what just said is not true, it will yield a negative value and will have no reason to exist.
This can also be seen as a proof of the fact that at that time, rating agencies were not carefully carrying out their charge. This rating arbitrage primarily rise due to the payment structure of rating agencies and because of the short-term incentive horizons for financial institution's managers.
Rather than being paid all at once, rating agencies faced periodical payments, that tended to foster a long-term relationship with their customers. On the other hand, managers' compensation mainly comprised end-of-the-year bonuses, making them weakly concerned about the future performance and consequences of their portfolios.
Two different classes of CDOs exist; cash flow and synthetic CDOs. The first category refers to the physical ownership of the bonds by the issuer. This involves a direct sensitivity to the default risk associated with the asset. As opposed to the latter, synthetic CDO's do not embed any purchase of bonds and are written on pools of credit default swaps, which creates the cash flows between the two parties.

The focus will be on synthetic CDOs.

Let us consider an example. Suppose a bank is currently holding a portfolio of bonds worth 150$M made up by 150 bonds each with a nominal of 1$M.

The SPV decides to share the credit risk with the latter in exchange of periodical payments.

150 swap contracts are issued by the purpose vehicle to provide protection for each bond holding.

CDSs are subsequently pooled and tranched. Without loss of generalities we will consider three tranches, even though usually many more are created.

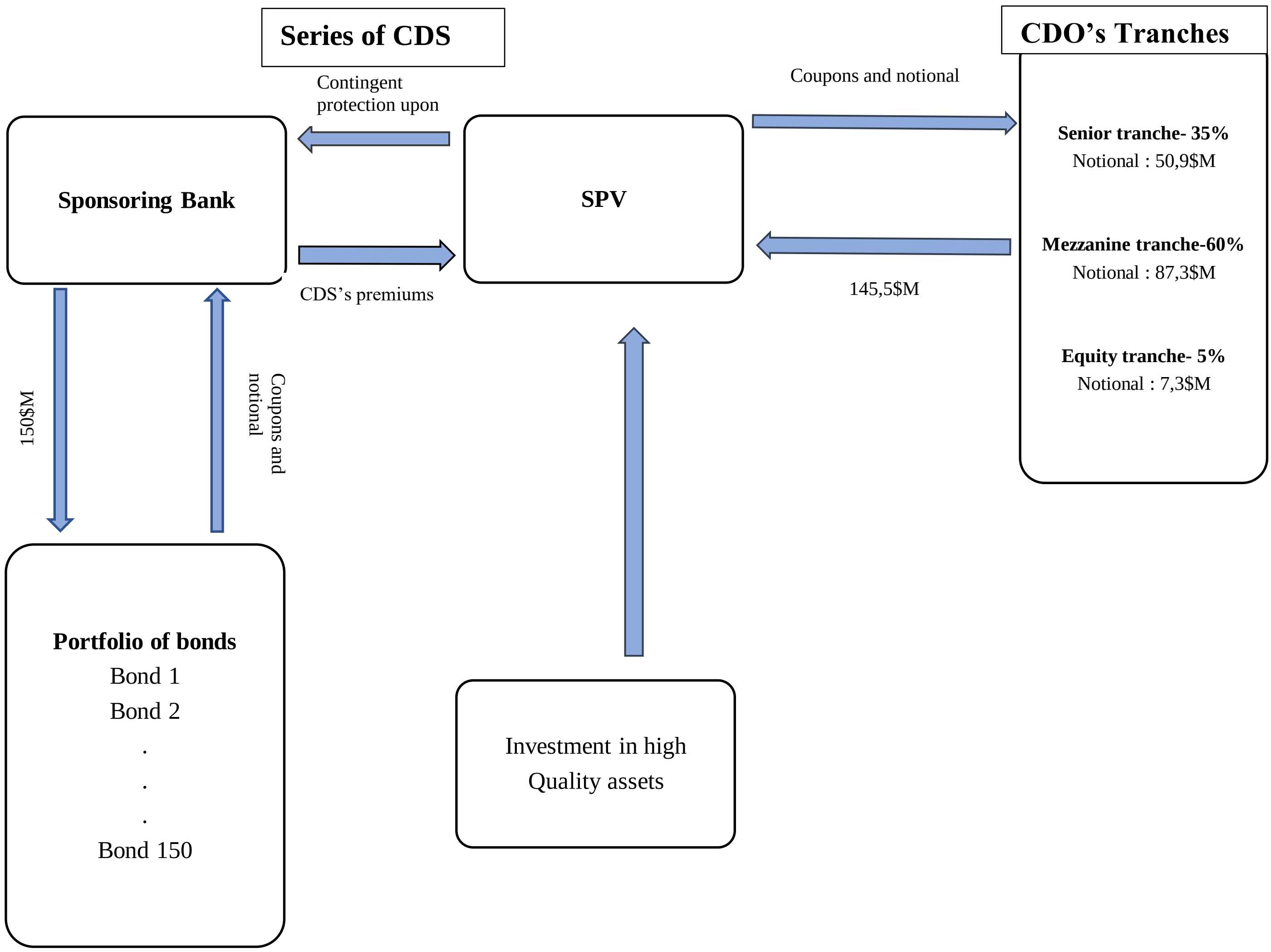
Senior, mezzanine and equity tranche are then sold to investors.

