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Pontryagin's Maximum Principle: some economic applications

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Introduction

In economics, as in many other scientific fields, theories are stated in a rigorous, non-ambiguous way, by formailizing them into mathematical expressions.

For example, the Fisher's equation: $MV = PY$, the consumer's *Utility function*, and the evolution law of capital in growth models: $k(t+1) = i(t) + k(t) - \delta k(t)$, have in common to estabilish cause-effect relationships between some *quantities*.

As happens in phisics, chemistry, engeneering etc., an economic theory is recognized as valid when its statement is confirmed by empirical evidence. For this purpose, mathematical and statistical tools, such as *regression models*, are employed.
On the converse, applying such techniques to empirical data allows to find regularities and patterns, so to formulate new theories.

The central assumption on which economic theory is built upon is scarcity of resources, and how these can be allocated in the most efficient way. Therefore, optimization and related mathematical theory is extensively applied.
Since the first year, each university student is expected to get familiar with utiliy maximization and profit maximization problems. As a student progresses, increasingly more complex optimization problems are studied: starting from *static optimization* problems encountered in microeconomic courses of the bachelor degree, till the more complex *dynamic optimization* problems in the most advanced master courses. As the student progresses, the necessary mathematical tools to solve these problems are taught, as constrained optimization theory, Lagrange method or the Kuhn-Tucker theorem. However, the mathematical theory behind dynamic optimization is not studied in a complete and extensive way, being considered outside the scope of the courses.

Fascinated by the complexity of these dynamic models, I wished I could get a better grasp of the underlying mathematical theory, and I had the opportunity to deepen my knowledge of it by writing a dissertation in *Mathematical Methods for Economics and Finance*.
After doing some research, I discovered Pontryagin's Maximum Principle, a fundamental instrument of optimal control theory, which is a mathematical branch that studies how to drive a controlled dynamical system towards a determined

objective.

My work has the following structure: in Chapter 1, I give an introduction about the theory of dynamical systems and optimal control theory; in Chapter 2 I present the precise statement of Pontryagin's Maximum Prinipcle; in Chapter 3 I show some applications of the principle to dynamic economic problems.

I want to thank everyone who supported me in this last phase of my journey as a university student: my parents, my brother, my closest friends, and my supervisor, Professor Gozzi.

Chapter 1

Dynamical systems and controls

1.1 Introduction

Optimal control theory is a field of mathematics that studies how dynamical systems are driven to pursue an objective.
Mathematically, a problem of optimal control is represented by the system:

$$\max_{\mathcal{C}} J(u) = \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \tag{1.1}$$

$$\dot{x}(t) = g(t, x(t), u(t)) \tag{1.2}$$

$$x(t_0) = p \tag{1.3}$$

The functional (1.1) is the *objective functional*; the differential system (1.2), and the equation (1.3), together, form a *controlled* dynamical system.

The set of functions $\mathcal{C}$, on which the functional (1.1) is maximized, is the set of *admissible controls.*

1.2 Dynamical systems

A dynamical system is a mathematical representation of the evolution of a quantity, or a set of quantities, on an time interval. For example, a dynamical system may describe fluctuations of the $S\&P500$ in a year, the price of iron, or increments in GDP.

A dynamical system is characterized by:

Dimension: denoted with letter n, it is the number of the quantities the dynamical system represents; for example, for a dynamical system modelling the evolution in time of the prices of 10 stocks, we set $n = 10$.

Time interval: it is the horizon where the dynamical system is defined, and is denoted with $[t_0, t_1]$; t_0 is defined *initial time*; t_1 denoted *final time*.

State space: a subset of $\mathbb{R}^n$ (where n is the dimension of the system), it is the space where the quantities represented by the system are allowed to take value. For example, if a dynamical system represents the evolution of the number of trees in an area, the *State space* must necessarly be only positive.
The state space is denoted with X; in the "tree" example, we should set $X = [0, +\infty)$.

Dynamics: it is the core of the dynamical system; its made of system of differential equations:

$$
\begin{aligned}
&\dot{x}_1 = g_1(t, x(t)) && \forall\ t \in [t_0, t_1] \\
&\dot{x}_2 = g_2(t, x(t)) && \forall\ t \in [t_0, t_1] \\
&\vdots \\
&\dot{x}_n(t) = g_n(t, x(t)) && \forall\ t \in [t_0, t_1]
\end{aligned}
$$

Initial values: a system of equations:

$$
\begin{aligned}
&x_1(t_0) = p_1 \\
&x_2(t_0) = p_2 \\
&\vdots \\
&x_n(t_0) = p_n
\end{aligned}
$$

$p_1, p_2, ..., p_n$ are the values the varaibles of the system assume at the *initial time* t_0.

Trajectories: n functions $x_1, x_2, ..., x_n$ that solve the *dynamics* and the initial values.

1.2.1 Example: bank balance

Immagine to model the time evolution of a bank balance where a continuously compounded nominal rate r is earned.

Denote with $x(t)$ the number of dollars on the bank account at any time $t \in [t_0, t_1]$; if the account can be negative, the state space X equals $\mathbb{R}$; otherwise, we would have $X = [0, +\infty)$.

The dynamic describing the evolution of the account is

$$\dot{x}(t) = rx(t) \quad \text{for a.e. } t \in [t_0, t_1] \tag{1.4}$$

The choice of the initial condition

$$x(t_0) = p \tag{1.5}$$

is the value of the bank account at the beginning of the period, and determines the trajectory x as

$$x(t) = pe^{t-t_0} \quad \forall\ t \in [t_0, t_1] \tag{1.6}$$

that gives the amount, in dollars, on the account at time t.

1.2.2 Autonomous dynamical systems

An autonomous dynamical system is a system whose dynamics $g_1, g_2, ..., g_n$ don't depend explicitely on time: $g(t, x) = g(x)$.

An example of autonomous dynamical system is

$$\begin{aligned} &\dot{x}(t) = 3x && \forall\ t \in [t_0, t_1] \\ &x(t_0) = p \end{aligned}$$

Autonomous dynamical systems evolution is independent of the choice of the initial time t_0, t_1, and the value of the trajectory x at time t depends exclusively on the difference $t - t_0$.

This same property holds for dynamical system that, even though depend on time (as the example in the former paragraph), only depend on the *difference* $t - t_0$ between time and initial time.

