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Options in asset allocation problems: an empirical model applied to the Italian FTSE MIB Index

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Introduction

Financial derivatives are a particular asset class whose value is dependent from another asset, the so called underlying, and which plays a very important role in the portfolio's choices of the investors.

These instruments can be used in a speculative manner, to try to catch early the future movements of the market and lock in the price of an asset, to hedge against the key risk factors within the market, or to exploit mispricing and general market inefficiencies through arbitrage.

The classic asset allocation problem as developed by Samuelson and Merton in 1969 does not take into consideration the role of derivatives due to their relative brief history (the first official security exchange was the Chicago Board Options Exchange in 1973) and so the lack of sufficient historical prices and data; the original problem was based only on the investor's selection between stocks, the risky asset, and bond, the risk free securities, with the aim of maximizing the expected utility.

Often the portfolios created under these constraints need a continuous rebalancing in order to maintain the risk and return trade-off at the right level to maximize the utility and cannot provide a reliable hedging against the key risk factors.

Starting from the explanation of the main characteristics of options, the key values that drive their prices, the most important types of options, from the classic European and American to the different exotic ones, and the most used trading strategies with combination of different type of options, with different maturities and different strike prices, this thesis contains an analysis of some academic papers that studied the importance of options in the asset allocation problem.

The working paper of Haugh and Lo is focused on the possibility of constructing a buy-and-hold portfolio of options to replicate a certain dynamic strategies based on the classic asset allocation problem (a portfolio of only stocks and bonds); the authors find a combination of options which, by considering the results of two different utility function, the constant relative risk aversion CRRA and the constant absolute risk aversion CARA, under three different leading cases, the Geometric Brownian motion, the Ornstein-Uhlenbeck process and a bivariate linear diffusion process, will come closest to the dynamic strategy in terms of certainty equivalence.

The closeness of the static portfolio with respect to the dynamic one is measured in three different ways: through the maximization of the expected utility, through the minimization of the mean-squared error and through the minimization of the weighted mean-squared error.

The second paper of Jun Liu and Jun Pan explains the importance of derivatives to complete the market and provide independent exposure to each of the three main risk factors incorporated in the model of the aggregate stock market: diffusive price shocks, price jumps, and volatility risks.

Typically, a risky stock provides risk exposure only to diffusive price risk; with the introduction of derivatives, the investor can take advantage of the risk-return trade-off provided by volatility risk and jump risk.

The third paper, written by Branger, Schlag and Schneider, consider the possibility of jumps not only related to prices but also to volatility; by deriving the optimal portfolio for an investor with a CRRA utility function, they find that the demand for jump also includes a hedging component, and ignoring the possibility of jump in volatility will lead to a loss in terms of utility for the investor.
The last paper introduces a dynamic option strategy applied to the S&P 500 Index, based on a risk-free security and monthly options selected from a range of possible strike prices. Options are held until maturity and each month the relative weight of them changes in response to the maximization of CRRA utility function with a risk aversion coefficient γ equal to 4 (10 for the optimization process).
The last part of this thesis is a back test process developed with Python 3.7 and applied to two of the most popular option strategies, the Straddle and the Iron Condor, a third strategy, called the Mixed strategy, that consists in a long Straddle plus a long Iron Condor (the two short options in the Iron Condor will reduce the total price of the structure), and finally the analysis of the performance offered by a simple strategy based on a volatility parameter.
The so-called V strategy consists in the selection between a long Straddle or a long Iron Condor in each period based on the difference between the implied volatility and the historical volatility: if the implied volatility on a 30 days basis (i.e. the market expectations regarding the possible future fluctuations in the underlying asset) is higher than the realized volatility of the last year, this means that investors in the market are expecting a potential big move of the underlying, so the V strategy will select the long Straddle position for that period, otherwise, if IV is less than HV, the market is expecting small movements, and the V strategy will select the Iron Condor, to buy a sort of insurance against potential big fluctuations and maximize the payoff in case the price underlying remains within a restricted price range.
Each of the four strategies analysed is made with quarterly options on the Italian FTSE MIB Index; the back test is performed during a period of ten years, from the first quarter of 2010 (starting the Monday after the third Friday of December 2009) to the last quarter of 2019.
The results of the Straddle, Iron Condor, Mixed strategy and V strategy are compared with each other and with the FTSE MIB Index performance, in terms of mean return, median, volatility, skewness and kurtosis.

1). What are options? History, definitions, and strategies.

1.1). Options: brief history and general definition.

Derivatives are financial instruments whose value is reliant upon (or derives from) another financial asset, or a group of assets, called the underlying.

An option is one of the main derivatives instruments, and it gives the holder the opportunity (i.e. the option) to buy or sell, depending on the type of option (Call or Put), a certain amount of the underlying asset (usually 1 option represents 100 shares of the underlying security) at a certain price (the strike price) in a certain time in the future. The time of the exercise differs from the two main type of options: the American options can be exercised only at maturity, while European options can be exercised at any date until maturity.

Options can be used for three main different purposes: hedging, speculation, arbitrage.

The hedging is aimed at reducing the risk arising from potential future movements in the value of some asset (it could be a stock, a commodity, etc…); if an investor wants to protect himself against the future movements in the price of an asset he can buy a Call Option to lock in the price at which he will buy the underlying asset.

The speculation is substantially a bet on the future direction of the market: if an investor thinks that the market will go up, he can buy a Call Option and, if the spot price of the underlying, at any date until the maturity of the option, will be greater than the strike price, he will gain from the purchase of the underlying at the strike price and the subsequent sale of it at the market price.

The arbitrage is an operation in which an investor tries to lock in a riskless profit by exploiting the market inefficiencies and by simultaneously entering opposite transactions in different markets.

Although options have been traded for centuries, the first options' appearance was in 1790, initially they were considered as "obscure and unimportant financial instruments"[1], and they were at times banned or considered illegal.

At that time options were only traded in small OTC markets with lack of intensive regulation until the '33 Act (the first federal legislation to regulate the disclosure on offer and sale of securities); there was not an official securities exchange until the creation of the Chicago Board Options Exchange (CBOE), the first registered securities exchange for trading in options, in 1973.

Before the 1933 the regulation of securities in the United States was based on the so called "blue sky securities law", state laws that require the registration of all securities offerings and sales to protect the investors from fraud based on the merit reviews: there were very strictly and specific qualitative requirements on offerings and if the company does not respect those ones it will not be allowed to register the offerings. The primary purpose of the '33 Act was to ensure that shareholders would have received key information regarding

[1] J. C. Cox – M. Rubenstein, *Options Markets*, Prentice – Hall Inc. 1985, p. vii.

securities in which they want to invest, and it was based on a disclosure philosophy, meaning that also a bad investment were legal as long as all the key information are accurately disclosed.
During 1929, after the stock market crash, the Congress had found that the granting of options to pool syndicate was one of the main elements at the bottom of the large-scale manipulative operations and was one of the causes of the market crash.
So, in 1934 the Securities Exchange Act, as initially drafted, banned all stocks options contract.
It was only thanks to the Committee of Put and Call Brokers and Dealers Association, formed to represent option dealers, that such a drastic step was not taken, and options was saved.
That Committee argued that options had a stabilizing effect on the markets and could be used by investors not only for speculation, but also to hedge their position (i.e. as insurance for portfolios). The Act of the 1934 did not ban options contracts but established the Securities and Exchange Commission (the SEC) and gave it the authority to regulate their secondary trading.
Initially, much of the stock options trading business was conducted solely by 25 members of the Put and Call Brokers and Dealer Association, so options' volume was very low: in 1968 the annual contract volume reached only the 1% of the New York Stock Exchange (NYSE) volume, and this was mainly due to the way in which OTC worked.
There were three different types of transaction costs that negatively affect a lot the volume of trading operations and the spread between the price paid by the buyer of a Call Option and the premium received by the writer (i.e. the seller): the endorsement fees to guarantee the performance of the option in the event of exercise, the dealer spread, and the buyer commission. For a Call Option for which a writer (the seller) typically received $200, a buyer will pay about $250. The effect of those transaction costs was even greater for the Put Options.
Moreover, due to the absence of a big secondary market and a market maker able to produce liquidity by granting every time the possibility to buy or sell options, investors were committed to their positions until the expiration date, and writers were highly exposed to the risk that the options could be exercised (with a negative effect on the supply of options).
Only after the creation of the CBOE the trading volumes in options started to increase.
It was created by a committee instituted by the Board of Trade to study the possibility of applying the main trading principles of commodity futures to options contracts on securities. The committee concluded that it was necessary to create a clearinghouse that would have the role of both buyer and seller of every options contract traded on the exchange.
Since options were considered as securities by the SEC, the CBOE Clearing Corporation, later renamed the Options Clearing Corporation (OCC), by virtue of the Security Exchange Act of 1933, was required to provide a prospectus in which it had to describe the main mechanism of options trading and the risks associated with these instruments.

Moreover, the CBOE agreed that all the underlying assets (the stocks on which the value of options relies), would be listed on a stock exchange to ensure the presence of a broad and not manipulated market for the underlying securities.
The CBOE began trading on April 26 of 1973 with initially only 305 seats sold for $10.000 and 16 Call options tradable, with a volume on the first day of just 900 contracts; Put were not even introduced until 1977.
Average daily volume (ADV) in the first full month of operation was 1.584 with an average of 1,7 contracts per trade.
The graphs below show the trading volume of U.S. options contracts from the inception in 1973.

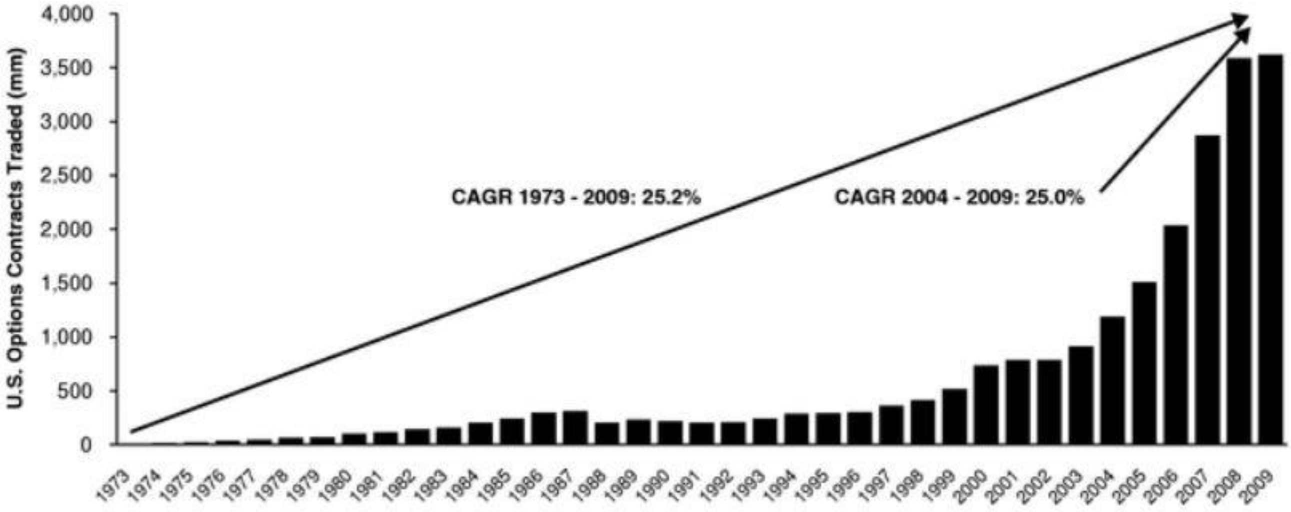

Figura 1. Number of options contracts traded from 1973 to 2009. [2]

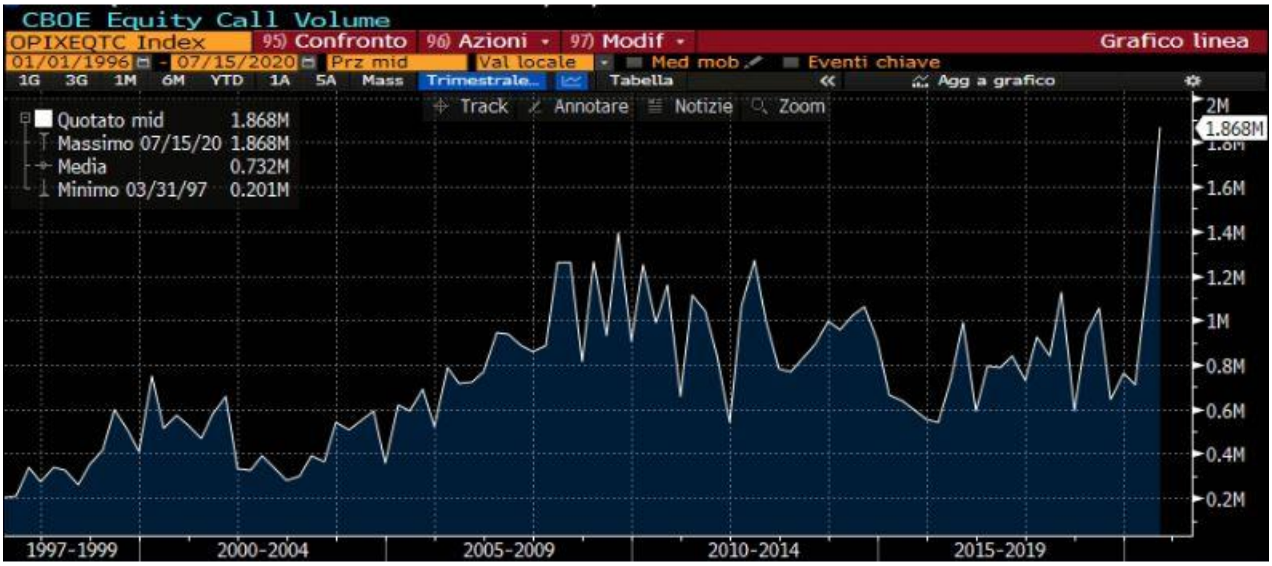

Figura 2. CBOE Equity Call Volume from '96 to today. Source: Bloomberg.

[2] Source: https://www.sec.gov/Archives/edgar/data/1374310/000104746910002050/a2197106zs-1.htm

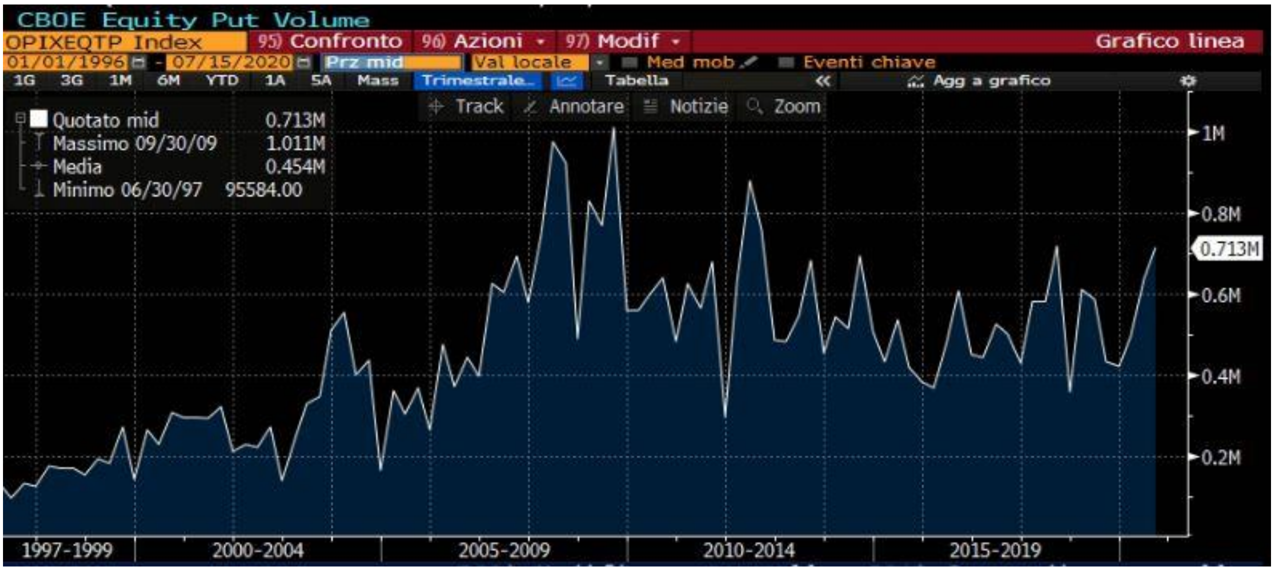

Figura 3. CBOE Equity Put Volume from '96 to today. Source: Bloomberg.

The increase in volumes was mostly due to the efficiency of the new mechanism introduced in the CBOE. Typically, on a security exchange, the limit order book is a record of the limit orders (buy orders or sell orders) maintained by the security specialist who works at the exchange and who executes the orders at or better than the given limit price.

Instead, the Market Maker system implemented by the CBOE involved a public limit order book in each option, with orders filled only by brokers who could not trade for their own accounts and were employees of the exchange (*Floor Brokers* and *Order Book Officials*), and *Market Makers*, each assigned to a specific option class, who provided liquidity to the market by trading any option at any time for their own accounts.

Market Makers also assume long-term positions based on their expectations and strategies, always respecting their primary obligation: “no more than 25% to 50% of any Market Maker’s trades (measured in number of contracts) can be outside his principal assignment – usually options on three underlying stocks – in any given quarter”[3]

Typically Floor Brokers fill *market orders* (i.e. an order that instructs the Floor Broker to immediately fil the order at the best possible price), while those who are self-employed to the exchange fill the *limit orders* (i.e. an order that instructs the Market Maker to fill it only at a specified price or better).

The incredible success of CBOE and the expansion of the options market brought competition, with the creation of other options floors, as the American Stock Exchange (AMEX) or the Philadelphia and Pacific Stock Exchange.

The SEC required a common clearing system and a common price-reporting system.

For this reason, the CBOE and AMEX agreed to use the OCC as common clearing agency, and a common price-reporting system managed by the Options Price Reporting Authority (OPRA).

The incredible success of the options exchange was due to three main different aspects:

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[3] J. C. Cox – M. Rubenstein, *Options Markets*, Prentice – Hall Inc. 1985, p. 81.

- **Size and profitability:** not only the huge increase in volume and open interest, but also the introduction of options on new stocks (from 32 stock in 1973 to 145 in 1983) and the number of total membership on the exchange (the price of a seat increased almost 10 times in the first 10 years).

Year	Cleared Contract Volume				Stocks Listed at Year End	Membership Profile at Year End	
	For Year	Daily Average	Daily High	Market Share		Total Membership	Last Sale ($000's)
1973	1,119,117	6,469	17,319	100.0%	32	520	25
1974	5,682,907	22,462	63,929	100.0%	40	684	39
1975	14,431,023	57,040	124,528	79.7%	79	1,281	64
1976	21,498,027	84,972	168,555	66.4%	86	1,337	62
1977	24,838,632	98,566	223,781	62.7%	95	1,293	45
1978	34,277,350	136,021	425,930	59.9%	95	1,253	75
1979	35,379,600	139,840	342,334	55.1%	95	1,240	99
1980	52,916,921	209,158	432,639	54.7%	120	1,640	152
1981	57,584,175	227,605	426,780	52.6%	120	1,666	180
1982	75,721,605	299,295	666,457	55.2%	140	1,697	174
1983	71,695,563	283,382	592,890	52.8%	145	1,753	212

Figura 4. CBOE Volumes, Listed Stocks and Membership in the first 10 years.[4]

- **Liquidity:** the new Market Maker system granted constant liquidity and was reflected in a great institutional participation in the marketplace.
- **Transaction speed:** execution times was enormously reduced by the introduction of high-speed communication equipment and the new electronic Order Support System (OSS) in 1981, with the first computerized limit order books in the history.

From 1973 options become more and more popular, often used as essential instruments in every investor's portfolio or by traders who want to build specific strategies to hedge risks and to build neutral positions in an underlying asset.

Below is explained the standard mechanism of options, the main factors that affect their price, and the most common option strategies used in the market today.

[4] *Ibidem*, p.83.

1.2). Plain Vanilla and Exotic options: main determinants, greeks, payoffs.

Plain Vanilla is a term used in the financial sector to indicate the most basic type of a specific financial instrument (options, futures, bonds). It represents the standard version of a financial security, without any extra that can alter the original nature of the instrument.

The Plain Vanilla options, the two basics type of options, are the Call and Put options.

A Call option is a financial security that gives the holder the right, not the obligation, to buy a particular asset, the underlying, at a pre specified amount, the strike price, at a specified time in the future.

A Put option, on the other side, is a financial security that gives the holder the right, not the obligation, to sell a particular asset, the underlying, at a pre specified amount, the strike price, at a specified time in the future.

Both these two basic options, as all other types of options, can be both bought or sold, so an investor could have a Long position in an option (he bought the right) or a Short position in it (he sold the right).

The amount that the buyer of an option must pay to the writer to obtain the right to buy or sell the underlying is the *premium*, indicated with *c* for Call options and *p* for Put options.

Every option has an Intrinsic Value and a Time Value, both reflected in its premium, and the *moneyness* represents a financial measure of the extent to which an option could have a positive value at maturity:

- If K = S the option is *at-the-money* (*ATM*).
- If K > S (K < S) the Call (Put) option is *out-of-the-money* (*OTM*).
- If K < S (K > S) the Call (Put) option is *in-the-money* (*ITM*).

The Intrinsic Value is the value of the option as if it would expire today; if the option is ITM it would have a positive value, if it is OTM its intrinsic value will be negative.

The Time Value is the difference between the price of the option and its intrinsic value. That's the reason why the value of an OTM option (i.e. an option that has no intrinsic value) will be equal to its time value.

The main elements that affect the Intrinsic Value of options are:

1. The Spot price (*S*): the market price of the underlying asset at any time until the maturity. The higher the Spot price, the higher (lower) the value of a Call (Put).

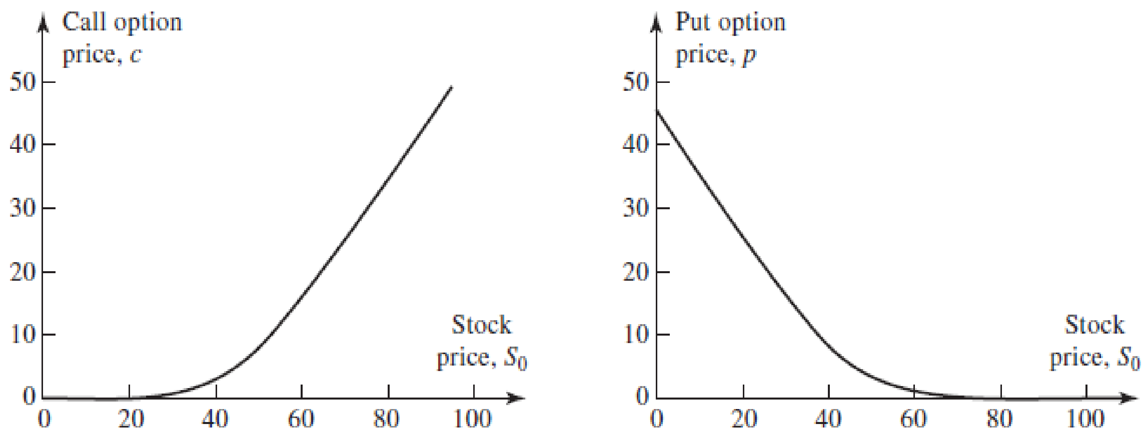

Figure 1. Source J. C. Hull, *Options, Futures and Other Derivatives*

2. The Strike price (K) or exercise price: the price at which the buyer of the option can buy (Call option) or sell (Put option) the underlying. The higher the Strike price, the lower (higher) the premium paid by the buyer of a Call (Put).

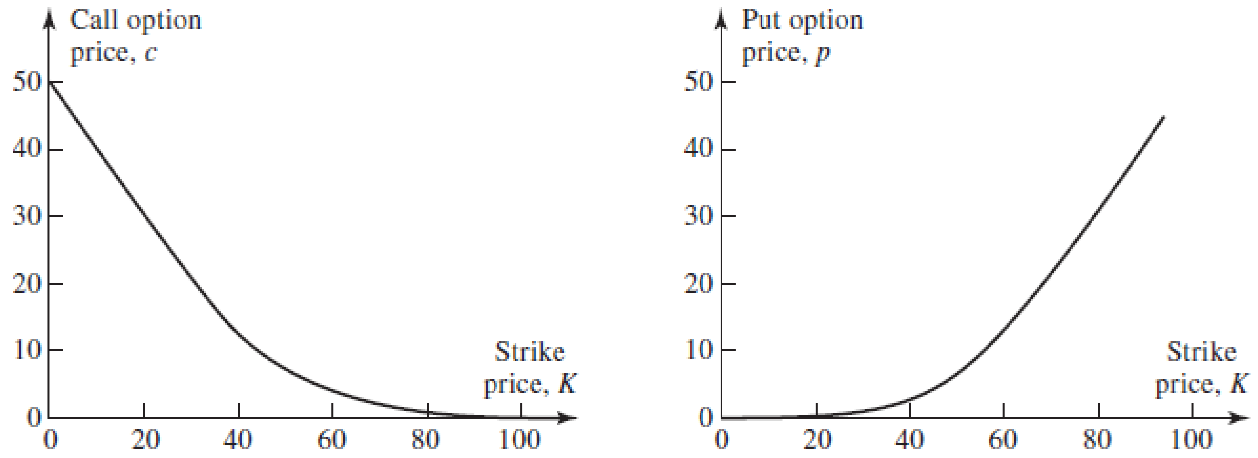

Figure 2. Source J. C. Hull, *Options, Futures and Other Derivatives*

The elements that affect the Time Value are:

1. The time to maturity (t): the fraction of time that remaining in the life of an option. The higher the time to maturity, the higher the probability that even a very low volatility stock can move, so the higher the premium of both Call and Put options.

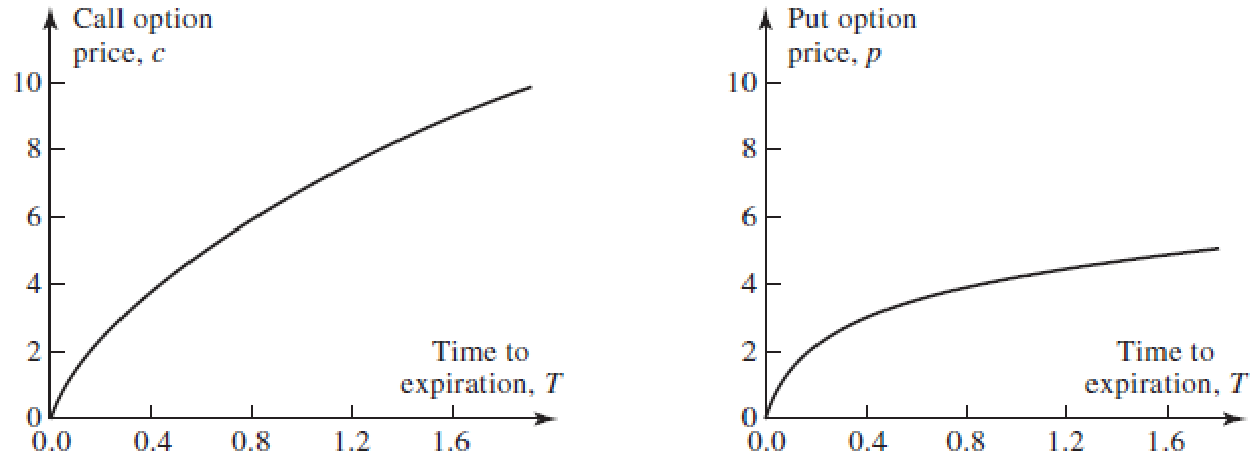

Figure 3. Source J. C. Hull, *Options, Futures and Other Derivatives*

2. The volatility (σ) as a measure of dispersion of future stock prices: it measures the uncertainty regarding the future value of a financial security, the higher it is, the higher the possibility of large price changes in the futures. It has typically a double effect for the owner of a stock, who can profit much more as loss much more in case of high volatility, while the owner of an option has a limited downside risk, due to the fact that, if the price goes too much down (for the owner of a Call option) or too much up (for the owner of a Put), he will not exercise the option and his maximum loss will be the premium paid. It should be noticed that, keeping all others aspects fixed, the premium of a Put is always

greater than the premium of a Call, since the up potential of a stock price is unlimited, while the minimum value is 0.