We know that the CDSs contracts will be constructed so that the coverage in case of default of the reference entity may not be full.

Suppose the Sponsoring banks agrees to be recompensated whenever the loss will exceed the 3% of the total pool of bonds' nominal. That is as to say that the notional of the CDS will be the 97% of the 150$M notional, 145,5$M.

Tranches will be divided as follows: the senior tranche will represent the 35% of the pool's notional, the mezzanine the 60% and the equity tranche the 5%.

Figure 1.2: example CDO's synthetic tranches



Following the structure described above, whenever the portfolio loss is below 3%, no changes to the tranche's notional will take place.
Conversely assume that 5 out of 150 bond issuers default, with 0 recovery rate. As we have just highlighted the first 3% of loss will not be counted; the other 2% has to be accounted through the cascade structure.
The equity tranche will absorb the loss and will suffer from a reduction in its notional of 2.9$M. Now on investors of this tranche will earn interest on a notional of 4.4$M and will receive this new amount at the end of the investment period. The reduction keeps moving up until the detachment point of the tranche, when the entire notional will be wiped out. (5%) For losses above this percentage (exceeding the 8% in aggregate) will be borne by the mezzanine tranche up to the 65% where the senior will be hurt as well.
In the end, the bank will pass through the credit risk associated with its investment while keeping the bonds in its balance sheets. The SPV will hedge the risk borne via the CDS contracts, through the issuance of the CDOs tranches. Investors preferences and needs will be satisfied through the purchase of tranches, with the possibility to eventually exploit any misprice caused by inefficiencies.
CDOs were highly traded during the years previous the turmoil because of their capacity to remodel junk bonds and lowly rated debt in general.
They could repackage a pool of junk bonds, for example, into different tranches of debt, where a senior tranche exists. This one will be rated AAA-, for example, and still yield a higher return than a similarly rated bond in the market.
This, helped overcoming the otherwise low demand for low grade debts, making equity tranches created from those instruments appreciated from investors.

2: Credit risk

The two pillar concepts of the investment world to be considered are risk and return.
We know that broadly speaking, whenever a rational investor is concerned in evaluating an investing opportunity, he or she will seek to obtain the highest achievable return while keeping risk at the minimum possible degree.
Nevertheless, when using the word "risk", we should be aware of the fact that many different classes do exist. Among the others, credit risk is defined as the unpredictability regarding the ability of a counterparty to fulfill promised payments due to a change in its credit quality. It is commonly also referred to as Default risk, as the trigger of the loss in value is the deficiency shown by one of the two party in repaying its debts.
The growth in popularity of ABS and of subprime lending, together with the increasing confusion brought by rating agencies have pushed analysts to model this source of risk as to account for it in the most accurate way when pricing derivatives.

Credit risk is composed by two ingredients: default risk and spread risk. The latter is the fear that a change in credit rating by the counterparty will lower the value of the position on hold. We have seen that during the crisis refinancing risk has constituted a huge slice of the cake.

Transition and default probabilities represent the major issues to be addressed when modeling the two sources of credit risk, that are identifies again in spread risk and default risk.

Since years different approaches are presented to fulfill the collection of information related to it and they all involve different assumptions and point of views.

We can summarize them in three major sections.

The historical approach has been highly used at the beginning of the credit derivative spread. The latter involves the use of historical default rates and the computation of the sample average, used as indicator of the riskiness of a certain obligor.

