**Department of Economics and Finance**

**Chair of Gambling: Probability and Decision**


# A Stochastic Analysis of The Binomial Asset Pricing Model


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*To my family and friends*

# Contents

# Introduction

The Binomial Option Pricing Model was first proposed by William Sharpe in 1978 [11] and then formalized in its most notable version by Cox, Ross and Rubenstein in 1979 [2]. It provides a simple framework to model stock price dynamics and to fairly price options and derivatives. Historically, this was not the first attempt to option valuation, but it offers indeed the simplest framework to do so. The first model that triggered the development of the binomial approach to option pricing was, in fact, the diffusion model underlying the famous Black-Scholes formula, which was published in 1973 [1]. Black and Scholes developed this formula on the principle that if options are correctly priced in the market, then it should not be possible to make profits by creating portfolios of long and short positions in options and their underlying stocks. The pricing model developed by Black and Sholes was undoubtedly seen as an innovation for the theory of finance, but it also caused a shock amongst the economists at the time of its introduction. This was due to the model involving a mathematical background which seemed to have appeared as too academic for the time. In fact, although the language of finance now widely relies on mathematics and stochastic calculus, management of risk in a quantifiable manner being the basis of modern theory and quantitative finance, not much time had passed since Markowitz's 1952 *Portfolio Selection* [12] had laid the groundwork for the mathematical theory of finance. This motivated various economists to search for simpler modeling frameworks that still preserved the economically relevant properties presented by the Black-Scholes formula. Among those models, we find the Cox, Ross and Rubenstein Binomial Asset Pricing Model. The main economic idea that this model preserves from Black-Scholes is that if an economy incorporating three securities can only attain two future states, one such security will be redundant; that is, each single security can be replicated by the other two, a fact later referred to as market completeness. One of the semplifications of the

Binomial Model is that it uses a discrete-time binomial tree framework to model the dynamics of the underlying stock price. Under special conditions, the binomial tree converges weakly to the Black-Scholes model in continuous time limit. The Arbitrage Pricing Theory [7] lies at the basis of both the Binomial Model and the Black-Scholes formula. It was developed as an alternative to the mean-variance Capital Asset Pricing Model [4] and it led to the proof of the existence of a risk-neutral measure, which is the other fundamental concept that underlies the Binomial Option Pricing Model. Indeed, the concept of risk-neutral probabilities underlies the whole area of asset pricing.

In general, the value of a derivative security is established by estimating the probability that its price overcomes a pre-determined level (for example, its initial price) at a future date. Most pricing models aim at calculating the *fair price* of a derivative security contract at the time of its stipulation by calculating the discounted expected value of its future pay-offs. The fair price of a derivative security is defined as its theoretical value obtained through the procedure of "subordinate pricing", that is, through the application of the no-arbitrage principle among the derivative security and its replicating portfolio. This theoretical price, however, is obtained by making three stringent assumptions concerning financial markets, which are assumed to be

1. competitive, meaning that agents in the market are assumed to be price-takers which maximize their utility and all are assumed to have the same information;

2. frictionless, that is, financial markets are assumed to be continuous, absent of transaction costs, with no limit to short-selling and without any presence of insolvency risk;

3. arbitrage free, that is, financial markets do not entail the possibility to make transactions which involve no negative cash flow at any probabilistic or temporal state and a positive cash flow in at least one state. In simple terms, it is impossible to make riskless profits.

The fundamental idea that underlies most derivative pricing models is that, in order to calculate the fair price of a derivative security, it is necessary to estimate the distribution of the future prices of the derivative, which are themselves a function of the prices of the underlying asset. By consequence, it is also necessary to estimate a law according to which the former vary. The Binomial Asset Pricing Model simply assumes them to increase or decrease by some $u$ and $d$ factors, respectively. Of course, there are also models which formulate this law using more advanced mathematical processes but, as we previously mentioned, the Binomial Model only focuses on mantaining the economic properties of such models, without engaging in complex mathematical procedures. However, it is important to highlight that all those models, including the Binomial one, calculate the price of the derivative security in a risk-neutral world, that is, a world in which the no-arbitrage assumption holds and the derivative security can be replicated by combining long and short positions in the underlying asset and in a risk-free one. This assumption justifies the usage of risk-neutral probabilities to price derivative securities. These risk-neutral probabilities are different from the real world ones and in no way should they be intepreted as probabilities which can be used to predict future prices. Indeed, risk-neutral probabilities are a pricing tool which only derives from the estimate of future possible values of the derivative security which result to be compatible with current market data. In other words, the information contained in the probabilistic scenarios shown by these models only constitute *ex-ante* projections of future prices of the derivative security which are coherent with market conditions and with the derivative security structure.

In order to construct probabilistic scenarios able to provide reliable information, one should use real world probabilities. However, doing so would require making assumptions about the risk premium that investors inevitably require to hold risky assets. This risk premium varies across investors, as it is based on personal preferences and risk-attitude. This means that every conclusion stemming from probabilistic scenarios which employ real world probabilities would be deemed as arbitrary and heavily dependent on hypotheses about investors' risk-aversion. Eventually, despite the stringent and nearly irrealistic assumptions on which the risk-neutral pricing method relies, it is still conceived as the most efficient one to price derivatives and it is still useful today, mostly when it comes to activities which have a risk-management scope.

This thesis presents the Binomial Asset Pricing Model and studies its properties through probabilistic instruments. Chapter 1, *The No-Arbitrage Binomial Asset Pricing Model*, introduces the concept of derivative securities and stock options to then present the no arbitrage method of option pricing in a binomial model. It mainly focuses on pricing European options, that is, path independent derivatives which can only be exercised at maturity. Chapter 2, *Martingales*, and Chapter 3, *Markov Processes*, provide definitions of two processes that lie at the basis of probability theory and that are used in the Binomial model to derive the fair price of a derivative security. Chapter 4, *General American Derivatives*, is made of four different sections. It begins with the application of the Pricing Algorithm developed in the previous chapters to path independent American derivatives, which are characterized by the feature of early exercise. In the last section, we develop a method to understand which is the optimal time to exercise a general American derivative. Before doing so, however, we need to introduce the concept of stopping times, which will be of fundamental importance to understand the algorithm developed in the last section. Finally, Chapter 5, *MATLAB codes*, contains two algorithms which can be used, respectively, to derive the price of both a European Call-Put Option and a Path-Independent American Put Option.

# Chapter 1

# The No-Arbitrage Binomial Asset Pricing Model

## 1.1 Stock Options

**Stock options** are contracts which give their holder the right, but not the obligation, to buy or sell a stock at a pre-determined price before a specified expiration date. Any option is a derivative security, that is a product whose value depends on the price movements of one or more underlying assets which can be stocks, bonds, commodities, currencies, indices or interest rates.
Options are traded both on exchanges and on over-the-counter markets for different reasons. They can be used, for example, as hedging instruments against their underlying asset. Indeed, one could set up a strategy such that a loss in one investment can be offset by a gain in a comparable derivative. Options can also be used to generate income, to speculate on the future price movements of an asset and, when given to traders as part of their salary, as an incentive to maximize the value of a particular underlying stock.

A **call option** gives its holder the right, but not the obligation, to buy a share of the underlying stock at a pre-determined price (the *strike price*) before the contract expires (*expiration date* or *maturity*), while a **put option** gives the holder the right to sell a share of the underlying stock at the strike price, still before the expiration date of the contract.

As for every contract, there are two sides to every option. The investor who buys the option is the one who takes a *long position* while, on the other side, the investor who takes a *short position* is the one who either sells or writes the option. This means that there are four types of option positions:

1. A long position in a call option

2. A long position in a put option

3. A short position in a call option

4. A short position in a put option

For convenience, it is useful to refer to the option payoff as to the one received by the purchaser of the option in case of its exercise.
Define $S_i$ as the stock price at time $i$ and $K$ as the strike price. The payoff of a call option will be given by $\max\{0; S_i - K\}$, while the payoff of a put option will be given by $\max\{0; K - S_i\}$. This means that an investor will purchase a call option if he believes that the price of the underlying stock is going to rise while, on the other hand, he will purchase a put option if he believes it is going to fall.
Options can be either European or American. The main difference between the two is that a **European option** can only be exercised on the expiration date of the contract, while **American Options** can be exercised at any time up to the expiration date.

The value of an option depends on different variables such as the current stock price, the volatility of the stock, the strike price, the time to expiration and the risk-free interest rate at which investors can borrow or lend. According to the **Option Pricing Theory**, the primary goal of option pricing is to calculate the probability that an option will be exercised, or be in-the-money, at expiration and assign a monetary value to it. Variables such as the price of the stock and time to expiration are typically employed to establish a fair value of the option through the use of probabilistic models. Indeed, the option being European or American is also taken into account to establish its price.

## 1.2 The Multiperiod Binomial Model

The Binomial Asset-Pricing Model is a risk-neutral method for valuing path-dependent options like, for example, American options. By path dependent, we mean that the value of the option depends on the previous movements in the underlying stock price. Such a model is a popular tool for stock options pricing and it is used by investors to evaluate the right to buy or sell at specific prices over time. The Multiperiod Binomial Model can be seen also as a basis to understand the Black-Scholes Model, one of the most famous instruments used for option pricing.

The Binomial Asset-Pricing model can be represented through a diagram which illustrates the path followed by the underlying stock price. We define the beginning of the period as *time zero* and we assume that the stock will rise or fall in the subsequent periods according, respectively, to some **up factor** $u$ and to some **down factor** $d$. We also assume that in the market there is a money market asset with a constant risk-free interest rate $r$, which is the same for borrowing and lending. This means that 1\$ invested in this asset will yield $(1+r)^n$ \$ in period $n$ and that, for 1\$ borrowed, $(1+r)^n$ \$ must be returned.

There is one fundamental assumption that must be made on these parameters, the so called **no-arbitrage condition**. In general, an **arbitrage opportunity** arises in a market if there is the possibility, at zero cost, to make a profit with positive probability without incurring in any risk of a loss. A mathematical model which admits arbitrage cannot be used for analysis as it would not be realistic. In fact, when markets are efficient, under the assumption that there is at least one intelligent investor, any arbitrage opportunity is soon discovered and eliminated through trading. If some asset is over-valued due to some form of mispricing, demand for that asset will decrease until its price decreases to reflect the fair value of the asset. If the asset is undervalued, the demand will increase and the outcome will be the same.

In this context the no-arbitrage condition is given by

$$u < 1 + r < d\,. \tag{1.2.1}$$

In fact, in any other case, there would be an arbitrage opportunity in the market. To show it, assume $1 + r < d < u$. This implies that in both possible states of the world, the return offered by the stock would be greater than the one offered by the money market asset. In this case, an investor could set up a portfolio in which he shorts the money market asset to buy a position in the stock. After one period, his portfolio would have positive value, regardless of the state of the world, without having to incur in any cost at time zero. A similar strategy can be employed in the case in which $d < u < 1 + r$ which would still result in the making of riskless profit. Thus, the only way to exclude arbitrage in the model is to construct it so that equation (1.2.1) holds. It is important to highlight that the inequalities are strict. Otherwise, there would still be arbitrage opportunities. Indeed, at the end of the period the portfolio would end up having zero value in one state of the world and positive value in the other.

Since there are only two possible states of the world, the stock price at any time can be seen as dependent on the outcome of a coin toss. The outcome will not be known in advance and, for this reason, it is regarded as *random*, as it is the result of a random experiment.
We denote by $S_0$ the stock price at time zero. At any time $i > 0$, we denote by $S_i(\omega_1, \ldots, \omega_i)$ the stock price at time $i$, where $\omega_j \in \{H, T\}$ for $j = 1, \ldots, i$ with $H$ and $T$ standing for 'head' and 'tail', respectively.
Denote by $p \in (0, 1)$ the probability of head and by $q = 1 - p \in (0, 1)$ the probability of tail. The model in case of two periods is illustrated in Figure 1.1 through a binomial tree, which is a graphical representation of the possible values that the stock may take at different nodes, which represent different time periods.