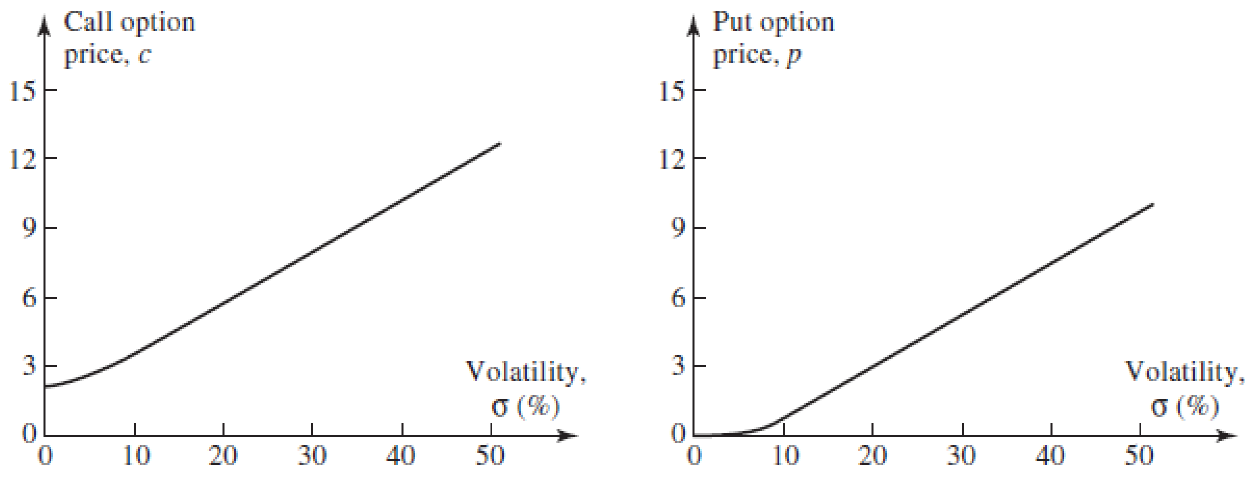

Figure 4. Source J. C. Hull, *Options, Futures and Other Derivatives*

3. The risk-free interest rate (r) affects the option value because an increase in it will reduce the present value of cash flows and also increase the expected return required by investors.

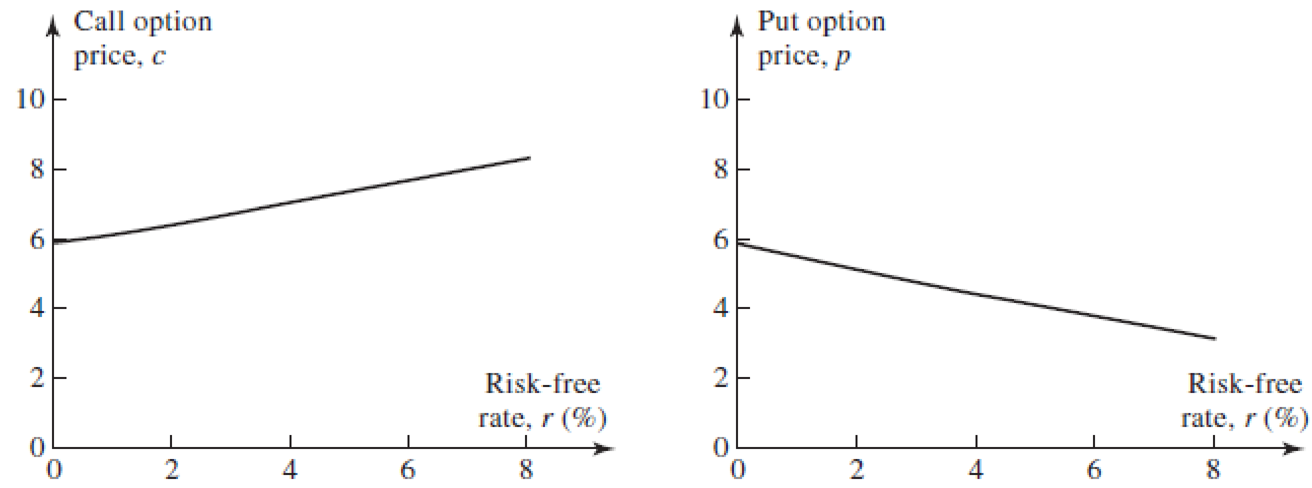

Figure 5. Source J. C. Hull, *Options, Futures and Other Derivatives*

4. The expected dividends paid by the underlying stock during the life of the option. Dividends reduce the stock price generating a negative effect for the Call option and a positive one for the Put option.

We have just exposed the main factors that affects the options value, now we need to understand the range into which this value moves.

Starting from a Call option, we know that it gives the holder the right to buy the underlying asset for a certain price, hence the price of it could never be greater than the stock: $\boldsymbol{c \leq S}$.

This condition represents an Upper Bound and must always hold, otherwise an arbitrageur could buy the Call and sell the stock, locking in a riskless profit.

The Lower Bound for the price of a Call is $\boldsymbol{S - Ke^{-rT}}$; if this condition doesn't hold an arbitrageur can short the stock (i.e. selling the stock without owning it by borrowing it from an intermediary with the intent to buy back later at a lower price and return it to the lender) and buy the Call, investing the amount obtained at the risk free rate for 1 year and at the expiry date he will lock in a secure profit.
Assuming that the Spot Price (*S*) is equal to \$20, the Strike Price (*K*) is \$18, the risk-free rate (*r*) is 10% and the maturity of the option (*t*) is in 1 year. In this case $\mathbf{S - Ke^{-rT} = 20 - 18e^{-0,1} = \$3,71.}$
Now we consider the situation in which the Call price is lower than the theoretical Lower Bound and equal to \$3.
The table below shows the payoff of an arbitrageurs in two cases:

Short the stock and buy the Call	Invest \$15 for 1 year at r	If at T S > K --> option is exercised	If at T S < K --> option is not exercised (S = \$13)
S - c = \$20 - \$3 = \$17	17*e^0,1 = \$18,79	\$18,79 - \$ 18 = \$0,79	\$18,79 - \$17 = \$1,79

In conclusion, we can say that the premium of a Call option should be $\boldsymbol{S \geq c \geq S - Ke^{-rT}}$.
On the other side, the Upper Bound for an America Put option, since it gives the holder the right to sell one share of a stock at price K, the option can never be worth more than K: $\boldsymbol{p \leq K}$; for an European Put option, which can be exercised at any time, the Upper Bound is: $\boldsymbol{p \leq Ke^{-rT}}$.
The Lower Bound instead is $\boldsymbol{Ke^{-rT} - S}$; if this condition doesn't hold an arbitrageur can borrow money needed to buy both the Put and the stock, and at maturity he will be able to repay the lender independently from the price of the underlying.
Assuming that the Spot Price (*S*) is equal to \$37, the Strike Price (*K*) is \$40, the risk-free rate (*r*) is 5% and the maturity of the option (*t*) is in 6 months. In this case $\mathbf{Ke^{-rT} - S = 40e^{-0,05*0,5} - 37 = \$2,01}$.
Now we consider the situation in which the Put price is lower than the theoretical Lower Bound and equal to \$1.
The table below shows the payoff of an arbitrageur in two cases:

Borrow \$38 to buy Put and Stock	Investor must repay in 6 months	If at T S < K --> option is exercised	If at T S > K --> option is not exercised (S = \$42)
S + p = 38\$	38*e^0,05*0,5 = \$38,96	\$40 - \$38,96 = \$1,04	\$42 - \$38,96 = \$3,04

In conclusion, we can say that the premium of an America Put option should be $\boldsymbol{K \geq p \geq Ke^{-rT} - S}$, while for a European Put option should be $\boldsymbol{Ke^{-rT} \geq p \geq Ke^{-rT} - S}$.
Value of options must be inside those boundaries to avoid arbitrage opportunities in the market.

Despite the fact that the price of options can only move within these bands and all the elements that affect it can be easily quantified, it is important to remember that, as other financial securities, options are risky instruments whose value could change rapidly in response to different aspects of the market.

There are some variables, the so-called ***Greeks***, that measure the dimension of risks the investor must take into consideration when he opens a Long or Short position.

Each Greek represents the sensibility of the value of the options in relation with underlying variables and are used by traders and investors to evaluate each instrument.

The most important Greek is the ***Delta (*Δ*)***.

It is the first derivatives of the premium of an option with respect to the Spot Price of the underlying: $\Delta \mathbf{c} = \frac{\partial \mathbf{c}}{\partial \mathbf{s}}$ where *c* is the price of a Call option and S the Spot price of the underlying. It represents the sensitivity of the option price with respect to changes in the value of the underlying (i.e. the rate of change between the price of an option and a $1 change in the price of the underlying asset). The Delta of a Call can assume value from 0 to 1, while the Delta of a Put can assume values from 0 to -1; for an OTM option the Delta is near to 0, while for an ATM option is near to 1.

For example, if a Call option has a Delta equal to 0,5, an increase in the price of the underlying asset of $1 will theoretically generate an increase by 50 cents in the option's price.

The usage of Delta is very important for traders who want to build a neutral position true the *Delta hedging*: in this view the Δ tells the trader how much of the underlying he has to buy/sell to hedge his position against moves in the premium of the option.

Assuming a trader who buy 100 Call options (C) with Δ = 0, each option gives the buyer the right to buy 100 shares (N) of the underlying; assuming also S = $10, c = $1and Δ = 0,4. It is possible for the trader to build a neutral position by shorting Δ * C * N = 4000 shares of the underlying.

The two tables below show the outcome of this neutral position in the case of both an increase and a decrease of $1 in the price of the underlying:

Increase of $1		
Loss from short position	**Increase in the option premium**	**Gain from the option position**
-4000 * 1 = -$4000	1 * Δ = $0,4	0,4 * C * N = +$4000

Decrease of $1		
Gain from short position	**Decrease in the option premium**	**Loss from the option position**
-4000 * -1 = +$4000	-1 * Δ = -$0,4	-0,4 * C * N = -$4000

This strategy protects the investor only from small movements in the price of the underlying. In case of big moves, the Δ changes and the trader will not be able to hedge his position.

The ***Gamma (Γ)***, also called the Delta of the Delta, is the second order derivative of the premium of the option with respect to the price of the underlying: $\boldsymbol{\Gamma = \frac{\partial c}{\partial S^2}}$.

It is the rate of change of the Δ with respect to a one-unit change in the price of the underlying and it represents the sensitivity of the Delta to changes in the price of the underlying.

A Gamma equal to 0,5 means that, if the price of the shares increase by $1 the option's Delta would decrease by 0,5.

Higher values of this variable indicate that the Delta could change a lot in response to minimum movements in the share's price; generally it assumes smaller values the higher the time to maturity of the option and higher values as expiration approaches.

The ***Vega (V)*** is the first derivative of the premium with respect to the volatility $\boldsymbol{V = \frac{\partial c}{\partial \sigma}}$ and represents the sensitivity of the option's premium to the underlying's implied volatility, a financial concept that differs from the historical volatility and that represents the expectations of investors regarding the likelihood of changes in a given security's price.

For example, an option with V = 0,3 indicates that the option's value is expected to change by 10 cents if the implied volatility changes by 1%.

For Long positions in plain vanilla options the Vega is always positive, for Short position the Vega is always negative.

The ***Theta (Θ)*** is the first derivative of the premium with respect to the maturity $\boldsymbol{\Theta = \frac{\partial c}{\partial t}}$.

It is the time decay of the option (i.e. the amount an option's price would decrease as the time to maturity decreases); the price of a Call option with Θ = -0,5 would decrease by $0,5 every day that passes.

Long Call and Put options usually have a negative Θ, while Short Call and Put have positive Θ.

The ***Rho (ρ)*** is the first derivative of the premium with respect to the risk-free rate $\boldsymbol{\rho = \frac{\delta c}{\delta r}}$ and represents the sensitivity with respect to the risk-free rate.

Besides these Greeks, that are the most important elements that affects the price of an option, there are some other variables, the *minor Greeks* who also have an impact on the value of these financial derivatives.

Here the list of the 4 most common minor Greeks used by derivatives traders.

The ***Lambda (Λ)*** is the leverage factor and it measure how much leverage an option is providing as the price of that option changes. The formula for the calculation of Lambda is $\boldsymbol{\Lambda = \frac{\partial lnc}{\partial lnS}}$, but it can be simplified as the value of Delta multiplied by the ratio between the stock price and the option price.

For example, assuming the stock price is $100, the option price is $5 and Δ is 0,4; Lambda will be equal to $\boldsymbol{\Lambda = \Delta * \frac{S}{c} = 0,4 * \frac{100}{5} = 8}$: this value indicates that a 1% increase in the dollar value of the stock would yield an 8% increase in the dollar value of option.

The ***Zomma*** is the third order derivative of the premium of the option with respect to the price of the underlying; it measures the sensitivity of the Gamma to price movements in the stock price.
The ***Vomma*** is the second order derivative of the price of the option with respect to the implied volatility of the underlying and it measures the sensitivity of the Vega to fluctuations in the implied volatility of the stock.
The ***Ultima*** is the third order derivative of the price of the option with respect to the implied volatility of the underlying and it measures the sensitivity of the Vomma to fluctuations in the implied volatility.
All those variables are often used by traders and investors for pricing of derivatives and are very important in the process of options' selection.
The analysis of these determinants allows the investor to understand the future possible scenarios for the option premium and the payoff that an option will grant.
"The understanding of options is helped by the visual interpretation of an option's value at expiry. We can plot the value of an option at expiry as a function of the underlying in what is known as a payoff diagram".[5]
The Payoff of an option is a formula that expresses the future gain or loss in which the investor would incur in all the possible scenarios on the price of the underlying, but is often figured with a simple diagram that allows the investor to have an immediate visualization of what will be his financial remuneration.
For a Long position in a Call option the payoff that the investor should expect is:

$$\mathbf{max}(\boldsymbol{S_t} - \boldsymbol{K}, \mathbf{0})$$

The option will be exercised if the price of the underlying is greater than the strike price, yield a gain to the investor equal to the difference between those two values, otherwise the option will not be exercised and the loss is 0 (without considering the premium c that the investor paid to buy the option).
For a Short position in a Call option the payoff that the investor should expect is:

$$\mathbf{min}(\boldsymbol{K} - \boldsymbol{S_t}, \mathbf{0})$$

The writer of the option, who has sold the right to buy the underlying, will have a gain equal to the premium he has received if the spot price S is lower than the strike price (i.e. if the buyer will not exercise the option), otherwise he would have a loss equal to the negative difference between the strike price that he receives by selling the underlying to the buyer and the spot price at which he had bought it.
The figures below show the payoff of a Long and a Short position in a Call option ATM on the FTSE MIB.

[5] R. Willmott, *R. Willmott introduces to Quantitative Finance – 2nd ed.*, Jhon Wiley & Sons, Ltd 2007, p. 34.

Figure 6. Payoff of a Long Call. Source: Bloomberg.

Figure 7. Payoff of a Short Call. Source: Bloomberg.

For a Long position in a Put option the payoff that the investor should expect is:

$$\max(K - S_t, 0)$$

The option will be exercised if the price of the underlying is lower than the strike price, yield a gain to the investor equal to the difference between those two values, otherwise the option will not be exercised and the loss is 0 (without considering the premium p that the investor paid to buy the option).

On the other side, for a Short position the payoff is:

$$\min(S_t - K, 0)$$

The writer of the option, who has sold the right to sell the underlying, will have a gain equal to the premium he has received if the spot price S is higher than the strike price (i.e. if the buyer will not exercise the option), otherwise he would have a loss equal to the negative difference between the strike price that he has to pay to buy the asset from the buyer and the spot price at which it could be possible to buy it on the market. Figures below show the diagram of payoff for these two positions in a Put option.

Figure 8. Payoff of a Long Put. Source: Bloomberg.

Figure 9. Payoff of a Short Put. Source: Bloomberg.

Besides the two classical Call and Put option there are several other types of options, the so-called *exotic options*, each with a specific characteristic.

These financial products are often much more profitable than plain vanilla one and they are designed and developed by derivatives dealer for different reasons.

"Sometimes they meet a genuine hedging need in the market; sometimes there are tax, accounting, legal, or regulatory reasons […]; sometimes the products are designed to reflect a view on potential future movements in particular market variables;…"[6].

Two of the most encountered exotics are the **Bermuda** and **Asian** options:

[6] J. Hull, *Options, Futures and Other Derivatives – 9th ed.*, Pearson Education, Inc. 2015, p. 598.

- The **Bermuda** option is between the America and European options because it could be exercised only at the expiration date or at pre-determined intermediate dates between the purchase and the maturity.
- The **Asian** options are financial instruments whose payoffs are determined not by the spot price of the underlying but by the average price of it during the period of existence of the option.

Next to these two exotics there are other important types that are often used by traders and investors in their portfolios.

Here we exposed only three of the most important: Barrier options, Basket options and Lookback options.

The **Barrier** options, Call and Put, are options whose payoff depends upon whether the price of the underlying reaches a predetermined level, precisely called *barrier*, during a certain period of time.

There are two different type of barrier: the *knock-in* and the *knock-out*: the knock-in Barrier option starts to exist if and only if the price of the underlying reaches that level, while the knock-out barrier option ceases to exist if the price of the underlying reaches that level.

Knock-in options can be both *up-and-in*, the option comes into existence only if the price of the underlying asset rises above the barrier (usually the strike price), or *down-and-in*, the option comes into existence only if the price of the underlying asset moves below the barrier.

Knock-out options can be both *up-and-out*, the option ceases to exist when the price of the underlying asset moves above the barrier set above the strike price, or *down-and-out*, the option ceases to exist when the price of the underlying moves below the barrier that is set below the underlying's initial price.

The **Basket** options have the strike price on the weighted value of its components; they are used to hedge portfolios against multiple assets in only one transaction rather than hedging each individual asset with different instruments.

The **Lookback** options initially do not have a specified strike price, it could be selected by a range of different exercise price only on the maturity date.

These are only some of the multitude of different options traded every day in the market by international traders, and it is possible to combine different kind of it, sometimes also with the purchase or sale of the underlying, to construct entire strategies with different payoff's structure.

1.3) The most common options strategies.

The objective of option' strategies is to provide a non-linear exposure to the underlying asset; the usage if different financial derivatives can provide payoff diagrams that generate returns in every kind of future scenarios.

Here a brief description of the main option strategies, with each payoff diagrams.

- **Covered Call:** a long position in a stock combined with a short position in European Call; this strategy provide a neutral exposure to the underlying and is usually used by investors who have a long position in the underlying with a short-term neutral view. If the price of the stock closes above the strike price the investor will gain from the long position in the stock and will lose from the short position in the Call, with a net positive payoff, while if the stock price closes below the strike price he will lose from the stock's leg but he has received the premium for selling the stock.

Figure 10. Payoff of a Covered Call. Source: Bloomberg.

- **Reverse Covered Call:** it is a specular strategy with respect to the first one. A Long Call combined with a Short position in the underlying stock.

Figure 11. Payoff of a Reverse Covered Call. Source: Bloomberg.

- **Protective Put**: this strategy involves buying a European Put on a stock and the stock itself.

Figure 12. Payoff of a Protective Put. Source: Bloomberg.

- **Reverse Protective Put**: this strategy involves selling short a European Put on a stock and the stock itself.

Figure 13. Payoff of a Reverse Protective Put. Source: Bloomberg

- **Bull Spread**: this is one of the most popular strategies among traders. It is composed by two different legs: a long position in a European Call with strike price K_1 and a short position in a European Call with a strike price $K_2 > K_1$. This strategy limits the both the investor's upside and downside risk, offering a profit in case of a lateral or slightly bullish scenario. A Bull Spread can be created also with Put options, by opening a long position in a European Put with strike price equal to K_1 and simultaneously selling short a European Put with strike price $K_2 < K_1$. In both cases the payoff of this strategy is the same.

Figure 14. Payoff of a Bull Spread. Source: Bloomberg.

- **Bear Spread**: it is a bearish option strategy composed by two legs. The firs one is a short position in a European Put with a strike price equal to K_1 and the second one is a long position in a European Put with strike price $K_2 > K_1$. This strategy, like the Bull Spread, limit both upside and downside risk for the investor and it offers a profit in case of lateral or slightly bearish scenario. A Bear Spread can be created also with Call options, by selling short a European call with strike price K_1 and by opening a long position in a European Call with strike price $K_2 > K_1$. In both cases the payoff of this strategy is the same.

Figure 15. Payoff of a Bear Spread. Source: Bloomberg.

- **Box Spread:** it is a combination of a Bull Call Spread and a Bear Put Spread with the same two strike prices (K_1 and K_2). This strategy is made of four legs: a long Call and a short Put, both with strike price equal to K_1, plus a long Put and a short Call with strike price equal to K_2.

It is also possible to selling short a Box Spread strategy by buying a European Put and selling a European Call with strike price K_1 and selling short a European Put and buying a European Call with strike price K_2.

Figure 16. Payoff of a Box Spread. Source: Bloomberg.

- **Butterfly Spread:** this is a three leg option strategy used by traders to limit the upside and downside risk (there is a maximum loss in both long and short directions) seeking to obtain a gain (capped) from a price movement that will eventually occur inside the range from the lower to the higher strike price (K_1 < Profit < K_2). The strategy is constructed by opening a long position in two Call options, one with strike price K_1 (the lower) and the other with strike price K_3 (the higher) and by opening a short position in two European Call with an intermediate strike price equal to K_2. It is possible to construct a diagram of payoff like this also using Put options, by going long with two Put options, one with strike K_1 and one with strike K_3, and by going short with one Put option with an intermediate strike price equal to K_2. In both cases the diagram of payoff will be the same.

Figure 17. Payoff of a Butterfly Spread. Source: Bloomberg.

- **Calendar Spread:** this strategy has an important different from the other listed above, because in this case the options used to build the strategy do not expire at the same date, but they have different maturities. The Calendar could be built both with two Call or two Put options and it consist in the short selling of the option with the lower maturity (T_1) and buying the options (with the same strike price K) with the longer maturity (T_2).

In both cases the trader will obtain a profit only if the stock price at the expiration date of the short-maturity option is close to the strike price of this option, since in this case the short-maturity option will cost the investor either a small amount or nothing while the long maturity options will be still valuable.

A Calendar Spread strategy could be *neutral* if the strike price of the two options is near to the current spot price of the underlying, *bullish* if the strike price is higher than the spot price, or *bearish* if it is lower than the spot price.

Figure 18. Payoff of a Calendar Spread. Source: Bloomberg.

- **Reverse Calendar Spread:** this strategy is the opposite of the Calendar Spread and consist in a long position in a short-maturity option and a short position in a long-maturity option (both with the same strike price K). In this case the payoff diagram offers a profit if the stock price of the underlying at the expiration of the short-maturity option is well above or well below the strike price of the short-maturity option; if the spot price is close to the strike price of the short-maturity option the trader will incur in a loss.

Figure 19. Payoff of a Reverse Calendar Spread. Source: Bloomberg.

- **Bottom Straddle:** this is one of the most popular strategy among traders and it is built by entering in a long position in both a European Call and a Put with the same strike price K. As it can be seen in the figure below this strategy leads to a loss in case of a small movement in the stock price of the underlying, while it offers a gain in case of large movements in either direction. This strategy allows the investor to limit his maximum loss (it will be the sum of the premiums paid to build it) but it is very useful for those investors who are expecting large price movements but they didn't know in which direction.

Figure 20. Payoff of a Bottom Straddle. Source: Bloomberg.

- **Top Straddle:** this is a straddle created by selling short both a Call and a Put with the same strike price K. This strategy will lead to a profit (the maximum profit is the two premiums received by selling the options) only if the options are not exercised during their existence. It is typically classified as a very risky strategy.

Figure 21. Payoff of a Top Straddle. Source: Bloomberg.

- **Strip:** it is a two legs strategy. The first one is a long position in a European Call option with strike K and expiration date T; the second one is a short position in two European Put options with the same strike price and expiration date. This strategy is used by traders who has positive expectations for a big movement in the stock price and who consider a decrease in the stock price more probable than an increase.

Figure 22. Payoff of a Strip. Source: Bloomberg.

- **Strap:** it is a two legs strategy. The first one is a long position in two European Call options with strike K and expiration date T; the second one is a short position in a European Put option with the same strike price and expiration date. This strategy is used by traders who has positive expectations for a big movement in the stock price and who consider an increase in the stock price more probable than a decrease.

Figure 23. Payoff of a Strap. Source: Bloomberg.

- **Bottom Strangle:** this strategy is very similar to the Bottom Straddle. The difference is the strike prices of the options. While in a Straddle the strike price is the same for both Call and Put, here there are two different strike prices (K_1 and K_2). The strategy consists in a long position in a Put option with strike K_1 and in a Call option with strike $K_2 > K_1$. It limits the downside risk (the maximum loss is the sum of the premiums paid to build it) that is maximize within the range price from K_1 to K_2. If the stock price increases above K_2 or decreases below K_1 the investor will profit from this position, otherwise he will not exercise the options.

Figure 24. Payoff of a Bottom Strangle. Source: Bloomberg.

- **Top Strangle:** this is a Strangle with two short positions, one in a Put with strike K_1 and the other in a Call with strike $K_2 > K_1$. In this case the strategy immediately leads to the maximum profit (the sum of the two premiums received from the selling of the options) while it generates a loss if the stock price moves below K_1 or above K_2.

Figure 25. Payoff of a Top Strangle. Source: Bloomberg.

- **Bullish Iron Condor:** this is a 4 legs strategy. The first leg is a long position in a OTM Put option with a strike price K_1 lower than the stock price of the underlying; the second leg is a short position in a ATM or OTM Put option with a strike price K_2 close to the spot price; the third leg is a short position in an OTM or ATM Call option with a strike price K_3 above the current stock price; the fourth leg is a long position in a OTM Call option with the higher strike price K_4. The long positions represent the so-called *wings* that limit the upside and downside risk. The trader will profit from a small move in the price of the underlying.

Figure 26. Payoff of a Bullish Iron Condor. Source: Bloomberg.

- **Bearish Iron Condor:** this is a 4 legs strategy. The first leg is a short position in a OTM Put option with a strike price K_1 lower than the stock price of the underlying; the second leg is a long position in a ATM or OTM Put option with a strike price K_2 close to the spot price; the third leg is a long position in an OTM or ATM Call option with a strike price K_3 above the current stock price; the fourth leg is a short position in a OTM Call option with the higher strike price K_4. The long positions represent the so-called *wings* that limit the upside and downside risk. The trader will profit from a small move in the price of the underlying.

Figure 27. Payoff of a Bearish Iron Condor. Source: Bloomberg.

2). The role of options in investors' portfolios.

2.1). The role of options in buy and hold portfolios to solve the classic asset allocation problem.

The classic Asset Allocation problem of modern finance, as introduced by Samuelson and Merton in 1969, is based on the formulation that the objective of an investor is "to maximize the expected utility $E[U(W_T)]$ of the end-of-period wealth W_T by allocating his wealth W_t between two assets, a risky security (the 'stock') and a riskless security (the 'bond'), over some investment horizon $[0, T]$"[7].

This problem can be expressed as a system of two main functions:

$$Max_{\{\omega_t\}} E[U(W_T)]$$

sub to

$$dW_t = [r + \omega_t(\mu - r)]W_t d_t + \omega_t W_t \sigma dB_t$$

with ω_t that is the portion of wealth invested in stocks at time t and dW_t is the budget constraint that must be satisfied in times $t \in [0, T]$.

In this classic problem the scope of the investor is to maximize the expected utility by constructing a dynamic investment policy with the combination of stocks and bonds in portfolio.

In 2001, Martin B Haugh and Andrew W Lo, in his paper "*Asset allocation and derivatives*" stated that it is possible for an investor, with the use of derivatives, to build a buy-and-hold kind of portfolio, with stocks, bonds and options, that can be an excellent substitute of complex dynamic strategies involving only the first two securities.

The objective of Haugh and Lo is to measure the closeness of the buy-and-hold portfolio with respect to the dynamic one, in three different ways: through the maximization of the expected utility and through the minimization respectively of the mean-squared error and the weighted mean-squared error.

