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Stochastic Processes with Application to Finance

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Stochastic Processes with Application to Finance

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To my Family.

Contents

Abstract

Knowledge about stochastic processes is the foundation needed in order to explore the market. These are used to model asset dynamics and their properties allow us to build model serving several purposes. One of these is pricing.

In this thesis we will explore some types of stochastic processes, the theory on which they are based on and some application related to finance. More precisely, we will see diffusion and jump diffusion processes, what makes them palatable for us and how we can play with them to perform different tasks. The focus will be on the theory and the steps needed in order to obtain a valid system to price options.

After some introductory, but necessary, topics we will see how pricing of barrier options can be carried out having the process at hand, and the legitimacy of this procedure.

Keywords: *Stochastic Processes*, *Option Pricing*, *Financial Application*, *Barrier Options*

Chapter 1

Introduction

Asset prices are quantities which are governed by uncertainty. Models to try to predict and forecast prices have been developed with more or less success, but the bottom line is that there is no possibility to do such thing. A reasonable starting point, tough, is to try to study the behaviour of such quantities in the most general form possible, backing up the treatment with solid mathematical ratiocination. This is the purpose of the theory of stochastic process in relation to finance. The need to identify and shape the quantities we are interested in, goes beyond the mere prediction aim, but rather an exploratory rationale. The idea of using stochastic processes to model asset price is an old one, even if the big leaps forward achieved on this matter are rather recent. The theory that permits the treatment is rather fresh, as well as the whole formalization of probability, but the efforts and development in this field have been growing continuously throughout the years. I wanted to explore this subject in more detail and its applicability in the finance field. The central idea of stochastic processes in finance is to use them as representation of the price evolution and try to study those to infer useful details that might be used for several purposes. The treatment of stochastic process does not fall under the umbrella of classic calculus, but one must use different calculus theories in order to handle these quantities. The reason why is quite straightforward: stochastic processes describe the evolution, over time in our case, of a random quantity, which, in turn, give rise to aleatory dynamics. Once the theory for the treatment is established, we can start

to operate on them, within the limits of it. We will mainly make use of the theory in order to arrive to a fair price for a derivative instrument, which is dependent from a stochastic process itself. The argument could be of different nature too. For example, how to perfectly hedge a position in an option or with an option might be the logical next step.
The theory stemmed within a framework which assumed asset prices to be properly represented by continuous stochastic processes. Later different types of those were investigated, also allowing for discontinuous versions of them to be considered as a reasonable modeling choice, sometimes even being more faithful to the actual state of things. We will see both of these cases for some type of processes and we will try to develop a general understanding aimed at moving from the theory to the applicability.

1.1 Structure

The structure of the thesis is as follows.

In **Chapter 2** we provide a general introduction to stochastic processes and the theory on which the treatment is based on. We will introduce some new concepts in order to understand this theory and, later on, an example using the results obtained. This will be the first introductory step to derive everything that comes next.

In **Chapter 3** we will explore the basics, introducing some useful and elegant results which will lead to the famous Black-Scholes pricing formula. We will illustrate the seminal work done toward this end and the alternative way to obtain the same result. Some concepts about probability measures will be introduced and we will see how to tackle the problems that arise when pricing a derivative instrument, complying with the idea of being in a market model.

In **Chapter 4** we will see a more sophisticated type of stochastic processes, obtained just tinkering what we introduced in the previous chapters. These new models will allow for more flexibility and a more faithful depiction of certain real life phenomena. We will again face some problems and see which are the viable solutions in this new framework. We will see an application and derive a specific stochastic process which will be used also in the experimental part of the thesis.

In **Chapter 5** we will set up everything we need to perform the experiment, defining a barrier option and illustrating the method we will use to price such option. We will see the benefits and the drawbacks of such method and discuss a bit on what we can make of it

In **Chapter 6** we will briefly review what we did and the effects and ideas that were provoked conducting this study

Appendix A contains all the matlab codes used to make this thesis.

Chapter 2

Stochastic Processes

Stochastic processes are an important topic in several fields of study, and finance is one of those. In finance, stochastic processes are used to model the assets' dynamics, which lend themselves perfectly for this purpose, due to the uncertainty that governs them. A stochastic process is a collection of random variables, indexed over some set, which live on the usual probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will index over $T = \mathbb{R}^+$, intended as the set of times. More precisely, we will mainly operate in a ***filtered*** probability space, i.e. a probability space endowed with a ***filtration*** $\{\mathcal{F}_t\}_{t\geq 0}$. A filtration is an increasing sequence of ordered σ-algebras. More formally, if $s, t \in T$ with $s \leq t$, we have $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$. A filtration can be the general one or the one generated by the stochastic process itself, the so called ***natural*** filtration. In our framework, we may intend the natural filtration of the stochastic process we chose to model the evolution of an asset, as the historical prices of the asset considered up to time t.

We want to model the random evolution of assets and try to study those to see what might be inferred from them, in order to take decision in the market. So we will assume that the asset dynamic will be driven by a certain stochastic process and model its behaviour with a ***stochastic di erential equation***.

In order to show which is the form of a stochastic differential equation we need to introduce one of the buiding blocks of this theory, the ***Brownian motion***, or ***Wiener process***

Definition 2.1. *A stochastic process W is called Wiener process if the following hold:*

- $W_0 = 0$
- *the process* W *has independent increments, namely if* $r < s \leq t < u$ *then* $(W_u - W_t) \perp (W_s - W_r)$
- *for* $s < t$, $(W_t - W_s) \sim \mathcal{N}(0, t-s)$
- *W has continuous trajectories*

So a Wiener process is a fascinating entity with some interesting features: the second property is called independency of increments, which allows us to make a nice use of the process when studying it, since we can always factorize the process partitioning it into increments; the third property is called stationarity of the increments, which tells us that the increments follows the same law as the original Wiener process and the difference depends only on the time interval considered. Another astounding feature is the continuity of trajectories: a Wiener process is nowhere differentiable, having kinks at every point, but, still, it is continuous. The first three properties(and *continuity in probability*) makes the Wiener process a *Lévy* process. Moreover it is the only proper Lévy process with continuous paths.

Now consider a stochastic process X_t. A stochastic differential equation is an equation of the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{2.1}$$

usually coupled with

$$X_0 = x_0, \tag{2.2}$$

where μ, σ are given functions called, respectively, *drift* and *di usion* term, W_t is the freshly introduced Wiener process, and 2.2 is an initial condition.
So we want to find a process X which satisfy the integral equation

$$X_t = x_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \ \forall t \geq 0 \tag{2.3}$$

What stands out in this equation is the fact that we have a dW_t term. That term should quantify the instantaneous change with respect to the quantity W_t, but W_t is a random quantity. This is the main problem with stochastic differential equation. Without the dW_t term we are up and running using calculus theory, but in this case classic calculus machinery cannot help us. So how do we proceed? Luckily for us an elegant and extensive theory have been devised. A theory which allow us to play with this quantity. It goes under the name of Itō calculus.

2.1 Itō Calculus

The theory was devised by Kiyoshi Itō in the '40s, and was later adopted in the financial field, becoming a turning point for the subject. I will limit myself to introducing the conditions needed and the ideas behind the construction, without showing the actual derivation of it, before presenting the important result. I believe that is something that must be done, due to the importance of the theory.
We said that we want to solve the stochastic integral 2.3, while facing the problem of a random differential term. So we study the more general integral given by

$$\int_0^t g(s)dW_s \tag{2.4}$$

The idea is to try to define the stochastic integral for a simple but large class of processes, and then by a limiting rationale, generalize this results. The complete construction of the stochastic integral can be found in any stochastic calculus in continuous time book, e.g. Shreve (2004) or Björk (2004). Let us see which are the conditions needed, starting from the ones on g.
In this study, we will work with the $\mathcal{L}^2$ class.

Definition 2.2. *We say that the process g belongs to the $\mathcal{L}^2$ class if the following conditions are satisfied*

- $\int_0^t \mathbb{E}[g^2(s)]ds < \infty$
- *The process g is **adapted** to the $\mathcal{F}_t^W$-filtration*

The first property is an integrability condition, which is needed to reach a well defined solution. The second one, for g being ***adapted*** to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, or, equivalently, g ***being*** $\mathcal{F}_t$***-measurable***, or again that $g(t) \in \mathcal{F}_t$, $\forall t \geq 0$, it simply means that at time t, we know the exact value of $g(t)$. This property is also called ***non-anticipating*** property. For $\mathcal{F}_t^W$ we intend the filtration generated by the stochastic process W_t.
The fact that we impose onto g to be in the $\mathcal{L}^2$ class bring us very close to the solution of our problem. What we need first is an important concept, which is crucial in developing everything that comes after: the concept of ***martingale***.

Definition 2.3. *A stochastic process X is called an $\mathcal{F}_t$-**martingale** if the following hold*

- X *is adapted to the filtration* $\mathcal{F}_t$, $\forall t \geq 0$*;*
- *for all* t
$$\mathbb{E}[|X_t|] < \infty;$$
- *for all* s, t *with* $s \leq t$
$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s.$$

The third condition is the most interesting one: it says, in words, that the best guess we can take with the information available up to time s, about the value of X_t in the future, is simply X_s. In our framework this translates into the fact that the process has no ***systematic*** drift. The concept of martingale will reveal itself as crucial both from a mathematical and a logical standpoint. It is easy to show that the Wiener process and the integral defined in 2.4 is indeed a martingale Björk (2004).
We are almost there, the other thing we need to know is a result which pertains the quadratic variation of the Wiener process. Informally we can say that the quadratic variation is the squared variation of a process, calculated over a partition for which the mesh tend to zero. The key result we want to make use of, is the fact that in the Brownian motion, the quadratic variation is ***finite*** and is equal to t, meaning that

the "Brownian motion accumulates quadratic variation at rate one per unit time" Shreve (2004). This property allow us some useful simplification when solving the, by now, infamous stochastic integral. For what interest us, this feature of the quadratic variation allow us to write the exceptional $dW_t dW_t = dt$.
We have everything we needed in order to arrive at the key result of this section, in which we dropped the unwieldy complete notation, but this creates no ambiguity.

Theorem 2.1. *Suppose we have a stochastic process X_t having a stochastic di erential 2.1, where μ, σ - we will use a less cumbersome notation - are in the $\mathcal{L}^2$. Let f be a $\mathcal{C}^{1,2}$ function. Define Z by $Z_t = f(t, X_t)$.Then Z has a stochastic di erential given by*

$$df(t, X_t) = \{\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\}dt + \sigma\frac{\partial f}{\partial x}dW_t \tag{2.5}$$

Proof. We give an heuristic sketch of the proof.
We have df. If we make a second order Taylor expansion what we get would be

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t\partial x}(dtdX)$$

By definition we have that

$$dX_t = \mu dt + \sigma dW_t$$

So squaring we would get

$$(dX)^2 = \mu^2(dt)^2 + 2\mu\sigma(dtdW) + \sigma^2(dW)^2$$

The idea is that the terms $(dt)^2, (dtdW)$ are negligible and tend to 0. Moreover we just said that $(dW)^2 = dt$ so pluggin everything in we get the result. □

This is clearly not the full proof and we should show also that all the remainder terms from the Taylor expansion go to 0, but this sketch of proof gives an intuitive idea about how we get such a powerful result. Anyway, the complete proof is outside the scope of this dissertation and is rather cumbersome.
What is exceptional about this result is the fact that we have no restriction on f except for it to be in the $\mathcal{C}^{1,2}$ space, meaning that this result is preserved under any transformation which satisfies the conditions, something which is not at all trivial.

We finally have a way to operate in such environment, thus is time to link the theory to our financial framework. We started by saying that we wanted to use the stochastic differential equation to model the behaviour of an asset. Now the Itō's formula gives us the tool in order to solve such problem for a certain type of stochastic processes.

Chapter 3

A First Financial Example: Black-Scholes Model

We want to finally start to see what everything we said means in the financial field. From now on we will assume to be in a market which has some standard properties:

- transactions take place in continuous time;
- positions can be fractional, positive or negative
- there are no transaction costs (*frictionless*);
- there are no dividends;
- there is a constant risk-free rate, r, both for borrowing and lending purposes;
- we need the market to be arbitrage free.

