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Chair of Gambling: Probability and Decision

Problems in Optimal Stopping Theory: the Prophet Inequality and the Secretary Problem

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Ai miei genitori, per avermi insegnato la bellezza dell'essere diversi.
Ai miei nonni, instancabili e perfetti, per il loro amore senza limiti.
A tutta la mia famiglia, per il supporto e l'immenso affetto.
A Clemente, fedele compagno, nella calma e nella tempesta.

"Everyone knew it was impossible,
until a fool who did not know
came along and did it."
(Albert Einstein)

Contents

Introduction

How do you hire the best employee or choose the best candidate for a job? Are there strategies that can increase the probability that you will choose the best one? And if these strategies exist, which is the best among them?
The main theme of this setting is the irrevocable and time-critical decision-making under the unpredictability of the possible future outcomes. This is what characterizes online selection problems, the spearheads of a field that combines probability theory and computer science.

In a classical online selection problem, items arrive in sequence and reveal, each at their turn, their value. These items must be accepted or rejected at the time of their arrival and thus they offer partial information about their value. This recalls the theory of optimal stopping, or early stopping, which is concerned with the problem of choosing a time to take a particular action, in order to maximize an expected reward or minimize an expected cost. This theory is covered precisely in Chapter 1, with the aim of providing the readers with a theoretical background that would allow them to fully understand the world of online selection problems. In particular, in the first chapter, the main goal is to explain what a stopping time is and how this is defined by a stopping rule, which at the same time is fundamental in the resolution of the problems mentioned above. A stopping rule outlines a probability of stopping at a certain time N that maximizes the expected revenue and which has been chosen based both on current and previous observations.

Going back to the previous description of the classic online selection problem, we define the agent as the one who observes the sequence of articles and decides when to accept one of them, and then to stop, or when to reject it, and therefore to continue observing, in the hope of maximizing his expected reward or minimizing the expected cost. It follows that, a stopping rule, together with the sequence of observations, determines the random time N at which the agent decides to stop, for N occurring between 0 and ∞.
It is not over here, since this thesis also analyzes how the theory of optimal stopping can be applied to problems with finite horizons, characterized by a situation in which the agent, or player, is required to stop observing the sequence once a certain stage is reached, for example X_T, for $T > 0$. Hence, the problem would have horizon T and the backward induction would be applied to find the maximum return one can obtain and the optimal rule associated with it. This hypothesis is reasonable since in real life problems the sequence is usually finite and does not contain a huge number of elements.

In the following chapters are then analyzed the most important problems in online selection, which are the secretary problem and the prophet inequality problem. In both problems a decision maker is presented with a sequence of numerical values, and for each value has to make an irreversible decision to accept or reject it. Only one value can be accepted, and a value that has been rejected is lost forever. In the secretary problem the objective is to maximize the probability of selecting the largest value of the entire sequence, while in the prophet problem it is to compare, in terms of expectations, the performance of a gambler with the one of a "prophet" that knows the entire sequence.
In Chapter 2, we deal with the secretary problem, in which a boss wants to hire the best secretary out of n rankable applicants. In the study of this problem we relied mainly on Ferguson [6]. As he explains, the boss can hire only one secretary among

the ones observed sequentially, in an order which is random and unknown to the boss himself. In this chapter, we have calculated the success probability of picking the best candidate for some value r, for $r = 1, 2, \dots, n$, of rejected candidates. More precisely, we have focused on computing the probability that the strategy S_r picks the best candidate, where the strategy rejects the first $r - 1$ applicants and then chooses the next applicant who is the best in the relative ranking of previous observations. Then, we have outlined the optimal value of r that maximizes this probability, obtaining as result that the optimal rule would be selecting the first candidate that appears among applicants from stage r on. From this analysis, we have found that, for large n, the optimal strategy would be waiting until about 37% of the applicants interviewed so far and then selecting the next best one, relative to the previous ones, and that the probability of success is also about 37%. To confirm such a result, our thesis contains a subsection which provides two simulations of the secretary problem, one where the secretaries sample values from a uniform distribution in $[0, 1]$ and the other one where secretaries sample values from a discrete uniform distribution on the set $\{1, 2, \dots, n\}$, that show how, the larger n, the better is the approximation of 37%. For doing this, we have constructed specific MATLAB codes and represented our results graphically. Then, this chapter contains a last subsection that analyzes a natural variation of the secretary problem, where the total number of candidates is divided in N groups and the boss must decide immediately whether to select the best member of the group just interviewed or to continue interviewing the members of the successive group, with no recall allowed. The analysis of this variation is made with the same aim of the one made on the classical problem, that is to find a strategy that maximizes the probability of selecting the best applicant. The result is similar as the one for the classical problem, except for the way the threshold r is defined, as you will see in the relative section.

In Chapter 3, we focus on the prophet inequality, trying to answer the following question: what is the best performance that a gambler is able to achieve when compared to a prophet who is capable of choosing the highest value? We base our study on the results that come from the study of Krengel, Sucheston and Garling [12]: the gambler can provide an expected reward at least equal to the half of the reward of a prophet able to forecast all the realizations and to choose the largest one. In mathematical terms, they showed that, given $X_1, \ldots, X_n$ be a sequence of independent, non-negative, real-valued random variables and $\mathbb{E}[\max_i X_i] < \infty$, then there exists a stopping rule T such that

$$\mathbb{E}[\max_i X_i] \leq 2\mathbb{E}[X_T].$$

This is known as the first of the prophet inequalities in optimal stopping theory and it aims at comparing the performance of online and offline algorithms involved in the selection of one or more elements from a random sequence. During the chapter, we also take care of providing evidence for this inequality. Moreover, we also deal with a variation of the classic prophet inequality, in which, for each i.id. real-valued variable observed, a non-negative fixed cost is charged to the gambler. Using the theory found in [15], we provide a prophet inequality in a “difference” form.

Chapter 4 contains a description of the economic view of the online selection problems just mentioned: it talks about the connection between pricing and prophet inequality and also about the connection between the secretary problem, stock prices and random walks. In particular, the first section of the chapter focuses on the link between post-price mechanisms and prophet inequality, following an article by Lucier [13] in which, defining Hajiaghayi [8] as the one who reintroduced the prophet inequality into the community of economics and computations, and the one who first took care to study the analogy between the latter and a simple pricing problem, he continues by exposing the theory explained in [8] and then gives us an overview of this connection through an economic-oriented proof of the prophet inequality. The last section presents as an argument the one of developing and testing models for stock price behavior, particularly relying on the approach deriving from the theory of random walks. Following the analysis by Fama [4], a random walk market implies that successive price changes in individual securities will be independent, meaning that it would not be possible to use past series to predict the behavior of stock prices. The consequence from this assumption is that a simple policy which consists of buying and holding the security is the same as any complicated mechanism that wants to plan purchases and sales. We then proceed with a problem that is found in a paper by Hlynka and Sheahan [9]: a stock analyst has to predict, for a client, the day in which a particular stock will have the highest price in a given month. If he makes the best decision on which day to choose, he receives as reward a major portfolio to manage, otherwise he receives nothing. In this paper is stated that all values of different stocks represent positions of a generalized one-dimensional random walk. In this section, we show how the secretary problem can be applied to find the best strategy that maximized the probability of choosing the largest value in n with no recall allowed.

Thus, this thesis provides an overview of how online selection problems can be applied to solve everyday life problems such as hiring the best candidate for a job, choosing the best place to park, or, again, predicting the day to choose to maximize the profits of your financial portfolio. In fact, although these problems are based on probability theory and computer science, and focus on the performance of online and offline algorithms, they can provide a considerable help in making the resolution of these daily life problems as efficient as possible, thus allowing us to maximize the reward that would result from the resolution of these or to minimize the cost associated with them.

Chapter 1

Stopping rule problems

A stopping time is a random variable that has a value associated with the time when a certain behavior of interest occurs in a stochastic process. It is defined by a stopping rule, which refers to a mechanism involving the decision of continuing the process or stopping it, basing this decision taking on the present position in the process and on past events. From this, the objective of optimal stopping theory is the one of choosing the exact time to take a given action, looking at the present position and past events, to maximize the expected revenue or to minimize the expected cost. The field of this problem's applications is very wide as it includes areas such as statistics and operations research.
In this section, we are going to define, mathematically, what is a stopping rule.

Definition 1.1. *Let* $(X_1, X_2, \dots)$ *be a sequence of random variables with known joint distribution and* $y_0, y_1(x_1), \dots, y_\infty(x_1, x_2, \dots)$ *be a sequence of real-valued reward functions. A stopping rule problem is defined as to choose a time to stop observing the sequence of random variables that maximizes the expected reward. In particular, a stopping rule outlines a probability of stopping at a certain time* n *that maximizes the expected revenue and which has been chosen based both on current and previous observations, that is, on the value attached to* $X_1, \dots, X_n$ *that have* *been observed so far. This probability is expressed by* $\phi(x_1, \ldots, x_n)$*. By this, it is possible to state that a stopping rule is defined as the sequence:*

$$\phi = (\phi_0, \phi_1(x_1), \phi_2(x_1, x_2), \ldots).$$

So, if an agent decides to observe the sequence $X_1, X_2, \ldots$, it would be possible for him to stop at time n, for $n = 1, 2, \ldots$, once observed $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$. In that case, he would receive a reward of $y_n(x_1, \ldots, x_n)$, which is known. The other available option for the player is to continue observing the sequence and to stop, for instance, at time $n + 1$, once the sequence is observed up to $X_{n+1} = x_{n+1}$, receiving the associated reward.
The value of the probabilty of choosing a stopping time that maximizes this reward is between 0 and 1 for all n and $x_1, \ldots, x_n$. Then, we know that ϕ_0 expresses the probability that the agent decides not to take part to the observation, while ϕ_1 defines the probability that the agent stops after the first observation $X_1 = x_1$, and so on for all the random variables of the sequence introduced above.

If the agent decides not to take part to the observation, his reward will be y_0, while, if he decides to observe the sequence and to not stop at any time, he will have a reward of $y_\infty(x_1, x_2, \ldots) = -\infty$.