The fundamental aim of option pricing is to find how much the option is worth at time zero before knowing whether the coin toss results in a head or a tail. In general, for any period $n$, $2^n$ outcomes are possible, even if not all of them result in different stock prices. At time 3, for example, the stock price would be one of the eight following outcomes:

$$S_3(HHH) = u^3 S_0 \,,$$

$$S_3(HHT) = S_3(HTH) = S_3(THH) = u^2 d S_0 \,,$$

$$S_3(HTT) = S_3(THT) = S_3(TTH) = u d^2 S_0 \,,$$

$$S_3(TTT) = d^3 S_0 \,.$$

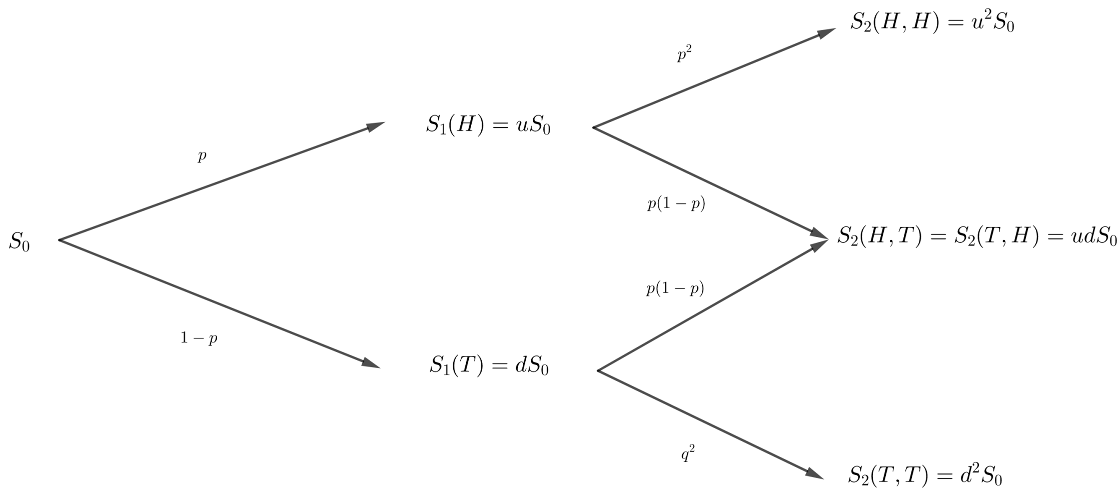


**Fig. 1.1:** General Binomial Model for two periods.

One of the main approaches to option pricing is the **Arbitrage Pricing Theory**, which consists in replicating the option by trading in the stock and money markets. This approach is actually based on the **Law of One Price**: if two investments yield the same return, then they must be priced the same, otherwise arbitrage opportunities would exist and the market would not be efficient.

As previously highlighted, an option is a particular type of a **derivative security**, which is a security that pays some amount $V_i(\omega_1, \ldots, \omega_i)$ at time $i$ if the first $i$ coin tosses are $\omega_1, \ldots, \omega_i$ with $\omega_j \in \{H, T\}$ for $j = 1, \ldots, i$. For example, the payoff of a European Call Option which expires at time $t = 1$ is given by $V_1 = (S_1 - K)^+$ and its result depends on the evolution of the price of the underlying stock.

The replicating portfolio technique is a procedure used to determine the price $V_0$ at time zero of a derivative security in order to replicate the derivative security at future periods by trading in the stock and the money market. We will now show how this technique works.

Let us use the replicating portfolio technique to determine the price $V_0$ at time zero of a derivative security.

Suppose that an investor begins with wealth $X_0$ and buys $\Delta_0$ shares of stock at time zero, leaving him with a position of $X_0 - \Delta_0$.
The value of his portfolio of stock and money market account at time one will be

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) = (1+r)X_0 + \Delta_0(S_1 - (1+r)S_0)\,. \quad (1.2.2)$$

In fact, at time one, the portion $\Delta_0$ of stock bought will be worth $S_1$ while the remaining wealth invested in the money market will yield $(1+r)(X_0 - \Delta_0 S_0)$.
A portfolio which perfectly agrees with the option value at time one for every possible state of the world has to satisfy $X_1(H) = V_1(H)$ and $X_1(T) = V_1(T)$. So $X_0$ and $\Delta_0$ must be chosen accordingly to these conditions. This yields to

$$X_0 + \Delta_0 \left( \frac{1}{1+r} S_1 - S_0 \right) = \frac{1}{1+r} V_1\,, \quad (1.2.3)$$

which is equivalent to (1.2.2) discounted back to time zero. This equation is actually a system of two different equations since $V_1$ and $S_1$ depend on the first coin toss and thus differ according to the two possible states of the world, that is

$$\begin{cases} X_0 + \Delta_0 \left( \frac{1}{1+r} S_1(H) - S_0 \right) = \frac{1}{1+r} V_1(H)\,, \\ \\ X_0 + \Delta_0 \left( \frac{1}{1+r} S_1(T) - S_0 \right) = \frac{1}{1+r} V_1(T)\,. \end{cases}$$

It can be proved that the solution of this system is

$$X_0 = \frac{1}{1+r} \left[ \tilde{p} S_1(H) + \tilde{q} S_1(T) \right]\,, \quad (1.2.4)$$

where

$$\tilde{p} = \frac{1+r-d}{u-d}\,, \qquad \tilde{q} = \frac{u-1-r}{u-d} = 1 - \tilde{p}\,. \quad (1.2.5)$$

The values $\tilde{p}$ and $\tilde{q}$ are called the **risk neutral probabilities** (of $H$ and $T$, respectively). Hence the previous equation can be rewritten as $X_0 = \frac{1}{1+r} \tilde{\mathbb{E}}[S_1]$, that is the expected value of $S_1$ under the risk-neutral probabilities discounted back to time zero.

If the system is solved for $\Delta_0$, the result is the **delta hedging formula**

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}\,, \quad (1.2.6)$$

that is, the portion of shares to buy at time zero to perfectly replicate (i.e. hedge) the option at time one.

In conclusion, if an agent begins with wealth given by (1.2.4) and at time zero buys $\Delta_0$ shares of stock, given by (1.2.6), then at time one, if the coin results in head, the agent portfolio will be worth $V_1(H)$, while if the coin results in tail, it will be worth $V_1(T)$. In this way, **the agent has hedged a position in the derivative security**, meaning that his portfolio at time one has the same value of the option. Consequently, according to the Law of One Price, it will also have the same price at any time, so that no arbitrage is introduced when the derivative security is added to the market comprising the stock and money market account. Indeed, the derivative security that pays $V_1$ at time one should be priced at

$$V_0 = \frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)] = \frac{1}{1+r}\tilde{\mathbb{E}}[S_1]\,. \tag{1.2.7}$$

As described before, $\tilde{p}$ and $\tilde{q}$ given by (1.2.5) are the risk neutral probabilities but they do not coincide with the **actual probabilities**, $p$ and $q$ introduced at the beginning. Under the risk neutral probabilities, the average growth rate of the stock is equal to the rate of an investment in the money market, meaning that any investor would be indifferent bewteen investing in the stock market or in the money market. However, this is typically not the case since, under the actual probabilities, the average rate of growth of the stock is strictly greater than the rate of an investment in the money market (that is, $r$). Specifically, while under $\tilde{p}$ and $\tilde{q}$

$$S_0 = \frac{1}{1+r}\tilde{\mathbb{E}}[S_1] = \frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)]\,, \tag{1.2.8}$$

under $p$ and $q$ we have

$$S_0 < \frac{1}{1+r}\mathbb{E}[S_1] = \frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)]\,. \tag{1.2.9}$$

If $\tilde{p}$ and $\tilde{q}$ were equal to the actual probabilities, investors would be neutral about risk, meaning that they would not require any compensation for assuming it, nor would they be willing to pay an extra for it. Since this is not the case, $\tilde{p}$ and $\tilde{q}$ cannot be the actual probabilities. The risk-neutral probabilities are values which naturally stem from solving the system of equations implied by (1.2.3) and can be regarded as a pricing tool needed to construct a portfolio whose value at time one is $V_1$. Indeed by choosing $X_0$ as in (1.2.4) and $\Delta_0$ as in (1.2.6), we get the identity (1.2.7) for the value of the portfolio $V_1$ also known as the **risk-neutral pricing formula**. A direct consequence of this formula is that the mean rate of growth of the value of the portfolio is the rate of growth of the money-market investment. This can only be true under the risk-neutral probabilities. Note also that, since $S_1(H) = uS_0$, $S_1(T) = dS_0$ and $\tilde{q} = 1 - \tilde{p}$, there exists a unique value $\tilde{p}$ (and hence a unique $\tilde{q}$) for which equation (1.2.8) holds. This tells us that the risk-neutral probabilities in the Multiperiod Binomial Model are unique.

The process of replication, which has now been applied only to one period, can be generalized to multiple ones. Let us use the same replication process as before to find the no-arbitrage price of this portfolio. Suppose an investor starts by selling the option at time zero for $V_0$ dollars, which will be his beginning wealth used to set up the hedging portfolio. He wants to buy $\Delta_0$ shares of stock at time zero. To finance it, he borrows $\Delta_0 S_0 - V_0$ from the money market.
At time one, the value of his portfolio of stock and money market account will be

$$X_1 = \Delta_0 S_1 + (1+r)(V_0 - \Delta_0 S_0)\,. \tag{1.2.10}$$

As with equation (1.2.3), since $X_1$ depends on the first coin toss, this equation actually implies

$$\begin{cases} X_1(H) = \Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0)\,, \\ X_1(T) = \Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0)\,. \end{cases} \tag{1.2.11}$$

At time one, the first coin toss has already happened and the investor knows its result. Suppose that, with this information, he decides to readjust his hedge by holding $\Delta_1$ shares of stock. He then invests the remainder in the money market. At time two, his portfolio will be valued

$$X_2 = \Delta_1 S_2 + (1+r)(X_1 - \Delta_1 S_1) \tag{1.2.12}$$

which will have to be equal to $V_2$ to hedge the option. This equation now implies a system of four equations since $V_2$ and $S_2$ depend on the first two toin tosses which can result in $2^2 = 4$ different outcomes. Indeed we have

$$V_2(H,H) = \Delta_1(H) S_2(H,H) + (1+r)(X_1(H) - \Delta_1(H) S_1(H)) \tag{1.2.13}$$

$$V_2(H,T) = \Delta_1(H) S_2(H,T) + (1+r)(X_1(H) - \Delta_1(H) S_1(H)) \tag{1.2.14}$$

$$V_2(T,H) = \Delta_1(T)S_2(T,H) + (1+r)(X_1(T) - \Delta_1(T)S_1(T)) \tag{1.2.15}$$

$$V_2(T,T) = \Delta_1(T)S_2(T,T) + (1+r)(X_1(T) - \Delta_1(T)S_1(T)) \tag{1.2.16}$$

Solving this four equations for $\Delta_1(T)$ or for $\Delta_1(H)$ with the two implied by equation (1.2.10) will give rise to two different *delta-hedging formulas* which indicate the portion of shares to buy at time one. The two equations correspond to the two possible outcomes of the first coin toss.

$$\Delta_1(T) = \frac{V_1(T,H) - V_1(T,T)}{S_1(T,H) - S_1(T,T)}, \tag{1.2.17}$$

$$\Delta_1(H) = \frac{V_1(H,H) - V_1(H,T)}{S_1(H,H) - S_1(H,T)}, \tag{1.2.18}$$

Substituting (1.2.17) into either (1.2.13) or (1.2.14) and (1.2.18) into either (1.2.15) or (1.2.16) will result in equations

$$X_1(H) = \frac{1}{1+r}[\tilde{p}V_2(H,H) + \tilde{q}V_2(H,T)] = \frac{1}{1+r}\tilde{\mathbb{E}}[V_2 \,|\, \omega_1 = H] \tag{1.2.19}$$

and

$$X_1(T) = \frac{1}{1+r}[\tilde{p}V_2(T,H) + \tilde{q}V_2(T,T)] = \frac{1}{1+r}\tilde{\mathbb{E}}[V_2 \,|\, \omega_1 = T]\,. \tag{1.2.20}$$

These two equations give, respectively, the value that the replicating portfolio should have if the stock goes up or down between time zero and time one. Since these values have to be equal to the price of the option at time one, $X_1(H)$, given in equation (1.2.19) has to be the *price of the option at time one if the first coin results in head.* The same applies for $X_1(T)$ in equation (1.2.20) that is the *price of the option at time one if the first coin results in tail.*
If we plug the values $X_1(H) = V_1(H)$ and $X_1(T) = V_1(T)$ into the two equations in (1.2.11) and then solve for $\Delta_0$ and $V_0$, the resulting equation will be the same as (1.2.6) and (1.2.7).

To summarize, if an agent begins with any initial wealth $X_0$ and specifies values for $\Delta_0$, $\Delta_1(H)$ and $\Delta_1(T)$, then he can compute the value of the portfolio that holds such number of shares of stock and, if necessary, finance these by borrowing or investing in the money market.

The value of this portfolio is defined recursively, beginning with $X_0$, through the **wealth equation**

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n), \tag{1.2.21}$$

This equation defines *random variables* whose values are not known until the outcomes of the coin tossing are revealed. However, already at time zero, equation (1.2.21) allows to compute the value of the portfolio at every subsequent time under every coin-toss scenario.
The process for a multiperiod binomial model is given in the following theorem.