The last property is something we must impose and we will see how it is done. Moreover the stochastic processes considered will have constant drift and diffusion terms, namely $\mu, \sigma \in \mathbb{R}$.

3.1 Basic Framework

I will introduce briefly the basic framework from which we start without going too deeply into it. It serves as starting point for the actual topic of the thesis and is something which must be enunciated for its importance. The model represents a market with derivative instruments, for which we want to determine the prices. Luckily for us this work was done by Black and Scholes (1973). Recall the adapted process g we used to build our stochastic integral above. That process might represents several things in finance, such as a portfolio position in one or more assets, or the value of an option, which is related to the price of the asset.

We assume that we have an asset S which is modeled as a ***geometric Brownian motion***. This means that the process S_t representing the asset considered will be of the form

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ S_0 &= s_0 \end{aligned} \tag{3.1}$$

Where $\mu, \sigma \in \mathbb{R}$ and W_t, with $t \in [0, T]$ for a fixed T, is a Wiener Process. The solution to this equation is easily obtained using Itō's formula - finally - on $Z_t := \ln S_t$ which yields

$$\begin{aligned} dZ &= \frac{1}{S} dS + \frac{1}{2} \left\{ -\frac{1}{S^2} \right\} (dS)^2 \\ &= \frac{1}{S} \{ \mu S dt + \sigma S dW \} + \frac{1}{2} \left\{ -\frac{1}{S^2} \right\} \sigma^2 S^2 dt \\ &= \{ \mu dt + \sigma dW \} - \frac{1}{2} \sigma^2 dt \end{aligned}$$

So that

$$dS_t = \exp \left[(\mu - \frac{\sigma^2}{2}) dt + dW_t \right] \tag{3.2}$$

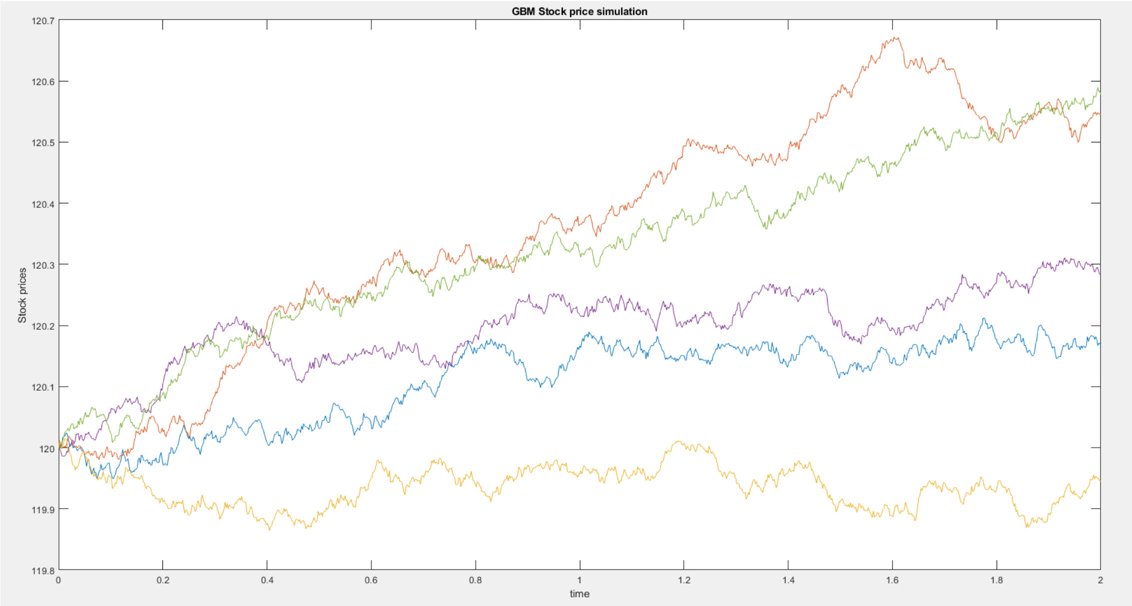

Figure 3.1: *Simulations of trajectories of 3.2 with* $\mu = 0.09, \sigma = 0.12, S_0 = 120$

Now, we want to evaluate the price of an option following the approach pursued by Black and Scholes in their famous paper. Reasonably, the option price would be a function of the discounted value of the payoff at maturity, something we can write, in case of a call option, as

$$e^{-r(T-t)}(S_T - K)^+$$

By a no arbitrage rationale, they started from the fact that the value of the option and the the value of an hedging portfolio, must be the same during the lifetime of the option. So the discounted values of these two quantity should be equal at any time t. In order to ensure this condition, having an equation which represents the option price according to the value of the underlying, we must match the evolutions of these two quantities. So they arrived to a partial differential equation, for which we will not see the derivation, of which the solution would be the value of the option. So, calling V the value of the option, the Black Scholes equation is

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

The financial idea behind it is that one can perfectly hedge the option by buying and selling the underlying asset.

Solving this equation, they arrived to a theoretical closed form, the Black-Scholes formula, for the price of the option.
There is another very elegant way to arrive to the price of an option, which does not involve the use of partial differential equation. The idea is to always start from a no arbitrage principle, but involves changing the way at which we look at the process. The condition we need for this to happen can be stated as the ***first fundamental theorem of asset pricing***. But first, we need some important concepts regarding the probability measure.

3.2 Change of measure

In our framework, a "measure" is a probability measure, which serves us the purpose of quantifying some uncertain quantities with which we have to deal with. The probability measure equipped in the probability space defined in chapter 1 is $\mathbb{P}$, which is also called physical measure, or, in our case, *market measure*. Following the thread as above, what we would like to do is to **change** the **probability measure**, so that, by evaluating the process under this new "point of view", we would satisfy the no arbitrage principle.
Reasonably, we must start from the idea that, the new measure must be consistent with the old one, that is what is impossible under the market measure, cannot be possible under the new one. This intention is represented by the notion of equivalence measure.

Definition 3.1. *Two probability measures, on the same space, are equivalent if they agree on the sets in $\mathcal{F}$ which have probability zero.*

Starting from here we now need the concept of **risk-neutral** measure, or **martingale** measure.

Definition 3.2. *For $\mathbb{Q}$ to be a risk neutral measure, we need that every bounded self-financing portfolio, has, under $\mathbb{Q}$, expectation of any discounted future value equal to the value as of today.*

It is called martingale measure because we are imposing for the process considered, to be a $\mathbb{Q}$-martingale. Since we do not want arbitrage opportunities to exist, the possibility of building a costless portfolio which have a positive probability to be greater than 0 for some t and a zero probability of being negative, must be excluded. So we want a measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$, and for which the process considered is a martingale. Knowing these definitions, we can enunciate the result we need.

Theorem 3.1. ***(First Fundamental Theorem of Asset Pricing)***
A market is arbitrage-free if and only if exists a risk-neutral probability measure, equivalent to the physical one.

We have now the definition of what we need, but we do not know how to obtain the desired equivalent risk-neutral measure. To do this we start an important result, which will lead to the so called *Radon-Nikodym derivative*.

Definition 3.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathbb{Q}$ a measure which is equivalent to $\mathbb{P}$, and Z an almost surely positive random variable, with $\mathbb{E}Z = 1$. For A define $\mathbb{Q}(A) = \int_A Z d\mathbb{P}$. Then Z is called the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ and we write $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$*

In words, it consist in defining a positive random variable Z with expected value under $\mathbb{P}$ equal to 1, and that for any $A \in \mathcal{F}$, $\mathbb{Q}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$. Then the Radon-Nykodim derivative is simply $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. The existence of this quantity is given by the *Radon-Nikodym* theorem.

Theorem 3.2. *Let $\mathbb{P}, \mathbb{Q}$ be equivalent probability measures. Then there exists an almost surely positive random variable Z such tat $\mathbb{E}Z = 1$ and*

$$\mathbb{Q}(A) = \int_A Z d\mathbb{P}, \quad \forall A \in \mathcal{F}.$$

The measures defined as above are equivalent, and this can be shown in the following way. The result we obtained draws up a relation for the expectations under the two measure. This can be easily seen by applying the definition we just provided above with $A = \Omega$

$$\mathbb{E}_{\mathbb{Q}}[X] = \int_{\Omega} X d\mathbb{Q} = \int_{\Omega} X \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{P} = \int_{\Omega} XZ d\mathbb{P} = \mathbb{E}_{\mathbb{P}}[XZ]$$

Using this result is easy to see that the two measures are equivalent. We now have a way to go from one measure to the other just multiplying, or dividing, by Z.
This is the first step we needed towards the change to the risk-neutral measure. We need the martingale measure part. We have our process S_t with drift μ. We want to change the drift of this process in a way that its discounted value is equal to 0, or equivalently, to obtain a process which has mean return equal to r. In order to do that without changing the process itself, we want to evaluate it with respect to the new measure. If we find the measure $\mathbb{Q}$ such that W_t is, under $\mathbb{Q}$, a Brownian motion, and the process has $\mu = r$, we then would have what we need.
In order to do that, we need the concept of exponential martingale, or, in the more general case, the so called *Doléans-Dade* exponential.

Definition 3.4. Let W_t be a Wiener process. The exponential martingale of W_t is defined to be

$$Z_t(\theta) = \exp[\theta W_t - \frac{1}{2}\theta^2 t] \tag{3.3}$$

Proposition 3.1. *For every $\theta \in \mathbb{R}$ the process $Z_t(\theta)$ defined as above is a positive martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$*

With this result in our hand we can go on. The exponential martingale defined above is the process that we will use in order to change the measure, using it as the Radon-Nikodym derivative. Indeed it is easy to see that we have $\mathbb{E}_{\mathbb{P}}[Z_t(\theta)] = 1$, and that $Z_t > 0$. Hence we can define $\mathbb{Q}$ using theorem 3.2.
The result we would get is that under $\mathbb{Q}$, the process $W_t^{\mathbb{Q}}$ has the same law as W_t with drift θ. Now we want to go the other way round, that is we want to obtain, upon a change of measure, a $W_t^{\mathbb{Q}}$ with drift equal to r. This trick is done thanks to a remarkable result which goes under the name of Girsanov theorem, which I will enunciate just for the one-dimensional case.

Theorem 3.3. *Let W_t be a Brownian motion, on $(\Omega, \mathcal{F}_t^W, \mathbb{P})$, with positive drift. Let $g(t) \in \mathcal{L}^2$. Define*

$$Z_t(g) = \exp\Big[- \int_0^t g(s)dW_s - \frac{1}{2}\int_0^t g^2(s)ds \Big]$$
$$W_t^{\mathbb{Q}} = W_t + \int_0^t g(s)ds \tag{3.4}$$

Set $Z = Z_T$.
Then $\mathbb{E}[Z] = \mathbb{E}^{\mathbb{P}}[Z] = 1$ and, under $\mathbb{Q}$, $W_t^{\mathbb{Q}}$ is a Brownian motion.

The Z_t process is the Doléans-Dade exponential cited above. Obviously we have that $Z_t(g(0)) = 1$, or $g(0) = 0$. The fact that $W_t^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$, implies that it is a martingale. This can be easily proved Shreve (2004).
Let us see how this work. We have our process which we can write as

$$S_t = s_0 \exp\big[\int_0^t (\mu - \frac{\sigma^2}{2})ds + \int_0^t \sigma dW_s \big]$$

or

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{3.5}$$

In turn, the discounted price process would be

$$\tilde{S}_t = e^{-rt} S_t = s_0 \exp\big[\int_0^t (\mu - r - \frac{\sigma^2}{2})ds + \int_0^t \sigma dW_s \big]$$

now we set $g = \theta = \frac{\mu - r}{\sigma}$, so that our new process $W_t^{\mathbb{Q}} = W_t + \int_0^t \frac{\mu - r}{\sigma} ds$.
This allow us to rewrite - easily seen by differentiating and substituting -

$$d\tilde{S}_t = e^{-rt}\sigma S_t dW_t^{\mathbb{Q}}$$

Which would give us

$$\tilde{S}_t = s_0 + \int_0^t e^{-rs}\sigma S_s dW_s^{\mathbb{Q}}$$

which is a martingale under $\mathbb{Q}$. Now returning to the undiscounted price process of 3.5, if we use $dW_t = dW_t^{\mathbb{Q}} - \theta dt$, we get

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}} \tag{3.6}$$

Finally reaching the desired result, since we obtained a process with drift equal to r.