Definition 1.2. *A stopping rule is said to be randomized if*

$$0 \leq \phi_n(x_1, \ldots, x_n) \leq 1.$$

On the other hand, it is said to be non-randomized if each $\phi_n(x_1, \ldots, x_n)$ *is either* 0 *or* 1*.*

It follows that, the stopping rule ϕ, together with the sequence of observations $X = (X_1, X_2, \ldots)$, determines the random time N at which the agent decides to stop. This N occurs between 0 and ∞, and in particular, if the player decides to never stop, we would have $N = \infty$.

The probability mass function (a function that gives the probability that a discrete random variable is exactly equal to some value) of N given $X = x = (x_1, x_2, \ldots)$ is denoted by $\psi = (\psi_0, \psi_1, \ldots, \psi_\infty)$ where

$$\psi_n(x_1, \ldots, x_n) = \mathbb{P}(N = n \mid X = x),$$

$$\psi_\infty(x_1, x_2, \ldots) = \mathbb{P}(N = \infty \mid X = x).$$

The first equation expresses the probability that the player decides to stop at a given time $N = n$, for $n = (0, 1, 2, \ldots)$, given all the observations, while the second one expresses the probability of never stopping. If we want to write the stopping rule in terms of the random stopping time N, we will have

$$\phi_n(X_1, \ldots, X_n) = \mathbb{P}(N = n \mid N \geq n, X = x) \text{ for all } n = 0, 1, \ldots.$$

Finally we need to define, mathematically, what is the expected reward that the agent wants to maximize, denoted by $V(\phi)$. For $0 \leq j \leq \infty$, it is defined as

$$\begin{aligned} V(\phi) &= \mathbb{E}[y_N(X_1, \ldots, X_N)] \\ &= \mathbb{E} \sum_{j=0}^{\infty} [\psi_j(X_1, \ldots, X_j) y_j(X_1, \ldots, X_j)]. \end{aligned}$$

1.1 Stopping rule problems with finite horizon

In this sections we discuss a particular case for stopping rule problems. We look at a situation in which the player is required to stop observing the sequence $X = (X_1, X_2, \ldots)$ once having observed up to X_T, where $T > 0$ is fixed. This problem is said to have horizon T.

Definition 1.3. *A stopping rule problem is said to have a finite horizon if there is a known upper bound on the number of stages at which an agent may stop.*

This special case can be analyzed through the general problem that we have discussed in the previous section. In particular, this special case arises when we set $y_{T+1} = \ldots = y_\infty = -\infty$.
Here, it is useful to apply the method of backward induction: we start from finding the optimal rule at stage $T-1$, since the highest step available to the player is T, and then, knowing this, we continue back forward to the initial stage, which is the stage 0. The expected reward that the agent wants to maximize is expressed by

$$V_T^{(T)}(x_1, \ldots, x_T) = y_T(x_1, \ldots, x_T).$$

Specifically, we need to apply a backward induction that goes from $j = T-1$ and arrives at $j = 0$, in order to write the equation above in the following way:

$$V_j^{(T)}(x_1, \ldots, x_j) = \max\{y_j(x_1, \ldots, x_j), \mathbb{E}[V_{j+1}^{(T)}(x_1, \ldots, x_j, X_{j+1}) \,|\, X_i = x_i \,\forall\, i \leq j]\}.$$

$V_j^{(T)}(x_1, \ldots, x_j)$ is the maximum return that can be obtain starting, inductively, from the $j-th$ position, once $X_1, \ldots, X_j$ have been observed. The two values compared are respectively the return for stopping at stage j and the expected return for stopping at another stage, like $j+1$, after deciding continuing the observation and using the optimal rule for stages $j+1$ through T.
As we know that the maximum of these two quantities is considered as the player's optimal return and that the $j-th$ step is an optimal stage to stop at if

$$V_j^{(T)}(x_1, \ldots, x_j) = y_j(x_1, \ldots, x_j),$$

we state that the value of the stopping rule problem for this special case is represented by $V_0^{(T)}$.

In the following chapters, we will analyze a particular problem in this framework, known as the "secretary problem". Moreover, we will be interested in a comparison between the optimal strategy and the perfect knowledge of a prophet that knows the entire sequence $X_1, X_2, \ldots, X_T$, that is revealed by the prophet inequality.

Chapter 2

The secretary problem

The secretary problem appeared in the late 1950s and early 1960s and became famous within the mathematical community for the ease with which it can be stated and for its striking solution. This is considered one of the problems that most characterizes the field of mathematics-probability-optimization: at the basis of its solution, in fact, there is a knowledge of optimal stopping theories and a close correlation with prophet inequalities.
In [6] Ferguson describes the framework in which the problem arises: suppose that a boss wants to hire the best secretary out of n rankable applicants. The administrator may hire only one secretary and the order the applicants follow to be interviewed is sequential, random and unknown to the boss. It is said that a weight is assigned to each applicant by some adversary unknown to the administrator, and that this adversary will also choose the order to reveal secretaries. Secretaries are revealed one at a time, the boss learns their weight (that is, their value), and he immediately and irrevocably decides whether to hire or not the applicant. The goal is to maximize the probability of selecting the secretary with the maximum weight and, in particular, here we want to maximize the expected weight of what the administrator will select.

As in [6], the classical secretary problem has the following features:

- There is one secretarial position available;
- The number n of applicants is known to the boss;
- The applicants are interviewed sequentially in a random order. All the orderings are equally likely;
- All applicants can be ranked from best to worst without ties. The decision to hire or not an applicant must be based only on current and past observations;
- Once rejected, an applicant cannot be recalled;
- The boss' payoff function is 1 if the best of the n applicants is chosen, 0 otherwise.

First, we need to calculate the success probability of picking the best candidate for some value r of rejected candidates. The success probability can be thought as the sum of the probabilities of finding the best candidate in position n. You might consider to make an arbitrary decision, such as to choose always the first applicant. This random strategy performs poorly. You only have a probability $1/n$ that the first applicant will be the best one. The same is true also when choosing always the last applicant or always the n–th one: your odds are always $1/n$ for any prearranged position. The random strategy gets worse as you increase the number of applicants.
We now consider a class of strategies indexed by a parameter $r = 1, \ldots, n-1$. In particular, given $r \geq 1$, define S_r the strategy that rejects the first $r-1$ applicants and then chooses the next applicant who is the best in the relative ranking of the observed applicants. We want to compute the probability that the strategy S_r chooses the best applicant.

Define the events

$$A_j := \{\text{the } j\text{-th applicant is the best}\}\,, \qquad B_j := \{S_r \text{ chooses the } j\text{-th applicant}\}\,.$$

We have the following proposition.

Proposition 2.0.1.

$$\mathbb{P}(A_j) = \frac{1}{n}\,, \qquad \mathbb{P}(B_j) = \frac{r-1}{j-1}\,.$$

Proof. Since $\sum_{j=1}^{n} \mathbb{P}(A_j) = 1$ and $\mathbb{P}(A_j) = \mathbb{P}(A_i)$ for all $i, j = 1, \ldots, n$, we have

$$\mathbb{P}(A_j) = \frac{1}{n}\,.$$

Moreover, if the j-th applicant is the best one, then it is selected if and only if the best applicant among the first $j-1$ is among the first $r-1$ applicants that were rejected. Hence

$$\mathbb{P}(B_j) = \frac{r-1}{j-1}\,.$$

□

By Proposition 2.0.1, we get

$$\begin{aligned}\mathbb{P}(S_r \text{ chooses the best applicant}) &= \sum_{j=r}^{n} \mathbb{P}(A_j \cap B_j) = \mathbb{P}(B_j \mid A_j)\mathbb{P}(A_j) = \\ &= \sum_{j=r}^{n} \frac{r-1}{j-1} \cdot \frac{1}{n} = \frac{r-1}{n} \sum_{j=r}^{n} \frac{1}{j-1}\,,\end{aligned} \tag{2.1}$$

where $(r-1)/(r-1)$ represents 1 if $r = 1$. Calling P_r the above probability, we have that the optimal r is the value that maximizes P_r. Note that

$$P_{r+1} \le P_r \iff \frac{r}{n} \sum_{j=r+1}^{n} \frac{1}{j-1} \le \frac{r-1}{n} \sum_{j=r}^{n} \frac{1}{j-1} \iff \sum_{j=r+1}^{n} \frac{1}{j-1} \le 1$$

and hence the optimal rule is to select the first candidate that appears among applicants from stage r on, where

$$r = \min\left\{ r \ge 1 \,\middle|\, \sum_{j=r+1}^{n} \frac{1}{j-1} \le 1 \right\}\,.$$

Note that, letting r and n go to infinity in equation (2.1) so that r/n converges to $x > 0$, defining $t = j/n$ and $dt = 1/n$, we get

$$\lim_{n\to\infty} \mathbb{P}(S_r \text{ chooses the best applicant}) = x \cdot \int_x^1 \frac{1}{t} dt = -x\ln(x)\,.$$

At this point, we need to find the value of x that maximizes this quantity. Note that $f''(x) = -1/x < 0$ for $x > 0$ and hence f is concave in $(0, +\infty)$. Moreover $f'(x) = -1 - \ln(x) = 0$ if and only if $x = e^{-1} \approx 0.37$. Hence for large n it is approximately optimal to wait until about 37% of the applicants that have been interviewed and then to select the next relatively best one. The probability of success is also about 37%.

2.0.1 Simulations of the Secretary Problem

We have used MATLAB to realize two simulations of the secretary problem, with $n = 1000$ secretaries. In the first one, each secretary samples, independently from the others, a value from a uniform distribution in $[0, 1]$ (see Fig. 2.1), while in the second one, each secretary samples, independently from the others, a value from a discrete uniform distribution on the set $\{1, 2, \ldots, n\}$ (see Fig. 2.2). In both figures is represented the probability that the strategy S_r picks the maximum for $r = 1, 2, \ldots, n$.

In the previous section, we have seen that when the number of secretaries n is large, approximately, the optimal strategy would be waiting until about 37% of the applicants interviewed so far and then to select the next relatively best one. Moreover, the probability of success is also about 37%. In the above simulations, we observe an approximation of such a result. The larger is n, the better the approximation of 37% is represented by the graphs in Fig. 2.1 and Fig. 2.2.

