Department of Economics and Finance

Major in Finance

Portfolio Optimization: a comparison among Markowitz, Black - Litterman and Robust Optimization approach

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Introduction

Investing in financial markets has always been seen as a kind of leap in the dark by people who do not work in the industry. Until the 1950s it was an activity based on naive strategies, intuition, and experience for practitioners. With the advent of the Markowitz model (MV), things have changed. He introduced the first quantitative model for the selection and construction of investment portfolios based on the maximization of expected returns and the simultaneous minimization of risk. From this model, despite its great limitations, all quantitative approaches have developed up to the present day.

One such model is the Black - Litterman model (BLM) developed in the 1990s by F. Black and R. Litterman. This model can reduce MV deficits through reverse optimization starting from a reference market portfolio. It introduces another important improvement, the possibility of introducing in the mathematical model the own views on the future trend of the assets we are considering, making the model versatile with the possibility of adapting to the needs of investors. Another model developed from the Markowitz model as a starting point is the Robust Optimization approach (RO) introduced by Soyster (1973). It differs from the BLM, because it does not intervene on the prior inputs but takes into account the uncertainty in the optimization objective function, giving a more consistent and systematic improvement to the limitations of MV.

The purpose of this thesis is to measure and compare the performance of these three models using the sectors that make up the S&P 500 as assets. We will use a ten-year data sample from 3rd January 2005 to 31st December 2014, called in-sample period, to create the optimal portfolios calculated by the three models. Afterwards, from 2nd January 2015 to 30th September 2020, called out-of-sample period, the three models will be compared with each other and with the market through multiple performance indicators used by the industry taking into consideration three different types of investor depending on the risk appetite. Two alternative approaches will be used: a no-rebalance approach, purely theoretical, in which the optimal portfolios of the various models will not be modified throughout the analysis period. A rebalance approach, in which, every 5 days, the optimal portfolios weights will be recalculated taking into account the new information deriving from the market data, as happens in reality.

From the results of the empirical chapters, we will see how the use of the two more complex models, BLM and RO, will lead to better results with differences depending on the risk appetite. In particular, we will see how BLM has better performance when we consider the lowest risk appetites and how it will not always be able to implement the views in the best way while RO will be characterized by constant stability which however costs something in terms of return. As expected, the rebalance approach increases the possibility of the models to perform better in the analysis period, thanks to a huge increment from return point of view and a contemporary stability of volatility and risk exposure. In particular, the first three chapters are dedicated to the theoretical explanation of MV, BLM and RO respectively. The fourth chapter describes the data used and the procedure followed for the analysis. In chapter five the analysis of the performance of the models will be presented following the two alternative approaches previously mentioned. The last chapter will be devoted to conclusions.

1 From Modern Portfolio Theory (MPT) to CAPM

1.1 Portfolio construction process

As a first thing, we try to give a definition to a concept that is one of the bases of the entire economic world: “What is investment?” We can start from away to offer a solution to this question. For most of our life, we will be earning and pocket money. Additionally, if the current income exceeds current consumption desires, we will tend to save lots of the excess; we call it savings. With our savings we can do almost two things: the first possibility is to place the money blocked until a moment in the future when consumption desires will exceed current income, with a wrong confidence that the amount will well worth the same; the second possibility is to give up the immediate possession of these saving for a future larger amount of cash which will be available for future consumption. At this point of our thought process, we can provide a simplistic definition of investment: what we do with our savings to make them increase over time.

From our discussion, we can give a formal definition of investment: specifically, it is the present commitment of dollars for a period to derive future payments that will compensate the investor for a) the time the funds are committed, b) the expected rate of inflation during this time period, and c) the uncertainty of the future payments. This definition includes all kinds of “investor”, like an individual, a government, a mutual fund or a corporation; similarly, it includes all types of investments, like investments by corporations in plant and equipment and investments in stocks, bonds, commodities etc... In all cases, the investor is trading a known dollar amount today for some expected future stream of payments that will be greater than the current dollar amount today.

The ways in which we can invest are infinite and it is very important to define a structured four-step portfolio management process. The first step consists in the construction of the policy statement. It is a road map, indeed in it the investors specify the kinds of risks they are willing to take and their investment goals and constraints; in the second step the portfolio manager studies current financial and economic conditions and forecasts future trends. The latter and therefore the investor’s needs will jointly determine the investment strategy; the third step is the Asset Allocation process. The investor's policy statement and financial market forecasts are used as inputs of the investment strategy formulated to determine how to allocate our wealth available among different countries, asset classes and securities. An asset class is formed from securities that have similar characteristics. The fourth step is the continual monitoring. It consists in overseeing the policy statement, the capital market conditions and the investment strategy to verify if are necessary some changes. A crucial component of the continual monitoring is the evaluation of the portfolio's performance and make a comparison with the requirements listed in the policy statements. At the end of the four steps, it is essential to revisit all the steps periodically, because the portfolio management process must be a continuous procedure.

About Asset Allocation process, we can make a distinction between two alternatives dependent on the investor's time horizon. Strategic Asset Allocation is suitable for investors with a long-term horizon because the portfolio constructed is typically reviewed annually but, in normal periods, it is not suffering by recent market changes and has a horizon of 5 years more or less. On the other hand, Tactical Asset Allocation is acceptable for investors with short- and medium-term horizon because the portfolio is modified more frequently to adapt it to temporary market changes. We can conclude that the choice between them is driven by the conditions of the market: in a volatile market is recommended the usage of the TAA, while during a trending and predictable market is advisable the utilization of the SAA. Nothing stops to mix the two strategies to create a portfolio that meets at the best the needs of the investor.

1.2 Markowitz and the Mean-Variance framework

1.2.1 Model description

Until the primary development of optimization techniques within the 1950's, portfolios were constructed with the so-called Naïve approach making only an analysis asset by asset. It consists in the creation of a portfolio following a criterion supported the overall concept of diversification and intuition about the future. A typical naïve strategy is "1/N Asset Allocation" (N is clearly the number of assets regarded), where the resulting portfolio consists by an equal amount of the assets that the investor is decided to take in consideration.

It is important clarify some basic and general concepts about portfolio theory before the treatment of the cornerstone of it. The first basic assumption is that investors want to maximize the returns from the entire set of investments for a given level of risk, so it implies that the portfolio should include all of your assets and liabilities because the returns from all of them interact and this relationship is extremely important.

Portfolio theory assumes (with the support of the empirical evidence of many studies) also that investors are basically risk averse, meaning that, given a choice between two assets with equal rates of return, they are going to select the asset with the lower level of risk. At this point we can give a general definition of risk with two different expressions that have the same meaning: for many investors, risk means the uncertainty of future outcomes, or the probability of an adverse outcome.

The cornerstone, before mentioned, of all the thousands different models used today is the Modern Portfolio Theory (MPT), that was developed in 1952 by Harry Markowitz.

It is supported by several assumptions regarding investor behaviour: 1) investors consider each investment alternative as being represented by a probability distribution of expected returns over some holding period; 2) investors maximize one-period expected utility and their utility curves demonstrate diminishing marginal utility of wealth; 3) investors estimate the risk of the portfolio on the basis of the variability of expected returns; 4) investors base decisions on expected return and risk, so their utility curves are a function only of them; 5) for a given risk level, investors prefer higher returns to lower returns and, similarly, for a given level of expected return, investors prefer less risk to more risk.

In general terms, MPT can be described as an investment framework for the choice and construction of investment portfolios based on the maximization of expected returns and the simultaneous minimization of investment risk through the diversification of the assets that compose the portfolio. Markowitz introduced a quantitative model to carry out MPT, called Mean-Variance Optimization; the elements taken in consideration for the optimization process are, respectively:

- expected returns of the assets (for asset i, μ_i)
- risk of the assets, measured by the variance of the expected returns (for asset i, σ_i^2)

It is necessary to highlight that, between them, the key element is risk because it is the driver of the returns indeed investors require higher returns if risk is higher and vice versa. We can say that MPT face a risk-return trade off in creating a portfolio.

Calculation of the expected returns of the assets is the first step in Markowitz' model. They can be defined as "the average of a probability distribution of possible returns". In a simple way, they can be viewed as the historic averages of the assets returns over a determined period of time. Calculations for a portfolio of securities involve calculating the weighted average of the individual expected returns.

$$E(R_p) = \sum_{i=1}^{n} w_i * \mu_i \tag{1}$$

The second step consists in calculating the risk of the assets. Before the introduction of the measures to calculate it, we have to underline that the risk of a security can be divided in two basic components: systematic risk and idiosyncratic risk. The former is the risk that virtually affects all securities with a different impact, so it cannot be eliminated; inflation, interest rates, unemployment levels are all examples of it. The latter is determined by risk factors that specifically affect a single asset or a narrow group of assets, so it can be significantly reduced by the diversification of securities within a portfolio; firm's credit rating, negative press reports about a business or a strike affecting a particular company are some examples of idiosyncratic risk factors.

The two most common measures of risk are variance and standard deviation. Variance is a "measure of the squared deviations of an asset's return from its expected return". Extending the calculation to a portfolio, the total variance is always lower that a simple weighted average of the individual asset variances because when many assets are held together in a portfolio, assets that suffer a value decreasing are often offset by assets that gain value, thereby minimizing risk.

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j \tag{2}$$

About standard deviation, can be defined as the deviation of an asset return from the expected return. It is the square root of the variance.

To understand (2) is necessary a brief dissertation about the concepts of covariance and correlation. Covariance is an absolute measure to which two variables move together relative to their individual mean values over time; in portfolio analysis we usually are concerned with the covariance of returns. For two assets, i and j, we define the covariance of rates of return as follows:

$$Cov_{ij} = E\{[R_i - E(R_i)][R_j - E(R_j)]\} \tag{3}$$

If the returns are positively related to each other, their covariance will be positive; if negatively related, the covariance will be negative; if they are unrelated, the covariance will be zero.

Covariance is affected by the variability of the two individual return indexes, so we want to standardize it as follows:

$$\rho_{ij} = \frac{Cov_{ij}}{\sigma_i \sigma_j} \tag{4}$$

The result is the correlation coefficient, which can vary only between -1 and +1. A value of +1 indicates a perfect positive linear relationship between the two asset returns. A value of -1 indicates a perfect negative relationship between the two asset returns. A value of 0 indicates the absence of relationship between the two asset returns.

Correlation concept is fundamental to understand a cornerstone of MPT, namely the diversification effect. Markowitz was able to prove that putting together assets that have different characteristics (different asset classes, different sectors, different geographical areas etc.) which react differently to same events (i.e. they have imperfect correlation, positive or negative), we can maximize returns and minimize risk.

An immediate conclusion that can arise is that will be sufficient to construct a portfolio composed by very diversified assets to totally eliminate the risk, but it is not so simple. Markowitz argues that diversification cannot eliminates all risk because, as discussed earlier, investors face two main sorts of risk: systematic risk and idiosyncratic risk. The latter is the part of the risk equation that can be reduced or eliminated, while the former cannot be nor reduced nor eliminated since it derives from external factors, as we have explained before.

At this point we can describe how the optimization process works in a qualitative way, and after we will describe it in mathematical terms. Going back to our framework of n assets available, we can invest our hypothetical budget across them in many ways, assigning all possible weights to all assets. Every portfolio constructable in this way is named achievable. Plotting all the achievable portfolios on the bases of their risk and return we obtain a graph like this:

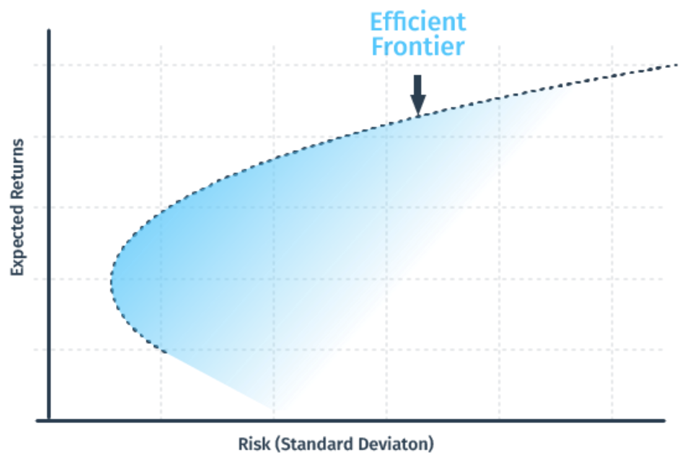


Figure 1: Efficient Frontier. Source:
https://meetinvest.com/insights/how/value-create-portfolio

Achievable portfolios are represented by those situated on the frontier and within the blue area, but only those on the frontier are efficient because they produce the maximum expected return for a given risk level or, the opposite way around, they face the minimum risk for a given expected return. As a consequence, rational investors will construct one of the portfolios that lie on the Efficient Frontier.

1.2.2 Utility functions and optimization problem

As we have seen, on the Efficient Frontier there are many efficient portfolios. What is the criterion to settle on the best point to be on it? The answer is that we need additional information represented by a utility function. Utility can be defined as the personal satisfaction of an individual, so utility functions have the objective to give a value to the degree of satisfaction.

In our case the satisfaction is a function of two inputs, expected return and risk. Now it is clearer the concept of risk-return trade off because it is very intuitive that the satisfaction increases if return is higher but at the same time higher return imply more risk and vice versa.

In the MPT framework a utility function: 1) provides a "rule" for the trade-off between risk and return; 2) using such rule, for each efficient portfolio computes their utility through the expected returns and variances; 3) the efficient portfolio for the investor will be the one with the highest utility.

In financial literature there are different utility functions, but the most common is the exponential utility function defined as follows:

$$U(W_{t+1}) = -exp(-\lambda W_{t+1}) \tag{5}$$

W_{t+1} is next period's wealth while $\lambda \geq 0$ is the absolute risk aversion coefficient.

Now we can treat the optimization process in mathematical terms. We present the problem as a risk minimization for a given target return using matrix notation since we are in a multivariate framework.

Single-Period Analysis

- m risky assets: $i = 1, 2, \ldots, m$
- Single-Period Returns: m-variate random vector

$$\mathbf{R} = [R_1, R_2, \ldots, R_m]'$$

- Mean and Variance/Covariance of Returns:

$$\mathrm{E}[\mathbf{R}] = \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}, Cov[\mathbf{R}] = \mathbf{\Sigma} = \begin{bmatrix} \Sigma_{1,1} & \cdots & \Sigma_{1,m} \\ \vdots & \ddots & \vdots \\ \Sigma_{m,1} & \cdots & \Sigma_{m,m} \end{bmatrix}$$

- Portfolio: m-vector of weights indicating the fraction of portfolio wealth held in each asset

$$\mathbf{w} = (w_1, \ldots, w_m) : \sum_{i=1}^{m} w_i = 1$$

- Portfolio Return: $R_w = \mathbf{w}'\mathbf{R} = \sum_{i=1}^{m} w_i R_i$ a r.v. with

$$\alpha_{\mathbf{w}} = E[R_w] = \mathbf{w}'\alpha$$
$$\sigma_{\mathbf{w}}^2 = var[R_w] = \mathbf{w}'\mathbf{\Sigma}\mathbf{w}$$

Evaluate different portfolios $\mathbf{w}$ using the mean-variance pair of the portfolio: $(\alpha_{\mathbf{w}}, \sigma_{\mathbf{w}}^2)$ with preferences for

- Higher expected returns $\alpha_{\mathbf{w}}$
- Lower variance $var_{\mathbf{w}}$

Risk minimization: For a given choice of target mean return α_0, choose the portfolio $\mathbf{w}$ to

$$\text{Minimize: } \frac{1}{2}\boldsymbol{w}'\mathbf{\Sigma}\boldsymbol{w}$$
$$\text{Subject to: } \mathbf{w}'\boldsymbol{\alpha} = \alpha_0$$
$$\mathbf{w}'\mathbf{1}_m = 1$$

Solution: Apply the method of Lagrange multipliers to the minimization problem subject to linear constraints:

- Define the Lagrangian

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}\boldsymbol{w}'\mathbf{\Sigma}\boldsymbol{w} + \lambda_1(\alpha_0 - \mathbf{w}'\alpha) + \lambda_2(1 - \boldsymbol{w}'\mathbf{1}_m)$$

- Derive the first order conditions

$$\frac{\partial L}{\partial \boldsymbol{w}} = \mathbf{0}_m = \mathbf{\Sigma}\boldsymbol{w} - \lambda_1\alpha - \lambda_2\mathbf{1}_m$$
$$\frac{\partial L}{\partial \lambda_1} = 0 = \alpha_0 - \mathbf{w}'\alpha$$
$$\frac{\partial L}{\partial \lambda_2} = 0 = 1 - \mathbf{w}'\mathbf{1}_m$$

- Solve for $\mathbf{w}$ in terms of λ_1, λ_2:

$$\mathbf{w}_0 = \lambda_1 \mathbf{\Sigma}^{-1}\alpha + \lambda_2 \mathbf{\Sigma}^{-1}\mathbf{1}_m$$

- Solve for λ_1, λ_2 by substituting for $\mathbf{w}$:

$$\alpha_0 = \mathbf{w}'_0\alpha = \lambda_1(\alpha'\mathbf{\Sigma}^{-1}\alpha) + \lambda_2(\alpha'\mathbf{\Sigma}^{-1}\mathbf{1}_m)$$
$$1 = \mathbf{w}'_0\mathbf{1}_m = \lambda_1(\alpha'\mathbf{\Sigma}^{-1}\mathbf{1}_m) + \lambda_2(\mathbf{1}'_m\mathbf{\Sigma}^{-1}\mathbf{1}_m)$$
$$\Rightarrow \begin{bmatrix} \alpha_0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} with$$
$$a = (\alpha'\mathbf{\Sigma}^{-1}\alpha), b = (\alpha'\mathbf{\Sigma}^{-1}\mathbf{1}_m), and\ c = (\mathbf{1}'_m\mathbf{\Sigma}^{-1}\mathbf{1}_m)$$

With the given values of $\lambda_1$and λ_2, the solution portfolio

$$\mathbf{w}_0 = \lambda_1 \mathbf{\Sigma}^{-1}\alpha + \lambda_2 \mathbf{\Sigma}^{-1}\mathbf{1}_m$$

has minimum variance equal to

$$\sigma_0^2 = \mathbf{w}'_0\mathbf{\Sigma}\mathbf{w}_0$$

Solving the problem as a risk minimization with a target expected return α_0, we have obtained the less risky portfolio that give us α_0.

1.2.3 Markowitz's limitations and improvement possibilities

Despite its importance, MPT is based on assumptions that are faraway from the reality and each of them compromises MPT. The main criticism regards inputs used in the optimization process: they are estimated using historical data (the model also supposed that the estimation is perfect) and they have the tendency to create portfolios called "estimation error maximizers" that overweight assets with high estimation error in returns and low estimation error in risk and vice versa (Michaud 1989). So, the process can be really optimal only if the true population parameters are known.
Additionally, optimal portfolios are very sensitive to small changes in the inputs and the immediate consequence is that they are highly concentrated in few assets, all the contrary to the diversification concept, milestone of MPT!
Other more technical criticisms regarding the assumptions of the model are:

1) Investor "Irrationality" – the assumption that investors are rational and seek to maximize returns while minimizing risk is contradicted by the observation of investors that routinely go for "hot" sectors, and markets regularly boom or bust due to speculative excesses (Morien).

2) Higher Risk = Higher Returns - the assumption that investors only accept higher risk if compensated by higher expected returns is not absolutely true because investors can have utility functions that may overweight distribution of returns.

3) Perfect Information – MPT assumes that investors receive all information during a prompt and complete manner while, actually, financial markets comprise situations of information asymmetry and insider trading.

4) No Taxes or Transaction Costs – MPT do not include taxes or transaction costs while real investment products are subject to both and factoring them may change the optimum portfolio selection.

It is clear that there is the need to modify the model, in order to promote a greater diversification and overtaking the assumption that the inputs are completely certain. To realize this, there are two possibilities available:

1) Heuristic Approach, that works on the adjustment of the optimization process
2) Bayesian Approach, that focuses on the adjustment of the estimated inputs (above all the expected returns)

About the first, one choice can be adding additional constraints (by limiting the weight that can be given to any singular asset) to force the optimization through an higher diversification; another choice is called $Resampling^{TM}$ that works in this way: firstly, a thousands of investment scenarios are created; secondly, the simulated expected returns, volatilities and correlation coefficients are used as input of a new Markowitz optimization; the final step consists in repeating the second step for every scenario created (i.e. thousands of Efficient Frontiers) and we will have the final Efficient Portfolios, called Resampled Portfolios, that have the composition of the "average" efficient portfolio. The third choice consists in running a Robust Optimization; it assumes that the estimated expected returns are random variables and seeks to create the optimal portfolio even when the realized values of inputs deviate from the estimated ones within some given set. It will be treated in the third chapter.

About the second, that it is based on Bayesian statistics, the most common and widely used is The Black-Litterman Model that will be treated in the next chapter of this work.

1.3 Capital Market Theory (CMT)

Capital market theory extends Markowitz portfolio theory by developing a model for pricing all risky assets. This development depends on the existence of a risk free asset which will lead to the designation of the market portfolio, a set of all the risky assets in the marketplace.
We will start, as usual, with the assumptions: 1) all investors are Markowitz-efficient in that they seek to invest in tangent points on the efficient frontier (the exact location of the portfolio selected will depend on the individual investor's risk-return utility function); 2) investors can borrow or lend any amount of money at the risk free rate of return; 3) all investors have homogeneous expectations; 4) all investors have the same one-period time horizon; 5) all investments are infinitely divisible; 6) there are no taxes or transaction costs involved in buying or selling assets; 7) there is no inflation or any change in interest rates, or inflation is fully anticipated; 8) capital markets are in equilibrium.

Now we will see what happens when we introduce the risk free asset into the risky world of the Markowitz portfolio model. First of all, we can demonstrate that the covariance of the risk free asset with any risky asset or portfolio of assets will always equal zero. Recalling equation (3), if we assume that the asset i is the risk free asset, because the returns of it are certain ($\sigma_{RF} = 0$), $R_i = E(R_i)$ during all periods; thus, $R_i - E(R_i)$ will equal zero and the final product will equal zero.

About expected return, a portfolio that includes a risk free asset with a collection of risky assets (Portfolio M) is the weighted average of the two returns:

$$E\left(R_{port}\right) = w_{RF}(RFR) + (1 - w_{RF})E(R_M) \tag{6}$$

w_{RF} = proportion of the portfolio invested in the risk free asset
$E(R_M)$ = expected rate of return on risky Portfolio M

About risk, because the correlation between the risk free asset and any risky asset is zero, the standard deviation of a portfolio that combines the risk free asset with risky assets is the linear proportion of the standard deviation of the risky asset portfolio:

$$\sigma_{port} = (1 - w_{RF})\sigma_M \tag{7}$$

Now we are able to develop the risk-return relationship between $E(R_{port})$ and σ_{port} by using algebraic manipulations, called Capital Market Line (CML):

$$
\begin{aligned}
E(R_{port}) &= (w_{RF})(RFR) + (1 - w_{RF})E(R_M) + \{RFR - RFR\} \\
&= RFR - (1 - w_{RF})RFR + (1 - w_{RF})E(R_M) \\
&= RFR + (1 - w_{RF})[E(R_M) - RFR] \\
&= RFR + (1 - w_{RF})\left\{\frac{\sigma_M}{\sigma_M}\right\}[E(R_M) - RFR]
\end{aligned}
$$

$$E(R_{port}) = RFR + \sigma_{port}\left[\frac{E(R_M) - RFR}{\sigma_M}\right] \tag{8}$$

Equation (8) holds for each combination of the risk free asset with any collection of risky assets. Now we assume that Portfolio M is the set of risky assets that maximize the risk premium, so it is called market portfolio and, by definition, it contains all risky assets held in the marketplace and receives the highest level of expected return per unit of risk for any available portfolio of risky assets. CML represents the set of portfolio possibilities that dominates all other feasible combinations that investors could form, so it represents a new efficient frontier that combines the Markowitz efficient frontier of risky assets with the possibility to invest in the risk free asset.

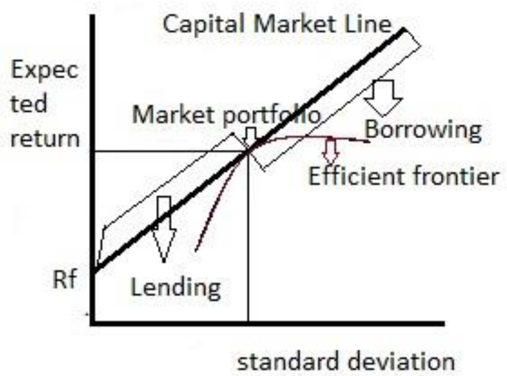


Figure 2: Combination of Efficient Frontier with CML.
Source: https://financenumericals.blogspot.com/2018/01/what-is-capital-market-line-cml.html

We have additional risk-return possibilities running along the CML. An investor may want invest a part of his/her money in the risk free asset (i.e. lend at the RFR) and the rest in the risky Portfolio M, or, alternatively, borrow at the RFR and invest these funds in the risky asset portfolio.