1.3 Controlled dynamical system

A controlled dynamical system is similar to a “classic” dynamical system (see par. 1.2), but the dynamics: g_1, g_2, ... ,g_n depend, other than t and $x(t)$, on a *function*, $u = u_1, u_2, ..., u_k$, that is called *control*:

$$\begin{aligned} &\dot{x}_1 = g_1(t, x(t), u(t)) && \text{for a.e. } t \in [t_0, t_1] \\ &\dot{x}_2 = g_2(t, x(t), u(t)) && \text{for a.e. } t \in [t_0, t_1] \\ &\vdots \\ &\dot{x}_n = g_n(t, x(t), u(t)) && \text{for a.e. } t \in [t_0, t_1] \end{aligned}$$

The trajectories $x_1, x_2, ..., x_n$, determined as solution of the system, plus initial conditions $x_1(t_0) = p_1$, $x_2(t_0) = p_2$, ... ,$x_n(t_0) = p_n$, are defined *associated trajectories* of the control $u_1, u_2, ..., u_k$.

1.3.1 Example: bank balance 2

Consider, again, the bank balance example of par. 1.2.1; now, suppose the owner plans a "withdraw" strategy, defined as a continuous function $w(t)$ over $[t_0, t_1]$.
The dynamic would become

$$\dot{x}(t) = rx(t) - w(t) \tag{1.7}$$

Depending on the withdraw strategy w, we shall have different trajectories computed for the bank balance $x(t)$, which are trends of the account associated to different withdrawal decisions.

1.4 Optimal control

A controlled dynamical system can be driven to purse pre-determined objectives.

Finding the control functions $u_1, u_2, ..., u_k$ that better aim at these objective is the object of study of Optimal Control Theory.

As introduced in 1.1, the *objective* is defined through a *functional*:

$$J(u) = \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \tag{1.8}$$

which takes the name of *objective functional*; the function f is called *cost function.*

An *optimal control* is a control function $u_1, u_2, ...u_k$ that verifies the inequality

$$J(u^*) \leq J(u) \tag{1.9}$$

for any other possible choice of a control $u_1, u_2, ..u_k$.

Among the candidates eligible for being optimal controls, we should restrict our choice to a set of *admissible* functions: $\mathcal{C}$; that is, functions which don't violate any condition posed by the problem.
As a general rule, a control is admissible when it satifies the constraint:

$$u(t) \in U \quad \forall t \in [t_0, t_1] \tag{1.10}$$

where U is called *control set.*
In some cases, constraints on the states, such as $x(t) \in X$, may be present.

1.4.1 Example: bank balance 2

Suppose the owner of the balance recieves an utility from his/her withdraw decision, $U(w)$; how should he/she choose his withdraw strategy so to maximize his utility over the time period $[t_0, t_1]$?

This is an optimal control problem with *objective functional*

$$\int_{t_0}^{t_1} U(w(t))\, dt$$

Dynamic:

$$\dot{x}(t) = rx(t) - w(t) \quad \text{for a.e. } t \in [t_0, t_1]$$

and initial condition

$$x(t_0) = p \tag{1.11}$$

The *admissible controls* $\mathcal{C}$ are all piecewise continous functions $w(t) : [t_0, t_1] \to [0, +\infty)$.

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Chapter 2

Pontryagin's Maximum Principle

Consider an optimal control problem with:

A state space X, a control set U and an admissible set $\mathcal{C}$, defined on a time period $[t_0, t_1]$.

Have also: An objective functional $J(u) = \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt$.

A dynamic $\dot{x}(t) = g(t, x(t), u(t))$ for a.e. $t \in [t_0, t_1]$.

An initial condition $x(t_0) = x_0$.

Define the *Hamiltonian function* as:

$$H(t, x, u, \lambda_0, \lambda) = \lambda_0 f(t, x, u) + < \lambda, g(t, x, u) >$$

2.1 The theorem

If $u \in \mathcal{C}$ is an optimal control, there exists a continuous function $\lambda_0, \lambda : [t_0, t_1] \to \mathbb{R}^{n+1}$ such that the following conditions are verified:

$$\lambda_0 = 1$$

$$u^*(t) \in \arg\max_{v \in U} H(t, x(t), v, 1, \lambda(t)) \quad \forall t \in [t_0, t_1] \tag{2.1}$$

$$\dot{\lambda}(t) = -\nabla_x H(t, x^*(t), u^*(t), 1, \lambda(t)) \quad \text{for a.e. } t \in [t_0, t_1] \tag{2.2}$$

$$\lambda(t_1) = 0 \tag{2.3}$$

We call (2.1) "Maximum Principle", (2.2) is the "adjoint equation" and (2.3) "transversality condition".

2.1.1 Example

Consider the optimal conttrol problem:

$$\begin{aligned}
&J(u) = \int_0^1 x(t) - u(t)^2 \, dt \\
&\dot{x}(t) = u \qquad\qquad\qquad\qquad \text{for a.e. } t \in [0,1] \\
&x(0) = 2
\end{aligned}$$

The Hamiltonian is

$$H(t, x, u, \lambda) = x - u^2 + \lambda u$$

From the Maximum principle (2.1) we obtain

$$u(t) = \frac{\lambda(t)}{2} \quad \forall\ t \in [0,1]$$

Then, from the adjoint equation, we have

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}(t, x(t), u(t), \lambda(t)) = -1$$

o we have $\lambda(t) = -t + c$; from the transverality condition (2.3) we obtain

$$\lambda(t_1) = \lambda(1) = -1 + c = 0$$

so $c = 1$, $\lambda(t) = 1 - t$.

For $u(t) = \lambda(t)/2$ we have

$$u(t) = \frac{1-t}{2}$$

We can obtain the associated trajectory x by integrating the dynamic $\dot{x}(t) = u(t)$:

$$x(t) = x_0 + \int_{t_0}^{t} u(s) \, ds = x_0 - \frac{t^2}{4} + \frac{t}{2}$$

By substituting x_0 with the initial condition $x_0 = 2$ we have

$$x(t) = -\frac{t^2}{4} + \frac{t}{2} + 2$$

2.2 General case

Consider a cost functional

$$J(u) = \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt + \phi(x(t_1))$$

where ϕ is called *savage value* or *terminal payoff*.