This result give us a way to move the drift of the process, leaving only the diffusion part unchanged. Indeed the diffusion tells us which paths are possible and under an equivalent measure, it is not possible to change this thing. So we just ***tilted*** our process recalibrating the existent paths. We can go on to the pricing for a European call option.

3.3 The Pricing

Let us go back to the basic framework above and let us apply the results just shown. We said that we need the discounted asset process to be a martingale under the risk-neutral measure. So we define the discounted price process $\tilde{S}_t := e^{-rt}S_t$ where r is the constant risk-free rate and S_t is the same as 3.1. Then

$$\tilde{S}_t = s_0 \exp\left[\int_0^t (\mu - r - \frac{1}{2}\sigma^2)ds + \int_0^t \sigma dW_s\right]$$

We define the quantity $\theta := \frac{\mu - r}{\sigma}$, which is called the market price of risk. Consider now the g of theorem 3.3, we set $\theta = g$ obtaining

$$S_t = s_0 \exp\left[\int_0^t \left(r - \frac{1}{2}\sigma^2\right)ds + \int_0^t \sigma dW_s^{\mathbb{Q}}\right] \tag{3.7}$$

So we changed the mean return but not the diffusion part, which remains as source of volatility. Indeed is easy to see that the discounted value of such price process is given only by the diffusion part. We just rebalanced the paths we had before, shifting the probability of occurrence.

Our initial purpose was to price a vanilla option on such asset. Consider a European call option. The payoff of such option is $P(S) = (S_T - K)^+$. The price of the option at any time $t < T$ must be a function of the discounted payoff. Given the information we have up to time t, we then have

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^+ \middle| \mathcal{F}_t\right]. \tag{3.8}$$

We also know that

$$S_t = s_0 \exp\left[\sigma W_t^{\mathbb{Q}} + \left(r - \frac{1}{2}\sigma^2\right)t\right]$$

So we can write

$$S_T = S_t \exp\left[\sigma(W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}}) + \left(r - \frac{1}{2}\sigma^2\right)(T - t)\right]$$

Now we define the function of the option price $c(t, S_t)$ as 3.8. Thus if we work on it, using all the nice properties we have seen so far we finally arrive to the Black-Scholes formula for pricing an European call option which in syntethic form is

$$c^{BS}(\tau, x; K, r, \sigma) = xN(d_+(\tau, x)) - e^{-r\tau}KN(d_-(\tau, x)) \tag{3.9}$$

where $\tau = T - t$, N denotes the cumulative normal distribution, and d_+, d_- are the parameters for which it must be calculated.

3.4 Moving forward

So we started from the asset price process for which we wanted to extrapolate the value of a European call option. We saw that we needed to change the measurement for the possible paths and change it in a way that made us work with a martingale process. Once we did that we finally found the price of the option. As we saw, there was no ambiguity with respect to the change we had to make. Indeed $\mathbb{Q}$ was univocally determined during the derivation. This lead us to the *second fundamental theorem of asset pricing*.

Theorem 3.4. *The market model is complete, i.e. every risk position can be hedged and security exchanged, if and only if the risk-neutral measure is unique.*

This theorem can be reformulated in an equivalent way, that is the market is complete if there is only one source of randomness in each asset dynamic. When we changed measure, we obtained that the discounted price process was driven only by the diffusion part. That part is perfectly replicable in the market since it is the driver of the asset price itself. Moreover when we changed measure we solved the market price of risk equation by the unknown parameter θ, i.e. $\theta\sigma = \mu - r$. This is an equation with one unknown, hence it has a unique solution (given that σ is different of zero, something that does not bother us).

This would be the ideal framework: every position can be hedged, every instrument can be properly priced, everything is Gaussian (well, this remains), and every person would be happy. Actually, this is not the case, since reality is much more tricky than that, so now we move on to try to shed some light on a more representative type of processes.

Chapter 4

Jump Diffusion Processes

Sometimes asset prices jump due to -positive or negative- shocks which affect the market. This in *normal* market conditions. In case of crashes, crises or bubbles, these jumps are even more severe, wrecking, at first glance, everything we said so far. So what we want to do now, is to allow for those jumps to happen, without losing all the nice properties and results we obtained until now. Luckily for us this is possible, and it is where we are going now.

But let us proceed step by step. We first need a way to introduce jumps in our model, and this purpose is served by the *Poisson process*.

4.1 Poisson Process and Surroundings

The Poisson process is a discrete time process which is constructed as a sequence of i.i.d. exponential random variables. We briefly recall the density of such random variable.

Definition 4.1. *Let τ be a random variable with density*

$$f(t) = \lambda e^{-\lambda t} 1_{\{t \geq 0\}}$$

where $\lambda > 0$. Then we say that τ is an exponential random variable. In notation $\tau \sim \varepsilon(\lambda)$.

The expectation is given by

$$\mathbb{E}[\tau] = \frac{1}{\lambda}$$

And the CDF

$$F(t) = 1 - e^{-\lambda t}$$

A most important feature of the exponential random variable is the memorylessness, i.e.

$$\mathbb{P}\{\tau > t + s | \tau > s\} = e^{-\lambda t} = \mathbb{P}\{\tau > t\} \tag{4.1}$$

This means that the probability of observing τ greater than $t + s$ given that τ is already greater than s, is the same as observing τ greater than t. It is called memoryless property because what already happened is not relevant for the computation of probability.

This concludes what we need to know about exponential random variables in order to move to the definition of the *Poisson process*.

Proposition 4.1. *Let $\{\tau_i\}_{i\geq 1}$ be a sequence of i.i.d. exponentially distributed random variable with parameter $\lambda > 0$.*
We call the Poisson Process with intensity λ the process

$$N_t = \sum_{n\geq 1} \mathbf{1}_{\{\tau_n \leq t\}} = \sum_{n\geq 1} n 1_{\{\tau_n \leq t < \tau_{n+1}\}} \tag{4.2}$$

The random variable N_t follows a Poisson law with parameter λt:

$$\mathbb{P}(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!},\ n \in \mathbb{N} \tag{4.3}$$

In particular

$$\mathbb{E}(N_t) = \lambda t = Var(N_t)$$

Moreover, for s>0

$$\mathbb{E}(s^{N_t}) = \exp\{\lambda t(s-1)\} \tag{4.4}$$

So a Poisson process represents the sum of the ***arrival times*** of a sequence of events, for which the ***inter-arrival*** times are modeled as i.i.d. exponential random variables. Namely, for every t, the density tells us which is the probability that the event occurred n times before time t.

The Poisson process has some nice intrinsic features which allow us to use it to model an asset dynamic, giving rise to tractable solutions and analyses:

- At jump times the process N_t is defined to be ***right-continuous***. In words, we know the jump time exactly at that moment. Just before the jump, the process takes the previous value.

- As the Brownian motion, the Poisson process has independent and stationary increments, the latter somewhat inherited thanks to the memorylessness property of the exponential distribution. These features will be crucial for the construction of the jump diffusion process and its tractability.

Remark. Notice that, while the (standard) Brownian Motion is a martingale, the Poisson Process itself is not.

Theorem 4.1. *Let N_t be a Poisson Process with intensity $\lambda > 0$. We define the Compensated Poisson Process*

$$M_t := N_t - \lambda t.$$

Then M_t is a martingale.

Proof. We want to show

$$\mathbb{E}[M_t|F_s] = M_s$$

Then,

$$\begin{aligned}
\mathbb{E}[M_t|F_s] &= \mathbb{E}[N_t - \lambda t|\mathcal{F}_s] \\
&= \mathbb{E}[N_t - N_s + N_s - \lambda t|\mathcal{F}_s] \\
&= \mathbb{E}[N_t - N_s|\mathcal{F}_s] + N_s - \lambda t \\
&= \mathbb{E}[N_{t-s}] + N_s - \lambda t \\
&= \lambda(t-s) + N_s - \lambda t \\
&= N_s - \lambda s
\end{aligned}$$

□

.

We are almost there. The Poisson process and the compensated Poisson process seem to have all the properties we need in order to start to build our model, but we

still have to face one constraint. Indeed the mentioned process allows only for jump of unitary size, something we do not like. We want to allow our process to have different jump sizes at each jump. That is where the compound Poisson process comes to assist us.
We build the compound Poisson process as the sum of jump sizes, which occur at Poisson distributed times.

Definition 4.2. *Let N_t be a Poisson process with intensity λ, and $U_1, U_2, ...$ be a sequence of i.i.d. random variables, each one independent of N_t.*
*We define the **compound Poisson process** as*

$$Q_t = \sum_{j=1}^{N_t} U_j, \; t \geq 0 \tag{4.5}$$

So in the compound Poisson process, the jumps will occur at the same time of the process N_t, but they will be of random size. Moreover, the fine properties of the Poisson process are still there, for us to work with it. In turn, this means that the compound Poisson process is not a martingale, so we establish the same result we had earlier on with the Poisson process.

Theorem 4.2. *Let Q_t be the same as 4.5, and $\mathbb{E}(U_j) = \beta$.*
Then the compensated compound Poisson process

$$Q_t - \beta\lambda t \tag{4.6}$$

is a martingale.

We finally defined a process which has all the features we needed. From now on we will assume that there are no jumps at time 0, namely $N_0 = 0$. Let us see how can we play with the new type of stochastic process we are going to build, in general terms.

4.2 Jump processes

Along the lines of chapter 2, we want to define a stochastic integral of the form

$$\int_0^t g(s) dX_s, \tag{4.7}$$

with X_t of the form

$$X_t = x_0 + I_t + L_t + J_t, \tag{4.8}$$

where x_0 is the initial condition; I_t is and Itō integral, i.e.

$$I_t = \int_0^t h(s)dW_s,$$

with h and adapted process - as all the others we are introducing -; L_t is a Lebesgue integral

$$L_t = \int_0^t l(s)ds;$$

and, finally J_t is a ***pure jump*** process.

Now we can split the process 4.8 in ***continuous*** part,namely

$$X_t^c = x_0 + I_t + L_t,$$

and pure jump part J_t.

The continuous part is the general form of what we saw so far and we know that

$$dX_t^c dX_t^c = g^2(t)dt.$$

Let us focus on the new quantity, the pure jump part. J_t is a ***right-continuous*** pure jump process with $J_0 = 0$. By right continuous we formally mean that

$$\lim_{s \downarrow t} J_s = J_t.$$

We define the ***left continuous*** type of this process as J_{t-}. This tells us that the J_{t-} is the value just before the jump.

Definition 4.3. *Let X_t be a jump process as 4.8. The stochastic integral 4.7 can be written as*

$$\int_0^t g(s)dX_s = \int_0^t g(s)h(s)dW_s + \int_0^t g(s)l(s)ds + \sum_{s\in(0,t]} g(s)\Delta J_s \tag{4.9}$$

where $\Delta J_s = J_s - J_{s-}$.

In di erential form

$$g(t)dX_t = g(t)(dX_t^c + dJ_t) = g(t)dI_t + g(t)dL_t + g(t)dJ_t \tag{4.10}$$

We finally have a definition of a jump process. Let us see how the Itō formula applies on this quantity.

4.2.1 Itō formula for jump processes

Before seeing the Itō formula for this type of processes we must see how the quadratic variation behave. starting from 4.8, we said we could divide the process in two parts. having this in mind we can write the following result

Theorem 4.3. *Let X_t as 4.8. Then, with notation for the quadratic variation up to T as $[X,X](T)$, we have*

$$[X,X](T) = [X^c,X^c](T) + [J,J](T) = \int_0^T h^2(s)ds + \sum_{s\in(0,T]} (\Delta J_s)^2$$

We will not see the proof for this theorem.

Now suppose we have a $\mathcal{C}^{1,2}$ function f. Then, in the same framework as above, we know that

$$df(X_s^c) = f'(X_s^c)dX_s^c + \frac{1}{2}f''(X_s^c)(dX_c^s)^2 = f'(X_s^c)h(s)dW_s + f'(X_s^c)l(s)ds + \frac{1}{2}f''(X_s^c)h^2(s)ds$$

in which we used the Itō formula as in chapter 2. Now we want to add the jump part. We know that jump occurs discretely in the interval $(0,T]$. So, in between jumps, we would have the formula above since the process behave like a continuous stochastic process. The only attention that we must have is at jump time. Indeed we can say that when there is a jump in X, from X_{s-} to X_s, there must be a jump also in $f(X)$, precisely from $f(X_{s-})$ to $f(X_s)$. Hence, when we integrate, we should add the jumps that occured.