We can conclude that the CML leads all investors to invest in the same risky asset Portfolio M, so they only differ regarding their position on the CML, which depends on their risk preferences. Specifically, to be somewhere on the CML efficient frontier, an investor initially decides to invest in the market Portfolio M (investment decision); then, based on his/her risk preferences, the investor makes a separate financing decision either to borrow or to lend to achieve the preferred risk position on CML. Tobin (1958) called this division of the investment decision from the financing decision the separation theorem.

Unfortunately, CMT is an incomplete explanation for the relationship that exists between risk and return. We recall that the CML defined the risk as the total volatility σ of the investment. However, since investors cannot expect to be compensated for any portion of risk that they could have diversified (idiosyncratic risk), the CML must be based on the assumption that investors only hold fully diversified portfolios for which total risk and systematic risk are the same. The limitation of the CML is that cannot provide an evidence for the risk-return trade off for individual risky assets because the standard deviation of them include an amount of unique risk.

1.4 Capital Asset Pricing Model (CAPM)

At this point we have the evolution of CMT in the Capital Asset Pricing Model (CAPM), whose allows investors to evaluate the risk-return trade off for both diversified portfolios and individual assets. CAPM redefines the relevant measure of risk from total volatility to only the non-diversifiable portion of the total volatility (systematic risk). This measure is named β coefficient and it represents the systematic risk level of a security compared to that of the market portfolio (by definition, it has a β of 1).

From equation (8) we are able to derive the mathematical formula of CAPM, extending the expression to allow for the evaluation of any individual risky asset i. To do this the logical reasoning is to replace σ_{port} with that of the single security σ_i. This approach overstates the relevant level of risk in the i-th security because it does not take under consideration how much of that volatility the investor could diversify by combining that asset with others. A solution is including only the portion of risk in security i that is systematically related to the risk in the market portfolio, multiplying σ_i

by the correlation coefficient between the returns to security i and the market portfolio r_{iM}. The resulting formula is the following:

$$E(R_i) = RFR + (\sigma_i r_{iM})\left[\frac{E(R_M) - RFR}{\sigma_M}\right]$$

Rearranging the terms:

$$E(R_i) = RFR + \beta_i[E(R_M - RFR)] \quad (9)$$

From equation (9) we can see that, instead of than calculate a different risk premium for every singular security that exists, the CAPM states that only the overall market risk premium $(E(R_M) - RFR)$ matters and this quantity can be adapted to any risky asset by scaling it up or down according to asset's beta β_i.

There are two ways to calculate the systematic risk (β_i). First, it can be calculated directly from the following formula:

$$\beta_i = \left(\frac{\sigma_i}{\sigma_M}\right)(r_{iM}) = \frac{Cov(R_i, R_M)}{\sigma_M^2} \quad (10)$$

Alternatively, β_i can be estimated as the slope coefficient in a linear regression between the security's returns (R_{it}) over time and the returns of the market portfolio (R_{Mt}):

$$R_{it} = \alpha_i + \beta_i(R_{Mt}) + \varepsilon_{it} \quad (11)$$

α_i = intercept of the regression
ε_{it} = random error term that accounts for the idiosyncratic risk of the security i

The graphical representation of equation (9) is called Security Market Line (SML):

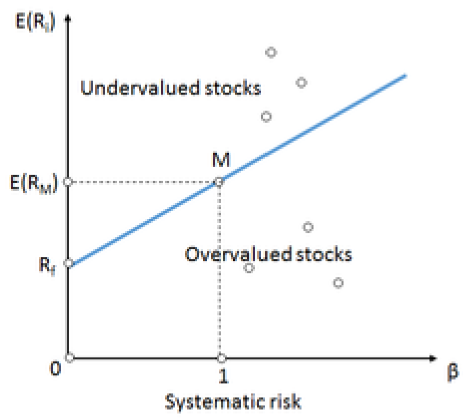


Figure 3: Security Market Line (SML). Source: https://en.wikipedia.org/wiki/Security_market_line

There are two differences between the CML and the SML.
First, the CML measures risk by the standard deviation (total risk) while the SML considers only the systematic component. Second, as a consequence of the first point, the CML can be applied only to portfolio holdings that are already fully diversified, whereas the SML can be applied to any individual asset or collection of assets.

In equilibrium, all assets and all portfolios of assets should plot on the SML because they should be priced in order that their estimated rates of return are consistent with their levels of systematic risk. Any security plotted above the SML, is considered undervalued because it implies that you forecast receiving a rate of return on the security that is above the required rate of return of the asset based on its systematic risk. On the other hand, the security is considered overvalued.
This difference between estimated return and expected return is called stock's expected alpha (α). If the stock is undervalued α is positive, if the stock is overvalued α is negative while if the stock is on the SML α is zero.

1.4.1 CAPM limitations

Although CAPM was a very good improvement in portfolio modelling, it is not immune to many criticism and limitations. First of all, CAPM has many unrealistic assumptions that undermine its correspondence to the reality. Secondly, Roll (1977) put in evidence the problem that the market portfolio at the core of the model is theoretically and empirically elusive, indeed is not clear which assets can be excluded from the market portfolio and, however, data availability limits the assets that are included.
Additionally, starting in the 1970s, many empirical works challenge the structure of CAPM and its functionality. Specifically, evidence highlights that much of the variation in expected return is unrelated to market beta. Basu's (1977) put in evidence that when common stocks are sorted on earning-price ratios, future returns on high E/P stocks are higher than predicted by CAPM; Banz (1981) reports a size effect when stocks are sorted on market capitalization, average returns on small stocks are higher than predicted by the CAPM; Statman (1980) and Rosenberg, Reid and Lanstein (1985) document that stocks with high book-to-market equity ratios (B/M) have high average returns that are not predicted by CAPM. More recent work by Fama and French (1992), demonstrated that

“value” stocks (those with high B/M) tend to produce larger risk-adjusted returns than “growth” stocks (those with low B/M).

In these four contradictions of the CAPM there is a common factor: ratios involving stock prices have information about expected returns missed by market betas.

Concluding, if betas are not enough to explain expected returns, the market portfolio is not efficient and so the majority of applications of CAPM are invalidate.

2 Black-Litterman Model

2.1 Bayesian background

Before the treating of Black-Litterman Model, we recognize the necessity about an introduction regarding the Bayesian approach because it represents the mathematical ground of the model.
In general, the Bayesian approach works in this way:

- A model for data analysis is defined that is thought to be suitable for the problem
- It is specified a prior probability distribution on the model parameters, reflecting our knowledge or beliefs about likely values of these parameters
- We look at the data we have collected, and we compute how likely our data are for different assumed values for the parameters; this yields the likelihood function that has to be computed making use of the specified model and its assumptions
- The combination of the prior distribution (representing the information outside the collected data) and the likelihood function (representing the information inside the collected data) returns the posterior probability distribution for the parameters; it reflects how likely different parameter values are true, after taking into account one's prior beliefs and the information within the collected data.

All the process can be represented in a mathematical way with the most important result in this statistic's field, Bayes' Theorem, formulated in this manner:

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)} \propto p(y|x)p(x) \tag{12}$$

The symbol $\propto$ means "proportional to" and give the possibility to eliminate the denominator of the fraction. In this context, x represents an event, y represents some observed data, $p(x)$ represents the prior probability of x, $p(y|x)$ is the likelihood function (the probability of y given that x is true) and $p(x|y)$ is the posterior probability of x (the new probability assigned to x given that we observed y).
To conclude this brief treatment, this methodology is very important for mainly two reasons that will lead to an overcoming of two limitations of MPT: firstly, observed data became useful for the estimation of future events (in economic terms, we can mix our inputs with our views to obtain

more trustworthy); secondly, it incorporates uncertainty about the true parameter value, so it is a more valid approach since in general we have no exact knowledge about the true parameter value.

2.2 Black-Litterman Model

2.2.1 Introduction

The Black-Litterman Model (BLM), created by Fischer Black and Robert Litterman, is a portfolio construction process which has its roots in MV optimization model and CAPM that overcomes the problem of highly concentrated portfolios, sensibility to inputs and estimation error maximization by using a Bayesian framework.

BLM has two fundamentally key assumptions behind. First, the model assumes that all asset returns follow the same probability distribution (usually normal distribution is selected). Second, variance of the prior and the conditional distribution about the true means of the assets and investor views are unknown.

Canonical BLM provided two significant contributions to the matter of asset allocation. First, it provides an intuitive prior, the equilibrium market portfolio, as a starting point for 'reverse optimization' to get a stable distribution of returns. Second, BLM provides a clear way to specify investors' views on returns and to blend the investors' views with prior information.

To start utilizing the models, investors must obtain implied market returns of the equilibrium market portfolio that are derived from the CAPM model. If them accept the implied returns, they can use the neutral weights given by the BLM to develop their optimal portfolio, but this model makes a step ahead because gives the possibility to investors to adjust the neutral weights according to their views about the market.

BLM expresses the investors' views and market equilibrium in terms of probability distributions. It uses the Bayesian approach, explained before, to develop a probability distribution for the expected returns by using CAPM equilibrium distribution as a starting point and then combining views into the distribution. Using the implied returns from CAPM as the prior and then adding the investors' views, can be obtained a posterior distribution which results in intuitive portfolios with sensible portfolio weights.

2.2.2 Model

The reference model for returns is the base upon which the rest of Black-Litterman model is built. We start with normally distributed expected returns:

$$r \sim N(\mu\ , \Sigma) \tag{13}$$

The fundamental goal of BLM is to model these expected returns, which are assumed to be normally distributed with mean μ and variance Σ.

We define μ, the unknown mean return, as a random variable itself distributed as:

$$\mu \sim N(\pi\ , \Sigma_\pi) \tag{14}$$

π is our estimate of the mean and Σ_π is the variance of the unknown mean, μ, about our estimate.

Another way to view this simple linear relationship is shown in the formula below:

$$\mu = \pi + \varepsilon \tag{15}$$

The prior returns are normally distributed around π with a disturbance value ε. ε is normally distributed with mean 0 and variance Σ_π and is assumed to be uncorrelated with μ.

We can complete the reference model by defining Σ_r as the variance of the returns about our estimate π. From formula (15) and the assumption above that ε and μ are not correlated, then the formula to compute Σ_r is:

$$\Sigma_r = \Sigma + \Sigma_\pi \tag{16}$$

Formula (16) tells us that the proper relationship between the variances is $(\Sigma_r \geq \Sigma, \Sigma_\pi)$.

In the absence of estimation error, e.g. $\varepsilon \equiv 0$, then $\Sigma_r = \Sigma$. As our estimate get worse, e.g. Σ_π increases, then Σ_r increases as well.

The canonical reference model for BLM expected return is:

$$r \sim N(\pi, \Sigma_r) \tag{17}$$

While (13) is the reference formula for the Alternate Reference Model, an alternative to approach the problem.

2.2.3 Equilibrium Returns

BLM starts with a neutral equilibrium portfolio for the prior estimate of returns. The model is based on General Equilibrium theory to state that if the aggregate portfolio is at equilibrium, each sub-portfolio must even be at equilibrium. Ideally, it can be used with any utility function but in practice is used the Quadratic Utility function assuming the existence of a risk free asset, and thus the equilibrium model coincide with the Capital Asset Pricing Model (CAPM) and the neutral portfolio is the CAPM Market portfolio.
Since we are starting with the market portfolio, we will start with a set of weights which are all greater than zero and sum to one. We will constrain the problem assuming that the covariance matrix of the returns, Σ, is known, but in the basic approach is estimated from historical return data.

Here we derive the equations for "reverse optimization" starting from the quadratic utility function:

$$U = w^T \Pi - \left(\frac{\lambda}{2}\right) w^T \Sigma w \tag{18}$$

U = Investors utility (objective function during optimization)
w = vector of weights invested in each asset
Π = vector of equilibrium excess returns for each asset
λ = risk aversion parameter
Σ = covariance matrix of the escess returns for the assets

U is a convex function, so if we maximize it with no constraints, there is a closed form solution. We find the exact solution by taking the first derivative of (18) with respect to w and setting it to zero.

$$\frac{\partial U}{\partial w} = \Pi - \lambda \Sigma \mathrm{w} = 0$$

Solving this for Π yields:

$$\Pi = \lambda \Sigma w \tag{19}$$

The risk aversion coefficient λ characterizes the expected risk-return trade-off; it is the speed at which an investor will forego expected return for less variance. In "reverse optimization" process, the risk aversion coefficient acts as a scaling factor for the reverse optimization estimate of excess returns; the weighted reverse optimized excess returns equal the specified market risk premium, so a higher excess return per unit of risk (larger lambda) increases the estimated excess returns.

One way to find λ is by multiplying both sides of (19) by w^T and replacing vector terms with scalar terms.

$$\lambda = (r - r_f)/\sigma^2 \tag{20}$$

$\mathrm{r} =$ total return on the market portfolio $(\mathrm{r} = \mathrm{w}^{\mathrm{T}}\Pi + \mathrm{r_f})$
r_f = risk free rate
σ^2 = variance of the market portfolio ($\sigma^2 = w^T\Sigma w$)

Many authors specify the value of lambda that they use. Bevan and Wickelmann (1998) describe their process of calibrating the returns to an average Sharpe Ratio based on their experience. Black and Litterman (1992) use a Sharpe ratio closer to 0.5 in the example in their paper.

We can rewrite formula (26) for lambda in terms of the Sharpe Ratio as

$$\lambda = \frac{SR}{\sigma_m} \tag{21}$$

Once we have a value for λ, we plug w, λ and Σ into formula (19) and generate the set of equilibrium returns.

Now we are missing the variance of our estimate of the mean, we need $\Sigma\pi$. Black and Litterman (1992) assumed that the structure of the covariance matrix of the estimate is proportional to the covariance of the returns Σ. More precisely, $\Sigma\pi = \tau\Sigma$, where τ is a scalar indicating the uncertainty of implied returns.

2.2.3.1 Specifying τ

τ is extremely important because controls how distinctly the optimized portfolio may depart from the market portfolio. For very small values ($\tau \rightarrow 0$) the combined returns converge to equilibrium returns and the BL optimized portfolio converges to the market portfolio.
For large values of ($\tau \rightarrow \infty$) the combined returns converge to the "views" and the BL optimized portfolio converges to the portfolio in which the "views" are the underlying return estimates.

The most used method to calibrate τ relies on basic statistics. When estimating the mean of distribution, the variance of the mean estimate will be proportional to the inverse of the number of samples. Given that we are estimating the covariance matrix from historical data, then:

$\tau = \frac{1}{T}$ the maximum likelihood estimator (22)

$\tau = \frac{1}{T-k}$ the best quadratic unbiased estimator (23)

T = number of samples
k = number of assets

In the literature, the values used typically range between 0.025 and 0.05 (Black and Litterman (1992); He and Litterman (1999); Drobetz (2001); Idzorek (2005)).

2.2.4 Investor's views

In the last section we have mentioned the so called "views". They are the main innovation of this model, indeed they allow to the investor to incorporate in the optimization process some outlooks about the future tendency of the expected return of the assets taken in consideration.
It is necessary to specify that the literature on the BLM does not provide a clear answer how to make subjective estimates and the reliability of these estimates. Several studies assume exogenously given estimates (He and Litterman (1999); Lee (2000); Drobetz (2001); Idzorek (2005)) and suggest confidence intervals of the return estimates as a measure of uncertainty (Black and Litterman (1992)). In this section we will describe the process of specifying the investors views on the estimated mean excess returns, defining the combination of the views as the conditional distribution. First, we will require that each view to be unique and uncorrelated with other views, giving to the conditional distribution the property that the covariance matrix will be diagonal, with all off-diagonal entries equal to zero. In this way we improve the stability of the results simplifying the analysis of the model. Second, we will require views to be fully invested, but we do not require a view on any or all assets.

We will represent the k views on n assets using the following matrices:

- P, a $k \times n$ matrix of the asset weights within each view. For a relative view the sum of the weights will be zero, for an absolute view the sum of the weights will be one. About the weights, He and Litterman (1999) and Idzorek (2005) use a market capitalization weightes scheme, whereas Satchell and Scowcroft (2000) use an equal weighted scheme. In practice weights will be a mixture depending on the process used to estimate the view returns.
- Q, a $k \times 1$ vector of the returns for each view.
- Ω, a $k \times k$ matrix of the covariance of the views. Ω is diagonal as the views are required to be independent and uncorrelated. Ω^{-1} is known as the confidence in the investor's views. The i-th diagonal element of Ω is represented as ω_i.

We describe how these matrices work with an example.

There are 3 assets each with its own expected return. The investor has an absolute view on asset 1 which states that the return of it will be 10%, and two relative views, the first that asset 2 will outperform asset 3 by 5% and the second that asset 3 will outperform asset 1 by 3%.

This can be summarized as follows:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \quad ; \quad Q = \begin{bmatrix} 10\% \\ 5\% \\ 3\% \end{bmatrix} \quad ; \quad \Omega = \begin{bmatrix} \omega_{11} & 0 & 0 \\ 0 & \omega_{22} & 0 \\ 0 & 0 & \omega_{33} \end{bmatrix}$$

2.2.5 Estimation model

In BLM, the prior distribution is based on the equilibrium implied excess returns and has this form:

$$P(A) \sim N(\Pi, \tau\Sigma), r_A \sim N(P(A), \Sigma) \tag{24}$$

It represents our estimate of the mean, which is expressed as a distribution of the actual unknown mean about our estimate.

The conditional distribution is based on the investor's views and has this form:

$$P(B|A) \sim N(P^{-1}Q, [P^T\Omega^{-1}P]^{-1}) \tag{25}$$

This representation is not usable practically. Incomplete views and relative views make the variance non-invertible. Additionally, $[P^T\Omega^{-1}P]^{-1})$ is not invertible. Luckily, we do not need to calculate this formula.

The posterior distribution from Bayes Theorem is the precision weighted average of the prior estimate and the conditional estimate. Now we can apply Bayes theory to create a new posterior distribution of the asset returns, called BL master formula.

$$P(A|B) \sim N([(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q][(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}, [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}) \tag{26}$$

An alternative representation of the same formula for the mean returns $\widehat{\Pi}$ and covariance M is:

$$\widehat{\Pi} = \Pi + \tau\Sigma P^T[(P\tau\Sigma P^T) + \Omega]^{-1}[Q - P\Pi] \tag{27}$$

$$M = ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1} \tag{28}$$

Remember that M is the variance of the posterior mean estimate about the actual mean, so it is the uncertainty in the posterior mean estimate and is not the variance of the returns.

Computing the posterior covariance of returns requires adding the variance of the estimate about the mean to the variance of the distribution about the estimate (He and Litterman (1999)).

$$\Sigma_p = \Sigma + M \tag{29}$$

Substituting the posterior variance (36) we get

$$\Sigma_p = \Sigma + ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1} \tag{30}$$

In the absence of the views this reduces to

$$\Sigma_p = \Sigma + (\tau\Sigma) = (1 + \tau)\Sigma \tag{31}$$

A brief recap of the BL process is given by the following scheme.

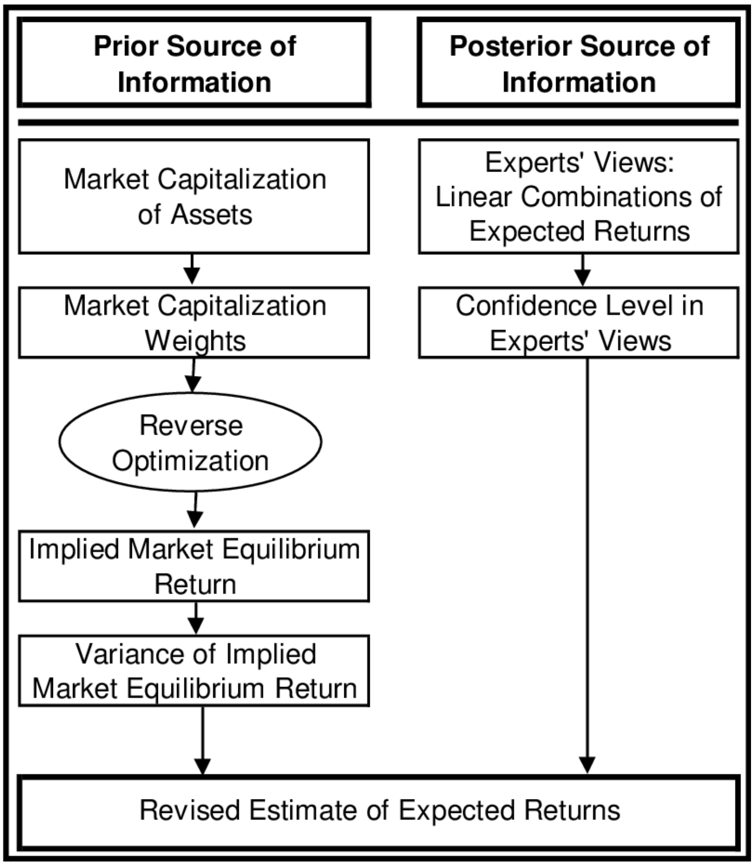


Figure 4: BL Process. Source: Lejeune, Miguel. (2009). A VaR Black–Litterman model for the construction of absolute return fund-of-funds. Quantitative Finance

2.2.5.1 Specifying Ω

Ω, the variance of the views is inversely related to the investor confidence in the views. It is intuitive that if the confidence is null (i.e. $\Omega \to \infty$), then the posterior distribution (26) equals the prior (24), while, at the other extreme, if the confidence is full (i.e. . $\Omega \to 0$), then the posterior distribution (26) equals the conditional distribution (25). We will briefly discuss two ways to calculate Ω.

1. Proportional to the variance of the prior. We can assume that the variance of the views will be proportional to the variance of the asset returns, like the variance of the prior distribution.
   - He and Litterman (1999) set the variance of the views as follows:

     $\omega_{ij} = p(\tau\Sigma)p^T \qquad \forall i = j$

     $\omega_{ij} = 0 \qquad \forall i \neq j$

     Or

     $$\Omega = diag(P(\tau\Sigma)P^T) \tag{32}$$

     This specification essentially equally weights the investor views and the market equilibrium weights. Including τ in the expression, the posterior estimate of the returns become independent of τ. This seems to be the most common method used in the literature.

   - Meucci (2006) does not take in consideration the diagonalization at all, setting:

     $$\Omega = \frac{1}{c} P\Sigma P^t \tag{33}$$

     He sets $c > 1$, and the obvious choice for c is τ^{-1}. This form lends itself to some simplifications of BL formulas.

2. Use a confidence interval. The investor can specify the variance using a confidence interval around the estimated mean return. For example, asset 1 has an estimated 5% mean return with the expectation it is 95% likely to be within the interval (4%, 6%). Knowing that 95% of the normal distribution falls within 2 standard deviation of the mean allows us to translate this into a variance for the view of $(5\%)^2$.

2.2.6 BLM limitations and improvement possibilities

Used as a part of an asset allocation process, BLM gives the possibility to overcome many of the limitations of previous approaches about portfolio theory. The result of the optimization process will be portfolios with a high level of diversification and much more stable compared to the resulting portfolios of MPT and CAPM, due to the fact that the starting point is the market portfolio that is already diversified. Additionally, traditional models ask investors to provide expected returns for all the assets in the portfolio, while investors often are specialized in certain areas and cannot develop knowledgeable expected returns for all assets in the portfolio. Therefore, we can connect this fact with the major improvement of BLM, the possibility to insert the views and the level of confidence. This gives the possibility to investors specialized in certain markets to introduce some views with a high level of confidence to obtain optimal portfolio weights. The resulting model is extremely versatile and practical with a high level of usability.

Nevertheless, all the improvements of BLM compared to previous approaches, it still has margins to make better. First, this model requires a broad sort of data, some of which are hard to seek out, indeed the investor need to identify the investable universe and find the market capitalization of each asset. Second, the investor must quantify their views, which will be derived from quantitative or qualitative processes, and they can be complete or incomplete or also conflicting; summarizing, views can be a double-edged sword. Linked to the previous point, the construction of the confidence matrix Ω is a weakness, because it is completely arbitrary and does not account for interdependence of the confidence over different views. Third, BLM is based on the assumption of normality of returns that is in conflict with the basic results in finance framework.