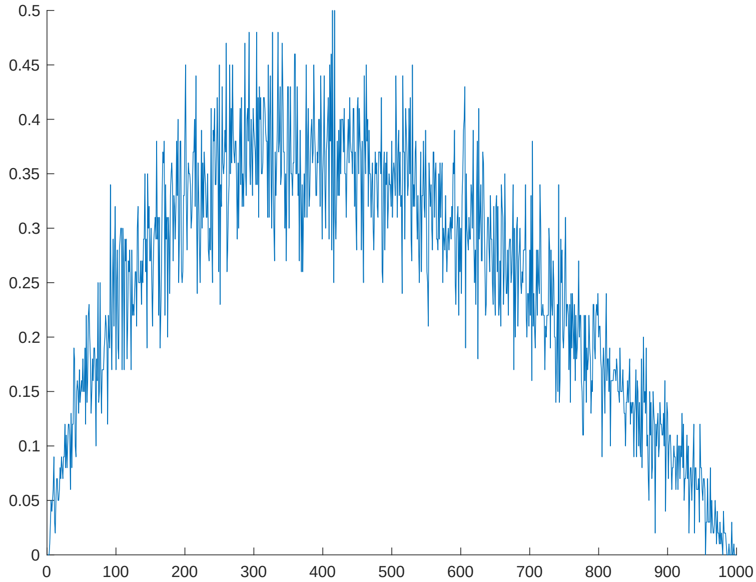

Fig. 2.1: Secretary problem with $n = 1000$ and each secretary assumes a value from a uniform distribution in $[0, 1]$, independently from the other secretaries. The x-axis reports the value of r, for $r = 1, 2, \ldots, n$, while the y-axis reports the value of S_r.

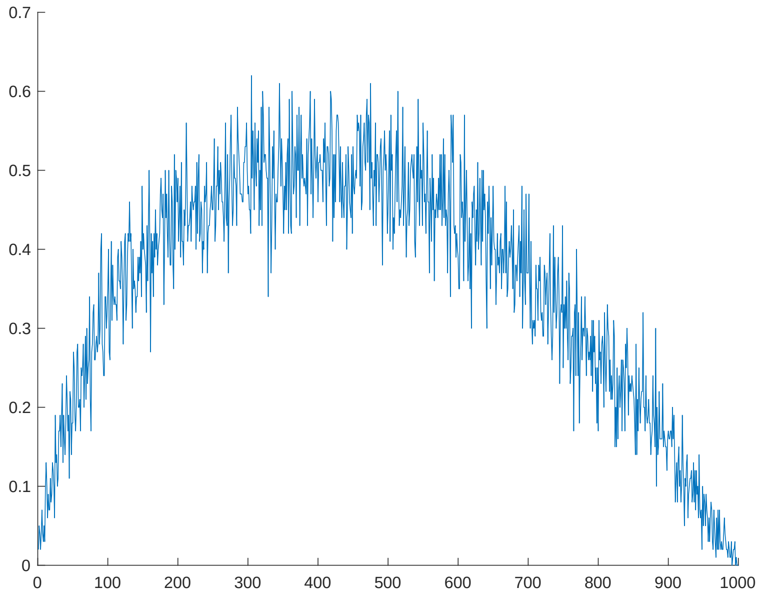

Fig. 2.2: Secretary problem with $n = 1000$ and each secretary assumes a value from a discrete uniform distribution on the set $\{1, 2, \ldots, n\}$, independently from the other secretaries. The x-axis reports the value of r, for $r = 1, 2, \ldots, n$, while the y-axis reports the value of S_r.

MATLAB code

The code below computes the probability $\mathbb{P}(S_T$ chooses the best applicant), for $T = 1, 2, \ldots, n$. To compute such a probability we approximate it with the fraction of success over the total number of samples of the same experiment (the experiment is "the secretary problem picks the maximum applying the strategy S_T"). Note that this approximation is justified by the Law of Large Numbers.

```
samples=100;
n=1000;

K=zeros(n,1);

for T=1:n
  c=0;
  for i=1:samples
    correct=secretary(n,T);
    c=c+correct;
  end
  K(T)=c/samples;
end

plot(1:n,K(1:n))
```

The function `secretary(n,T)` gives 1 if the strategy S_T picks the maximum over the n secretaries, otherwise, it gives 0.

Below, we write the code of such a function for the Fig. 2.1, that is when the secretaries samples value from the uniform distribution in $[0, 1]$.

```
v=zeros(n,1);

for i=1:n
  v(i)=rand(1);
end

M=v(1);
for i=2:n
  if v(i)>M
    M=v(i);
  end
end

max=v(1);
for i=2:T
  if v(i)>max
    max=v(i);
  end
end



if T<n
  j=T+1;
  while v(j)<max && j<n
    j=j+1;
  end
  S=v(j);
```

```
end

if T==n
  S=v(n);
end

if S==M
  correct=1;
else
  correct=0;
end
```

Similarly, we write below the code of such a function for the Fig. 2.2, that is when the secretaries samples value from a discrete uniform distribution on the set $\{1, 2, \ldots, n\}$.

```
v=zeros(n,1);

for i=1:n
  v(i)=randi(1000);
end

M=v(1);
for i=2:n
  if v(i)>M
    M=v(i);
  end
end
```

```
max=v(1);
for i=2:T
  if v(i)>max
    max=v(i);
  end
end



if T<n
  j=T+1;
  while v(j)<max && j<n
    j=j+1;
  end
  S=v(j);
end

if T==n
  S=v(n);
end

if S==M
  correct=1;
else
  correct=0;
end
```

2.1 A natural variation of the secretary problem

Suppose now to have N groups of people, where the i-th group contains l_i people. At the i-th round a manager interviews the i-th group and then he must decide immediately whether to select the best member in the i-th group. If so, then the interview ends, otherwise he continues to interview the members of the $(i+1)$-th group. At each step the manager knows only the relative ranks of the applicants who have been interviewed so far. No recall is permitted.

The aim is to find a strategy that maximizes the probability of selecting the best applicant. It is not surprising that the optimal strategy is similar to the one in the classical problem, that is to reject the applicants in the first $r-1$ groups (for some r) and accept the best one in the next group which contains the one that is preferable to all the predecessors. What is new is that the threshold r is determined by the following simple formula

$$r = \min\left\{ n \,\middle|\, \sum_{k=n+1}^{N} \frac{l_k}{b_{k-1}} \leq 1 \right\}, \tag{2.2}$$

where

$$b_k = \sum_{i=1}^{k} l_i \,. \tag{2.3}$$

Below we prove the above result.

2.1.1 Proof of formula (2.2)

Recall the definition of b_k in (2.3) for $k = 1, \ldots, N$. Note that b_k is exactly the number of people interviewed up to step k. Denote by $P_1, \ldots, P_{b_N}$ the people that are in the N groups and suppose that the manager has a list $\mathcal{L}_0$ with their names ranked exactly in the above order, that is $\mathcal{L}_0 := (P_1, \ldots, P_{b_N})$. Suppose that the manager has interviewed the first k people $P_1, \ldots, P_k$. Then he is able to construct a new list, that we denote by $\mathcal{L}_k$, in which he ranks the interviewed candidates looking at their performance (from the best to the worst). So $\mathcal{L}_k$ is a permutation of $P_1, \dots, P_k$. Note in particular that $\mathcal{L}_{b_N}$ is the list that contains all the people ranked in terms of their performances (from the best to the worst).

We define Y_k as the relative rank of P_k among the first k people, that is Y_k is the position of P_k in the list $\mathcal{L}_k$. Moreover we define A_k as the absolute rank of P_k, that is A_k is the position of P_k in the list $\mathcal{L}_{b_N}$. Let Z_n be the absolute rank of the best applicant in the n-th group, that is $Z_n := \min\{A_{b_{n-1}+1}, \dots, A_{b_n}\}$. Note that if $Z_n = 1$, then the best applicant of the n-th group is the best of all the applicants.

Note that the optimal strategy is determined by the optimal stopping rule that solves the problem in which we want to maximize $\mathbb{E}[X_n]$, where $X_n := \mathbb{P}(Z_n = 1 \,|\, Y_1, \dots, Y_{b_n})$. Indeed if τ is the optimal stopping rule for such a problem, then $\mathbb{E}[X_\tau] = \mathbb{P}(Z_\tau = 1)$ and hence it is equivalent to maximize $\mathbb{E}[X_\tau]$ or $\mathbb{P}(Z_\tau = 1)$.

Below we state two lemmas that we are going to use. The proof of the first lemma can be found in [1], while the second lemma is a consequence of the first one.

Lemma 2.1.1. $Y_1, \dots, Y_{b_N}$ *are independent random variables and for each* $k = 1, \dots, b_N$

$$\mathbb{P}(Y_k = j) = \frac{1}{k}\,, \qquad \text{for } j = 1, \dots, k\,.$$

Lemma 2.1.2. *For any* $n = 1, \dots, N$

$$X_n = \mathbb{P}(Z_n = 1 \,|\, Y_1, \dots, Y_{b_n}) = \begin{cases} c_n = \frac{b_n}{b_N}\,, & \text{if } Y_k = 1 \text{ for some } k \in [b_{n-1}+1, b_n]\,, \\ 0, & \text{if } Y_k \neq 1 \text{ for all } k \in [b_{n-1}+1, b_n]\,, \end{cases}$$

where we use the convention $b_0 = 0$.