Than, Pontryagin Maximum Principle is re-stated as:

$$\lambda_0 \in \{0, 1\} \tag{2.4}$$

$$(\lambda_0, \lambda(t)) \neq (0, 0) \quad \forall\ t \in [t_0, t_1] \tag{2.5}$$

$$u^*(t) \in \arg\max_{v \in U} H(t, x(t), v, \lambda_0, \lambda(t)) \quad \forall t \in [t_0, t_1] \tag{2.6}$$

$$\dot{\lambda}(t) = -\nabla_x H(t, x^*(t), u^*(t), \lambda_0, \lambda(t)) \quad \text{for a.e. } t \in [t_0, t_1] \tag{2.7}$$

$$\lambda(t_1) = \lambda_0 \frac{\partial \phi}{\partial x}(x(t_1)) \tag{2.8}$$

where (2.5) is called *non-triviality* condition ((2.6), (2.7) and (2.8) take the same names as par. 2.1).

If on the state is asked a condition

$$x(t_1) = \beta \tag{2.9}$$

(called *final condition*) then the *transversality condition* (2.8) is not required.

Free final time
If the final time t_1 is free, then there's the additional condition

$$H(t_1, x(t_1), u(t_1), \lambda_0, \lambda(t_1)) + \frac{\partial \phi}{\partial t}(t_1, x(t_1)) = 0$$

Chapter 3

Applications of the PMP

3.1 Cost minimization with general price function

Suppose a woolen mill recieves the order to produce $F \in \mathbb{R}$ tonnes of whool in $T \in \mathbb{R}$ days from now, with $0 < F < \alpha \in \mathbb{R}$.

We denote with $x(t)$ tons of wool producedd at time $t \in [0, T]$ (for example, $x(3) = 4$ means 4 tonnes produced after exactly 4 days from now).

We denote with $u(t)$ the *production rate* at time t, such that

$$\dot{x}(t) = u(t) \quad \text{for a.e. } t \in [0, T] \tag{3.1}$$

Suppose also $0 \leq u(t) \leq \alpha$, for all $t \in [0, T]$; therefore we have a one dimensional ($n = 1$) controlled dynamical system, whose state variable is x and whose control variable is $u(t)$; the set of *admissible controls* is the set of *piecewise continuos* functions defined on $[0, T]$, with range contaied in the set $U = [0, \alpha]$, that is the *control set*.

Suppose $x(0) = 0$.
The objective of the firm is to minimize

$$\int_0^T p(t)u(t)\, dt \tag{3.2}$$

where $p(t)$ is an instant cost function, $p(t) > 0$ for all $t \in [0, T]$, $p \in C^1$, $p'(t) \neq 0$ a.e. in $[0, T]$ (that is, there are no intervals on which the function is flat, so $p(t)$ is alternatively striclty increasing/decreasing).

This is an optimal control problem, with cost functional

$$J(u) = \int_0^T p(t)u(t)\,dt \tag{3.3}$$

so we have the *cost function*, $f(t,x,u)$, equal to $p(t)u$;
the dynamic is

$$\dot{x}(t) = u(t) \quad \text{for a.e. } t \in [0,T] \tag{3.4}$$

so we have $g(t,x,u) = u$;
then the initial condition, $x(0) = 0$, and the final condition, $x(T) = F$.

The control set is $U = [0,\alpha]$, and the admissible controls are all piecewise continuous functions on $[0,T]$ with range contained in U.

$$\begin{aligned}
&\min_{u\in\mathbb{C}} J(u) = \int_0^T p(t)u(t)\,dt\\
&\dot{x}(t) = u(t) \quad \text{for a.e. } t \in [0,T]\\
&x(0) = 0\\
&x(T) = F\\
&U = [0,\alpha] \cap \mathbb{R}
\end{aligned}$$

We can apply Pontryagin's Maximum Principle, referring to the version reported in *par.* 2.2 and fixing the *savage value* $\phi(t) \equiv 0$.
Since we have a *final condition* like (2.9), that is, $(x(T) = F)$, the last equation of the theorem (2.8) does not hold.

We start by writing down the Hamiltonian:

$$H(t,x,u,\lambda_0,\lambda) = \lambda_0 p(t)u + \lambda u$$

The first condition of the theorem (2.4) tell's us λ_0 is either 0 or 1. See the Appendix to see why, in this problem, it is necessairly $\lambda_0 = 1$.

Our Hamiltonian therefore becomes::

$$H(t,x,u,1,\lambda) = p(t)u + \lambda u$$

The Maximimum Principle requires an optimal control $u(t)$ to minimize the Hamiltonian, which is, in this case, a linear function of $u(t)$ with slope $p(t)+\lambda(t)$ (*Figure* 3.1).

This implies that the minimum is reached for $u(t) = \alpha$ if the slope is negative, and for $u(t) = 0$ if the slope is positive. That is, we must ask $u(t)$ to follow the rule

$$u(t) = \alpha \quad \text{when}\, p(t) < -\lambda(t) \tag{3.5}$$

$$u(t) = 0 \quad \text{when}\, p(t) > -\lambda(t) \tag{3.6}$$

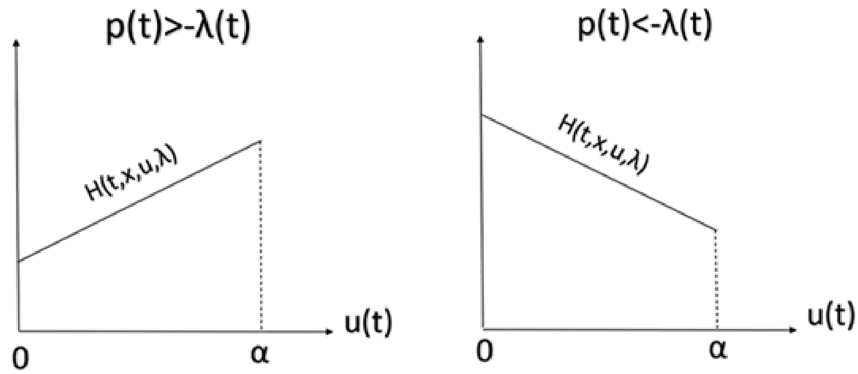

Figure 3.1:

What happens when $p(t) = -\lambda(t)$? Here the PMP does not impose a rule on $u(t)$, or better, since the slope of the Hamiltonian is zero and $H(t, x, u\lambda) = 0$ for any choice of $u(t)$.