Many authors tried to overcome these weak spots with the introduction of many extension to the traditional BLM. Some examples are: Idzorek (2005) presents a method to calibrate the confidence of variance of the investor view in a simple method; Jay Walters (2014) describes how use relative entropy to measure quality of the views; Krishnan and Mains (2006) presents a method to incorporate additional factors in the model; Qian and Gorman (2001) propose a way to integrate views on the covariance matrix as well as views on the returns.

3 Robust Optimization

3.1 Introduction

As we have explained in the first chapter of this work, Markowitz's work was the ground for the modern portfolio construction theory, but its application has been disappointing for several drawbacks explained yet. The main problem of MVO is that it fails to take into consideration the uncertainty in the estimation process of expected returns and also tends to increase it; this is a big problem because Chopra and Ziemba (1993), Kallberg and Ziemba (1984) and Ya et al. (2016) find that the uncertainty in expected returns is roughly ten times as important as that in the covariance matrix for the sensitivity of the solution.

In this chapter is treated one of the approaches that have been proposed to mitigate the drawbacks of MVO, the RO approach, first introduced by Soyster (1973) and developed by El Ghaoui et al. (1997, 1998) and Ben-Tal and Nemirovsky (1998). As opposed to the MVO that treats the estimated expected returns in a deterministic manner, RO assumes that the estimated expected returns are random variables and seeks to seek out the optimal portfolio even when the realized values of inputs deviate from the estimated ones within some given set. The latter is called uncertainty set and defines the degree of deviation one desire to be protected from.

In RO literature applied to portfolio construction, two major forms of uncertainty set are considered. Ben-Tal and Nemirovski (1998 and 2000), El-Ghaoui et al. (1997, 1998) and Goldfarb and Iyengar (2003) analyse the quadratic uncertainty set for expected returns $(\mu - \hat{\mu})^T \Omega_\mu^{-1} (\mu - \hat{\mu}) \leq k^2$, with μ the vector of expected returns, T transpose, $\hat{\mu}$ estimated expected returns vector, Ω_μ the uncertainty matrix and k the level of uncertainty.

Tütüncü and König (2004) introduce the box uncertainty set $|\mu_i - \widehat{\mu_i}| \leq \xi_i, i = 1,2,\ldots,n$ with μ_i the expected return of asset I, $\widehat{\mu_i}$ the estimated expected return of asset I and ξ_i the level of uncertainty for the expected return estimation of asset i.

The choice of uncertainty sets, the selection of uncertainty matrices for the quadratic uncertainty set as well as the calibration of k are the three challenges to implement robust portfolio optimization. The final objective of RO is to improve the MVO by reducing the sensitivity to inputs and increasing the diversification.

3.2 Model

RO can be reformulated by modifying the MVO through a max-min process (Scherer 2006). Before to start treating the model, we can assume without loss of generality, that the expected returns are estimated by sample mean $\hat{\mu} = \bar{\mu}$ because the draw backs of MVO exist no matter the estimated expected returns used. First, we have to find the worst-case expected returns of assets (defined as the realized returns that deviate most negatively form the estimated expected returns $\hat{\mu}$), within the uncertainty set U_μ. At this point the optimization process maximizes the portfolio returns, computed with the worst case expected returns, under the risk constraint.

$$\max_{w}\left(\min_{\mu\in U_\mu}(w^T\mu) - \frac{\lambda}{2}w^T\Sigma w\right) \tag{34}$$

w the vector of portfolio weights, μ the vector of expected returns, λ the risk aversion parameter and Σ the covariance matrix of asset returns.

Now we formulate the robust portfolio optimization assuming the choice of the quadratic uncertainty set (this assumption will be explained in the next section):

$$\max_{w}\left(\min_{\mu\in U_\mu}(w^T\mu) - \frac{\lambda}{2}w^T\Sigma w\right), U_\mu = (\mu - \bar{\mu})^T\Omega^{-1}(\mu - \bar{\mu}) \leq k^2 \tag{35}$$

The first step consists in finding the worst case realized returns from the confidence region derived from the uncertainty set. Minimizing $w^T\mu$ for μ within the uncertainty set defined by U_μ is equivalent to maximizing $w^T\bar{\mu} - w^T\mu$ for μ within the uncertainty set:

$$\max_{\mu}(w^T\bar{\mu} - w^T\mu) \; s.t.(\mu - \bar{\mu})^T\Omega^{-1}(\mu - \bar{\mu}) \leq k^2 \tag{36}$$

Rewriting equation (36) with the Lagrangian:

$$\mathcal{L}(\bar{\mu}, \delta) = w^T\bar{\mu} - w^T\mu - \delta((\mu - \bar{\mu})^T\Omega^{-1}(\mu - \bar{\mu}) - k^2) \tag{37}$$

Solving equation (35) with the help of Lagrangian yields:

$$\mu = \bar{\mu} - \sqrt{\frac{k^2}{w^T\Omega w}}\Omega w \tag{38}$$

Substitute the formula for μ in equation (41):

$$\max_w \left(w^T \bar{\mu} - \sqrt{\frac{k^2}{w^T \Omega w}} w^T \Omega w - \frac{\lambda}{2} w^T \Sigma w \right) \tag{39}$$

We note the optimal robust portfolio weights as w^*_{rob}:

$$w^*_{rob} = argmax \left(w^T \bar{\mu} - k\sqrt{w^T \Omega w} - \frac{\lambda}{2} w^T \Sigma w \right) \tag{40}$$

3.2.1 Form of uncertainty set

As mentioned in the introduction, in RO literature there are two most common forms of uncertainty sets: quadratic and box. About financial applications, Goldfarb and Iyengar (2003) demonstrate analytically that the quadratic uncertainty set comes out naturally from the estimation process using regression when the expected returns are estimated from a linear factor model, while Fabozzi et al. (2007) point out that the box uncertainty set assumes that all assets will deliver their worst case returns at the same time, an assumption that is not good in practice.

According to results above, the quadratic uncertainty set in RO is chosen as more suitable for applications in finance as it is less conservative and more in line with the characteristics of the distribution of returns of financial assets. We will derive the robust formulation of it.

The quadratic uncertainty set includes the uncertainty matrix Ω_μ. It is assumed that the expected returns μ are normally distributed with mean vector $\bar{\mu}$. So, the uncertainty $\mu - \bar{\bar{\mu}}$ follow a multivariate normal distribution with mean 0 and covariance matrix of uncertainty in mean return Ω_μ.

$$U_\mu = \left\{ \mu \,|\, (\mu - \hat{\mu})^T \Omega_\mu^{-1} (\mu - \hat{\mu}) \leq k^2 \right\} \tag{41}$$

k^2 represents the level of uncertainty and this formulation gives the possibility to define the expected returns that deviate most negatively from the estimated returns within the level of uncertainty k.

3.3 RO with quadratic uncertainty set vs MVO

Now we take a look to the potential improvements of RO compared to MVO. At the optimum the gradient of equation (40) is zero and the optimal robust weights w_{rob}^* satisfy the following equality when at least one of the optimal weights is different from zero:

$$\bar{\mu} - \frac{k}{\sqrt{w_{rob}^{*T}\Omega w_{rob}^*}}\Omega w_{rob}^* - \lambda\Sigma w_{rob}^* = 0 \tag{42}$$

MVO optimal portfolio weights w_{MVO}^* can be obtained by setting to zero the derivative of $w^T\bar{\mu} - \frac{\lambda}{2}w^T\Sigma w$, with respect to w:

$$\bar{\mu} - \lambda\Sigma w_{MVO}^* = 0 \tag{43}$$

Rearranging the terms and inverting Σ, we get:

$$w_{MVO}^* = \frac{1}{\lambda}\Sigma^{-1}\bar{\mu} \tag{44}$$

Roncalli (2013) points out that the inversion of a covariance matrix Σ with small eigenvalues is the main cause of the high sensitivity to inputs and counter-intuitive long-short position suffered by MVO. But in a covariance matrix there are two elements, correlation coefficients and volatilities. Are they both responsible or only one is?

To answer this question, we make an equivalence between optimizing on vector of weights w, covariance matrix Σ and vector of expected returns $\bar{\mu}$ and optimizing on risk budgets x, pair-wise correlation matrix P and vector of Sharpe ratios $\overline{SR}$. This equivalence is obtained by decomposing the covariance matrix Σ into the pair-wise correlation matrix P and variances σ^2. Expressing MVO with risk budgets and correlation matrix is that the effect of correlation on the small eigenvalues of Σ is separated from that of volatilies.

Equation (44) can be reformulated in terms of $\overline{SR}, P\ and\ X_{MVO}^*$, assuming $\lambda = 1$:

$$X_{MVO}^* = P^{-1}\overline{SR} \tag{45}$$

Equation (45) shows that Sharpe ratios and the correlation matrix are liable for the drawbacks of MVO. Indeed, MVO aims to take advantage of the differences in Sharpe ratios while taking under consideration the correlations among assets. Once the MVO optimal risk budgets are determined, the volatilities are there to produce the final portfolio weights but this step is linear and does not involve any matrix inversion.

Because P is symmetric and positive semi-definite, it can be decomposed into $P = ZLZ^T$, with Z the matrix of eigenvectors of P and L the diagonal matrix with eigenvalues of P on the diagonal.

Equation (45) can be transformed as:

$$X^*_{MVO} = ZL^{-1}Z^T\overline{SR} \tag{46}$$

By expressing equation (46) in the spaces spanned by the eigenvectors of P, we get:

$$\ddot{X}^*_{MVO} = L^{-1}\ddot{\overline{SR}} \tag{47}$$

Equation (47) shows two origins of the drawbacks of the MVO:

1) Inversion of small eigenvalues in L^{-1} which is the diagonal matrix of eigenvalues of correlation matrix P.
2) Non-negligible expected returns in $\ddot{\overline{SR}}$ of the eigenvectors of P associated with small eigenvalues.

RO improves MVO in these two ways. Equation (42) modify the MVO optimality condition with the introduction of uncertainty in the objective function.

There are two ways to interpret equation (42) compared to equation (43). These two interpretations represent two ways in which RO mitigates the drawbacks of the MVO.

Modification of Σ:
$$\bar{\mu} - \lambda\left(\frac{k}{\lambda\sqrt{w^{*T}_{rob}\Omega w^*_{rob}}}\Omega + \Sigma\right) w^*_{rob} = 0 \tag{48}$$

Factoring w^*_{rob}, equation (42) illustrates the modification of covariance matrix when the uncertainty is introduced in the objective function. We can see that the robustness of the RO depends on the choice of Ω because it can have a big impact on the final covariance matrix that will be inverted. In the next section we analyse four different uncertainty matrices.

Modification of $\bar{\mu}$:
$$\left(\bar{\mu} - \frac{k}{\sqrt{w^{*T}_{rob}\Omega w^*_{rob}}}\Omega w^*_{rob}\right) - \lambda\Sigma w^*_{rob} = 0 \tag{49}$$

Grouping the first two terms on the left-hand side, equation (42) represents the modification on expected returns by the uncertainty. In this formulation the original covariance matrix is not modified, however, the $\bar{\mu}$ are adjusted so that the expected returns of eigenvectors of Σ associated with small eigenvalues are reduced. Later we will discuss the choice of k.

3.3.1 Choice of uncertainty matrix

In the RO literature, four types of uncertainty matrices are proposed.

1) Scherer (2006) proposed Ω proportional to Σ $\quad \Omega = \Sigma$
2) Stubbs and Vance (2005) suggested to consider only the diagonal of Σ $\quad \Omega \propto \text{diag}(\Sigma)$

Heckel et al. (2016) proposed two others Ω:

3) The uncertainty matrix equal to the identity matrix $\quad \Omega = I_n$
4) The uncertainty matrix equal to the diagonal matrix of sample volatilities $\Omega = \text{sqrt}(\text{diag}(\Sigma))$

The analysis of uncertainty matrix starts with equation (48), since at the optimum $\sqrt{w_{rob}^{*T}\Omega w_{rob}^{*}}$ is just a number. Noting $\frac{k}{\sqrt{w_{rob}^{*T}\Omega w_{rob}^{*}}}$ as β and $\frac{\beta}{\lambda+\beta}$ as η, we get:

$$\bar{\mu} = (\beta\Omega + \lambda\Sigma)w_{rob}^{*} \tag{50}$$

$$\frac{\bar{\mu}}{(\lambda + \beta)} = (\eta\Omega + (1 - \eta)\Sigma)w_{rob}^{*} \tag{51}$$

Instead of inverting Σ, the solution to RO with a quadratic uncertainty set requires the inversion of a modified covariance matrix $\eta\Omega + (1 - \eta)\Sigma$.

<u>**Case 1**</u>: $\mathbf{\Omega} = \mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \vdots & \ddots & \vdots \\ \rho_{1n}\sigma_1\sigma_n & \cdots & \sigma_n^2 \end{pmatrix}$

Replacing Ω by Σ in the modified covariance matrix:

$$\eta\Omega + (1 - \eta)\Sigma = \Sigma \tag{52}$$

There is no change to the original covariance matrix, so in this case RO cannot mitigate the drawbacks of MVO.

Case 2: $\boldsymbol{\Omega} = \mathbf{diag}(\boldsymbol{\Sigma}) = \begin{pmatrix} \sigma_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n^2 \end{pmatrix}$ with σ_1^2, σ_n^2 the variances of asset 1 and asset n

Replacing Ω by $diag(\Sigma)$:

$$\eta\Omega + (1-\eta)\Sigma = \eta \begin{pmatrix} \sigma_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n^2 \end{pmatrix} + (1-\eta) \begin{pmatrix} \sigma_1^2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \vdots & \ddots & \vdots \\ \rho_{1n}\sigma_1\sigma_n & \cdots & \sigma_n^2 \end{pmatrix} \tag{53}$$

The new covariance matrix is now a weighted average between the original covariance matrix and the diagonal matrix of sample variances.

Case 3: $\boldsymbol{\Omega} = \boldsymbol{I_n}$, with I_n the $n \times n$ identity matrix

Replacing Ω by I_n:

$$\eta\Omega + (1-\eta)\Sigma = \eta \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} + (1-\eta) \begin{pmatrix} \sigma_1^2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \vdots & \ddots & \vdots \\ \rho_{1n}\sigma_1\sigma_n & \cdots & \sigma_n^2 \end{pmatrix} \tag{54}$$

The new covariance matrix is a weighted average between the original covariance matrix and the identity matrix.

$$\textbf{Case 4}: \boldsymbol{\Omega} = \mathbf{sqrt}(\mathbf{diag}(\boldsymbol{\Sigma})) = \eta \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n \end{pmatrix} + (1-\eta) \begin{pmatrix} \sigma_1^2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \vdots & \ddots & \vdots \\ \rho_{1n}\sigma_1\sigma_n & \cdots & \sigma_n^2 \end{pmatrix} \tag{55}$$

The new covariance matrix is a weighted average between the original covariance matrix and the diagonal matrix of sample volatilities.

As we have said before, equation (45) shows that the part of covariance matrix that is liable for the high sensitivity to inputs is the correlation matrix, while the volatilities, in the solution to MVO in terms of risk budgets, are only a scaling factor to define the final portfolio weights. Considering that, the objective is the elimination of small eigenvalues from the correlation matrix, according to equation (47), to scale back the sensitivity, while, if volatilities are unchanged, the relative magnitude of the Sharpe ratios is preserved. Kirby and Ostdiek (2012), Perchet et al. (2016) and Santos (2018) provide empirical evidence about the advantages of volatilities and the negative contribution of off-diagonal covariance elements to the results of optimal portfolios.

At the end of this reasoning, we can propose the subsequent criteria for the selection of the uncertainty matrix:
The ideal uncertainty matrix reduces the sensitivity to inputs by shrinking the original correlation coefficients towards zero and keeping the original volatilities unchanged.

About the sensitivity reduction, we note that in the case 1, where the uncertainty matrix is equal to the covariance matrix, there is no reduction of sensitivity to inputs of the optimization solution because the correlation coefficients are unmodified. On the other hand, in case 2,3 and 4 the resulting covariance matrix is a weighted average between the uncertainty matrix that is diagonal and covariance matrices; additionally, all the off-diagonal terms in the uncertainty matrices are zero so the original correlation coefficients are tighten towards zero achieving the sensitivity reduction.
About keeping volatilities unchanged, we see immediately that in case 3 and 4 the diagonal terms of both uncertainty matrices are different from those of the original covariance matrix, consequently using those uncertainty matrices would cause a changes in volatilities. On the other hand, introducing the uncertainty matrices of case 1 and 2 keep the diagonal terms of the new covariance matrix unchanged.

We can summarize the above results with a table that will give an idea about the final choice:

	Case 1: $\boldsymbol{\Omega = \Sigma}$	**Case 2**: $\boldsymbol{\Omega = \mathrm{diag}(\Sigma)}$	**Case 3**: $\boldsymbol{\Omega = I_n}$	**Case 4**: $\boldsymbol{\Omega = \mathrm{sqrt}(\mathrm{diag}(\Sigma))}$
Reducing sensitivity	NO	YES	YES	YES
Volatilities unchanged	YES	YES	NO	NO

Table 1: Uncertainty matrix choice

It is evident that the final best choice will be the case 2, namely $\Omega = \mathrm{diag}(\Sigma)$. Choosing Ω as the diagonal of the original covariance matrix Σ, we will have a new covariance matrix that reduces the condition number of the correlation matrix, eliminates the small eigenvalues, and reduces the sensitivity of the solution to inputs keeping unchanged the volatility structure.

3.3.2 Choice of the uncertainty level: k

The simplest approach to determine the level of uncertainty k is the utilisation of a rule of thumb to calibrate it. We recall that the solution to an MVO, viewed on the basis defined by the eigenvectors of the correlation matric and assuming λ equal to 1, is given by: $\ddot{X}_{MVO}^{*} = L^{-1}\ddot{\overline{SR}}$.

Both the uncertainty matrix and k address the sensitivity to inputs. About the uncertainty matrix we have determined an optimal solution yet, while k can be viewed as reducing the expected returns of the eigenvectors corresponding to the small eigenvalues.

The proposed rule of thumb, that can be demonstrated with a multi asset example (C. Yin, R. Perchet, F. Soupè 2020), consists in choosing k as half of the average Sharpe ratios without a dependence to the amount of assets in the universe. In this way, k is able to reduce the returns of eigenvectors that correspond to 40% smallest eigenvalues. The mathematical explanation is tricky and is left to the curious reader in Appendix A where it is described all process (C. Yin, R. Perchet, F. Soupè 2020).

4 Empirical application

4.1 Introduction

This chapter will be dedicated to the application of the optimization approaches described in the first 3 chapters with real financial data. First it is necessary to obtain data and, luckily, today data sources are easily accessible through platforms such as Bloomberg and Thomson Reuters. We have downloaded data from Bloomberg, more specifically we have chosen as object analysis the sectoral decomposition of the most important north American equity index, S&P 500, to discover the relationship among them and the best combination that give birth to optimal portfolios. When we have all data that we need, we have to manage them to make them suitable for the application of the various models. To perform all the analysis, we have used MATLAB software.

4.2 Data

In this section we will go through a deeper description of the data used in the empirical application. S&P 500 contains 500 stocks of listed companies on NYSE and Nasdaq that represent the 80% of the market capitalization. It is divided in 11 sectors:

- Information Technology sector
- Healthcare sector
- Consumer discretionary sector
- Telecommunication sector
- Financials sector
- Industrials sector
- Consumer staples sector
- Utilities sector
- Materials sector
- Energy sector
- Real estate sector

Thanks to a University license it was very simple the collection of data from Bloomberg, that also give the possibility to import them directly in Excel to manage them in an easier way. First, we choose to take in consideration SP&500 because it can be considered one of the major indices all over the world, so it ensures an adequate dimension and liquidity to be considered investable and reliable. As a consequence, we think that it is the best choice to conduct a sector analysis because it included many different sectors and the companies that are included in them are key players on the world market and so provide significant data.

To keep the analysis congruent and meaningful, we choose to take in consideration only the first 10 sectors because the Real estate sector has some data inconsistencies for the time period that we will take in consideration.

Namely, the frequency of data used is daily, to have a big quantity of information. The data starts from 3rd January 2005 and ends on 30th September 2020. From 3rd January 2005 to 31st December 2014 is called the in-sample set, namely the data used to obtain the optimal portfolios weights.

From 2nd January 2015 to 30th September 2020 is called the out-of-sample set, namely the time period where we will see the results, in terms of risk/return using statistical techniques and different performance indicators, obtained by the portfolios created with in-sample data. We have to highlight the fact that the dataset include data about three critical financial crises. The subprime crisis of 2008 is included in our in-sample period as well as the sovereign debts crisis of 2010, while COVID-19 crisis is included in our out-of-sample period until September 2020.

4.3 Procedure

The analysis will be done with MATLAB software and could be summarized in this way:

1) Data import
2) MV optimization
3) BL optimization without views
4) BL optimization with three different scenarios
5) RO optimization with different choice of Ω
6) Analysis out-of-sample without rebalancing
7) Analysis out-of-sample with 5 days rebalancing
8) Comparison of the models in both points 6) and 7)

About the optimization models we have talked about them in the previous chapters of this work yet, while it is necessary specify what means the term "rebalancing". In point 6), the analysis out-of-sample is done without changing the portfolios composition resulting from the optimization process conducted in the in-sample period, namely without rebalancing the weights in the portfolios; on the other hand, in point 7) the analysis out-of-sample is built changing the portfolio composition every 5 days, namely the optimization routine runs at interval of 5 days incorporating in the process all data of the sample (also out-of-sample information) hand to hand we go forward through time. This constant adjustment in portfolios weights is called rebalancing.

What we expect is that the analysis with rebalancing will be more efficient compared to that without rebalancing since, in the former, the optimizer processes a higher quantity of information and, especially, incorporating new information going ahead gives the possibility to react to market events in the out-of-sample period, whether good or bad.

Given this clarification, we specify that we decide to set 10 different levels of risk (1 level = lower risk, 10 level = higher risk), and for each level we will have an optimal allocated portfolio by each model.

5 Results

5.1 In-sample analysis

In this section we will present the optimal weights provided by the four model, comparing the output allocations. Portfolios obtained are 10 for each model, due to the different level of risk taken in consideration.

The four optimization methods are:

1) Markowitz optimization
2) Black-Litterman without views
3) Black-Litterman with views
4) Robust Optimization

Making a brief recap of the four methods, the first performs an optimization using the mean of the assets as expected return and the historical covariance matrix for risk. The second case is characterized by the usage of the neutral expected returns instead of the simple mean, while as risk is still used the historical covariance matrix. Comparing these first two methods we can see already important differences about optimal allocation. The third method started to be very interesting because the inputs are dependent from the views of the investor and the confidence in those. The fourth method is formulated as a resolver of Markowitz problems, and is based on the calculation of the uncertainty matrix to obtain more stable results.

The graphs representing the weights are organized as follows: on the x-axis we have the risk propensity (from 1 to 10), while on the y-axis we have the weight (from 0 to 1); the sum of the weights will sum to 1 and each asset has a different colour on the graph.

We can start with Markowitz allocation:

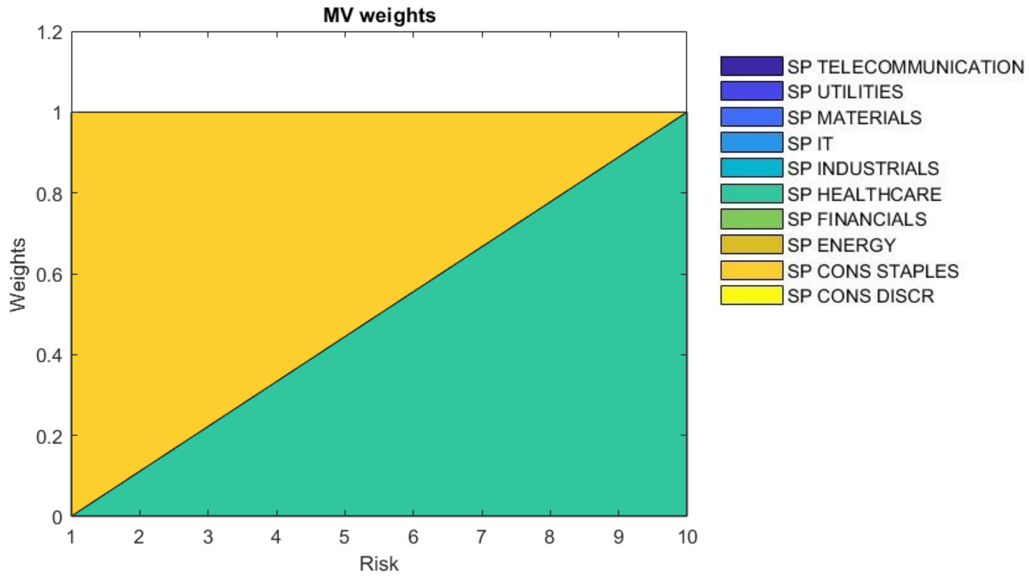


Figure 5: MV weights

From figure 5 we can see that we are fully invested only in two assets: healthcare sector and consumer staples sector. In this situation we are very exposed to shocks that could involve individual assets and we are not taking advantage from diversification, so the common problems of Markowitz optimization are recognised.