**Theorem 1.2.1.** ***(Replication in the multiperiod binomial model)*** *Consider an $N$-period binomial asset pricing model with $0 < d < 1+r < u$ and with*

$$\tilde{p} = \frac{1+r-d}{u-d}, \quad \tilde{q} = 1 - \tilde{p} = \frac{u-1-r}{u-d}\,. \tag{1.2.22}$$

*Let $V_N$ be a random variable (a derivative security paying off at time $N$) depending on the first $N$ coin tosses $\omega_1, \omega_2 \ldots, \omega_N$. Define recursively backward in time the sequence of random variables $V_{N-1}, V_{N-2} \ldots, V_0$ by*

$$V_n(\omega_1, \omega_2, \ldots, \omega_n) = \frac{1}{1+r}[\tilde{p}V_{n+1}(\omega_1, \omega_2, \ldots, \omega_n, H) + \tilde{q}V_{n+1}(\omega_1, \omega_2, \ldots, \omega_n, T)]\,, \tag{1.2.23}$$

*so that $V_n$ depends on the first $n$ coin tosses $\omega_1, \omega_2, \ldots, \omega_n$ where $n$ ranges between $N-1$ and $0$. Next define*

$$\Delta_n(\omega_1, \omega_2, \ldots, \omega_n) = \frac{V_{n+1}(\omega_1, \omega_2, \ldots, \omega_n, H) - V_{n+1}(\omega_1, \omega_2, \ldots, \omega_n, T)}{S_{n+1}(\omega_1, \omega_2, \ldots, \omega_n, H) - S_{n+1}(\omega_1, \omega_2, \ldots, \omega_n, T)}\,, \tag{1.2.24}$$

*where again $n$ ranges between 0 and $N-1$. Set $X_0 = V_0$ and define recursively forward in time the portfolio values $X_1, X_2, \ldots, X_N$ by (1.2.21). Then*

$$X_N(\omega_1, \omega_2, \ldots, \omega_N) = V_N(\omega_1, \omega_2, \ldots, \omega_N) \;\; \forall\, \omega_1, \omega_2, \ldots, \omega_N\,. \tag{1.2.25}$$

**Definition 1.2.1.** *For $n = 1, 2, \ldots, N$, the random variable $V_n(\omega_1, \ldots, \omega_n)$ in Theorem 1.1 is defined as the price of the derivative security at time $n$ if the outcome of the first $n$ tosses are $\omega_1, \ldots, \omega_n$. The price of the derivative security at time zero is defined to be $V_0$*

*Proof of Theorem 1.2.1.* The aim is to prove by forward induction on $n$ that

$$X_N(\omega_1, \omega_2, \dots, \omega_N) = V_N(\omega_1, \omega_2, \dots, \omega_N) \quad \forall \omega_1, \omega_2, \dots, \omega_N \,,$$

where $n$ ranges between 0 and $N$. When $n = 0$, $X_0 = V_0$ is true by definition. We have to prove it for $n = N$.
Assume that (1.2.25) holds for some value $n < N$, we want to show that it also holds for $n + 1$.
Let $\omega_1, \omega_2, \dots, \omega_n$ be fixed and assume that (1.2.25) holds for $\omega_1, \omega_2, \dots, \omega_n$. For the rest of the proof, when it is not misleading, we will omit the dependence of all the quantities on $\omega_1, \omega_2, \dots, \omega_n$. First, we consider the case where $\omega_{n+1} = H$ and we use (1.2.21) to compute $X_{n+1}(H)$. Since $S_{n+1} = uS_n$ when $\omega_{n+1} = H$, we get

$$X_{n+1}(H) = \Delta_n u S_n + (1 + r)(X_n - \Delta_n S_n) \,. \tag{1.2.26}$$

Moreover, from (1.2.24), we get

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{(u - d)S_n} \,. \tag{1.2.27}$$

Note that equation (1.2.26) can be restated as

$$X_{n+1}(H) = (1 + r)X_n + \Delta_n S_n(u - (1 + r))$$

and hence, by (1.2.27) and by the identity $X_n = V_n$, we get

$$X_{n+1}(H) = (1 + r)V_n + \frac{(V_{n+1}(H) - V_{n+1}(T))(u - (1 + r))}{u - d} \,. \tag{1.2.28}$$

Recalling that $\tilde{q} = \frac{u-1-r}{u-d}$ and by (1.2.23), (1.2.28) becomes

$$X_{n+1}(H) = \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) + \tilde{q}V_{n+1}(H) - \tilde{q}V_{n+1}(T) = V_{n+1}(H) \,, \tag{1.2.29}$$

which, for completeness, can be restated as

$$X_{n+1}(H) = V_{n+1}(H).$$

A similar argument shows that

$$X_{n+1}(T) = V_{n+1}(T) \,.$$

Consequently, regardless of whether $\omega_{n+1} = H$ or $\omega_{n+1} = T$, we have proved

$$X_{n+1}(\omega_1, \omega_2, \dots, \omega_n, \omega_{n+1}) = V_{n+1}(\omega_1, \omega_2, \dots, \omega_n, \omega_{n+1})$$

Since $\omega_1, \omega_2, \dots, \omega_n, \omega_{n+1}$ is arbitrary, the induction step is complete. $\square$

Theorem 1.2.1 applies both to options whose payoff depends on the final stock price and to path-dependent options, whose price depends on the previous price movements.

The multiperiod model of this section is said to be **complete** since every derivative security can be replicated by trading in the underlying stock and the money market. The completeness of the model stems from the First and the Second Fundamental Theorem of Asset Pricing. Indeed the **First Fundamental Theorem of Asset Pricing** ensures that in an arbitrage free market model there exists a risk neutral probability, while the **Second Fundamental Theorem of Asset Pricing** states that if a market model is arbitrage free, then it is complete if and only if the risk neutral probability $\tilde{p}$ is unique. We have already discussed the uniqueness of $\tilde{p}$. In the next chapter we will see that, if $d < 1 + r < u$, the multiperiod binomial model is a martingale model.

**Example 1.** *To practically illustrate how this process works, let us try to find the price at time zero of a European Call Option with strike price* $K = 8$*. Let* $S_0 = 10$ *with* $u = 1.2$ *and* $d = 0.8$*, so that the price of the stock will either rise or fall by 20%. Moreover, assume that* $r = 0.1$*.*

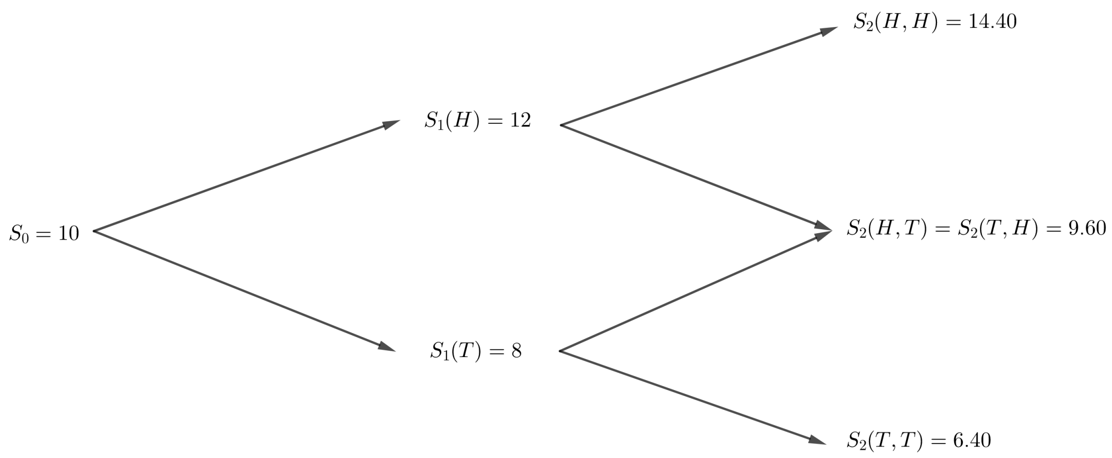


**Fig. 1.2:** Example of a two-period model.

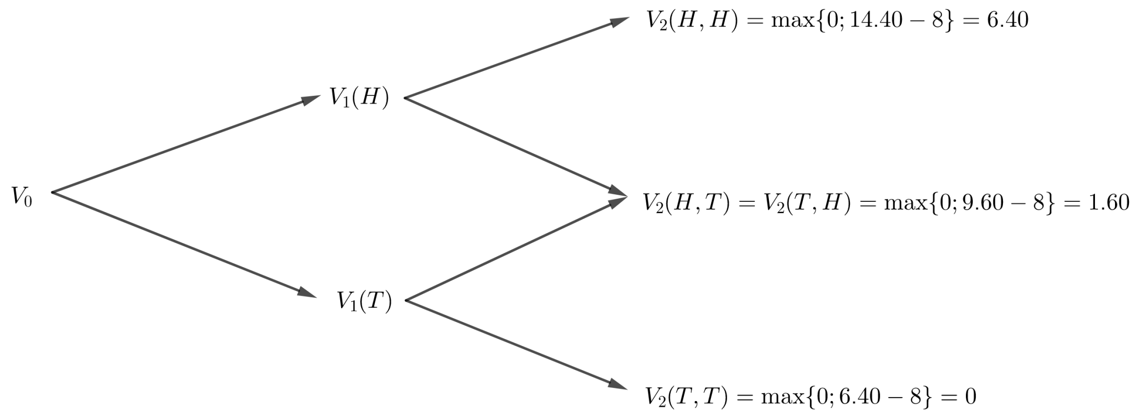


**Fig. 1.3:** Call option payoff.

*First, we find the risk-neutral probabilities* $\tilde{p}$ *and* $\tilde{q}$ *:*

$$\tilde{p} = \frac{1 + r - d}{u - d} = \frac{1.1 - 0.8}{1.2 - 0.8} = 0.75\,, \qquad \tilde{q} = (1 - \tilde{p}) = 1 - 0.75 = 0.25\,.$$

*The values that* $V_2 = \max\{0; S_2 - K\}$ *can assume are computed in Figure 1.3, while the possible values of* $S_2$ *and* $S_1$ *are computed in Figure 1.2. We calculate the option value at time one according to the risk-neutral pricing formula. Notice that, since the model is for two periods, this step will imply an equation for each state of the world*

$$V_1(T) = \frac{1}{1+r}[\tilde{p}V_2(T,H) + \tilde{q}V_2(T,T)] = \frac{1}{1.1}[0.75 \cdot 1.6 + 0.25 \cdot 0] = 1.09,$$

$$V_1(H) = \frac{1}{1+r}[\tilde{p}V_2(H,H) + \tilde{q}V_2(H,T)] = \frac{1}{1.1}[0.75 \cdot 0.64 + 0.25 \cdot 1.6] = 4.73.$$

*Finally, compute the value of the option at time zero:*

$$V_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] = \frac{1}{1.1}[0.75 \cdot 4.73 + 0.25 \cdot 1.09] = 3.47.$$

# Chapter 2

# Martingales

In the previous chapter we have introduced the binomial model in which the randomness is given by the outcomes of $n$ independent and identically distributed coin tosses in which $p$ and $q = 1 - p$ are respectively the probabilities of getting head ($H$ for short) and tail ($T$ for short). In this chapter we will speak about martingales and related properties in this framework. To this aim, we define the probability space $(\Omega, \mathbb{P})$, where $\Omega$ is the set of all the possible outcomes for the sequence of the $n$ coin tosses and $\mathbb{P}$ is the probability measure that, at each coin toss, gives probability $p$ to the outcome $H$ and $q$ to the outcome $T$. More precisely, denoting by $\omega_i$ the outcome of the $i$-th coin toss, we define

$$\Omega = \{\omega = (\omega_1, \ldots, \omega_n) \,|\, \omega_i \in \{H, T\} \text{ for } i = 1, \ldots, n\}\,,$$

$$\mathbb{P}(\omega_i = H) = p\,, \qquad \mathbb{P}(\omega_i = T) = 1 - p = q\,.$$

We will refer to $\mathbb{P}$ as the **actual probability measure**.

In the same space $\Omega$ we will also work with another probability measure $\widetilde{\mathbb{P}}$, that we call **risk neutral probability measure**, defined as

$$\widetilde{\mathbb{P}}(\omega_i = H) = \tilde{p}\,, \qquad \widetilde{\mathbb{P}}(\omega_i = T) = \tilde{q}\,,$$

where $\tilde{q} = 1 - \tilde{p}$ and $\tilde{p}$ are defined as in (1.2.5). Such probability measures have been already introduced in the previous chapter, but now we will give a more detailed analysis.

The risk neutral probabilities are those for which, at every time $n$ and for every coin toss sequence $\omega_1, \ldots, \omega_n$, the following equation holds

$$S_n(\omega_1, \ldots, \omega_n) = \frac{1}{1+r}\left[\tilde{p}S_{n+1}\left(\omega_1, \ldots, \omega_n H\right) + \tilde{q}S_{n+1}\left(\omega_1, \ldots, \omega_n T\right)\right], \quad (2.0.1)$$

that is, the stock price at time $n$ is the discounted weighted average of the two possible stock prices at time $n+1$, where $\tilde{p}$ and $\tilde{q}$ are the weights used in averaging. To simplify notation, equation (2.0.1) can be restated as

$$S_n = \frac{1}{1+r}\tilde{\mathbb{E}}_n\left[S_{n+1}\right]. \quad (2.0.2)$$

$\tilde{\mathbb{E}}_n\left[S_{n+1}\right]$ is defined as the *conditional expectation of* $S_{n+1}$ *based on the information at time n*, which can be regarded as an estimate of the value of $S_{n+1}$ based on the knowledge of the first $n$ coin tosses.