In this process they take into consideration the results of two different utility function, the constant relative risk aversion CRRA and the constant absolute risk aversion CARA, under three leading cases for the stock-price process: the geometric Brownian motion, the trending Ornstein-Uhlenbeck process, and a bivariate linear diffusion process with a stochastic mean-reverting drift.

In the maximization of the expected utility, Haugh and Lo allows investors to include up to n European Call options at date 0 which expire at date T in their portfolios (with a payoff equal to $D_i = (P_T - k_i)^+$) without the possibility to trade after setting the initial allocation of stocks, bonds and options.

So, the asset allocation problem to be solved is

$$Max_{\{a,b,c_i,k_i\}} E[U(V_T)]$$

sub to

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[7] M. B. Haugh and A. W Lo, *Asset allocation and derivatives*, Quantitative Finance Volume 1, 2001, MIT Sloan School of Management and Operations Research Center, p. 47.

$$V_T = ae^{(rT)} + bP_T + c_1 D_1 + c_2 D_2 + \cdots + c_n D_n$$
$$W_0 = e^{(-rT)} E^Q [V_T]$$

where a and b denoted the allocation in bond and stock, and c and k denoted the number of different options with different strike prices; V_T is the investor's end-of-period wealth and W_0 is the budget constraint.

To solve this problem, we must assume fixed strike prices and select the best set of it by specifying a large number N of possible strike prices representative of the distribution of prices P_T (we must include 4 to 6 standard deviations of P_T in this interval).

In the minimization of the mean-squared error the buy-and-hold portfolio problem becomes:

$$Min_{\{a,b,c_{i,}k_i\}} E[(W_T^* - V_T)^2]$$

sub to:

$$V_T = ae^{(rT)} + bP_T + c_1 D_1 + c_2 D_2 + \cdots + c_n D_n$$
$$W_0 = e^{(-rT)} E^Q [V_T]$$

The last is a hybrid of the first two approaches, and it involves the approximation of W_T^*:

$$Min_{\{a,b,c_{i,}k_i\}} E[-U^{"}(W_T^*)(W_T^* - V_T)^2]$$

sub to:

$$V_T = ae^{(rT)} + bP_T + c_1 D_1 + c_2 D_2 + \cdots + c_n D_n$$
$$W_0 = e^{(-rT)} E^Q [V_T]$$

After the definition of the three different problems to be solved in three different approaches, Haugh an Lo explain the importance of options in asset allocation through the evaluation of the certainty equivalence. They impose different assumptions:

$$U(W_T) = \frac{W_T^{\gamma}}{\gamma};$$
$$\gamma = 0, -1, -1, -9, -14, -19;$$
$$W_0 = \$100.000;$$
$$T = 20\ years;$$
$$P_0 = \$50;$$
$$r = 0{,}05;$$
$$E\left[\log\left(\frac{P_t}{P_{t-1}}\right)\right] = 0{,}15;$$
$$Var\left[\log\left(\frac{P_t}{P_{t-1}}\right)\right] = 0{,}20^2;$$

where γ corresponds to the relative risk-aversion coefficients 1, 2, 5, 10, 15 and 20 and the n options are selected from a set of 45 different strike prices.

Starting from the geometric Brownian motion (GBM) with the maximization of the expected utility, it is shown in the paper that, for RRA = 1 (investors who are not very risk averse), the certainty equivalent of the optimal buy-and-hold portfolio CE(V_T^*) reported as a percentage of the certainty equivalent of the optimal dynamic stock/bond policy $\mathrm{CE}(W_T^*)$, increases as the number of option is increased, while the root mean-squared error RMSE decreases at a slow rate; this means that "it may be possible to obtain an excellent approximation to the optimal dynamic strategy – in terms of expected utility – without being able to approximate W_T^* very well in mean-square"[8].

n	Options (%)	Stock (%)	$\mathrm{CE}(V_T^*)$ (%)	RMSE (%)
		$\mathrm{CE}(W_T^*) = \$9\,948\,433$		RRA = 1
0	0.0	100.0	20.2	3 659.6
1	60.4	39.6	68.7	3 653.4
2	80.0	20.0	87.7	3 642.4
3	99.3	0.7	92.2	3 642.4

Figure 28. Source: Haugh and Lo, *Asset Allocation and Derivatives*.

For these investors, the use of options to increase their risk exposure, increases as the number of options allowed increases (from an allocation of 60,4% with *n* = 1 to a distribution of 99,3% of wealth when *n* = 3). Instead, the second group of investors, those who are risk averse and want to hold less than 100% of their wealth in the risky asset, are net sellers of ITM Call options.
As shown in the figure below, with *n* = 3, in the case of RRA=20, the certainty equivalents $\mathrm{CE}(V_T^*)$ is 99,8%, with a short option positions equal to -24%, used to hedge risks (gains of one security offsets to some degree the losses of the other).

		$\mathrm{CE}(W_T^*) = \$325\,437$		RRA = 20
0	0.0	11.6	97.7	101.0
1	−30.0	60.7	99.3	66.5
2	−24.0	53.0	99.8	13.3
3	−24.0	53.1	99.8	3.2

Figure 29. Source: Haugh and Lo, *Asset Allocation and Derivatives*.

In case of the mean-square-optimal policy, it is possible to observe portfolios with a small RMSE and a small certainty equivalent at the same time, with $\mathrm{CE}(V_T^*)$ that doesn't increase monotonically with the number of options *n*.

[8] *Ibidem*, p.54.

The two figures below show the asset allocation and certainty equivalents in case of RRA=1 and RRA=20 for an investor who wants to minimize the mean-square error.

n	Options (%)	Stock (%)	$\mathrm{CE}(V_T^*)$ (%)	RMSE (%)
				$\mathrm{CE}(W_T^*)$
0	0.0	100.0	20.2	3 659.6
1	0.2	99.8	20.4	2 889.6
2	62.9	−4.0	10.2	2 886.7
3	2.8	97.2	23.1	2 870.6
4	8.0	92.0	28.1	2 869.5
5	15.5	84.5	34.9	2 869.3

Figure 30. RRA=1. Source: Haugh and Lo, *Asset Allocation and Derivatives*.

0	0.0	2.5	91.7	17.5
1	−0.2	10.0	97.5	3.9
2	−0.9	14.1	98.5	1.7
3	−0.8	13.2	98.4	1.1
4	−37.8	75.7	98.9	0.8
5	−40.7	80.2	98.5	0.5

Figure 31. RRA=20. Source: Haugh and Lo, *Asset Allocation and Derivatives.*

If we apply the Ornstein-Uhlenbeck process, the set of strike prices will be in a narrower range, and the benchmark for the certainty equivalent of the buy-and-hold portfolio will be the certainty equivalent in case of an infinite number of options $\mathrm{CE}(V_T^\infty)$.

The maximization of utility leads to a better $\mathrm{CE}(V_T^*)$ with low levels of risk aversion, and worst levels in case of higher risk aversion.

The most important aspect of this process is that now the stock price is predictable; this can be view in the certainty equivalent of the optimal dynamic strategy that is considerably larger than that of the GBM case; moreover, in all the RRA scenarios form 1 to 10, the investor will not allocate any of his wealth in bonds: predictability in stock returns make stocks less risky instruments, and they are adaptable also to a very risk averse investor.

Moreover, the optimal buy-and-hold portfolios created with this process contains short positions in some options, unlike the GBM case.

The figures below show the results in the scenario of RRA = 1 and RRA = 20 respectively in case of the utility maximization and mean-square optimization.

n	Options (%)	Stock (%)	$\mathrm{CE}(V_T^*)$ (%)	RMSE (%)
	$\mathrm{CE}(W_T^*) = \$13\,162\,500$		$\mathrm{CE}(V_T^\infty) = \$12\,417\,350$	
0	0.0	100.0	16.2	115.7
1	96.6	3.4	89.1	42.3
2	98.8	1.2	96.2	40.3
3	98.8	1.2	97.8	12.6

Figure 32. Utility-optimal RRA=1. Source: Haugh and Lo, *Asset Allocation and Derivatives*.

		$\mathrm{CE}(W_T^*) = \$560\,880$		$\mathrm{CE}(V_T^\infty) = \$537\,074$
0	0.0	54.1	87.9	138.6
1	−13.9	89.0	93.2	5.1
2	−15.5	91.7	93.7	11.6
3	−15.7	92.0	93.8	4.0

Figure 33. Utility-optimal RRA=20. Source: Haugh and Lo, *Asset Allocation and Derivatives.*

n	Options (%)	Stock (%)	$\mathrm{CE}(V_T^*)$ (%)	RMSE (%)
		$\mathrm{CE}(W_T^*) = \$13\,162\,500$		
0	0.0	100.0	16.2	115.7
1	91.7	8.3	87.4	37.5
2	86.8	13.2	81.3	9.3
3	87.0	13.0	81.5	6.3
4	88.7	−3.4	56.7	4.5
5	90.2	9.8	56.7	2.7

Figure 34. Mean-square-optimal RRA=1. Source: Huagh and Lo, *Asset Allocation and Derivatives.*

0	0.0	15.0	73.7	22.6
1	−1.5	37.6	86.8	3.0
2	−3.0	46.4	89.9	1.2
3	−4.7	53.6	91.8	0.6
4	−8.5	66.4	94.4	0.4
5	−7.8	63.9	94.1	0.3

Figure 35. Mean-square-optimal RRA=20. Source: Huagh and Lo, *Asset Allocation and Derivatives.*

Finally, in case of a bivariate linear diffusion process, the optimal dynamic asset-allocation strategy is path-dependent and Haugh and Lo use again the upper bound V_T^{∞} as the benchmark.
Through the maximization of utility, the certainty equivalents of V_T^{∞} are lower than that for W_T^* but declines monotonically as risk aversion increases, while the results of the mean-square-optimal portfolios match closely those of the GBM process.
Below the figures of optimal portfolios in both cases with RRA=1 and RRA=20.

n	Options (%)	Stock (%)	$\mathrm{CE}(V_T^*)$ (%)	RMSE (%)
	$\mathrm{CE}(W_T^*) = \$11\,861\,394$		$\mathrm{CE}(V_T^{\infty}) = \$10\,142\,498$	
0	0.0	100.0	19.8	4 346.6
1	60.3	39.7	68.2	4 341.5
2	79.9	20.1	87.5	4 332.2
3	99.3	0.7	91.9	4 332.2

Figure 36. Utility-optimal RRA=1. Source: Huagh and Lo, *Asset Allocation and Derivatives.*

		$\mathrm{CE}(W_T^*) = \$327\,732$		$\mathrm{CE}(V_T^{\infty}) = \$325\,182$
0	0.0	11.6	97.7	101.2
1	−29.8	60.3	99.3	67.1
2	−23.8	52.8	99.8	13.3
3	−23.9	52.8	99.8	3.3

Figure 37. Utility-optimal RRA=20. Source: Huagh and Lo, *Asset Allocation and Derivatives.*

n	Options (%)	Stock (%)	$CE(V_T^*)$ (%)	RMSE (%)
			$CE(W_T^*) = \$11\,861\,394$	
0	0.0	100.0	19.8	4 346.6
1	0.2	99.8	20.0	3 579.8
2	100.0	0.0	0.0	3 578.5
3	2.8	97.2	22.7	3 564.7
4	8.1	91.9	27.7	3 563.8
5	15.6	84.4	34.5	3 563.7

Figure 38. Mean-square-optimal RRA=1. Source: Huagh and Lo, *Asset Allocation and Derivatives.*

			$CE(W_T^*) = \$327\,732$	
0	0.0	2.5	91.8	17.5
1	−0.2	9.9	97.6	3.9
2	−0.9	14.0	98.6	1.7
3	−0.8	13.2	98.4	1.1
4	−38.2	76.2	98.9	0.8
5	−40.9	80.5	98.5	0.5

Figure 39. Mean-square-optimal RRA=20. Source: Huagh and Lo, *Asset Allocation and Derivatives.*

The paper of Haugh and Lo show the importance of options in asset allocation problems and, through the comparison of optimal buy-and-hold portfolios of stocks, bonds and options to the standard optimal dynamic portfolios of stock and bonds they demonstrate that “buy-and-hold portfolios are excellent approximations – in terms of certainty equivalence and mean-squared-error of end-of-period wealth – to their dynamic counterparts, suggesting that in those cases, dynamic trading strategies may be ‘automated’ by simple buy-and-hold portfolios with just few options”[9].

[9] *Ibidem*, p.69.

2.2). Benefit from including derivatives in optimal dynamic strategies.

The classic dynamic asset allocation problem, under the assumptions of completeness of the market, is based on the construction of the optimal investment portfolio made just of risky stock and riskless bond, with the exclusion of derivatives.

However, assuming the incompleteness of the market, because of infrequent trading or uncertainty, the exclusion of derivatives leads to suboptimal allocation of wealth for investors.

Jun Liu and Jun Pan, in their paper "*Dynamic derivative strategies*" published in 2003 on the *Journal of Financial Economics*, explain the importance of derivatives to complete the market and provide independent exposure to each of the three main risk factors incorporated in the model of the aggregate stock market: diffusive price shocks, price jumps, and volatility risks.

Typically, a risky stock provides risk exposure only to diffusive and jump risk, not on the volatility risk; with the introduction of derivatives, the investor can take advantage of the risk-return trade-off provided by volatility risk or exploit the time-varying nature of investment opportunity set.

Liu and Pan solve the dynamic asset allocation problem through three steps:

1. They solve the optimal wealth dynamics;
2. They find the optimal exposure to each of the three risk factors that support the optimal allocation of wealth;
3. They find the optimal positions in stocks and options to obtain the optimal exposure to those risk factors.

The model for stock price process *S* is set as:

$$\mathrm{d}S_t = (r + \eta V_t + \mu(\lambda - \lambda^Q)V_t)S_t\mathrm{d}t + \sqrt{V_t S_t \mathrm{d}B_t} + \mu S_t - (\mathrm{d}N_t - \lambda V_t \mathrm{d}t)$$

$$\mathrm{d}V_t = \kappa(v - V_t)\mathrm{d}t + \sigma\sqrt{V}_t(\rho\mathrm{d}B_t + \sqrt{1} - p^2\mathrm{d}Z_t)$$

with *B* and *Z* that are standard Brownian motions, and *N* is a jump process, and the price of options and their payoffs set respectively as:

$$O_t = g(S_t, V_t);$$

$$g(S_t, V_t) = (S_t - K)^+ \text{ for a Call option;}$$

$$g(S_t, V_t) = (K - S_t)^+ \text{ for a Put option;}$$

where O_t is the price of options, represented as a function *g* of price and volatility, and *K* is the strike price.

The approach of Liu and Pan is based on the construction o a pricing kernel "to price all of the risk factors in this economy, and consequently any derivative securities"[10], set as:

$$\mathrm{d}\pi_t = -\pi_t\left(r\mathrm{d}t + \eta\sqrt{V_t\mathrm{d}B_t} + \xi\sqrt{V_t\mathrm{d}Z_t}\right) + \left(\frac{\lambda^Q}{\lambda} - 1\right)\pi_t - (\mathrm{d}N_t - \lambda V_t\mathrm{d}t)$$

---

[10] J. Liu and J. Pan, *Dynamic derivative strategies*, Journal of Financial Economics 69, 2003, Anderson School of Business at UCLA, MIT Sloan School of Management, Cambridge, p. 406.

where η represents the control premium for the diffusive price risk *B*, ξ represents the control premium for the additional volatility risk *Z*, and $\frac{\lambda^Q}{\lambda}$ represents the control premium for the jump risk *N*.

The price dynamics for the *i*-th derivative security include three parameters that measure the exposure to each risk factors:

- $g_s^{(i)}$ that measures the sensitivity to changes in the stock price. If it is different from zero it means that the option provides exposure to the diffusive price shock *B*;
- $g_t^{(i)}$ that measures the sensitivity to changes in the stock volatility. If it is different from zero it means that the option provides exposure to the additional volatility risk *Z*;
- $\Delta g^{(i)}$ that measures the change in the price of derivative for each jump in the underlying stock price. If it is not zero it means that the option provides exposure to the jump risk *N*.

With these elements as inputs, Liu and Pan try to solve the classic investment problem of the maximization of the expected utility of an investor's terminal wealth W_T; the investor will allocate a portion ϕ_t of his wealth in the stock S and a portion $\psi_t^{(1)}$ and $\psi_t^{(2)}$ in the two derivatives.

The weights of risk factor exposure are expressed by θ^B for the diffusive price shock *B*, θ^Z for the additional volatility risk *Z*, and θ^N for the jump risk *N*.

The first part of the paper shows the result of the use of derivatives as vehicles to stochastic volatility. Liu and Pan assume an economy with volatility risk but no jump risk in which a derivative is necessary to complete the market, and they consider the portfolio choices of both a myopic and non-myopic investor ("myopic behavior represents the behavior of an investor who acts in accordance with what he or she wants right now. […] they care only about short-term results"[11]).

The optimal positions for options and stock are defined as:

$$\phi_t^* = \frac{\eta}{\gamma} - \frac{\xi\rho}{\gamma\sqrt{1-\rho^2}} - \psi_t^* \frac{g_s S_t}{O_t};$$

$$\psi_t^* = \left(\frac{\xi}{\gamma\sigma\sqrt{1-\rho^2}} + H(T-t)\right)\frac{O_t}{g_v}$$

where H represents the absence of price jumps.

It is possible to observe that the portion of wealth allocated to risky stock ψ_t^* is inversely proportional to $\frac{g_v}{O_t}$ (i.e. the volatility exposure to each dollar invested in derivatives); the higher this parameter, the higher the efficiency of derivative security in terms of vehicle to volatility risk, so the demand of it will decrease.

For a myopic investor the demand of derivative security will be led by the possible benefits in terms of risk-and-return trade-off: if the volatility risk is negatively priced, $(\xi < 0)$ the investor will have a short position

[11] https://corporatefinanceinstitute.com/resources/knowledge/trading-investing/myopic-behavior/

in options with positive volatility exposure; if the volatility risk is positively priced ($\xi > 0$) the investor will have a long position.
For a non-myopic investor instead, the demand for options depends on the second term of ψ_t^* so, a risk averse investor ($\gamma < 1$) will sell derivative to hedge against uncertainty, while an investor with $\gamma > 1$ will take a long position in volatility to speculate and obtain profits.
The demand for stock is instead related only to the risk-return trade-off and the delta exposure to stock introduced by the derivative (if an investor has a long position in a Call, he is investing a portion of his wealth, the delta of the option, on the underlying stock).
The empirical proof of the model is made under different assumptions regarding the long-run mean $v = (0.13)^2$, the mean reversion rate $\kappa = 5$, the volatility coefficient $\sigma = 0.25$ and the correlation between price and volatility risk $\rho = -0.40$, moreover assuming that the volatility-risk premium coefficient ξ can vary, by considering the investment choices of an investor who can select a riskless asst, a risky stock, and a delta-neutral straddle.
It is possible to observe that the demand for derivatives is driven mainly by the myopic component, and the optimal portfolio weight in the straddle increases as the volatility-risk premium increases. However, also when ξ is equal to zero (no myopic demand due to the absence of benefit from including derivatives) there is still a non-myopic component.
The figure below shows the portfolio improvement R^W in terms of the annualized, continuously compounded return in certainty-equivalent wealth with respect to different variables.

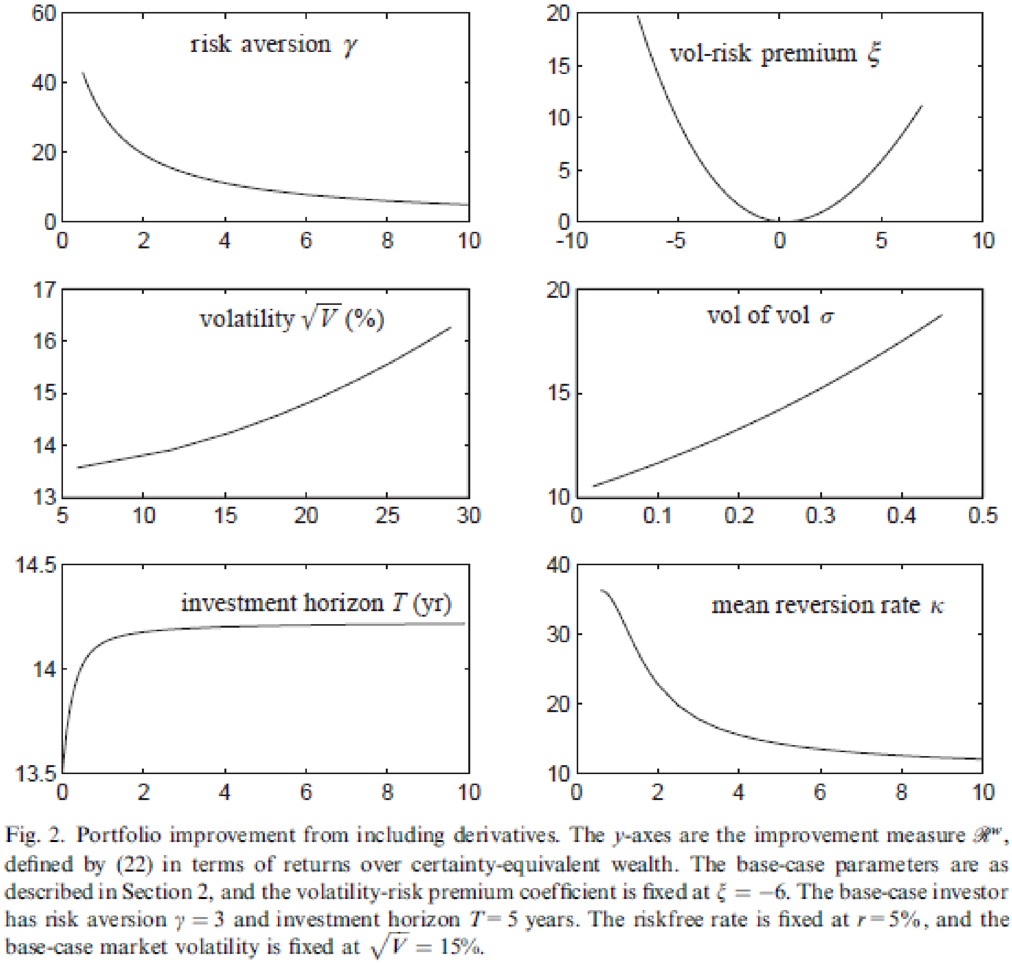


Figure 40. Source: J. Liu and J. Pan, *Dynamic derivative strategies.*

"Under normal market conditions with a conservative estimate of the volatility-risk premium $\xi = -6$ […] the portfolio improvement from including derivatives is about 14.2% per year in certainty-equivalent wealth for an investor with risk aversion $\gamma = 3$"[12].

In the last part of the paper the authors suppose an economy with jump risk and no volatility risk; in this case the optimal portfolio weights for options and stock are expressed as:

$$\phi_t^* = \frac{\eta}{\gamma} - \psi_t^* \frac{g_s S_t}{O_t};$$

$$\psi_t^* = \left(\frac{\Delta g}{\mu O_t} - \frac{g_s S_t}{O_t}\right)^{-1}\left(\frac{1}{\mu}\left[\left(\frac{\lambda}{\lambda^Q}\right)^{\frac{1}{\gamma}} - 1\right] - \frac{\eta}{\gamma}\right);$$

and the role of derivatives is to separate jump risk from diffusive price risk, so the investor will allocate less of his wealth to derivative security the lower will be its ability to do that: if an option is equally sensitive to infinitesimal and large price movements, so as $\frac{\delta g}{\delta S} = \frac{\Delta g}{\Delta S}$, it will be not efficient to disentangle the two risk factors.

[12] J. Liu and J. Pan, *Dynamic derivative strategies*, Journal of Financial Economics 69, 2003, Anderson School of Business at UCLA, MIT Sloan School of Management, Cambridge, p. 418-419.

Moreover, the asset allocation problem takes into account the attractiveness of those risk factors for the investor: if those are equally attractive, he would not like to separate them, and his demand will decrease. However, the risk factors have different nature and they are not typically equally price: the diffusive risk can be much easily controlled by the investor via continuous trading, while jump risk cannot be controlled.
The demand for stock, instead, depends on the risk-and-return trade-off and is corrected by the delta exposure introduced by the option.
Liu and Pan set different assumptions (risk-free rate $r = 5\%$, market volatility fixed at 15% a year, equity risk premium is fixed at 8% a year) and apply the model above in three scenarios ($\mu = -10\%$ jumps once every 10 years; $\mu = -25\%$ jumps once every 50 years; and $\mu = -50\%$ jumps once every 200 years) to find the optimal portfolio weight of an investor for both the case of no derivatives or the combination of stock and an European Put 5% OTM. Moreover, for each jump case, the authors let the jump-risk premium $\lambda^Q/_\lambda$ from 1 to 5.
The table below shows the results in term of portfolio weights in each scenario.

Jump Cases		$\mu=-10\%$ Every 10 yr			$\mu=-25\%$ Every 50 yr			$\mu=-50\%$ Every 200 yr		
			Stock & put			Stock & put			Stock & put	
γ	λ^Q/λ	Stock only	ϕ^*	$\psi^*(\%)$	Stock only	ϕ^*	$\psi^*(\%)$	Stock only	ϕ^*	$\psi^*(\%)$
	1	6.74	9.34	4.33	4.00	8.52	2.28	2.00	8.38	1.85
0.5	2	6.74	6.25	−0.67	4.00	7.59	1.40	2.00	7.94	1.54
	5	6.74	1.95	−5.63	4.00	5.88	0.85	2.00	7.10	1.53
	1	1.17	1.56	0.72	1.12	1.42	0.38	0.99	1.40	0.31
3	2	1.17	0.82	−0.66	1.12	1.22	0.12	0.99	1.31	0.21
	5	1.17	−0.44	−3.38	1.12	0.85	−0.34	0.99	1.13	0.09
	1	0.70	0.93	0.43	0.68	0.85	0.23	0.62	0.84	0.18
5	2	0.70	0.48	−0.43	0.68	0.73	0.06	0.62	0.78	0.13
	5	0.70	−0.35	−2.28	0.68	0.50	−0.25	0.62	0.67	0.04
	1	0.35	0.47	0.22	0.34	0.43	0.11	0.32	0.42	0.09
10	2	0.35	0.23	−0.23	0.34	0.36	0.03	0.32	0.39	0.06
	5	0.35	−0.21	−1.24	0.34	0.24	−0.15	0.32	0.33	0.01

Figure 41. Source: J. Liu and J. Pan, *Dynamic derivative Strategies.*

It is possible to observe how the jump risk affects the portfolio choices of the stock only investor, that become more and more cautious in the presence of jump risk: is stock allocation shifts from 6.74% to only 2% in the case of a risk aversion coefficient equal to 0.5.
On the other side, the allocation of wealth for the investor who can invest in both stock and put would depend on how the two risk factors (diffusive price risk and jump risk) are compensated; he will buy put options to hedge his exposure to jump risk (risky stock exposes him to negative jump risk) for low levels of compensation of this risk factor until the relative attractiveness of this one will be higher than that of the diffusive risk; in that situation the investor will write put options to earn the higher premium.

The portfolio improvement R^W in terms of the annualized continuously compounded return in certainty-equivalent wealth with the introduction of derivative security is represented in the figure below.

Jump cases		$\mu = -10\%$ Every 10 yr	$\mu = -25\%$ Every 50 yr	$\mu = -50\%$ Every 200 yr
γ	λ^Q/λ	$\mathcal{R}^W$ (%)	$\mathcal{R}^W$ (%)	$\mathcal{R}^W$ (%)
	1	2.11	8.62	16.74
0.5	2	0.13	5.97	15.14
	5	11.78	1.84	11.28
	1	0.26	0.43	0.71
3	2	0.28	0.06	0.46
	5	7.68	0.46	0.09
	1	0.15	0.24	0.37
5	2	0.19	0.02	0.22
	5	5.12	0.36	0.02
	1	0.08	0.12	0.17
10	2	0.10	0.01	0.09
	5	2.77	0.22	0.003

Figure 42. Source: J. Liu and J. Pan, *Dynamic derivative Strategies.*

The paper of Liu and Pan empirically demonstrates the importance of the inclusion of financial derivatives in the classic asset allocation problem and it was also replicate by the professor Massimo Guidolin in his paper "*The Economic Value of Derivatives and Structured Products in Long-Horizon, Dynamic Asset Allocation*" in which he applied the same model of Liu and Pan to the FTSE MIB index instead of the S&P 500.