Now we compare MV results with the allocation resulting from BL without views. Substantially, the difference regards the expected return input because BL introduced the neutral expected returns (calculated using the capitalization weights of S&P 500) that represents the beliefs of the market.

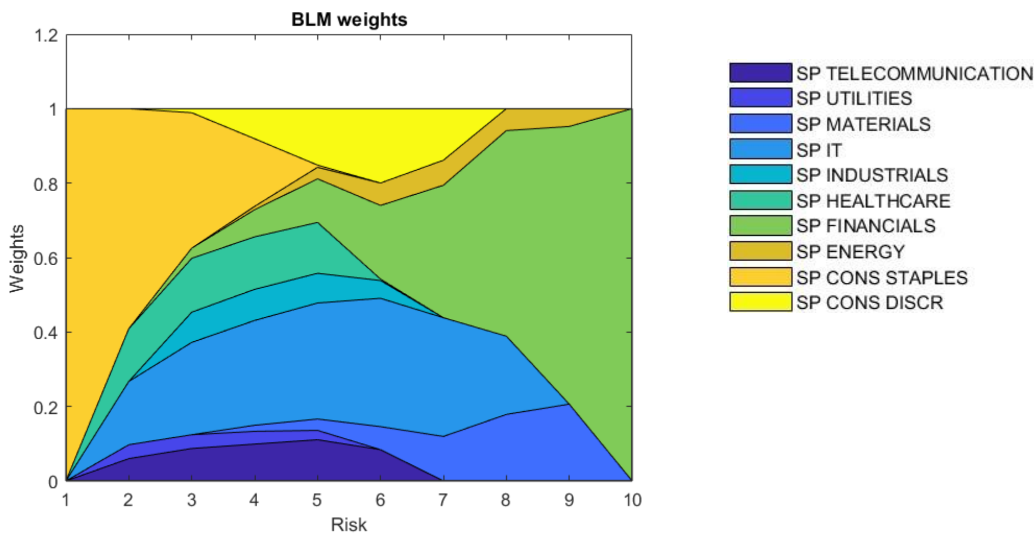


Figure 6: BL weights

It is easy to see that we have a very different situation. With the exceptions of the extremes (minimum and maximum risk) portfolios are very diversified across all the sectors. In particular we can observe that, taking in consideration this universe of assets, we can identify two assets that are relevant: consumer staples sector is the most conservative asset that will reduce the overall risk, like bonds, while financial sector represents the most aggressive asset that provides higher risk, like stocks. As a consequence, in this situation risk adverse investors will have a big position on consumer staples sector, while whenever we take in consideration investors more incline to risk, they will overweight the position in financials sector.

The introduction of views in the BLM gives the possibility to make a step ahead for the investor. We define a particular scenario to give a view about the functioning of the process.

Basic Scenario

1) IT sector performs 20% (uncertainty 5%)
2) Energy sector performs -15% (uncertainty 5%)
3) Industrials sector overperforms Financials sector by 12% (uncertainty 5%)

These views and the confidence on them are defined in an arbitrary way, like simply an opinion based on the personal views of an investor. Now we will expect that the model creates an optimal portfolio with higher position in the assets that will perform good in the opinion of the investor.

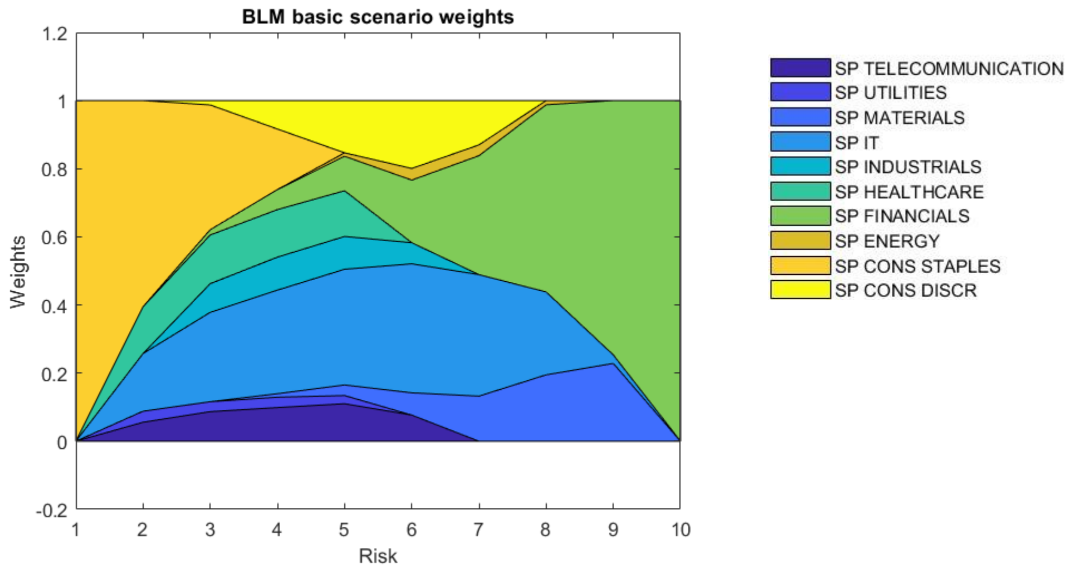


Figure 7: BLM basic scenario weights

We can see immediately that the model works in the right direction, but limited to the intermediate levels of risk, mainly from Portfolio 3 to Portfolio 8. Making a comparison with the BL allocation without views, we have a higher allocation in IT sector and Industrials sector and the presence of Energy sector is reduced in the graph. If the views are correct, this investor will obtain better results than an investor that does not provide any outlook and vice versa.

We want to highlight that uncertainty level assumed is 5% (i.e., high confidence in the views). As a matter of fact, the confidence will have a high impact on the output.

If we change the uncertainty level from 50% (high unconfident) to 0% (full confidence) we have these results:

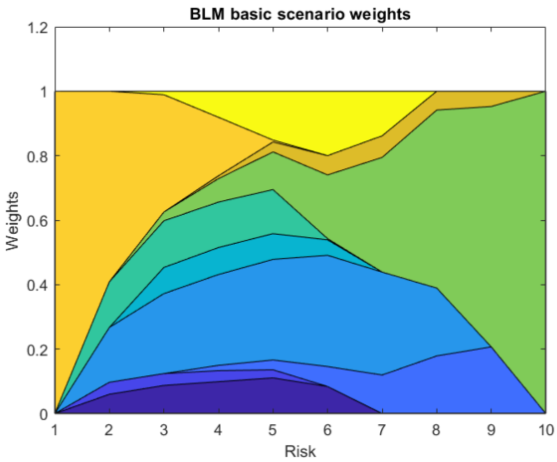
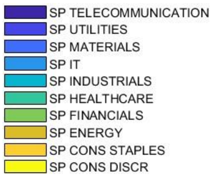


Figure 8: Uncertainty 50%



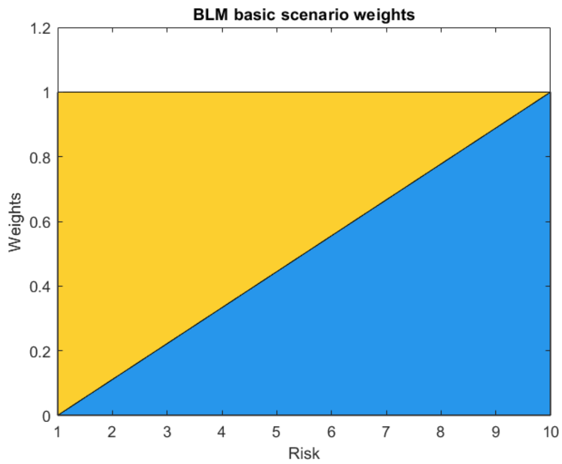


Figure 9: Uncertainty 0%

In figure 8, it is evident that the views are not considered so much due to the high uncertainty, that tilts back the allocation towards the market weights. Instead in figure 9, we can say that the views are taken as a “clairvoyant opinion”, because the mixture of BLM is eliminated, and we are going back to a Markowitz scenario where we take in consideration only our inputs.

We can conclude that the role of the uncertainty matrix is very important and can conduct to different results.

Now we define two other scenarios that will be very important to provide an idea of views power.

Best scenario

1. Healthcare sector overperforms Financials sector by 34,78% (uncertainty 5%)
2. IT sector overperforms Consumer Discretionary sector by 84,97% (uncertainty 5)
3. Materials sector overperforms Utilities sector by 5,81% (uncertainty 5%)

This scenario is defined using real market data and we will use, as a view, three comparisons between returns of the various sectors calculated using the out-of-sample data. In particular, this scenario simulates a situation where the investor has argued perfect views about the future.

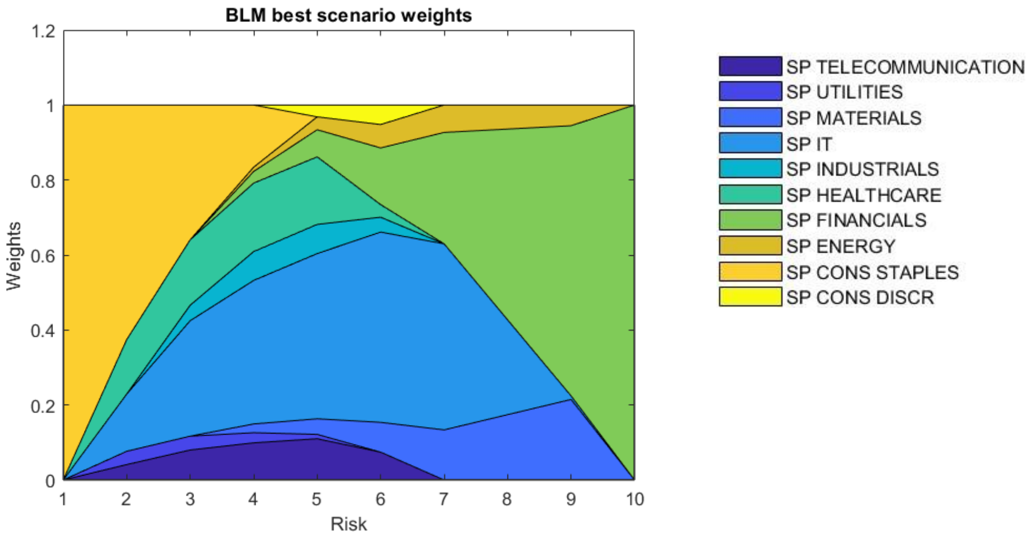


Figure 10: BLM best scenario weights

It is evident that also in this case the model is working very well, indeed all sectors that overperform are over allocated in the portfolios compared to the BLM without views. Even in this case, limited to the intermediate levels of risk, mainly from Portfolio 3 to Portfolio 8.

Worst scenario

1. Healthcare sector underperforms Financials sector by 34,78% (uncertainty 5%)
2. IT sector underperforms Consumer discretionary sector by 84,97% (uncertainty 5%)
3. Materials sector underperforms Utilities sector by 5,81% (uncertainty 5%)

This scenario is the exact opposite of the previous, to obtain an idea of the differences that arise when the investor has good or bad views.

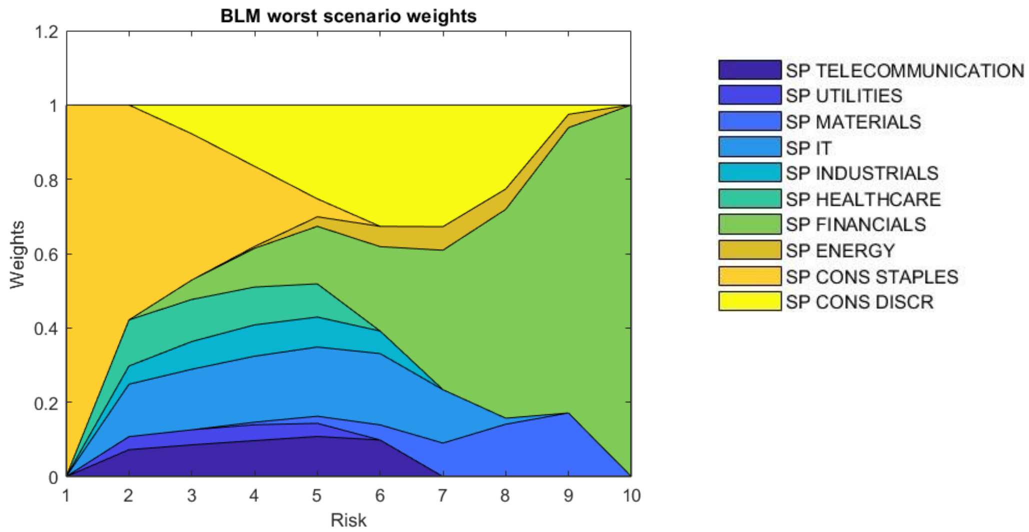


Figure 11: BLM worst scenario weights

As expected, the weights are very different from the best case and also from the other BLM situations.

We have considered two extremes cases to highlight the fact that the allocations will be completely different according to the views expressed.

Now we come back to a model that does not give the possibility to provide views but should have the power to provide very stable allocations compared to the previous models.

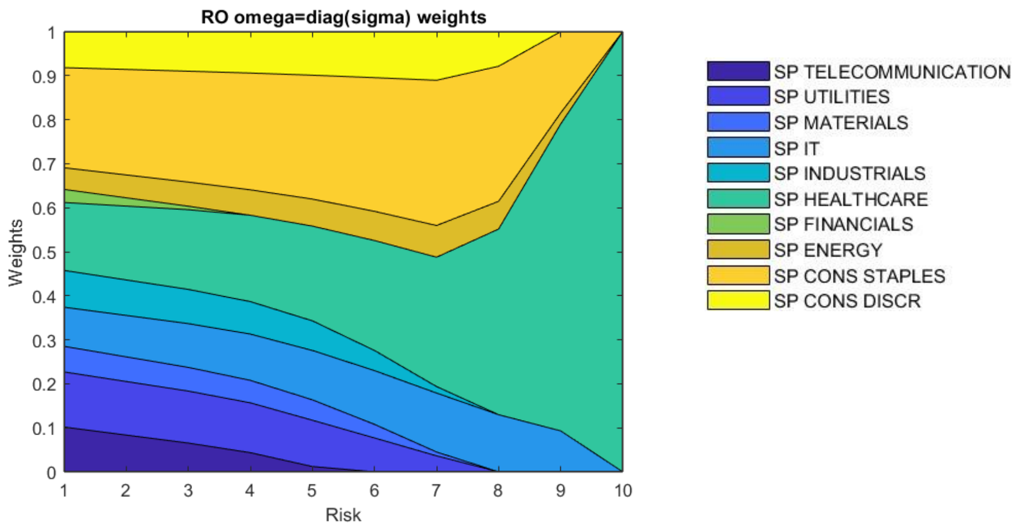


Figure 12: RO $\Omega = diag(\Sigma)$ weights

With this approach is evident the differences with the previous approaches. First, the weights change in a very smooth manner, that is an indication of the stability of the portfolios, so the model reaches his objective to avoid corner-solutions, in particular for low risk levels. Additionally, the model gives a predominant role to the Healthcare sector and to the Consumer Staples sector, distinguishing from the precedent models.

In figure 12 are plotted the weights considering, as uncertainty matrix, the best choice provided by C. Yin et al. 2020, namely $\Omega = diag(\Sigma)$.

It will be interesting take a look to the other 3 cases to see the differences in terms of allocation.

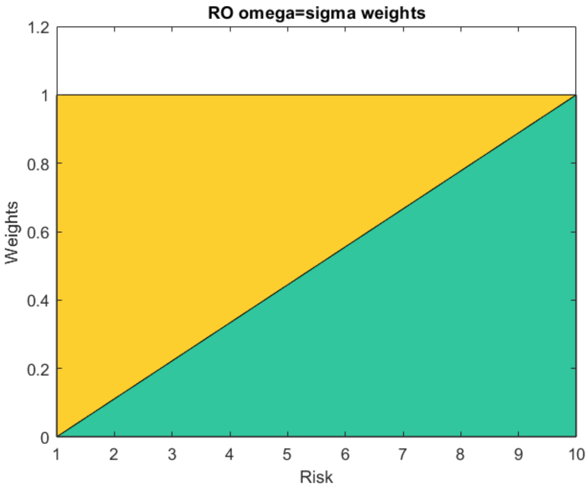


Figure 13: RO $\Omega = \Sigma$ weights

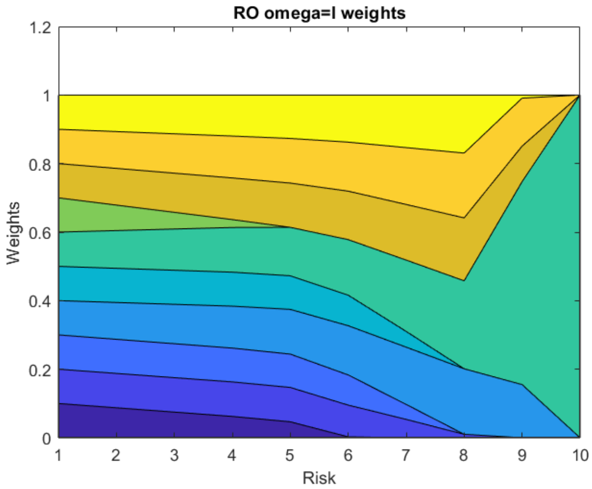


Figure 14: RO $\Omega = I$ weights

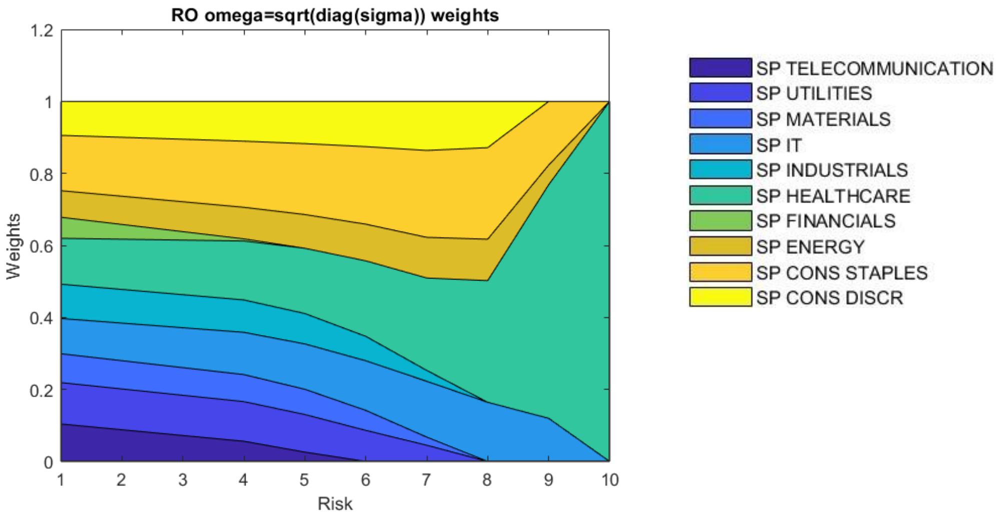


Figure 15: RO $\Omega = sqrt(diag(\Sigma))$ weights

About Figure 13, we can see the results that we expect, indeed using as uncertainty matrix the historical covariance matrix we obtain the same results of Markowitz optimization with the allocations divided between two assets. About figure 14 and 15, instead we can see results that differ

from the case of $\Omega = diag(\Sigma)$ not so much as figure 11, but it is clear that we have allocations characterized by something like an "equal weighted approach" compared to figure 12.
We will check with no-rebalance approach if taking $\Omega = diag(\Sigma)$ will be the right choice in performance terms.

First of all, we have to specify that the portfolios taken in consideration will be three: Low volatility (Portfolio 2), Middle volatility (Portfolio 5), High volatility (Portfolio 9). What do they mean these three entries? We recall that we have defined ten different level of risk that corresponds to ten different optimal portfolios. With the help of a MATLAB function, these ten optimal portfolios are placed on the efficient frontier of each model equally spaced one from the other. This procedure helps us to obtain optimal portfolios with standardized risk levels and it is necessary because each model presents a different efficient frontier.

About the low volatility portfolio and high volatility portfolio, we have not chosen Portfolio 1 and 10 to avoid extreme allocations and, as a consequence, to be more near to real allocations that will be chosen by investors. Here we present a table that summarizes the output weights for every model taking in consideration the three portfolios specified above:

LOW VOLATILITY

	MV	BLM	BLM-basic	BLM-best	BLM-worst
Telecomm.	0	5.99%	5.57%	4.12%	7.24%
Utilities	0	3.74%	3.24%	3.54%	3.51%
Materials	0	0	0	0	0
IT	0	17%	16.91%	15.25%	14.10%
Industrials	0	0	0	0	4.90%
Healthcare	11.11%	14.16%	13.76%	14.54%	12.40%
Financials	0	0	0	0	0
Energy	0	0	0	0	0
Cons. Staples	88.89%	59.11%	60.52%	62.55%	57.85%
Cons. Discr.	0	0	0	0	0

Table 2a: Low volatility portfolios weights

	RO Ω = diag(Σ)	RO Ω = Σ	RO Ω = I	RO Ω = sqrt(diag(Σ))
Telecomm.	8.36%	0	8.75%	8.80%
Utilities	12.16%	0	10.03%	11.33%
Materials	5.60%	0	9.95%	7.85%
IT	9.41%	0	10.75%	10.44%
Industrials	8.07%	0	9.97%	9.35%
Healthcare	16.77%	11.11%	11.01%	13.95%
Financials	1.88%	0	7.43%	4.10%
Energy	5.20%	0	10.71%	7.84%
Cons. Staples	23.96%	88.89%	10.75%	16.36%
Cons. Discr.	8.59%	0	10.66%	9.98%

Table 2b: Low volatility portfolios weights

From table 2a and 2b we have a numerical view of the figures presented before. We recall that these two tables refer to the level of risk 2. We can notice that, going from Markowitz optimization to RO, diversification among all sectors increases. BLM variants are characterized by over allocating IT and Consumer staples sector. On the other hand, RO differentiates from BLM because it over allocates Utilities and Healthcare sectors, with Consumer staples sector which is common to both models. Coming back to BLM, it is clear that, in low volatility framework, views are not well incorporated because the allocations in sectors considered by the views go down comparing BLM without views with the other variants.

MIDDLE VOLATILITY

	MV	BLM	BLM-basic	BLM-best	BLM-worst
Telecomm.	0	11.07%	11.02%	11.01%	10.79%
Utilities	0	2.51%	2.39%	1.16%	3.54%
Materials	0	3.05%	3.11%	4.16%	1.90%
IT	0	31.16%	33.97%	44.05%	18.62%
Industrials	0	7.98%	9.62%	7.78%	8.07%
Healthcare	44.44%	13.66%	13.42%	18.02%	8.93%
Financials	0	11.72%	10.12%	7.24%	15.48%
Energy	0	3.10%	1.01%	3.44%	2.57%
Cons. Staples	55.56%	0.59%	0	0	4.83%
Cons. Discr.	0	15.16%	15.34%	3.16%	25.26%

Table 3a: Middle volatility portfolios weights

	RO $\Omega = \text{diag}(\Sigma)$	RO $\Omega = \Sigma$	RO $\Omega = \text{I}$	RO $\Omega = \text{sqrt}(\text{diag}(\Sigma))$
Telecomm.	1.21%	0	4.66%	2.59%
Utilities	10.51%	0	10.01%	10.44%
Materials	4.63%	0	9.74%	7.01%
IT	11.21%	0	13.04%	12.60%
Industrials	6.73%	0	9.8%	8.44%
Healthcare	21.52%	44.44%	14.14%	18.12%
Financials	0	0	0	0
Energy	6.14%	0	12.91%	9.37%
Cons. Staples	28.13%	55.56%	13.01%	19.69%
Cons. Discr.	9.90%	0	12.68%	11.73%

Table 3b: Middle volatility portfolios weights

Table 3a and 3b proposed us a different situation. Increasing the risk propensity, MV increases the weight of Healthcare sector, BLM variants increase the diversification in their optimal portfolios while RO confirmed its variegated allocation. Differently from the low volatility situation, views are very well incorporated by BLM variants, easily seen by over/under allocation of the sectors interested by the views (IT, Energy, Industrials for basic scenario, IT, Healthcare, Materials for best/worst scenarios).