By dividing both sides of (2.0.2) by $(1+r)^n$, we get the equation[1]

$$\frac{S_n}{(1+r)^n} = \tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right]. \quad (2.0.3)$$

Equation (2.0.3) tells us that, under the risk-neutral measure $\widetilde{\mathbb{P}}$, *based on the information at time n, the average value of the discounted stock price at time* $n+1$ *is the discounted stock price at time n.* In other words, if we are at time $n$ and we only know how the stock price has moved up to today's time, the value of the stock price that we will observe tomorrow (i.e., time $n+1$) will be on average equal to the price of today. This is true because the average of the two possible future stock prices, $S_{n+1}\left(\omega_1, \ldots, \omega_n, H\right)$ and $S_{n+1}\left(\omega_1, \ldots, \omega_n, T\right)$ , is weighted by the risk-neutral probabilities, which are indeed chosen to enforce this fact.

Processes satisfying this property are called **martingales** and their formal definition in this context is given below.

**Definition 2.0.1.** *Consider the binomial asset-pricing model. Let* $M_0$ *be a constant and ,* $M_1, \ldots, M_n$ *be a sequence of random variables, with each* $M_n$ *depending only on the first n coin tosses. Such a sequence of random variables is called an* adapted stochastic process.

[1] For the purpose of this model, the term $(1+r)^{n+1}$ could be included either outside or inside of the conditional expectation since interest rates are constant by assumption.

*(i) If*

$$M_n = \mathbb{E}_n[M_{n+1}]\,, \qquad n = 0, 1 \ldots N-1\,, \tag{2.0.4}$$

*we say this process is a* martingale*.*

*(ii) If*

$$M_n \leq \mathbb{E}_n[M_{n+1}]\,, \qquad n = 0, 1 \ldots N-1\,,$$

*we say this process is a* submartingale

*(iii) If*

$$M_n \geq \mathbb{E}_n[M_{n+1}]\,, \qquad n = 0, 1 \ldots N-1\,,$$

*we say this process is a* supermartingale*.*

**Proposition 2.0.1.** *The expectation of a martingale is constant over time, that is, if $M_0, M_1, \ldots, M_n$ is a martingale, then*

$$\mathbb{E}[M_n] = M_0\,, \qquad n = 0, 1 \ldots N-1\,. \tag{2.0.5}$$

*Proof.* If $M_0, M_1, \ldots, M_n$ is a martingale, by taking the expectations on both sides of (2.0.4) and by using property (iii) in Proposition A.0.1, we obtain

$$\mathbb{E}[M_n] = \mathbb{E}[M_{n+1}]$$

for every $n$. By iterating this identity, we get

$$\mathbb{E}[M_0] = \mathbb{E}[M_1] = \mathbb{E}[M_2] = \ldots = \mathbb{E}[M_{N-1}] = \mathbb{E}[M_N]\,.$$

Since $M_0$ is a constant, we have $M_0 = \mathbb{E}[M_0]$ and hence the thesis. □

In order to have a martingale, (2.0.4) must hold for all possible coin toss sequences. This means that at every node the binomial tree, the stock price shown is the average of the two possible subsequent stock prices. This shows how a martingale has no tendency to rise or fall since the average of its next period values is always its value at the current time. However, in real markets stock prices have a tendency to rise and, on average, they rise faster than the money market in order to compensate investors for their inherent risk. This tells us that under the actual probabilities the discounted stock price is a submartingale while, on the other hand, the risk-neutral probabilities are chosen to make it a martingale.

**Theorem 2.0.1.** *Under the risk-neutral measure, the discounted stock price is a martingale, i.e.,* (2.0.4) *holds at every time n and for every sequence of coin tosses.*

*Proof.* The proof relies on Proposition A.0.1 in Appendix. Note that

$$\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] = \tilde{\mathbb{E}}_n\left[\frac{S_n}{(1+r)^{n+1}} \cdot \frac{S_{n+1}}{S_n}\right].$$

Since $\frac{S_n}{(1+r)^n}$ is known at time $n$, it can be taken out from the conditional expectation, hence

$$\tilde{\mathbb{E}}_n\left[\frac{S_n}{(1+r)^{n+1}} \cdot \frac{S_{n+1}}{S_n}\right] = \frac{S_n}{(1+r)^n}\tilde{\mathbb{E}}_n\left[\frac{1}{1+r} \cdot \frac{S_{n+1}}{S_n}\right].$$

We have assumed that interest rates are constant, hence

$$\frac{S_n}{(1+r)^n}\tilde{\mathbb{E}}_n\left[\frac{1}{1+r} \cdot \frac{S_{n+1}}{S_n}\right] = \frac{S_n}{(1+r)^n} \cdot \frac{1}{1+r}\tilde{\mathbb{E}}\left[\frac{S_{n+1}}{S_n}\right].$$

Since $\frac{S_{n+1}}{S_n}$ equals $u$ and $d$ with probabilities $p$ and $q$, respectively, we have

$$\frac{S_n}{(1+r)^n} \cdot \frac{1}{1+r}\tilde{\mathbb{E}}\left[\frac{S_{n+1}}{S_n}\right] = \frac{S_n}{(1+r)^n} \cdot \frac{\tilde{p}u + \tilde{q}d}{1+r} = \frac{S_n}{(1+r)^n},$$

where in the last equality we have used the definition of $\tilde{p}$ and $\tilde{q}$. □

The replicating portfolio technique illustrated in the previous chapter requires that an investor takes a position of $\Delta_n$ shares of stock at each time $n$ and holds this position until time $n+1$, when he takes a new position of $\Delta_{n+1}$ shares. This portfolio rebalancing is financed by investing or borrowing from the money market. The portfolio process $\Delta_0, \Delta_1, \dots, \Delta_{N-1}$ is *adapted*, meaning that each variable $\Delta_i$ depends on the first $i$ coin tosses, with $i \in [0, N-1]$. If the investor begins with initial wealth $X_0$ and $X_n$ denotes his wealth at each time $n$, then the evolution of his wealth is governed by the *wealth equation* (1.2.21) introduced in Chapter 1, which we repeat here

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n). \tag{2.0.6}$$

It is important to highlight that $X_n$ depends only on the first $n$ coin tosses, that is, the *wealth process* is adapted.

The average rate of growth of the investor's wealth depends on the probabilities

under which this average is computed. Under the actual probabilities, such rate of growth depends on the portfolio used (that is, the chosen combination of the positions in the stock and in the money market account). Since a stock generally has a higher average rate of growth than the money market, the investor can achieve a higher rate of growth for his wealth by taking long positions in the stock, which he can finance by borrowing from the money market. Of course, such leveraged positions can be extremely risky.
Under the risk neutral probabilities, the rate of growth of the investor's wealth does not depend on the portfolio used. Indeed, under $\tilde{p}$ and $\tilde{q}$, the average rate of growth of the stock is equal to the interest rate. How the investor decides to divide his wealth between the stock and the money market account is not relevant since, on average, he will achieve a rate of growth equal to the interest rate. We state this result as the following theorem.

**Theorem 2.0.2.** *Consider the binomial model with $N$ periods. Let $\Delta_0, \Delta_1, \ldots, \Delta_{N-1}$ be an adapted portfolio process, let $X_0$ be a real number and let the wealth process $X_1, \ldots, X_N$ be generated recursively by* (2.0.6)*. Then the discounted wealth process $\frac{X_n}{(1+r)^n}$, $n = 0, 1, \ldots, N$ is a martingale under the risk-neutral probability measure, that is*

$$\frac{X_n}{(1+r)^n} = \tilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right], \qquad n = 0, 1, \ldots, N-1. \tag{2.0.7}$$

*Proof.* By (2.0.6), we have

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \tilde{\mathbb{E}}_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}} + \frac{X_n - \Delta_n S_n}{(1+r)^n}\right] = \\
&= \tilde{\mathbb{E}}_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}}\right] + \tilde{\mathbb{E}}_n\left[\frac{X_n - \Delta_n S_n}{(1+r)^n}\right] = \\
&= \Delta_n \tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] + \frac{X_n - \Delta_n S_n}{(1+r)^n} = \\
&= \Delta_n \frac{S_n}{(1+r)^n} + \frac{X_n - \Delta_n S_n}{(1+r)^n} = \\
&= \frac{X_n}{(1+r)^n},
\end{aligned}$$

where we have used Proposition A.0.1 and Theorem 2.0.1. □

**Corollary 2.0.1.** *Under the conditions of Theorem 2.0.2, we have*

$$\tilde{\mathbb{E}}\left[\frac{X_n}{(1+r)^n}\right] = X_0\,, \quad n = 0, 1, \ldots, N. \tag{2.0.8}$$

Corollary 2.0.1 stems from Proposition 2.0.1 (see also Remark 2.0.1).

Theorem 2.0.2 and its corollary have two important consequences. The first is that there cannot be an arbitrage opportunity in the binomial model. If this were the case, an investor could begin with wealth $X_0 = 0$ and find a portfolio process whose corresponding wealth process $X_1, X_2, \ldots, X_N$ satisfies $X_N(\omega) \geq 0$ for all coin toss sequences $\omega$ and $X_N(\bar{\omega}) > 0$ for at least one coin toss sequence $\bar{\omega}$. But then he would have $\tilde{\mathbb{E}}[X_0] = 0$ and $\tilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right] > 0$, which violates Corollary 2.0.1. *The First Fundamental Theorem of Asset Pricing*, that we mentioned in the previous chapter, stems from this fact. Indeed, in general, there cannot be arbitrage in a model in which we can find a risk-neutral measure, that is, a measure that agrees with the actual proability measure about which price paths have zero probability and under which the discounted prices of all primary assets are martingales.
The other consequence of Theorem 2.0.2 is the following version of the *risk-neutral pricing formula*. Let $V_n$ be a random variable (a derivative security payoff at time $N$) depending on the first $N$ coin tosses. We know from Theorem 1.2.1 of Chapter 1 that there is an initial wealth $X_0$ and a replicating portfolio process $\Delta_0, \ldots, \Delta_{N-1}$ that generates a wealth process $X_1, \ldots, X_N$ satisfying $X_N = V_N$, no matter how the coin tossing turns out. Because $\left\{\frac{X_n}{(1+r)^n}\right\}_{n=0}^{N}$ is a martingale, then, by Proposition A.0.2 in Appendix, we have

$$\frac{X_n}{(1+r)^n} = \tilde{\mathbb{E}}_n\left[\frac{X_N}{(1+r)^N}\right] = \tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^N}\right]. \tag{2.0.9}$$

According to Definition 1.2.1 in Chapter 1, the price of the derivative security at time $n$ is defined as $X_n$ and denoted by $V_n$. Thus, (2.0.9) can be rewritten as

$$\frac{V_n}{(1+r)^n} = \tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^N}\right] \tag{2.0.10}$$

or, equivalently,

$$V_n = \tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right]. \tag{2.0.11}$$

We summarize this relation in the following theorem.

**Theorem 2.0.3 (Risk-neutral pricing formula).** *Consider an $N$-period binomial asset pricing model with $0 < d < 1 + r < u$ and with risk-neutral probability measure $\tilde{\mathbb{P}}$. Let $V_n$ be a random variable (a derivative security payoff at time $N$) depending on the coin tosses. Then, for $n = 0, \ldots, N$, the price of the derivative security at time $n$ is given by the risk-neutral pricing formula* (2.0.11)*. Furthermore, the discounted price of the derivative security is a martingale under $\tilde{\mathbb{P}}$; that is,*

$$\frac{V_n}{(1+r)^n} = \tilde{\mathbb{E}}_n \left[ \frac{V_{n+1}}{(1+r)^{n+1}} \right], \qquad n = 0, 1, \ldots, N-1. \tag{2.0.12}$$

# Chapter 3

# Markov Processes

In Chapter 1, we have made the distinction between *European Options*, which can only be exercised at the maturity date of the option, and *American Options*, which can be exercised at any date up to maturity. We have developed a pricing algorithm for the former and, in the following chapter, we will do the same also for the latter. However, to better understant how to price American Options, we will first need to introduce Markov processes and their relation to martingales.

**Definition 3.0.1.** *Consider the binomial asset-pricing model. Let $X_0, X_1, \ldots, X_N$ be an adapted process. If, for every $n$ between 0 and $N-1$ and for every function $f(x)$, there is another function $g(x)$ (depending on $n$ and $f$) such that*

$$\mathbb{E}_n \left[ f(X_{n+1}) \right] = g(X_n) \,, \tag{3.0.1}$$

*then $X_0, X_1, \ldots, X_N$ is a Markov process.*

Equation (3.0.1) is known as **Markov property** and it states that a stochastic process is Markov if the conditional expectation of future states of the process (conditional on both past and present values) depends only upon the present state; that is, given the present, the future does not depend on the past.