"As a vehicle to additional risk factors such as stochastic volatility and price jumps in the stock market, derivative securities play an important role in expanding the investor's dimension of risk-return tradeoffs. In addition, by providing access to volatility risk, derivatives are used by non-myopic investors to take advantage of the time-varying nature of their opportunity set. Similarly, by providing access to jump risk, derivatives are used by investors to disentangle their simultaneous exposure to diffusive and jump risks in the stock market"[13].

[13] *Ibidem*, p.425.

2.3). Optimal Portfolio's choices whit jumps in volatility.

The possibility for an investor to select derivative securities to allocate part of his wealth leads to different benefits, as we have seen before: it is possible to replicate a dynamic strategy based on the use of only risky stocks and riskless bonds with the construction of a simple buy-hold strategy, or to improve the risk-return trade-off by obtaining exposure to each of the three main risk factors (diffusive price shocks, price jumps and volatility risk).

The introduction of derivatives, specifically options, generates a significant improvement in the investor's utility, measured as the annualized percentage difference in certainty equivalent wealth (a sort of "additional interest rate").

The academic paper of Liu and Pan (2003) shows has the role that derivative securities have in the disentanglement of each risk component and how it is possible to obtain portfolio improvements, by considering two cases, one in which there are only diffusive price shock and volatility risk, and one wit diffusive price shock and jump risk, without volatility risk.

Nicole Branger, Christian Schlag and Eva Schneider, in their paper "*Optimal Portfolios when Volatility can Jump*" published in 2005 introduce a more complex model for an investor with constant relative risk aversion with one more risk factor: they assume that jumps are possible not only in the asset price, but also in volatility.

This paper represents a sort of step forward with respect to that of Liu and Pan, in which the two authors did not consider the possibility of jumps in volatility; in this way, as exposed by Branger, Schlag and Schneider, the investor will suffer a utility loss by ignoring volatility jumps in his process of portfolio selection, and the overall performance of the portfolio will have a deterioration.

"One main finding is that the demand for jump risk now also includes a hedging component, which is not present in models without jumps in volatility"[14].

The authors separate the overall demand for risk factors into two main components:

- a speculative one that is the investor's desire to earn the risk premium;
- a hedging component, that was absent in the paper of Liu and Pan in relation to the jump risk, to protect the investor from unfavourable changes in the opportunity set.

The hedging component in case of no volatility jumps was built by trading only the diffusion risk, with the omission of the volatility risk, leading to "an overestimation of the hedging demand for the diffusion risk"[15].

The dynamics of sock price and instantaneous variance are defined as follows:

$$dS_t = [r + \eta^{B1}V_t + \mu_X(\lambda^P V_t - \lambda^Q V_t)]S_t dt + \sqrt{V_t}S_t dB_t^{(1)} + S_{t-\mu_X}(dN_t - \lambda^P V_t dt);$$

[14] N. Branger, C. Schlag and E. Schneider, *Optimal Portfolios when Volatility can Jump*, 2005, Finance Department of Goethe University, p. 1.

[15] *Ibidem*, p.3.

$$dV_t = \kappa^P(v^P - V_t)dt + \sigma_V\sqrt{V_t}(pdB_t^{(1)} + \sqrt{1-p^2}dB_t^{(2)} + \mu_Y(dN_t - \lambda^P V_t dt);$$

while the pricing kernel for the market prices of risk is:

$$d\xi_t = -\xi_t\left(rdt + \eta^{B1}\sqrt{V_t}dB_t^{(1)} + \eta^{B2}\sqrt{V_t d}B_t^{(2)}\right) + \xi_{t-}(\lambda^Q/\lambda^P - 1)(dN_t - \lambda^P V_t dt);$$

where the market price of risk $\eta^{B1}V_t$ is the reward receive by the investor per unit of $\sqrt{V_t}dB_t^{(1)}$, while $\eta^{B2}V_t$ is the compensation for one unit of $\sqrt{V_t}dB_t^{(2)}$, and $(\lambda^P - \lambda^Q)V_t\mu_X$is the extra expected return per unit of risk $\mu_X dN_t$.

The price of options is defined as

$$dO_t^{(i)} = rO_t^{(i)}dt + \left(g_s^{(i)}S_t + \sigma_V p g_v^{(i)}\right)\left(\eta^{B1}V_t dt + \sqrt{V_t}dB_t^{(i)}\right) + \sigma_V\sqrt{1-p^2}g_v^{(i)}\left(\eta^{B2}V_t dt + \sqrt{V_t d}B_t^{(2)}\right) +$$
$$\Delta g^{(i)} * [(\lambda^P - \lambda^Q)V_t dt + dN_t - \lambda^P V_t dt];$$

where $g_s^{(i)}$, $g_v^{(i)}$, and $\Delta g^{(i)}$ represent respectively the stock price diffusion risk, the volatility diffusion risk, and the jump risk.

The authors define the model by the setting the objective of the investor as the maximization of his expected utility of terminal wealth that is exposed in terms of the exposure to the fundamental risk factors $B^{(1)}$, $B^{(2)}$ and N instead of portfolio weights as below:

$$dW_t = rW_t + \theta_t^{B1}W_t\left(\eta^{B1}V_t dt + \sqrt{V_t}dB_t^{(1)}\right) + \theta_t^{B2}W_t\left(\eta^{B2}V_t dt + \sqrt{V_t}dB_t^{(2)}\right)$$
$$+ \theta_t^N W_t\mu_X[(\lambda^P - \lambda^Q)V_t dt + dN_t - \lambda^P V_t dt]$$

with the coefficients of each risk exposure set as θ_t^{B1}, θ_t^{B2}, and θ_t^N.

The optimal position in those coefficients is:

$$\theta_t^{*B1} = \frac{\eta^{B1}}{\gamma} + \rho\sigma_V H(\tau)$$

$$\theta_t^{*B2} = \frac{\eta^{B2}}{\gamma} + \sqrt{1-\rho^2}\sigma_V H(\tau)$$

$$\theta_t^{*N} = \frac{1}{\mu_X}\left[\left(\frac{\lambda^P}{\lambda^Q}\right)^{\frac{1}{\gamma}} - 1\right] + \frac{1}{\mu_X}\left(\frac{\lambda^P}{\lambda^Q}\right)^{\frac{1}{\gamma}}\left[e^{H(\tau)\mu_Y} - 1\right], \quad 1 + \theta_t^{*N}\mu_X \geq 0$$

In this case, as in the academic paper of Liu and Pan, the demand of the investor is driven by two components: a speculative or myopic component, that arises because the investor wants to earn the risk premium for each factors, and a hedging or non-myopic component that is driven by the will of the investor to protect against unfavourable changes in the variables determining the investing opportunity set.

It is important to remember that the optimal exposure to jump risk is dictated also by a hedging component, which was absent in an economy without variance jumps, like that described by Liu and Pan.

This demand is given by the second element in the equation of θ_t^{*N} that is $\frac{1}{\mu_X}\left(\frac{\lambda^P}{\lambda^Q}\right)^{\frac{1}{\gamma}}\left[e^{H(\tau)\mu_Y}-1\right]$; this means that even if $\lambda^P = \lambda^P$ (i.e. even if the jump risk premium is zero and so there is no speculative demand for it), if $\mu_Y > 0$ then $\theta_t^{*N} \neq 0$.

Branger, Schlag and Schneider solve the following investor's optimization problem:

$$\max_{\{\theta_t^{B1},\theta_t^{B2},\theta_t^N,0\leq t\leq T\}} E\left[\frac{1}{1-\gamma}W_T^{1-\gamma}\right]$$

under the indirect utility function:

$$J(t,w,v) = \max_{\{\theta_s^{B1},\theta_s^{B2},\theta_s^N,t\leq s\leq T\}} E\left[\frac{1}{1-\gamma}W_T^{1-\gamma}|W_t = w, V_t = v\right]$$

The numerical results of the economic benefits of derivatives are exposed as in the case of Liu and Pan (2003) as the portfolio improvement R^W (i.e. the annualized percentage difference in certainty equivalent wealth):

$$R^W = \frac{\ln\left(W/\widehat{W}\right)}{T}$$

where W $\left(\widehat{W}\right)$ is the certainty equivalent wealth for the case with (without) derivatives defined through the respective indirect utility function:

$$J(0,W_0,V_0) = \frac{W^{1-\gamma}}{1-\gamma} \text{ and } \hat{J}(0,W_0,V_0) = \frac{\widehat{W}^{1-\gamma}}{1-\gamma}$$

The main results are illustrated in the graphs below, in which it is possible to observe the portfolio improvement from including derivatives with respect to four different variables.

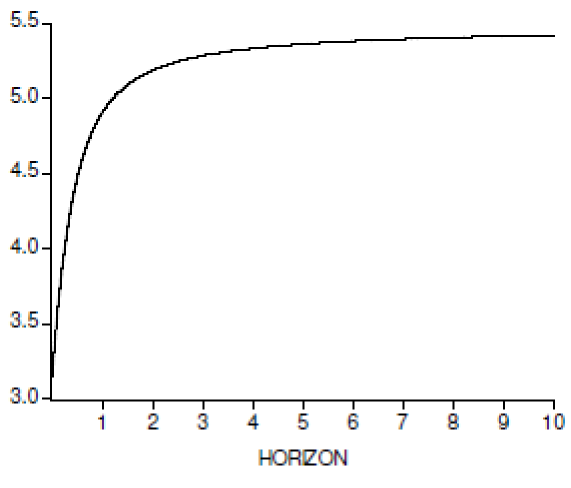


Figure 43. Portfolio Improvements under varying horizon. Source: N. Branger, C. Schlag and E. Schneider, *Optimal Portfolios when Volatility can Jump.*

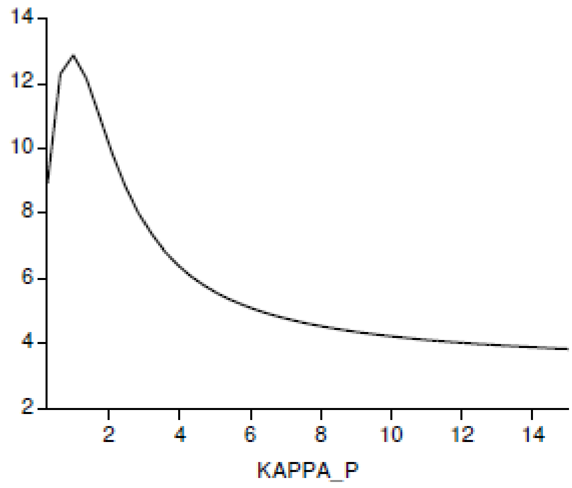


Figure 44. Portfolio Improvements under varying mean reversion. Source: N. Branger, C. Schlag and E. Schneider, *Optimal Portfolios when Volatility can Jump.*

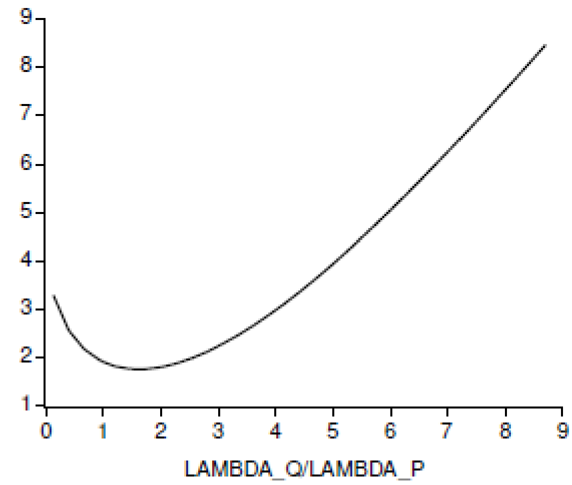


Figure 45. Portfolio Improvements under varying risk neutral jump intensity. Source: N. Branger, C. Schlag and E. Schneider, *Optimal Portfolios when Volatility can Jump.*

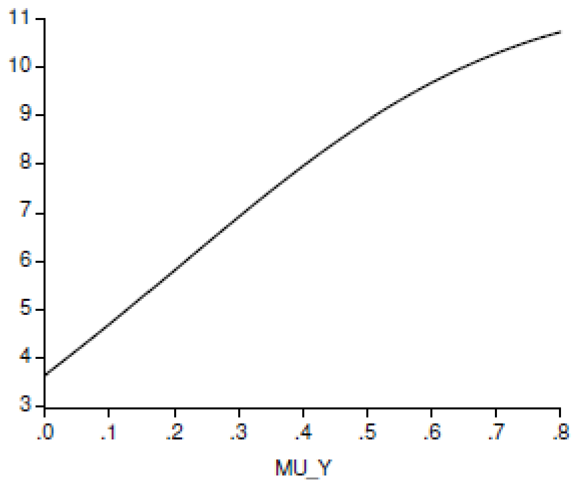


Figure 46. Portfolio Improvements under varying variance jump size. Source: N. Branger, C. Schlag and E. Schneider, *Optimal Portfolios when Volatility can Jump.*

Figure 43 shows the portfolio improvements with different horizon time τ: if is equal to zero there is only the myopic component of the investor's demand, and the portfolio improvement will be equal to 3%, while for increasing time horizons this improvement stabilizes at nearly 5,5%.
Figure 44 shows the "impact of the speed of the mean reversion on the portfolio improvement. The higher the κ^P, the less impact shocks in variance have, and the lower the variance of variance"[16]; the higher this variable, the lower will be the portfolio improvement of trading derivatives.
Figure 45 shows the impact of the jump intensity λ^Q: the larger the compensation of jump risk (i.e. the larger the difference between λ^Q and λ^P.
Finally, figure 46 shows the portfolio improvement for varying variance jump size: the higher μ_Y the larger the hedging demand of the investor and the portfolio improvement in trading derivatives.
The publication of Branger, Schlag and Schneider further illustrates the important role of options to complete the market and obtain the optimal exposure to each risk factor; the introduction of jump in volatility extends the study of Liu and Pan (2003) and shows that the demand for jump risk in such case include also a hedging component in addition to the myopic speculative part.

[16] *Ibidem*, p. 12.

2.4). A myopic portfolio to exploit the mispricing.

The option portfolio's optimization is a tough problem that must overcome three different limitations: the distribution of option returns, that cannot be described by mean-variance analysis, the shortage of data about option returns available, due to the short history of that instrument, and the high transaction costs.

To solve these problems J. A. Faias and P. Santa-Clara, in their paper "*Optimal Option Portfolio Strategies: Deepening the Puzzle of Index Option Mispricing*", proposed a simple optimization method, the optimal option portfolio strategies "OOPS", focused on the maximization of a utility function that penalizes the negative skewness and high kurtosis of returns.

Starting from the historical data of the underlying, in this case the S&P 500 Index, from February 1950 to August 2013, and knowing the structure of option payoffs, they simulate option returns for the portfolio based on their current prices.

They build a portfolio through the allocation "between a risk-free asset and four European options on the S&P 500 Index with one-month to maturity: a ATM call, a ATM put, a 5% OTM call, and a 5% OTM put option"[17] hold to maturity, to incur in transaction costs only at the interception and they impose a no-short selling restriction: each option has a long and a short side (the long is initiated at the *ask* quote, the short at the *bid* quote) and each month only one of them is traded by the investor.

Regarding the transaction costs, they considered only the bid-ask spread, measured as around 5% for ATM options and 10% around for OTM options.

The figure below shows the bid-ask spread express in percentage difference between the bid and ask quote for each of the options used in the portfolio, from 1996 to 2013.

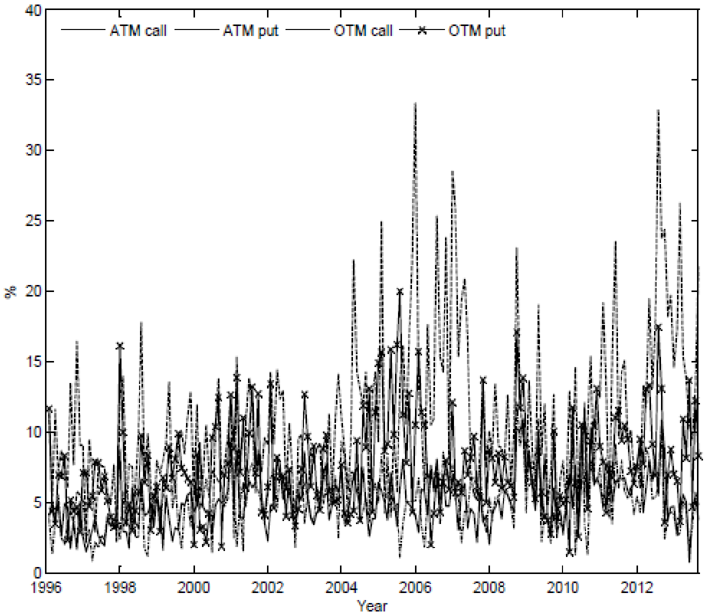


Figure 47. Source: J. Faias and P. Santa-Clara, *Optimal Option Portfolio Strategies: Deepening the Puzzle of Index Option Mispricing*.

[17] J. Faias and P. Santa-Clara, *Optimal Option Portfolio Strategies: Deepening the Puzzle of Index Option Mispricing*, 2011, p. 2.

The average moneyness of ATM call and put option is respectively 0,24% and -0,10%, while for OTM call and put options the average is respectively -4,09% and 4,37% as summarized in the table below also for other statistics.

	ATM Call	ATM Put	OTM Call	OTM Put
Option moneyness	0.24%	−0.10%	−4.09%	4.37%
Price	27.31	24.97	6.79	10.86
Bid-ask spread	1.47	1.37	0.65	0.84
Relative bid-ask spread	5.22%	5.27%	10.91%	7.68%
Volume	12,298	13,322	13,092	17,342
Open interest ($000)	15,356	15,756	15,721	15,591
Implied volatility	19.48%	18.41%	16.89%	21.83%
Delta	0.53	−0.50	0.19	−0.23
Beta	25.32	−26.57	47.33	−30.60
Gamma	0.01	0.01	0.01	0.01
Vega	1.33	1.34	0.84	0.99

Figure 48. Source: J. Faias and P. Santa-Clara, *Optimal Option Portfolio Strategies: Deepening the Puzzle of Index Option Mispricing*.

They start with the simulation of the underlying log-returns, computing the next period asset's value from the previous period value as $S^n_{t+1|n} = S_t \exp\left(r^n_{t+1}\right)$; they study standardized returns sr as the ratio between the raw returns rr and their standard deviation, and then scale up or down them by multiplying with their current realized volatility.

The value selected for realized volatility "is calculated from the last *d* trading days and scaled by 21 days (…) to get monthly units. We consider alternatives values of *d* to be 1, 5, 10, 20, 30 and 60 days. In each month, we only use the realized volatility length that maximizes the expected utility in sample"[18].

From these, using the known strike prices for the options, they simulate the option payoffs at maturity with the typical formula:

$$C^n_{t+1|t,c} = \max\left(S^n_{t+1|t} - K_{t,c}; 0\right) \text{ and } P^n_{t+1|t,p} = \max\left(K_{t,p} - S^n_{t+1|t}; 0\right)$$

and, consequently, the options return as:

$$r^n_{t+1|t,c} = \frac{C^n_{t+1|t,c}}{C_{t,c}} - 1 \text{ and } r^n_{t+1|t,p} = \frac{P^n_{t+1|t,p}}{P_{t,p}} - 1$$

In order to select the optimal portfolio weights for call and put options the portfolio is built through the maximization of the expected CRRA (constant-relative risk-aversion) utility function given by the following formula:

[18] *Ibidem*, p.8.

$$U(W) = \begin{cases} \frac{1}{1-\gamma} W^{1-\gamma}, & if\ \gamma \neq 1 \\ \ln(W), & if\ \gamma = 1 \end{cases}$$

with γ as the coefficient of relative risk aversion.

The return of the portfolio is then calculated as:

$$rp_{t+1|t}^{n} = rf_t + \sum_{c=1}^{C} \omega_{t,c}\left(r_{t+1|t,c}^{n} - rf_t\right) + \sum_{p=1}^{P} \omega_{t,p}\left(r_{t+1|t,p}^{n} - rf_t\right)$$

Finally, they compute the certainty equivalent of an investor with CRRA utility function with $\gamma = 4$ by the formula:

$$CE = [(1-\gamma)U]^{1/(1-\gamma)} - 1.$$

The out-of sample results (from 1993 to 2013) of the OOPS method shows a distribution of one-month returns after transaction costs similar to a normal one, with symmetric shape and low tail risk, a shown in the figure below.

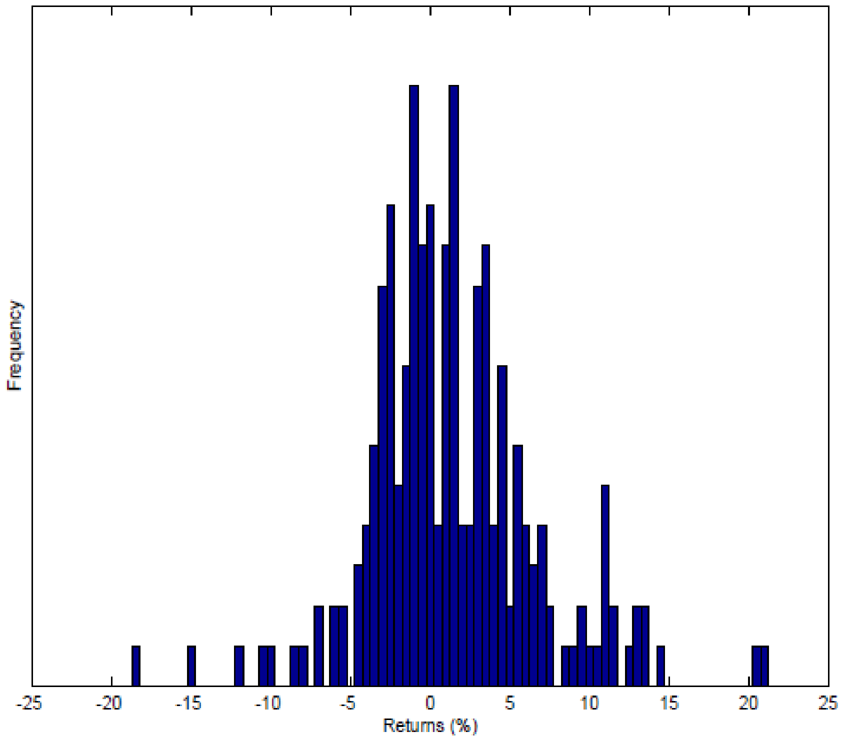


Figure 49. Source: J. Faias and P. Santa-Clara, *Optimal Option Portfolio Strategies: Deepening the Puzzle of Index Option Mispricing.*

Here is a table of comparison of the key results obtained by Faias and Santa-Clara OOPS model and the benchmark index S&P 500. It is possible to observe the improvement in terms of annualized certainty equivalence and Sharpe Ratio with respect to the S&P 500.

	Ann Mean	Ann Std Dev	Min	Max	Skew	Exc Kurt	Ann CE	Ann SR
S&P 500	6.4%	18.2%	−29.5%	18.6%	−1.24	5.57	−1.31%	0.29
OOPS	16.1%	18.4%	−18.4%	20.8%	0.35	2.03	9.94%	0.82

Figure 50. Source: J. Faias and P. Santa-Clara, *Optimal Option Portfolio Strategies: Deepening the Puzzle of Index Option Mispricing.*

The figure below shows the OOPS cumulative returns after transaction costs, compare with the risk-free asset return and the S&P 500.

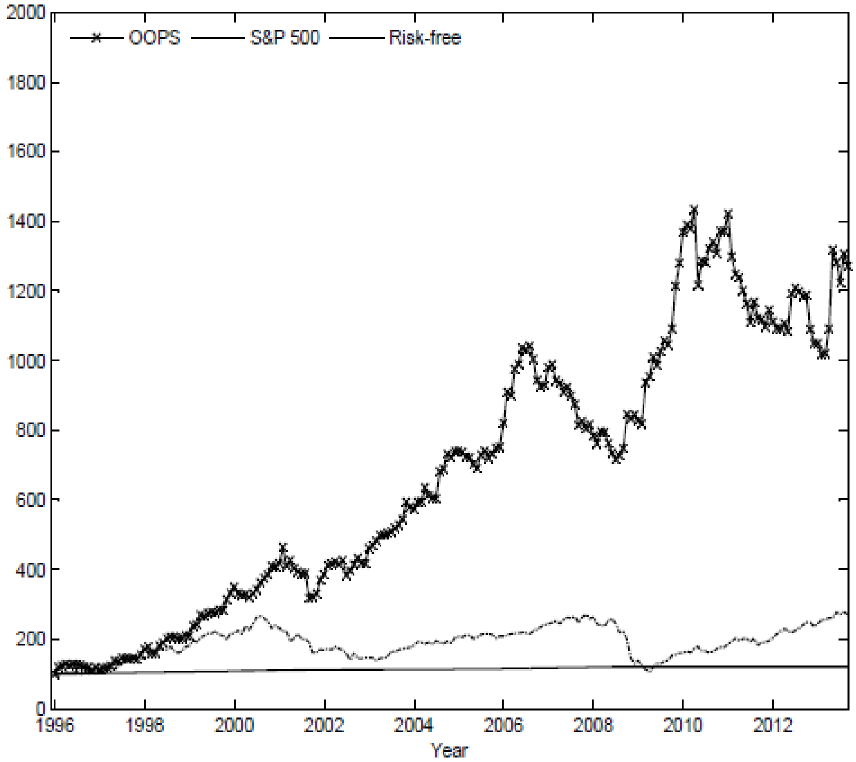


Figure 51. Source: J. Faias and P. Santa-Clara, *Optimal Option Portfolio Strategies: Deepening the Puzzle of Index Option Mispricing.*

Such a result is obtained thanks to the variables monthly weights assigned to the different options and the risk-free security; the optimal weights for call and put vary within a range from -87% to 3,5%, while the mean risk-free weight is 104,47%, with a minimum of 94% and a maximum of 116% (weights higher than 100%, given by 100% minus the sum of weights invested in options, are possible because in some cases, precisely in the 73% of the total months, the investor is a net seller of options).

Graphs below shows the optimal weights of each security in portfolio

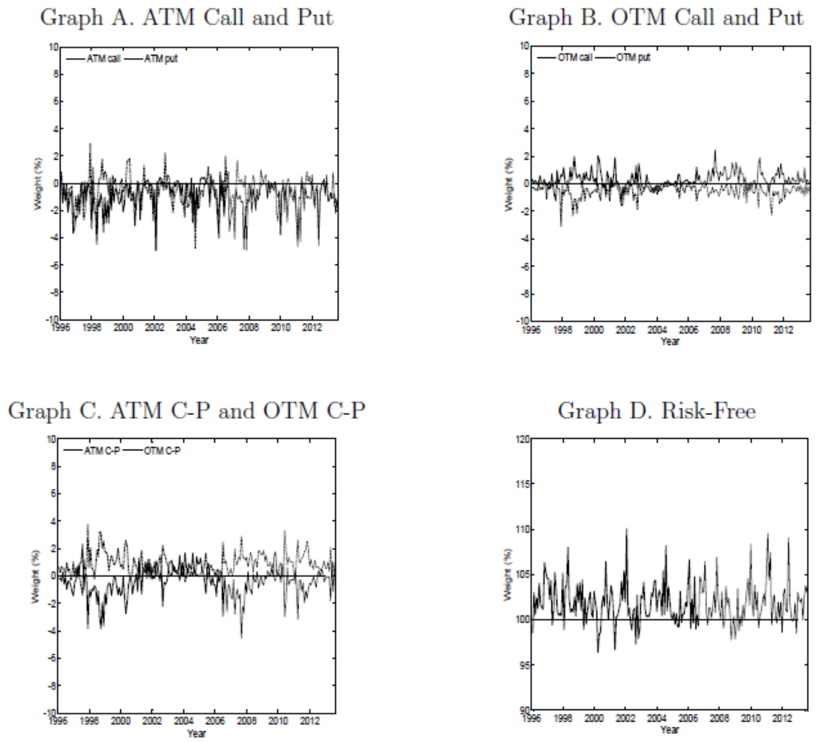


Figure 52. Source: J. Faias and P. Santa-Clara, *Optimal Option Portfolio Strategies: Deepening the Puzzle of Index Option Mispricing.*

However, the most interesting result of the Faias and Santa-Clara model is that the OOPS has a very low exposure to the three main risk factors (market risk, volatility risk and jump risk) expressed respectively through the computation of the *Beta*, the *Percentage Vega* and the *Market Jump of -5%*.