Nevertheless, as it doesn't allow to account for external factors, such as the economic environment, other approaches have been proposed.

Structural models exploit the balance sheet distributions of firms for the pricing of defaultable products.

We consider a defaultable ZCB and we assume that an entity embedded with default risk is issuing those products with face value $\overline{DD}$.

The firm issues today debt instruments with no coupons, meaning that the only (defaultable) payment will be transferred at the end of the holding period, that is at T.

The assets underlying the business are funded via equity and debt and displays a value equal to V(T).

The key idea behind the model is that, being shareholders residual claimants, their payoff is related to the probability of occurrence that the asset value at the end of the year exceeds the amount of debt to be repaid. In fact, whenever the latter happens, they will get the difference between the asset value and debt and receive 0 otherwise. This can be characterized with a call option on the assets value of the firm with strike price equal to the face value of debt, that is $\overline{DD}$.

The payoff for shareholders at T can then be written as

$$C(T) = \max\,(V(T) - \overline{DD}, 0)$$

Under the put call parity and making some further assumptions regarding the debt value, Asset value is written as

$$\mathrm{V(T)} = \overline{\mathrm{DD}} - \max(\,\overline{\mathrm{DD}} - \mathrm{V(T)}; 0) + \max\,(\mathrm{V(T)} - \overline{\mathrm{DD}}, 0)$$

Assuming the latter follows a brownian motion process we can exploit the Black and Scholes formula to reprice the bond so that in the end we arrive at an estimate for the bond default rate, to be equal to $\phi(-\mathrm{d}2)$.

On the other hands intensity based model do exist and exploit the Poisson process for the modelling of the default probability, with an intensity equal to the hazard rate.

Assumed that the default rate process is orthogonal to the interest rate, we can define the probability of survivance until time T (probability that default τ has not yet occurred by time T) as

$$P(\tau > T) = \exp[-\lambda(T - t)]$$

That being said, under the risk-neutral probability measure Q, the conditional probability of default, given no earlier default is given by

$$E_Q[1 - 1_{DEF}] = P(\tau > T) = \exp[-\lambda(T - t)]$$

From which the default probability is extracted by computing the inverse of the above.
This will be the model used in our application.

3: One factor copula model to account for the underlyings' correlation

Recalling that CDO's are tranched portions of debt backed by a pool of underlyings, another issue arises in our modelling. The implicit construction of those contracts suggests that we now have to handle with more than one obligor exposed to the risk of default.
conclusion as some interrelationships between default do exist.
We need to come up with a technique able to model different defaults probabilities jointly. Copula functions allow us to do.
Models to provide correlation estimates between assets, are divided in three groups: *Stochastic Financial correlation models*, where, as the name suggests, the dependence between defaults times is considered to follow a stochastic process with some randomness; *Statistical correlations models* which involves parameters such as the Kendall's tau; in the end Deterministic financial correlation models.
Multi-asset Gaussian copula (Li) and the One-factor Gaussian Copula (Vasicek) fall under the latter category. More specifically they are part of the bottom-up models that, as opposed to the top-down ones, first collect single asset data, quantify it and then find a model to join them together to come up with a comprehensive correlation result.
A Copula function, technically speaking, is a function able to transform an n-dimensional function into a unit-dimensional one.
Let us consider n univariate marginal distributions, defined as $Q_i(x_i) = P(X_i \leq x_i)$ with $i = 1,2, \dots, n$. In the previous paragraph we have defined the latter probability in terms of strike price and stock price and we have denote it by $F(K_1)$ and $F(K_2)$.
Each marginal probability belongs to the unit-dimension [0,1] and we know that there exists a copula function such that

$$F(X_1, X_2, \dots, X_n) = P[X_1 \le x_1;\ X_2 \le x_2; \dots;\ X_n \le x_n) \qquad \in [0,1]^n$$

$$F(X_1, X_2, \dots, X_n) = C(\ F(X_1), F(X_2), \dots, F(X_n))$$

To express the above in terms of multivariate density function we change the variable $x_i \to F_i^{-1}(x_i)$

$$C(x_1, x_2, \dots, x_n) = F_n(F_1^{-1}(x_1), F_2^{-1}(x_2), \dots, F_n^{-1}(x_n))$$

When we deal with a Gaussian copula, the above multivariate function will be Gaussian.
Let us make an example in a bivariate copula case.
We have two speculative-grade (BBB-) corporate bonds (two entities), with given cumulative default probabilities over the holding period.
The cumulative default probabilities represent the likelihood of default every year, not conditional on any prior default or survivance, to rephrase it they are unconditional.
Suppose the time to maturity of the bonds is T= 5y. The table below synthetizes the cumulative probabilities over the years.