By definition, $\mathbb{E}_n \left[ f(X_{n+1}) \right]$ is random: it depends on the first $n$ coin tosses. The Markov property says that this dependence on the coin tossess occurs through $X_n$, that is, the information about the coin tossess one needs to evaluate $\mathbb{E}_n \left[ f(X_{n+1}) \right]$ is summarized by $X_n$. The existence of the function $g$ tells us that if the payoff of a derivative security is random only through its dependence on $X_N$, then there is a version of the derivative security pricing algorithm in which we do not need to store path information, which will be given below in Theorem 3.0.1. In the following example, we will develop a method to find the function $g$ for the stock price process.

**Example 2.** *In the binomial model, the stock price at time $n+1$ in terms of the stock price at time $n$ is given by the formula*

$$S_{n+1}(\omega_1,\dots,\omega_n,\omega_{n+1}) = \begin{cases} uS_n(\omega_1,\dots,\omega_n), & \text{if } \omega_{n+1}=H \\ dS_n(\omega_1,\dots,\omega_n), & \text{if } \omega_{n+1}=T \end{cases}$$

*Therefore,*

$$\mathbb{E}_n\left[f(S_{n+1})\right](\omega_1,\dots,\omega_n) = pf(uS_n(\omega_1,\dots,\omega_n)) + qf(dS_n(\omega_1,\dots,\omega_n)).$$

*where the right-hand side depends on $\omega_1,\dots,\omega_n$ only through the value of $S_n(\omega_1,\dots,\omega_n)$. This equation can be rewritten as*

$$\mathbb{E}_n\left[f(S_{n+1})\right] = g(S_n),$$

*where the function $g(x)$ is defined by*

$$g(x) = pf(ux) + qf(dx).$$

*This shows that the stock price process is Markov. Indeed, it is Markov under either the actual or the risk-neutral probability measure.*
*To determine the price $V_n$ at time $n$ of derivative security whose payoff at time $N$ is a function $v_N$ of the stock price $S_N$ (that is, $V_N = v_N(S_N)$), we use the risk-neutral pricing formula* (2.0.12), *which reduces to*

$$V_n = \frac{1}{1+r}\tilde{\mathbb{E}}_n[V_{n+1}], \qquad n = 0,1,\dots,N-1.$$

*But the stock price process is Markov and $V_N = v_N(S_N)$, so*

$$V_{N-1} = \frac{1}{1+r}\tilde{\mathbb{E}}_{N-1}[v_N(S_N)] = v_{N-1}(S_{N-1})$$

*for some function $v_{N-1}$. In general, $V_n = v_n(S_n)$ for some function $v_n$.*
*Morevover, we can compute these functions recursively by the algorithm*

$$v_n(s) = \frac{1}{1+r}[\tilde{p}v_{n+1}(us) + \tilde{q}v_{n+1}(ds)]. \qquad n = N-1, N-2, \dots, 0. \tag{3.0.2}$$

*This algorithm works in the binomial model for every derivative security whose payoff at time $N$ is a function only of the stock price at time $N$. It is the same for puts and calls, the only difference is in the formula of $v_N(s)$ which is equal to $(s-K)^+$ for calls and to $(K-s)^+$ for puts.*

In order for a process to be Markov, it is necessary that for *every* function $f$ there is a corresponding function $g$ such that (3.0.1) holds. The martingale property is the special case of (3.0.1) with $f(x) = g(x) = x$, so if a process is a martingale, the corresponding $g$ function to $f(x) = x$ is $g(x) = x$. However, if we take $f$ to be some other function, the definition of martingale does not guarantee that we can find any corresponding function $g$. This means that *not every martingale is Markov.* On the other hand, even when considering the function $f(x) = x$, the Markov property requires only that $E_n[M_{n+1}] = g(M_n)$ for some function $g$, but it does not require $g$ to be given by $g(x) = x$. This means that *not every Markov process is a martingale.* Indeed, Example 2 shows that the stock price is Markov under both the actual and the risk-neutral probability measures, but it will be both a martingale and a Markov process under the actual probability measure only if $pu + qd = 1$.

In the binomial pricing model, suppose we have a Markov process $X_0, X_1, \ldots, X_N$ under the risk-neutral probability measure $\tilde{\mathbb{P}}$, and we have a derivative security whose payoff $V_N$ at time $N$ is a function $v_N$ of $X_N$, that is, $V_N = v_N(X_N)$. The following theorem can be proved.

**Theorem 3.0.1.** *Let $X_0, X_1, \ldots, X_N$ be a Markov process under the risk-neutral probability measure $\tilde{\mathbb{P}}$ in the binomial model. Let $v_N(x)$ be a function of the dummy variable $x$, and consider a derivative security whose payoff at time $N$ is $v_N(X_N)$. Then, for each $0 < n < N$, the price $V_n$ of this derivative security is some function $v_n$ of $X_n$, that is*

$$V_n = v_n(X_n)\,. \qquad n = 0, 1, \ldots, N-1\,. \tag{3.0.3}$$

*There is a recursive algorithm for computing $v_n$ whose exact formula depends on the underlying Markov process $X_0, X_1, \ldots, X_N$.*

# Chapter 4

# American Derivative Securities

## 4.1 Introduction

As we already mentioned in the previous chapters, American options differ from European ones because of their so called *early exercise feature*, that is, the fact that they can be exercised at any time up to and including their expiration date. Because of this feature, any American option is always at least as valuable as its European counterpart. Indeed, since an American option can be exercised at any time prior to its expiration, it can never be worth less than the payoff associated with immediate exercise. This is defined as the *intrinsic value* of the option. We shall see in this chapter that the discounted price of an American option is a supermartingale under the risk-neutral measure, in contrast to the case for a European option, for which we have seen that, under this measure, the discounted price process is a martingale. However, we will se that if the holder of an American option fails to exercise at the optimal exercise date, the option has a tendency to lose value, hence, the supermartingale property. Indeed, the discounted stock price of an American option behaves as a martingale during any period of time in which it is not optimal to exercise.

To price an American option, we shall imagine selling the option in exchange for some initial capital and then consider how this capital can be used to hedge the short position in the option. In this case, we need to be ready to pay off the option at all times prior to expiration date, since we don't know when it will be exercised. We determine, from our point of view (that is, the one of the seller of the option) which is the worst time for the owner to exercise the option. From the owner's point of view, this will be the *optimal exercise time.* We then compute the initial capital we need in order to be hedged against exercise at the optimal exercise time. Finally, we show how to invest this capital so that we are hedged even if the owner exercises at a time which is not optimal. We conclude that the initial price of the option is the capital required to be hedged against optimal excercise.

## 4.2 Path-Independent American Derivatives

In this section, we will develop a pricing algorithm for American derivative securities when the payoff is not path dependent. We first summarize the pricing algorithm for European derivative securities with a non-path dependent payoff. In an $N$-period model binomial model with up factor $u$, down factor $d$ and interest rate $r$ satisfying the no-arbitrage condition $0 < d < 1+r < u$, consider a derivative security that pays off $g(S_N)$ at time $N$ for some function $g$. Since the stock price is a Markov process, the value $V_n$ of this derivative security can be written at each time $n$ as a function $v_n$ of the stock price at that time, that is $V_n = v_n(S_n), \quad n = 0, 1 \dots, N$ (Theorem 3.0.1 of Chapter 3). The risk-neutral pricing formula (2.0.9) of Chapter 2 implies that for $0 \le n \le N$, the function $v_n$ is defined by the *European algorithm*, that is

$$v_N(s) = \max\{g(s), 0\}\,, \tag{4.2.1}$$

$$v_n(s) = \frac{1}{1+r}\Big[\tilde{p}v_{n+1}(us) + \tilde{q}v_{n+1}(ds)\Big], \qquad n = N-1, N-2 \dots, 0\,. \tag{4.2.2}$$

where $\tilde{p}$ and $\tilde{q}$ are the risk-neutral probabilities that the stock moves up and down, respectively. The replicating portfolio which hedges a short position in the option is given by (see (1.2.24) of Chapter 1)

$$\Delta_n = \frac{v_{n+1}(uS_n) - v_{n+1}(dS_n)}{(u-d)S_n}\,, \qquad n = 0, 1, \dots, N\,. \tag{4.2.3}$$

Now consider an *American derivative security.* In any period $n \le N$ the holder of the security can exercise and receive payment $g(S_n)$. Consequently, the portfolio that hedges a short position should always have value $X_n$ satisfying

$$X_n \ge g(S_n)\,, \qquad n = 0, 1, \dots, N\,. \tag{4.2.4}$$

This means that the value of the derivative security at each time $n$ is at least as much as the intrinsic value $g(S_n)$. Moreover, the value of the replicating portfolio at that time must equal the value of the derivative security.

This suggests that to price an American security, we should replace the European algorithm (4.2.1) by the *American algorithm*

$$v_N(s) = \max\{g(s), 0\}\,, \tag{4.2.5}$$

$$v_n(s) = \max\left\{g(s), \frac{1}{1+r}\left[\tilde{p}v_{n+1}(us) + \tilde{q}v_{n+1}(ds)\right]\right\}, \qquad n = N-1, N-2\ldots, 0\,. \tag{4.2.6}$$

Then $V_n = v_n(S_n)$ would be the price of the derivative security at time $n$.

In the following example we will se how to use the American algorithm (4.2.5) to price a path-independent American put option.

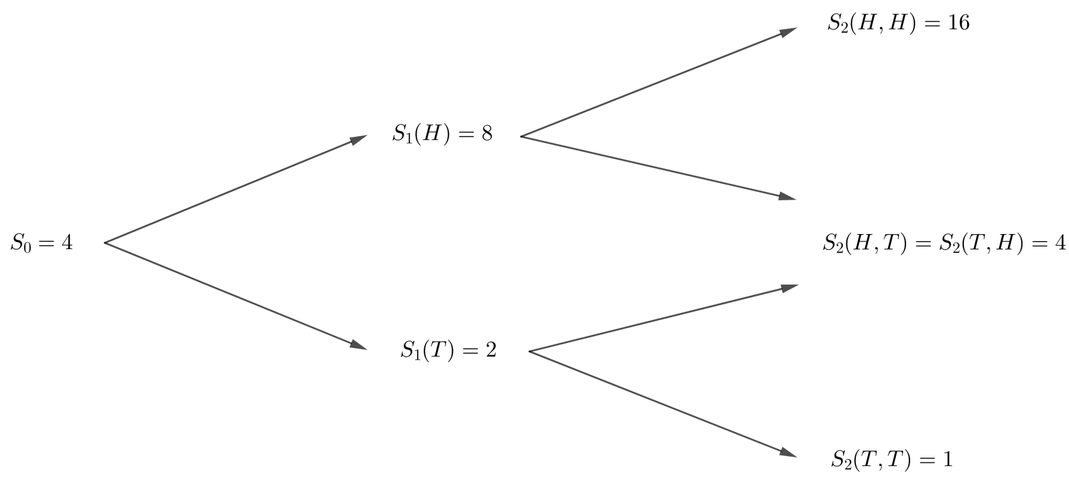


**Fig. 4.1:** A two-period model.

**Example 3.** *Consider the two-period model in Figure 4.1 with $S_0 = 4$, $u = 2$ and $d = \frac{1}{2}$ and let the interest rate be $r = \frac{1}{4}$, so that the risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$. Consider an American put option, expiring at time two, with strike* *price* 5*. Since the security is a put, if the owner exercises at time* $n$*, he receives* $5 - S_n$*, so we take* $g(s) = 5 - s$*. By consequence, the American algorithm* (4.2.5) *becomes*

$$v_2(s) = \max\{5 - s, 0\}\,,$$
$$v_n(s) = \max\left\{5 - s, \frac{2}{5}\Big[v_{n+1}(2s) + v_{n+1}\left(\frac{s}{2}\right)\Big]\right\}, \qquad n = 1, 0\,.$$

*In particular, with reference to the values of* $S_n$ *in Figure 4.1,*

$$\begin{aligned}
v_2(16) &= \max\{5 - S_2(H,H), 0\} = \max\{5 - 16, 0\} = 0\,,\\
v_2(4) &= \max\{5 - S_2(H,T), 0\} = \max\{5 - 4, 0\} = 1\,,\\
v_2(1) &= \max\{5 - S_2(T,T), 0\} = \max\{5 - 1, 0\} = 4\,,\\
v_1(8) &= \max\left\{(5 - S_1(H)), \frac{2}{5}(v_2(16) + v_2(4))\right\} = \max\left\{(5-8), \frac{2}{5}(0+1)\right\} =\\
&= \max\{-3, 0.40\} = 0.40\,,\\
v_1(2) &= \max\left\{(5 - S_1(T)), \frac{2}{5}(v_2(4) + v_2(1))\right\} = \max\left\{(5-2), \frac{2}{5}(1+4)\right\} =\\
&= \max\{3, 2\} = 3\,,\\
v_0(4) &= \max\left\{(5 - S_0), \frac{2}{5}(v_2(8) + v_2(2))\right\} = \max\left\{(5-4), \frac{2}{5}(0.40+3)\right\} =\\
&= \max\{1, 1.36\} = 1.36\,,
\end{aligned}$$