In optimal control theory, this kind of points are called *switching* points, and the standard way to define $u(t)$ in this cases is to exploit the discontinuity of $u(t)$ to make it "jump" from a value (in this case, α or 0) to another in this points.
More precisely (*Figure* 3.2) if in a neighbourhood a point $\tau \in [0, T]$ such that

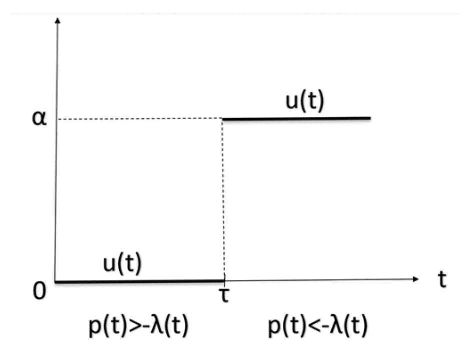

Figure 3.2:

$p(\tau) = -\lambda(\tau)$, with $p(t) > -\lambda(t)$ on a left neghbourhood of τ and $p(t) < -\lambda(t)$ on the right, we would set $u(t)$ such that

$$u(t) = 0 \quad \text{in } [..., \tau] \tag{3.7}$$

$$u(t) = \alpha \quad \text{in } (\tau, ...] \tag{3.8}$$

or, equivantely

$$u(t) = 0 \quad \text{in}\, [.., \tau) \tag{3.9}$$
$$u(t) = \alpha \quad \text{in}\, [\tau, T] \tag{3.10}$$

Now, consider the second statement of the PMP:

$$-\dot{\lambda}(t) = \frac{\partial H}{\partial x}(t, x, u, \lambda) \tag{3.11}$$

In our problem, the variable $x(t)$ does not appear in the Hamiltonian ($H = c(t)u(t) + \lambda(t)u(t)$), so its partial derivative with respect to x is 0:

$$\frac{\partial H}{\partial x} = 0 \tag{3.12}$$

that

$$-\dot{\lambda}(t) = 0 \quad \forall\, t \in [0, T] \tag{3.13}$$

This tells us that the function λ is a constant to be determined.

If $b = -\lambda$ than conditions (3.5) and (3.6) imply

$$u(t) = \alpha \quad \text{when}\, p(t) < b \tag{3.14}$$
$$u(t) = 0 \quad \text{when}\, p(t) > b \tag{3.15}$$

with

$$u(t) = 0 \quad \text{in}\, [..., \tau] \tag{3.16}$$
$$u(t) = \alpha \quad \text{in}\, (\tau, ...] \tag{3.17}$$

in the neighbourhood of points τ such that if $p(\tau) = b$ with $p(t) > b$ on the left and $p(t) < b$ on the right (that is, $p'(\tau) < 0$, see *Figure* 3.3), and

$$u(t) = 0 \quad \text{in}\, [.., \tau) \tag{3.18}$$
$$u(t) = \alpha \quad \text{in}\, [\tau, T] \tag{3.19}$$

for the opposite case.

Look at *Figure* 3.3, where the graphs of the function $p(t)$ and the constant $b = -\lambda$ are shown.

The meaning of this representation is clear: the firm should produce at the maximum rate α in periods where the price $p(t)$ is lower, while stop completely ($u(t) = 0$) when its higher. This firm, while remaining in the limits given by this elementary model, should employ a astrategy $u(t)$ so to solve the maximum problem.

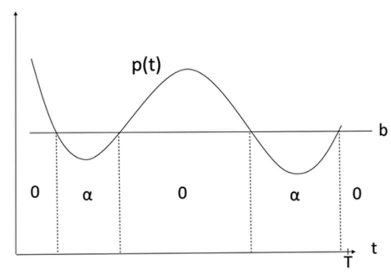

Figure 3.3:

Now, if $u(t)$ respect the specified rule, and $\lambda(t)$ as well, than the only remaing condition we can employ to find a solution is the final state $x(T) = B$.

Knowing that the dynamic of $x(t)$ is $\dot{x} = u$, and its initial point $x(0) = 0$, we can integrate $\dot{x} = u$ to obtain:

$$x(t) = \int_0^t \dot{x}\, ds = \int_0^t u(s)\, ds \tag{3.20}$$

The condition $x(T) = B$ is equivalent to ask

$$x(T) = \int_0^T \dot{x}\, ds = \int_0^T u(s)\, ds = B \tag{3.21}$$

that is, the integral of $u(t)$ over the whole time period $[0, T]$ should equal B.

The integral of $u(t)$ can be re-written as:

$$\int_0^T u(t)\, dt = \int_0^{\tau_1} u(t)\, dt + \int_{\tau_1}^{\tau_2} u(t)\, dt + \int_{\tau_2}^{\tau_3} u(t)\, dt + \int_{\tau_3}^{\tau_4} u(t)\, dt + \dots \tag{3.22}$$

$$= \sum_{p(t)<b} \int \alpha\, dt + \sum_{p(t)>b} \int 0\, dt = \sum_{p(t)<b} \int \alpha\, dt \tag{3.23}$$

$$=\alpha \sum_{p(t)<b} (\tau_i - \tau_{i-1}) \tag{3.24}$$

After last result, a natural question arises: how to determine (if possible) b such that (3.24) equals the order B?

Consider *Figure 3.4*: if we choose the constant $b = b_1$ (the higher of the four horizontal lines), we would obtain a control $u(t) = alpha$ on the whole horizon $[0, T]$.

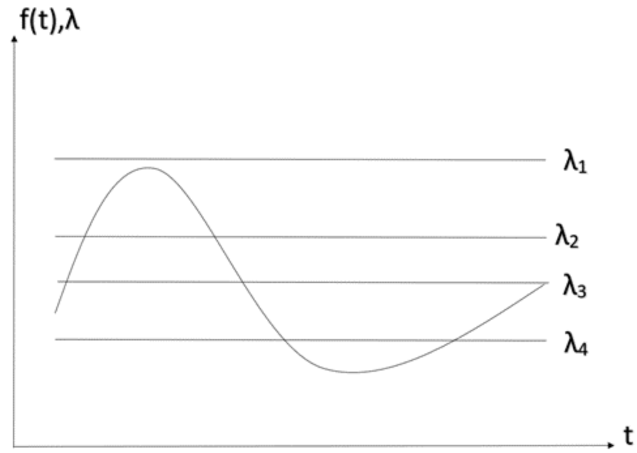

Figure 3.4:

This would imply $u(t) = \alpha$ for all $t \in [0, T]$, so that

$$\int_0^T u(t)\, dt = \int_0^T \alpha = \alpha T \tag{3.25}$$

so, unless we had $B = \alpha T$, this wouldn't be a solution.