Table 3.1

TIME	$\boldsymbol{Q_A(t)}$	$\boldsymbol{Q_B(t)}$
1	6,3%	2,6%
2	13,4%	7,3%
3	21,3%	14,2%
4	29,8%	23,6%
5	39,8%	38,9%

Having just two companies, our copula will have two arguments, and it is defined as

$$C^{GA}\big(Q_A(t), Q_B(t)\big) = M_2\big[N^{-1}\big(Q_A(t)\big), N^{-1}\big(Q_B(t)\big); \rho_M\ \big]$$

Applying our Gaussian copula, we can map the above marginal probabilities to the standard normal through the inverse of the standard normal distribution N_2^{-1}. Notice that in our example it will be a bivariate distribution.
We do so using the excel function "normsinv($Q_i(t)$)". The results are displayed in the following table

Table 3.2

Time	$Q_A(t)$	$N^{-1}(Q_A(t))$	$Q_B(t)$	$N^{-1}(Q_B(t))$
1	6,3%	-1,5301	2,6%	-1,9431
2	13,4%	-1,1077	7,3%	-1,4538
3	21,3%	-0,7961	14,2%	-1,0714
4	29,8%	-0,5302	23,6%	-0,7192
5	39,8%	-0,2585	38,9%	-0,2819

Since we are dealing with two entities, the correlation between them can be expressed as a single number, as opposed to the multivariate case, in which a correlation matrix is involved.

Let us assume their Gaussian correlation for the next year is 0,5, that is their defaults are subject to positive comovements, due to similar economic conditions, and so similar ratings.

The joint cumulative probability of both company A and B defaulting in year 1, given the correlation rate, can be defined as

$$Q(t_A \leq 1 \cap t_b \leq 1) = M_2(x_a \leq -1{,}5301 \cap x_b \leq -1{,}9431, \rho = 0{,}5)$$

In the above we express the joint probability of having the default before the end of year 1, by considering the joint probability of having the mapped values x_a and x_b lower or equal than the abscise corresponding to each mapped distribution.

We can solve it in excel through the bivariate distribution code, so to have

$$M_2(x_a \leq -1{,}5301 \cap x_b \leq -1{,}9431, \rho = 0{,}5) = 0{,}0088 = 0{,}88\%.$$

Pricing large portfolio of assets with the Gaussian copula do provide a simplicist way of modelling correlated default probabilities of each single product, but, on the other hand implies a lot of calculation and an abundant set of information.

Often market practitioners make strong assumptions in trying to simplify pricing models.

In 1987 Oldrich Vasicek introduced a model to price credit risk in homogeneous portfolios, called the one-factor copula function. In the latter two main assumptions defining the homogeneity of the asset pool are made: the first is that, when we are dealing with bonds with similar ratings or belonging to analogous sectors, the pairwise correlation among the assets may be considered to be unique. In addition to that, having comparable characteristics, their probability of default could be thought to be the same.

Three ingredients are then required for modeling. They are the *recovery rate* RR, the *default intensity* λ, and a default correlation, that in this specific case will be defined as *conditionally independent default correlation*.

We consider a portfolio of N entities issuing assets that join the homogeneous portfolio. We define a latent variable $x_i, i = 1,2, \dots, N$ that is a default indicator for the i^{th}obligor. The variable x exhibits a negative relationship with the default probability, as the higher it is the farer in years the default time
will be, and viceversa. It can also be interpreted as the company's overall health, so that the weaker it is the sooner default will occur.
Each asset is considered to be affected by a common factor M, that interprets the role of the systematic market factor, and is an indicator of the whole economy conditions.
Furthermore, an idiosyncratic component Z_i, representing each firm particular position, is included in the model. Both are distributed with o mean and unit-variance.
All the above can be summarized in the formula for the calculation of the latent variables