Theorem 4.4. *Let X_t as 4.8 and $f \in \mathcal{C}^{1,2}$. Then*

$$f(X_t) = f(x_0) + \int_0^t f'(X_s)dX_s^c + \frac{1}{2} + \int_0^t f''(X_s)(dX_s^c)^2 + \sum_{s\in(0,T]} [f(X_s) - f(X_{s-})] \tag{4.11}$$

This is the Itō formula for jump processes. We can see that the result is the same that we saw in chapter 2 with the difference that we sum the jump parts when they occur. This result is the for the most general version of jump processes, indeed we just assume for the processes to be adapted and to have a stochastic integral similar to the one already introduced. Now we will go more into detail and see the jump diffusion processes, by always using as a starting point the stochastic process we use earlier, i.e. the geometric Brownian motion.

4.3 A second Financial Application

We want to incorporate this new process into the old one. We said that the compound Poisson is a discrete process, which at random times t will tell us if there is a jump. So the idea is to have a process which behave as the diffusion we saw in the previous chapter characterized by jumps here and there. We assume to have a Brownian motion W_t, a Poisson process N_t with intensity λ and a sequence of i.i.d. random variables U_j on the usual probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume for these quantities to be independent of each other. Thanks to the properties of the processes chosen, everything we said is retained with respect to the $\sigma - algebra$ considered, which is the one generated by the two processes. So we still have independence and stationarity of increments, adaptivity and measurability, these last two holding true for the random variables U_j too, meaning that the amplitude of the jumps before of or at time t are known.

A simple and intuitive idea about the construction Lamberton and Lapeyre (2008), is to assemble the process step by step. So we know that in the intervals between jumps, the process behave as in 3.1, namely

- for $t \in [\tau_j, \tau_{j+1})$,

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{4.12}$$

- At jump time τ_j, the jump part of S_{τ_j}

$$S_{\tau_j} - S_{\tau_{j-}} = S_{\tau_{j-}} U_j \tag{4.13}$$

  which yields $S_{\tau_j} = S_{\tau_{j-}}(1 + U_j)$

So if now consider the first interval before the jump - we assume no jumps at time 0, i.e. $N_0 = 0$ - that is $t \in [0, \tau_1)$, and calculate it at the left limit of τ_1, we have

$$S_{\tau_{1-}} = s_0 \exp\left[(\mu - \frac{1}{2}\sigma^2)\tau_1 + \sigma W_{\tau_1}\right] \tag{4.14}$$

And at time τ_1

$$S_{\tau_1} = s_0 \exp\left[(\mu - \frac{1}{2}\sigma^2)\tau_1 + \sigma W_{\tau_1}\right]\Big(1 + U_1\Big).$$

Iterating,

$$S_t = s_0 \exp\left[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right]\left(\prod_{j=1}^{N_t} 1 + U_j\right) \tag{4.15}$$

Where we set $\prod_{j=1}^{0} = 1$.

We must clarify some aspects of this derivation:

- We did not assume anything about the distribution of the jump sizes U_j except for them to be i.i.d., measurable, hence adapted, with respect to the filtration generated by the other two processes and to be right continuous. This leave us with some flexibility for the upcoming developments, and, reasonably, makes us ignorant to future jumps amplitude even just right before the jump's occurrence.

- The geometric Brownian motion is a continuous process. This allow us to evaluate the continuous part at the left hand limit or at the point, without making a difference. This justifies the right-hand side of 4.14.

Equation 4.15 tells us that the new process we introduced is simply the same S_t we discussed in the previous chapters - look at the first part of the right and side of 4.15 - that, at jump times, is increased by the jump size, e.g. $S_t^J = S_t^B + S_t^B U_T$ (where J and B indicated the new and the old processes). So when there is no jump, U is equal to 0, and we have the usual geometric Brownian motion process, and when there is a jump we have the increase we just described.

S_t can be also rewritten in integral form

$$S_t = s_0 + \int_0^t S_u(\mu du + \sigma dW_u) + \sum_{j=1}^{N_t} S_{\tau_j -} U_j, \tag{4.16}$$

and in differential form

$$\frac{dS_t}{S_{t^-}} = \mu dt + \sigma dW_t + dZ_t,$$

where $Z_t = \sum_{j=1}^{N_t} U_j$ is the compound Poisson process.

So we finally built our jump diffusion process. We want now move to the pricing of options on an asset which follows a jump diffusion. Let us see the seminal work done on the subject matter.

4.3.1 Merton Model

The first one to propose the idea was Robert Merton in his paper Merton (1976). Merton proposed a stochastic differential equation of the form

$$S_t = s_0 \exp\left[\mu t + \sigma W_t + \sum_{j=1}^{N_t} U_j\right] \tag{4.17}$$

Where U_j are i.i.d normal random variables, namely $U_j \sim \mathcal{N}(\mu_J, \sigma_J^2)$, and everything else is the same as above. The form of this equation is slightly different from the one we showed but, in the end, the results are the same, so we stick with our derivation. When faced with the problem of choosing the risk-neutral measure, Merton proposed a change of measure analogous of the one used in the Black-Scholes model, hence just changing the drift and leave everything else as it is. This is one way to obtain a risk-neutral measure, which we recall is the only path to follow in order to price a derivative instrument. The rationale behind this choice was that jump risk is an idiosyncratic risk, pertaining just the assets chosen, and in a well diversified portfolio this would be diversified away. This hypotesis is far from true both in normal and in severe market conditions. As a counter-example just consider the market indexes: according to this idea, they should not exhibit jumps in their value, but we know this occurs in normal market conditions too. Let us see which is the measure he proposed

Proposition 4.2. *Under $\mathbb{Q}_M$ the process become*

$$S_t = s_0 \exp\left[\mu^M t + \sigma W_t^M + \sum_{j=1}^{N_t} U_j\right] \tag{4.18}$$

Where W_t^M is a Brownian motion, N_t, U_j are unchanged, and μ^M is chosen in a way such that $\tilde{S}_t$ is a $\mathbb{Q}_M$-martingale.

$$\mu^M = r - \frac{\sigma^2}{2} - \lambda \mathbb{E}[e^{U_j} - 1] = r - \frac{\sigma^2}{2} - \lambda[e^{\mu_J + \frac{\sigma_J^2}{2}} - 1] \tag{4.19}$$

We will do something similar but the rationale behind it will not be the same. Along the lines of the previous chapter, in order to move to the pricing, we must define a change of measure, as Merton did. Here we face our first problem.

4.4 Change of measure

If we recall, in the previous chapter we enunciated the second fundamental theorem of asset pricing, and said that an analogous way to express it was that completeness is achieved if each asset has only one source of randomness. This makes us face a new issue in the framework just introduced, since the jump diffusion process has more than one random processes as drivers of the price. Indeed we could change the drift, as we did before, but we could also change the intensity of the jumps. This would lead to a market price of risk equation which has more than one unknowns Shreve (2004), giving rise to an infinite set of solutions. So we must find a way to tackle that. Ascertained the fact that we cannot satisfy the second fundamental theorem, we must comply with the first one. thus, we must find an equivalent measure which is risk-neutral. Recall the measure change chosen by Merton

$$\mu^M = r - \frac{\sigma^2}{2} - \lambda \mathbb{E}[e^{U_j} - 1] = r - \frac{\sigma^2}{2} - \lambda[e^{\mu_J + \frac{\sigma_J^2}{2}} - 1]. \tag{4.20}$$

This is the Girsanov transform, indeed Merton changed only the drift in order to work with a risk neutral measure, as in the previous chapter.
In this dissertation we will follow a different approach which will lead to the same result and start from this important result.

Theorem 4.5. *Suppose* $\mathbb{E}|U_1| < \infty$. *The process* $\tilde{X}_t = e^{-rt}X_t$ *is a martingale if and only if*

$$\mu = r - \lambda \mathbb{E}(U_1) \tag{4.21}$$

Proof. We must show that $\mathbb{E}[\tilde{X}_t|\mathcal{F}_s] = \tilde{X}_s$

We know that

$$\begin{aligned}\tilde{X}_t &= x_0 \exp\left[(\mu - r - \frac{\sigma^2}{2})t + \sigma W_t\right]\left(\prod_{j=1}^{N_t} 1 + U_j\right) \\ &= \tilde{X}_s \exp\left[(\mu - r - \frac{\sigma^2}{2})(t-s) + \sigma(W_t - W_s)\right]\left(\prod_{j=N_s+1}^{N_t} 1 + U_j\right)\end{aligned}$$

So

$$\begin{aligned}
\mathbb{E}[\tilde{X}_t|\mathcal{F}_s] &= \tilde{X}_s\mathbb{E}\Bigg[\exp\Big[(\mu - r - \frac{\sigma^2}{2})(t-s) + \sigma(W_t - W_s)\Big]\Big(\prod_{j=N_s+1}^{N_t} 1 + U_j\Big)\Big|\mathcal{F}_s\Bigg] \\
&= \tilde{X}_s\mathbb{E}\Bigg[\exp\Big[(\mu - r - \frac{\sigma^2}{2})(t-s) + \sigma(W_t - W_s)\Big]\Big(\prod_{j=1}^{N_t-N_s} 1 + U_j\Big)\Big|\mathcal{F}_s\Bigg] \\
&= \tilde{X}_s\mathbb{E}\Bigg[\exp\Big[(\mu - r - \frac{\sigma^2}{2})(t-s) + \sigma(W_{t-s})\Big]\Big(\prod_{j=1}^{N_{t-s}} 1 + U_j\Big)\Big|\mathcal{F}_s\Bigg] \\
&= \tilde{X}_s\mathbb{E}\Bigg[\exp\Big[(\mu - r - \frac{\sigma^2}{2})(t-s) + \sigma(W_{t-s})\Big]\Big(\prod_{j=1}^{N_{t-s}} 1 + U_j\Big)\Bigg]
\end{aligned}$$

The last equality is given by the fact that the two processes W_t, N_t have independent $\sigma - algebras$ Lamberton and Lapeyre (2008). We can take out the expectations of $(\mu - r - \frac{\sigma^2}{2})(t-s)$ since it is deterministic. Moreover we know that the remaining terms are independent, so we can take the individual expectations.

Now we must introduce a concept that is called ***moment generating function*** which is an alternative specification for the distribution of a random variable. It is defined as

$$\phi_X(u) = \mathbb{E}[e^{uX}]$$

which in the case of a Brownian motion, being it a normal random variable with mean zero, it is equal to

$$\phi_{W_t}(u) = \mathbb{E}[e^{uW_t}] = \exp[\frac{u^2}{2}t]$$

With this result, we can write

$$\mathbb{E}[\sigma W_{t-s}] = \frac{\sigma^2}{2}(t-s)$$

Hence,

$$\begin{aligned}
\mathbb{E}[\tilde{X}_t|\mathcal{F}_s] &= \tilde{X}_s\mathbb{E}\Bigg[\exp\Big[(\mu - r - \frac{\sigma^2}{2})(t-s) + \sigma(W_{t-s})\Big]\Big(\prod_{j=1}^{N_{t-s}} 1 + U_j\Big)\Bigg] \\
&= \tilde{X}_s e^{(\mu-r)(t-s)}\mathbb{E}\Bigg[\prod_{j=1}^{N_{t-s}} 1 + U_j\Bigg]
\end{aligned}$$

Set $u = t - s$ So we are left to solve

$$\begin{aligned}\mathbb{E}\Big[\prod_{j=1}^{N_{t-s}}(1+U_j)\Big] &= \mathbb{E}\Big[\prod_{j=1}^{N_u}(1+U_j)\Big] = \\ &= \sum_{m\geq 1}\mathbb{E}\Big[\prod_{j=1}^{N_u}(1+U_j)\Big]\mathbb{P}(N_u = m) = \sum_{m\geq 1}(1+\mathbb{E}[U_j])^m\mathbb{P}(N_u = m) \\ &= \sum_{m\geq 1}(1+\mathbb{E}[U_1])^m\frac{(\lambda u)^m e^{-\lambda u}}{m!} = e^{-\lambda u}\sum_{m\geq 1}\frac{(\lambda u(1+\mathbb{E}[U_1]))^m}{m!} \\ &= e^{-\lambda u}e^{\lambda u(1+\mathbb{E}[U_1])} = e^{\lambda u(1+\mathbb{E}[U_1]-1)} \\ &= e^{\lambda(t-s)\mathbb{E}[U_1]}\end{aligned}$$

Where we used the law of iterated expectation, the fact that we assume $\mathbb{P}(N_t = 0 = 0$, and

$$\sum_{i\geq 0}\frac{(\alpha x)^i}{i!} = e^{\alpha x}$$

Hence, we finally reach

$$\mathbb{E}[\tilde{X}_t|\mathcal{F}_s] = \tilde{X}_s e^{(\mu-r)(t-s)}e^{\lambda(t-s)\mathbb{E}[U_1]},$$

and it is easy to see that $\tilde{X}_t$ is a martingale if and only if $\mu = r - \lambda\mathbb{E}[U_1]$ □

We obtained the transformation desired. So far, we always assumed that the two process are independent, but since we introduced the concept of moment generating function we will show an use we can make of it, in conjuction with the new Itō formula, in order to prove independence of Brownian motion and Poisson process with respect to a general filtration.