Note that by Lemma 2.1.2 X_n is a function of $Y_{b_{n-1}+1}, \dots, Y_{b_n}$. Define

$$\gamma_N = X_N\,; \qquad \gamma_n = \max\{X_n\,,\ \mathbb{E}[\gamma_{n+1} \,|\, Y_1, \dots, Y_{b_n}] \quad \text{for } n = N-1, \dots, 1\,. \tag{2.4}$$

Since the Y's are independent (see Lemma 2.1.1) and $\gamma_N = X_N$ is a function of $Y_{b_{N-1}+1}, \ldots, Y_{b_N}$, then

$$\mathbb{E}[\gamma_N \,|\, Y_1, \ldots, Y_{b_{N-1}}] = \mathbb{E}[\gamma_N]\,.$$

Consequently $\gamma_{N-1} = \max\{X_{N-1}, \mathbb{E}[\gamma_N]\}$ is a function of X_{N-1} and hence a function of $Y_{b_{N-2}+1}, \ldots, Y_{b_{N-1}}$. By a similar argument and backward induction, we have that γ_n is a function of $Y_{b_{n-1}}, \ldots, Y_{b_n}$ and hence

$$\mathbb{E}[\gamma_{n+1} \,|\, Y_1, \ldots, Y_{b_n}] = \mathbb{E}[\gamma_{n+1}] := V_{n+1} \qquad \text{for } n = 1, \ldots, N-1\,.$$

Then if $V_{N+1} = 0$, we have

$$\gamma_n = \max\{X_n, V_{n+1}\}\,, \qquad \text{for } n = 1, \ldots, N\,. \tag{2.5}$$

Observe that $\mathbb{E}[\gamma_1] = V_1 \geq V_2 \geq \ldots \geq V_N \geq V_{N+1} = 0$ and $0 := c_0 < c_1 < c_2 < \ldots < c_N = \frac{b_N}{b_N} = 1$. So we can find a unique positive integer r between 1 and N such that $c_i \geq V_{i+1}$ whenever $r \leq i \leq N$ and $c_{r-1} < V_r$.

Let us consider the following stopping rule τ

$$\tau = \min\{n \geq r \,|\, Y_k = 1 \text{ for some } k \in [b_{n-1}+1, b_n]\}\,.$$

In [10] it is shown that the stopping rule τ is the optimal stopping rule. By Lemma 2.1.2, equation (2.5) and the definition of r for each $n = r, \ldots, N$

$$\gamma_n = \max\{X_n, V_{n+1}\} = \begin{cases} \frac{b_n}{b_N}\,, & \text{if } Y_k = 1 \text{ for some } k \in [b_{n-1}+1, b_n]\,, \\ V_{n+1}\,, & \text{if } Y_k \neq 1 \text{ for all } k \in [b_{n-1}+1, b_n]\,. \end{cases}$$

For $r \leq n \leq N$ the V's satisfy the following recursive formula with $V_{N+1} = 0$ and

$$\begin{aligned} V_n = \mathbb{E}[\gamma_n] &= \frac{b_n}{b_N}\mathbb{P}(Y_k = 1 \text{ for some } b_{n-1}+1 \leq k \leq b_n) + \\ &+ V_{n+1}\mathbb{P}(Y_k \neq 1 \text{ for all } b_{n-1} \leq k \leq b_n) = \\ &= \frac{b_n}{b_N} \cdot \frac{l_n}{b_n} + V_{n+1} \cdot \frac{b_{n-1}}{b_n} = \frac{l_n}{b_N} + \frac{b_{n-1}}{b_n} \cdot V_{n+1}\,. \end{aligned} \tag{2.6}$$

Solving the above equation yields

$$V_n = \frac{b_{n-1}}{b_N} \sum_{k=n}^{N} \frac{l_k}{b_{k-1}} \,, \qquad r \le n \le N\,. \tag{2.7}$$

If $r = 1$ then, from (2.6) we have $V_1 = \frac{l_1}{b_N}$ (though some ambiguity arises at (2.7)).

By (2.7) and the definition of r, we have

$$\frac{b_r}{b_N} = c_r \ge V_{r+1} = \frac{b_r}{b_N} \sum_{k=r+1}^{N} \frac{l_k}{b_{k-1}} \quad \text{and} \quad \frac{b_{r-1}}{b_N} = c_{r-1} < V_r = \frac{b_{r-1}}{b_N} \sum_{k=r}^{N} \frac{l_k}{b_{k-1}} \,,$$

that is equivalent to

$$1 \ge \sum_{k=r+1}^{N} \frac{l_k}{b_{k-1}} \qquad \text{and} \qquad 1 < \sum_{k=r}^{N} \frac{l_k}{b_{k-1}}\,.$$

Hence

$$r = \min \left\{ n \,\middle|\, \sum_{k=n+1}^{N} \frac{l_k}{b_{k-1}} \le 1 \right\}. \tag{2.8}$$

On the other hand by (2.5) and the definition of r, for each $n = 1, \ldots, r-1$, $\gamma_n = \max\{X_n, V_{n+1}\} = V_{n+1}$, which implies that

$$V_1 = V_2 = \ldots = V_r\,.$$

The above equation together with equation (2.7), Lemma 2.1.1 and the optimality of the stopping rule τ, we have

$$\mathbb{P}(Z_\tau = 1) = \mathbb{E}[X_\tau] = \mathbb{E}[\gamma_1] = V_1 = V_r = \frac{b_{r-1}}{b_N} \sum_{k=r}^{N} \frac{l_k}{b_{k-1}}\,.$$

This establishes the following proposition.

Proposition 2.1.3. *Let r be defined as in (2.8). Then the strategy which maximizes the probability of selecting the best applicant is as follows: the manager should reject the applicants in the first $r-1$ groups and accept the best one in the next group which contains the one who is preferable to all his/her predecessors. Under this strategy, the probability of selecting the best applicant is*

$$\frac{b_{r-1}}{b_N} \sum_{k=r}^{N} \frac{l_k}{b_{k-1}}\,.$$

Remark 1. *Note that if $l_k = l$ for all $k = 1, \ldots, N$, then $b_k = \sum_{i=1}^{k} l_i = kl$ and*

$$r = \min \left\{ n \,\middle|\, \sum_{k=n+1}^{N} \frac{1}{k-1} \leq 1 \right\} ,$$

as in the classical problem.

Moreover, in [10] it is proved that, if $l_k = k$ for $1 \leq k \leq N$, then $b_k = \sum_{i=1}^{k} \frac{k(k+1)}{2}$ and

$$r = \min \left\{ n \,\middle|\, \sum_{k=n+1}^{N} \frac{1}{k-1} \leq \frac{1}{2} \right\} .$$

Under the optimal strategy, the probability of selecting the best one is

$$\mathbb{P}(Z_\tau = 1) = \frac{2r(r-1)}{N(N+1)} \cdot \sum_{k=r}^{N} \frac{1}{k-1} \approx \frac{1}{e} ,$$

as in the classical problem.

Chapter 3

The prophet inequality

In the first part of this chapter we are going to analyze the classic prophet inequality, which interest is mainly devoted to optimal stopping rules and pure online algorithms, but also to mechanism design.
First of all, it is necessary to define the framework in which this prophet inequality is considered. The classical setting describes a gambler who is dealing with a sequence of independent, non–negative random variables with finite expectations. Following an indexed order, a value is drawn from each distribution, and after every draw the gambler may choose to accept the value and end the game, or discard the value permanently and continue the game. The gambler is free to choose any stopping rule and she claims a reward equal to the last observation. The objective of the gambler is to achieve the highest reward possible, maximizing the expected value related to it. At this point, one question arises: what is the best performance that a gambler is able to achieve when compared to a prophet who is capable of choosing the highest value?
The performance measure, which is the comparative ratio between the performance of the gambler and the one of the prophet, was deeply analyzed by Krengel, Sucheston and Garling (see [12]): they stated that a gambler who finds herself in

a framework with the same characteristics of the one described above can realize a reward, in expectation terms, that is at least equal to the half of the reward of a prophet who is able to see all the realizations in advance and, for this reason, to choose the largest one.
In [12] Krengel, Sucheston and Garling show a very important result based on online and offline algorithms in Bayesian settings. The Bayesian setting examined is the one in which the online algorithm knows the distribution from which the sequence will be sampled whereas the offline optimum knows the values of the samples themselves and chooses the maximum among them. They showed that a prophet, who can foretell the entire sequence and stop at its maximum value, can gain at most twice as much payoff as a player who must choose the stopping time based only on the current and past observations. In mathematical terms, they showed that, given $X_1, \ldots, X_n$ be a sequence of independent, non–negative, real–valued random variables and $\mathbb{E}\left[\max_i X_i\right] < \infty$, then there exists a stopping rule τ such that

$$\mathbb{E}\left[\max_i X_i\right] \leq 2\mathbb{E}[X_\tau] .$$

This is known as the first of the prophet inequalities in optimal stopping theory and it aims at comparing the performance of online and offline algorithms involved in the selection of one or more elements from a random sequence.

To provide a proof of this prophet inequality for independent random variables, we proceed as in [11].
Let $X_1, \ldots, X_n$ be a sequence of independent, non–negative, real–valued random variables and assume that

$$\mathbb{E}\left[\max_i X_i\right] < \infty . \tag{3.1}$$

Define $T = \mathbb{E}\left[\max_i X_i\right]/2$. We will show that an algorithm that stops at the first time τ such that $X_\tau \geq T$ makes at least T in expectation. If $X_i < T$ for all

$i = 1, \ldots, n$, we assume $\tau = \infty$ and we define $X_\infty := X_n$. Let $p = \mathbb{P}(\max_i X_i \geq T)$. Then for any $x > T$ we have

$$\begin{aligned}
\mathbb{P}(X_\tau > x) &= \sum_{i=1}^{n} \mathbb{P}(X_\tau > x \,|\, \tau = i)\mathbb{P}(\tau = i) + \mathbb{P}(X_\tau > x \,|\, \tau = \infty)\mathbb{P}(\tau = \infty) = \\
&= \sum_{i=1}^{n} \mathbb{P}(X_i > x \,|\, \tau = i)\mathbb{P}(\tau = i) + 0 \cdot \mathbb{P}(\tau = \infty) = \sum_{i=1}^{n} \mathbb{P}(X_i > x \,|\, \tau = i)\mathbb{P}(\tau = i) = \\
&= \sum_{i=1}^{n} \mathbb{P}\left(X_i > x \,|\, \cap_{j=1}^{i-1} \{X_j < T\} \cap \{X_i > T\}\right) \mathbb{P}\left(\cap_{j=1}^{i-1}\{X_j < T\} \cap \{X_i > T\}\right) .
\end{aligned} \tag{3.2}$$