*The American put prices we have just computed are represented in Figure 4.2. The American algorithm gives a different result than the European algorithm in the computation of* $v_1(2)$*, where the discounted expectation of the option price,* $\frac{2}{5}(1+4)$*, is strictry smaller than the intrinsic value. Since* $v_1(2)$ *is strictly greater than the price of a comparable European put, the initial price* $v_0(4)$ *for the American put is also strictly greater than the initial price of a comparable European put.*

*Let us now construct the replicating portfolio. We begin with initial capital* $X_0 = 1.36$ *and compute* $\Delta_0$ *so that the value of the portfolio at time one is equal to the option value. By substituting into* (4.2.3)*, we find that*

$$\Delta_0 = \frac{v_1(8) - v_1(2)}{8 - 2} = -0.43\,.$$

*This means that, in any case, if we begin with initial capital* $X_0 = 1.36$ *and take a position of* $\Delta_0 = -0.43$ *shares of stock at time zero, then at time one we will have*

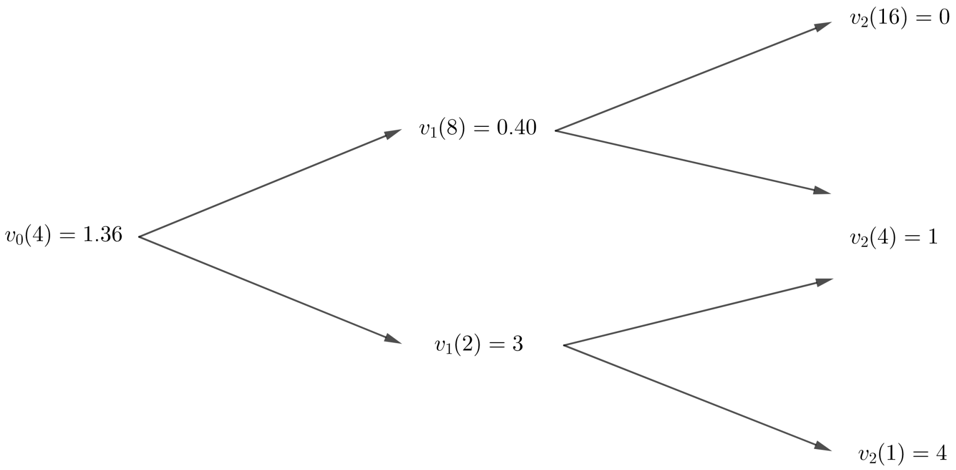


**Fig. 4.2:** American put prices.

$X_1 = V_1 = v_1(S_1)$. *We continue by first analyzing the case in which the first coin toss results in a tail and that the owner of the option has decided to not exercise at time one. We note that the following period the option will be worth* $v_2(4) = 1$ *if the second toss results in head and* $v_2(1) = 4$ *otherwise. According to the risk-neutral pricing formula, to construct a hedge against these two possibilities, our portfolio at time one should be valued at*

$$\frac{2}{5}(v_2(4) + v_2(1)) = \frac{2}{5}(1 + 4) = 2$$

*but we have a hedging portfolio valued at* $v_1(2) = 3$. *Thus, we may consume* \$1 *and continue the hedge with the remaining* \$2 *value in our portfolio. As this suggests, the option holder has let an optimal exercise time go by. More specifically, after consuming* \$1 *we change our position to* $\Delta_1(T)$ *shares of stock, where*

$$\Delta_1(T) = \frac{v_2(4) - v_2(1)}{4 - 1} = \frac{1 - 4}{4 - 1} = -1\,.$$

*We now analyze the case in which the first coin toss results in head. At time one, our portfolio is valued at* $X_1(H) = 0.40$. *We choose*

$$\Delta_1(H) = \frac{v_2(16) - v_2(4)}{16 - 4} = \frac{0 - 1}{16 - 4} = -\frac{1}{12}\,,$$

*so that if the second toss results in head, at time two our hedging portfolio will be valued at* $X_2(H,H) = 0 = v_2(16)$*, while if the toss results in tail, our portfolio will be valued at* $X_2(H,T) = 1 = v_2(4)$

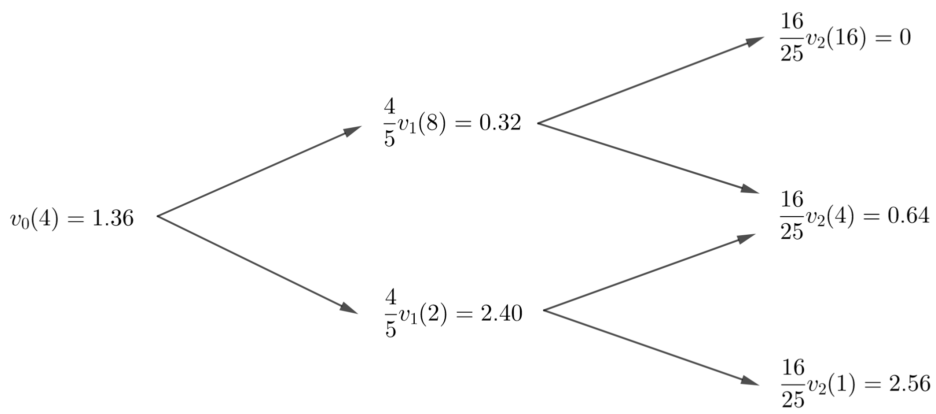


**Fig. 4.3:** Discounted American put prices.

*Finally, we consider the discounted American put prices in Figure 4.3. These constitute a supermartingale under the risk-neutral probabilities* $\tilde{p} = \tilde{q} = \frac{1}{2}$*. At each node, the discounted American put price is greater than or equal to the average of the discounted prices at the two subsequent nodes. This price is not a martingale because the inequality is strict at the time-one node corresponding to a tail on the first toss.*

The following theorem formalizes what we have seen in Example 3 and justifies the American algorithm (4.2.5).

**Theorem 4.2.1 (Replication of path-independent American derivatives).** *Consider an* $N$*-period binomial asset pricing model with* $0 < d < 1 + r < u$ *and with*

$$\tilde{p} = \frac{1+r-d}{u-d} \qquad \tilde{q} = \frac{u-1-r}{u-d},$$

*Let a payoff function* $g(s)$ *be given and define recursively backward in time the sequence of functions* $v_N(s), v_{N-1}(s), \ldots, v_0(s)$ *by* (4.2.5)*. Next define*

$$\Delta_n = \frac{v_{n+1}(uS_n) - v_{n+1}(dS_n)}{(u-d)S_n}, \qquad n = 0, 1, \ldots, N. \tag{4.2.7}$$

$$C_n = v_n(s) - \frac{1}{1+r}\left[\tilde{p}v_{n+1}(uS_n) + \tilde{q}v_{n+1}(dS_n)\right], \tag{4.2.8}$$

*where $0 \leq n \leq N-1$. We have $C_n \geq 0$ for all $n$. If we set $X_0 = v_0(S_0)$ and define recursively forward in time the portfolio values $X_1, X_2, \ldots, X_N$ by*

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - C_n - \Delta_n S_n) \tag{4.2.9}$$

*then we will have*

$$X_n(\omega_1, \ldots, \omega_n) = v_n(S_n(\omega_1, \ldots, \omega_n)) \tag{4.2.10}$$

*for all $n$ and all $\omega_1, \ldots, \omega_n$. In particular, $X_n$ is a supermartingale and $X_n \geq g(S_n)$ for all $n$.*

## 4.3 Stopping Times

Before introducing the pricing algorithm for general American derivatives, including also path-dependent ones, we need to briefly analyse the concept of stopping times. We will begin doing so by referring to Example 3. In general, the time at which an American derivative should be exercised is random since it depends on the price movements of the underlying asset. In Example 3, we claimed that if the first coin toss results in tail, then the owner should exercise at time one. On the other hand, if the first toss results in head, then the owner of the put should not exercise at time one but rather wait for the outcome of the second toss. Indeed if the first toss results in head, then $S_1(H) = 8$ and the put is out of the money. If the second toss results in head, then $S_2(H,H) = 16$ and the put is still out of the money, so the owner should let it expire without exercising it. If instead the first is a head and the second is a tail, then $S_2(H,T) = 4$, the put is in the money and the owner should exercise. We can describe this exercise rule by the following random variable $\tau$, which represents the time of exercise

$$\tau(H,H) = \infty\,, \qquad \tau(H,T) = 2\,, \qquad \tau(T,H) = 1\,, \qquad \tau(T,T) = 1\,. \tag{4.3.1}$$

This exercise rule is displayed in Figure 4.4.

The random variable $\tau$ defined on $\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$ takes values in the set $\{0,1,2,\infty\}$. The owner of the put in this example will regret

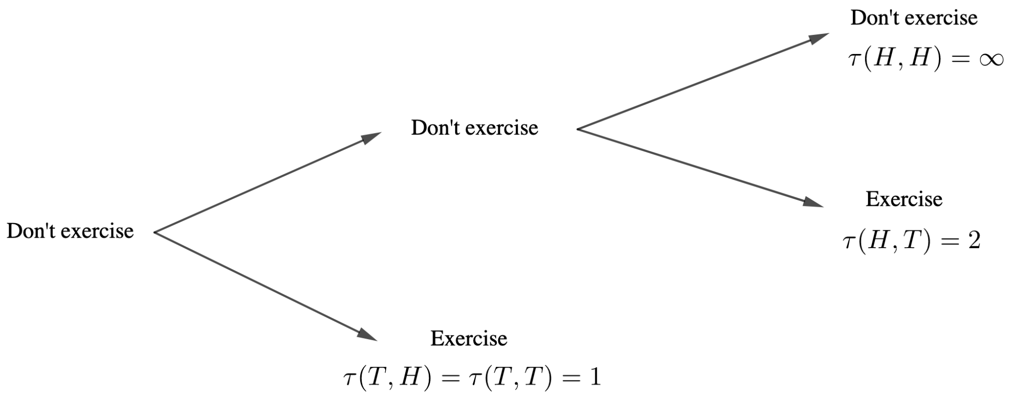


**Fig. 4.4:** Exercise rule $\tau$.

not exercising the put at time zero if the first coin toss results in a head [1]. In particular, if he already knew the outcome of the coin tosses, he would prefer using the following exercise rule $\alpha$, which is displayed in Figure 4.5

$$\alpha(H,H) = 0\,, \qquad \alpha(H,T) = 0\,, \qquad \alpha(T,H) = 1\,, \qquad \alpha(T,T) = 2\,. \qquad (4.3.2)$$

If he could use this exercise rule, then regardless of the coin tossing, he would exercise the put in the money. However, the problem with the exercise rule $\alpha$ is that it cannot be implemented without "insider information", indeed the decision whether or not to exercise at time $n$ is always based on the outcome of the coin toss at time $n+1$. This means that $\alpha$ is not a stopping time, whose definition in this context is given below

**Definition 4.3.1.** *In an $N$-period binomial model, a stopping time is a random variable $\tau$ that takes values $0, 1, \ldots, N$ or $\infty$ and satisfies the condition that if $\tau(\omega_1, \omega_2, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_N) = n$, then $\tau(\omega_1, \omega_2, \ldots, \omega_n, \omega'_{n+1}, \ldots, \omega'_N) = n$ for all $\omega'_{n+1}, \ldots, \omega'_N$.*

This definition ensures that stopping is based only on available information. If stopping occurs at time $n$, then this decision is based only on the first $n$ coin tosses and not on the outcome of any subsequent toss.

[1] Even if $S_2(H,T) = S_0 = 4$, the owner would rather exercise today than in two periods due to the time value of money.