HIGH VOLATILITY

	MV	BLM	BLM-basic	BLM-best	BLM-worst
Telecomm.	0	0	0	0	0
Utilities	0	0	0	0	0
Materials	0	20.63%	22.84%	21.48%	17.11%
IT	0	0	2.49%	0.95%	0
Industrials	0	0	0	0	0
Healthcare	88.89%	0	0	0	0
Financials	0	74.59%	74.67%	72.05%	76.79%
Energy	0	4.78%	0	5.52%	3.58%
Cons. Staples	11.11%	0	0	0	0
Cons. Discr.	0	0	0	0	2.52%

Table 4a: High volatility portfolios weights

	RO $\Omega = \text{diag}(\Sigma)$	RO $\Omega = \Sigma$	RO $\Omega = I$	RO $\Omega = \text{sqrt}(\text{diag}(\Sigma))$
Telecomm.	0	0	0	0
Utilities	0	0	0	0
Materials	0	0	0	0
IT	9.30%	0	15.51%	11.98%
Industrials	0	0	0	0
Healthcare	69.72%	88.89%	59.20%	64.88%
Financials	0	0	0	0
Energy	2.51%	0	10.37%	5.46%
Cons. Staples	18.47%	11.11%	14.01%	17.68%
Cons. Discr.	0	0	0.91%	0

Table 4b: High volatility portfolios weights

Table 4a and 4b provide a very particular situation. MV continues with the tendency to over allocate Healthcare sector. BLM variants have very concentrated portfolios, mainly in Materials and Financials sector, implying that also for this risk propensity views do not work in the proper way. Even for RO we have a surprising result, indeed the optimal portfolio is composed only by four sectors with a 69.72% allocated in Healthcare sector.

We can make a brief recap of models behaviour. Increasing the risk propensity:

- MV over allocates gradually Healthcare sector, deducing that it may be the return driver for Markowitz approach, against Consumer staples sector.
- BLM variants start with good diversified portfolios with a poor implementation of the views, passing to very diversified portfolios with views that work in a proper way and concluding with portfolios concentrated in four assets with the absence of views consideration.
- RO present largely diversified portfolios for the first two level of risk, while, in the high return situation, RO loses the good diversification for a concentrated portfolio in four assets. The common factor is the increment of the allocation in Healthcare and Consumer staples sector. From this fact we can immediately see that, in contrast with MV approach, RO presents more stability because it is true that is going to search more return with aggressive sectors (i.e. Healthcare) but it does not "forget" to maintain in its optimal portfolio more defensive sectors (i.e. Consumer staples) to maintain a good ratio between risk and return.

5.2 Out-of-sample analysis

Now we will focus on the evaluation of the performance of these models with back testing. After the construction of the portfolios done with the models in the in-sample period, we will assess the results of them during the out-of-sample period, as if all data after 31st December 2014 is unknown. We recall that the latter goes from 2nd January 2015 to 30th September 2020 with a daily interval, so we will have 1447 observations that give us the possibility to obtain a good analysis.

We will start with the evaluation of Markowitz portfolios, going on with BLM without specifying any views. Then we will consider portfolios come out from BLM with the three different scenarios explained yet:

1) **Basic Scenario**
   a. IT sector performs 20% (uncertainty 5%)
   b. Energy sector performs -15% (uncertainty 5%)
   c. Industrials sector overperforms Financials sector by 12% (uncertainty 5%)

2) **Best Scenario**
   a. Healthcare sector overperforms Financials sector by 34,78% (uncertainty 5%)
   b. IT sector overperforms Consumer Discretionary sector by 84,97% (uncertainty 5%)
   c. Materials sector overperforms Utilities sector by 5,81% (uncertainty 5%)

3) **Worst Scenario**
   a. Healthcare sector underperforms Financials Sector by 34,78% (uncertainty 5%)
   b. IT sector underperforms Consumer discretionary sector by 84,97% (uncertainty 5%)
   c. Materials sector underperforms Utilities sector by 5,81% (uncertainty 5%)

The confidence level has been calibrated at 5% for all scenarios, following the level mostly used in the literature to produce meaningful results. In general, the range between 5% and 25% is acceptable. We remark that the two extreme cases of BLM, best and worst scenario, will be useful to understand if there is a margin to impact the performances with views formulation, but they are constructed in an artificial way so they will not be considered as best models to be picked. Finally, we will evaluate the performance of RO portfolios, including all the alternatives about the choice of the uncertainty matrix. This will permit us to check if the choice proposed by the literature is the best also in our empirical application. We recall that out-of-sample analysis will be conducted in two alternative ways: without portfolios rebalancing and with a 5 days rebalance, explained yet. Before to start with them it is necessary an analysis of the behaviour of single sectors that represent our asset universe.

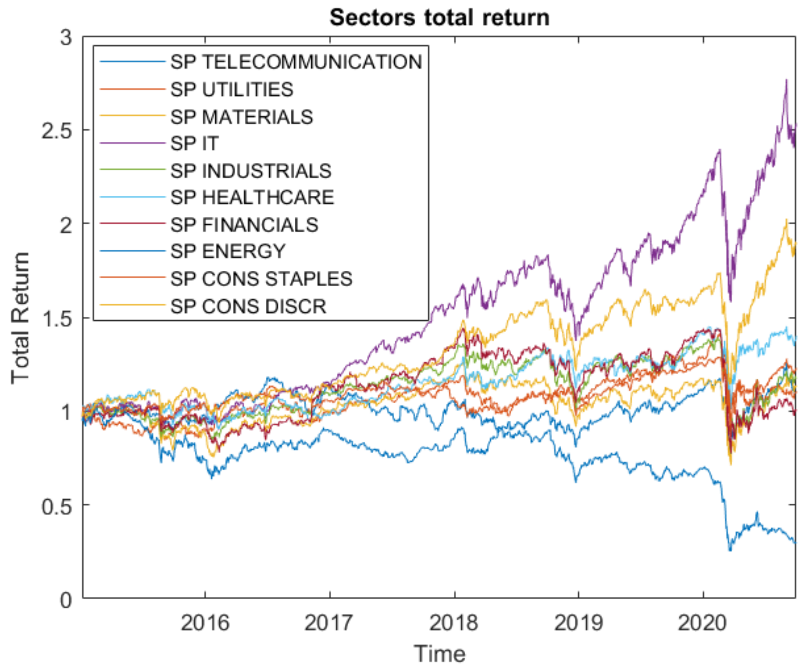


Figure 16: Sectors total return

From figure 16 we can see that until 2017 there are not big differences in sectors' performances. In the last three years IT, Consumer discretionary and Healthcare sectors stand out in a positive manner compared to the others, in particular they gain a lot during the pandemic period. On the other hand, Energy sector has suffered a lot Covid 19 crisis.

	Telecomm.	Utilities	Materials	IT	Industrials
Total return	13%	10%	13%	153%	17%
Mean return	4.23%	3.89%	4.68%	18.99%	5.09%
Volatility (σ)	20.12%	20.71%	21.96%	23.51%	21.42%
Sharpe ratio	14.02%	11.96%	14.89%	74.80%	17.21%
	Healthcare	Financials	Energy	Cons. staples	Cons. Discr.
Total return	40%	1%	-71%	23%	90%
Mean return	7.62%	3.19%	-16.58%	4.89%	13.13%
Volatility (σ)	18.64%	24.37%	30.47%	15.95%	19.57%
Sharpe ratio	33.30%	7.29%	-59.04%	21.83%	59.87%

Table 5: Sectors basic statistic

Table 5 represents the basic statistic of the single assets. We are able to see that the good results of SP&500 are mainly due to IT, Consumer discretionary and Healthcare sectors, indeed they have the best Sharpe ratios, a good summarizer of risk/return results (a brief description of it will be provided in the next section). Energy sector is the worst performer among all.

Considering these results, we can better understand the differences in allocation terms between MV and RO. We recall that both models are focused on Healthcare and Consumer staples sectors, but between the two models, RO demonstrates a higher diversification and stability with a higher homogeneity in allocation between the two sectors. We can explain this approach with the fact that Consumer staples sector has the lowest volatility among all sectors, so it contributes to maintain the risk level under control against the weight increment of Healthcare sector, the aggressive one. At the same time, MV is not able to implement this fact and increments the allocation of Healthcare sector increasing the risk propensity without a counterweight.

	Telec.	Util.	Mat.	IT	Indus.	Health.	Financ.	Ener.	Cons. Stap.	Cons. Disc.
Telec.	1	0.5252	0.6494	0.7160	0.6826	0.6475	0.6517	0.5649	0.6753	0.7434
Util.	0.5252	1	0.5453	0.5130	0.5692	0.5752	0.5161	0.4380	0.7475	0.5052
Mat.	0.6494	0.5453	1	0.7751	0.8927	0.7431	0.8412	0.7745	0.6862	0.7935
IT	0.7160	0.5130	0.7751	1	0.8030	0.7944	0.76	0.6291	0.6992	0.8874
Industr.	0.6826	0.5692	0.8927	0.8030	1	0.7715	0.8945	0.7784	0.7162	0.8323
Health.	0.6475	0.5752	0.7431	0.7944	0.7715	1	0.7465	0.5982	0.7360	0.7733
Financ.	0.6517	0.5161	0.8412	0.76	0.8945	0.7465	1	0.7648	0.6739	0.7930
Ener.	0.5649	0.4380	0.7745	0.6291	0.7784	0.5982	0.7648	1	0.5335	0.6614
Cons. Stap.	0.6753	0.7475	0.6862	0.6992	0.7162	0.7360	0.6739	0.5335	1	0.7020
Cons. Discr.	0.7434	0.5052	0.7935	0.8874	0.8323	0.7733	0.7930	0.6614	0.7020	1

Table 6: Sectors correlation

We can conclude the analysis of the single sectors with the correlation matrix, that is very important to understand the relation among all the assets.

We observe that the correlations are mainly included between 0.5 and 1. This is an expected result because all these sectors are contained in the same index, but nevertheless there are sectors more correlated than others and will be interesting to see how this will influence the performance of the various models. In particular, we observe a high correlation with other sectors for Consumer discretionary, Financials and Industrials sector.

5.2.1 No-rebalance approach

In this section we will evaluate performances of the portfolios constructed with in-sample data without modify the weights for all the out-of-sample period.

We will go through the analysis making an initial comparison among the models grouped for level of risk (low volatility, middle volatility and high volatility) and at the end we will select the final best choice on the basis of risk level. In the comparison we will include also the S&P 500, as a benchmark, to understand if the models are able to beat the market.

We will analyse portfolios taking in consideration the basic statistics like total return, mean, volatility and Sharpe ratio. Additionally, we will calculate the Information ratio. Sharpe ratio is a very popular measure of performance that measures the amount of return per unit of risk taken, measured with standard deviation of returns. Information Ratio is an indicator calculated as the ratio between the excess return of the portfolio compared to the benchmark and the Tracking Error Volatility (volatility of the differential returns of the portfolio with respect to a benchmark). It gives the possibility to evaluate the manager's ability to outperform the benchmark in relation to the risk assumed (represented by the deviation from the benchmark).

After this, we will go on with additional measures of risk exposure.

Maximum drawdown (MDD) measures the maximum fall in the value of the portfolio, as given by the difference between the value of the lowest trough and that of the highest peak before the trough. Downside volatility takes into account the fact that volatility is a symmetric measure. It only focuses on the negative returns when computing volatility and disregards the "positive" volatility (due to positive returns). Value at Risk, on the other hand, answers to the question: what is the return that I will observe in the worst 5% of possible scenarios?". To proxy this quantity, we use the empirical 5% quantile of the distribution. Expected Shortfall (ES) is a concept very close to VaR. If the VaR tells us what the 5% worst scenario is, ES tells us what the expected loss is given that we are in the worst 5% of possible outcomes. Using the downside risk measure we also provide another performance measure, the Sortino Ratio. The concept is analogous to the Sharpe Ratio but uses downside volatilities instead of normal standard deviations.

Remark: all the indicators are annualized (considering 252 business days in a year) and expressed in percentage, with the exception of Sortino ratio that is in absolute value.

5.2.1.1 Low volatility

Starting from low volatility portfolios, we present a table that summarizes the total return of them.

	Total return
MV	24.95%
BLM	46.58%
BLM-basic	46.50%
BLM-best	44.56%
BLM-worst	42.13%
RO $\Omega = \mathrm{diag}(\Sigma)$	35.33%
RO $\Omega = \Sigma$	24.95%
RO $\Omega = \mathrm{I}$	30.81%
RO $\Omega = \mathrm{sqrt}(\mathrm{diag}(\Sigma))$	33.94%
Market	47.40%

Table 7: Low volatility portfolios total return

From table 7 is evident that MV is the worst among them, the only one under 30% of total return. Concerning BML framework, we can confirm that views impact is not very relevant, indeed we can see that among the total return of all BLM optimizations is very similar. About the various RO optimizations, from a total return point of views the literature is confirmed indeed the best approach is RO with $\Omega = \mathrm{diag}(\Sigma)$. Concluding, among low volatility portfolios, the best model in terms of total return results the BLM without views, but the basic scenario case is very close. We highlight that no one model is able to overperform the market, only BLM is very close to S&P500.

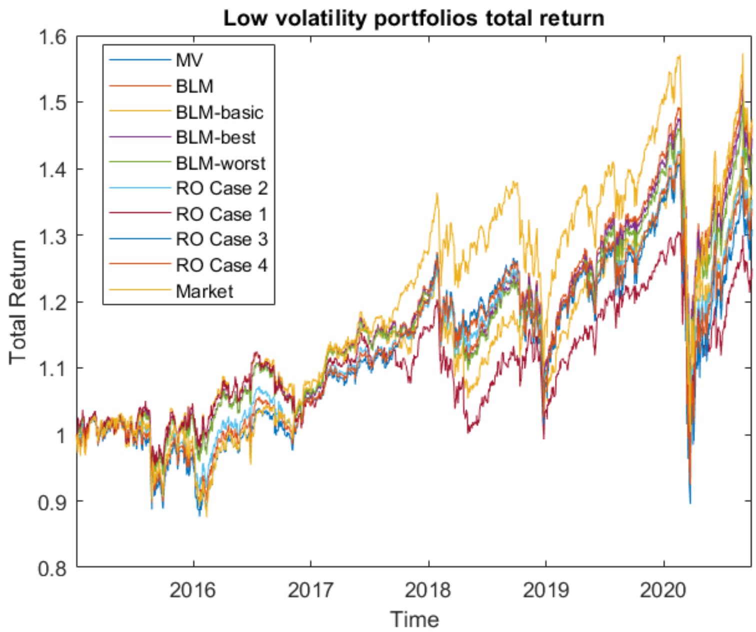


Figure 17: Low volatility portfolios total return

Figure 17 presents the evolution of total return of low volatility portfolios on the out-of-sample period.

	Mean return	Volatility (σ)	Sharpe ratio	Information ratio
MV	5.19%	15.76%	24.01%	-32.97%
BLM	7.60%	16.24%	38.10%	-13.59%
BLM-basic	7.58%	16.23%	38.03%	-13.67%
BLM-best	7.37%	16.16%	36.92%	-15.98%
BLM-worst	7.15%	16.21%	35.39%	-20.55%
RO $\Omega = \text{diag}(\Sigma)$	6.06%	17.02%	27.33%	-60.04%
RO $\Omega = \Sigma$	5.19%	15.76%	24.01%	-32.97%
RO $\Omega = \text{I}$	5%	18.40%	19.51%	-119.65%
RO $\Omega = \text{sqrt}(\text{diag}(\Sigma))$	5.64%	17.68%	23.94%	-90.21%
Market	8.56%	18.86%	37.90%	

Table 8: Low volatility portfolios basic statistic

Table 8 evidence that among all the low volatility portfolios, the best picking will be indifferent between BLM without views or BLM-basic. About RO variants, also from a risk/return point of view RO $\Omega = \text{diag}(\Sigma)$ is the best choice, so we will consider it the relevant RO. We can observe that MV is clearly more conservative compared to RO, despite it has a lower Sharpe ratio. In every case, they are united by the fact that BLM and BLM basic exploit completely them from return and risk point of view, clearly summarized by the Sharpe ratios. It is interesting the fact that BLM and BLM basic overperform the market from a Sharpe ratio, while their Information ratios are negatives. This means that two BLM variants perform better per unit of total volatility, but it is not the case per unit of volatility of the portfolio's differential returns relative to a benchmark (i.e. TEV). In other words, BLM approaches do not over perform the market.

	Maximum drawdown	Downside risk	VaR (95%)	ES (95%)	Sortino ratio
MV	26.95%	0.62%	-22.3%	-38.21%	5.94
BLM	29.47%	0.67%	-23.47%	-39.94%	9.10
BLM-basic	29.38%	0.67%	-23.53%	-39.92%	9.03
BLM-best	29.24%	0.66%	-23.53%	-39.67%	8.82
BLM-worst	29.97%	0.66%	-23.64%	-39.98%	8.48
RO $\Omega = \text{diag}(\Sigma)$	33.65%	0.70%	-23.29%	-42.69%	6.52
RO $\Omega = \Sigma$	26.95%	0.62%	-22.3%	-38.21%	5.94
RO $\Omega = I$	36.38%	0.78%	-25.85%	-46.84%	4.55
RO $\Omega = \text{sqrt}(\text{diag}(\Sigma))$	35.03%	0.74%	-24.75%	-44.74%	5.58
Market	36.10%	0.79%	-27.68%	-48.41%	8.95

Table 9: Low volatility portfolios risk exposure measures

Table 9 provide us a clear ranking in terms of risk exposure. We start with MV, that presents the lowest values for every indicator. Then we go through BLM variants, which have similar values among themselves. RO variants have the worst values with the exception of VaR, that is lower than BLM variants one. Additionally, all models present a lower risk exposure compared to the market but, despite this, only BLM variants present a higher Sortino ratio. This means that they have the better trade-off between negative outcomes and average returns.

5.2.1.2 Middle volatility

Now we present the total return of middle portfolios in the following table:

	Total return
MV	30.62%
BLM	68.55%
BLM-basic	74.52%
BLM-best	78.81%
BLM-worst	57.84%
RO $\Omega = diag(\Sigma)$	39.96%
RO $\Omega = \Sigma$	30.62%
RO $\Omega = I$	35.69%
RO $\Omega = sqrt(diag(\Sigma))$	38.94%
Market	47.40%

Table 10: Middle volatility portfolios total return

In this risk framework, the power of views comes out, indeed from the worst scenario to the best scenario there is a big difference in total return, namely 20,97%. BLM basic scenario performs somewhat below BLM best, it means that views of basic scenario are very good. About other models, RO and Markowitz increase their performances, but not sufficiently to beat the market. On the other hand, all BLM approaches are able to beat the benchmark.

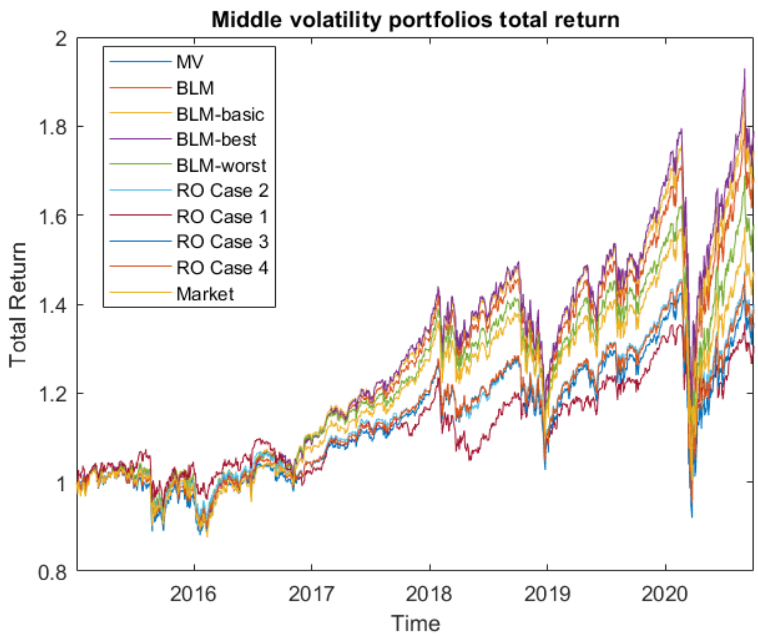


Figure 18: Middle volatility portfolios total return

Figure 18 presents the evolution of total return on the out-of-sample period.

	Mean return	Volatility (σ)	Sharpe ratio	Information ratio
MV	6.10%	15.98%	29.38%	-30.20%
BLM	9.95%	19.51%	43.78%	76.70%
BLM-basic	10.84%	19.56%	48.20%	106.68%
BLM-best	10.92%	20%	47.54%	85.64%
BLM-worst	8.93%	18.95%	39.70%	20.65%
RO $\Omega = \text{diag}(\Sigma)$	6.45%	17.01%	29.61%	-49.89%
RO $\Omega = \Sigma$	6.10%	15.98%	29.38%	-30.20%
RO $\Omega = \text{I}$	5.26%	18.26%	21.07%	-107.89%
RO $\Omega = \text{sqrt}(\text{diag}(\Sigma))$	6%	17.62%	26.03%	-78.08%
Market	8.56%	18.86%	37.90%	

Table 11: Middle volatility portfolios basic statistic

In the middle volatility framework, despite BLM approaches volatility levels increase, they have high Sharpe ratios. Additionally, positive Information ratios demonstrate that they over perform the market with conviction. In particular we will pick BLM basic scenario. About RO, the best performer is still RO $\Omega = \text{diag}(\Sigma)$ and it is interesting the fact that middle portfolio presents the same volatility of low volatility portfolio. Taking a look to the high return portfolio we will see if RO $\Omega = \text{diag}(\Sigma)$ is able to maintain this trend of low volatility level and obtain better returns at the same time.

	Maximum drawdown	Downside risk	VaR (95%)	ES (95%)	Sortino ratio
MV	28.04%	0.62%	-22.72%	-38.85%	7.22
BLM	35.49%	0.81%	-28.33%	-50.24%	10.16
BLM-basic	35.28%	0.81%	-28.38%	-50.28%	11.20
BLM-best	34.91%	0.82%	-29.19%	-51.38%	11.09
BLM-worst	35.83%	0.78%	-27.50%	-48.80%	9.24
RO $\Omega = \text{diag}(\Sigma)$	33.03%	0.68%	-23.29%	-42.56%	7.10
RO $\Omega = \Sigma$	28.04%	0.62%	-22.72%	-38.85%	7.22
RO $\Omega = \text{I}$	35.50%	0.75%	-25.67%	-46.42%	4.95
RO $\Omega = \text{sqrt}(\text{diag}(\Sigma))$	34.31%	0.72%	-24.33%	-44.54%	6.15
Market	36.10%	0.79%	-27.68%	-48.41%	8.95

Table 12: Middle volatility portfolios risk exposure measures

Table 12 presents a different situation compared to low volatility framework. Risk exposure of all models increases, with the exception of RO $\Omega = \text{diag}(\Sigma)$, confirming the stability noted before also from volatility. Despite this, it is not sufficient to obtain a better result in terms of Sortino ratio compared to one of the market. However, BLM variants confirm their good results for this risk propensity considering Sortino ratio values.

5.2.1.3 High volatility

Now we present the total return of high volatility portfolios in the following table:

	Total return
MV	38.19%
BLM	0.27%
BLM-basic	7.77%
BLM-best	1.29%
BLM-worst	2.92%
RO Ω = diag(Σ)	44.70%
RO $\Omega = \Sigma$	38.19%
RO Ω = I	44.24%
RO Ω = sqrt(diag(Σ))	44.59%
Market	47.40%

Table 13: High volatility portfolios total return

For a higher propensity to risk, it is impressive to see that BLM fails completely the task. From the graph below it is clear that COVID-19 crisis impacted in a big manner BLM approach, no matter the views included. Equally impressive is the fact that Markowitz approach is close to the RO model, which in this case is the best in terms of total return. Additionally, it is interesting the fact that, for aggressive portfolio, all types of RO are very close.