If, instead, we chose $b = b_2$, we would obtain a set of intervals $[0, \tau_1]$, $[\tau_1, \tau_2]$,... where $u(t)$ takes alternatively the values of α and zero, depending on whether $p(t) < b$ or $p(t) > b$ on those intervals; in particular, it is α on intervals $[\tau_1, \tau_2]$ and $[\tau_3, \tau_4]$, while it be zero on $[\tau_2, \tau_3]$.

If we chose $b = b_3$, we would obtain a new set of intervals $[0, \theta_1]$, $[\theta_1, \theta_2]$, ... Note how the intervals $[\theta_1, \theta_2]$ and $[\theta_3, \theta_4]$, which are the "analog" of where $u(t) = \alpha$, have been restricted, while the interval $[\theta_2, \theta_3]$, (the "analogous" of$[\tau_2, \tau_3]$), where $u(t) = 0$, has widened.

This holds for all intervals $[\theta_{i-1}, \theta_i]$, analogous of $[\tau_{i-1}, \tau_i]$. This implies that

$$\int_0^T u(t)\, dt = x(T) = \sum_{p(t)<b} [\tau_{i-1}, \tau_i] = B$$

in (3.24) must decrease as we choose lower b, that is

$$\int_0^T u(t)\, dt = x(T)$$

is a (continuous) increasing function of b, with

$$\begin{array}{ll} x(T) = \alpha T & \text{if } b > p(t) \quad \forall\, t \in [0, T] \\ x(T) = 0 & \text{if } b < p(t) \quad \forall\, t \in [0, T] \end{array}$$

This implies that there exists a b such that $x(T) = B$ if and only if $0 \leq B \leq \alpha T$.

3.1.1 Periodic price function

Suppose the cost function $p(t)$ is:

$$p(t) = c \sin(kt) + d \tag{3.26}$$

with $c, k > 0$, $d \in \mathbb{R}$

We are looking for $b \in \mathbb{R}$ such that

$$\alpha \sum_{p(t)<b} (\tau_i - \tau_{i-1}) = F \tag{3.27}$$

Can we identify the intervals in the *sum* in (3.27) knowing b?

We are looking for all the subintervals of $[0, T]$ where conditions

$$c \sin(kt) + d < b$$

holds. By manipulating the expression, we obtain the equivalent condition

$$\sin(kt) < \frac{b-d}{c} \tag{3.28}$$

If we choose b such that $\frac{b-d}{c} > 1$ (3.28) is always verified, and we have $u(t) \equiv \alpha$, while if $\frac{b-d}{c} < -1$ (3.28) is never verified and $u(t) \equiv 0$.

In general, for any $a \in (-1, 1]$, the condition

$$\sin(\theta) < a$$

holds on the set of intervals (see *Figure 3.5*) $[\sigma_1 + 2n\pi, \sigma_2 + 2n\pi]$, $n \in \mathbb{Z}$, with $\sigma_2 = \arcsin(a)$, $\sigma_1 = -\sigma_2 - \pi$ .Here we have $\theta = kt$ and $a = \frac{b-d}{c}$; so we obtain

$$\sin(kt) < \frac{b-d}{c} \quad \text{for } kt \in [\sigma_1 + 2n\pi, \sigma_2 + 2n\pi] \quad n \in \mathbb{Z}$$

with $sigma_2 = \arcsin(b-d)/c$.
We therefore have

$$p(t) < b \quad \text{for} t \in [\tau_1(n), \tau_2(n)] \cap [0, T] \quad n \in \mathbb{Z}$$

with

$$\tau_1(n) = \frac{1}{k}(\sigma_1 + 2n\pi)$$
$$\tau_2(n) = \frac{1}{k}(\sigma_2 + 2n\pi)$$

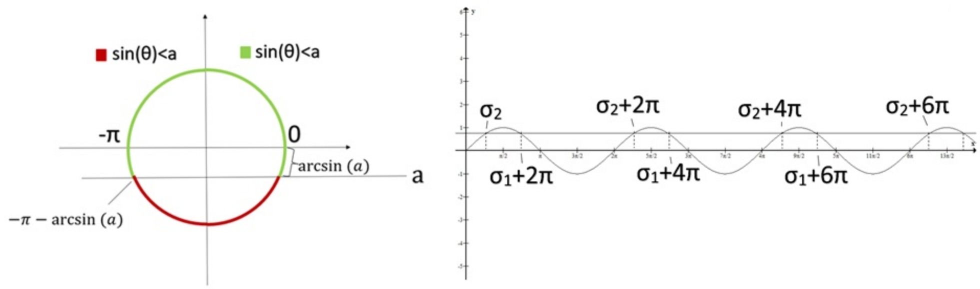

Figure 3.5:

3.2 A moving car

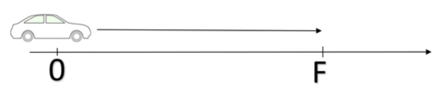

Figure 3.6:

Consider a car driving along a straight road; denote with with $s(t)$ the distance (in kilometers) the car has travelled at time t and with $v(t)$ the velocity (in km/h)the car has at time t.

Being the velocity the derivative of the distance, we have

$$\dot{s}(t) = v(t)$$

Denote with $a(t)$ the intensity of acceleration of the car at time t, with $a(t) \in [0, \alpha]$ and with $b(t)$ the braking intensity of the car at time t, with $b(t) \in [0, \beta]$ the derivative of velocity, we have

$$\dot{v}(t) = a(t) - b(t) \tag{3.29}$$

Moreover, suppose the car starts its journey at time 0, with velocity 0:

$$s(0) = v(0) = 0$$

and that $t \in [0, +\infty) \cap \mathbb{R}$ are the *hours* passed from the start of the journey.

Now, define as F the distance in kilometers from the point where the car starts,

to a *finish line*, and suppose the car wants to reach such finisch line the minimum possible time; moreover, suppose also the car has to stop exactly at the finish line.
If we denote by T the number of hours passed from the start of the "race" when the car reaches the finish line, the last two requirements can be formalized as: $s(T) = F$, $v(T) = 0$, and asking to minimize T.

Euristically, a solution which comes to mind is to accelerate at the maximumum rate (β) until the distance that remains to percour till F is such that, braking with the maximum force (α) would allow to stop in F with a velocity of 0 Pontryagin's Maximum Principle (as shown in this section) it turns out that this is actually the optimal strategy.