Corollary 4.1. *Let W_t be a Brownian motion and N_t a Poisson process with intensity $\lambda > 0$, on the same probability space and relative to the same filtration. Then $W_t \perp N_t$.*

Proof. Notice that the moment generating function of a Poisson process is given by

$$\phi_N(u) = \mathbb{E}[e^{uN}] = \exp[\lambda t(e^u - 1)] \tag{4.22}$$

Now define

$$Y_t = \exp[u_1 W_t + u_2 N_t - \frac{1}{2}u_1^2 t - \lambda t(e^{u_2} - 1)] \tag{4.23}$$

Let us call the argument of the exponential X_t and apply the Itō formula. First let us divide in continuous part and jump part and see what we can extrapolate

$$dX_s^c = u_1 dW_s - \frac{1}{2}u_1^2 ds - \lambda ds(e^{u_2} - 1) \tag{4.24}$$

Then notice that if there is a jump at time s

$$Y_s = \exp[u_1 W_s + u_2(N_{s-} + 1) - \frac{1}{2}u_1^2 t - \lambda t(e^{u_2} - 1)] = Y_{s-}e^{u_2}, \tag{4.25}$$

so that

$$Y_s - Y_{s-} = (e^{u_2} - 1)Y_{s-}\Delta N_s.$$

Now setting $f(x) = e^x$ and applying the Itō formula we can write

$$\begin{aligned} Y_t = f(X_t) = f(x_0) + \int_0^t f'(X_s)dX_s^c + \frac{1}{2}\int_0^t f''(X_s)(dX_s^c)^2 + \sum_{s\in(0,t]}[f(X_s) - f(X_{s-})] \\ = 1 + u_1\int_0^t Y_s dW_s - \frac{1}{2}u_1^2\int_0^t Y_s ds - \lambda(e^{u_2} - 1)\int_0^t Y_s ds + \frac{1}{2}u_1^2\int_0^t Y_s ds \\ + \sum_{s\in(0,t]}[Y_s - Y_{s-}] \\ = 1 + u_1\int_0^t Y_s dW_s - \lambda(e^{u_2} - 1)\int_0^t Y_{s-}ds + e^{u_2} - 1)\int_0^t Y_{s-}dN_s \\ = 1 + u_1\int_0^t Y_s dW_s - \lambda(e^{u_2} - 1)\int_0^t Y_{s-}dM_s \end{aligned} \tag{4.26}$$

where $M_s = N_s - \lambda s$. The integral in the last line of 4.26 is a martingale, hence Y is a martingale. Now, since $\mathbb{E}Y_0 = 1$ and we just shown that Y is a martingale, then $\mathbb{E}Y_t = 1$ for every t, or equivalently

$$\mathbb{E}[\exp[u_1 W_t + u_2 N_t - \frac{1}{2}u_1^2 t - \lambda t(e^{u_2} - 1)] = 1.$$

Now we can write

$$\mathbb{E}[\exp[u_1 W_t + u_2 N_t]] = \exp[\frac{1}{2}u_1^2 t]\exp[\lambda t(e^{u_2} - 1)]. \tag{4.27}$$

This is the product of the moment generating functions of a Brownian motion and a Poisson process, and since they factorize, this mean that the two process are independent. We can complete the proof by showing that the vectors of random variable $(W_{t_1}, ..., W_{t_n})$is independent of $(N_{t_1}, ..., N_{t_n})$ for any finite set of times $0 \leq t_1 < ... < t_n$ □

To link it to the construction we had in the previous chapter, consider a European option. We see that under the new probability $\mathbb{Q}$ defined as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\theta W_t - \frac{\theta^2}{2}t} \tag{4.28}$$

with $\theta = \frac{r-\mu-\lambda\mathbb{E}[U_1]}{\sigma}$, the discounted price process is a martingale.
Indeed $W_t^{\mathbb{Q}} = W_t - \theta t$ is a standard Brownian motion under $\mathbb{Q}$. Notice that we have a minus sign, differently from the plus sign in the previous chapter, since θ is set differently. Thus under $\mathbb{Q}$, the process become

$$S_t = s_0 \exp\big[(\mu^{\mathbb{Q}} - \frac{\sigma^2}{2})t + \sigma W_t^{\mathbb{Q}}\big]\big(\prod_{j=1}^{N_t} 1 + U_j\big) \tag{4.29}$$

where $\mu^{\mathbb{Q}} = \mu + \sigma\theta$, and, consequently, $\mu^{\mathbb{Q}} = r - \lambda\mathbb{E}[U_1]$.
From now on we will assume to be under this new risk neutral probability. Before going onto the pricing, we will introduce some new concepts, to explain why we must choose this type of measure, in line with what is done in Lamberton and Lapeyre (2008)
For technical reasons we assume that the U_j are square integrable. We note that

$$\mathbb{E}[S_t^2] = s_0^2\mathbb{E}\Big[\exp\big((\mu - \frac{\sigma^2}{2})t + \sigma W_t\big)\prod_{j=1}^{N_t} 1 + U_j\Big]$$

Hence, using the reasoning of the proof above,

$$\mathbb{S}_t^2 = s_0^2 \exp((\sigma^2 + 2r)t)\exp(\lambda t\mathbb{E}[U_1^2])$$

Thus the process $\tilde{S}_t$ is a square integrable martingale. Now consider a European option defined by a random variable c which is measurable and square integrable. In order to hedge the option, the agent, let us assume she is the writer of the option, will follow a strategy V_t which is based on the amount of the risky asset in the portfolio. Assume the strategy V_t is a square-integrable martingale. Its discounted value is a martingale too, namely $\mathbb{E}\tilde{V_t} = V_0$. A way to measure the risk of hedging mismatch between the value of the option and the strategy undergone, is to introduce the notion of quadratic risk. Quadratic risk is simply a loss function calculated on the square of the expected loss. Calling R_t this risk process, at any t, the value of this quantity will be the discounted expected value of the hedging mismatch squared. So at time 0, considering the perspective of the writer of an option, we would have,

$$R_0^T = \mathbb{E}[(e^{-rT}(c - V_T))^2] \tag{4.30}$$

We want to check the value, at any time t, which minimizes this risk, namely the quantity

$$R_t^T = \mathbb{E}\big[(e^{-r(T-t)}(h - V_T))^2 | \mathcal{F}_t\big]. \tag{4.31}$$

The value we find would be the premium asked by an agent selling the option. We can write

$$R_t^T = (\mathbb{E}[e^{-r(T-t)}c|\mathcal{F}_t] - V_t)^2 + \mathbb{E}\big[e^{-r(T-t)}c - \mathbb{E}[e^{-r(T-t)}c|\mathcal{F}_t] - e^{-r(T-t)}V_T + V_t|\mathcal{F}_t]^2$$

It is clear that an agent who wants to minimize this quantity will ask for a premium $V_t = \mathbb{E}[e^{-r(T-t)}c|\mathcal{F}_t]$.

If we consider this, the result tells us that on average the chosen strategy hedges the option, where we calculated the expectation under the risk neutral measure. It can be shown, due to the strict convexity of the option price function, that the delta hedging strategy outperforms the option between jumps, where the process behave like in the Black-Scholes framework, but at jump times, the option outperforms the strategy Shreve (2004). This is reasonable since we could only perfectly hedge the volatility of the model, adjusting for the average of "jump risk". Before moving onto the pricing we will make a small excursus trying to understand something more about the idea of change of measure.

4.4.1 Esscher Transform

The method we just saw, is not the only one that can be used in order to obtain an equivalent martingale measure.

Definition 4.4. *Let X_t be a jump di usion process as we defined it. We change the probability measure from $\mathbb{P}$ to $\mathbb{Q}$ by means of the Esscher transform, defined as*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{e^{\theta X}}{\mathbb{E}[e^{\theta X}]} \tag{4.32}$$

We already notice that the denominator is the moment generating function for the process X_t. Now defining Z_t as 4.32, we must prove that this is actually a change of measure, namely that $\mathbb{E}Z_t = 1$ and that it is a martingale. In order to do that we notice that

$$\begin{aligned} \phi_X(\theta) &= \mathbb{E}[e^{\theta X_t}] = \mathbb{E}[\exp(\theta(X_t + X_s - X_s)] \\ &= \mathbb{E}[e^{\theta X_{t-s}}]\mathbb{E}[e^{\theta X_s}]. \end{aligned} \tag{4.33}$$

Z_t is integrable by thanks to the assumption we have on the process X_t and once we integrate is easy to see that $\mathbb{E}Z_t = 1$.
About the $\mathbb{P}$-martingality,

$$\begin{aligned} \mathbb{E}[Z_t|\mathcal{F}_s] &= \mathbb{E}\left[\frac{e^{\theta X_t}}{\mathbb{E}[e^{\theta X_t}}\middle|\mathcal{F}_s\right] \\ &= \frac{1}{\mathbb{E}[e^{\theta X_t}]}\mathbb{E}[^{\theta(X_t+X_s-X_s)}|\mathcal{F}_s] \\ &= e^{\theta X_s}\frac{e^{\theta X_{t-s}}}{\mathbb{E}[e^{\theta X_t}]} \\ &= e^{\theta X_s}\frac{e^{\theta X_{t-s}}}{\mathbb{E}[e^{\theta(X_t+X_s-X_s)}]} \\ &= e^{\theta X_s}\frac{e^{\theta X_{t-s}}}{\mathbb{E}[e^{\theta X_{t-s}}]\mathbb{E}[e^{\theta X_s}]} \\ &= \frac{e^{\theta X_s}}{\mathbb{E}[e^{\theta X_s}]} \\ &= Z_s \end{aligned} \tag{4.34}$$

Thus, we showed that it is a change of measure. We still have the problem of how to choose θ. A necessary and sufficient condition on θ to obtain a risk-neutral measure $\mathbb{Q}$ is that

$$\phi_X(\theta) = \phi_X(\theta + 1) \tag{4.35}$$

which admits a unique solution.

Let us see now, how we can use it in the Black Scholes framework.