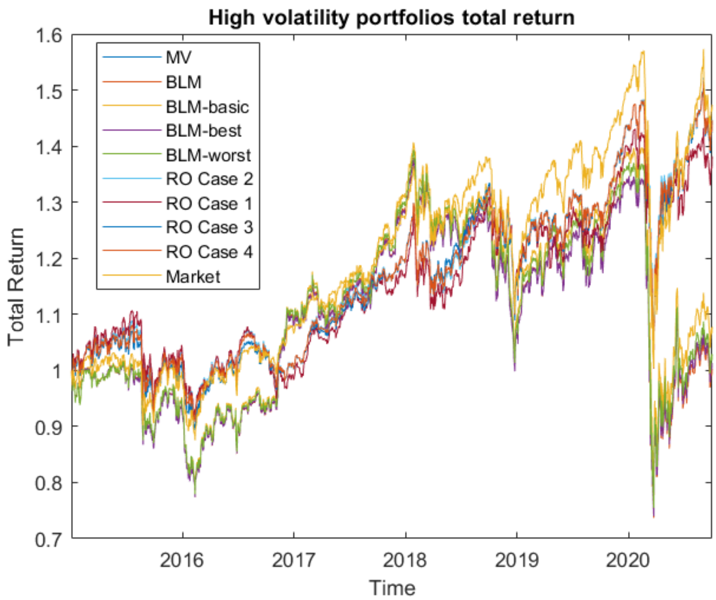


Figure 19: High volatility portfolios total return

	Mean return	Volatility (σ)	Sharpe ratio	Information ratio
MV	7.32%	17.92%	32.96%	-14.63%
BLM	2.55%	23.29%	4.89%	-63.04%
BLM-basic	3.92%	23.04%	10.90%	-50.67%
BLM-best	2.56%	23.20%	4.98%	-64.20%
BLM-worst	2.98%	23.24%	6.77%	-58.97%
RO Ω = diag(Σ)	7.57%	17.60%	34.97%	-14.81%
RO Ω = Σ	7.32%	17.92%	32.96%	-14.63%
RO Ω = I	6.54%	18.19%	28.20%	-39.45%
RO Ω = sqrt(diag(Σ))	7.18%	17.76%	32.47%	-23.02%
Market	8.56%	18.86%	37.90%	

Table 14: High volatility portfolios basic statistic

From the table above we can see that BLM results are in free fall, with an increase in volatility and a fall of mean return. The result is that Sharpe ratios are very bad. About RO Ω = diag(Σ), the stable tendency noted before is confirmed indeed the volatility is slightly higher than low volatility and middle portfolio framework, but is a little increment compared to the return one. We can conclude that increasing the risk propensity, it is evident the RO exploit compared also to Markowitz optimization, but, taking a look to Information ratios, it is still insufficient for RO to overperform the benchmark.

	Maximum drawdown	Downside risk	VaR (95%)	ES (95%)	Sortino ratio
MV	29.90%	0.71%	-26.53%	-44.05%	8.07
BLM	46.57%	0.92%	-34.25%	-58.62%	1.20
BLM-basic	44.50%	0.92%	-34.07%	-58.01%	2.67
BLM-best	46.21%	0.93%	-34.28%	-58.45%	1.22
BLM-worst	45.81%	0.93%	-33.88%	-58.49%	1.66
RO Ω = diag(Σ)	30.17%	0.70%	-25.04%	-43.43%	8.44
RO Ω = Σ	29.90%	0.71%	-26.53%	-44.05%	8.07
RO Ω = I	31.74%	0.74%	-25.64%	-45.71%	6.67
RO Ω = sqrt(diag(Σ))	30.73%	0.71%	-25.42%	-44.12%	7.82
Market	36.10%	0.79%	-27.68%	-48.41%	8.95

Table 15: High volatility portfolios risk exposure measures

Table 15 presents predictable results, with an increasing of the risk exposure due to the higher risk propensity. RO Ω = diag(Σ) continues to be an exception in this sense, with values in line with them of lower risk propensities and in particular the maximum drawdown is the lowest. RO $\Omega = \text{diag}(\Sigma)$ obtains a good Sortino ratio but not enough to beat the market.

Now we can summarize the choices that we have done in risk/return terms.

	Total return	Mean return	Volatility (σ)	Sharpe ratio	Information ratio
BLM (low volatility)	46.58%	7.60%	16.24%	38.10%	-13.59%
BLM-basic (middle volatility)	74.52%	10.84%	19.56%	48.20%	106.68%
RO $\Omega = \text{diag}(\Sigma)$ (high volatility)	44.7%	7.57%	17.60%	34.97%	-14.81%
Market	47.4%	8.56%	18.86%	37.90%	

Table 16: Basic statistic summary

From this table we notice some interesting facts. First, we observe that low volatility BLM portfolio overperform the RO high volatility portfolio due to the fact that RO volatility is higher while the mean return is practically the same. Second, only BLM-basic overperform the market with conviction, while BLM low volatility performs mainly the same from Sharpe ratio point of view. Third, total return of all approaches perfectly reflects the Sharpe ratios. Fourth, RO high volatility portfolio has a lower volatility compared to the middle volatility portfolio obtained with BLM, but there is no challenge about the return. Usually, increasing the risk propensity, the volatility increases in a significant manner, while, in the RO case, we observe that the increment in volatility is very moderate. We can find the cause of this phenomena in the correlation among assets and the particular feature of RO, namely the stability and the high level of diversification. We have seen before that the correlations among assets are very high because they are included in the same index. Given that, analysing the allocation weights of RO, we can see that going from low volatility portfolio (Portfolio 2) to high volatility portfolio (Portfolio 9), we have an increasing concentration in Healthcare and Consumer Staples sectors and these two sectors have a correlation of 0.7360, so they go mainly in the same direction. Additionally, they have the lowest values of volatility among all the assets. The combination of these two facts conducts to argue that, in this case, a lower diversification is a good fact and permits to RO model to maintain stable the overall risk increasing risk propensity.

In the following graph we plot total return of chosen models.

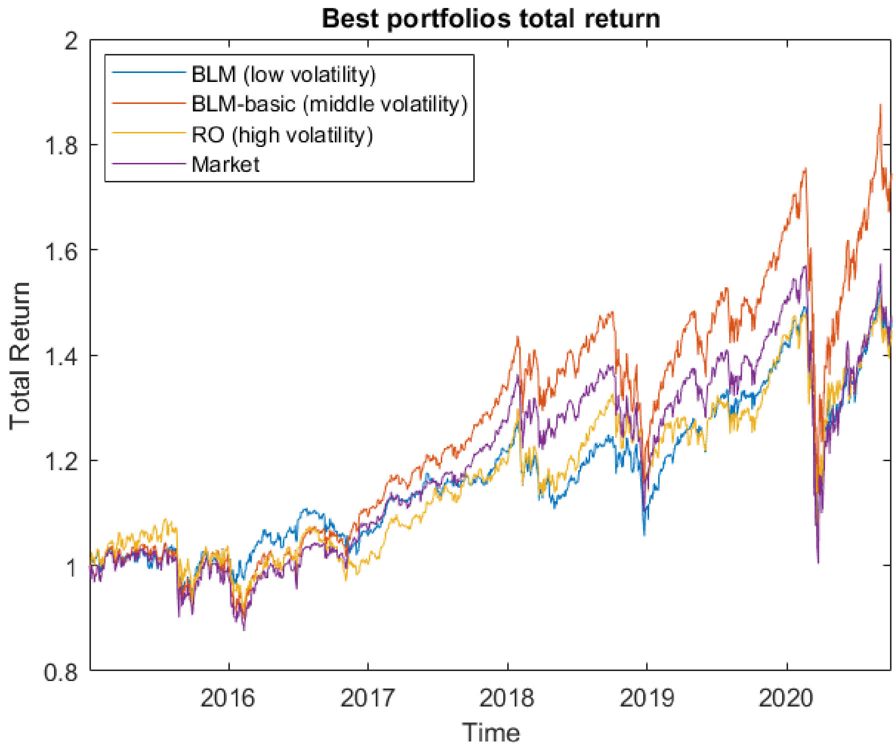


Figure 20: Best portfolios total return

	Maximum drawdown	Downside risk	VaR (95%)	ES (95%)	Sortino ratio
BLM (low volatility)	29.47%	0.67%	-23.47%	-39.94%	9.10
BLM-basic (middle volatility)	35.28%	0.81%	-28.38%	-50.28%	11.20
RO $\Omega = \text{diag}(\Sigma)$ (high volatility)	30.17%	0.70%	-25.04%	-43.43%	8.44
Market	36.10%	0.79%	-27.68%	-48.41%	8.95

Table 17: Risk exposure measures summary

We notice from the table above that we have higher values for the middle volatility portfolio. This is a direct consequence of the RO exception. However, taking a look to the Sortino ratio, that summarizes the trade-off between negative outcomes and average returns, we observe that the best result is provided by BLM-basic middle volatility portfolio.

5.2.1.4 Summary

From the analysis above we can summarize some general facts about all models. First, among all four variants of RO the choice falls on the alternative indicated by the literature, indeed it overperforms the other options for every risk propensity. Second, only BLM approaches in middle volatility framework overperform with conviction the market, while low volatility BLM is very similar to the benchmark. This is highlighted by the fact that views work in the proper way in middle volatility situation. Third, we have a performance escalation of RO from low volatility to high volatility portfolios. Fourth, Markowitz approach is always beaten by other models.

- In the low volatility environment, MV is the most conservative model while RO is the most exposed portfolio despite the high level of diversification. BLM variants outstanding MV and RO models, and in particular BLM standard provides very good results so we can conclude that views are not relevant due to the composition of the BLM optimal portfolios. All information ratios negative mean that all models are not able to beat the market. On the other hand, from a risk exposure point of view, BLM variants perform better than the market thanks to Sortino ratios higher than the benchmark one.
- In the middle volatility framework, we can see a big exploit of BLM approaches, despite the ascent of MV and RO models. In particular, all models increase volatility and return with the exception of RO that maintain a stable level of risk. In this case we can see a great influence of the views expressed, indeed all BLM with views are the best models on every level. BLM variants overperform the market, testified by Information ratios positive and Sortino ratios higher than the market one.
- In the high volatility context, we can see a fully debacle of BLM approaches, mainly due to COVID-19 crisis, in favour of MV and especially of RO. It is interesting to observe that in this situation BLM come back to a situation where the views do not have a big effect on the performances. We observe a RO exploit thanks to the ongoing stability of volatility and risk exposure but is still not sufficient to beat the market.

Concluding, increasing the risk propensity we can see that we go from good results of BLM to an outstanding of RO. In a low volatility environment, it is better making a BL optimization starting from neutral weights with a negligible influence of the views. In this optimal portfolio we detect three main sectors: Consumer staples, IT and Healthcare.

About middle volatility, the importance of the views arises, and we will shift to BLM-basic. The optimal allocation is high diversified, and we find five main sectors: IT, Consumer discretionary, Healthcare, Financials and Telecommunication.

Finally, when the risk is higher, RO stability becomes the winning strategy despite the poor diversification of the optimal portfolio. In particular, we note two main sectors: Healthcare and Consumer staples. These two sectors are identified also in the optimal low volatility portfolio, so this is another point in favour of the RO capacity to maintain a less risky portfolio despite the increment of risk propensity.

Without considering risk propensity distinction, our best choice from all points of view is middle volatility portfolio constructed with BLM-basic, marked by an elevate diversification that gives the possibility to work in a proper manner to the views.

	Low volatility	Middle volatility	High volatility
Choice	BLM	BLM-basic	RO $\Omega = diag(\Sigma)$

Table 18: Best models choice

5.2.2 Rebalance approach

In this section we will provide a performance analysis making a step ahead. We will not use the same weights calculated in the in-sample period, instead we will make a periodic rebalance of the optimal weights, in particular every 5 days. In practical terms, after the first 5 days of the out-of-sample period the optimizer recalculates the weights on the basis of data of the in-sample period plus the 5 days additional and this process go ahead for all the out-of-sample period. It is intuitive that with approach we will have a more dynamic process that gives the possibility to the models to work in a better way, due to the higher quantity of information that they will process. We highlight that we provide a starting value of 100 to the portfolios to evaluate their performances.
We will apply the same schedule of the previous section: before we make a comparison of the models grouped by risk propensity, then we will make a final confrontation among the best models for each level of risk using the performance indicators used before. Concerning the four different cases of RO, we decided to limit the analysis only to the RO $\Omega = \text{diag}(\Sigma)$ because we have seen in the previous section that it is optimal, compared to other cases, in all situations. This conclusion is supported by the literature, as we mentioned.

5.2.2.1 Low volatility

As usual, we will start with less risky portfolios providing the total return of them.

	Total return
MV	34.55%
BLM	62.85%
BLM-basic	62.78%
BLM-best	61.74%
BLM-worst	61.31%
RO $\Omega = \text{diag}(\Sigma)$	52.14%
Market	47.40%

Table 19: Low volatility portfolios total return

Looking at table 19, we will observe many similarities with the results of no-rebalance approach. The proportions among the models results are very similar, indeed BLM still performs very well in all of his variants, the role of the views is not very relevant and the differences compared to Markowitz and RO are mainly the same. However, in absolute terms the results are very different, because all models have performed better than the previous approach, in particular Markowitz makes an increment in the order of 10%, while BLM and RO in the order of 16%. Considering these numbers, the other important difference is that all the models overperformed the market, with the only exception of Markowitz.

Below we will provide, as usual, the graph representing the trend during the out-of-sample period.

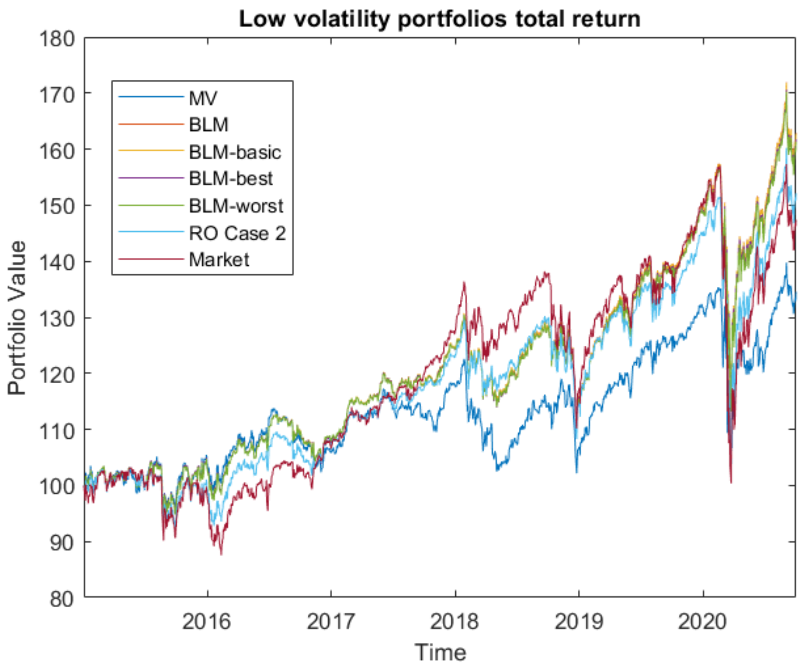


Figure 21: Low volatility portfolios total return

In the following graphs we will see how the various sectors contribute to the portfolio value for each model.

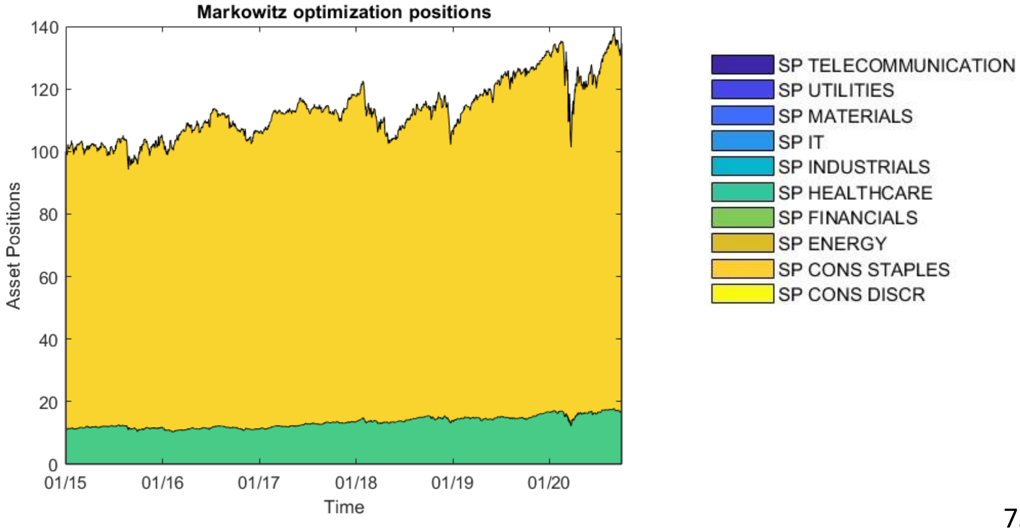


Figure 22: MV low volatility sectors contribution

About MV optimization, we can see that the dynamic is provided mainly by the Consumer staples sector, while the Healthcare sector is practically constant overtime. It is immediately evident the advantage of rebalancing: in this case it gives the possibility to overweight the portfolio in the Consumer staples sector to obtain better results in terms of total return.

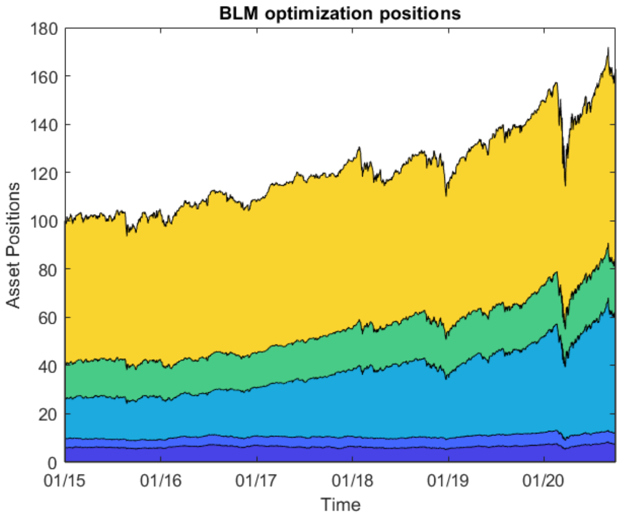


Figure 23: BLM low volatility sectors contribution

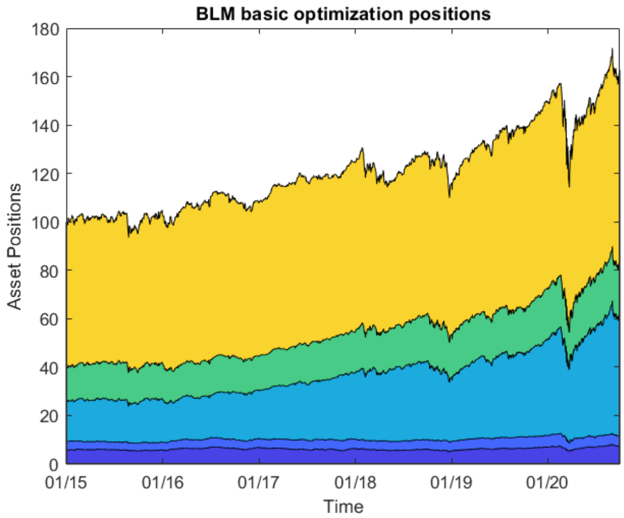


Figure 24: BLM basic low volatility sectors contribution

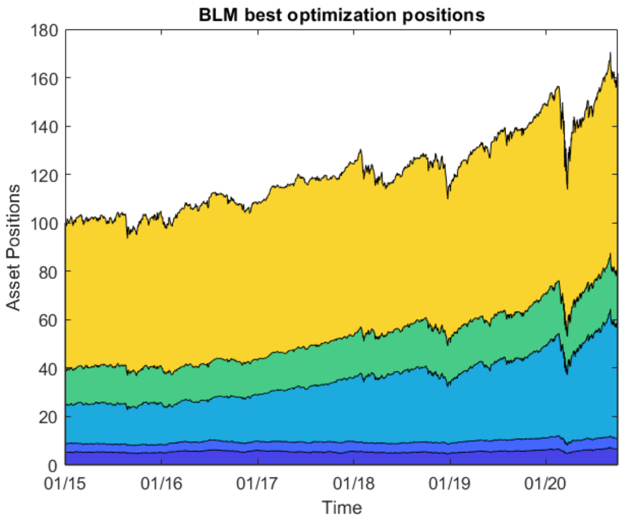


Figure 25: BLM-best low volatility sectors contribution

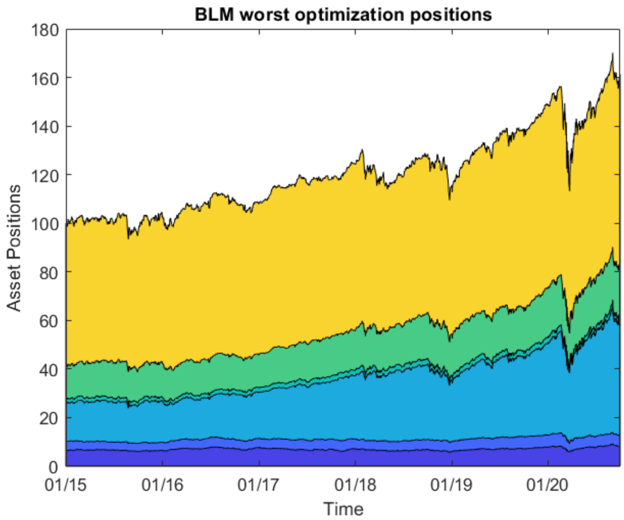


Figure 26: BLM worst low volatility sectors contribution

We have regrouped all the four graphs because the influence of the views is irrelevant for this risk propensity, as a consequence all four variants of BLM are mainly the same. We immediately notice that the main contributors are IT sectors, Healthcare sector and Consumer staples sector, with the latter especially relevant. Even for BLM, the dynamical rebalancing provides better results.

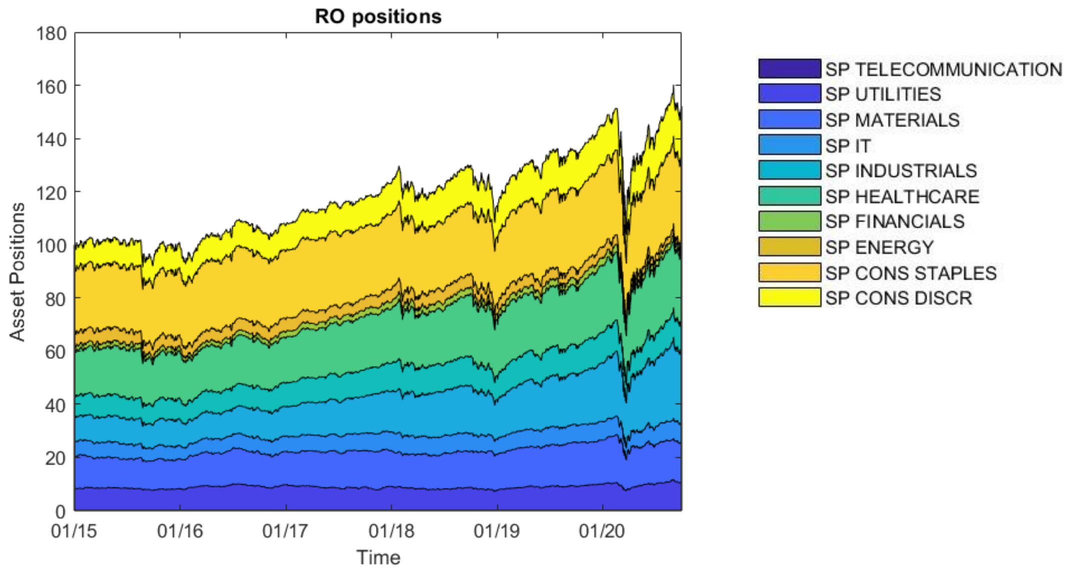


Figure 27: RO low volatility sectors contribution

About RO, we can observe a higher level of diversification with the involvement of all sectors in the optimal portfolio. All sectors present a high level of stability, but also for RO we can identify three driving assets, namely IT, Healthcare and Consumer staples sectors.

In the table below we provide the same indicators used in the previous section to understand models performances from a risk/return point of view.