Since the random variables $\{X_i\}_{i=1}^{n}$ are independent, we get

$$\begin{aligned}
\mathbb{P}\left(X_i > x \,|\, \cap_{j=1}^{i-1} \{X_j < T\} \cap \{X_i > T\}\right) &= \mathbb{P}(X_i > x \,|\, X_i > T) = \\
&= \frac{\mathbb{P}(X_i > x, X_i > T)}{\mathbb{P}(X_i > T)} = \frac{\mathbb{P}(X_i > x)}{\mathbb{P}(X_i > T)} .
\end{aligned} \tag{3.3}$$

So by independence of the random variables $\{X_i\}_{i=1}^{n}$ and by (3.3), (3.2) becomes

$$\begin{aligned}
\mathbb{P}(X_\tau > x) &= \sum_{i=1}^{n} \frac{\mathbb{P}(X_i > x)}{\mathbb{P}(X_i > T)} \cdot \mathbb{P}\left(\cap_{j=1}^{i-1}\{X_j < T\} \cap \{X_i > T\}\right) = \\
&= \sum_{i=1}^{n} \frac{\mathbb{P}(X_i > x)}{\mathbb{P}(X_i > T)} \cdot \mathbb{P}(X_i > T) \cdot \mathbb{P}\left(X_j < T \text{ for } j = 1, \ldots, i-1\right) = \\
&= \sum_{i=1}^{n} \mathbb{P}(X_i > x) \cdot \mathbb{P}\left(X_j < T \text{ for } j = 1, \ldots, i-1\right) .
\end{aligned} \tag{3.4}$$

Note that the event $\{\max_i X_i < T\}$ implies the event $\{X_j < T \text{ for } j = 1, \ldots, i-1\}$. So we have

$$\mathbb{P}\left(X_j < T \text{ for } j = 1, \ldots, i-1\right) \geq \mathbb{P}\left(\max_i X_i < T\right) = 1 - p \,,$$

by definition of p. Hence (3.4) becomes

$$\mathbb{P}(X_\tau > x) \geq \sum_{i=1}^{n} \mathbb{P}(X_i > x) \cdot (1 - p) = (1 - p) \sum_{i=1}^{n} \mathbb{P}(X_i > x) \,. \tag{3.5}$$

Moreover

$$\mathbb{P}\left(\max_i X_i > x\right) = \mathbb{P}(\exists\, i \text{ such that } X_i > x) = \mathbb{P}\left(\cup_{i=1}^n \{X_i > x\}\right) \leq \sum_{i=1}^n \mathbb{P}(X_i > x)\,.$$

Then (3.5) becomes

$$\mathbb{P}(X_\tau > x) \geq (1-p)\mathbb{P}\left(\max_i X_i > x\right)\,.$$

Note that if $x \leq T$, then

$$\begin{aligned}\mathbb{P}(X_\tau > x) &= \sum_{i=1}^n \mathbb{P}(X_\tau > x \,|\, \tau = i)\mathbb{P}(\tau = i) + \mathbb{P}(X_\tau > x \,|\, \tau = \infty)\mathbb{P}(\tau = \infty) = \\ &= \sum_{i=1}^n 1 \cdot \mathbb{P}(\tau = i) + \mathbb{P}(X_\tau > x \,|\, \tau = \infty)\mathbb{P}(\tau = \infty) \\ &\geq \sum_{i=1}^n \mathbb{P}(\tau = i) + 0 = \mathbb{P}(\tau < \infty) = \mathbb{P}\left(\max_i X_i \geq T\right) = p\,.\end{aligned} \tag{3.6}$$

Moreover applying Lemma A.1

$$\begin{aligned}2T = \mathbb{E}\left[\max_i X_i\right] &= \int_0^T \mathbb{P}\left(\max_i X_i > x\right)\,dx + \int_T^{+\infty} \mathbb{P}\left(\max_i X_i > x\right)\,dx \\ &\leq \int_0^T 1\,dx + \int_T^{+\infty} \mathbb{P}\left(\max_i X_i > x\right)\,dx = T + \int_T^{+\infty} \mathbb{P}\left(\max_i X_i > x\right)\,dx\end{aligned}$$

and so

$$2T \leq T + \int_T^{+\infty} \mathbb{P}\left(\max_i X_i > x\right)\,dx\,,$$

that is

$$T \leq \int_T^{+\infty} \mathbb{P}\left(\max_i X_i > x\right)\,dx\,.$$

Hence, applying (3.6) and Lemma A.1, we get

$$\begin{aligned}\mathbb{E}[X_\tau] &= \int_0^{+\infty} \mathbb{P}(X_\tau > x)\,dx = \int_0^T \mathbb{P}(X_\tau > x)\,dx + \int_T^{+\infty} \mathbb{P}(X_\tau > x)\,dx \\ &\geq \int_0^T p\,dx + \int_T^{+\infty} (1-p)\mathbb{P}\left(\max_i X_i > x\right)\,dx \geq pT + (1-p)T = \\ &= T = \frac{1}{2}\mathbb{E}\left[\max_i X_i\right]\,.\end{aligned}$$

So we have obtained the following result.

Theorem 3.0.1 (Prophet inequality for independent random variables)**.** *Let* $\{X_i\}_{i=1}^n$, T *and* τ *be defined as above and assume that* (3.1) *holds. Then*

$$\mathbb{E}\left[\max_i X_i\right] \leq 2\mathbb{E}[X_\tau]\,.$$

The constant 2 that appears in Theorem 3.0.1 is the optimal constant, that is it is not possible to find a constant $C \in (0,2)$ for which $\mathbb{E}\left[\max_i X_i\right] \leq C\mathbb{E}[X_\tau]$ for any stopping rule τ. To prove it, suppose to have $n = 2$ (that is, we work with two random variables X_1 and X_2). Given $\varepsilon \in (0,1)$, we define $X_1 := 1$ and

$$X_2 := \begin{cases} \frac{1}{\varepsilon}\,, & \text{with probability } \varepsilon\,, \\ 0, & \text{with probability } 1-\varepsilon. \end{cases}$$

Note that $\mathbb{E}[X_2] = \mathbb{E}[X_1] = 1$. Hence, under any stopping rule τ, we have $\mathbb{E}[X_\tau] = 1$. Moreover

$$\max\{X_1, X_2\} = \begin{cases} \frac{1}{\varepsilon}\,, & \text{with probability } \varepsilon\,, \\ 1\,, & \text{with probability } 1-\varepsilon\,. \end{cases}$$

Hence

$$\mathbb{E}[\max\{X_1, X_2\}] = 2 - \varepsilon\,.$$

So when $\varepsilon \to 0$ we get

$$\mathbb{E}[\max\{X_1, X_2\}] = 2 = 2 \cdot 1 = 2 \cdot \mathbb{E}[X_\tau]\,.$$

3.1 A prophet inequality with cost for observations

In the previous section, we have analyzed the ratio prophet inequality for independent random variables $\{X_i\}_{i=1}^n$, without considering any cost of sampling. In [15] such a cost is introduced with the additional hypothesis that all the random variables $\{X_i\}_{i=1}^n$ are i.i.d. and take values in $[0,1]$. In this particular case, the comparison is made between the performance of a prophet with complete foresight and the performance of a gambler, who uses a strategy made of non–anticipating stopping times, observing a sequence of i.i.d., real–valued random variables with a non–negative fixed cost charged for each observation.

Given the sequence $\{X_i\}_{i=1}^n$, we define $Y_i := X_i - ic$ for $i = 1, \dots, n$, where $c > 0$ represents the cost of each observation. As in [15] we want to prove the *difference prophet inequality*, that we state in the following theorem.

Theorem 3.1.1. *Let $\{X_i\}_{i=1}^n$ be a sequence of i.i.d. with $0 \leq X_i \leq 1$. Define $Y_i = X_i - ic$ and*

$$V(Y_1, \dots, Y_n) = \sup\{\mathbb{E}[Y_\tau] \,:\, \tau \textit{ is a stopping rule}\}\,.$$

The following inequalities hold:

(a) fixed $0 < c \leq 1$ and a positive integer n, we have

$$\mathbb{E}\left[\max_{1\leq i\leq n} Y_i\right] - V(Y_1, \dots, Y_n) \leq \left\lfloor \frac{1}{c} \right\rfloor \; c \;\, (1-c)^{\lfloor \frac{1}{c} \rfloor + 1}\,, \tag{3.7}$$

where for any $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the largest integer smaller than x.

(b) fixed $c \geq 0$ and a positive integer n, we have

$$\mathbb{E}\left[\max_{1\leq i\leq n} Y_i\right] - V\left(Y_1, \dots, Y_n\right) \leq \left(1 - \frac{1}{n}\right)^{n+1}\,. \tag{3.8}$$

(c) fixed $c \geq 0$ and for any positive integer n (possibly $n = +\infty$), we have

$$\mathbb{E}\left[\sup_{1\leq i\leq n} Y_i\right] - V\left(Y_1, Y_2, \dots, Y_n\right) \leq e^{-1}. \tag{3.9}$$

All bounds are the best possible.

This theorem shines the light on three different cases, which depend on the different restrictions which can be assigned on the cost and the length of the sequence, and states the best possible bound that can be applied to each situation.

Remark 2. *Note that in Theorem 3.1.1 it is enough to consider the case $c \leq 1$ since for $c > 1$ the maximum and the optimal stopping value are obtained for $n = 1$ and the difference is 0.*

3.1.1 Proof of Theorem 3.1.1

We can assume $n \geq 2$, otherwise the inequalities are trivial. Chow, Robbin and Siegmund in [2] have shown that an optimal stopping rule is given by

$$s = \inf\{i \mid X_i \geq \beta\}\,, \tag{3.10}$$

where β is the unique value for which $\mathbb{E}\left[\max\{X_1 - \beta, 0\}\right] = c$. Moreover they show that $\mathbb{E}[Y_s] = \beta$.

Define $s_n = \min\{s, n\}$. Our first goal is to maximize $\mathbb{E}\left[\max_{1\leq i\leq n} Y_i\right] - \mathbb{E}\left[Y_{s_n}\right]$. We will show that this difference is always less or equal to the right-hand side of (3.7) and (3.8). Actually it is possible to show that (3.7) and (3.8) become identities when $\{X_i\}_{i=1}^n$ are particular Bernoulli random variables (see [15]). For this reason we restrict to consider the case in which $\{X_i\}_{i=1}^n$ are i.i.d. Bernoulli random variables of parameter $p = c/(\beta - 1)$.