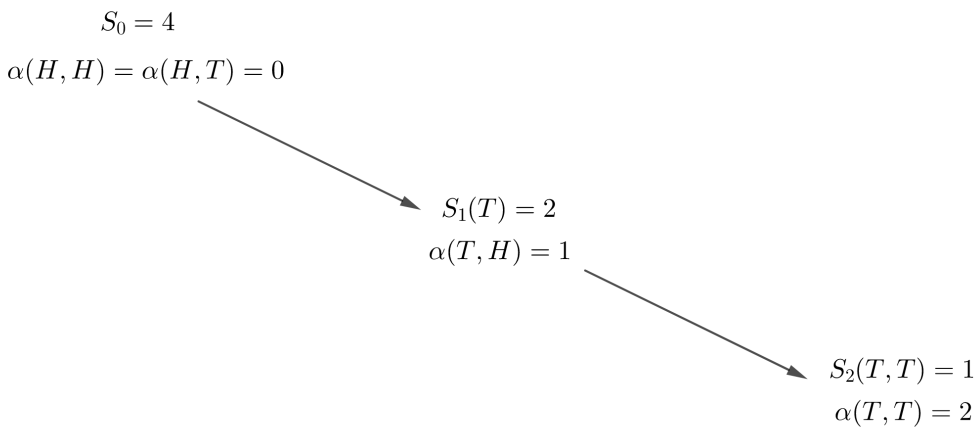


**Fig. 4.5:** Exercise rule $\alpha$.

Whenever we have a stochastic process and a stopping time, we can define a *stopped process*. For example, let $Y_n$ be the process of discounted American put prices in Figure 4.3, that is

$$Y_0 = 1.36\,, \qquad Y_1(H) = 0.32\,, \qquad Y_1(T) = 2.40\,,$$
$$Y_2(H,H) = 0\,, \qquad Y_2(H,T) = Y_2(T,H) = 0.64\,, \qquad Y_2(T,T) = 2.56\,.$$

Let $\tau$ be the stopping time defined in (4.3.1). We define the stopped process $Y_{n\wedge\tau}$, where the notation $n \wedge \tau$ denotes the minimum between $n$ and $\tau$, by the formulas below.
We set

$$Y_{0\wedge\tau} = Y_0 = 1.36$$

because $0 \wedge \tau = 0$ regardless of the coint tossing. Similarly,

$$Y_{1\wedge\tau} = Y_1$$

because $1 \wedge \tau = 1$ regardless of the coin tossing. However, $2 \wedge \tau$ depends on the coin tossing. If the coin tossing results in $(H,H)$ or $(H,T)$, then $2 \wedge \tau = 2$, but if we get $(T,H)$ or $(T,T)$, we have $2 \wedge \tau = 1$. Therefore there are four possible cases

$$Y_{2\wedge\tau}(H,H) = Y_2(H,H) = 0\,, \qquad Y_{2\wedge\tau}(H,T) = Y_2(H,T) = 0.64\,,$$

$$Y_{2\wedge\tau}(T,H) = Y_1(T) = 2.40\,, \qquad Y_{2\wedge\tau}(T,T) = Y_1(T) = 2.40\,.$$

Notice how the value of the process is *stopped* at time $\tau$, even if in this construction the process continues on past time one.

We have seen that the discounted American put price process in Figure 4.3 is a supermartingale but not a martingake under the risk-neutral probabilities $\tilde{p} = \tilde{q} = \frac{1}{2}$ since

$$2.40 = Y_1(T) > \frac{1}{2}Y_2(T,H) + \frac{1}{2}Y_2(T,T) = \frac{1}{2}\cdot 0.64 + \frac{1}{2}\cdot 2.56 = 1.60\,.$$

However, the stopped process $Y_{n\wedge\tau}$ is a martingale. In particular

$$2.40 = Y_{1\wedge\tau}(T) = \frac{1}{2}Y_{2\wedge\tau}(T,H) + \frac{1}{2}Y_{2\wedge\tau}(T,T) = \frac{1}{2}\cdot 2.40 + \frac{1}{2}\cdot 2.40 = 2.40\,.$$

This observation is true in general. Indeed, under the risk-neutral probabilities, a discounted American derivative security price process is a supermartingale. However, if this process is stopped at the optimal exercise time, it becomes a martingale. If the owner of the security permits a time to pass in which the supermartingale inequality is strict, he has failed to exercise optimally.

## 4.4 General American Derivatives

In this section, we introduce path-dependent American derivative securities. We define the price process for such a security and develop its properties. We also show how to hedge a short position in the derivative security and study the optimal exercise time. We define $B_n$ to be the set of all stopping times $\tau$ that take values in the set $\{n, n+1, \ldots, N, \infty\}$. For example, the set $B_0$ contains every stopping time, while a stopping time in $B_N$ can take the value $N$ on some paths, the value $\infty$ on others, and can take no other value.

**Definition 4.4.1.** *For each $0 \le n \le N$, let $G_n$ be a random variable depending on the first $n$ coin tosses. An american derivative security with intrinsic value process $G_n$ is a contract that can be exercised at any time prior to and including time $N$ and, if exercised at time $n$, pays off $G_n$. We define the price process $V_n$ for this contract by the American risk-neutral pricing formula*

$$V_n = \max_{t\in S_n} \tilde{\mathbb{E}}_n\left[\mathbb{I}_{\{\tau\le N\}}\frac{1}{(1+r)^{\tau-n}}G_\tau\right]\,, \qquad n = 0,1,\ldots,N\,. \tag{4.4.1}$$

*where*

$$\mathbb{I}_{\{\tau\le N\}} = \begin{cases} 1 & \textit{if } \tau \le N\,, \\ 0 & \textit{otherwise}\,. \end{cases}$$

The idea behind (4.4.1) is the following. Suppose the American derivative security is not exercised at times $0, 1, \ldots, n-1$ and we want to determine its value at time $n$. At time $n$, the owner can decide to exercise immediatly or to postpone exercise to some later date. The date at which he exercises, if he does, can depend on the path of the stock price up to the exercise time but not beyond it, so the exercise date will be a stopping time $\tau$ which has to be in $B_n$, since exercise was not done before $S_n$. Of course the owner might also not exercise ($\tau = \infty$) and let the security expire. The term $\mathbb{I}_{\{\tau\le N\}}$ appears in (4.4.1) to tell us that $\mathbb{I}_{\{\tau\le N\}}\frac{1}{(1+r)^{\tau-n}}G_\tau$ should be replaced by zero on those paths for which $\tau = \infty$. When the owner exercises according to a stopping time $\tau \in B_n$, the value of the derivative to him at time $n$ is the risk-neutral dicounted expectation of its payoff. The $\max_{t\in S_n}$ term tells us that he should choose $\tau$ to make this as large as possible. One of the immediate consequences of this definition is that

$$V_N = \max\{G_N, 0\}\,, \tag{4.4.2}$$

which simply means that the owner of the security is going to exercise at time $N$ only if the payoff is greater than zero, otherwise he will let the security expire.

We will now develop the properties of the American derivative security price process of Definition 4.4.1. These properties justify calling $V_n$ in Definition 4.4.1 the price of the security.

**Theorem 4.4.1.** *The American derivative security price process given by Definition 4.4.1 has the following properties:*

*(i)* $V_n \ge \max\{G_n, 0\}$ *for all* $n$*;*

*(ii) the discounted price process* $\frac{1}{(1+r)^n}V_n$ *is a supermartingale;*

*(iii) if* $Y_n$ *is another process satisfying* $Y_n \ge \max\{G_n, 0\}$ *for all* $n$ *and for which* $\frac{1}{(1+r)^n}Y_n$ *is a supermartingale, then* $Y_n \ge V_n$ *for all* $n$*.*

*We summarize property (iii) by saying that* $V_n$ *is the smallest process satisfying (i) and (ii).*

Property (ii) in Theorem 4.4.1 guarantees that an agent beginning with initial capital $V_0$ can construct a hedging portfolio whose value at each time $n$ is $V_n$. Property (i) guarantees that if an agent does this, he has hedged a short position in the derivative security; no matter when he exercises it, the agent's hedging portfolio value is sufficient to pay off the derivative security. Thus, properties (i) and (ii) guarantee that the derivative security price is acceptable to the seller. Condition (iii) says that the price is no higher than necessary in order to be acceptable to the seller, meaning that the price is fair for the buyer. Items (i) and (ii) of Theorem 4.4.1 can be proved using only the definition of $V_n$ and the properties of conditional expectation, while item (iii) is based on the Optional Stopping Theorem [3] which assures that a stopped supermartingale is a supermartingale. For details on the proof see [13].

We can now generalize the American pricing algorithm given by (4.2.5) to path-dependent securities.

**Theorem 4.4.2.** *We have the following American pricing algortihm for the path-dependent derivative security price process given by Definition 4.4.1*

$$V_N(\omega_1,\dots,\omega_N) = \max\{G_N(\omega_1,\dots,\omega_N),0\}\,, \tag{4.4.3}$$

$$\begin{aligned} V_n(\omega_1,\dots,\omega_N) = \max\{&G_n(\omega_1,\dots,\omega_n)\,,\\ &\frac{1}{1+r}\left[\tilde{p}V_{n+1}(\omega_1,\dots,\omega_n,H)+\tilde{q}V_{n+1}(\omega_1,\dots,\omega_n,T)\right]\} \end{aligned} \tag{4.4.4}$$

$n = N-1,\dots,0.$

This result is a direct consequence of Theorem 4.4.1 (see [13]).

In order to justify Definition 4.4.1 for American derivative security prices, we must show that a short position can be hedged using these prices. This requires a generalization of Theorem 4.4.2 to the path-dependent case.

The proof of the following result is a consequence of Theorem 4.4.1 and can be proved through induction (see [13]).

**Theorem 4.4.3 (Replication of path-dependent American derivatives).** *Consider an $N$-period binomial asset-pricing model with $0 < d < 1+r < u$ and with*

$$\tilde{p} = \frac{1+r-d}{u-d} \qquad \tilde{q} = \frac{u-1-r}{u-d}\,,$$

*For each $0 \le n \le N$, given by Definition 4.4.1 we define*

$$\Delta_n(\omega_1, \ldots, \omega_n) = \frac{V_{n+1}(\omega_1, \ldots, \omega_n H) - V_{n+1}(\omega_1, \ldots, \omega_n T)}{S_{n+1}(\omega_1, \ldots, \omega_n H) - S_{n+1}(\omega_1, \ldots, \omega_n T)}, \tag{4.4.5}$$

$$C_n(\omega_1, \ldots, \omega_n) = V_n(\omega_1, \ldots, \omega_n) - \frac{1}{1+r}[\tilde{p}V_{n+1}(\omega_1, \ldots, \omega_n H) + \tilde{q}V_{n+1}(\omega_1, \ldots, \omega_n T)], \tag{4.4.6}$$

*where $0 \le n \le N-1$. We have $C_n \ge 0$ for all $n$. If we set $X_0 = V_0$ and define recursively forward in time the portfolio values $X_1, X_2, \ldots, X_N$ by*

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - C_n - \Delta_n S_n), \tag{4.4.7}$$

*then we have*

$$X_n(\omega_1, \ldots, \omega_n) = V_n(\omega_1, \ldots, \omega_n) \tag{4.4.8}$$

*for all $n$ and all $\omega_1, \ldots, \omega_n$. In particular, $X_n \ge G_n$ for all $n$.*

Theorem 4.4.3 shows that the American derivative security price given by (4.4.1) is acceptable to the seller because he can construct a hedge for the short position. We next argue that it is also acceptable to the buyer. We fix $n$ and imagine we have got to time $n$ without the derivative security being exercised. We denote $\tau^* \in S_n$ the stopping time that attains the maximum in (4.4.1), so that

$$V_n = \tilde{\mathbb{E}}_n[\mathbb{I}_{\{\tau^* \le N\}} \frac{1}{(1+r)^{\tau^*-n}} G_\tau]. \tag{4.4.9}$$

For $k = n, n+1 \ldots, N$, we define

$$C_k = \mathbb{I}_{\{\tau^* = k\}} G_k .$$

If the owner of the security exercises it according to the stopping time $\tau^*$, then she will receive the *cash flows* $C_n, C_{n+1}, \ldots, C_N$ at times $n, n+1, \ldots, N$ respectively. Actually, at most one of these $C_k$ values is non-zero. If the option is exercised at or before the expiration time $N$, then the $C_k$ at such an exercise time is the only nonzero payment among them. However, this payment will come at different times on different paths.

In any case, (4.4.9) becomes

$$V_n = \tilde{\mathbb{E}}_n \left[ \sum_{k=n}^{N} \mathbb{I}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*-n}} G_\tau \right] = \tilde{\mathbb{E}}_n \left[ \sum_{k=n}^{N} \frac{1}{(1+r)^{k-n}} C_k \right] .$$

which is the value at time $n$ of the cash flows $C_n, C_{n+1}, \ldots, C_N$, received at times $n, n+1, \ldots, N$, respectively. Once the option holder decides on the exercise strategy $\tau^*$, this is exactly the contract he holds, thus the American derivative security price $V_n$ is acceptable to him.
Now we need to provide a method for the American derivative security owner to choose an optimal exercise time. We consider this problem by seeking a stopping time $\tau^* \in B_0$ that achieves the maximum in (4.4.1) when $n = 0$.

**Theorem 4.4.4** (**Optimal exercise**). *The stopping time*

$$\tau^* = \min\{n : V_n = G_n\} , \tag{4.4.10}$$

*maximizes the right-hand side of* (4.4.1) *when* $n = 0$*, that is*

$$V_0 = \tilde{\mathbb{E}} \left[ \mathbb{I}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*}} G_{\tau^*} \right] . \tag{4.4.11}$$

We know that the value of an American derivative security is always greater than or equal to its intrinsic value. The stopping time $\tau^*$ of (4.4.10) is the first time these two are equal. However, it can happen that they are never equal. For example, the value of an American put is always greater than or equal to zero, but the put can always be out of money (that is, with negative intrinsic value). In this case, the minimum in (4.4.10) is over the empty set, and we follow the matematical convention that the minimum over the empty set is $\infty$. For us, $\tau^* = \infty$ is synonymous with the derivative security expiring unexercised. We now prove Theorem 4.4.4.