	Mean return	Volatility (σ)	Sharpe ratio	Information ratio
MV	6.41%	15.74%	31.77%	-20.64%
BLM	9.88%	16.62%	50.98%	21.05%
BLM-basic	9.87%	16.61%	50.94%	20.83%
BLM-best	9.75%	16.56%	50.37%	18.66%
BLM-worst	9.71%	16.59%	50.04%	18.60%
RO $\Omega = \mathrm{diag}(\Sigma)$	8.76%	17%	43.25%	5.16%
Market	8.56%	18.86%	37.90%	

Table 20: Low volatility portfolios basic statistic

From table 20 we can see that the model ranking is the same of no-rebalance approach, but there are some interesting facts. First, better performances are due to the return increase because volatility levels are mainly the same among BLM variants and RO. Second, RO reduces the distance from BLM variants, that still remain the best models for this risk propensity. Additionally, all models, with the exception of MV, are able to beat the market.

	Maximum drawdown	Downside risk	VaR (95%)	ES (95%)	Sortino ratio
MV	25.12%	0.60%	-22.18%	-37.62%	7.94
BLM	27.39%	0.65%	-24.19%	-40.47%	12.31
BLM-basic	27.36%	0.65%	-24.22%	-40.47%	12.30
BLM-best	27.31%	0.65%	-24.10%	-40.47%	12.16
BLM-worst	27.60%	0.65%	-24.18%	-40.47%	12.12
RO $\Omega = diag(\Sigma)$	31.57%	0.68%	-23.07%	-42.10%	10.43
Market	36.10%	0.79%	-27.68%	-48.41%	8.95

Table 21: Low volatility portfolios risk exposure measures

Table 21 provides us a non-uniform situation. About maximum drawdown and downside risk we note slightly better results than no-rebalance approach, while from VaR and ES point of view we identify a worst situation. This tells us that the stability of the models is improved, but at the same time, if one of the worst scenarios occur, we will have greater losses. The increased stability is confirmed by Sortino ratio values that overperform the market with conviction.

5.2.2.2 Middle volatility

	Total return
MV	42.38%
BLM	93.79%
BLM-basic	96.72%
BLM-best	99.40%
BLM-worst	87.96%
RO $\Omega = \text{diag}(\Sigma)$	56.97%
Market	47.40%

Table 22: Middle volatility portfolios total return

From table 22 we can see that the exploit of BLM compared to Markowitz and RO is confirmed, and the distances are even greater. A difference from the no-rebalance approach is that views are less relevant, indeed the difference between the best case and the worst case is 11,44% compared to 20,97%. Additionally, deserves to highlight the fact that rebalance approach gives the possibility to RO, also in middle volatility framework, to beat the benchmark. We can conclude that all models are taking advantage from rebalancing considering this level of risk, more than other BLM.

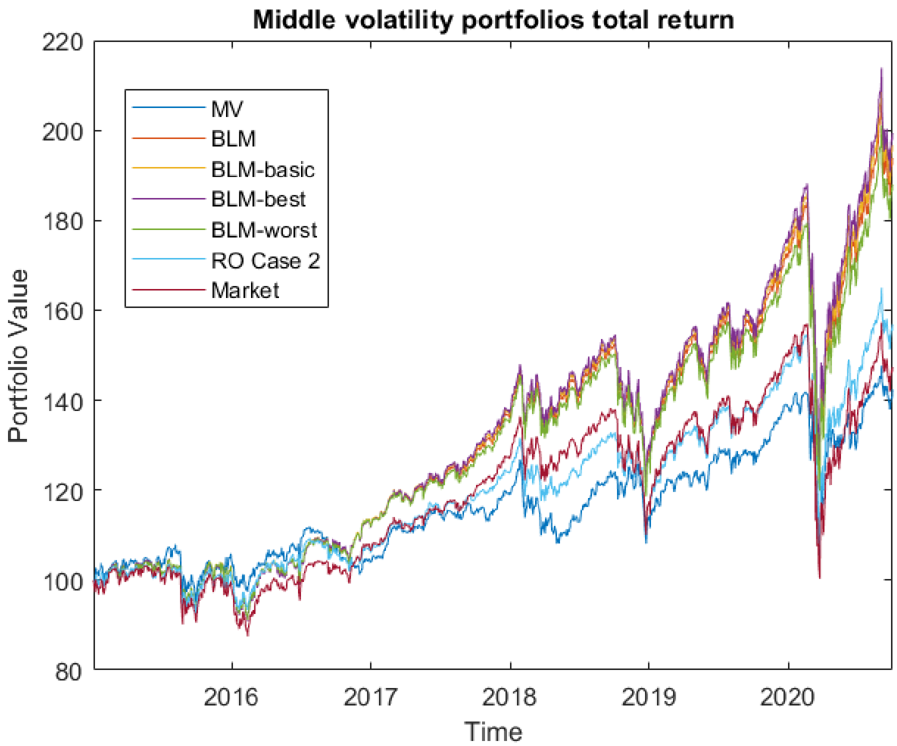


Figure 28: Middle volatility portfolios total return

Also for middle volatility we provide dynamical sectors positions graphs.

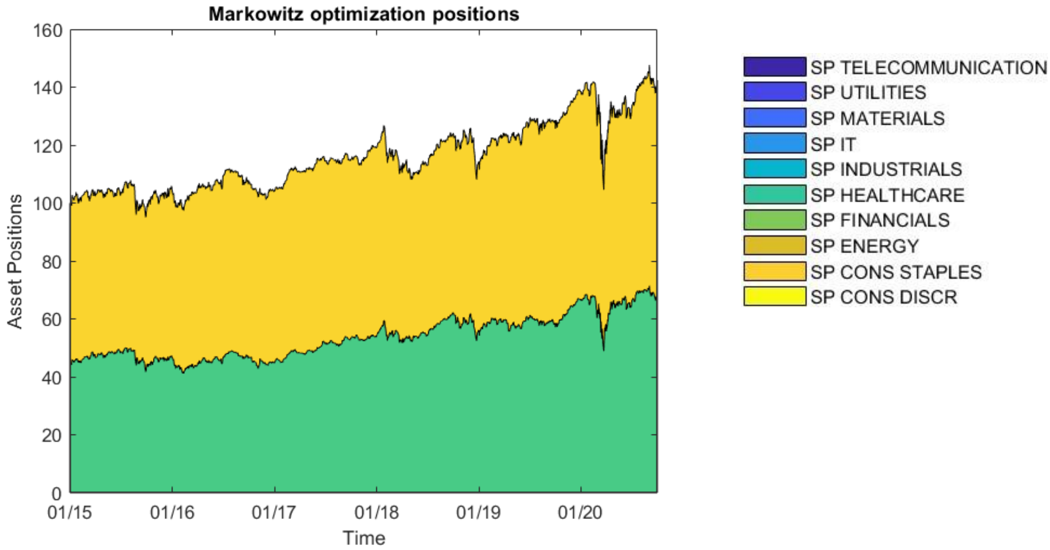


Figure 29: MV middle volatility sectors contribution

We can see that MV takes in consideration the same sectors, but there is an evident difference. In middle volatility case there is not only Consumer staples as main driver of the return, but also the value of the position in Healthcare sector is increasing overtime. We can argue that Markowitz optimizer consider, as aggressive asset, Healthcare sector.

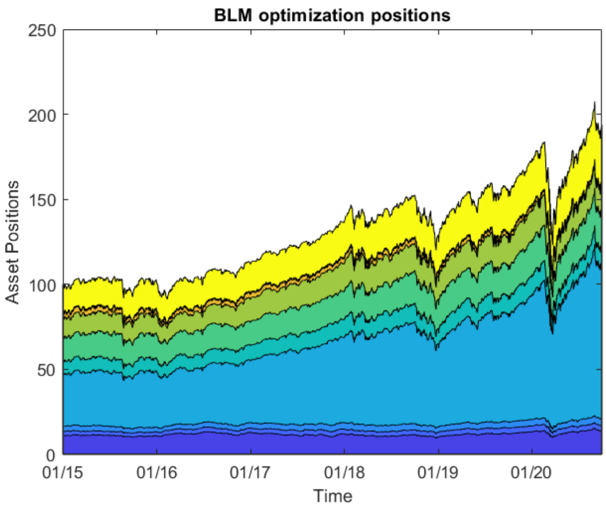


Figure 30: BLM middle volatility sectors contribution

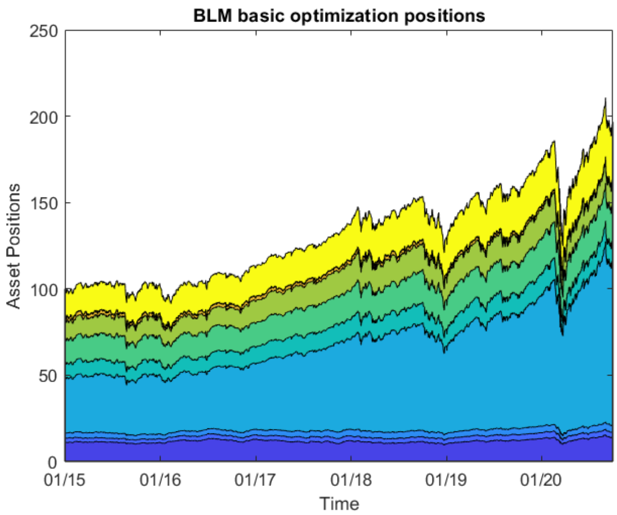
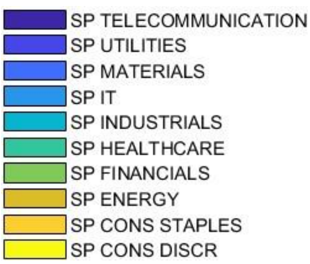


Figure 31: BLM-basic middle volatility sectors contribution



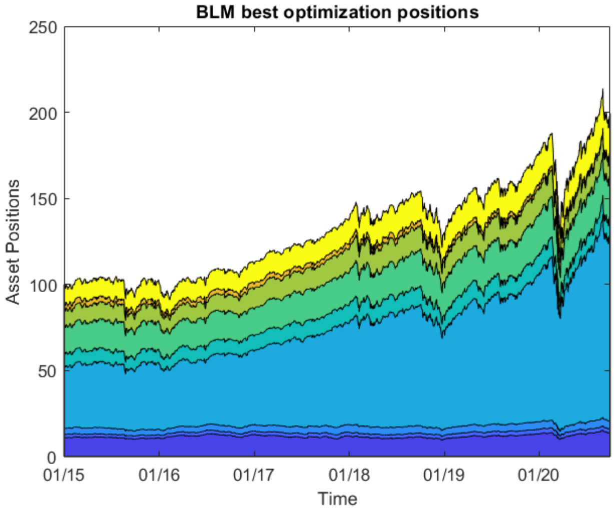


Figure 32: BLM-best middle volatility sectors contribution

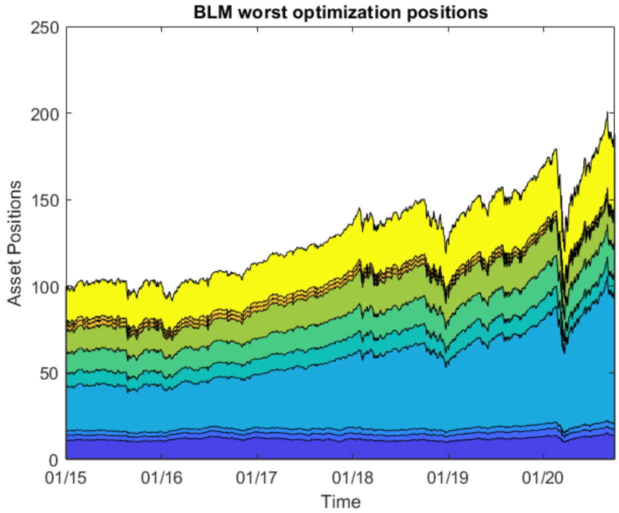


Figure 33: BLM-worst middle volatility sectors contribution

From figures 30, 31, 32, 33 we can see that the four variants of BLM have a common factor. The views influence comes back in a strong way and all four allocations have a high position in IT sector while Consumer staples sector has practically vanished from the optimal allocations. This is mainly due to the fact that, increasing the risk propensity, the model shifts towards less conservative sectors. We can define it a winning strategy from the absolute return point of view.

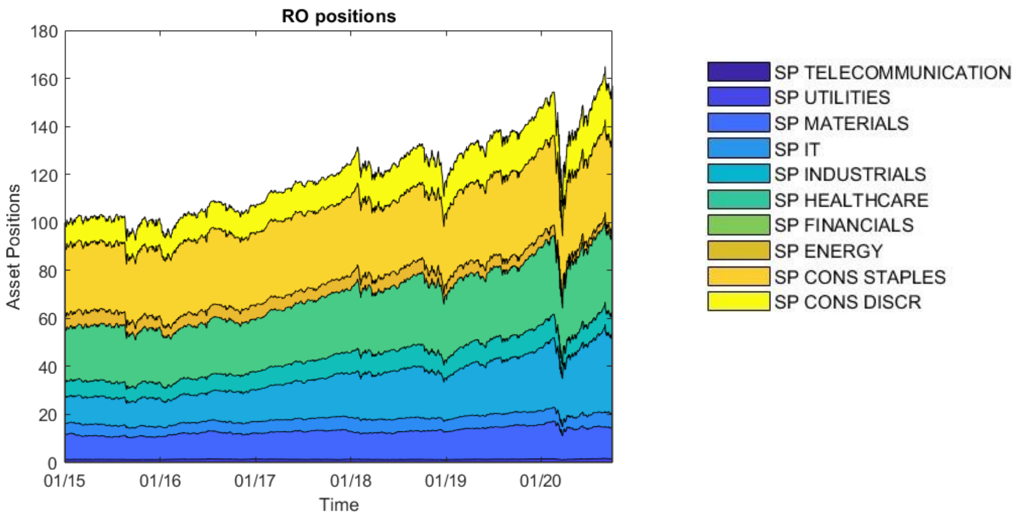


Figure 34: RO middle volatility sectors contribution

About RO, we can observe that the typical diversification of this model is confirmed but we have also to highlight that there is a tendency in overweighting some sectors compared to low volatility portfolio. With the traditional stability, we have mainly three sectors that drive the return: Consumer staples, Healthcare and IT.

	Mean return	Volatility (σ)	Sharpe ratio	Information ratio
MV	7.43%	15.99%	37.68%	-13.78%
BLM	13.48%	19.71%	61.24%	190.14%
BLM-basic	13.75%	19.76%	62.46%	189.92%
BLM-best	14.03%	19.99%	63.16%	179.08%
BLM-worst	12.89%	19.42%	59.13%	195.42%
RO $\Omega = \mathrm{diag}(\Sigma)$	9.31%	17.03%	46.41%	19.01%
Market	8.56%	18.86%	37.90%	

Table 23: Middle volatility portfolios basic statistic

From table 23 we can see that BLM variants are the best choices also in terms of risk/return. We observe a difference of 2% about volatility between BLM and RO, but thanks to higher levels of return BLM variants exploit with conviction RO, in particular BLM-basic. Comparing with no-rebalance results, we find a convinced increment in performance because the levels of volatility are mainly the same while returns are much higher. Additionally, it is interesting to observe that RO presents the same level of volatility of low vol portfolio. Concluding, the capacity of BLM variants and RO to overperform the market is confirmed.

	Maximum drawdown	Downside risk	VaR (95%)	ES (95%)	Sortino ratio
MV	26.28%	0.62%	-22.74%	-38.37%	9.42
BLM	33.06%	0.79%	-28.10%	-49.73%	14.45
BLM-basic	32.97%	0.79%	-28.19%	-49.73%	14.73
BLM-best	32.80%	0.80%	-28.38%	-49.73%	14.95
BLM-worst	33.27%	0.77%	-27.72%	-49.73%	14.20
RO $\Omega = \mathrm{diag}(\Sigma)$	30.96%	0.67%	-23%	-42.06%	11.30
Market	36.10%	0.79%	-27.68%	-48.41%	8.95

Table 24: Middle volatility portfolios risk exposure measures

Table 24 tells us some important results. First, RO confirms is capacity to maintain a low and stable risk exposure going through all risk propensities, indeed present even lower values compared to low volatility portfolios. On the other hand, all other models present higher values as expected, but making a comparison with the same risk propensity of no-rebalance approach the situation is different. About maximum drawdown and downside risk we observe considerably smaller values, while VaR and ES values are slightly smaller. Concluding, the introduction of rebalancing has provided a reduction in the risk exposure, also evidenced by higher Sortino ratios.

5.2.2.3 High volatility

In the table below we provide the total return for each model considering high volatility framework.

	Total return
MV	52.83%
BLM	18.46%
BLM-basic	21.47%
BLM-best	18.23%
BLM-worst	18.66%
RO $\Omega = \text{diag}(\Sigma)$	61.33%
Market	47.40%

Table 25: High volatility portfolios total return

Table 25 provides a very different situation compared to the two other risk propensities. It is evident the exploit of Markowitz and RO, especially the latter one. On the other hand, BLM approaches completely fail their work, even considering the views impact, with a total return very far from Markowitz and RO, and, additionally, they are not able to overperform the market. However, we have to highlight that rebalancing gives a big hand to BLM variants to reduce the damages, indeed BLM total return is 18% higher than the no-rebalancing one, while for BLM-basic is 14% higher.

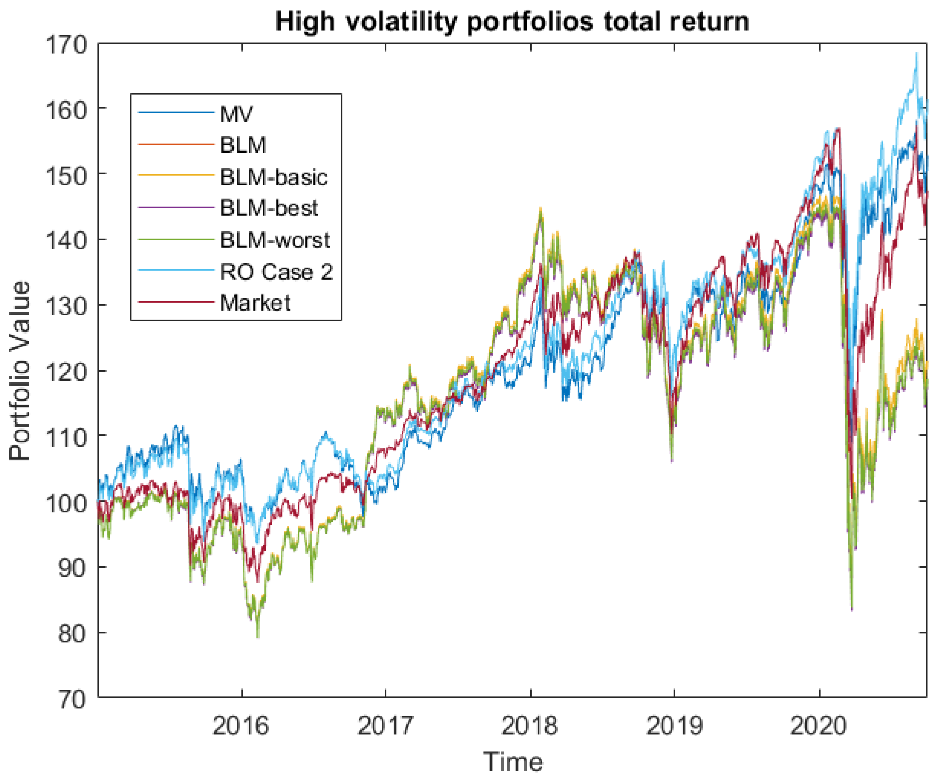


Figure 35: High volatility portfolios total return

About dynamical sectors positions, we provide below all the graphs concerning all the models.

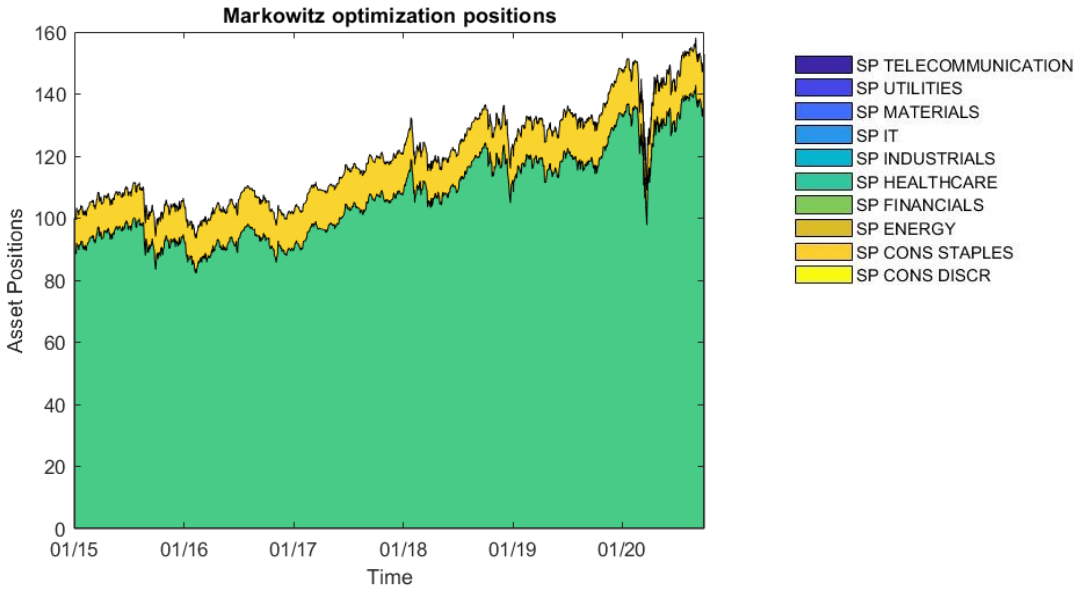


Figure 36: MV high volatility sectors contribution

About MV optimization, we observe that moving from low volatility to a high volatility situation the model increases the overweighting in Healthcare sector, that is much more aggressive compared to Consumer staples sector.

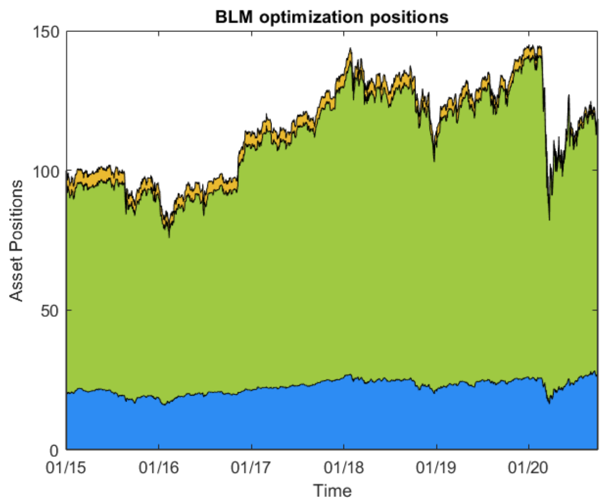


Figure 37: BLM high volatility sectors contribution

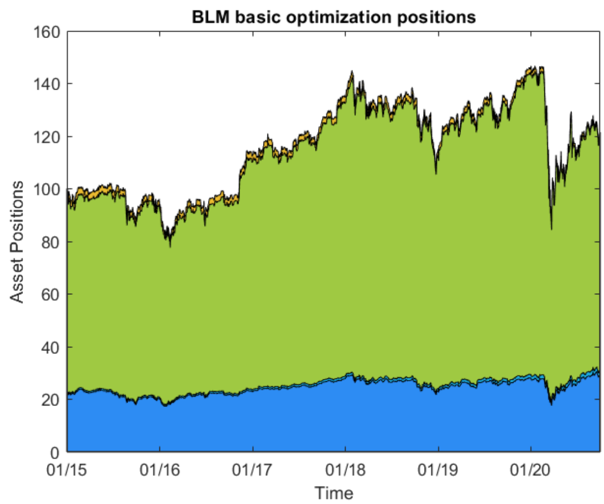
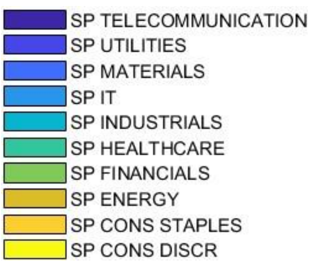


Figure 38: BLM-basic high volatility sectors contribution



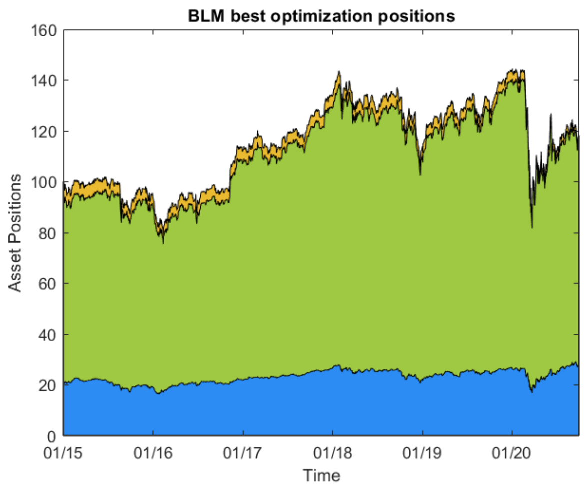


Figure 39: BLM-best high volatility sectors contribution

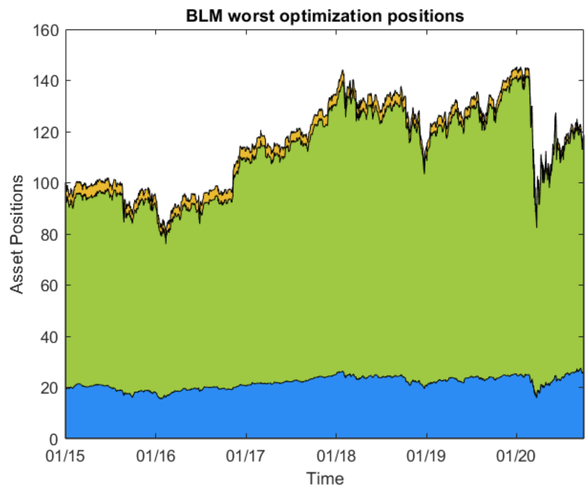


Figure 40: BLM-worst high volatility sectors contribution

The failure of BLM for the highest risk propensity is shown also graphically. We observe a high concentration in Financials sector, that is the worst sector in terms of risk/return. Additionally, the introduction of the views is not able to modify in a relevant way the allocations because sectors involved are others, namely Financials and Materials sectors.