We know that

$$\tilde{S}_t = s_0 \exp[(\mu - r - \frac{\sigma^2}{2})t + \sigma W_t] \tag{4.36}$$

So calling X_t the diffusion process, which is the argument of the exponential, we get

$$\begin{aligned} \mathbb{E}[e^{\theta X_t}] &= \mathbb{E}[\exp(\theta(\mu - r - \frac{\sigma^2}{2})t + \sigma W_t)] \\ &= e^{(\mu - r - \frac{\sigma^2}{2})t}\mathbb{E}[e^{\theta\sigma W_t}] \\ &= \exp[\theta(\mu - r - \frac{\sigma^2}{2})t + \frac{\sigma^2\theta^2}{2}t] \end{aligned} \tag{4.37}$$

Now, in order to find θ^* such that 4.35 holds, we must solve the expression for θ. Hence, we would get

$$\begin{aligned} \exp[\theta(\mu - r - \frac{\sigma^2}{2})t + \frac{\sigma^2\theta^2}{2}t] &= \exp[(\theta+1)(\mu - r - \frac{\sigma^2}{2})t + \frac{\sigma^2(\theta+1)^2}{2}t] \\ \theta(\mu - r - \frac{\sigma^2}{2}) + \frac{\sigma^2\theta^2}{2} &= (\theta+1)(\mu - r - \frac{\sigma^2}{2}) + \frac{\sigma^2(\theta+1)^2}{2} \\ (\theta^2 - (\theta+1)^2)\frac{\sigma^2}{2} &= (\theta + 1 - \theta)(\mu - r - \frac{\sigma^2}{2}) \\ \theta^* &= -\frac{\mu - r}{\sigma^2}. \end{aligned} \tag{4.38}$$

Now, evaluating Z_t we get

$$\begin{aligned} Z_t &= \frac{e^{\theta X_t}}{\mathbb{E}[e^{\theta X_t}]} \\ &= \frac{\exp\Big((-\frac{\mu-r}{\sigma^2})(\mu - r - \frac{\sigma^2}{2})t + (-\frac{\mu-r}{\sigma^2}\sigma W_t)\Big)}{\mathbb{E}\Big[\exp\Big((-\frac{\mu-r}{\sigma^2})(\mu - r - \frac{\sigma^2}{2})t + (-\frac{\mu-r}{\sigma^2}\sigma W_t)\Big)\Big]} \\ &= \frac{e^{-(\frac{\mu-r}{\sigma})W_t}}{\mathbb{E}[e^{-(\frac{\mu-r}{\sigma})W_t}]} \\ &= \exp\Big(-\frac{\mu-r}{\sigma}W_t - \frac{t}{2}\Big(\frac{\mu-r}{\sigma}\Big)^2\Big) \end{aligned} \tag{4.39}$$

As we can see the expression obtained is the same defined by the Girsanov transform (3.4) with the only difference given by the sign of the market price of risk. This is something we would expect since we know that the measure in the Black Scholes framework is unique and the market is complete, so consistency between the two approach was expected.

After this excursus we can go on to pricing.

4.5 The Pricing

We finally have all the ingredient we need in order to proceed onto the pricing of an option. Consider the usual European call option with payoff $P(S) = (S_T - K)^+$. As saw in the previous chapter, at any time t, the price of the option is $\mathbb{E}[e^{-r(T-t)}(S_T - K)^+|\mathcal{F}_t]$. In this new framework, this translates into

$$\mathbb{E}[e^{-r(T-t)}(S_T - K)^+|\mathcal{F}_t] =$$

$$\mathbb{E}\left[e^{-r(T-t)}\left(S_t \exp\left[(\mu - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)\right]\left(\prod_{j=N_t+1}^{N_T} 1 + U_j\right) - K\right)^+ \middle| \mathcal{F}_t\right] =$$

$$\mathbb{E}\left[e^{-r(T-t)}\left(S_t \exp\left[(\mu - \frac{\sigma^2}{2})(T-t) + \sigma W_{T-t}\right]\left(\prod_{j=1}^{N_{T-t}} 1 + U_j\right) - K\right)^+ \middle| \mathcal{F}_t\right]$$

Which under our choice for the risk neutral measure become

$$\mathbb{E}\left[e^{-r(T-t)}\left(S_t \exp\left[(r - \lambda\mathbb{E}[U_1] - \frac{\sigma^2}{2})(T-t) + \sigma W_{T-t}\right]\left(\prod_{j=1}^{N_{T-t}} 1 + U_j\right) - K\right)^+ \middle| \mathcal{F}_t\right] \tag{4.40}$$

Now, using dummy variables and calling again c^{BS} the Black-Scholes option price function of the form

$$c^{BS}(t, s) = \mathbb{E}\left[e^{-r(T-t)}\left(se^{(r-\sigma^2/2)(T-t)+\sigma W_{T-t}} - K\right)^+\right]$$

We have that, setting $c^{JD}(t, s) =$ as 4.40, we can write

$$c^{JD}(t, s) = \mathbb{E}\left[c^{BS}\left(t, se^{-\lambda\mathbb{E}[U_1](T-t)}\left(\prod_{j=1}^{N_{T-t}} 1 + U_j\right)\right)\right] \tag{4.41}$$

Moreover, having that $N_{T-t} \perp U_j$, for every j, and knowing $N_{T-t} \sim Pois(\lambda(T-t))$ we can write

$$c^{JD} = \sum_{n\geq 0} \mathbb{E}\left[c^{BS}\left(t, se^{-\lambda\mathbb{E}[U_1](T-t)}\left(\prod_{j=1}^{n} 1 + U_j\right)\right)\right]\frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!} \tag{4.42}$$

Where we disintegrated the function according to the Poisson law of the number of jumps. This formula allow us to have closed form for some distribution of the jump amplitudes U_j, something we will show now.

4.6 An Illustration

Consider the usual jump diffusion process we have discussed so far. As we said already, we derived our result without assuming the distribution of the jump amplitudes. We just showed the model proposed by Merton, with the log-normal (considering our construction) jump sizes.

Now assume that our U_j are i.i.d. as a binomial with support$\{a, b\}$. Formally

$$U_j = \begin{cases} a \text{ with probability} & p \\ b & q = 1 - p \end{cases} \tag{4.43}$$

Using the reasoning of the previous section, we want to reach a closed form for the price of an option. Assume we are under the risk-neutral probability and we want to find the price of the option at time t, which we denote as $c(t, S_t) = \mathbb{E}[e^{-r(T-t)}(S_T - K)^+|\mathcal{F}_t]$ and call $P(S_T)$ the payoff inside the round brackets. We can write

$$e^{-r(T-t)}\mathbb{E}[P(e^{r(T-t)}S_t(\frac{S_T}{S_t}e^{-r(T-t)})|\mathcal{F}_t]$$

in which we can isolate the ratio $Z := \frac{S_T}{S_t}e^{-r(T-t)} = \frac{\tilde{S}_T}{\tilde{S}_t}$ which is independent of the $\sigma - algebra$. Moreover, the remaining part $e^{r(T-t)}S_t = y$ is $\mathcal{F}_t$-measurable, thus is known at time t. We can write

$$e^{-r(T-t)}\mathbb{E}[P(y\tilde{S}_{T-t})|F_t].$$

Let us call the time increment $(T - t) = \tau$. Now, thanks to the Lévy property of our process, namely independence and stationarity of the increments, developing the expectation, and noting that $Z_0 = \frac{S_0}{S_0} = 1$, we would get

$$\mathbb{E}[P(yZ)] = \sum_{n\geq 0} \mathbb{E}[P\big(ye^{(\mu-r-\sigma^2/2)\tau+\sigma W_\tau}\prod_{j=1}^{n}(1+U_j)\big)]\frac{(\lambda\tau)^n}{n!}e^{-\lambda\tau} \tag{4.44}$$

We can now disintegrate the U_j according to their distribution. Then 4.44 becomes

$$\sum_{n\geq 0}\sum_{k=0}^{n}\mathbb{E}\left[P\left(ye^{(\mu-r-\sigma^2/2)\tau+\sigma W_\tau}(1+a)^k(1+b)^{n-k}\right)\right]\binom{n}{k}p^k(1-p)^{n-k}\frac{(\lambda\tau)^n}{n!}e^{-\lambda\tau} \tag{4.45}$$

We developed the expectation. Now discounting back and calling $n-k=h$ we would get

$$e^{-r\tau}\sum_{n\geq 0}\sum_{k=0}^{n}\mathbb{E}\left[P\left(ye^{(\mu-r-\sigma^2/2)\tau+\sigma W_\tau}(1{+}a)^k(1{+}b)^h\right)\right]\frac{(\lambda p\tau)^k}{k!}e^{-\lambda p\tau}\frac{(\lambda(1-p)\tau)^h}{h!}e^{-\lambda(1-p)\tau} \tag{4.46}$$

Notice that, If we move to the risk neutral measure $\mu = r - \lambda\mathbb{E}[U]$ and we take the discount factor inside the expectation we would get the price of an option with maturity τ with asset price $y(1+a)^k(1+b)^h$. Expanding things more it would become

$$e^{-r\tau}\sum_{n\geq 0}\sum_{k=0}^{n}\mathbb{E}\left[P\left(S_te^{(\mu-\sigma^2/2)\tau+\sigma W_\tau}(1{+}a)^k(1{+}b)^h\right)\right]\frac{(\lambda p\tau)^k}{k!}e^{-\lambda p\tau}\frac{(\lambda(1-p)\tau)^h}{h!}e^{-\lambda(1-p)\tau} \tag{4.47}$$

And now evaluating everything under the risk neutral measure

$$\sum_{n\geq 0}\sum_{k=0}^{n}\mathbb{E}\left[P\left(S_te^{(-\lambda\mathbb{E}[U]-\sigma^2/2)\tau+\sigma W_\tau}(1{+}a)^k(1{+}b)^h\right)\right]\frac{(\lambda p\tau)^k}{k!}e^{-\lambda p\tau}\frac{(\lambda(1-p)\tau)^h}{h!}e^{-\lambda(1-p)\tau} \tag{4.48}$$

Which is the closed formula we wanted after 4.42. Indeed, we may also rewrite the last expression as

$$c^{JD}(t,s)=\sum_{n\geq 0}\sum_{k=0}^{n}\mathbb{E}\left[c^{BS}\left(t,se^{-\lambda\mathbb{E}[U]\tau}(1{+}a)^k(1{+}b)^h\right)\right]\frac{(\lambda p\tau)^k}{k!}e^{-\lambda p\tau}\frac{(\lambda(1-p)\tau)^h}{h!}e^{-\lambda(1-p)\tau} \tag{4.49}$$

This formula tells us that in this jump diffusion setup what we would get in terms of the price of the option is a weighted average of the Black-Scholes price calculated according to the distributions of jumps' times and amplitudes. This result allow us to compute this term numerically if we have a way to generate the laws involved. This is where we are heading.

Chapter 5

The experiment

It is time to see how we can use everything we learned so far, having now all the ingredients needed to set up a simulation. I decided to apply the concept we discussed about on the pricing of a barrier option. The underlying asset, obviously, will follow a jump diffusion process. The pricing will be done under two scenarios: the Merton model and the process we showed in the previous section. There are several ways to price barrier options. We will use Monte Carlo methods, which I will explain below. First things first let us see what is a barrier option.

5.1 Barrier Options

A barrier option is an option which has a payoff similar to the plain vanilla counterparty, except for the presence a "barrier" which affect the value of the option. This additional condition, makes the pricing to be path-dependent, since it does not rely only on the maturity, but also on the probability of hitting the barrier before T. As usual there are call and put barrier options but we need another specification:

- A **knock-in** barrier option represent the situation in which the option, upon hitting the barrier, comes into life. This means that before this event, the option represent a worthless claim, even if its plain vanilla analogue has a positive payoff. For example consider a call option with maturity T with a barrier $H > K > S_0$ where K is the strike price and S_0 is the value of the underlying as of today. This is called up-and-in call option. So during the lifetime of the option we, holders of the option, have a worthless claim until the price $S_t = H$. After this event we have a plain vanilla call. This holds true also in the interval $S_t \in (H, K)$.

- A **knock-out** barrier options represent the opposite scenario, that is the option gives right to a claim, as long as the price of the underlying does not hit the barrier. So a call as above would be call an up-and-out call option

- Of course we can consider all the possible permutations of such scenarios: considering a put, a different level for the barrier and the strike price, and the different type (in or out) of the option.

As we can see, other than being interested to the value of the underlying at maturity, in order to evaluate our payoff, we also want to know if the event - hitting the barrier - occurs. One thing must be noted: the time of recording of the asset price option is relevant for the price of the option. Suppose we only look at the closing value, so once a day. The underlying might have been at or over the barrier during the day. So computing the trajectories at a more and more granular time grid, will increase the probability for the asset to breach the barrier, at a given time of the day, making the price of the option higher or lower, according to the type considered. Indeed there are different methods and prices for continuously monitored barrier with respect to discretely monitored ones.
The payoff of a barrier option can be written in the following way. Consider a up-and-out call option. Let us call $P(S_T)$ the payoff of a European call option, H the barrier level and K as usual the strike price. Then, for $t \in [0, T]$,

$$\mathcal{B}_{CUO} = \begin{cases} P(S_T) = (S_T - K)^+ & if\ S_t < H, \\ 0 & S_t \geq H \end{cases} \tag{5.1}$$

As we can see we must consider the possibility for the price to hitting the barrier before maturity when considering the pricing. So the idea would be to include the probability of the underlying reaching the barrier into the computation of the price.