This is an optimal control problem, where the dimension of the controlled dynamical system is 2 (so $n = 2$); the dimension of the control vector is 2 as well ($k = 2$).
The dynamic is $g(t, x, u) = (x_2, u_1 - u_2)$, the *cost function* is $f(t, x, u) = 1$ equals 1, and the *savage value* is $\phi(t, x) \equiv 0$[1]

The set $\mathcal{C}$ of *admissible controls* is made up of all piecewise continuous functions on$[0, +\infty) \cap \mathbb{R}$ with image subset of $U = [0, \alpha] \times [0, \beta] \cap \mathbb{R}^2$. We can now state the problem in the standard form

$$\begin{aligned}
&\min_{u \in \mathbb{C}} J(u) = \int_0^T dt \\
&\dot{x}(t) = u_1(t) - u_2(t) \quad \text{for a.e. } t \in [0, T] \\
&x_1(0) = x_2(0) = 0 \\
&x_1(T) = F \\
&x_2(T) = 0 \\
&U = [0, \alpha] \times [0, \beta] \cap \mathbb{R}^2
\end{aligned}$$

We start by writing the Hamiltonian:

$$H(t, x, u, \lambda_0, \lambda) = \lambda(u_1 - u_2)$$

In the Appendix is shown why $\lambda_0 \equiv 1$; the Hamiltonian becomes

$$H(t, x, u, \lambda) = 1 + \lambda_1(t)x_2(t) + \lambda_2(t)u_1(t)$$

For the first condition of PMP (Minimum Principle), we have

$$\begin{cases} u_1(t) = \alpha \\ u_2(t) = 0 \end{cases} \quad \text{for } \lambda_2(t) < 0 \begin{cases} u_1(t) = 0 \\ u_2(t) = \beta \end{cases} \quad \text{for } \lambda_2(t) > 0 \tag{3.30}$$

[1] Since $T = \int_0^T 1\, dt$.

For the second condition (adjoint equation), we have

$$\dot{\lambda}_1(t) = -\frac{\partial}{\partial s} H(t, x, u, \lambda) = 0$$
$$\dot{\lambda}_2(t) = -\frac{\partial}{\partial v} H(t, x, u, \lambda) = -\lambda_1(t)$$

So we have that λ_1 has to be a constant, which we denote by d, and λ_2 takes the form

$$\lambda_2(t) = -dt + f,\ f \in \mathbb{R}$$

Now, set $d, f < 0$[2]; we have

$$\lambda_2(t) < 0 \quad \text{for}\, t \in [0, \tau)$$
$$\lambda_2(t) > 0 \quad \text{for}\, t \in (\tau, T]$$

where $\tau = \frac{f}{d}$.

For (3.30) we would have (*Figure* 3.7)

$$\begin{cases} u_1(t) = \alpha \\ u_2(t) = 0 \end{cases} \quad \text{for } t \in [0, \tau] \begin{cases} u_1(t) = 0 \\ u_2(t) = \beta \end{cases} \quad \text{for}\, t \in (\tau, T] \tag{3.31}$$

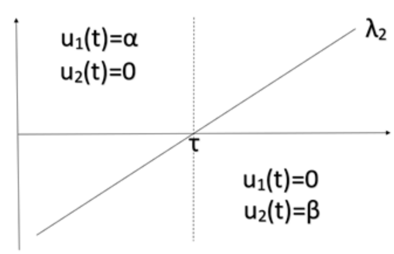

Figure 3.7:

Integrating (A.5) we obtain

$$x_2(t) = \begin{cases} \alpha t & t \in [0, \tau] \\ \tau(\alpha + \beta) - \beta t & t \in (\tau, T] \end{cases} \tag{3.32}$$

and

$$x_1(t) = \begin{cases} \frac{\alpha}{2} t^2 & t \in [0, \tau] \\ \frac{\alpha}{2}\tau^2 + \tau(\alpha + \beta)(t - \tau) - \frac{\beta}{2}(t^2 - \tau^2) & t \in (\tau, T] \end{cases} \tag{3.33}$$

[2]Proof in *Appendix*

Condition[3]

$$H(t, x, u, \lambda) = 0 \quad \text{for all}\, t \in [0, T]$$

implies $f = -\frac{1}{\beta}$.

From $x_2(T) = 0$ and using (3.49) we obtain

$$\frac{\beta}{\alpha + \beta} T = \tau \tag{3.34}$$

and from $x_1(T) = F$, (3.34) and (3.50) we obtain[4]

$$T = \sqrt{2F \frac{\alpha + \beta}{\alpha\beta}} \tag{3.35}$$

from $x_1(T) = F$.

3.2.1 Analysis of the results

These last two results ((3.34) and (3.35)), allow us to make some observations.

From (3.34), we see that (not surprisingly) the switching point τ increases if the braking force β increases, and vice versa; moreover, it also tells us, that the division of the total time T between acceleration time $[0, \tau]$ and braking time $[\tau, T]$ is exactly equal to the proportion between the acceleration force and the braking force (*Figure 3.8*):

$$\frac{\tau}{T} = \frac{\beta}{\alpha + \beta}, \qquad \frac{T - \tau}{T} = \frac{\alpha}{\alpha + \beta}$$

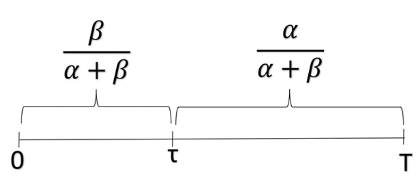

Figure 3.8:

[3] See section *Appendix*
[4] Appendix

The minimum necessary time T to reach F

$$T = \sqrt{2F\frac{\alpha+\beta}{\alpha\beta}}$$

is the square root of a term, $2F\frac{\alpha+\beta}{\alpha\beta}$, which is directly proportional to the distance F, and the ratio between the sum of the two coefficients (α and β) and their product.
Computing the partial derivatives of this term ($\frac{\alpha+\beta}{\alpha\beta}$) with respect to the two coefficients, we get

$$\frac{\partial}{\partial\alpha}\frac{\alpha+\beta}{\alpha\beta} = -\frac{1}{\alpha}$$
$$\frac{\partial}{\partial\beta}\frac{\alpha+\beta}{\alpha\beta} = -\frac{1}{\beta}$$

This implies that the two coefficients have a marginal effect on the total time, which is inversely proportional to their value (that is, if for example α is realy law, it will impact T more strongly than if it was higher); morover, being the term under square root, the effect also decreases as T is bigger.