The following lemma is a crucial result that we will use many times in the proof of Theorem 3.1.1.

Lemma 3.1.2.

$$\mathbb{E}[Y_{s_n}] = \beta - (1-u)^n(\beta - \mathbb{E}[X_1 \mid X_1 < \beta])\,,$$

where $u = \mathbb{P}(X \geq \beta)$.

Proof. We know that $\mathbb{E}[\max\{X_1 - \beta, 0\}] = c$. So

$$\begin{aligned} c &= \mathbb{E}[\max\{X_1 - \beta, 0\}] = \\ &= \mathbb{E}[X_1 - \beta \mid X_1 \geq \beta]\mathbb{P}(X_1 \geq \beta) + \mathbb{E}[0 \mid X_1 < \beta]\mathbb{P}(X_1 < \beta) = \\ &= (\mathbb{E}[X_1 \mid X_1 \geq \beta] - \beta)\mathbb{P}(X_1 \geq \beta)\,, \end{aligned}$$

that is, defining $u = \mathbb{P}(X_1 \geq \beta)$

$$\mathbb{E}[X_1 \mid X_1 \geq \beta] - \beta = \frac{c}{u}\,.$$

Applying this identity, the thesis follows straightforwardly. □

Define $r = \sup\{i \,:\, 1 - ic > -c\}$, that is $r = 1 + \lfloor 1/c \rfloor$. For all $i > r$ we have $Y_i \leq 1 - ic < -c \leq 0$. Let

$$D_n = \mathbb{E}[\max_{1\leq i\leq n} Y_i] - \mathbb{E}[Y_{s_n}]\,.$$

Proposition 3.1.3. *Fixed c and p, for all $n \geq r$ we have*

$$D_n \leq D_r = (1-p)^r(r-1)c\,. \tag{3.11}$$

Proof. For $n \geq r$, we have

$$\begin{aligned}\mathbb{E}\left[\max_{1\leq i\leq n} Y_i\right] &= 1 - (1-p)^r - c\sum_{i=1}^{r} ip(1-p)^{i-1} - c(1-p)^r = \\ &= \beta + (1-p)^r(c(r-1) - \beta)\,,\end{aligned}$$

where in the last equality we have used the definition of p and the identity

$$\sum_{i=1}^{r} ip(1-p)^{i-1} = \frac{1-(1-p)^{r+1}}{p} - (r+1)(1-p)^r\,.$$

Applying then Lemma 3.1.2 and observing that Y_{s_n} is non-decreasing in n yields the thesis. □

Proposition 3.1.4. *Fixed c, for all p and all n we have*

$$D_n \leq D_r = \lfloor 1/c \rfloor c(1-c)^{\lfloor 1/c \rfloor + 1}\,. \tag{3.12}$$

Proof. Let c and r be fixed. For $n \geq r$ it follows that (3.11) is maximized when p is minimized, that is when $\beta = 0$ and $p = c$. With such a choice of p the right-hand side of (3.11) becomes the right-hand side of (3.12).

For $n < r$, we have

$$\mathbb{E}\left[\max_{1\le i\le n} Y_i\right] = \beta + (1-p)^n[c(n-1)-\beta]$$

and hence

$$D_n = (1-p)^n c(n-1)\,.$$

Note that when $n = 1$, we have $D_n = 0$ and for $n > 1$ we have

$$\frac{D_{n+1}}{D_n} > 1 \iff n < \frac{1}{p}\,.$$

Since $\frac{1}{p} = \frac{1}{c} > \lfloor \frac{1}{c} \rfloor = r-1 \ge n$, we have $D_n < D_r$ for all $n < r$, that is the thesis. □

Since the case $\beta < 0$ is of no concern, (3.12) implies (3.7). Moreover (3.7) holds also when $n = \lceil 1/c \rceil + 1$ and X_i is a Bernoulli random variable of parameter c.

The same argument can be used to prove (3.8). For $p = c$ we can rewrite (3.11) as

$$D_n \le D_r = (1-p)^r p(r-1) \qquad \text{for all } n\,. \tag{3.13}$$

Since $r = \lceil 1/c \rceil + 1$, we have $1/r \le p < 1/(r-1)$. Moreover, $(1-p)^r p$ is increasing for $p \in [0, 1/(r+1)]$ and decreasing for $p \in [1/(r+1), 1]$. Hence the right-hand side of (3.13) is therefore maximal for $p = 1/r$ and assumes value $(1-1/r)^{r+1}$, which is the right hand-side of (3.8) for $n = r$. Since $(1-1/n)^{n+1}$ increases to e^{-1}, $\mathbb{E}\left[\sup_{i\ge 1} Y_i\right] = \mathbb{E}\left[\max_{1\le i\le n} Y_i\right]$ for n large and by Lemma 3.1.2 $V(Y_1, Y_2, \ldots, Y_n) \le \beta = V(Y_1, Y_2, \ldots)$, we have (3.9).

To complete the proof of the theorem it remains to consider the case $\beta < 0$. Hence by (3.10), we have $s = 1$ and hence $\mathbb{E}[Y_{s_n}] = \beta$ for all n. Moreover $\max\{X_1 - \beta, 0\} = X_1 - \beta$ and hence $\mathbb{E}[\max\{X_1 - \beta, 0\}] = c$ implies $\mathbb{E}[X_1] = \beta + c$. Since X_1 is a Bernoulli random varioable of parameter p, we have $\beta + c = p$. Thus for all $n \le r$ it holds

$$\mathbb{E}\left[\max_{1\le i\le n} Y_i\right] = \beta/p + (1-p)^n[c(n-1) - \beta/p]$$

and for $n \geq r$ we have

$$\mathbb{E}\left[\sup_{i\geq 1} Y_i\right] = \mathbb{E}\left[\max_{1\leq i\leq n} Y_i\right] = \mathbb{E}\left[\max_{1\leq i\leq r} Y_i\right] .$$

Thus for $n \leq r$

$$D_n = \beta(1-p)/p + (1-p)^n[(p-\beta)(n-1) - \beta/p] ,$$

$D_n = D_r$ for $r \leq n$ and $D_1 = 0$. Hence it is sufficient to consider $n \leq r$. Fixed p we can write $D_n = \beta A_n + B_n$, where $A_n > 0$ and B_n do not involve β. Hence D_n is increasing in β, that is, for $\beta \leq 0$, D_n is maximal when $\beta = 0$. This completes the proof of Theorem 3.1.1.

Chapter 4

An economic view of the prophet inequality and the secretary problem

4.1 Connection between pricing and the prophet inequality

The optimal stopping theory gives us, among many things, the evidence of a connection between post-priced mechanisms and prophet inequality. The existence of such connection began to receive attention only two decades ago, mainly because of how it could be applied to make the allocation of resources as efficient as possible. In [13] Lucier gives us an overview of this connection through an economic-oriented proof of the prophet inequality. In fact, he goes over the mere direct applications to pricing and mechanism design and firmly believes that an economic perspective could be useful to make the prophet inequality a tool for stochastic optimization and online algorithms.

In his article, Lucier defines Hajiaghayi (see [8]) as the one who reintroduced the prophet inequality into the community of economics and computations, and the one who first took care to study the analogy between the latter and a simple pricing problem. As in [8], the analogy is characterized by a setting where there are k identical indivisible goods, or units, for sale, and there are n agents, or bidders, each of whom wants to purchase one unit. Each agent i, for $0 \leq i \leq n$, is defined by three components that are called arrival time a_i, departure time d_i and value v_i. We assume that if an agent receives one or more units during the time interval $[a_i, d_i]$ with a payment p_i, her utility for this allocation is $v_i - p_i$; for all other allocations her utility is 0. The value the agent assigns to the good is drawn from a distribution D_i which is known to the seller and can vary among bidders. The agent and the seller can negotiate using an arbitrary protocol, which leads to the decision to sell or not and, if yes, at what price. To evaluate the performance of an online mechanism, two measures of solution quality are used: efficiency and revenue. The efficiency of an output is defined as the combined welfare of all the agents, that is $\sum_i q_i v_i$, where q_i is an allocation rule. The revenue is the sum of the payments made by the agents, that is $\sum_i p_i$. A mechanism is defined as ρ–competitive with respect to efficiency if the expected efficiency of the outcome computed by the mechanism is at least $1/\rho$ times the expectation of the maximum efficiency over all outcomes. On the other hand, we say that the mechanism is ρ–competitive with respect to revenue if the expected revenue of the outcome computed by the mechanism is at least $1/\rho$ times the expectation of the maximum revenue that can be obtained by setting a single fixed price p and selling to all agents whose value is at least p.

We need a direct-revelation mechanism which is truthful, that is strategyproof, in dominant strategies, which means that the utility of an agent i is maximized if she bids truthfully, regardless of what other agents report.

In order to do that, agents are considered as treasure chests: in this way, the prophet inequality can be applied to establish a sales protocol to guarantee at least half of the optimal gains from trade, which is social welfare, in expectation. In particular, the prophet inequality makes sure that each bidder's value is completely revealed to the seller upon arrival. The method would require to accept the first buyer who's value exceeds a fixed threshold (see [14]).

Now, as in [13], we want to prove that there exists a threshold policy that gives, in expectation, at least half of the expected maximum value. First of all, we indicate as V^* the random variable that has $\max_i v_i$ as value and which is the maximum of the n realized prizes. We consider a threshold policy that accepts the first prize whose value exceeds $\frac{1}{2}\mathbb{E}[V^*]$, if any, where the expectation is over the realizations of the prizes. We want to show that the expected prize generated by this policy is at least $\frac{1}{2}\mathbb{E}[V^*]$. This is a direct consequence of the Theorem 3.0.1 in Chapter 3, but, as in [13], we give below another proof of this fact in terms of the quantities introduced up to now.

Theorem 4.1.1. *The policy that accepts the first prize that is at least* $\frac{1}{2}\mathbb{E}[V^*]$ *has expected reward* $\frac{1}{2}\mathbb{E}[V^*]$. *This is true regardless of the decision made when a prize is equal to* $\frac{1}{2}\mathbb{E}[V^*]$.