*Proof.* We first observe that the stopped process

$$\frac{1}{(1+r)^{n \wedge \tau^*}} V_{n \wedge \tau^*} \tag{4.4.12}$$

is a martingale under the risk-neutral probability measure. This is a consequence of (4.4.1). Indeed, if the first $n$ coin tosses result in $\omega_1, \ldots, \omega_n$ and along this path

$\tau^* \geq n+1$, then we know that $V_n(\omega_1, \dots, \omega_n) > G_n(\omega_1, \dots, \omega_n)$ and (4.4.1) implies

$$
\begin{aligned}
V_{n\wedge\tau^*}(\omega_1, \dots, \omega_n) &= V_n(\omega_1, \dots, \omega_n) = \\
&= \frac{1}{1+r}[\tilde{p}V_{n+1}(\omega_1, \dots, \omega_n H) + \tilde{q}V_{n+1}(\omega_1, \dots, \omega_n T)] = \\
&= \frac{1}{1+r}[\tilde{p}V_{(n+1)\wedge\tau^*}(\omega_1, \dots, \omega_n H) + \tilde{q}V_{(n+1)\wedge\tau^*}(\omega_1, \dots, \omega_n T)]\,.
\end{aligned}
$$

This is the martingale property for the process (4.4.12). On the other hand, if along path $\omega_1, \dots, \omega_n$ we have $\tau^* \leq n$, then

$$
\begin{aligned}
V_{n\wedge\tau^*}(\omega_1, \dots, \omega_n) &= V_{\tau^*}(\omega_1, \dots, \omega_{\tau^*}) = \\
&= \tilde{p}V_{\tau^*}(\omega_1, \dots, \omega_{\tau^*}) + \tilde{q}V_{\tau^*}(\omega_1, \dots, \omega_{\tau^*}) = \\
&= \tilde{p}V_{(n+1)\wedge\tau^*}(\omega_1, \dots, \omega_n H) + \tilde{q}V_{(n+1)\wedge\tau^*}(\omega_1, \dots, \omega_n T)\,.
\end{aligned}
$$

Again we have the martingale property. Since the stopped process (4.4.12) is a martingale, we have

$$
\begin{aligned}
V_0 &= \tilde{\mathbb{E}}\left[\frac{1}{(1+r)^{N\wedge\tau^*}}V_{N\wedge\tau^*}\right] = \\
&= \tilde{\mathbb{E}}\left[\mathbb{I}_{\{\tau^*\leq N\}}\frac{1}{(1+r)^{\tau^*}}G_{\tau^*}\right] + \tilde{\mathbb{E}}\left[\mathbb{I}_{\{\tau^*=\infty\}}\frac{1}{(1+r)^{N}}V_N\right]\,.
\end{aligned}
\tag{4.4.13}
$$

But on those paths for which $\tau^* = \infty$, we must have $V_n > G_n$ for all $n$ and, in particular, $V_n > G_N$. In light of (4.4.3), this can be simplified to

$$
V_0 = \tilde{\mathbb{E}}\left[\mathbb{I}_{\{\tau^*\leq N\}}\frac{1}{(1+r)^{\tau^*}}G_{\tau^*}\right]\,. \tag{4.4.14}
$$

This is (4.4.11). □

As it can be noted, we have only used the American Algorithm (4.2.5) to make examples about American put options. This is because it can be proved that the early exercise of an American call contributes nothing to its value, making an American call exactly as valuable as its European counterpart. This is because the discounted intrinsic value of the call is a submartingale (that is, it has a tendency to rise) under the risk-neutral probabilities. Moreover, for a call option whose payoff is $g^+(s) = (K - s)^+)$, the owner pays $K$ and prefers the value of this payment to be discounted away before exercise, making early exercise optimal.

# Chapter 5

# MATLAB codes

## 5.1 Path Independent European Put-Call Options

The following MATLAB code is used to compute an European Option price as described in Theorem 1.2.1 in Chapter 1. Fixed the up and down factors `u` and `d`, the interest rate `r`, the price of the underlying stock at time zero `S0`, the option's strike price `K` and the number of periods `n`, the algorithm gives

- `V0`: the price of the option at time zero;
- `Delta0`: the portion of shares to buy in the replicating portfolio at time zero in order to hedge a short positiion in the option.

The variable **`type`** can assume two values: **`type`**`=1` in the case of a call option, **`type`**`=0` in the case of a put option.

```
u=1.2;
d=0.8;
r=0.1;
S0=10;
K=8;
n=20;
p=(1+r-d)/(u-d);
```

```
q=1-p;
type=1; %if call, type=1, otherwise type=0


S=zeros(2^(n+1)-1,1);
S(1)=S0;

for i=1:2^n-1
    S(2*i)=d*S(i);
    S(2*i+1)=u*S(i);
end

V=zeros(2^(n+1)-1,1);

for i=2^n:2^(n+1)-1
    if type==1
        V(i)=max([0,S(i)-K]);
    else
        V(i)=max([0,K-S(i)]);
    end
end

for i=2^n-1:-1:1
    V(i)=1/(1+r)*(p*V(2*i+1)+q*V(2*i));
end

Delta=zeros(2^n-1,1);

for i=2^n-1:-1:1
    Delta(i)=(V(2*i+1)-V(2*i))/(S(2*i+1)-S(2*i));
end

V0=V(1)
Delta0=Delta(1)
```

The output of the algortihm with the above inputs is

```
V0 =

    8.8152


Delta0 =

    0.9982
```

Note that the results are coherent with those given in Example 1 in Chapter 1. Actually, the algorithm computes the price of the derivative security and the portion of shares to buy at any period and for any outcome. However, the algorithm only returns the relevant values `V0` and `Delta0` for short.

## 5.2 Path Independent American Put Options

The following MATLAB code is used to compute an American Option price as described in Theorem 4.2.1 in Chapter 4. Fixed the up and down factors `u` and `d`, the interest rate `r`, the price of the underlying stock at time zero `S0`, the option's strike price `K` and the number of periods `n`, the algorithm gives

- `V0`: the price of the option at time zero;
- `Delta0`: the portion of shares to buy in the replicating portfolio at time zero in order to hedge a short positiion in the option;
- `tau`: the stopping time in which it is optimal to exercise the option (according to Theorem 4.4.4 in Chapter 4).

The variable **`type`** can assume two values: **`type`**`=1` in the case of a call option, **`type`**`=0` in the case of a put option.

```
u=2;
d=0.5;
r=0.25;
S0=4;
K=5;
n=2;
p=(1+r-d)/(u-d);
q=1-p;
type=0; %if call, type=1, otherwise type=0


S=zeros(2^(n+1)-1,1);
S(1)=S0;

for i=1:2^n-1
    S(2*i)=d*S(i);
    S(2*i+1)=u*S(i);
end

G=zeros(2^(n+1)-1,1);
if type==1
    for i=1:2^(n+1)-1
        G(i)=S(i)-K;
    end
else
    for i=1:2^(n+1)-1
        G(i)=K-S(i);
    end
end

V=zeros(2^(n+1)-1,1);

for i=2^n:2^(n+1)-1
    V(i)=max([0,G(i)]);
```

```
end

for i=2^n-1:-1:1
    V(i)=max([1/(1+r)*(p*V(2*i+1)+q*V(2*i)),G(i)]);
end

Delta=zeros(2^n-1,1);

for i=2^n-1:-1:1
    Delta(i)=(V(2*i+1)-V(2*i))/(S(2*i+1)-S(2*i));
end

t=0;
i=1;
while t==0 && i<=2^(n+1)-1
    if G(i)==V(i)
        t=i;
    end
    i=i+1;
end

V0=V(1)
Delta0=Delta(1)
tau=t
```

The output of the algorithm with the above inputs is

```
V0 =

    1.3600



Delta0 =

   -0.4333
```

```
tau =

      2
```

Note that the results are coherent with those given in Example 3 in Chapter 4. Actually, the algorithm computes the price of the derivative security and the portion of shares to buy at any period and for any outcome. However, the algorithm only returns the relevant values `V0` and `Delta0` for short.

# Appendix

**Proposition A.0.1. (Fundamental Properties of Conditional Expectations)** *Let $N$ be a positive integer, and let $X$ and $Y$ be random variables depending on the first $N$ coin tosses. Let $0 \leq n \leq N$ be given. The following properties hold.*

*(i)* ***Linearity of conditional expectations****. For all constants $c_1$ and $c_2$, we have*

$$\mathbb{E}_n[c_1 X + c_2 Y] = c_1 \mathbb{E}_n[X] + c_2 \mathbb{E}_n[Y] \,.$$

*(ii)* ***Taking out what is known****. If $X$ actually depends only on the first $n$ coin tosses, then*

$$\mathbb{E}_n[XY] = X \cdot \mathbb{E}_n[Y] \,.$$

*(iii)* ***Iterated conditioning*** *. If $0 \leq n \leq m \leq N$, then*

$$\mathbb{E}_n\left[\mathbb{E}_m[X]\right] = \mathbb{E}_n[X] \,.$$

*(iv)* ***Independence****. If $X$ depends only on tosses $n+1$ through $N$, then*

$$\mathbb{E}_n[X] = \mathbb{E}[X] \,.$$

*(v)* ***Conditional Jensen's Inequality****. If $\varphi(x)$ is a convex function of the dummy variable $x$ then*

$$\mathbb{E}_n[\varphi(x)] \geq \varphi \mathbb{E}_n[X] \,.$$

The proof of Theorem A.0.1 is provided by [13]

The martingale property in (2.0.1) discussed in Chapter 2 is a "one-step-ahead"condition. However, it can be generalized for any number of steps.

**Proposition A.0.2** (**Martingale "Multistep-Ahead" Property**). *If* $M_0, M_1, \ldots, M_N$ *is a martingale and* $m$ *is a positive integer such that* $0 \leq n \leq m \leq N$*, then*

$$M_n = \mathbb{E}_n[M_m] \, .$$

*Proof.* We first show the "two-step ahead" property, that is

$$M_n = \mathbb{E}_n[M_{n+2}] \, .$$

By the martingale property (2.0.1) we know that

$$M_{n+1} = \mathbb{E}_{n+1}\left[M_{n+2}\right] \, .$$

Hence by property (iii) of Theorem A.0.1, we have

$$M_n = \mathbb{E}_n[M_{n+1}] = \mathbb{E}_n\left[\mathbb{E}_{n+1}\left[M_{n+2}\right]\right] = \mathbb{E}_n\left[M_{n+2}\right] \, .$$

By iterating this argument, we can prove the "multi-step ahead" property, that is, for any $m$ such that $0 \leq n \leq m \leq N$ ,, we have

$$M_n = \mathbb{E}_n\left[M_m\right] \, ,$$

□

# Bibliography


[1] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy **81** (1973), no. 3, 637–654.

[2] J. C. Cox, S. A. Ross, and M. Rubinstein, *Option pricing: A simplified approach*, Journal of Financial Economics **7** (1979), no. 3, 229–263.

[3] S. N. Ethier, *The doctrine of chances: Probabilistic aspects of gambling*, Probability and Its Applications, Springer, Berlin and Heidelberg, 2010.

[4] E. F. Fama and K. R. French, *The capital asset pricing model: Theory and evidence*, Journal of Economic Perspectives, American Economic Association **18** (2004), no. 3, 25–46.

[5] L. Giordano and G. Siciliano, *Probabilità reali e probabilità neutrali al rischio nella stima del valore futuro degli strumenti derivati*, Quaderni di finanza (CONSOB) **74** (2013), no. 3, 1–11.

[6] X. Z. He and L. Shi, *A binomial model of asset and option pricing with heterogeneous beliefs*, Journal of Management Science and Engineering **1** (2016), no. 1, 94–113.

[7] G. Huberman, *A simple approach to arbitrage pricing theory*, Journal of Economic Theory **28** (1982), no. 1, 183–191.

[8] J. C. Hull, *Options, futures and other derivatives*, Boston: Prentice Hall, 2012.

[9] R. Korn and S. Muller, *Binomial trees in option pricing: History, practical applications and recent developments*, Recent Developments in Applied Probability and Statistics, Physica-Verlag HD, 2010.

[10] D. Lamberton and B. Lapeyre, *Introduction to stochastic calculus applied to finance*, Chapman and Hall/CRC, 2007.

[11] R. H. Litzenberger, *William F. Sharpe's contributions to financial economics.*, The Scandinavian Journal of Economics **93** (1991), no. 1, 37–46.

[12] H. Markowitz, *Portfolio selection*, The Journal of Finance **7** (1952), no. 1, 77–91.

[13] S. Shreve, *Stochastic calculus for finance I: The binomial asset pricing model*, Springer Finance, Springer New York, 2005.