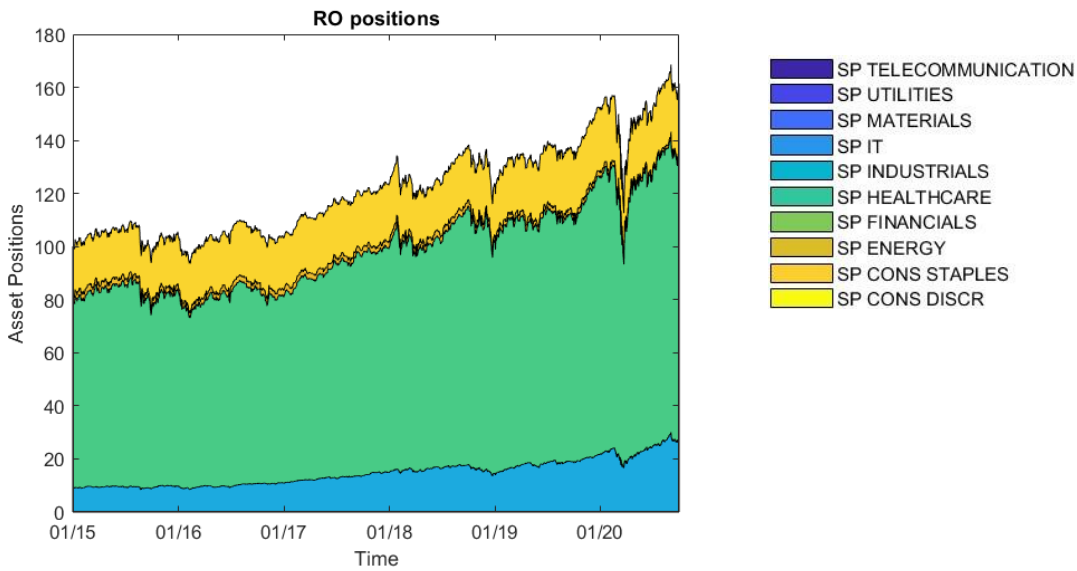


Figure 41: RO high volatility sectors contribution

From figure 41 we can see that RO reduces the diversification in the high volatility optimal portfolio. We observe the presence of three main sectors: Healthcare, Consumer staples and IT. The difference between Markowitz and RO is the presence of IT sector, that gives a boost to the performances from a total return point of view.

	Mean return	Volatility (σ)	Sharpe ratio	Information ratio
MV	9%	17.92%	42.35%	4.57%
BLM	5.65%	23.17%	18.31%	-30.50%
BLM-basic	6.06%	23.07%	20.18%	-26.66%
BLM-best	5.61%	23.14%	18.16%	-31.11%
BLM-worst	5.69%	23.20%	18.44%	-29.99%
RO $\Omega = \text{diag}(\Sigma)$	9.90%	17.71%	47.97%	19.96%
Market	8.56%	18.86%	37.90%	

Table 26: High volatility portfolios basic statistic

Table 26 confirms the problems of BLM variants considering high volatility situation. The reduced level of diversification provides high values for the volatility, that combined with low levels of mean return, provides low Sharpe ratios. On the other hand, RO presents a level of volatility very similar to those detected for low volatility and middle volatility, that combined with a great level of return declare it the winning strategy in this risk framework. It deserves to say that, in this situation Markowitz is able to beat the market together with RO.

	Maximum drawdown	Downside risk	VaR (95%)	ES (95%)	Sortino ratio
MV	28.15%	0.70%	-26.34%	-43.42%	10.53
BLM	42.42%	0.90%	-33.83%	-57.23%	4.61
BLM-basic	41.96%	0.89%	-33.88%	-57.23%	5.06
BLM-best	42.42%	0.89%	-33.86%	-57.23%	4.56
BLM-worst	42.43%	0.90%	-33.74%	-57.23%	4.64
RO $\Omega = \text{diag}(\Sigma)$	28.30%	0.69%	-25.09%	-43.09%	11.73
Market	36.10%	0.79%	-27.68%	-48.41%	8.95

Table 27: High volatility portfolios risk exposure measures

Table 27 confirms the tendency that we have seen in the previous risk propensity, indeed all values are lower than those of no-rebalance approach. It is impressive the improvement of MV considering its closeness to RO, especially it presents the lowest maximum drawdown among all models. RO continue to be virtuous from risk exposure point of view, even it is able to obtain its lowest drawdown and highest Sortino ratio in the higher risk propensity.

As in the no-rebalance approach, we take the best models in terms of risk/return considering all the risk propensities to make a comparison among them.

	Total return	Mean return	Volatility (σ)	Sharpe ratio	Information ratio
BLM (low volatility)	62.85%	9.88%	16.62%	50.98%	21.05%
BLM-basic (middle volatility)	96.72%	13.75%	19.76%	62.46%	189.92%
RO $\Omega = \text{diag}(\Sigma)$ (high volatility)	61.33%	9.90%	17.71%	47.97%	19.96%
Market	47.4%	8.56%	18.86%	37.90%	

Table 28: Basic statistic summary

First of all, we can observe that best models for each risk propensity are the same of no-rebalance approach, but at the same time are all able to beat the market. This proves the fact that rebalancing approach gives the possibility to the models to work in a better way and make better performances. As a common factor, we notice that volatility level is mainly the same of no-rebalance procedure while return indicators are much higher, means return in the order of 2-3% and total return in the order of 16-22%. As in no-rebalance approach, we can explain the stability of volatility level of RO with the explanation provided yet considering that the relevant sectors are the same. Considering that, RO high volatility portfolio is able to be riskless than BLM-basic middle volatility portfolio, but this is not sufficient to make it the best among them in terms of risk/return because BLM-basic provides a huge level return with a resulting Sharpe ratio of 62.46% compared to 47.97%. At the same time, BLM low volatility portfolio is surprising because it obtains approximately the same total return of RO with lower volatility.

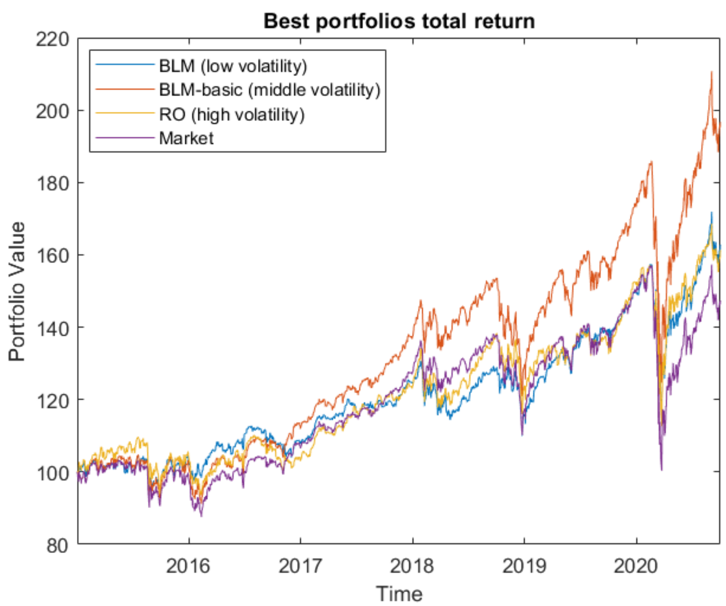


Figure 42: Best portfolios total return

	Maximum drawdown	Downside risk	VaR (95%)	ES (95%)	Sortino ratio
BLM (low volatility)	27.39%	0.65%	-24.19%	-40.47%	12.31
BLM-basic (middle volatility)	32.97%	0.79%	-28.19%	-49.73%	14.73
RO $\Omega = \mathrm{diag}(\Sigma)$ (high volatility)	28.30%	0.69%	-25.09%	-43.09%	11.73
Market	36.10%	0.79%	-27.68%	-48.41%	8.95

Table 29: Risk exposure measures summary

Table 29 highlights the fact that also the risk exposure indicators remain stable compared to those of no-rebalance approach, conducting to a general increment of Sortino ratios thanks to the returns increase. Despite its stability, RO fails to obtain a better Sortino ratio compared to BLM-basic and it is also overperformed by BLM low volatility portfolio because it has the lowest risk exposure compared to the two other portfolios. Concluding, BLM-basic can be considered the best if we take into account that if a worst scenario happens the suffer loss is higher than other models, testified by VaR and ES values.

5.2.2.4 Summary

In this section we will present some general facts about the application of rebalancing approach and then we focus on the single risk propensities. First, performances obtained by the models are much better of those obtained by them without a periodic rebalance. It is confirmed that making available to the optimizers more information, they will be able to create dynamical portfolios with the capacity to face various market phases. As a consequence of this, all models are capable to beat the market in various situations, but only RO has the capacity to overperform the benchmark for every risk propensity. Another general fact about RO, is that its optimal portfolios maintain a stable level of volatility and risk exposure reducing the diversification level going from low volatility to high volatility and at the same time is confirmed the performance increase. Second, MV is still the taillight among all models but, going through the three risk levels, it demonstrates a better performance growth than no-rebalance approach.

- In the low volatility environment, we notice a volatility level similar to that of no-rebalance approach and an increasing trend about returns. As a consequence, all models overperform the market with the only exception of MV. We observe also a reduction of maximum drawdown and downside risk, while at the same time a slightly increase of VaR and ES.
- In the middle volatility framework, it is confirmed the predominance of BLM variants and, thanks to rebalance technique, they obtain much better performance. An interesting aspect is that we detect a lower impact of the views compared to no-rebalance approach. Additionally, we observe a reduction of risk exposure indicators, confirmed by higher Sortino ratios.
- About high volatility portfolios, rebalancing gives a hand to BLM variants, helping them to obtain decent performances despite the increment in volatility and risk exposure. Even for the highest risk level, we note a lower risk exposure. Concluding, for the first time MV is able to beat the benchmark.

Concluding, the best models are the same of no-rebalance approach but in a "powerful" version, indeed they provide better returns without an increasing in volatility and risk exposure. Optimal portfolios are still characterized by the same main sectors: IT, Healthcare and Consumer staples. We can conclude that they are the best picking from S&P 500, also confirmed by the statistics provided at the begin of the out-of-sample analysis, and the high correlation among these sectors provides a boost to the performances.

	Low volatility	Middle volatility	High volatility
Choice	BLM	BLM-basic	RO $\Omega = \mathrm{diag}(\Sigma)$

Table 30: Best models choice

6 Conclusion

The thesis deals with the application of three different models, MV, BLM and RO, using the S&P 500 sectors as reference assets. The first part of our sample, namely in-sample period, will be used to compose the optimal portfolios, while in the second part, called out-of-sample period, we will analyse the performances with multiple risk and return indicators. Specifically, the first three chapters are dedicated to the theoretical background, the fourth chapter describes the data used and the analysis procedure, while the fifth chapter is the core of the work as it presents the results of the models by comparing them with each other and with the market with two different approaches: the first does not involve the rebalancing of the optimal portfolios for the entire period of analysis, consequently it can be considered a theoretical approach as we know how dynamic the financial markets are. The second approach involves a 5 days rebalancing of the portfolios, a common technique used in reality. Taking into consideration the empirical results, we can see several interesting points. Considering the no-rebalance approach, the challenge is always between BLM and RO. BLM appears to be more efficient in the first two risk appetites, achieving better returns having volatility levels similar to RO. Furthermore, thanks to the high level of diversification of the optimal portfolio, the views are crucial in the middle volatility framework. It is no coincidence that it is the only model that manages to beat the market using the no-rebalance approach. At the same time, RO has the peculiar characteristic of maintaining a stability in terms of risk exposure and volatility and will be the winning card considering the highest risk propensity. In the rebalance approach, the best models still remain BLM and RO but there is a generalized increase in return. Even MV manages to exploit rebalance to get closer to the other two models. Another point in favour of rebalance is the fact that the increase in return is not followed by an increase in volatility, giving the possibility to BLM and RO to beat the market more frequently, but only RO can do it in every situation, emphasizing its performance stability. The predominant sectors are the same for the two approaches: IT, Healthcare and Consumer staples. They actually pushed the S&P 500 higher over the period with an average level of volatility considering all the sectors, so it is not surprising. Interesting is the fact that BLM and RO improve MV by following two different paths. BLM achieves better results with a well diversified portfolio, while RO achieves them with a relatively concentrated portfolio. In conclusion, we have very clear results. As for low and middle volatility portfolios, BLM over performs MV and RO in both approaches. The views are relevant in the middle volatility portfolio, thanks to its high diversification, and indeed it is the best performing portfolio, confirmed by its ability to beat the market both with rebalance and not. Conversely, concerning the high volatility portfolio, BLM completely fails while we detect a RO exploit, thanks to its ability to keep the risk constant despite the increased risk appetite. Like the low volatility BLM portfolio, portfolio rebalancing is required to beat the market. Further empirical applications, for example using assets belonging to different asset classes or introducing a dynamic manner to manage BLM views during the out-of-sample period performance analysis, can provide further insights for the comparison between these models.

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Appendix A

In this appendix we show the derivation of the value of k related to the Sharpe ratios of eigenvectors starting from the calculation of an upper bound.

We reformulate the RO stated in equation (47) in terms of the estimated Sharpe ratios $\overline{SR}$, risk budget $\boldsymbol{X}$ and correlation matrix **P**. Please note that in MVO as well as in RO, λ is a parameter that measures the overall risk aversion and can be scaled so that the ex-ante risk reaches a given target level.

If we solve equations (55) or (56), we get:

$$w_{rob}^* = \frac{1}{\lambda}\Sigma^{-1}\left(\bar{\mu} - \frac{k}{\sqrt{w_{rob}^{*T}\Omega w_{rob}^*}}\Omega w_{rob}^*\right)$$

The optimal robust weights are homothetic with respect to λ. We observe that the change of κ does not affect this homothetic relationship. For simplicity and without loss of generality, we assume that λ is equal to one in this unconstrained case.

We note the optimal robust risk budget as X_{rob}^*:

$$X_{rob}^* = argmax\left(X^T\overline{SR} - k\sqrt{X^T I_n X} - \frac{1}{2}X^T PX\right)$$

Deriving the optimality condition:

$$\overline{SR} - \left(\frac{k}{\sqrt{X_{rob}^{*T} X_{rob}^*}} I_n + P\right) X_{rob}^* = 0$$

In addition, rearranging it yields the following expression:

$$\overline{SR} = \left(\frac{k}{\sqrt{X_{rob}^{*T} X_{rob}^*}} I_n + P\right) X_{rob}^*$$

The above formulation sheds light on the role of κ as the parameter that tackles the high sensitivity to inputs suffered by MVO. In fact, the greater κ is, the more $\frac{k}{\sqrt{X_{rob}^{*T} X_{rob}^*}} I_n + P$ shifts towards $\boldsymbol{In}$.

The shift of the modified correlation matrix towards $\boldsymbol{In}$ helps reduce the high sensitivity caused by the small eigenvalues but the benefit does not come without cost: a large κ may distort completely the correlation structure that makes assets indistinguishable from a risk perspective.

Taking the L2-Norm on both sides' yields:

$$\overline{SR}^T\overline{SR} = k^2 + X_{rob}^{*T} P^T P X_{rob}^* + 2 \times \frac{k}{\sqrt{X_{rob}^{*T} X_{rob}^*}} X_{rob}^{*T} P^T X_{rob}^*$$

Note that the second term on the right-hand side ${X^*_{rob}}^T P^T P X^*_{rob}$, as the square of the L2 Norm of PX^*_{rob} is non-negative; the third term on the right-hand side is also non-negative because **P** is positive semi-definite, the L2 Norm of X^*_{rob} is non-negative and k^2 is always non-negative. Therefore, the following upper bound for k holds:

$$k \leq \sqrt{\overline{SR}^T \overline{SR}}$$

Note that if k is set higher than the upper bound, the first order derivative of the optimization with respect to ***X*** will always be negative. Therefore, the solution to the RO will be no-investment, i.e., ***X***=0.

Provided this upper bound, now we derive a rule of thumb that will be helpful to calibrate k.

Recalling that RO is a max-min process, the objective is to maximize the objective function even under the worst return realization. Thus, the RO uses penalized returns μ instead of the traditional expected returns from the sample mean $\bar{\mu}$:

$\mu = \bar{\mu} - \sqrt{\frac{k^2}{w^T \Omega w}} \Omega w$, with $\Omega = diag(\Sigma)$

Re-expressing just the above equation in terms of Sharpe ratios and risk budget yields the following expression:

$$SR = \overline{SR} - \frac{k}{\|X\|_2} X$$

The expected returns on the eigenvectors can be found easily when we apply the L2 Norm of SR expressed in the spaces spanned by the eigenvectors of correlation matrix P.

Consider the eigenvalues-eigenvectors decomposition of the correlation matrix $P = ZLZ^T$, with **Z** the matrix of the eigenvectors and **L** the diagonal matrix with the eigenvalues on the main diagonal, the expected returns on the eigenvectors $\ddot{\overline{SR}}$ of **P** can be found with the upper limit of $\ddot{SR}$:

$$\|\ddot{SR}\|_2 = \sqrt{\ddot{\overline{SR}}^T \ddot{\overline{SR}} + k^2 - 2 * k \frac{\ddot{\overline{SR}}^T \ddot{X}}{\|\ddot{X}\|_2}} \leq \sqrt{\ddot{\overline{SR}}^T \ddot{\overline{SR}} - k^2}$$

With $\ddot{\overline{SR}} = Z^T \overline{SR}$, $\ddot{SR} = Z^T SR$ and $\ddot{X} = Z^T X$, see Appendix B for details.

$$\|\ddot{SR}\|_2 \leq \sqrt{(Z^T SR)^T (Z^T SR) - k^2}$$

The right-hand side of the precedent equation can be rewritten in two ways:

$$\sqrt{(Z^T SR)^T (Z^T SR) - k^2} = \sqrt{(Z_1^T SR)^2 + \ldots + (Z_n^T SR)^2 - k^2}$$

$$\sqrt{(Z^T SR)^T (Z^T SR) - k^2} = \sqrt{SR^T ZZ^T SR - k^2} = \sqrt{SR^T SR - k^2}$$

The vectors of Z are ordered following the order of eigenvalues (from the largest to smallest).

The key for calibrating κ is to make use of the equivalence between equation (36) and equation (37): when k is calibrated in terms of the Sharpe ratios (equation 37), it is able to reduce or even neutralize the cumulative sum of "returns" on eigenvectors that correspond to the small eigenvalues (equation 36). Namely, $k = \sqrt{(Z_1^T SR)^2 + \ldots + (Z_n^T SR)^2}$ with i to n the indices that correspond to small eigenvalues. Small is a rather abstract term and it does not tell us how to choose the cut-off number i. In the empirical experiment below, we propose a rule of thumb to help us determine the cut-off number and thus to calibrate κ.

C. Yin et al. 2020 demonstrate, with a practical application in a multi-asset universe, that the proposed rule of thumb consists of choosing κ as half of the average of Sharpe ratios. This rule of thumb applies for multi-asset portfolios regardless of the number of assets they comprise and regardless of the assumptions on Sharpe ratios.

Appendix B

In this appendix, we derive the upper limit of the L2 Norm of Sharpe ratios used in the RO in terms of the "returns" on the eigenvectors of the correlation matrix. Recall that the Sharpe ratio used for the RO is written as follows: $SR = \overline{SR} - \frac{k}{\|X\|_2} X$

Applying the change of basis according to the coordinate defined by the eigenvectors $\boldsymbol{Z}$ of the correlation matrix **P**, we get:

$$Z^T SR = Z^T \overline{SR} - \frac{k}{\|Z^T X\|_2} Z^T X$$

$$\ddot{SR} = \ddot{\overline{SR}} - k \frac{\ddot{X}}{\|\ddot{X}\|_2}$$

Note that $Z^T SR$ can be viewed as the "return" of the eigenvectors. Again, the exact solution for κ is not feasible because the above equation involves $\ddot{X}$, which is the solution of the RO itself. However, it is important to note that κ should be chosen so that the "returns" of the eigenvectors that correspond to the small eigenvalues could be reduced. Following this guideline, we consider the two terms on the right-hand side of the above equation separately by taking the L2 Norm.

$$\left\|\ddot{\overline{SR}}\right\|_2 = \sqrt{\ddot{\overline{SR}}^T \ddot{\overline{SR}}} = \sqrt{\ddot{\overline{SR}}_1^2 + \ldots + \ddot{\overline{SR}}_n^2}$$

The $\ddot{\overline{SR}}_i^2$, for i=1, …, n, follow the order of eigenvalues and the last $\ddot{\overline{SR}}_n^2$ corresponds to the "returns" of the eigenvector that corresponds to the smallest eigenvalue: $\left\|k \frac{\ddot{X}}{\|\ddot{X}\|_2}\right\|_2 = k$

We take the L2 Norm on both side of equation (46) as follows:

$$\left\|\ddot{SR}\right\|_2 = \left\|\ddot{\overline{SR}} - k \frac{\ddot{X}}{\|\ddot{X}\|_2}\right\|_2 = \sqrt{\ddot{\overline{SR}}^T \ddot{\overline{SR}} + k^2 - 2 * k \frac{\ddot{\overline{SR}}^T \ddot{X}}{\|\ddot{X}\|_2}}$$

Recall that the optimality condition of the RO mentioned earlier is written as:

$$\left(\frac{k}{\sqrt{X^T X}} I_n + P\right) X = \overline{SR}$$

Multiplying both sides of equation (47) by X^T, we get:

$$k\sqrt{X^T X} + X^T P X = X^T \overline{SR}$$

Expressing both sides in the space spanned by the eigenvectors, our calculations could read as follows:

$$k\sqrt{\ddot{X}^T \ddot{X}} + \ddot{X}^T P \ddot{X} = \ddot{\overline{SR}}^T \ddot{X}$$

Diving both sides by $\sqrt{\ddot{X}^T \ddot{X}} = \|\ddot{X}\|_2$, the equivalent to the previous equation is given by:

$$k + \frac{\ddot{X}^T P \ddot{X}}{\|\ddot{X}\|_2} = \frac{\overline{\ddot{SR}}^T \ddot{X}}{\|\ddot{X}\|_2}$$

As $\ddot{X}^T P \ddot{X} \geq 0$ and $\|\ddot{X}\|_2 > 0$, the optimality condition gives rise to the following inequality:

$$k \leq \frac{\overline{\ddot{SR}}^T \ddot{X}}{\|\ddot{X}\|_2}$$

The inequality just derived can be used to yield an upper limit of $\|\ddot{SR}\|_2$:

$$\|\ddot{SR}\|_2 = \sqrt{\overline{\ddot{SR}}^T \overline{\ddot{SR}} + k^2 - 2 * k \frac{\overline{\ddot{SR}}^T \ddot{X}}{\|\ddot{X}\|_2}} \leq \sqrt{\overline{\ddot{SR}}^T \overline{\ddot{SR}} - k^2}$$

End of proof.