$$x_i = \sqrt{\rho}M + \sqrt{1-\rho}Z_i$$

with ρ equal and constant for each pair of assets.
Here we are not considering the conditional default probabilities directly, but, instead, we are conditioning the defaults on M. For example, whenever $\rho = 1$, assets are perfectly dependent, and in fact the above formula will simplify to the common factor M. Contrarily, if $\rho = 0$, the only driver of the default indicator will be the firm specific factor.
M and Z_i are found drawing samples of random variables from standard normal distributions.
Once the x_is are calculated on the given correlation, they have to be subsequently transformed into cumulative probabilities through the cumulative standard normal distribution. For that reason, the model is called Gaussian copula model.
De facto, important divergences exist between the one-factor model and the model we described in the last paragraph.
First of all, in the latter, we have to deal with an $n \times n$ correlation matrix, while here we just use a single parameter. In that regard we do not use the matrix to correlate defaults directly and we cannot allow for different pairwise correlations. It follows that the standard normal distribution considered here is one-dimensional as opposed to the n-dimensional used before.
It can be notice that in the bivariate case the two models coincide as the correlation matrix will be made of identical entries. As a consequence, the random sampling conducted through the application of the Cholesky decomposition, will yield the same result for x_i, as the one we achieved under the one-factor Gaussian copula.

4: Pricing Techniques for CDOs: Application of the one factor copula model to price tranches on the CDX NA IG

Synthetic CDOs are tranches written on a pool of CDS, those contracts, as such, are said to be “funded” in that a sale of the underlying swap contracts is provided via credit linked notes.
With the increase in trading and appreciation for such products, Credit Default Index Swaps have been created. The latter are indexes on a pool of standardized CDS contracts that trade on the OTC market, and they have become the most liquid credit derivative.
The pricing technique of CDIS index tranches is still the same as the one for synthetic CDOs as they are economically equivalent in terms of cash flows.
Two main families of indices do exist and those are the CDX type and the iTraxx type, with the former making reference to north American markets and emerging ones, and the latter to the European and Asian markets.
Both kinds of indices share the same composition of the underlyings, that is each title in the pool is characterized by the same weight, and the total of contracts is usually 125, so that the equal weights are 0,8%. Furthermore, the standard maturities to which they make reference are 3years,5years,7 years and 10 years (where 5 and 10 are the most liquid).
The product chosen for our application is the CDX NA IG 34 V1 for 5 years.
The Index is traded with a specified spread which should represent the average credit risk associated with the underlying contracts; even thought it could be tempting to compute it as a weighted average of the underlying spreads using the portfolio weights of 0,8%, it can be seen that the two do not coincide, this is due to an adjustment made in order to consider the correlation that does link the defaults and so the realization of each contract, so that the weight used is equal to the risky duration of asset i over the sum of all the risky durations in the portfolio.
It can be shown that in our case the weighted average (with reference to portion of tot Nominal) should be a CDS par spread equal to 86,96, while the index is currently trading with a par spread of 69,7.
The total notional on which the index trades is assumed to be 125.000.000.000 $, so that each contract represents a notional of 1.000.000.000 $. The CDX NA IG version 34 is currently trading with an index factor of 1 indicating that the contingent payments upon credit events occurrence will cover the tot loss perceived by the underlying.
Indexes have become really famous since the very beginning mainly because of the standardization and transparency embedding their trades.
Let us now suppose that there exist a special purpose vehicle willing to tranche this index and to transfer the credit exposure of each obligor in the pool to external investors.
In constructing tranches, we will make reference to the standard attachment and detachment point associated to the CDX investment grade. The latter are collected in the below table.

Table 4.1

TRANCHE	Attachment (%)	Detachment (%)	Attachment (M$)	Detachment (M$)	Tranche Notional (M$)
Equity	0	3%	0	37,5$	37,5$
Junior Mezzanine	3%	7%	37,5$	87,5$	50$
Senior mezzanine	7%	10%	87,5$	125$	37,5$
Senior	10%	15%	125$	187,5$	62,5$
Supersenior	15%	30%	187,5$	375$	187,5$

We can immediately recognize that the total value of the underlying index's Nominal that is tranched is the 30% and not the 100%. This is because we are dealing with investment grade assets that have a low probability of default, so that the probability that all the assets will default by the expiration, that in our case is T=5 years, is equal to 0 on average.

Let us remind that one main assumption on which the model is based is called the Large Homogeneous Portfolio Approximation (Vasicek-1987). The latter refers to the fact that since we are dealing with a portfolio of n>100 assets and each title in the pool shares the same characteristics with the others, instead of modelling the parameters for Recovery rates, hazard rates and correlation separately, we can assume they take on the same value for all components of the portfolio.