Remark. We might rewrite the payoff $\mathcal{B}_{CUO}$ as

$$\mathcal{B}_{CUO} = (S_T - K)^+ 1_{(M(S_T)<H)} \tag{5.2}$$

Where $M(S_t)$ represent the maximum value the underlying reached upt to time t. Equivalently we can write the payoff of a up-and-in call option as

$$\mathcal{B}_{CUI} = (S_T - K)^+ 1_{(M(S_T)\geq H)} \tag{5.3}$$

Now consider the prices of such option on the same underlying, with the same barrier, same strike price and same maturity. Then, under the risk-neutral measure, calling c the price function,

$$\begin{aligned} c_{CUO} + c_{CUI} &= e^{-rT}\mathbb{E}[(S_T - K)^+ 1_{(M(S_T)<H)}] + e^{-rT}\mathbb{E}[(S_T - K)^+ 1_{(M(S_T)\geq H)}] \\ &= e^{-rT}\mathbb{E}[(S_T - K)^+ (1_{(M(S_T)<H)} + 1_{(M(S_T)\geq H)})] \\ &= e^{-rT}\mathbb{E}[(S_T - K)^+] \end{aligned} \tag{5.4}$$

Thus, by the usual no arbitrage principle, the sum of two identical up-and-in and up-and-out options must be equal to the price of an European call option with same strike price and maturity.

Something that must be noted is that there is the possibility to obtain closed forms for the price of the barrier options using different method, but as we said already, in this dissertation we will see how the pricing works using Monte Carlo methods. In this study I considered a discretely monitored *up-and-out call option*, but the reasoning used to reach the price is the same for whatever option we would choose.

5.2 Methodology

Monte Carlo methods are a widespread practice in finance. Also called Monte Carlo experiments, they are a broad class of computational algorithms that rely on repeated random sampling to obtain numerical results. Starting from the late '90s they have been started to be used in order to price options. The idea is to generate a large numbers of realizations of a stochastic process, then calculate the payoff value for each simulation, and taking the average. They rely on the risk-neutral valuation of the payoff, since we said is the only way to properly price. To make use of such methodology we must introduce a well-known concept on which the legitimacy of this practice is based on.

Theorem 5.1. ***Kolmogorov's strong law***
Let X_n be a collection of i.i.d. random variables with $\mathbb{E}[|X_n|] < \infty$ for every n. Then, the sample average converges almost surely to the expected value. More formally,

$$\bar{X}_n \xrightarrow{a.s.} \mu \quad as \ n \to \infty \tag{5.5}$$

With this result in mind we can explain better what we are going to do. First we generate the stochastic process with the random quantities generated, according to their distribution, by the software. Once we have enough realizations, we can start to compute the payoff described above for each of the realization. The realizations are i.i.d random variables at any time t, so calculating the payoff and averaging them at a fixed time point will reveal the true mean, or the proper price of the barrier option. One disadvantage of Monte Carlo methods is that sometimes the process can be quite time-demanding since we must generate a large number of simulations. On the other hand, they give the possibility to obtain results for questions which might not have a closed form solution, thus the need for the analytical answer might be overlooked.

5.3 The simulations

We will analyze two scenarios with different jump size distributions. Everything else would be the same. We start by simulating the jump times which follows a Poisson distribution. In order to build such process we start by simulating n Poisson random variables with parameter λT, with n being the number of simulations we want to perform. The output of this, which we call $N_T^i \ for \ i = 1, .., n$, will be the number of total jumps spanning the lifetime of the option for each simulation. In order to simulate the exact moment of occurrence of the jumps, we exploit the fact that, conditionally on N_T, the jump times have the same distribution as N_T independent random numbers, uniformly distributed on this interval Cont and Tankov (2004). So we generate a matrix that for each simulation gives us a time series of exact jump times, for the different values of N_T^i and we sort it in ascending order. This procedure allows us to witness the occurrence of more than one jump at a given time, something we do not want to restrict. Once we have that, we can move onto the simulation of the stochastic price process itself. The two scenarios considered will be:

- Merton scenario: we are assuming that the price is driven by a jump diffusion process as in the Merton model,i.e.

$$S_t = s_0 \exp\left[\mu t + \sigma W_t + \sum_{j=1}^{N_t} U_j\right] \tag{5.6}$$

  with $U_j \sim \mathcal{N}(\mu_J, \sigma_J^2)$. So in our simulation, when one or more jump occur, the amplitude of those will be generated as normal random variables.

- Binomial scenario: we assume a price stochastic process which behave as a jump diffusion where the jump sizes follows a binomial distribution with probability p and support $\{a, b\}$, as we already saw in section 4.6.

The jump times will be the same for the two scenarios. Regarding the parameters used in the simulation, assuming a starting price is $s_0 = 120$

- Merton jump diffusion: we assumed that $\mu = 0.9, \sigma = 0.12, \lambda = 16$, and that $U_j \sim N[0.0116, 0.06]$, where, using this notation, the second parameter represent the standard deviation and not the variance.

- Binomial jump diffusion: μ, σ, λ are as above. About the binomial we have $a = 0.08, b = -0.04$ and $p = 0.43$.

Here we can see simulations of the two processes.

Figure 5.1: *Simulations of trajectories of the two jump di usion processes. Top: Merton. Bottom: Binomial*

As we can see comparing these two with 3.1 it is clear the presence of jumps, or discontinuities, that occur over the time interval considered. We now move to the pricing.

5.4 Pricing

To do the pricing we must evaluate the realizations according to the risk-neutral measure chosen. We then build a payoff matrix and see for each time t what would be the discounted payoff of the option. In order to build the payoff matrix for a out-and-up option we proceed in the following way. The payoff, given the fact that the underlying has never hit the barrier, is the same as a plain vanilla call, i.e. $(S_T - K)^+$. so we evaluate the discounted value of the payoff for each time t for each realization, given that the stock does not hit the barrier. Whenever the asset price reach the barrier, we set the payoff to 0, from that time t_H onward. The more realizations reach the barrier the slimmer becomes the payoff, since the averages are calculated on a smaller pool of values. In this simulation the time interval is one year recorded twice a day. This is purely explanatory since it could be 6 months monitored 4 times a day or whichever is the granularity preferred. Of course if we want to increase or decrease the time interval, the parameter of the Poisson process must change accordingly, since we said that the jump occurrence's time is distributed as a Poisson with parameter λT. We will perform 2000 simulations, which is more than enough to get an accurate estimate for the price of the option. All the codes written to perform this experiment will be at the end of this dissertation. Here we can see the price of an option at each time t under the two scenarios

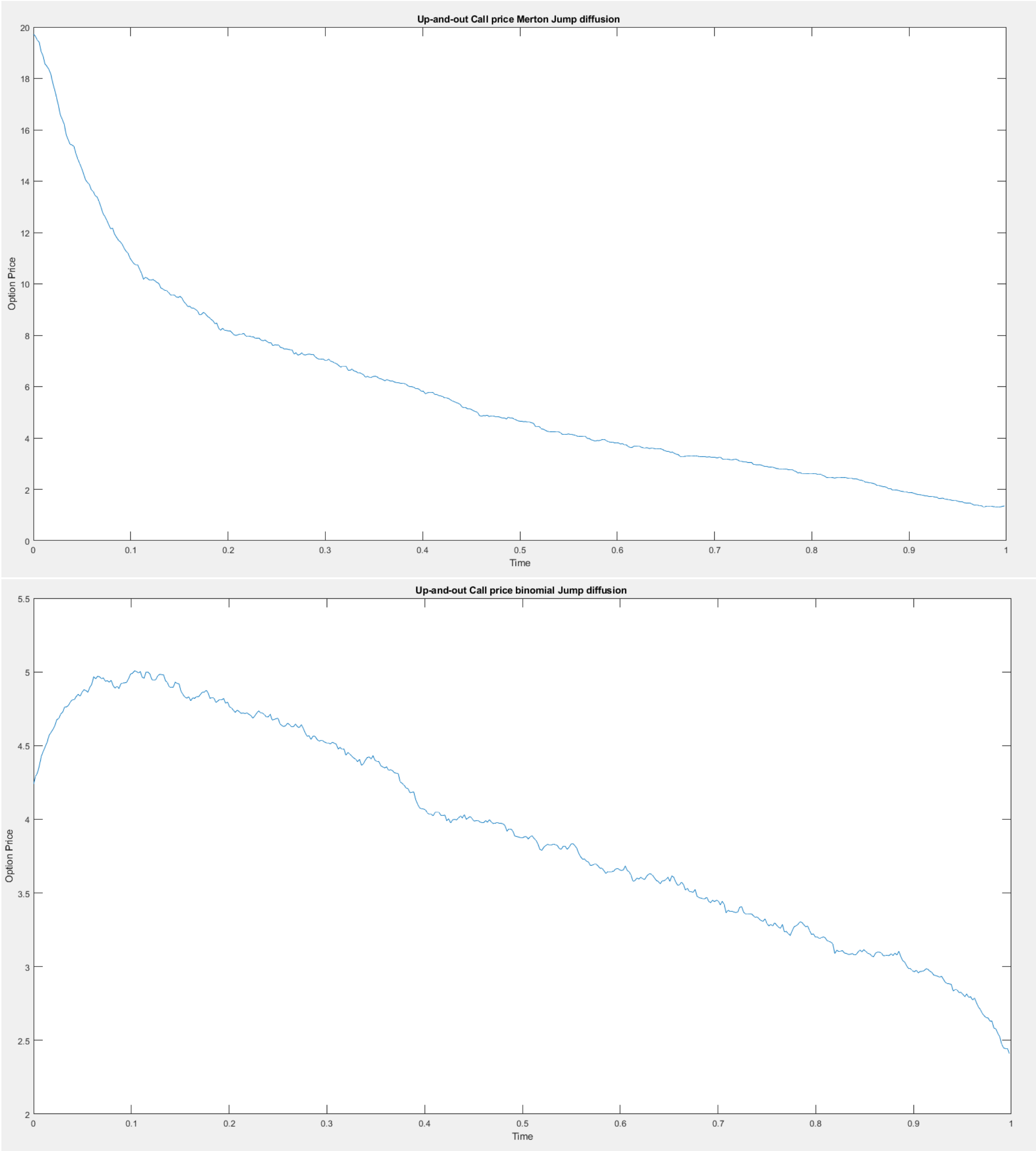

Figure 5.2: *Option price obtained through MC methods under the two scenarios. Top: Merton. Bottom: Binomial*

Chapter 6

Conclusion

We started from the idea of trying to analyze and study stochastic processes in order to model the behaviour of asset in the market. The joint action of randomness and evolution over time posed us a challenging task. We explored the instruments needed to tackle this problem and the inception of the theory, to then move gradually to shaping the random dynamics in a way that we could work with and observe the implication of the choices made. Of course at a tangible level, the goodness of our ideas are as good as the goodness of our intuitions, and, paraphrasing a great probabilist, models do work as long as they do not work anymore. We proceeded step by step, starting from the more classic process for modeling asset behaviour and then allowing for a more varied type of evolution. The jump diffusion model seem to depict a little bit better the reality of thing, rather than the classic Black-Scholes framework. Several comparative studies on the goodness of fit of the former type of processes have been done, and they allow for a more accurate representation of what is going on. A widespread example is the ability of jump diffusion models to recognize and exhibit the so called "volatility smile", something that the Black-Scholes model fails to do. Moreover, the increased volatility stemming from the jump part, translates into a bigger kurtosis, or "heavier tails", in words, it allows for more extreme events to happen.