Proof. The threshold policy we take into account is the one setting $p = \frac{1}{2}\mathbb{E}[V^*]$ on the indivisible good that we had in [8]. Then, we adjust the setting as follows. The expected revenue of this policy is simply equal to

$$\frac{1}{2}\mathbb{E}[V^*]\mathbb{P}(\text{item is sold})\,. \tag{4.1}$$

The expected utility of agent i is at least

$$(v_i - p)^+ := \max\{v_i - p, 0\} \tag{4.2}$$

In the event {item is not sold before buyer i has a chance to purchase}. So the expected agent's surplus will be at least equal to

$$\sum_i \mathbb{E}[(v_i - p)^+]\mathbb{P}(i \text{ has a chance to purchase}). \tag{4.3}$$

Note that if the item is left unsold, then every customer has a chance to purchase it. So

$$\mathbb{P}(\text{item is unsold}) \leq \mathbb{P}(i \text{ has a chance to purchase})\,,$$

and hence the summation in (4.3) is at least

$$(\sum_i \mathbb{E}[(v_i - p)^+])\mathbb{P}(\text{item is unsold})\,. \tag{4.4}$$

Note that

$$\sum_i \mathbb{E}[(v_i - p)^+] \geq \mathbb{E}[\max_i (v_i - p)^+] \geq \mathbb{E}[\max_i v_i] - p \geq \frac{1}{2}\mathbb{E}[V^*]$$

and hence (4.4) becomes

$$\frac{1}{2}\mathbb{E}[V^*]\ \mathbb{P}(\text{item is unsold})\,. \tag{4.5}$$

This means that, summing (4.1) (that is the expected revenue) and (4.5) (that is the expected agent's surplus), the expected welfare which comes from this sales mechanism is at least $\frac{1}{2}\mathbb{E}[V^*]$, as we wanted to prove. □

The value $\frac{1}{2}\mathbb{E}[V^*]$ represents the equilibrium price that preserves the creation of welfare: setting a lower price would be a disadvantage for higher-valued agents because lower-valued agents would be more likely to purchase the good before them, while setting a higher price increases the probability of not selling the item at all; both cases are covered by revenue and agent surplus.
In [13] Lucier underlines the comparison of the prophet inequality with market–clearing price. Considering the same setting as before but in a deterministic, full–information version, the optimal price to set would be obviously $\max_i v_i$, which,

supposing all customers are indifferent, gives the fully efficient outcome. Compared to the prophet inequality price above, we know that the latter is the best approximation to the optimal welfare but it does not completely guarantee the complete satisfaction of each customer as it is characterized by the uncertainty of the market.

4.2 The secretary problem for a random walk: connection with the stock prices

In this section, we look at developing and testing models for stock price behavior. There is one important approach, which is the theory of random walks, that has disrupted the other two important theories that were significantly considered in this research: they were the chartist or technical analysis theory and the theory of fundamental intrinsic value analysis. Briefly, chartists generally believe that price movements in a security are not random but can be predicted through a study of past trends and other technical analysis, while the theory of fundamental intrinsic value analysis believes that an individual security has an intrinsic value, also known as equilibrium price, that basically depends on the earning potential of the security which, in turn, depends on several factors, both tangible and intangible, such as business models, management quality, governance and other market factors. Both these theories are based on the assumption that there is a systematic behavior in price series and that past patterns can be used to study and predict the behavior of present and future price series.
In [4] Fama has analyzed how the competition that characterizes an efficient market makes the full effects caused by new information of intrinsic value, to be instantaneously reflected on actual prices. This implies that successive price changes in individual securities will be independent, which means that the market is, by defi-

nition, a random walk market. The theory of random walks firmly sustains that a series of stock price changes has no memory, which implies that past series cannot be used to predict the behaviors of stock prices. The independence assumption that underlies this theory is valid as long as the past series of stock price changes cannot be used to increase profitability: in fact, as long as successive price changes for a security are independent, a simple policy of buying and holding the security will be as good as any complicated mechanism for timing purchases and sales.
This theory can be proved following two different approaches:

a) Relying on common statistical tools as serial correlation coefficients and analysis of runs of consecutive price changes of the same sign. If this statistical test supports the assumption of independence, it means that there are no mechanical trading rules or chartist techniques based on past patterns that can make higher expected profits to investors.

b) Testing directly different mechanical trading rules to observe if they give higher profits that the simple policy of buying and holding.

These approaches tend to confirm the theory of random walks, considering also the fact that there is no evidence of dependence in successive price changes.

The analysis by Fama is the basis for the problem that arises in the paper by Hlynka and Sheahan (see [9]). In this article there is a stock analyst who is asked to pick the day in which a particular stock will be the highest during a given month, immediately notifying his view to the client. If the analyst has taken the best decision on which day to pick, he receives, as reward, a major portfolio to manage, otherwise he receives nothing. If stock prices behave independently during the month, the random walk theory comes into play.
In this paper, all values represent positions of a generalized one-dimensional ran-

dom walk. The objective is to find a strategy that maximizes the probability of picking the largest value in n steps with no recall allowed. In this framework, the secretary problem can be applied to find the best strategy.

Definition 4.1. *For $i = 1, \ldots, n$, let $\{X_i\}_{i=1}^n$ be i.i.d. random variables. Take $Y_0 = 0$ and let $Y_i = Y_{i-1} + X_i$. The process $\{Y_i\}_{i=1}^n$ is called random walk and we define $Y^* = \max\{Y_i : 0 \le i \le n\}$ the maximum value that it assumes.*

We will not discuss the properties of random walks and more details can be found in [3]. Our objective is to find a strategy S that maximizes the probability of picking Y^*. More precisely, calling $Y^\sharp$ the picked value under strategy S, we want to choose such a strategy to maximize the probability $\mathbb{P}(Y^\sharp = Y^*)$.

Definition 4.2. *Let S_0 be the strategy that picks $Y_0 = 0$ as its choice for Y^*. For $0 < k \le n$, let S_k be the strategy which examines $Y_0, \ldots, Y_{k-1}$ and picks the next Y_i (for $i \ge k$) such that $Y_i > \max\{Y_0, \ldots, Y_{k-1}\}$. If such Y_i does not exist, the value Y_n is picked.*

In the following we will assume that $\{X_i\}_{i=1}^n$ are discrete random variables with symmetric distribution. More precisely, we make the following assumption.

Assumption 1. *For $i = 1, 2, \ldots, n$ and $m = 0, 1, \ldots, \infty$, let $\{X_i\}_{i=1}^n$ be i.i.d. discrete random variables such that*

$$\mathbb{P}(X_i = m) = \mathbb{P}(X_i = -m) = p(m)\,,$$

that is, the distribution of X_i is symmetric about 0.

Actually in [9] is considered also the case of continuous distribution, but we restrict here only to the discrete case for simplicity.

Proposition 4.2.1. *Let $Y^\sharp$ be the picked value under strategy S_k. Then $\mathbb{P}(Y^\sharp = Y^*)$ is independent on the index k, that is $\mathbb{P}(Y^\sharp = Y^*)$ is the same for all $k = 0, \ldots, n$.*

Proof. We give here a sketch of the proof (we refer to [9] for more details). Define $Y_{i,k} := Y_i$ in order to underline that we use the strategy S_k on the sequence $\{Y_{i,k}\}_{i=0}^n$. Moreover define $Y_k^* := \max\{Y_{i,k} \,|\, 0 \le i \le n\}$ and let $Y_k^\sharp$ be the value picked from $\{Y_{i,k}\}_{i=0}^n$ using the strategy S_k. It is possible to construct a bijective function φ that maps the sequence $\{Y_{i,k}\}_{i=1}^n$ into a sequence $\{Y_{i,0}\}_{i=1}^n$ with the following properties:

(i) $\{Y_{i,0}\}_{i=1}^n$ is a permutation of $\{Y_i\}_{i=1}^n$;

(ii) $Y_k^* - Y_k^\sharp = Y_0^* - Y_0^\sharp$, that is the distance between the actual maximum of the sequence and the "guess" of the maximum of the sequence for the respective strategies is preserved. Obviously, by item (i) above, we have $Y^* = Y_k^* = Y_0^*$.

Property (ii) above together with Assumption 1 assure that $\{Y_{i,0}\}_{i=1}^n$ and $\{Y_{i,k}\}_{i=1}^n$ are equally probable and $\mathbb{P}(Y_k^\sharp = Y_k^*) = \mathbb{P}(Y_0^\sharp = Y_0^*)$. Since the index k can vary among 0 and n, we have that $\mathbb{P}(Y_k^\sharp = Y^*) = \mathbb{P}(Y_0^\sharp = Y^*)$ for all $k = 0, \ldots, n$, that is the thesis. □

Proposition 4.2.2. *If Assumption 1 holds, then S_0 is an optimal strategy.*

Proof. To prove what this proposition states, we refer to [7]. Consider the element $Y^\sharp$ picked under strategy S_k from the sequence $\{Y_i\}_{i=1}^n$. This problem can be thought of as a game and the following approach can be applied:

(i) The event "$Y_i = Y^*$" HO SISTEMATO VIRGOLETTE, which means that the element Y_i is equal to the maximum of the sequence $\{Y_i\}_{i=1}^n$, is considered as "win with Y_i";

(ii) The event "$Y^\sharp = Y^*$", that states that the value picked under strategy S_k is the maximum of the sequence $\{Y_i\}_{i=1}$, is defined as a "win".