The reasoning is even more based when we deal with an index, as the market practice suggests to use the index CDS par spread as proxy for each underlying company's spread.

As a result, a unique hazard rate can be recovered and used in the model, together with a unique recovery.

Regarding the latter parameter, it can be seen from market data that on average it is always equal to RR=40% for each asset with a low variability in value. A further assumption is that it stays constant over the holding period so we can calculate the tranches price assuming the percentage as an exogenous factor.

The first data we have to recover is the CDX CDS par spread, which can be easily collected in the trading info of the index. In our case it is equal, again, to 69,7.

We recover the hazard rates from our CDX spread by diving the latter by (1-RR).

In our case the estimate for hazard is equal to *0,01162*. We will assume, again that the given value is constant over time, that means that the intensity of default will be the same each year.

Given that, we can easily recover the cumulative probability of survivance until T=10 by simply computing the inverse of the hazard and exponentiating it to the power of 10, so that the probability of not having any default until T is 94,33%.

We compute the inverse (1-94,33%) and we get the cumulative probability of default of the index over the time period, which turns out to be   5,68%.
It can be checked that the market given default probability is the same as the one we have just calculated (see in data stream under "Default Probability").
Let us recall the standard one-factor Gaussian copula equation

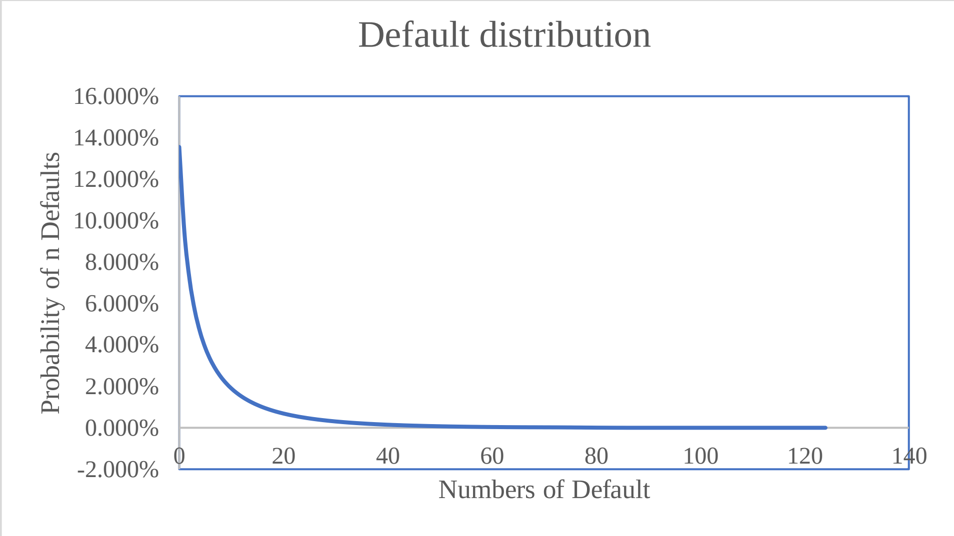
$$x_i = \sqrt{\rho}M + \sqrt{1-\rho}Z_i$$

The process we will follow here is slightly different.
The one-factor equation can be, in fact, also used to recover a more market oriented (realistic) loss distribution. Basing our modelling on the homogeneity of the single assets, that we have seen implies same constant hazard rate, same recovery, same probability of default and an equally weighted portfolio (same notional for each contract), we can solve the above equation for the idiosyncratic factor Zi.

$$Zi = \left(\frac{x_i - \sqrt{\rho}M}{\sqrt{1-\rho}}\right)$$

We now map the cumulative probability of default to a standard normal cumulative distribution by computing $N^{-1}\big(Q_i(t)\big)$.
 Remember that we already have the cumulative probability of default, 5,68%, we now get its corresponding value in standard normal terms, that is -1,5827 (it is the x-axis value corresponding to the mapped probability in the cumulative standard normal distribution).
We set our auxiliary variable x_i equal to our $N^{-1}\big(Q_i(t)\big),$ as we know that through the LHP it will be the same for each asset.
Zi can be used to calculate the cumulative default probability of the idiosyncratic factor. And this is what we will be doing by using the Normdist function.
The above is actually a representation of the cumulative probability of default of the i^{th} asset, conditional on M. We use the conditioning as it will allow us to define the probability of default as mutually independent.
We then derive the unconditional ones, by integrating over the common factor M.
Historical daily data for Johnson & Johnson and Macy's have been collected in order to have a measure of correlation. The 0,3535 result is then considered to be the same for each pair of assets in our index.
500 different values for the common factor are then considered, and the corresponding conditional probabilities are computed and then transformed into unconditional through the piecewise integration.