In fact, as shown in the figure below, the *Beta* of the OOPS, computed as the average Beta of all options (i.e. the Delta of the portfolio multiplied by the ratio between the underlying asset value and the option value) varies between -0,75 a 0,62, with an average value of 0,03, is lower than the Beta of individual options, so there is a lower exposure to the market risk.

The *Percentage Vega* (the % change in the portfolio value due to a 1% change in volatility) is very low, ranging from a minimum value of -1,4% to a maximum value of 0,7%, with a mean value of -0,3%, meaning that, on average, an increase in volatility of 1% leads to a 0,3% fall in the portfolio's value.

The *Market Jump of -5%* is "the OOPS return from a sudden drop of 5% in the stock market"[19] and ranges from a minimum value of -4,8% to a maximum value of 3,7%, with an average value of -0,6%, meaning that, in case of drops of 5% in the S&P 500, the OOPS loses only 0,6%.

[19] *Ibidem*, p.21.

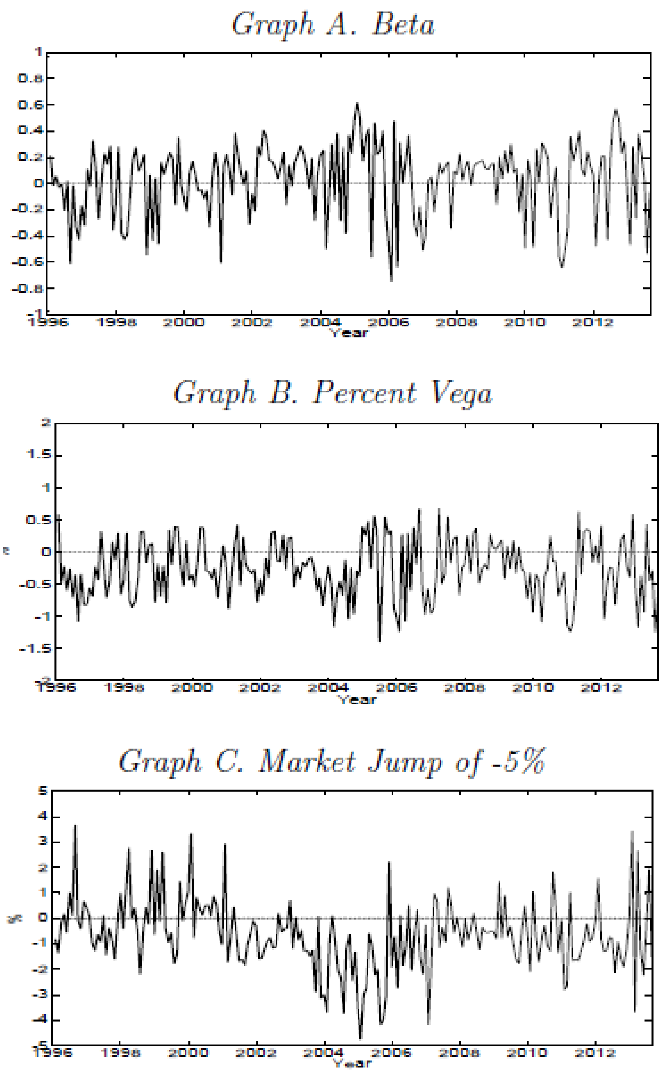


Figure 53. Source: J. Faias and P. Santa-Clara, *Optimal Option Portfolio Strategies: Deepening the Puzzle of Index Option Mispricing.*

These low values of exposure to the three main risk suggest that the returns offered by the OOPS are not a compensation for the risk embedded by the investor in his asset allocation problem, but “are mostly based on the exploiting mispricing between options”[20].

[20] *Ibidem*, p.23.

3). Option strategies applied to the FTSE MIB Index.

3.1). Objective.

The objective of the model is to analyse the performance offered by different quarterly option strategies applied to the Italian FTSE MIB Index through a period of 10 years, from the 21st of December in 2009 to the 19th of December in 2019, and compare the result of each strategy with the performance of the underlying asset, in terms of average return, variance, standard deviation, skewness and kurtosis.

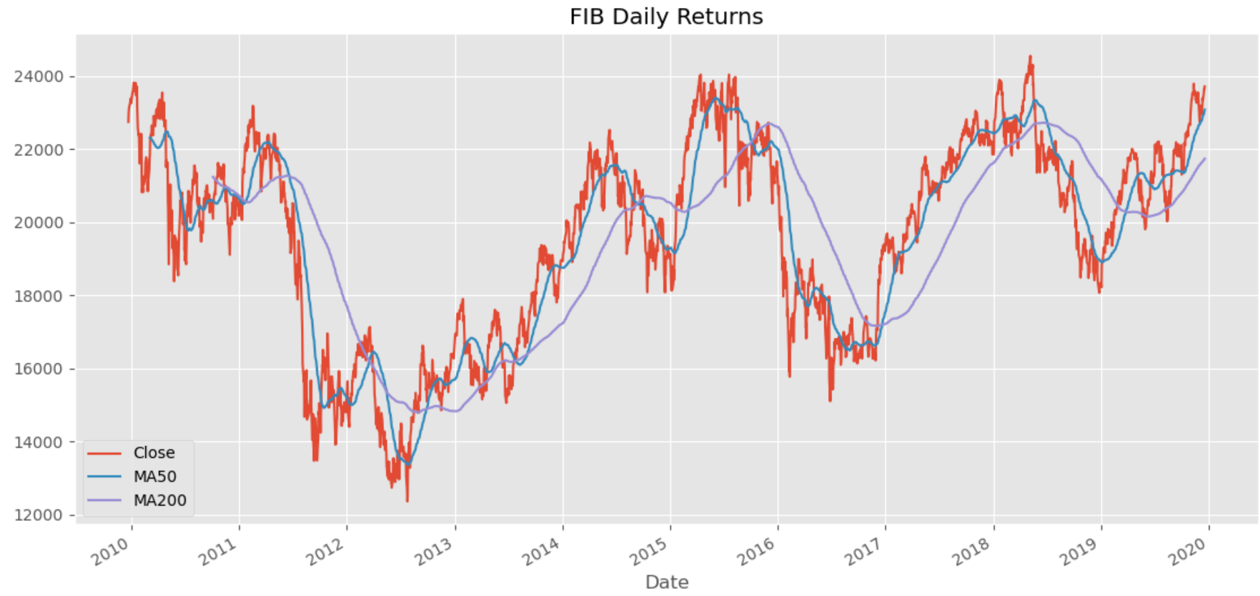


Figure 54. Source: Python. Daily Closing Price of FTSE MIB Index from 21/12/2009 to 19/12/2019.

Each strategy tested has a quarterly time frame: the options bought or sold, that has a premium of 2,5€ each index points, are held until maturity (i.e. until the expiration at 9:05 of the third Friday of the month of maturity) so to avoid transaction costs, and the next Monday new options are bought or sold to roll the strategy.

The model is developed with the use of the programming language Python 3.7, through different libraries as *pandas*, *pandas.Series*, *numpy*, *matplotlib*, *yfinance*, *scipy*.*Stats* and *seaborn*; the study of each strategy is based on the creation of a *pandas DataFrame* made of columns that summarize the key elements, as the strike price selected for the single options to buy or sell, the premium of options, the quarterly payoff offered by each of them and percentage return.

The figure below shows the modules imported and the main setting determined for the model.

```python
import pandas as pd
from pandas import Series
import yfinance as yf
import numpy as np
from scipy.stats import kurtosis
from scipy.stats import skew
import matplotlib.pyplot as plt
import seaborn as sns


plt.style.use('ggplot')

pd.set_option('display.max_columns', 30)
pd.set_option('display.width', 100)
```

Figure 55. Source: Python. Import phase of different libraries and define settings for plots and pandas.DataFrame.

The first step consists in the analysis of the quarterly returns offered by the FTSE MIB Index from 2009 to 2019, with a focus on the key statistics.

The second step consists in the analysis of the returns offered by a simple Straddle strategy, a classic option strategy that consist in a long position in a Call option and a long position in a Put option, both with the same strike price and the same maturity; this strategy is used to exploit big movements in the price of the underlying asset, with a positive return in case of a high volatility.

The third step is focused on the analysis of the performance of an Iron Condor, that consist in 2 short positions, one in a Call and one in a Put, both OTM, and 2 long positions, one in a Call option and one in a Put option, both more OTM than the options sold; this strategy has a maximum profit, due to the net credit received from the options sold, that is obtained in case the underlying asset remains within a certain price range, determined by the difference between the strike of a short Call and that of a short Put. The percentage return computed for this strategy in the model is based on a hypothetical starting capital of 5.000 €.

The third strategy tested is a Mixed one, assuming the acquisition of both a long Straddle and a long Iron Condor each quarter. The payoff offered by this strategy reduces both the loss typical of a Straddle in case of a small move in the underlying asset and the possible gain in case of big movements, but also reduces the cost of the implementation, thanks to the two shorted options.

The last strategy tested, the V Strategy, is a simple strategy that consists in the selection of a Straddle or an Iron Condor each quarter based on a simple volatility parameter: if the implied volatility of the Index, on a 30 days basis, is higher than the historical volatility, on a 1 year basis, assuming that the historical volatility will adapt to the values of the implies, the investor must choose a Straddle, to exploit possible big moves in the underlying, otherwise he must select an Iron Condor; the return on this strategy is computed based on a hypothetical initial capital of 5.000 €.

3.2). Data Selection.

The data used for the model developed are taken from different sources.

The daily and quarterly open and close prices of FTSE MIB Index are download from Yahoo Finance through the Python 3.7 module *yfinance* imported in the code as *yf*.

The data of options are downloaded on Excel from two sources: the data from 2014 to 2019 (i.e. the historical price of options with the different strike price selected) are downloaded from Eikon Thomson Reuters Datastream, while the data from December 2009 to December 2013 are download from Bloomberg Terminal; these are not historical prices as that from Thomson Reuters, but they represent a theoretical price for the call and put selected for the different strategies.

Volatility data used for the implementation of V Strategy are downloaded from the Bloomberg Terminal.

The strike prices chose for the long Straddle are the most ATM strike prices possible for each quarter; as it is possible to observe from the results obtained on Python, the average moneyness (the distance from the strike and the underlying) is 0,09% for call options and -0,09% for put options, while the median value is respectively 0,22% and -0,22%. The figure below shows the moneyness of each Call option used for the Straddle (that of the Put is specular).

Moneyness	0,35%	0,30%	0,23%	0,02%	0,50%	-0,20%	0,88%	1,07%	-0,22%	0,40%
	0,22%	-1,22%	-0,81%	0,76%	0,88%	-0,51%	0,62%	-0,23%	-0,31%	-0,90%
	0,45%	0,77%	0,50%	-0,12%	-1,11%	0,30%	1,15%	1,61%	-0,17%	-0,03%
	-0,05%	-0,96%	1,01%	-0,69%	0,44%	-0,26%	-0,88%	0,43%	-0,87%	0,40%

Figure 56. Source: Excel. Moneyness of Call option used each quarter.

The strike prices for the Iron Condor strategy are selected in order to maintain the same distance from the ATM strike price used for the Straddle; for example, in the first quarter taken into consideration, the ATM strike price is 22.500 index points, the strike of the short Call and short Put are respectively 23.500 and 21.500 index points, with a distance equal to 1.000 index points in absolute value, while the strike for the long Call and long Put are respectively 24.500 and 21.500 index points, with a distance from ATM strike in absolute value equal to 2.000 index points.

The figure below shows the average moneyness of each of four options selected each quarter for the Iron Condor.

The range of moneyness for the short Call is from -3,20% to -6,10%, for the long Call is from -6,34% to -9,78%, for the short Put is from -1,24% to -8,14% and for the long Put is from -4,57% to -11,67%.

```
mean Short Call moneyness is:  -0.0448494241200905
mean Long Call moneyness is:  -0.08189135274796508
mean Short Put moneyness is:  -0.048653562964468176
mean Long Put moneyness is:  -0.08569549159234276
```

Figure 57. Source: Python. Mean moneyness of options for the IronCondor.

The same options are used for both the Mixed and V Strategy.

3.3). FTSE MIB Index.

The first step of the model is to analyse the quarterly performance of the FTSE MIB Index, in terms of the key statistics.

Daily data for the open and close price are downloaded from Yahoo Finance into a .csv file that has been modified in Excel to obtain the quarterly data.

The code below describes the initial uploading phase and the calculations performed to plot the daily data with a 50 days and 200 days Moving Average and the quarterly returns; moreover is possible to observe the definition of a KeyStatistics function that compute the mean, median, standard deviation, variance, skewness, kurtosis, and both the maximum and minimum value of returns.

```
#import Data
FIB_daily_data = yf.download('FTSEMIB.MI', start='2009-12-21', end='2019-12-20', interval='1d')
FIB_daily-data.to_csv('FIB_daily.csv')
FIB_daily = pd.read_excel (r'C:\Users\Gino\Desktop\FIB Daily.xlsx')
FIB_quarterly = pd.read_excel (r'C:\Users\Gino\Desktop\FIB Quarterly.xlsx')
Straddle = pd.read_excel (r'C:\Users\Gino\Desktop\Data for Straddle.xlsx')
IronCondor = pd.read_excel (r'C:\Users\Gino\Desktop\Data for Iron Condor.xlsx')
Volatility = pd.read_excel (r'C:\Users\Gino\Desktop\Data for Volatility.xlsx')


#plotting FIB daily returns with MA and quarterly returns
FIB_daily['MA50'] = FIB_daily['Close'].rolling(50).mean()
FIB_daily['MA200'] = FIB_daily['Close'].rolling(200).mean()
FIB_daily[['Close', 'MA50', 'MA200']].plot(title='FIB Daily Returns')
FIB_quarterly[['Close']].plot(title='FIB Quarterly Returns')
for i in FIB_quarterly:
    FIB_quarterly['Return'] = (FIB_quarterly.Close / FIB_quarterly.Open) - 1


#define function of key statistics
def KeyStatistics(x):
    print("mean: ", np.mean(x))
    print("median: ", np.median(x))
    print("std dev: ", np.std(x))
    print("variance: ", np.var(x))
    print("skewness: ", skew(x))
    print("kurtosis: ", kurtosis(x))
    print("The maximum return is: ", max(x))
    print("The minimum return is: ", min(x))


print('FIB quarterly returns have the following key statistics:')
KeyStatistics(FIB_quarterly.Return)
```

The performance of the FTSE MIB Index through the 10 years of the analysis shows a lateralization within a large range that goes from around 12.000 to 24.000 index points and is mostly affected by the macroeconomic conditions: the effect of the financial crisis of the 2008, the crisis of the sovereign debt in the Euro Area between 2010 and 2013, and the consequently slow recovery until the end of 2019.

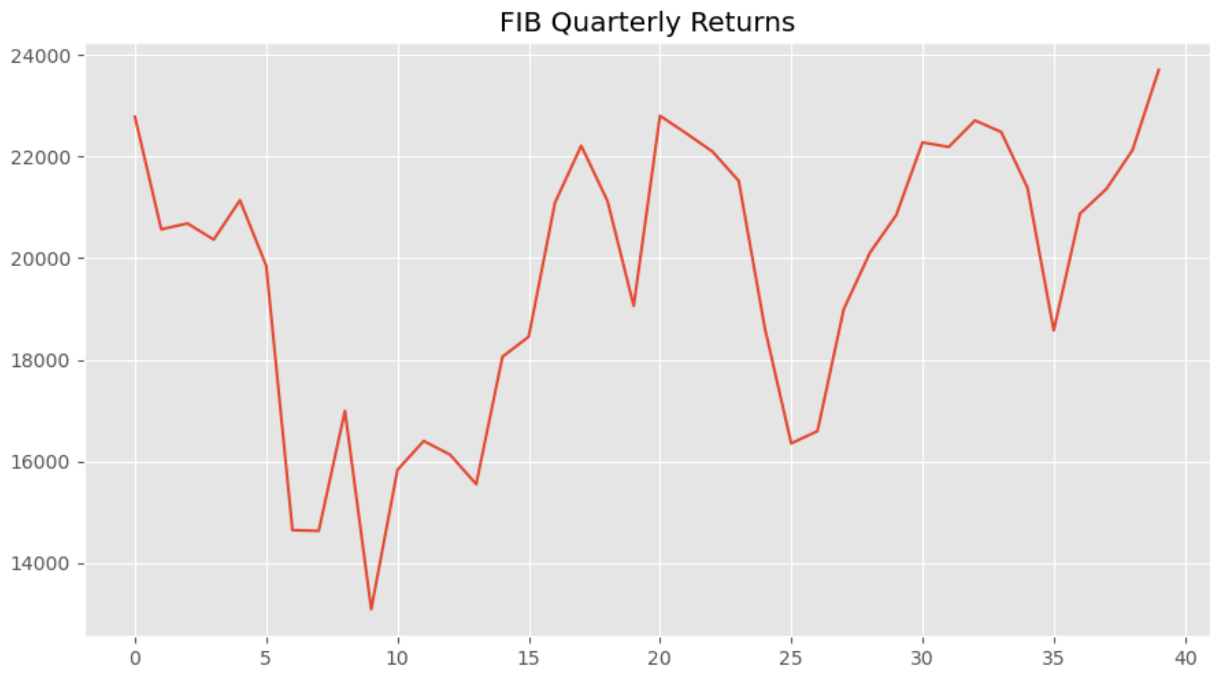


Figure 58. Source: Python. Quarterly closing prices of FTSE MIB Index.

During the period of analysis there are 19 out of 40 quarters that have registered a negative percentage return, with a consequently "winning rate" of 52,5%.

The worst quarter, always determined from the Monday after the third Friday of the month through the third Friday of the third next month, is that from 20/06/2011 to 15/09/2011, that registered an opening value of 19.673 index points and a closing value of 14.643 index points, with a negative performance of -25,57%.

The best quarter is that from 22/12/2014 to 19/03/2015 with a positive increase of +19,49%, from the starting 19.086 index points to the final value of 22.805 index points.

```
FIB quarterly returns have the following key statistics:
mean:  0.008823746209270487
median:  0.005680330006262113
std dev:  0.10360425774450563
variance:  0.010733842222789955
skewness:  -0.28989493167281927
kurtosis:  0.3245466489813822
The maximum return is:  0.19485486744210423
The minimum return is:  -0.2556803741168099
```

Figure 59. Source: Python. Key statistics of quarterly FIB returns.

The figure above shows the results of the KeyStatistics() function defined in the Python model developed. The negative value of the skewness is a positive aspect of the Index performances, because it implies that the returns are distributed mostly to the right of the mean value, with a sort of hump, so in the positive part of the Cartesian axis; the volatility of returns is 10,36%.

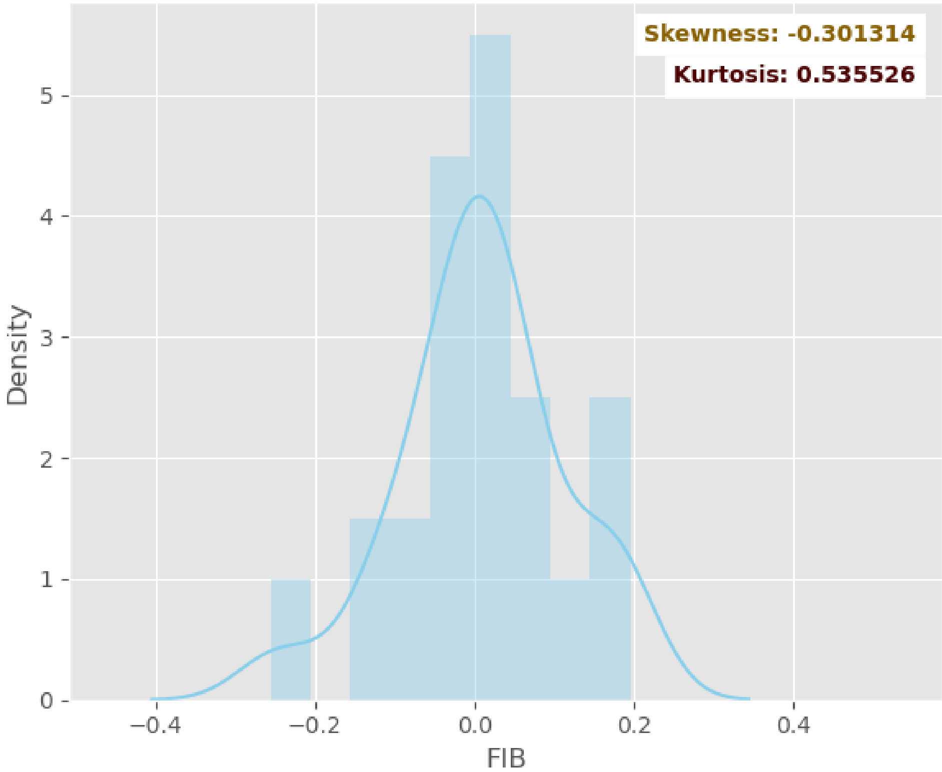


Figure 60. Source: Pyhton. Distribution of quarterly returns for FTSE MIB Index.

3.4). Straddle strategy.

The second part of the model is focused on the analysis of the performance offered by a simple Straddle strategy applied quarterly to the underlying. It is based on the acquisition of both a call and a put option with the same strike price and maturity and it offers a payoff diagram that guarantees the investor a profit in case the underlying asset has a sufficient big movement, both up or down, while he will incur in a loss in case of small movements: the return offered by the strategy is mostly related to the volatility risk.

The strike prices selected for the strategy are the most near to the actual spot price of the underlying at the starting day of each quarter, with an average moneyness of -1,22% for the call option and 1,22% for the put option.

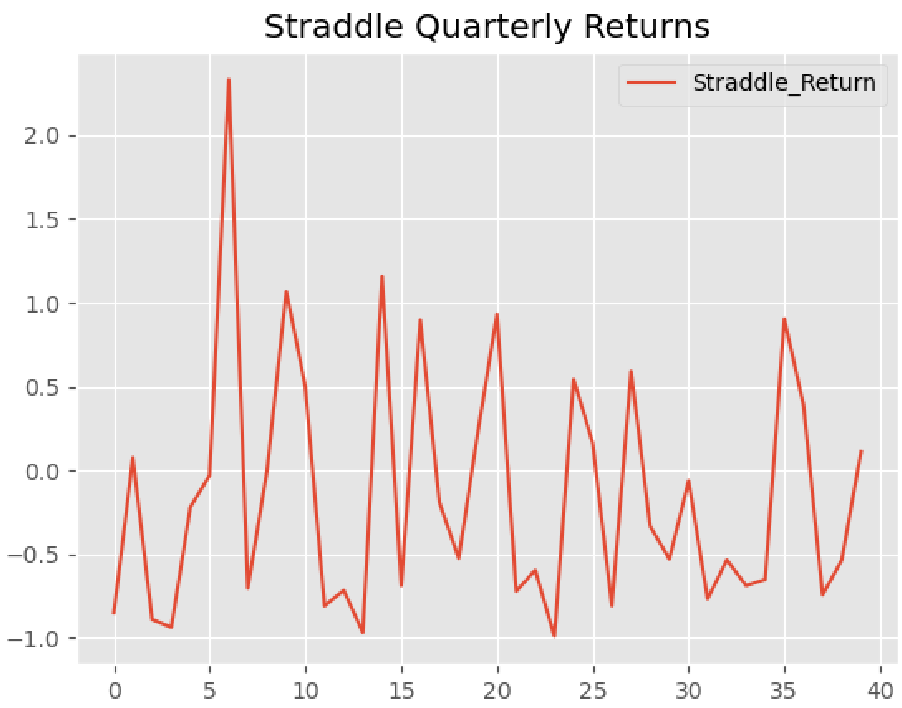


Figure 61. Source: Python. Quarterly returns of a Straddle.

The Straddle offers a “winning rate” of 37,5%, with 15 out of 40 quarters with positive performance.

Due to the nature of this strategy, the quarters with higher percentage returns are that with higher movement in the underlying: the average performance, expressed in absolute value, of the FTSE MIB in those quarters in which the Straddle strategy would have positive returns is equal to 15,23%, while on average, the absolute value of the Index performance in those quarters in which the Straddle would have suffered a loss is 7,48%.

The maximum percentage return is equal to 233%, in the quarter form 20/06/2011 to 15/09/2011, during which the Index registered a loss of 25,57%, from a starting value of 19.673 index points to a closing value of 14.643 index points; on the other side, the worst performance offered by the Straddle is that of the last quarter of 2015, with an up movement of the underlying equal to only 0,23%.

The Straddle offers a negative average return of -13,88% due to the high losses occurred in those period with small movements of the underlying: in those cases the payoff offered by the options are not sufficient to reach the break-even and the investor will incur in a loss on the initial capital invested to build the strategy.

As it is possible to observe from the summary figure below, the Straddle has a very high value of volatility (74,27%) with a high skewness (>1): the returns are mostly distributed to the left side of the mean value, in the negative part of the Cartesian axis.

```
Straddle quarterly returns have the following key statistics:
mean:  -0.13880148919499807
median:  -0.4285257330747527
std dev:  0.7426875176979061
variance:  0.5515847489442776
skewness:  1.1470427663348661
kurtosis:  1.0993402762767488
The maximum return is:  2.3299738785248563
The minimum return is:  -0.9894784995425434
mean Call moneyness is:  0.0009337619992708601
mean Put moneyness is:  -0.0009337619992708601
```

Figure 62. Source: Python. Key statistics of Straddle quarterly returns.

The code below shows the definition of another function, named Plotting(), used to plot the graph of the returns' distribution of the Straddle compared with the FTSE MIB.

```
#plotting distribution of returns
list_FIB = list(FIB_quarterly.Return)
list_Straddle = list(Straddle.Straddle_Return)


#creating a function to plot distribution of returns
def Plotting(x):
    f, axes = plt.subplots(1, 2, figsize=(12, 5), sharex=True)
    sns.distplot(x.iloc[:,0], color="skyblue", ax=axes[0])
    sns.distplot(x.iloc[:,1], color="olive", ax=axes[1])
    for i, ax in enumerate(axes.reshape(-1)):
        ax.text(x=0.97, y=0.97, transform=ax.transAxes, s="Skewness: %f" % x.iloc[:,i].skew(),\
            fontweight='demibold', fontsize=10, verticalalignment='top', horizontalalignment='right',\
            backgroundcolor='white', color='xkcd:poo brown')
        ax.text(x=0.97, y=0.91, transform=ax.transAxes, s="Kurtosis: %f" % x.iloc[:,i].kurt(),\
            fontweight='demibold', fontsize=10, verticalalignment='top', horizontalalignment='right',\
            backgroundcolor='white', color='xkcd:dried blood')
    plt.tight_layout()
    plt.show()


new_df = pd.DataFrame({'FIB':list_FIB,
                       'Straddle':list_Straddle})
Plotting(new_df)
```

The results obtained through the implementation of this function are observable in the comparable figure below.