3.3 A moving car 2

Consider, again, the time minimization problem as that of the former section: a moving car on a straight road, with distance function $s(t)$ and velocity function $v(t)$ characterized by the dynamics:

$$\dot{s}(t) = v(t) \tag{3.36}$$
$$\dot{v}(t) = a(t) - b(t) \tag{3.37}$$
$$a(t) \in [0, \alpha] \quad \forall\, t \in [0, T] \tag{3.38}$$
$$b(t) \in [0, \beta] \quad \forall\, t \in [0, T] \tag{3.39}$$

and whose objective is to minimize the time it takes to reach a point F.

Now, suppose the car's runs into cost, equal to $c(a(t) + b(t))$, which equally penalizes acceleration and braking; moreover, we denote with s the "weight" we assign to T to manage its optimization in the problem.

We write the objective functional as:

$$sT + \int_0^t c((a(t) + b(t)))\, dt = \int_0^T s + c((a(t) + b(t)))\, dt$$

with $(a(t), b(t)) \in [0, \alpha] \times [0, \beta]$.

This is therefore an optimal control problem with *cost function* $f(t, x, u) = s + c(u_1 + u_2)$ and *savage value* $\phi(t, x) \equiv 0$; the dimension of the controlled dynamical system is 2 (so $n = 2$); the dimension of the control vector is 2 as well ($k = 2$).
The dynamic is $g(t, x, u) = (x_2, u_1 - u_2)$, the *cost function* is $f(t, x, u) = 1$ equals 1, and the *savage value* is $\phi(t, x) \equiv 0$[5]

The set $\mathcal{C}$ of *admissible controls* is made up of all piecewise continuous functions on$[0, +\infty) \cap \mathbb{R}$ with image subset of $U = [0, \alpha] \times [0, \beta] \cap \mathbb{R}^2$. We can now state the problem in the standard form

$$\begin{aligned}
&\min_{u \in \mathbb{C}} J(u) = \int_0^T s + c(u_1(t) + u_2(t))\, dt \\
&\dot{x}(t) = u_1(t) - u_2(t) \quad \text{for a.e. } t \in [0, T] \\
&x_1(0) = x_2(0) = 0 \\
&x_1(T) = F \\
&x_2(T) = 0 \\
&U = [0, \alpha] \times [0, \beta] \cap \mathbb{R}^2
\end{aligned}$$

[5] Since $T = \int_0^T 1\, dt$.

We start by writing down the Hamiltonian:

$$H(x(t), u(t), \lambda(t)) = \lambda_0(s + b(u_1(t) + u_2(t))) + \lambda_1 x_2(t) + \lambda_2(u_1(t) - u_2(t))$$

We can immediately exclude the case $\lambda_0 = 0$ (see *See Appendix*) for the same reasons as in the previous problem; so we can set $\lambda_0 = 1$; the Hamiltonian becomes

$$H(t, x(t), u(t), \lambda(t)) = s + c(u_1(t) + u_2(t)) + \lambda_1(t)x_2(t) + \lambda_2(t)(u_1(t) - u_2(t))\, dt$$

We start by verifying the first condition of the PMP:

$$u(t) \in \arg\max_{u(t) \in U} H(x(t), u(t), \lambda(t)) \quad \forall\, t \in [0, T]$$

that is

$$u(t) \in \arg\max_{u(t) \in U} a + b(u_1(t) + u_2(t)) + \lambda_1(t)x_2(t) + \lambda_2(t)(u_1(t) - u_2(t))$$

Since the part of the Hamiltonian that depends on $u(t)$ is

$$b(u_1(t) + u_2(t)) + \lambda_2(t)(u_1(t) - u_2(t))$$

finding the *arg min* of (3.3) is equivalent to find the *arg min* of the function

$$y(u(t)) = b(u_1(t) + u_2(t)) + \lambda_2(t)(u_1(t) - u_2(t))$$

From these three we have the following values for $u(t)$:

$$\begin{matrix} u_1(t) = \alpha \\ u_2(t) = 0 \end{matrix} \qquad \lambda_2(t) < -c \tag{3.40}$$

$$\begin{matrix} u_1(t) = 0 \\ u_2(t) = 0 \end{matrix} \qquad -c < \lambda_2(t) < c \tag{3.41}$$

$$\begin{matrix} u_1(t) = 0 \\ u_2(t) = \beta \end{matrix} \qquad \lambda_2(t) > c \tag{3.42}$$

We now know how $\lambda_2(t)$ determines $u(t)$, and we can use the second condition of the PMP (adjoint equation) to know $\lambda_2(t)$; as in the former problem, we have:

$$\dot{\lambda}_1(t) = -\frac{\partial}{\partial x_1} H(x(t), u(t), \lambda(t)) = 0$$
$$\dot{\lambda}_2(t) = -\frac{\partial}{\partial x_2} H(x(t), u(t), \lambda(t)) = -\lambda_1(t)$$

so that λ_1 is a constant, which we denote by d, and λ_2 is the straight line $-dt + f$, $f \in \mathbb{R}$.

Since we can exlude[6] the case $d \geq 0$, we fix $d < 0$: $\lambda_2(t)$ is therefore a line with positive slope $(-d)$ such that

$$\lambda_2(t) < -b \qquad \text{for}\, t \in [0, \tau_1) \tag{3.43}$$

$$-b \leq \lambda_2(t) < b \qquad \text{for}\, t \in [\tau_1, \tau_2) \tag{3.44}$$

$$\lambda_2(t) \geq c \qquad \text{for}\, t \in [\tau_2, T] \tag{3.45}$$

with $\tau_1 = \frac{f+c}{d}$ and $\tau_2 = \frac{f-c}{d}$ (see *Figure 3.9*).