Figure 4.1



Now that we have the cumulative probabilities of default, we can derive the expected loss on the portfolio. Recall that the expected loss can be recovered as (1-RR), that is the Loss Given Default, times the default probability. We first of all consider the percentage of loss expected in case of each combination of defaults, that is for example, for 5 defaults

$$Portfolio\ Loss = \frac{5}{125}(1 - 40\%)$$

Having the total pool loss, each single tranche expected loss as a percentage can be expressed considering the boundaries represented by the corresponding attachment and detachment points.

Each tranche under a given combination of defaults will suffer a loss equal to

$$Tranche\ Loss(T)\% = min\left[\max\left(\frac{L(T) - Attachmment}{Detachment - Attachment}; 0\right); 1\right]$$

$L(T)$ represents the portfolio percentage loss under the same combination, and the above can be shown to be true by considering different scenarios.

We calculate the loss given the number of defaults for each tranche. We then use the probabilities of n default to weight them and to have a unique estimates of Tranche percentage expected loss over 5 years.

Table 4.2

Attach	**Detach**	**EL**
0	3%	51,77%
3%	7%	22,41%
7%	10%	11,68%

10%	15%	6,23%
15%	30%	1,65%

What we need now in order to price the tranche is to have an estimate of the percentage losses for each year. The latter can be recovered from the cumulative expected loss per tranche over different time horizons.
What can be done, is to run the whole process T times (in our case 5) changing the time horizon each time and so plugging the market given cumulative probability of defaults (from CDS spreads) accordingly into the initial Z estimate.
Once we have them we can use the no-arbitrage argument used for swaps contracts and calculate the fair spread that makes the PV equal to 0.

$$PV_{Tranche_{A,D}}(t) = UP_{A,D} + s_{A,D}\sum_{t=1}^{N} On_t D(0,t) + \sum_{t=1}^{N} EL_t\, D(0,t)$$

For the equity tranche what is calculated is the upfront payment made as a percentage of notional, and the on the run spread is considered to be 500 basis points by convention.
Below the results for the tranches are displayed.

Table 4.3

Equity Tranche	0-3%	20,86%
Junior Mezzanine	3%-7%	472,46 bps
Senior Mezzanine	7%-10%	213,07 bps
Senior	10%-15%	104,75bps
Supersenior	15%-30%	24,64 bps

In our example for the achievement of the tranche prices we have assumed that the latter are all described by the same correlation coefficient of the underlying asset, that is the historical correlation. In reality other type of proxies for the comovement of the entities are used, such as the implied correlation. This one, is really similar to the volatility parameter in the Black and Scholes as both are characterized by a smile and both shows evidence that the model used is not efficient in considering all the factors affecting the output.
Implied correlation is defined as the correlation level that makes our calculated spread equal to the market one, and as such can be extrapolated setting the spread used in the PV equation equal to the observed one and solving for the correlation that makes it equal to 0.
Below are the results related to our tranches, assuming fictitious market quotes