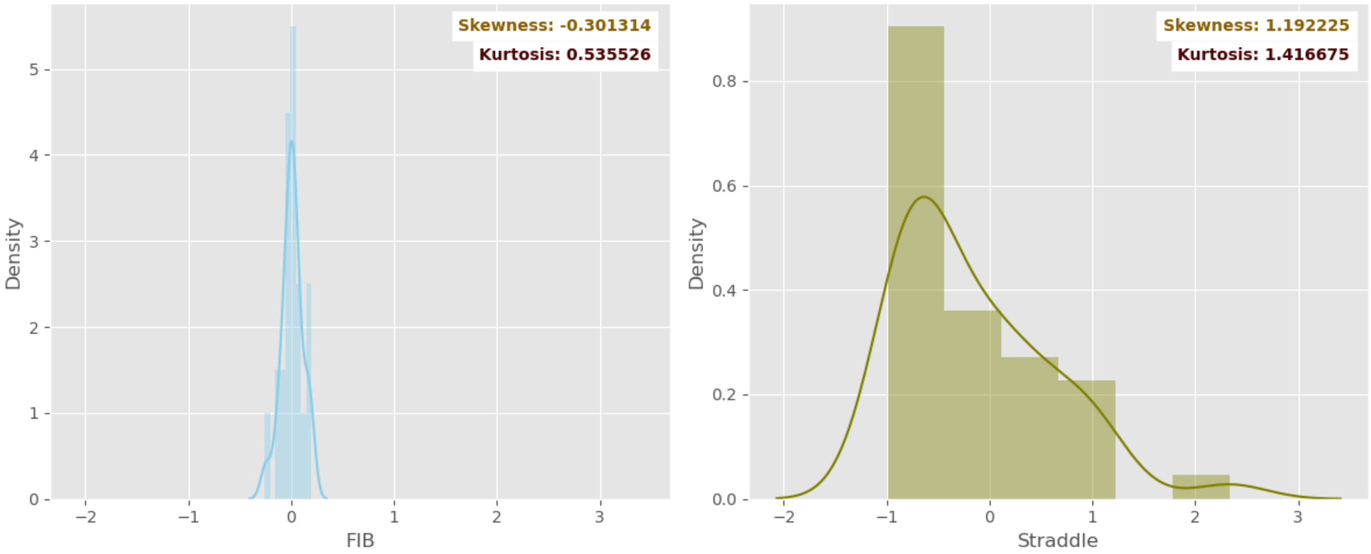


Figure 63. Source: Python. Distribution of quarterly returns of a Straddle compared with that of the FTSE MIB.

It is quite evident the effect of the skewness on the return of the Straddle: it is not an efficient strategy on the long term and it must be used only occasionally to exploit possible big up or down moves of the underlying.

3.5). Iron Condor strategy.

The third part of the model is based on the analysis of the performance offered by a classic option strategy that has the opposite objective of the Straddle: the Iron Condor.

This strategy consists in two short positions in OTM options, one Call and one Put, that constitute the initial gross credit received, and two long positions in more OTM options, one Call and one Put.

The maximum profit obtainable by this strategy is the net credit received (i.e. the gross credit minus the price of the two long positions), and it is obtainable if the price of the underlying remains within a certain price range defined by the distance between the strike price of the short Call and that of the short Put; the maximum loss, that occurs if the price of the underlying goes above the strike price of the long Call or below the strike price of the long Put, is determined by the difference between the Gross Margin required, computed as the distance between the strike price of the two Call options (or that of the two Put options, since they are the same) multiplied by 2,5€ for each index points, and the Net Credit.

An investor who buys an Iron Condor is expecting a small movement, up or down, of the underlying: he is exposed to the risk of high values of volatility but he can limit the loss to an amount initially known.

As explained in the "Data selection" paragraph, the strike prices of each option are selected to maintain fixed the distance between each one of them and the ATM strike price (that of the options bought to build the Straddle).

The returns of this strategy are computed as the ratio between the capital at the end of the period (the starting capital plus the payoff obtained) and the starting capital each quarter, assuming an initial amount of 5.000 €.

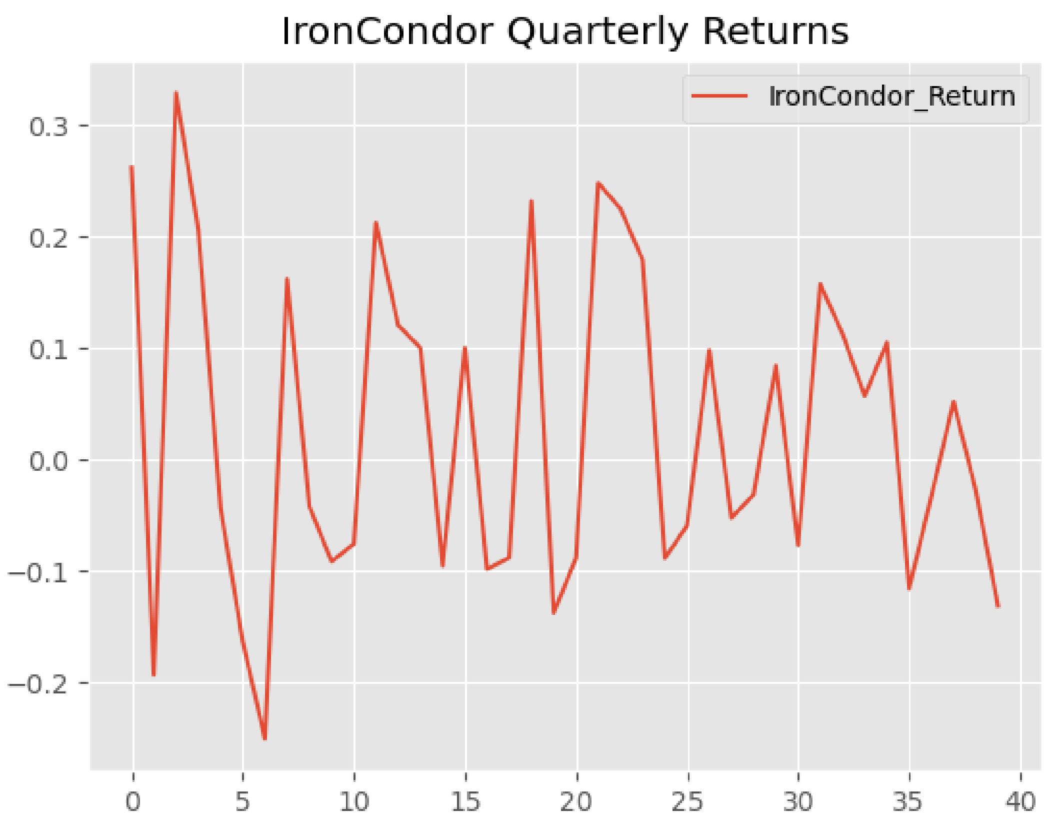


Figure 64. Source: Python. Quarterly returns of an Iron Condor.

The Iron Condor offers a “winning rate” of 47,5%, with 19 out of 40 quarters with positive performance; it is a better result with respect to the Straddle.
Due to the nature of the strategy, the quarters with higher percentage returns are that with lower movement in the underlying: the average performance, expressed in absolute value, of the FTSE MIB in those quarters in which the Iron Condor strategy would have positive returns is equal to 2,27%, while on average, the absolute value of the Index performance in those quarters in which the Iron Condor would have suffered a loss is 12,53%.
The maximum percentage return is equal to 32,87%, in the quarter from 21/06/2010 to 16/09/2010, with a corresponding Index performance equal to -1,73%, from a starting value of 21.048 index points to a closing value of 20.684 index points.
The worst performance of the Iron Condor is equal to -25,04%, registered in the third quarter of 2011, during which the FTSE MIB registered a loss of 25,57% from the starting value of 19.673 index points (this quarter is the corresponding best quarter in terms of performance of the Straddle strategy).
This strategy offers a positive average return equal to 2,63%, with a volatility of 14,29% (lower if compared to that of the Straddle but higher than that of the FTSE MIB) and a skewness > 0: as with the Straddle strategy the distribution of returns has a hump to the left, with a longer right tail.

```
Iron Condor has the following key statistics:
mean:  0.026323416511159815
median:  -0.029896145525963 87
std dev:  0.14290109542993584
variance:  0.020420723075075 63
skewness:  0.23670352285216162
kurtosis:  -0.9833223251844769
The maximum return is:  0.3287168682686783
The minimum return is:  -0.2504059001026237 6
mean Short Call moneyness is:  -0.0448494241200905
mean Long Call moneyness is:  -0.08189135274796508
mean Short Put moneyness is:  -0.048653562964468176
mean Long Put moneyness is:  -0.08569549159234276
```

Figure 65. Source: Key statistics of Iron Condor quarterly returns.

After the upload of the Excel file with strike and price data for options and the construction of the *pandas DataFrame* used for the analysis, the returns are computed assuming a starting capital of 5.000 €, and the figure below shows the code implemented to calculate them.

```
payoff_list = list(IronCondor.IronCondor_Payoff)
net_credit_list = list(IronCondor.Net_Credit)
net_margin_list = list(IronCondor.Net_Margin)

for n, i in enumerate(payoff_list):
    if i == 0:
        payoff_list[n] = net_credit_list[n]
    else:
        payoff_list[n] = (max(payoff_list[n], net_margin_list[n]))
for i in IronCondor:
    IronCondor['Final_Value'] = [n for n in payoff_list]


starting_capital = 5000
final_value_list = list(IronCondor.Final_Value)
final_value_list.insert(0, starting_capital)
series1 = pd.Series(final_value_list)
series2 = list(series1.cumsum())
series3 = []
for i in range(len(series2)-1):
    n = ((series2[i+1] / series2[i]) - 1)
    series3.append(n)
IronCondor['IronCondor_Return'] = (series3)
IronCondor[['IronCondor_Return']].plot(title='IronCondor Quarterly Returns')
Iron_equity_line = pd.DataFrame({'Equity_Line':series2})
Iron_equity_line.plot(title='Iron Condor Equity Line')
```

Finally, as the last step of the Iron Condor analysis, the returns distribution is plotted against that of the FTSE MIB Italian Index.

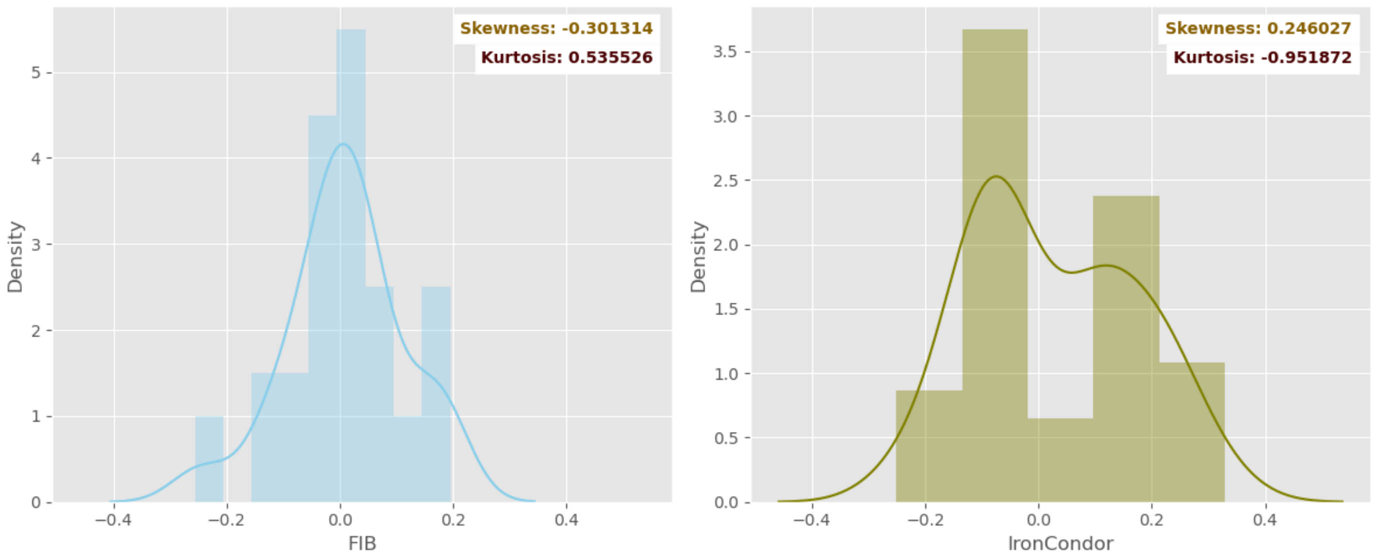


Figure 66. Source: Python. Distribution of quarterly returns of an Iron Condor compared with that of the FTSE MIB.

This strategy shows better results of the Straddle, with positive mean returns, and lower skewness. Moreover, thanks to the maximum potential loss known at the start of each period, the average performance offered by the strategy, considering only the “losing quarters”, is equal to only -9,46%, a huge improvement if compared to the corresponding average negative return of the Straddle equal to -61,79%.

Finally, the graph below shows the equity line from the initial value of 5.000 €.

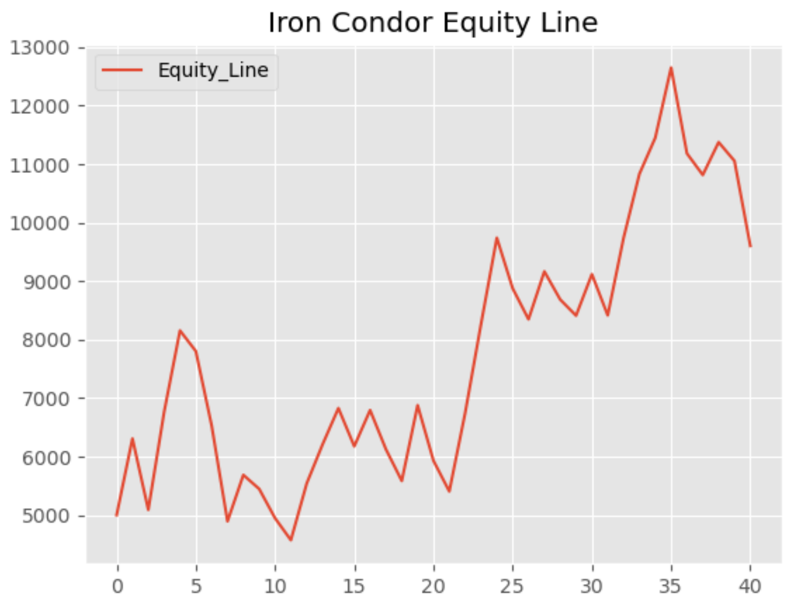


Figure 67. Source: Phyton.

3.6). Mixed strategy.

The fourth part of the model consist in a mixed strategy developed by buying each quarter a Straddle and an Iron Condor. Each period the investor will buy two ATM options, a Call and a Put, two OTM options, a Call and a Put, and will sell two less OTM options, a Call and a Put.

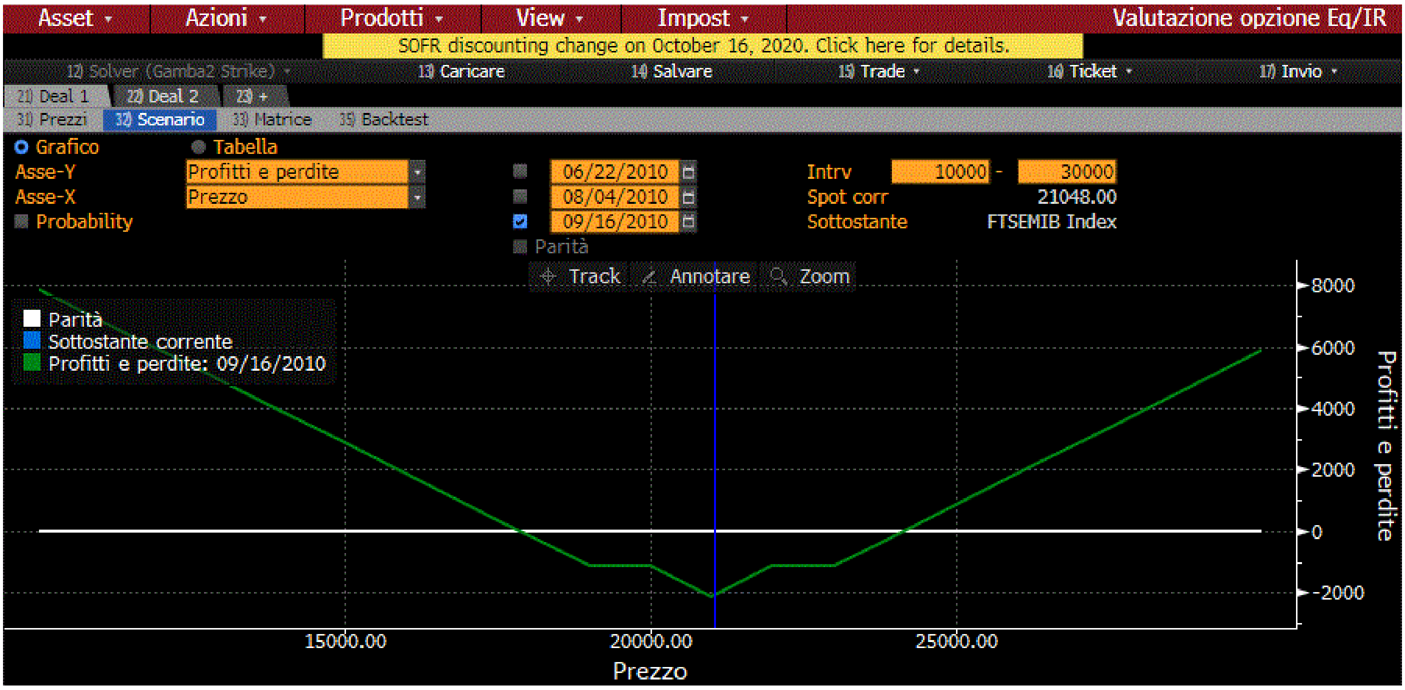


Figure 68. Source: Python.

The payoff offered by this kind of strategy allows the investor to reduce the loss in case of small movements in the underlying while it amplifies the positive performance if compared to the returns of a simple Straddle strategy (it weights more in the Mixed strategy returns than the Iron Condor): due to the premium received from the two short options of the Iron Condor component, the cumulative price of the strategy will be reduced, and the final cumulative payoff would generate higher returns and lower losses.

The performance of this strategy is computed quarter by quarter as the cumulative payoff divided by the cumulative price (the price of the Straddle plus the price of the two long options of the Iron Condor, minus the two short options).

Here is the code implemented in Python to analyse the performance of this strategy.

```
#combining the two strategies
straddle_price_list = list(Straddle.Straddle_Price)
straddle_payoff_list = list(Straddle.Straddle_Payoff)
Mixed = pd.DataFrame({'Straddle_Price': straddle_price_list,
                               'Straddle_Payoff': straddle_payoff_list,
                               'IronCondor_Net_Credit': net_credit_list,
                               'IronCondor_Payoff': payoff_list})
for i in Mixed:
    Mixed['Cumulative_Price'] = Mixed.Straddle_Price - Mixed.IronCondor_Net_Credit
    Mixed['Cumulative_Payoff'] = Mixed.Straddle_Payoff + Mixed.IronCondor_Payoff
    Mixed['Cumulative_Return'] = (Mixed.Cumulative_Payoff / Mixed.Cumulative_Price) - 1
Mixed[['Cumulative_Return']].plot(title='Mixed Strategy Quarterly Returns')


#statistics
print('Mixed strategy has the following key statistics: ')
KeyStatistics(Mixed.Cumulative_Return)


#plot distribution of returns
list_Mixed = list(Mixed.Cumulative_Return)
new_df3 = pd.DataFrame({'FIB':list_FIB,
                        'Mixed':list_Mixed})
Plotting(new_df3)
```

The graph below shows the quarterly returns of this Mixed approach, that obviously does not differ so much from that of the Straddle and the Iron Condor.

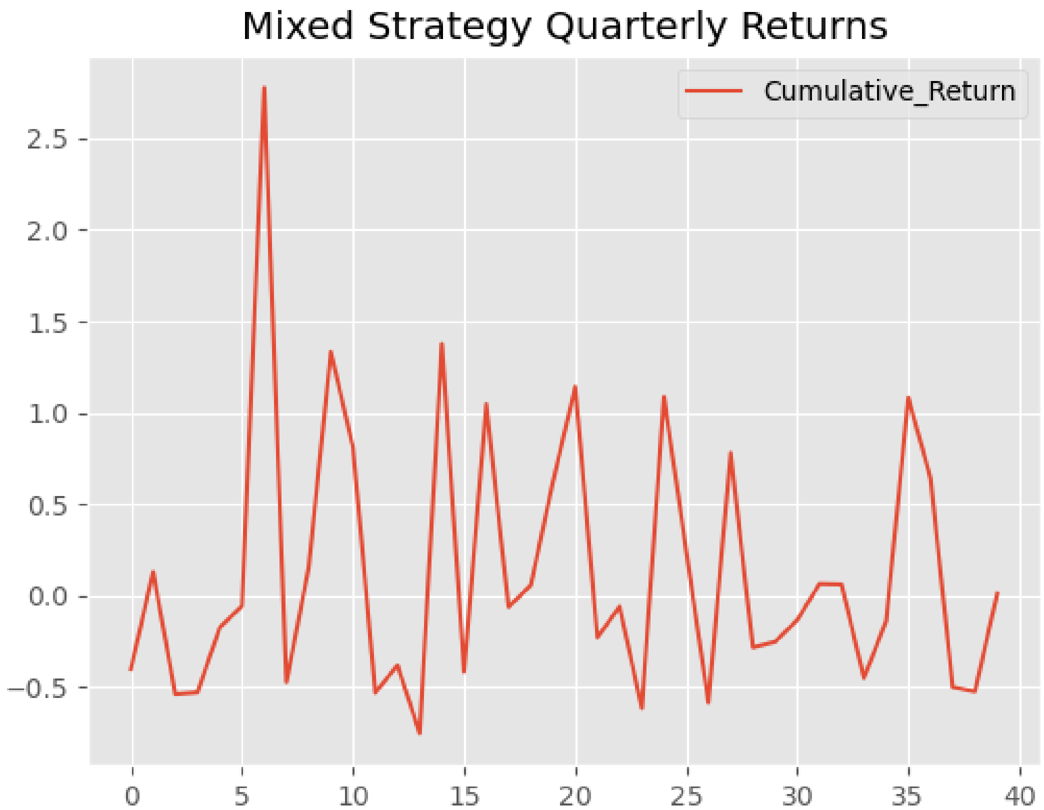


Figure 69. Source: Python. Quarterly returns of the Mixed strategy.

The Mixed strategy offers a “winning rate” of 45%, with 18 out of 40 quarters with positive performance. Due to the presence of two short positions the positive performance obtained by the Straddle component are amplified, and at the same time the negative performances are discounted.
The average performance, expressed in absolute value, of the FTSE MIB in those quarters in which the Mixed strategy would have positive returns is equal to 14,8%, while on average, the absolute value of the Index performance in those quarters in which the Mixed strategy would have suffered a loss is 3,25% (the Straddle component have a higher weight in the performance of the Mixed strategy).
The average return in positive quarters is +74,53%, amplified with respect to the Straddle (+67,31%), while the average return in negative quarters is -36,80%, reduced if compared to that of the Straddle (-57,11%).
The maximum percentage return is equal to 277,8%, in the quarter from 21/06/2010 to 16/09/2010, compared to the 233% of the Straddle.
The worst performance is equal to -75,28%, registered in the first quarter of 2013, compared to a worst performance for the Straddle of -99,12%.
This strategy offers a positive average return equal to 13,3%, with a very high volatility value of 73,25% (lower if compared to that of the Straddle) and a skewness > 1: as with the Straddle strategy the distribution of returns has a hump to the left, with a longer right tail.

```
Mixed strategy has the following key statistics:
mean:  0.13300441917399333
median:  -0.061450062814116246
std dev:  0.7325148266388302
variance:  0.5365779712457154
skewness:  1.4278536221995872
kurtosis:  2.2123797794240883
The maximum return is:  2.778040449438202
The minimum return is:  -0.7527999134004479
```

Figure 70. Source: Python. Key statistics of Mixed strategy quarterly returns.

Finally, the distribution of returns is plotted compared with that of the FTSE MIB.

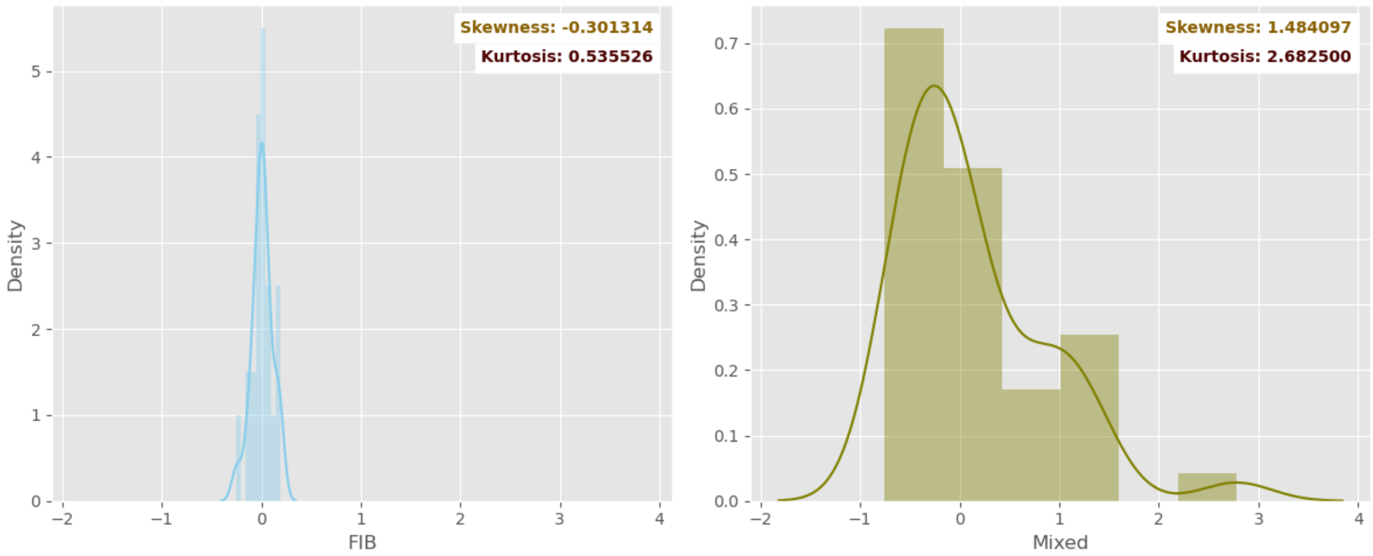


Figure 71. Source: Python. Distribution of quarterly returns of a Mixed strategy compared with that of the FTSE MIB.

It is possible to observe, also in this case, the effect of the high skewness value on the returns' distribution.

3.7). Volatility strategy.

The last part of the model is focused on the analysis of a simple strategy that consists in the selection of one of the two main strategy, the Straddle or the Iron Condor, based on the difference between two volatility values of the underlying, the Implied Volatility on a 30 days basis (IV) and the Historical Volatility on a year basis (HV): if the IV at the starting date of each quarter is higher than the corresponding value of the yearly HV, the investor will buy the Straddle (he will open a long position in a Call and a Put, both ATM with the same strike price and maturity).

The difference between these two measures of volatility is known as the *volatility risk premium* (VRP): this is the premium that option sellers receive from option buyers to bear the risk of an increase in the volatility during the life of the option. This premium is used by the writer of options to cover himself from possible unfavourable moves in the underlying.

For example, in the case of a Straddle, the buyer is "buying" volatility, he will profit if there will be a big fluctuation in the price of the underlying asset, in this case the volatility will not represent a risk for the trader or investor, but it represent the opportunity to obtain a profit.

The seller, on the other side, must protect himself from the risk of volatility (if the underlying remains in a certain price range the option will not be exercised, so the buyer will incur in a loss and the writer locks the profit, that is the premium received) so he will ask for a higher premium the higher is the risk of volatility jump.

Since the implied volatility represents a market expectation of the future possible movement of the underlying, it historically tends to exceed the realized one, and one of the classic profit opportunity is to short a Straddle, so "sell" volatility.

However, the logic behind the model developed is that negative values of VRP (i.e. the case in which the IV on a 30 days basis, the market expectations on possible future fluctuations in the price of the underlying asset, is higher than the HV realized over the last year) anticipates a possible partially realignment of these two values.

If IV is higher than HV a trader or an investor is willing to pay the VRP to the seller of options because he believes that the volatility of the next period (a quarter) will be higher and tend to reduce the gap with the implied one.

When, in the opposite case, the IV is lower than the HV, this means that the market is expecting a lower volatility if compare to the one year realize volatility, and, through a long position in an Iron Condor, the investor can lock in a profit in case of small movements.

Before the implementation of the model explained below, a test was carried out on Excel to verify the best alternative in relation to a time parameter: the same strategy exposed below was tested with different daily basis for the historical volatility.