The approach I wanted to undergo was a probabilistic one. Starting to study and exploring the power of such subject is something really fascinating. I believe that

probability is the foundation needed to dive in a world governed by uncertainty, not for the presumption to understand it but for the charm of losing oneself in it. In that fashion, I decide to not explore a predictional point of view, which would have involved estimates of fitness and reliability of the model chosen, but only to investigate the legitimacy and the derivation of the results shown. Obviously I just saw the "pin" of the iceberg, and further deepening of the subject requires much more than this. Nevertheless it was very informative and educational.
Following this thread, I really appreciated learning Monte Carlo methods, in a more and more data-filled environment, the possibility to experiment the replication of such unfathomable systems with just having the access to a notebook and to look at the realizations,"in flesh and blood", of something unknown is something that really enthused me.
About the experiment, the simulations are intended to be purely illustrative. We decided everything about the distributions involved and the parameters, something that cannot be done in real life analysis where the model must be calibrated, which is not so trivial to do. This kind of analysis might belong to the decision making process of an analysis department, in order to search for profitable situations in the market. A problem we face with this method of pricing is the numerical instability of calculation of hedging strategy, since errors compound, giving rise to unreliable responses. Nonetheless, all the results presented still hold true.
What might be difficult to model are the jump amplitudes: we might consider the discontinuities as being shocks caused by some external information or some other exogenous factors. The magnitudes of the impact caused by those events can be highly irregular and assuming certain distributions might be far off the reality most of the times. Nonetheless, I believe that the case for the inclusion is stronger than the drawbacks which may arise.
Following this thread, is quite clear that the second model has mainly an explanatory purpose: we cannot expect jumps of the same sizes all the time, even if it might happen that a test of fitness would give good result. On the other hand, one thing which was desirable, that is achieved in this study, is the possibility to allow several

jumps for each time increment and not limit ourselves to just one. Moreover it is an intuitive example which show how we can allow for different behaviors rather than the classic geometric Brownian motion just tinkering a bit the process.
Other studies with different jump size distribution are available, most notably the one by Kou and Wang (2004) which assumes a double exponential distribution for the jumps, leading to a more sophisticated but still closed form for the price of an option.

I really enjoyed myself exploring and learning this topics. I believe stochastic processes are crucial in finance, since they allow us to get closer to the state of the things and give us a way to improve our decision-making and knowledge of the market and the dynamics which govern it, and that this is something that must be in every finance enthusiast toolkit. However, one must not fall for the belief of confusing the "map from the actual territory": market are still an unforeseeable environment, and the goodness of models must not distract ourselves from the intangible forces which are present. Nevertheless, this analyises give us valuable insights and are food for thought if we are able to read trough it.

Appendix A

Matlab Codes

```
%Black Scholes model function

function [BSsim, dt] = BSfun(mu, sigma, x0, t0, T, nsim)
rng(5);%fix the seed for the simulation
N=252*2*T;
dt = (T-t0)/N;%discretize
NP = N + 1;
sqrtdt = sqrt(dt);
muddt = (mu - sigma^2/2)*dt;% dt part of the equation
BSsim = zeros(NP, nsim);%initialize matrix
dW = sqrtdt*randn(N, nsim); %dWt part of the equation
for k=1:nsim
BSsim(1,k) = x0; %set value at time 0
    for i = 1:N
        BSsim(i+1,k) = BSsim(i,k) + muddt+sigma*dW(i,k); %
            generate the trajectories.
    end
end
```

```
%Figure of the simulation of trajectories of a Geometric
    Brownian motion
%Code for figure 1
mu=0.09; sigma=0.12; x0=120; t0=0; T=2; nsim=5;
rng(5);
[BSsim,dt]=BSfun(mu,sigma,x0,t0,T,nsim);
timegrid=t0:dt:T;
plot(timegrid,BSsim)
title('GBM Stock price simulation')
xlabel('Time')
ylabel('Stock prices')
```

```
%Merton jump diffusion function
function [XS,Nt,N] = JDMerton(mu,sigma,lambda,muj,sigmaj,x0,
    t0,T,nsim)
rng(5); %fix the seed for the simulation
N=252*2*T;
dt = (T-t0)/N; %discretize
sqrtdt = sqrt(dt);
muddt = (mu - sigma^2/2)*dt; % dt part of the equation
P=poissrnd(lambda*T,1,nsim); %generate nsim Poisson rv's
    with intensity lambdaT
jtime=zeros(max(P),nsim); %initialize jump time matrix
for i=1:nsim
jtime(1:end,i)=[randi(N,P(i),1);zeros(max(P)-P(i),1)];%
    generate uniformly distributed jump time over 0,T
end
Jtimesorted=sort(jtime); %sort the result
Nt=zeros(N,nsim); %initialize matrix of poisson processes
```

```
for t=1:N
    for i=1:nsim
        for k=1:max(P)

            if Jtimesorted(k,i)==t
                Nt(t,i)=sum(Jtimesorted(:,i)==Jtimesorted(k,
                    i)); %create a matrix that have for each
                    time between 0,T
                %the exact moment of the jump
            end
        end
    end
end

XS = zeros(N,nsim); %initialize matrix for simulations
DW = sqrtdt*randn(N,nsim); %dWt part of the equation
YS=zeros(N,nsim); %initialize matrix for the exponential
for k=1:nsim
XS(1,k) = x0; %set value of processes at time 0
    for i = 1:N-1
        YS(i+1,k) = YS(i,k) + muddt+sigma*DW(i,k);
        if Nt(i,k) >0
                YS(i+1,k) = YS(i+1,k)+(Nt(i,k)*sum(normrnd(
                    muj,sigmaj,Nt(i,k),1))); %if there is one
                    or more jumps
                % generate the normal random variable(s) and
                    add it(them) to
                % the exponential
        end
        XS(i+1,k) = x0*exp(YS(i+1,k)); %put things together
```

```
    end
end
```

```
%Binomial jump diffusion function
function [XS,Nt,N] = JDbinomial(mu,sigma,lambda,a,b,p,x0,t0,
   T,nsim)
rng(5);
N=252*2*T;
dt = (T-t0)/N;
sqrtdt = sqrt(dt);
muddt = (mu - sigma^2/2)*dt;% dt part of the equation
P=poissrnd(lambda*T,1,nsim);%generate nsim Poisson rv's with
    intensity lambdaT
jtime=zeros(max(P),nsim);%initialize jump time matrix
for i=1:nsim
jtime(1:end,i)=[randi(N,P(i),1);zeros(max(P)-P(i),1)];%
   generate uniformly distributed jump time over 0,T
end
Jtimesorted=sort(jtime);
Nt=zeros(N,nsim); %initialize matrix of Poisson processes
for i=1:nsim
    for t=1:N

        for k=1:max(P)

            if Jtimesorted(k,i)==t
                Nt(t,i)=sum(Jtimesorted(:,i)==Jtimesorted(k,
                   i)); %create a matrix that have for each
                   time between 0,T
```

```
                %the exact moment of the jump
            end
        end
    end
end

XS = zeros(N,nsim); %initialize matrix for simulations
DW = sqrtdt*randn(N,nsim); %dWt part of the equation
YS=zeros(N,nsim);%initialize matrix for the exponential
for k=1:nsim
    XS(1,k) = x0;%set value of processes at time 0
    for i = 1:N-1
        YS(i+1,k) = YS(i,k) + muddt+sigma*DW(i,k);
        if Nt(i,k) >0
            YS(i+1,k) = YS(i+1,k)*log(1+(Nt(i,k)*b+(a-b)*
               binornd(Nt(i,k),p)));%if there is one or
               more jumps
            % generate the binomial random variable(s)
               and add it(them) to
            % the exponential
        end
        XS(i+1,k) = x0*exp(YS(i+1,k));%put things together
    end
end
```

```
%Code for figure 2.1 - Simulation of Merton jump diffusion
mu=0.09; sigma=0.12; lambda=16; muj=0.0116; sigmaj=0.06; x0
   =120; t0=0; T=1; nsim=5;
```

```
rng(5);%Define parameters we chose muj as the mean of the Uj
    of the binomial scenario
[LNSim,AH]=JDMerton(mu,sigma,lambda,muj,sigmaj,x0,t0,T,nsim)
    ;%generate realizations
N=252*2*T;%define N for time grid
dt=(T-t0)/N;
timegrid=t0:dt:T;
plot(timegrid,LNSim)
title('Jump diffusion Merton stock price simulation')
xlabel('Time')
ylabel('Stock prices')
```

```
%Code for figure 2.2 - Simulation of binomial jump diffusion
mu=0.09; sigma=0.12; lambda=16; a=0.08; b=-0.04; p=0.43; x0
    =120; t0=0; T=1; nsim=5;
rng(5);%Define parameters, same as the other simulations
[BinSim,AH]=JDbinomial(mu,sigma,lambda,a,b,p,x0,t0,T,nsim);%
    generate realizations
N=252*2*T;%define N for time grid
dt=(T-t0)/N;
timegrid=t0:dt:T;
plot(timegrid,BinSim)
title('Jump diffusion binomial stock price simulation')
xlabel('Time')
ylabel('Stock prices')
```

```
%Code for pricing of Merton jump diffusion
```

```
mu=0.09; sigma=0.12; lambda=16; muj=0.0116; sigmaj=0.06; x0
    =120; t0=0; T=1; nsim=2000;
rng(5)
r=0.02; strike=115; barrier= 125; %set the parameters for
    the simulations
mutilda=r-lambda*muj; %prepare the change to the risk-
    neutral measure
[Xsim,Nt,N]=JDMerton(mu,sigma,lambda,muj,sigmaj,x0,t0,T,nsim
    ); %generate the realizations
XsimRN=JDMerton(mutilda,sigma,lambda,muj,sigmaj,x0,t0,T,nsim
    ); %generate the realizations under the RN measure
dt=(T-t0)/N;
tv=[t0:dt:T].';%allocate vector for discounting
payoffsRN=zeros(N,nsim);%initiate payoff matrix
for k=1:nsim
    for t=1:N
        if XsimRN(t,k)<barrier %If the price is below the
           barrier compute the discounted payoff
           payoffsRN(t,k)=exp(-r*(T-tv(t)))*max((x0*XsimRN(
              end,k))/XsimRN(t,k) - strike, 0);
        elseif k<nsim %if the price has hit the barrier go
           to the next iteration
              k=k+1;
        elseif k==nsim
             break %break the simulation at the last
              iteration
        end
    end
end
CpRN=zeros(N,1);% initiate option price matrix
```

```
for t=1:N
    CpRN(t,1)=mean(payoffsRN(t,:)); %calculate averages over
        realizations for each time t
end

%Code for figure 3.1 - Barrier Option price Merton
timegrid=t0:dt:T-dt;
plot(timegrid,CpRN)
title('Up-and-out Call price Merton Jump diffusion ')
xlabel('Time')
ylabel('Option Price')
```

```
%Code for pricing of binomial jump diffusion
mu=0.09; sigma=0.12; lambda=16; a=0.08; b=-0.04; p=0.43; x0
   =120; t0=0; T=1; nsim=2000;
rng(5)
umean = a*p+b*(1-p);
r=0.02; strike=115; barrier= 125;%set the parameters for the
    simulations
mutilda=r-lambda*umean;%prepare the change to the risk-
   neutral measure
[Xsim,Nt,N]=JDbinomial(mu,sigma,lambda,a,b,p,x0,t0,T,nsim);%
   generate the realizations
XsimRN=JDbinomial(mutilda,sigma,lambda,a,b,p,x0,t0,T,nsim);%
   generate the realizations under the RN measure
dt=(T-t0)/N;
tv=[t0:dt:T].';%allocate vector for discounting
payoffsRN=zeros(N,nsim);%initiate payoff matrix
for k=1:nsim
```

```
    for t=1:N
        if XsimRN(t,k)<barrier %If the price is below the
            barrier compute the discounted payoff
            payoffsRN(t,k)=exp(-r*(T-tv(t)))*max((x0*XsimRN(
                end,k))/XsimRN(t,k) - strike, 0);
        elseif k<nsim %if the price has hit the barrier go
            to the next iteration
                k=k+1;
        elseif k==nsim
             break %break the simulation at the last
                iteration
        end
    end
end
CpRN=zeros(N,1);% initiate option price matrix
for t=1:N
    CpRN(t,1)=mean(payoffsRN(t,:));%calculate averages over
        realizations for each time t
end

%Code for figure 3.1 - Barrier option price binomial
timegrid=t0:dt:T-dt;
plot(timegrid,CpRN)
title('Up-and-out Call price binomial Jump diffusion ')
xlabel('Time')
ylabel('Option Price')
```

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