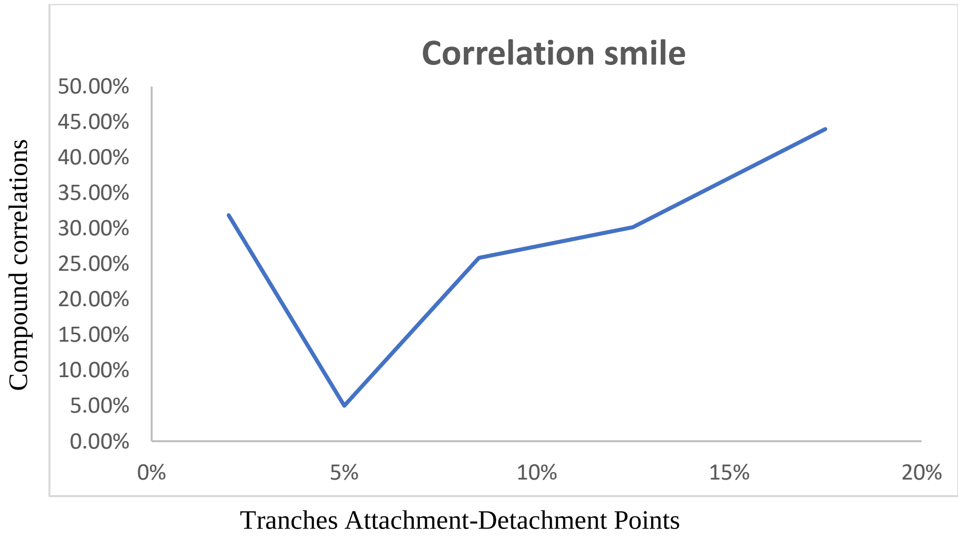
Table 4.4

Implied correlation (equity)	31,87%
implied correlation junior mezzanine	5,000%-78,535%
implied correlation senior mezzanine	25,81%
implied correlation senior	30,18%
implied correlation supersenior	44,00%

The first thing that falls in the spotlight is that there is no unique solution for the Junior Mezzanine tranche. We can see how the market spread can be equated either via 5% or 78,534%. This result implies that the risk profile embedded in the investment in those tranches can vary significantly. When the correlation is high mezzanine investors are said to be long on correlation, that is on credit risk, while when the latter is at low levels they are short on credit risk. This changing relationship is due to the position of the tranche in between high default-probability investments (equity tranche), and low probability ones (Senior tranches). The market practice is to consider the lower level as it is reasonably nearer to the average historical equity returns correlations.

If we plot the values against the attachment-detachment points we can see that the curve creates a smile, the famous correlation smile.

Figure 4.2



To overcome the inefficiencies enclosed by considering the compound (implied) correlation, as the market of CDSs and CDOs tranches became more saturated, quotes for the standard tranches became more available. As a consequence, they started to consider base correlation, that takes on its name from the fact that base tranches (equity tranches) are used for the pricing of higher tranches. With base tranche is meant a tranche with 0 attachment. From that it follows that for example, the mezzanine tranche (that is the one usually causing more troubles) with attachment 3% and detachment 7% can be thought to be a composition of a long position in a base tranche with detachment 7% and a short position in our equity tranche with detachment 3%.

We recalculate the value of the equity tranche with the given market spread of the 3%-7%; we can then achieve the PV that has to be subtracted from the new base (0%-7%) tranche's PV in order to reach the PV of the mezzanine tranche and solve it for the correlation that makes it equal to 0, after having adjusted the notional and expected loss.

After the analysis of all the factors embedded in the financial collapse, there are some personal considerations I would like to draw.

When the economy is in a downturn, each single default is mostly likely driven by external factors, we know that in 2007-2009 even well established companies (characterized for example by low leverage and high liquidity of assets) suffered because of the unexpected difficulties in borrowings and in the temporaneous decline in the willingness to trade. That have been said, even if an entity had enough theoretical liquidity, if there's no market, then the assets cannot be turned into cash, even if they are the highest demanded ones. As a consequence, whatever the company position was, they were all associated by the same struggle.

My opinion is that, as this was the case, considering a common pairwise correlation could turn out to be an acceptable compromise, as the dependence between each pair of companies was more driven by common factors, affecting the entirety of the portfolio.

We have seen how one of the main features of CDSs contracts is that they are off balance sheet, meaning that they cannot be included in the risk evaluation of a company. We also have seen that in the years prior to the crisis and during the first period, such market was at the peak of its expansion, so it is a natural consequence that the credit ratings could not have been precise as they should, since there was a high probability of dealing with a company involved in many of such contracts, hiding the risk of such transactions, as they were not required to furnish any detail about them.

Notwithstanding the simplicity embedded in it, marketers preferred to avoid other sophisticated models, as the latter required the modelling of huge amounts of parameters with few information effectively available.

For the reasons just exposed above, my opinion is that the main issue behind the crisis cannot be blamed to be the model, nor even the parameters estimations, but rather the surrounding environment.

As all the recipes, even the one for disaster requires more than one ingredient, so even though it didn't permit fair precise estimates it is true that it had to calculate factors based on available data that were not accurately reflecting the reality. The lack of control by the government and the speculative use of asset backed securities made by institutions, together with the misalignment of interests, have for sure caused the whole economy to suffer, and to keep hiding the real risk profile behind the transactions.