Both the case with HV computed on a 90 days basis or a 30 days basis will lead to worst outcome, with lower average return and higher volatility of the strategy, due to the fact that in both cases the Straddle has a higher weight: with HV on a 90 days basis the Straddle was selected 18 out of 40 quarters, with a weight of 45% and a winning rate of 44,44% (8 out of 18), with HV on a 30 days basis the Straddle was selected 14 out of 40 quarters, with a weight of 35% and a winning rate of 35,71% (5 out of 14), while in the model analysed, with HV on a year basis, the Straddle was selected 14 out of 40 quarters, with a weight of 35% and a winning rate of 57,14% (8 out of 14).

```
#Volatility strategy --> if implied(30days) > historical(252days)
#buy the straddle, otherwise the iron condor
Volatility['Straddle'] = Straddle.Straddle_Payoff
Volatility['IronCondor'] = IronCondor.Final_Value
s_list = list(Volatility.Straddle)
i_list = list(Volatility.IronCondor)
hist_list = list(Volatility.Historical_V)
impl_list = list(Volatility.Implied_V)
vol_dif = []
strategy = []
for i in range(len(hist_list)):
    a = (impl_list[i] - hist_list[i])
    vol_dif.append(a)
for i in range(len(vol_dif)):
    if vol_dif[i] > 0:
        strategy.append(s_list[i])
    else:
        strategy.append(i_list[i])
strategy.insert(0, starting_capital)
strategyseries1 = pd.Series(strategy)
strategyseries2 = list(strategyseries1.cumsum())
strategyseries3 = []
for i in range(len(strategyseries2)-1):
    z = ((strategyseries2[i+1] / strategyseries2[i]) - 1)
    strategyseries3.append(z)


equity_line = pd.DataFrame({'Equity_Line':strategyseries2})
equity_line.plot(title='V Strategy Equity Line')
Volatility['V_Strategy'] = (strategyseries3)
Volatility[['V_Strategy']].plot(title='V Strategy Quarterly Returns')


#statistics
print('V Strategy has the following key statistics: ')
KeyStatistics(Volatility.V_Strategy)


list_VStrategy = list(Volatility.V_Strategy)
new_df4 = pd.DataFrame({'FIB':list_FIB,
                        'V_Strategy':list_VStrategy})
Plotting(new_df4)
```

So, after this test, the concept behind this strategy is that, if the IV of the last 30 days is higher than the HV of the last year, there is a higher probability of a big movement in the underlying, with a consequently increase in the value of the HV, so the investor would like to “buy” volatility; otherwise he would like to “sell” volatility, assuming more probable a small movement in the underlying.

On the 40 quarters of the analysis period, the strategy will “select” the Straddle (so the IV is higher than the HV) 14 times (35%), while in the other 26 cases (65%) the Iron Condor will be selected.

The returns of this strategy are computed as the ratio between the capital at the end of the period (the starting capital plus the payoff obtained) and the starting capital each quarter, assuming an initial amount of 5.000 €. The average return offered when the Straddle is selected (IV>HV) is 23,88%, with a “winning rate” of 100%, while the average return offered when the Iron Condor is selected (HV>IV) is 1,96%, with a “winning rate” of 53,85% (14 out of 26 Iron Condors generate a positive payoff).
The V strategy offers a cumulative “winning rate” of 70%, with 28 out of 40 quarters with positive performance: it is the best result if compared to the other three strategies and also the direct investment in the FTSE MIB Index.
The maximum percentage return is equal to 185,99%, in the third quarter of 2011, while the worst performance is equal to -19,31% of the first quarter of 2010: in the first case the strategy selected is the Straddle, in the second one is the Iron Condor.
The average return of the strategy is 9,63%, with a high volatility value of 30,22% (this is an expectable parameter due to the alternate selection between the Straddle, with large positive and negative returns, and the Iron Condor, with more contained returns).
The skewness value of 4,94 determines a left-shaped curve, with long right tail; the huge value of kurtosis indicates a leptokurtic distribution, with a very high concentration around the mean value.

```
V Strategy has the following key statistics:
mean:  0.09632330093912662
median:  0.020457099341013807
std dev:  0.3021677444335206
variance:  0.09130534577604145
skewness:  4.943121553676873
kurtosis:  26.066437564075706
The maximum return is:  1.8598649041922588
The minimum return is:  -0.19311836990247777
```

Figure 72. Source: Python. Key statistics of Volatility strategy quarterly returns.

The figure below shows the comparable graphs of the distribution of returns of the Volatility strategy with respect to the FTSE MIB Index.

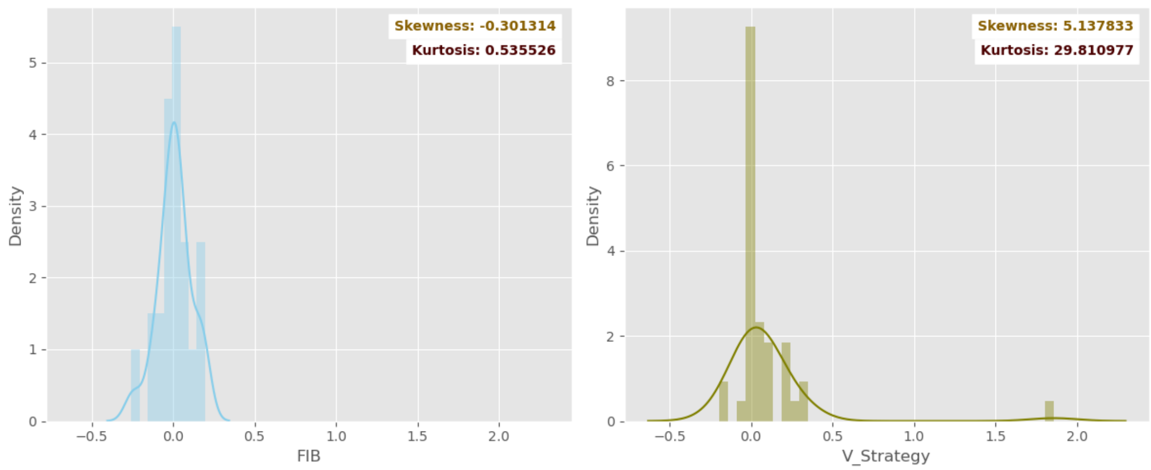


Figure 73. Source: Python. Distribution of quarterly returns of the V strategy compared with that of the FTSE MIB.

The implementation of a simple strategy like this, base on a simple volatility parameter, represents a right compromise between the big up potential offered by the Straddle and the downside limit risk offered by an Iron Condor strategy.

The simple parameters selected leads the investor to the right decision in the majority of the quarters under observations, offering a better mean performance of the FTSE MIB Index and more stability than the other strategies proposed, as it is possible to observe in the equity line below.

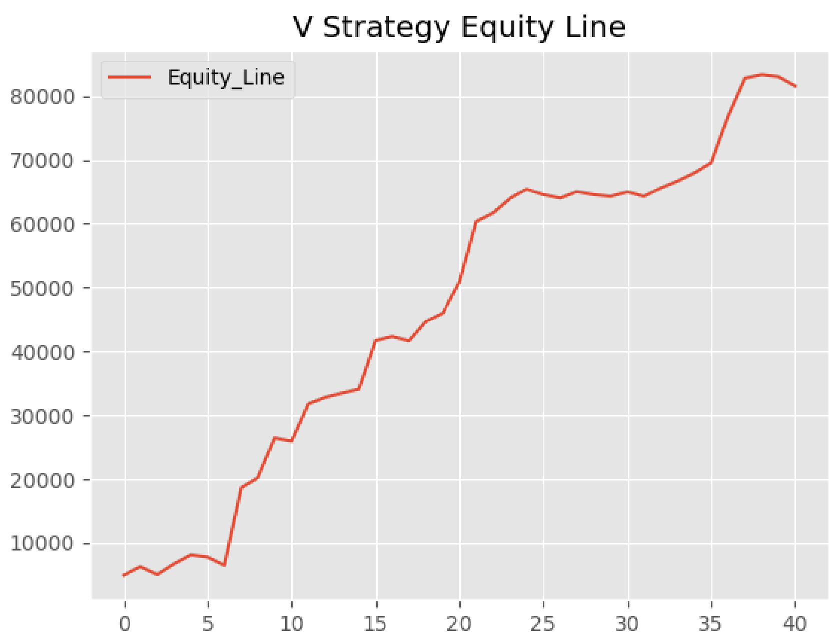


Figure 74. Source: Python.

The table below shows a summary of the key statistics of the 4 strategies tested and the FTSE MIB Index.

	Mean return	ann.	Median	Volatility	Min	Max	Skew	Kurt
FTSE MIB	0,88%	0,09%	0,57%	10,36%	-25,57%	19,49%	-0,29	0,32
Straddle	-13,88%	-1,48%	42,85%	74,27%	-98,95%	232,99%	1,15	1,10
Iron Condor	2,63%	0,26%	-2,99%	14,29%	-25,04%	32,87%	0,24	-0,98
Mixed	13,30%	1,26%	-6,15%	73,25%	-75,28%	277,80%	1,43	2,21
V Strategy	9,63%	0,92%	2,05%	30,22%	-19,31%	185,98%	4,94	26,07

Conclusion

Options are one of the most important type of derivatives instruments in the market nowadays; the academic papers analysed demonstrate the different possible uses of these assets and show that they have a very important role in the modern asset allocation problem.

Haugh and Lo demonstrate that it is possible to obtain an excellent approximation of a dynamic portfolio built under the classic asset allocation problem, with only stocks and bonds, in terms of certainty equivalence of the end-of-period wealth of the investor; the construction of a static portfolio of stocks, bonds and options, allows the investor to avoid the continuous rebalancing of a classic portfolio that does not include derivatives.

They studied three different leading cases, the Geometric Brownian motion, the Ornstein-Uhlenbeck process and a bivariate linear diffusion process, under both a CRRA and CARA utility function, under the assumptions of different level of risk aversion; the results shown that, under certain conditions, it is possible to automatize dynamic trading strategies.

Moreover, as stated by the Liu and Pan in their paper "*Dynamic Derivatives Strategies*" and by Branger, Schlag and Schneider in "*Optimal Portfolios when Volatility can Jump*", options can complete the market and expand investor's dimension of risk-return trade-offs by providing access to fundamentals risk factors as the volatility risk, and the jump risk, both in the stock price and in the volatility.

The paper of Branger, Schlag and Schneider can be considered as an expansion of that of Liu and Pan as it discusses the impact of volatility jumps in the overall risk exposure and composition of an investor portfolio.

Moreover, as exposed by Faias and Santa-Clara in the last paper analysed in the second chapter, the market of options often presents mispricing that can be exploited to obtain significant returns.

The authors develop an optimization process performed with the use of different OTM options, with weights determined each period by a CRRA utility function, applied to the S&P 500 Index; results shows that the return obtained by this strategy (OOPS) are way better than that of the underlying and of the risk-free asset, and these returns are not a compensation for the risk embedded by the investor in his asset allocation problem, but derives from the exploitation of the mispricing implicit in the market.

Thanks to the payoffs at maturity offered by the combination of various options, it is possible to obtain a different kind of exposure to the main risk factors in the market.

The model developed in the last chapter is focused on the volatility risk and represents an empirical back test of the returns generated by two of the most famous options strategies among traders, the Straddle and the Iron Condor.

The first one allows investors to gain from high value of volatility and possible jumps, while the second is a very useful hedging instruments that locks in a maximum profit if the volatility remains low.

Both those strategies applied each quarter in the period taken into consideration don't represent a valuable alternative to the performance offered by the Italian FTSE MIB Index, both in terms of mean return and variance, and both shows a distribution of returns higher than 1, with returns distributed mostly to the negative side of the mean.

The combination of a Straddle and an Iron Condor (i.e. assuming that an investor would have a long Straddle position and a long Iron Condor position at the start of each quarter), the so-called Mixed strategy, reduces the overall cost of implementation, amplifies the positive returns and cushions the negative performances, thanks to the two short position in OTM call and put options.

However, also in this case the strategy does not represent a valid alternative to the FTSE MIB due to the excess volatility: the weight of the Straddle performance is far greater than that of the Iron Condor. This one indeed has only the role of an “airbag” in the strategy.

Finally, as it is possible to observe in the last part of the chapter 3, the V strategy shows interesting results in terms of mean return, with higher volatility compare to the Italian Index (this was expectable due to the alternating selection of two strategies with totally opposite payoffs) but lower if compared to the Straddle and the Iron Condor.

The selection of one of this two structures based on divergencies between implied and historical volatility allows investors to exploit the market expectations of the future fluctuations of the underlying’ price and represents one of the possible use of options in a portfolio.

Appendix

```
import pandas as pd
from pandas import Series
import yfinance as yf
import numpy as np
from scipy.stats import kurtosis
from scipy.stats import skew
import matplotlib.pyplot as plt
import seaborn as sns


plt.style.use('ggplot')

pd.set_option('display.max_columns', 30)
pd.set_option('display.width', 100)


#import Data
#FIB_daily_data = yf.download('FTSEMIB.MI', start='2009-12-21', end='2019-12-20', interval='1d')
#FIB_daily-data.to_csv('FIB_daily.csv')
FIB_daily = pd.read_excel (r'C:\Users\Gino\Desktop\FIB Daily.xlsx')
FIB_quarterly = pd.read_excel (r'C:\Users\Gino\Desktop\FIB Quarterly.xlsx')
Straddle = pd.read_excel (r'C:\Users\Gino\Desktop\Data for Straddle.xlsx')
IronCondor = pd.read_excel (r'C:\Users\Gino\Desktop\Data for Iron Condor.xlsx')
Volatility = pd.read_excel (r'C:\Users\Gino\Desktop\Data for Volatility.xlsx')


#plotting FIB daily returns with MA and quarterly returns
FIB_daily['MA50'] = FIB_daily['Close'].rolling(50).mean()
FIB_daily['MA200'] = FIB_daily['Close'].rolling(200).mean()
FIB_daily[['Close', 'MA50', 'MA200']].plot(title='FIB Daily Returns')
FIB_quarterly[['Close']].plot(title='FIB Quarterly Returns')
for i in FIB_quarterly:
    FIB_quarterly['Return'] = (FIB_quarterly.Close / FIB_quarterly.Open) - 1


#define function of key statistics
def KeyStatistics(x):
    print("mean: ", np.mean(x))
    print("median: ", np.median(x))
    print("std dev: ", np.std(x))
    print("variance: ", np.var(x))
    print("skewness: ", skew(x))
    print("kurtosis: ", kurtosis(x))
    print("The maximum return is: ", max(x))
    print("The minimum return is: ", min(x))


print('FIB quarterly returns have the following key statistics:')
KeyStatistics(FIB_quarterly.Return)

#Straddle dataframe
for i in Straddle:
    Straddle['Call_Premium'] = Straddle.Call_Price * 2.5
    Straddle['Put_Premium'] = Straddle.Put_Price * 2.5
    call_list = list(Straddle.Close - Straddle.Strike)
    put_list = list(Straddle.Strike - Straddle.Close)
    Straddle['Call_Payoff'] = [0 if i < 0 else i for i in call_list]
    Straddle['Put_Payoff'] = [0 if i < 0 else i for i in put_list]
    Straddle['Call_Moneyness'] = 1 - (Straddle.Strike / Straddle.Open)
    Straddle['Put_Moneyness'] = - Straddle.Call_Moneyness
    Straddle['Straddle_Price'] = Straddle.Call_Premium + Straddle.Put_Premium
    Straddle['Straddle_Payoff'] = (Straddle.Call_Payoff + Straddle.Put_Payoff) * 2.5
    Straddle['Straddle_Return'] = (Straddle.Straddle_Payoff / Straddle.Straddle_Price) - 1
Straddle[['Straddle_Return']].plot(title='Straddle Quarterly Returns')
```

```
#Statistics of Straddle and returns distribution
print('Straddle quarterly returns have the following key statistics:')
KeyStatistics(Straddle.Straddle_Return)
print("Average Call moneyness is: ", np.mean(Straddle.Call_Moneyness))
print("Average Put moneyness is: ", np.mean(Straddle.Put_Moneyness))



#plotting distribution of returns
list_FIB = list(FIB_quarterly.Return)
list_Straddle = list(Straddle.Straddle_Return)


#creating a function to plot distribution of returns
def Plotting(x):
    f, axes = plt.subplots(1, 2, figsize=(12, 5), sharex=True)
    sns.distplot(x.iloc[:,0], color="skyblue", ax=axes[0])
    sns.distplot(x.iloc[:,1], color="olive", ax=axes[1])
    for i, ax in enumerate(axes.reshape(-1)):
        ax.text(x=0.97, y=0.97, transform=ax.transAxes, s="Skewness: %f" % x.iloc[:,i].skew(),\
            fontweight='demibold', fontsize=10, verticalalignment='top', horizontalalignment='right',\
            backgroundcolor='white', color='xkcd:poo brown')
        ax.text(x=0.97, y=0.91, transform=ax.transAxes, s="Kurtosis: %f" % x.iloc[:,i].kurt(),\
            fontweight='demibold', fontsize=10, verticalalignment='top', horizontalalignment='right',\
            backgroundcolor='white', color='xkcd:dried blood')
    #plt.tight_layout()
    #plt.show()


new_df = pd.DataFrame({'FIB':list_FIB,
                       'Straddle':list_Straddle})
#Plotting(new_df)

#Iron Condor premiums, payoffs and returns
for i in IronCondor:
    IronCondor['Short_Call_Premium'] = IronCondor.Price_Short_Call * 2.5
    IronCondor['Long_Call_Premium'] = IronCondor.Price_Long_Call * 2.5
    IronCondor['Short_Put_Premium'] = IronCondor.Price_Short_Put * 2.5
    IronCondor['Long_Put_Premium'] = IronCondor.Price_Long_Put * 2.5
    short_call_list = list((IronCondor.Strike_Short_Call - IronCondor.Close) * 2.5)
    long_call_list = list((IronCondor.Close - IronCondor.Strike_Long_Call) * 2.5)
    short_put_list = list((IronCondor.Close - IronCondor.Strike_Short_Put) * 2.5)
    long_put_list = list((IronCondor.Strike_Long_Put - IronCondor.Close) * 2.5)
    IronCondor['Short_Call_Payoff'] = [0 if i > 0 else i for i in short_call_list]
    IronCondor['Long_Call_Payoff'] = [0 if i < 0 else i for i in long_call_list]
    IronCondor['Short_Put_Payoff'] = [0 if i > 0 else i for i in short_put_list]
    IronCondor['Long_Put_Payoff'] = [0 if i < 0 else i for i in long_put_list]
    IronCondor['Short_Call_Moneyness'] = 1 - (IronCondor.Strike_Short_Call / IronCondor.Open)
    IronCondor['Long_Call_Moneyness'] = 1 - (IronCondor.Strike_Long_Call / IronCondor.Open)
    IronCondor['Short_Put_Moneyness'] = (IronCondor.Strike_Short_Put / IronCondor.Open) - 1
    IronCondor['Long_Put_Moneyness'] = (IronCondor.Strike_Long_Put / IronCondor.Open) - 1
    IronCondor['Gross_Margin'] = (IronCondor.Strike_Short_Put - IronCondor.Strike_Long_Put) * 2.5
    IronCondor['Gross_Credit'] = (IronCondor.Short_Call_Premium + IronCondor.Short_Put_Premium)
    IronCondor['Net_Credit'] = (IronCondor.Gross_Credit - IronCondor.Long_Call_Premium - IronCondor.Long_Put_Premium)
    IronCondor['Net_Margin'] = -(IronCondor.Gross_Margin - IronCondor.Net_Credit)
    IronCondor['IronCondor_Payoff'] = (IronCondor.Short_Call_Payoff + IronCondor.Long_Call_Payoff + IronCondor.Short_Put_Payoff + IronCondor.Long_Put_Payoff)


payoff_list = list(IronCondor.IronCondor_Payoff)
net_credit_list = list(IronCondor.Net_Credit)
net_margin_list = list(IronCondor.Net_Margin)

for n, i in enumerate(payoff_list):
    if i == 0:
        payoff_list[n] = net_credit_list[n]
    else:
        payoff_list[n] = (max(payoff_list[n], net_margin_list[n]))
for i in IronCondor:
    IronCondor['Final_Value'] = [n for n in payoff_list]


starting_capital = 5000
final_value_list = list(IronCondor.Final_Value)
final_value_list.insert(0, starting_capital)
series1 = pd.Series(final_value_list)
series2 = list(series1.cumsum())
series3 = []
for i in range(len(series2)-1):
    n = ((series2[i+1] / series2[i]) - 1)
    series3.append(n)
IronCondor['IronCondor_Return'] = (series3)
IronCondor[['IronCondor_Return']].plot(title='IronCondor Quarterly Returns')
Iron_equity_line = pd.DataFrame({'Equity_Line':series2})
Iron_equity_line.plot(title='Iron Condor Equity Line')
```

```python
#Volatility strategy --> if implied(30days) > historical(252days)
#buy the straddle, otherwise the iron condor
Volatility['Straddle'] = Straddle.Straddle_Payoff
Volatility['IronCondor'] = IronCondor.Final_Value
s_list = list(Volatility.Straddle)
i_list = list(Volatility.IronCondor)
hist_list = list(Volatility.Historical_V)
impl_list = list(Volatility.Implied_V)
vol_dif = []
strategy = []
for i in range(len(hist_list)):
    a = (impl_list[i] - hist_list[i])
    vol_dif.append(a)
for i in range(len(vol_dif)):
    if vol_dif[i] > 0:
        strategy.append(s_list[i])
    else:
        strategy.append(i_list[i])
strategy.insert(0, starting_capital)
strategyseries1 = pd.Series(strategy)
strategyseries2 = list(strategyseries1.cumsum())
strategyseries3 = []
for i in range(len(strategyseries2)-1):
    z = ((strategyseries2[i+1] / strategyseries2[i]) - 1)
    strategyseries3.append(z)

#statistics
print('Iron Condor has the following key statistics: ')
KeyStatistics(IronCondor.IronCondor_Return)
print("mean Short Call moneyness is: ", np.mean(IronCondor.Short_Call_Moneyness))
print("mean Long Call moneyness is: ", np.mean(IronCondor.Long_Call_Moneyness))
print("mean Short Put moneyness is: ", np.mean(IronCondor.Short_Put_Moneyness))
print("mean Long Put moneyness is: ", np.mean(IronCondor.Long_Put_Moneyness))


#plot distribution of returns
list_IronCondor = list(IronCondor.IronCondor_Return)
new_df2 = pd.DataFrame({'FIB':list_FIB,
                        'IronCondor':list_IronCondor})
#Plotting(new_df2)


#combining the two strategies
straddle_price_list = list(Straddle.Straddle_Price)
straddle_payoff_list = list(Straddle.Straddle_Payoff)
Mixed = pd.DataFrame({'Straddle_Price': straddle_price_list,
                      'Straddle_Payoff': straddle_payoff_list,
                      'IronCondor_Net_Credit': net_credit_list,
                      'IronCondor_Payoff': payoff_list})
for i in Mixed:
    Mixed['Cumulative_Price'] = Mixed.Straddle_Price - Mixed.IronCondor_Net_Credit
    Mixed['Cumulative_Payoff'] = Mixed.Straddle_Payoff + Mixed.IronCondor_Payoff
    Mixed['Cumulative_Return'] = (Mixed.Cumulative_Payoff / Mixed.Cumulative_Price) - 1
Mixed[['Cumulative_Return']].plot(title='Mixed Strategy Quarterly Returns')


#statistics
print('Mixed strategy has the following key statistics: ')
KeyStatistics(Mixed.Cumulative_Return)


#plot distribution of returns
list_Mixed = list(Mixed.Cumulative_Return)
new_df3 = pd.DataFrame({'FIB':list_FIB,
                        'Mixed':list_Mixed})
Plotting(new_df3)
```

```
equity_line = pd.DataFrame({'Equity_Line':strategyseries2})
equity_line.plot(title='V Strategy Equity Line')
Volatility['V_Strategy'] = (strategyseries3)
Volatility[['V_Strategy']].plot(title='V Strategy Quarterly Returns')


#statistics
print('V Strategy has the following key statistics: ')
KeyStatistics(Volatility.V_Strategy)


list_VStrategy = list(Volatility.V_Strategy)
new_df4 = pd.DataFrame({'FIB':list_FIB,
                        'V_Strategy':list_VStrategy})
Plotting(new_df4)
plt.show()
```

Summary

Derivatives are financial instruments whose value is reliant upon (or derives from) another financial asset, or a group of assets, called the underlying.

An option is one of the main derivatives instruments, and it gives the holder the opportunity (i.e. the option) to buy or sell, depending on the type of option (Call or Put), a certain amount of the underlying asset (usually 1 option represents 100 shares of the underlying security) at a certain price (the strike price) in a certain time in the future. The time of the exercise differs from the two main type of options: the American options can be exercised only at maturity, while European options can be exercised at any date until maturity.

Options can be used for three main different purposes: to hedge against hostile fluctuations in the price of the underlying asset, to speculate on the future possible price changes, or to do arbitrage operations to exploit mispricing and market inefficiencies.

Initially options were only seen as complex and obscure financial instruments without an important role within the market, and they were also banned or considered illegal.

The first option appearance was in 1790; at that time options were only traded in small OTC markets with lack of intensive regulation.

In fact, before the 1933 the regulation of securities in the United States was based on the so called “blue sky securities law”, state laws that require the registration of all securities offerings and sales to protect the investors from fraud based on the merit reviews: there were very strictly and specific qualitative requirements on offerings and if the company does not respect those ones it will not be allowed to register the offerings. The primary purpose of the ’33 Act was to ensure that shareholders would have received key information regarding securities in which they want to invest, and it was based on a disclosure philosophy, meaning that also a bad investment were legal as long as all the key information are accurately disclosed.

After that, the Act of the 1934 established the Securities and Exchange Commission (the SEC) and gave it the authority to regulate the secondary trading of options.

Initially, much of the stock options trading business was conducted solely by 25 members of the Put and Call Brokers and Dealer Association, so options’ volume was very low: in 1968 the annual contract volume reached only the 1% of the New York Stock Exchange (NYSE) volume, and this was mainly due to the way in which OTC worked.

Only after the creation of the first official exchange, the CBOE, in 1973, the option’s market started to increase. The CBOE began trading on April 26 of 1973 with initially only 305 seats sold for $10.000 and 16 Call options tradable, with a volume on the first day of just 900 contracts; Put were not even introduced until 1977.

Average daily volume (ADV) in the first full month of operation was 1.584 with an average of 1,7 contracts per trade.

The increase in volumes was mostly due to the efficiency of the new mechanism introduced in the CBOE.

Typically, on a security exchange, the limit order book is a record of the limit orders (buy orders or sell orders) maintained by the security specialist who works at the exchange and who executes the orders at or better than the given limit price.

Instead, the Market Maker system implemented by the CBOE involved a public limit order book in each option, with orders filled only by brokers who could not trade for their own accounts and were employees of the exchange (*Floor Brokers* and *Order Book Officials*), and *Market Makers*, each assigned to a specific option class, who provided liquidity to the market by trading any option at any time for their own accounts.

Market Makers also assume long-term positions based on their expectations and strategies, always respecting their primary obligation: "no more than 25% to 50% of any Market Maker's trades (measured in number of contracts) can be outside his principal assignment – usually options on three underlying stocks – in any given quarter"[21]

Typically Floor Brokers fill *market orders* (i.e. an order that instructs the Floor Broker to immediately fil the order at the best possible price), while those who are self-employed to the exchange fill the *limit orders* (i.e. an order that instructs the Market Maker to fill it only at a specified price or better).

Thanks to this mechanism and to the increasing volumes traded, options become more and more popular, with investors that started to considerate them as valuable alternatives to classic financial assets as stocks and bonds.

There are different kind of options, each one with different characteristic, different payoff diagram and different impact of price drivers. The two main categories of options are the Plain Vanilla options and the Exotic options.

Plain Vanilla is a term used in the financial sector to indicate the most basic type of a specific financial instrument (options, futures, bonds): it represents the standard version of a financial security, without any extra that can alter the original nature of the instrument.

The Plain Vanilla options are the Call option and the Put option, and in both cases an investor could have a long or a short position.