Then we define

$$g(i){:=}\mathbb{P}(\text{win with } Y_i \,|\, Y_i \geq Y_0, Y_1, \dots, Y_{i-1}),$$

which refers to the conditional probability that Y_i is the maximum of the sequence given that it is greater than or equal to the values observed before.
Moreover we define

$$h(i){:=}\mathbb{P}(\text{win with best strategy from } i+1 \text{ on} \,|\, Y_i \geq Y_0, Y_1, \dots, Y_{i-1}),$$

which reflects the conditional probability of winning with the best strategy waiting for an other element of the set that comes after Y_i, given that this one is grater or equal to all the values observed. Moreover, we know that $g(0) = \mathbb{P}(\text{win with } Y_0)$ and that $h(n) = 1$, which means that the win with the best strategy is ensured when choosing the n-th element of the sequence, knowing that $Y_n \geq Y_0, \dots, Y_{n-1}$. Following this proof, we consider $h(i)$ as a nondecreasing function (see [9]) and not as an increasing function as in [7].
We analyze the following two cases:

- $g(i) > h(i)$
- $g(i) = h(i)$

In the first case, the best strategy would be choosing Y_i as $Y^\sharp$. In the second case, as the probability of the two events are equal, the player is indifferent between taking Y_i as $Y^\sharp$ or just waiting for an other element of the set $\{Y_i\}_{i=1}^n$, such as i', which satisfies $g(i') \geq h(i')$.
From this, we know that the strategy S_0, which involves picking Y_0 as $Y^\sharp$, can be defined as the best strategy if we are able to show that $g(0) \geq h(0)$. We start

from defining for $0 \leq r \leq n$

$$u(r) = \mathbb{P}(Y_0 \text{ is a maximum for the random walk sequence } Y_0, \ldots, Y_r)\,,$$

which basically expresses the probability that S_0 is the best strategy. From the previous definitions we derive that $g(n-1) = h(n-1)$, which means that, when considering the second-last element of the sequence, the player is indifferent between choosing Y_{n-1} as $Y^\sharp$ or waiting for the next and last element, and that $g(n) = h(n) = 1$, that is, the two events are equivalent and assured when taking the last element of the sequence Y_n.

Then, we choose from $\{Y_i\}_{i=1}^n$ an element Y_k for $0 \leq k \leq n$, to be the subject of the two functions previously introduced, in order to analyze $g(k)$ and $h(k)$. Furthermore, define $W_i := Y_{k+i} - Y_k$ which represents the difference between the i-th element of the sequence $Y_k, Y_{k+1}, \ldots, Y_n$ and the element Y_k. Specifically, we are going to use a random walk sequence that goes from W_0 to W_{n-k}, where the first element is $W_0 = Y_{k+0} - Y_k = 0$ and the last one is $W_{n-k} = Y_n - Y_k$. Then, we have

$$\begin{aligned} g(k) &= \mathbb{P}(\text{win with } Y_k \,|\, Y_k \geq Y_0, \ldots, Y_{k-1}) \\ &= \mathbb{P}(Y_k \text{ is a maximum for the sequence } Y_k, \ldots, Y_n) \\ &= \mathbb{P}(W_0 \text{ is the maximum for the sequence } W_0, \ldots, W_{n-k}) \\ &= u(n-k)\,, \end{aligned}$$

$$\begin{aligned} h(k) &= \mathbb{P}(\text{win with the best strategy from } k+1 \text{ on} \,|\, Y_k \geq Y_0, \ldots, Y_{k-1}) \\ &= \mathbb{P}(\text{win with the best strategy from 1 on in the sequence } W_0, \ldots, W_{n-k}) \\ &\geq \mathbb{P}(\text{win by picking the last element of the sequence } W_0, \ldots, W_{n-k}) \\ &= \mathbb{P}(\text{win by picking the first element } W_0 \text{ of the sequence } W_0, \ldots, W_{n-k}) \\ &= u(n-k)\,. \end{aligned}$$

The second last equality comes from 4.2.1.

Surely, from this derives that $g(k) \leq h(k)$, for $0 \leq k \leq n$, but this does not prove that S_0 would be an optimal strategy. By the way, such an inequality implies that it would be better to wait for the last value and choose it. Hence the above inequality is actually an equality, that is $h(k) = u(n-k) = g(k)$, and hence S_0 is an optimal strategy. □

The following corollary is a direct consequence of Proposition 4.2.1 and Proposition 4.2.2.

Corollary 4.2.3. *If Assumption 1 holds, then S_k is an optimal strategy.*

Remark 3. *Note that the random walk $\{Y_i\}_{i=1}^n$ is a martingale with respect to the filtration $\{X_i\}_{i=1}^n$. Indeed, since $\{X_i\}_{i=1}^n$ are i.i.d. random variables and Y_i is known when conditioning to $X_1, \ldots, X_i$, we have*

$$\mathbb{E}[Y_{i+1} \mid X_1, \ldots, X_i] = \mathbb{E}[Y_i + X_{i+1} \mid X_1, \ldots, X_i] = Y_i + \mathbb{E}[X_{i+1}] = Y_i\,,$$

where, in the last equality, we have used the symmetry of the probability distribution of X_{i+1}. This implies that $\mathbb{E}[Y^*] = \mathbb{E}[Y_0] = 0$ and $\mathbb{E}[Y^\sharp] = \mathbb{E}[Y_0] = 0$.

If the distribution of X_1 is such that $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 0.5$, then we can compute the probability of picking the maximum of the random walk $Y_0, \ldots, Y_n$ for even values of n. More precisely we have the following result.

Proposition 4.2.4. *Let n be an even positive integer. If Assumption 1 holds with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 0.5$, then, for all $k = 0, \ldots, n$, using strategy S_k*

$$\mathbb{P}(Y^\sharp = Y^*) = \binom{n}{n/2} \frac{1}{2^n} \underset{n\to\infty}{\sim} \sqrt{\frac{2}{\pi n}}\,.$$

Proof. We know that $\mathbb{P}(Y^\sharp = Y^*)$ expresses the probability that the value picked under strategy S_k is the maximum value of the sequence $\{Y_i\}_{i=1}^n$. Moreover by Proposition 4.2.1 we know that such a probability does not depend on k and hence, since $Y^\sharp = Y_0 = 0$ when $k = 0$, we have

$$\mathbb{P}(Y^\sharp = Y^*) = \mathbb{P}(Y^* = 0)\,.$$

Since Y^* is the maximum value of the random walk, if $Y^* = 0$ we have that all the Y_i are smaller or equal to zero, that is

$$\mathbb{P}(Y^* = 0) = \mathbb{P}(Y_i \leq 0 \text{ for all } i = 1, \ldots, n)\,.$$

The thesis follows by Lemma A.2. □

The maximum of a random walk sequence is more probable to occur near the endpoints rather than near the middle (see page 94 in [5]). This means that a stock analyst can predict that the maximum price of the month will occur on the first day of the month itself. By Proposition 4.2.4, if $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 0.5$, and for the market being opened for 30 days, then the probability of the analyst being correct is about 0.15 and there is no other strategy for him to improve the probability of success. Moreover, this proposition tells us that, although all strategies are equivalent, this does not mean all elements have the same probability to be chosen.

The results discussed in this section tell us that choosing the first value of a random walk sequence is an optimal strategy when we are trying to maximize the probability of choosing the largest element of the sequence with no recall allowed. This probability is greater than $\frac{1}{n}$ (see Proposition 4.2.4) and this is due to the considerations in [5] about the maximum of a random walk. Looking at this result, we immediately notice that it is different from the result of the "classical" secretary problem, in which the probability of success tends to $\frac{1}{e}$ as $n \to \infty$.

Appendix

In this appendix we list some technical basic results used in the previous chapters.

Lemma A.1. *If X is a non-negative and real-valued random variable. Then*

$$\mathbb{E}[X] = \int_0^{+\infty} \mathbb{P}(X > x)\, dx\,.$$

Lemma A.2. *Let n be a positive even integer. Let $\{X_i\}_{i=0}^n$ be a sequence of i.i.d. random variables with*

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}\,,$$

and define $Y_i = Y_0 + \sum_{j=0}^{i-1} X_j$ for $i = 0, \dots, n$, with $Y_0 = 0$. Then

$$\mathbb{P}(Y_i \leq 0 \text{ for } i = 0, \dots, n) = \binom{n}{n/2}\frac{1}{2^n} \underset{n\to\infty}{\sim} \sqrt{\frac{2}{\pi n}}\,.$$

Proof. We follow [5] for the proof. The outcomes of the X_i can be thought as prizes associated to a sequence of tossings of a fair coin realized: at time i a fair coin is tossed and in case of head we win 1 unit, while in case of tail we loose 1 unit. Hence Y_i represents what we gain in the first i tossings.

Now we analyze the event $\{Y_n = r\}$, that is at time n we have r units. Define $p_{n,r} := \mathbb{P}(Y_n = r)$. We want to compute such a probability. Denoting by p and q, respectively, the number of heads and tails obtained in the first n tossings, we have that

$$Y_n = r \iff \begin{cases} p - q = r\,, \\ p + q = n\,. \end{cases}$$

So we get $p = \frac{n+r}{2}$ and $q = \frac{n-r}{2}$. So the number of paths that start with Y_0 and arrive with $Y_n = r$ are given by

$$N_{n,r} := \binom{n}{\frac{n+r}{2}}\,,$$

where the binomial coefficient is due to the fact that we have to count the number of ways in which we choose the tossings that give head (and consequently the ones that give tail). Note now that each path $Y_0, \ldots, Y_n$ is given obtained through the sequence $(X_1, \ldots, X_n)$ that has 2^n possible outcomes (since each X_i can assume values ± 1). Hence, fixed a path for $Y_0, \ldots, Y_n$, such a path has probability $\frac{1}{2^n}$ to occur. Then we have that

$$p_{n,r} = N_{n,r} \cdot \frac{1}{2^n} = \binom{n}{\frac{n+r}{2}} \cdot \frac{1}{2^n}.$$

where the binomial coefficient is zero if $(n+r)/2 > n$ (that is $r > n$). So in particular we have

$$p_{n,0} = \binom{n}{\frac{n}{2}} \cdot \frac{1}{2^n}\,.$$

Applying equation (3.4) in [5] and noting that the distribution of X_i is symmetric about 0, we have that such a probability corresponds also to the probability of having a path with $Y_0 = 0$ and $Y_i \leq 0$ for $i = 1, \ldots, n$. So

$$\mathbb{P}(Y_i \leq 0 \text{ for } i = 0, \ldots, n) = p_{n,0} = \binom{n}{n/2} \frac{1}{2^n}\,.$$

By Stirling's formula we have that

$$p_{n,0} \underset{n \to \infty}{\sim} \sqrt{\frac{2}{\pi n}}\,,$$

and hence we get the thesis. $\square$

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