For a Long position in a Call option the payoff that the investor should expect is:

$$\mathbf{max}(\boldsymbol{S_t - K, 0})$$

The option will be exercised if the price of the underlying is greater than the strike price, yield a gain to the investor equal to the difference between those two values, otherwise the option will not be exercised and the loss is 0 (without considering the premium c that the investor paid to buy the option).

For a Short position in a Call option the payoff that the investor should expect is:

$$\mathbf{min}(\boldsymbol{K - S_t, 0})$$

The writer of the option, who has sold the right to buy the underlying, will have a gain equal to the premium he has received if the spot price S is lower than the strike price (i.e. if the buyer will not exercise the option),

[21] J. C. Cox – M. Rubenstein, *Options Markets*, Prentice – Hall Inc. 1985, p. 81.

otherwise he would have a loss equal to the negative difference between the strike price that he receives by selling the underlying to the buyer and the spot price at which he had bought it.
For a Long position in a Put option the payoff that the investor should expect is:

$$\mathbf{max}(\boldsymbol{K} - \boldsymbol{S_t}, \mathbf{0})$$

The option will be exercised if the price of the underlying is lower than the strike price, yield a gain to the investor equal to the difference between those two values, otherwise the option will not be exercised and the loss is 0 (without considering the premium p that the investor paid to buy the option).
On the other side, for a Short position the payoff is:

$$\mathbf{min}(\boldsymbol{S_t} - \boldsymbol{K}, \mathbf{0})$$

The writer of the option, who has sold the right to sell the underlying, will have a gain equal to the premium he has received if the spot price S is higher than the strike price (i.e. if the buyer will not exercise the option), otherwise he would have a loss equal to the negative difference between the strike price that he has to pay to buy the asset from the buyer and the spot price at which it could be possible to buy it on the market.
Besides the two main options there exist several other types, the so-called *Exotic* options:

- The **Bermuda** option is between the America and European options because it could be exercised only at the expiration date or at pre-determined intermediate dates between the purchase and the maturity.
- The **Asian** options are financial instruments whose payoffs are determined not by the spot price of the underlying but by the average price of it during the period of existence of the option.
- The **Barrier** options, Call and Put, are options whose payoff depends upon whether the price of the underlying reaches a predetermined level, precisely called *barrier*, during a certain period of time.

  There are two different type of barrier: the *knock-in* and the *knock-out*: the knock-in Barrier option starts to exist if and only if the price of the underlying reaches that level, while the knock-out barrier option ceases to exist if the price of the underlying reaches that level.

  Knock-in options can be both *up-and-in*, the option comes into existence only if the price of the underlying asset rises above the barrier (usually the strike price), or *down-and-in*, the option comes into existence only if the price of the underlying asset moves below the barrier.

  Knock-out options can be both *up-and-out*, the option ceases to exist when the price of the underlying asset moves above the barrier set above the strike price, or *down-and-out*, the option ceases to exist when the price of the underlying moves below the barrier that is set below the underlying's initial price.
- The **Basket** options have the strike price on the weighted value of its components; they are used to hedge portfolios against multiple assets in only one transaction rather than hedging each individual asset with different instruments.
- The **Lookback** options initially do not have a specified strike price, it could be selected by a range of different exercise price only on the maturity date.

Every option has an Intrinsic Value and a Time Value, both reflected in its premium, and the *moneyness* represents a financial measure of the extent to which an option could have a positive value at maturity:

- If K = S the option is *at-the-money* (*ATM*).
- If K > S (K < S) the Call (Put) option is *out-of-the-money* (*OTM*).
- If K < S (K > S) the Call (Put) option is *in-the-money* (*ITM*).

The Intrinsic Value is the value of the option as if it would expire today; if the option is ITM it would have a positive value, if it is OTM its intrinsic value will be negative.

The Time Value is the difference between the price of the option and its intrinsic value. That's the reason why the value of an OTM option (i.e. an option that has no intrinsic value) will be equal to its time value.

The main elements that affect the Intrinsic Value of options are:

1. **The Spot price (*S*)**: the market price of the underlying asset at any time until the maturity. The higher the Spot price, the higher (lower) the value of a Call (Put).
2. **The Strike price (*K*)** or exercise price: the price at which the buyer of the option can buy (Call option) or sell (Put option) the underlying. The higher the Strike price, the lower (higher) the premium paid by the buyer of a Call (Put).

The elements that affect the Time Value are:

1. The **time to maturity (*t*)**: the fraction of time that remaining in the life of an option. The higher the time to maturity, the higher the probability that even a very low volatility stock can move, so the higher the premium of both Call and Put options.
2. The **volatility (*σ*)** as a measure of dispersion of future stock prices: it measures the uncertainty regarding the future value of a financial security, the higher it is, the higher the possibility of large price changes in the futures. It has typically a double effect for the owner of a stock, who can profit much more as loss much more in case of high volatility, while the owner of an option has a limited downside risk, due to the fact that, if the price goes too much down (for the owner of a Call option) or too much up (for the owner of a Put), he will not exercise the option and his maximum loss will be the premium paid. It should be noticed that, keeping all others aspects fixed, the premium of a Put is always greater than the premium of a Call, since the up potential of a stock price is unlimited, while the minimum value is 0.
3. The **risk-free interest rate (*r*)** affects the option value because an increase in it will reduce the present value of cash flows and also increase the expected return required by investors.
4. The **expected dividends** paid by the underlying stock during the life of the option. Dividends reduce the stock price generating a negative effect for the Call option and a positive one for the Put option.

The factors just exposed affects the option values but each of them could have a different impact, measured by some variables called ***Greeks***, that represents the sensibility of the value of the options in relation with underlying factors.

The ***Delta*** is the first derivatives of the premium of an option with respect to the Spot Price of the underlying. It represents the sensitivity of the option price with respect to changes in the value of the underlying (i.e. the rate of change between the price of an option and a $1 change in the price of the underlying asset). The Delta of a Call can assume value from 0 to 1, while the Delta of a Put can assume values from 0 to -1; for an OTM option the Delta is near to 0, while for an ATM option is near to 1.

The ***Gamma (Γ)***, also called the Delta of the Delta, is the second order derivative of the premium of the option with respect to the price of the underlying.

It is the rate of change of the Δ with respect to a one-unit change in the price of the underlying and it represents the sensitivity of the Delta to changes in the price of the underlying.

The ***Vega (V)*** is the first derivative of the premium with respect to the volatility and represents the sensitivity of the option's premium to the underlying's implied volatility, that represents the expectations of investors regarding the likelihood of changes in a given security's price.

The ***Theta (Θ)*** is the first derivative of the premium with respect to the maturity.

It is the time decay of the option (i.e. the amount an option's price would decrease as the time to maturity decreases).

The ***Rho (ρ)*** is the first derivative of the premium with respect to the risk-free rate and represents the sensitivity to the risk-free rate.

The ***Lambda (Λ)*** is the leverage factor and it measure how much leverage an option is providing as the price of that option changes.

The ***Zomma*** is the third order derivative of the premium of the option with respect to the price of the underlying; it measures the sensitivity of the Gamma to price movements in the stock price.

The ***Vomma*** is the second order derivative of the price of the option with respect to the implied volatility of the underlying and it measures the sensitivity of the Vega to fluctuations in the implied volatility of the stock.

The ***Ultima*** is the third order derivative of the price of the option with respect to the implied volatility of the underlying and it measures the sensitivity of the Vomma to fluctuations in the implied volatility.

Each of these greeks has different impact on the price of options and are often analysed by traders to set neutral factor strategies.

Thanks to the combination of different options, each one with a particular diagram of payoff, traders and investors can obtain a non-linear exposure to the market, both on the upside and the downside.

Here is a brief description of the most common strategies used by practitioners in the market.

- **Covered Call:** a long position in a stock combined with a short position in European Call; this strategy provide a neutral exposure to the underlying and is usually used by investors who have a long position in the underlying with a short-term neutral view.
- **Reverse Covered Call:** a long Call combined with a Short position in the underlying stock.
- **Protective Put**: this strategy involves buying a European Put on a stock and the stock itself.

- **Reverse Protective Put**: a short position both in a European Put on a stock and the stock itself.
- **Bull Spread**: a long position in a European Call (Put) with strike price K_1 and a short position in a European Call (Put) with a strike price $K_2 > K_1$ ($K_2 < K_1$). This strategy limits the both the investor's upside and downside risk, offering a profit in case of a lateral or slightly bullish scenario.
- **Bear Spread**: a short position in a European Put (Call) with a strike price equal to K_1 and the second one is a long position in a European Put (Call) with strike price $K_2 > K_1$ ($K_2 < K_1$). This strategy, like the Bull Spread, limit both upside and downsize risk for the investor and it offers a profit in case of lateral or slightly bearish scenario.
- **Box Spread:** it is a combination of a Bull Call Spread and a Bear Put Spread with the same two strike prices (K_1 and K_2). This strategy is made of four legs: a long Call and a short Put, both with strike price equal to K_1, plus a long Put and a short Call with strike price equal to K_2.
- **Butterfly Spread:** this is a three leg option strategy used by traders to limit the upside and downside risk (there is a maximum loss in both long and short directions) seeking to obtain a gain (capped) from a price movement that will eventually occur inside the range from the lower to the higher strike price (K_1 < Profit < K_2). The strategy is constructed by opening a long position in two Call options, one with strike price K_1 (the lower) and the other with strike price K_3 (the higher) and by opening a short position in two European Call with an intermediate strike price equal to K_2.
- **Calendar Spread:** it could be built both with two Call or two Put options and it consist in the short selling of the option with the lower maturity (T_1) and buying the options (with the same strike price K) with the longer maturity (T_2).
- **Reverse Calendar Spread:** a long position in a short-maturity option and a short position in a long-maturity option (both with the same strike price K).
- **Bottom Straddle:** a long position in both a European Call and a Put with the same strike price K. This strategy leads to a loss in case of a small movement in the stock price of the underlying, while it offers a gain in case of large movements in either direction. This strategy allows the investor to limit his maximum loss (it will be the sum of the premiums paid to build it) but it is very useful for those investors who are expecting large price movements but they didn't know in which direction.
- **Top Straddle:** this is a straddle created by selling short both a Call and a Put with the same strike price K. This strategy will lead to a profit (the maximum profit is the two premiums received by selling the options) only if the options are not exercised during their existence.
- **Strip:** it is a two legs strategy. The first one is a long position in a European Call option with strike K and expiration date T; the second one is a short position in two European Put options with the same strike price and expiration date.

- **Strap:** it is a two legs strategy. The first one is a long position in two European Call options with strike K and expiration date T; the second one is a short position in a European Put option with the same strike price and expiration date.
- **Bottom Strangle:** a long position in a Put option with strike K_1 and in a Call option with strike $K_2 > K_1$. It limits the downsize risk (the maximum loss is the sum of the premiums paid to build it) that is maximize within the range price from K_1 to K_2. If the stock price increases above K_2 or decreases below K_1 the investor will profit from this position, otherwise he will not exercise the options.
- **Top Strangle:** this is a Strangle with two short positions, one in a Put with strike K_1 and the other in a Call with strike $K_2 > K_1$.
- **Bullish Iron Condor:** this is a 4 legs strategy. The first leg is a long position in a OTM Put option with a strike price K_1 lower than the stock price of the underlying; the second leg is a short position in a ATM or OTM Put option with a strike price K_2 close to the spot price; the third leg is a short position in an OTM or ATM Call option with a strike price K_3 above the current stock price; the fourth leg is a long position in a OTM Call option with the higher strike price K_4. The long positions represent the so-called *wings* that limit the upside and downsize risk. The trader will profit from a small move in the price of the underlying.
- **Bearish Iron Condor:** this is a 4 legs strategy. The first leg is a short position in a OTM Put option with a strike price K_1 lower than the stock price of the underlying; the second leg is a long position in a ATM or OTM Put option with a strike price K_2 close to the spot price; the third leg is a long position in an OTM or ATM Call option with a strike price K_3 above the current stock price; the fourth leg is a short position in a OTM Call option with the higher strike price K_4. The long positions represent the so-called *wings* that limit the upside and downsize risk. The trader will profit from a small move in the price of the underlying.

As it is possible to understand by analysing the main option strategies, combination of options can lead to a big variety of payoff, each one with impacted differently by the greeks and the main factors that affect the price of options.

As exposed in the second chapter of the thesis the role of options in portfolios of investors and traders plays an important role not only due to the possibility to combine them in order to speculate from the possible future fluctuations or to hedge against some kind of risk.

Different academic paper stated the importance of options in relation to other aspects and the necessity to include them in the classic asset allocation problem.

The classic Asset Allocation problem of modern finance, as introduced by Samuelson and Merton in 1969, is based on the formulation that the objective of an investor is to maximize its expected utility $E[U(W_T)]$ of the end-of-period wealth W_T by allocating his wealth W_t between a risky security (the 'stock') and a riskless security (the ‘bond’), over some investment horizon $[0, T]$; the creation of a dynamic investment strategy of stocks and bonds can lead to the maximum utility.

In 2001, Martin B Haugh and Andrew W Lo, in his paper “*Asset allocation and derivatives*” stated that it is possible for an investor, with the use of derivatives, to build a buy-and-hold kind of portfolio, with stocks, bonds and options, that can be an excellent substitute of complex dynamic strategies involving only the first two securities.

Their objective is to measure the closeness of the buy-and-hold portfolio with respect to the dynamic one, in three different ways: through the maximization of the expected utility and through the minimization respectively of the mean-squared error and the weighted mean-squared error.

In this process they take into consideration the results of two different utility function, the constant relative risk aversion CRRA and the constant absolute risk aversion CARA, under three leading cases for the stock-price process: the geometric Brownian motion (GBM), the trending Ornstein-Uhlenbeck process, and a bivariate linear diffusion process with a stochastic mean-reverting drift.

For example, taking the GBM case with the maximization of the expected utility, it is shown in the paper that, for RRA = 1 (investors who are not very risk averse), the certainty equivalent of the optimal buy-and-hold portfolio $\mathrm{CE}(V_T^*)$ reported as a percentage of the certainty equivalent of the optimal dynamic stock/bond policy $\mathrm{CE}(W_T^*)$, increases as the number of options allowed is increased (from an allocation of 60,4% with *n* = 1 to a distribution of 99,3% of wealth when *n* = 3), while the root mean-squared error RMSE decreases at a slow rate (from 3659,6 with *n* = 1 to 3.642,4 when *n* = 3) .

n	Options (%)	Stock (%)	$\mathrm{CE}(V_T^*)$ (%)	RMSE (%)
		$\mathrm{CE}(W_T^*) = \$9\,948\,433$		RRA = 1
0	0.0	100.0	20.2	3 659.6
1	60.4	39.6	68.7	3 653.4
2	80.0	20.0	87.7	3 642.4
3	99.3	0.7	92.2	3 642.4

On the other side, with *n* = 3, in the case of RRA=20, the certainty equivalents $\mathrm{CE}(V_T^*)$ is 99,8%, with a short option positions equal to -24%, used to hedge risks (gains of one security offsets to some degree the losses of the other).

		$CE(W_T^*) = \$325\,437$		RRA = 20
0	0.0	11.6	97.7	101.0
1	−30.0	60.7	99.3	66.5
2	−24.0	53.0	99.8	13.3
3	−24.0	53.1	99.8	3.2

The paper of Haugh and Lo demonstrate that "buy-and-hold portfolios are excellent approximations – in terms of certainty equivalence and mean-squared-error of end-of-period wealth – to their dynamic counterparts, suggesting that in those cases, dynamic trading strategies may be 'automated' by simple buy-and-hold portfolios with just few options".

Another important aspect of including derivatives in investment strategies has been studied by Liu and Pan in their paper "*Dynamic derivative strategies*", in which they explain the importance of derivatives to complete the market and provide independent exposure to each of the three main risk factors incorporated in the model of the aggregate stock market: diffusive price shocks, price jumps, and volatility risks. Typically, a risky stock provides risk exposure only to diffusive and jump risk, not on the volatility risk; with the introduction of derivatives, the investor can take advantage of the risk-return trade-off provided by volatility risk or exploit the time-varying nature of investment opportunity set.

Liu and Pan solve the dynamic asset allocation problem through three steps:

1. They solve the optimal wealth dynamics;
2. They find the optimal exposure to each of the three risk factors that support the optimal allocation of wealth;
3. They find the optimal positions in stocks and options to obtain the optimal exposure to those risk factors.

The price dynamics for the *i*-th derivative security include three parameters that measure the exposure to each risk factors:

- $g_s^{(i)}$ that measures the sensitivity to changes in the stock price. If it is different from zero it means that the option provides exposure to the diffusive price shock B;
- $g_t^{(i)}$ that measures the sensitivity to changes in the stock volatility. If it is different from zero it means that the option provides exposure to the additional volatility risk Z;
- $\Delta g^{(i)}$ that measures the change in the price of derivative for each jump in the underlying stock price. If it is not zero it means that the option provides exposure to the jump risk N.

The first part of the paper shows the result of the use of derivatives as vehicles to stochastic volatility. Liu and Pan assume an economy with volatility risk but no jump risk in which a derivative is necessary to complete the market, and they consider the portfolio choices of both a myopic and non-myopic investor

For a myopic investor the demand of derivative security will be led by the possible benefits in terms of risk-and-return trade-off: if the volatility risk is negatively priced, ($\xi < 0$) the investor will have a short position in options with positive volatility exposure; if the volatility risk is positively priced ($\xi > 0$) the investor will have a long position.

For a non-myopic investor instead, the demand for options depends on the second term of ψ_t^* so, a risk averse investor ($\gamma < 1$) will sell derivative to hedge against uncertainty, while an investor with $\gamma > 1$ will take a long position in volatility to speculate and obtain profits.

The demand for stock is instead related only to the risk-return trade-off and the delta exposure to stock introduced by the derivative (if an investor has a long position in a Call, he is investing a portion of his wealth, the delta of the option, on the underlying stock).

The empirical proof of the model is made under different assumptions; the results shows that the demand for derivatives is driven mainly by the myopic component.

"Under normal market conditions with a conservative estimate of the volatility-risk premium $\xi = -6$ [...] the portfolio improvement from including derivatives is about 14.2% per year in certainty-equivalent wealth for an investor with risk aversion $\gamma = 3$".

In the last part of the paper the authors suppose an economy with jump risk and no volatility risk; the role of derivatives is to separate jump risk from diffusive price risk, so the investor will allocate less of his wealth to derivative security the lower will be its ability to do that.

Liu and Pan set different assumptions and apply the model above in three scenarios ($\mu = -10\%$ jumps once every 10 years; $\mu = -25\%$ jumps once every 50 years; and $\mu = -50\%$ jumps once every 200 years).

Results shown that the jump risk affects the portfolio choices of the stock only investor, that become more and more cautious: its stock allocation shifts from 6.74% to only 2% in the case of a risk aversion coefficient equal to 0.5.

The jump risk taken into consideration by Liu and Pan is related exclusively to the price of the underlying.

Branger, Schlag and Schneider, in their paper "*Optimal Portfolios when Volatility can Jump*" introduce a more complex model for an investor with constant relative risk aversion with one more risk factor: they assume that jumps are possible not only in the asset price, but also in volatility.

The authors separate the overall demand for risk factors into two main components:

- a speculative one that is the investor's desire to earn the risk premium;
- a hedging component, that was absent in the paper of Liu and Pan in relation to the jump risk, to protect the investor from unfavourable changes in the opportunity set.

The authors define the model by the setting the objective of the investor as the maximization of his expected utility of terminal wealth that is exposed in terms of the exposure to the fundamental risk factors $B^{(1)}$, $B^{(2)}$ and N instead of portfolio weights as below:

$$dW_t = rW_t + \theta_t^{B1} W_t \left(\eta^{B1} V_t dt + \sqrt{V_t} dB_t^{(1)} \right) + \theta_t^{B2} W_t \left(\eta^{B2} V_t dt + \sqrt{V_t} dB_t^{(2)} \right)$$
$$+ \theta_t^N W_t \mu_X [(\lambda^P - \lambda^Q) V_t dt + dN_t - \lambda^P V_t dt]$$

with the coefficients of each risk exposure set as θ_t^{B1}, θ_t^{B2}, and θ_t^N.

The optimal position in those coefficients is:

$$\theta_t^{*B1} = \frac{\eta^{B1}}{\gamma} + \rho \sigma_V H(\tau)$$

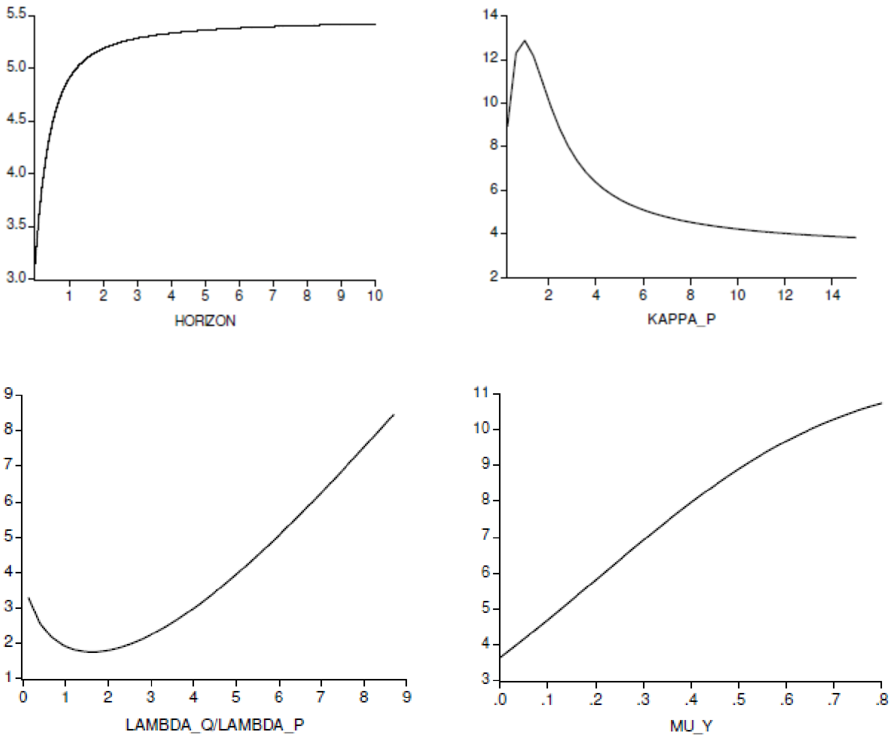
$$\theta_t^{*B2} = \frac{\eta^{B2}}{\gamma} + \sqrt{1 - \rho^2} \sigma_V H(\tau)$$

$$\theta_t^{*N} = \frac{1}{\mu_X} \left[\left(\frac{\lambda^P}{\lambda^Q} \right)^{\frac{1}{\gamma}} - 1 \right] + \frac{1}{\mu_X} \left(\frac{\lambda^P}{\lambda^Q} \right)^{\frac{1}{\gamma}} \left[e^{H(\tau)\mu_Y} - 1 \right], \quad 1 + \theta_t^{*N} \mu_X \geq 0$$

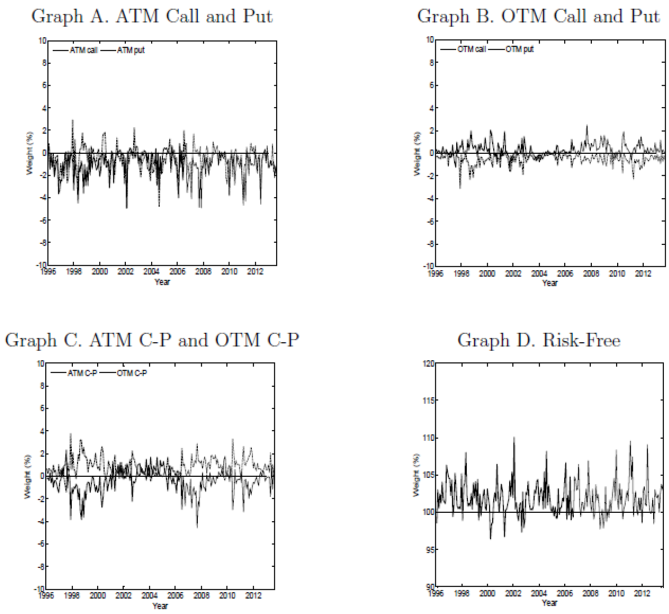
The numerical results of the economic benefits of derivatives are exposed as in the case of Liu and Pan (2003) as the portfolio improvement R^W (i.e. the annualized percentage difference in certainty equivalent wealth):

$$R^W = \frac{\ln\left(W / \widehat{W} \right)}{T}$$

The main results are illustrated in the graphs below, in which it is possible to observe the portfolio improvement from including derivatives with respect to four different variables: horizon time, mean reversion speed, jump intensity and jump size.



The last paper analysed in the second chapter, "*Optimal Option Portfolio Strategies: Deepening the Puzzle of Index Option Mispricing*" by Faias and Santa-Clara, is based on a simple optimization method, the optimal option portfolio strategies "OOPS", focused on the maximization of a utility function that penalizes the negative skewness and high kurtosis of returns. The dynamic option strategy is applied to the S&P 500 Index, based on a risk-free security and four monthly options held until maturity, two call and tow put, two ATM and two OTM, selected from a range of possible strike prices, with weight of each option defined month by month to maximize the CRRA utility function, as it is possible to observe in the graph below.



The model is applied in-sample from 1950 to 1995 and out-of-sample from 1996 to 2013; the results show that, assuming no transaction costs, the OOPS is a better alternative to both the direct investment in the S&P 500 Index and in the risk-free asset.

	Ann Mean	Ann Std Dev	Min	Max	Skew	Exc Kurt	Ann CE	Ann SR
S&P 500	6.4%	18.2%	−29.5%	18.6%	−1.24	5.57	−1.31%	0.29
OOPS	16.1%	18.4%	−18.4%	20.8%	0.35	2.03	9.94%	0.82

The last part of this thesis is a back test process developed with Python 3.7 and applied to two of the most popular option strategies, the Straddle and the Iron Condor, a third strategy, called the Mixed strategy, that consists in a long Straddle plus a long Iron Condor (the two short options in the Iron Condor will reduce the total price of the structure), and finally the analysis of the performance offered by the V strategy, a simple strategy derived from the selection between a long Straddle or a long Iron Condor in each period based on the difference between the implied volatility and the historical volatility: if the implied volatility on a 30 days basis (i.e. the market expectations regarding the possible future fluctuations in the underlying asset for the next 30 days) is higher than the realized volatility of the last year, this means that investors in the market are expecting a potential big move of the underlying, so the V strategy will select the long Straddle position for that period, otherwise, if IV is less than HV, the market is expecting small movements, and the V strategy will select the Iron Condor, to buy a sort of insurance against potential big fluctuations and maximize the payoff in case the price underlying remains within a restricted price range.

Each of the four strategy analysed are made with quarterly options on the Italian FTSE MIB Index; the back test is performed during a period of ten years, from the first quarter of 2010 (starting the Monday after the third Friday of December 2009) to the last quarter of 2019.

The results of the Straddle, Iron Condor, Mixed strategy and V strategy are compared with each other and with the FTSE MIB Index performance, in terms of mean return, median, volatility, skewness and kurtosis. The graph below is a summary table of the key statistics measured in the Python model by the definition of the *KeyStatistics()* function.

	Mean return	ann.	Median	Volatility	Min	Max	Skew	Kurt
FTSE MIB	0,88%	0,09%	0,57%	10,36%	-25,57%	19,49%	-0,29	0,32
Straddle	-13,88%	-1,48%	42,85%	74,27%	-98,95%	232,99%	1,15	1,10
Iron Condor	2,63%	0,26%	-2,99%	14,29%	-25,04%	32,87%	0,24	-0,98
Mixed	13,30%	1,26%	-6,15%	73,25%	-75,28%	277,80%	1,43	2,21
V Strategy	9,63%	0,92%	2,05%	30,22%	-19,31%	185,98%	4,94	26,07

As it is possible to observe, the Straddle has the worst values, both in terms of volatility and mean return; it is a strategy that cannot be applied constantly in the long term, as the Mixed one, that shows a massive weight of the Straddle component with respect to the Iron Condor.

The V strategy allows investor to obtain a good mean return, higher than that of the Iron Condor, with a lower volatility than the Straddle or the Mixed strategy. The negative aspects of all those option combinations is that they show very high values for skewness, with returns distributed to the left of the average value, and so leptokurtic distributions with hump to the left.

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