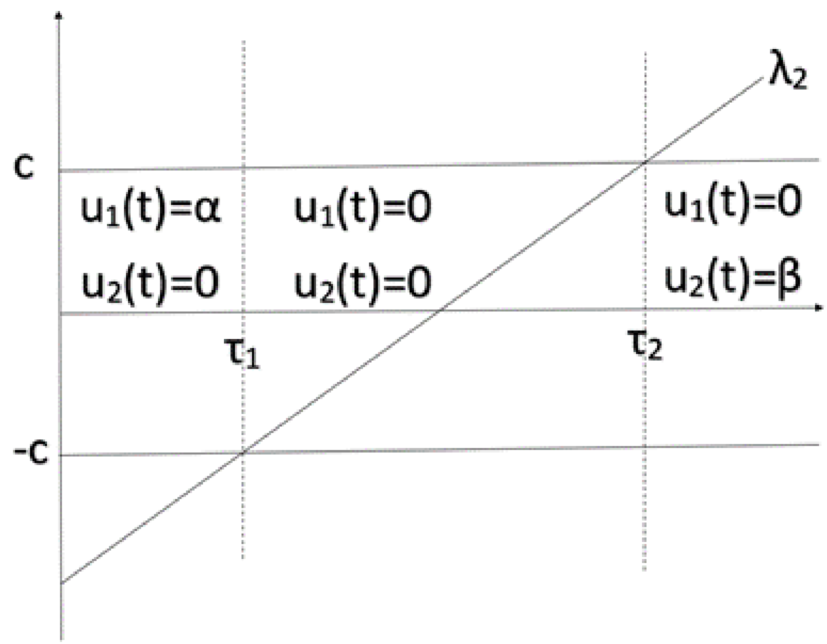

Figure 3.9:

By (3.40), (3.41) and (3.42), we have:

$$\begin{matrix} u_1(t) = \alpha \\ u_2(t) = 0 \end{matrix} \qquad t \in [0, \tau_1) \tag{3.46}$$

$$\begin{matrix} u_1(t) = 0 \\ u_2(t) = 0 \end{matrix} \qquad t \in [\tau_1, \tau_2) \tag{3.47}$$

$$\begin{matrix} u_1(t) = 0 \\ u_2(t) = \beta \end{matrix} \qquad t \in [\tau_2, T \tag{3.48}$$

[6]See Appendix, where is also shown $f < -b$.

Note how this time, differently from the previous problem, λ_2 splits the time interval $[0, T]$ in three segments (instead of two).

Similarly to the previous case, the control $u(t)$ takes a *bang-bang* form; however, we note tha in the central interval $[\tau_1, \tau_2]$ the car neither accelerates or brakes, proceding on the whole interval with the same the velocity $(\beta\tau_1)$ that was reached at the end of the first interval $([0, \tau_1])$.

We therefore ascertain that the introduciton of the new parameters a and b has changed the solution.

Now, we can integrate $\dot{x}_2$ on the interval $[0, T]$, with u_1 and u_2 following the law given by (3.46), (3.47) and (3.48) and get

$$x_2(t) = \begin{cases} \alpha t & t \in [0, \tau_1] \\ \alpha\tau_1 & t \in [\tau_1, \tau_2] \\ \alpha\tau_1 + \beta\tau_2 - \beta t & t \in (\tau_2, T] \end{cases} \tag{3.49}$$

then integrate $\dot{x}_1(t) = x_2(t)$ and obtain

$$x_1(t) = \begin{cases} \frac{\alpha}{2}t^2 & t \in [0, \tau_1] \\ \alpha t\tau_1 - \frac{\alpha}{2}\tau_1^2 & t \in [\tau_1, \tau_2] \\ \frac{\alpha}{2}\tau_1^2 + (\alpha\tau_1 + \beta\tau_2)(t - \tau_2) - \frac{\beta}{2}(t^2 - \tau_2^2) & t \in (\tau_2, T] \end{cases} \tag{3.50}$$

Now, we can use the two remaining conditions ($x_2(T) = 0$ and $x_1(T) = F$) to gain further information.
The first ($x_2(T) = 0$), together with (3.49) provides

$$\beta T = \alpha\tau_1 + \beta\tau_2 \tag{3.51}$$

and the second ($x_1(T) = F$), together with (3.51) and (3.50) and[7] $f = -\frac{s}{\alpha} - c$, provides

$$T = \sqrt{2F\delta\frac{s^2 + 4c^2 + 4sc}{s^2 + 4sc}}$$

with

$$\delta = \frac{\alpha + \beta}{\alpha\beta}$$

3.3.1 Analysis of results

Respect to the previous problem, where the only objective was to minimize time, here a "trade-off" is introduced by assigning a cost, parametrized by c, to the use of acceleration $a(t) > 0$ and braking $(b(t) > 0)$.

[7] See Appendix

The "bang-bang" situation of the previous problem here is partially modified: this time the control $u(t)$ takes three different values on three intervals ($[0, \tau_1]$, $[\tau_1, \tau_2]$ and $[\tau_2, T]$), where the optimal strategy is to use maximal acceleration ($u(t) = (\alpha, 0)$) in the first segment, then keeping a constant speed ($u(t) = (0, 0)$) on the second, and finally brake at maximum force in the third segment ($u(t) = (0, \beta)$).

In "a moving car 2", he "appearance" of the central interval where the car proceeds at constant velocity explained by the fact that, introducing a penalty for accelaration and braking, it may be more convenient to both accerate less and brake less, even if this implies increasing the final time T (depending also on the weight a put on time).
Indeed, it turns out that the proportion of the length $(\tau_2 - \tau_1)$ central interval over the whole time interval is[8]

$$\frac{\tau_2 - \tau_1}{T} = \frac{2c}{s\delta + 2c} \tag{3.52}$$

which goes to 1 as b dominates a.

A further observation is that, by (3.3) since, in the previous problem, we had

$$T^* = \sqrt{2F\frac{\alpha + \beta}{\alpha\beta}} = \sqrt{2F\delta} \tag{3.53}$$

the minimal time T of this problem can be expressed as

$$T = \sqrt{T^* \frac{s^2 + 4c^2 + 4sc}{s^2 + 4sc}} \tag{3.54}$$

where T^* is the optimal time of the old problem.

The term under square root converges to T^* as s grows respect to c; on the converse, it diverges to $+\infty$ as c dominates s.

[8] See Appendix

3.4 General cost function on state and control, free final time

Suppose a firm has recieved the request to produce amount F of some good; denote with $x(t)$ the amount the firm has produced at time t, and starting with $x(0) = 0$; the firm ends its production at the first time $T > 0$ such that $x(T) = F$; the production rate is $u(t)$, which determines $x(t)$ throught the relation

$$\dot{x}(t) = u(t)$$

Suppose $u(t) \in [0, \alpha]$ for all $t > 0$.

When the firm delivers the order, it recieves a payment

$$P(T) = P^* - \phi(T)$$

with $P^* > 0$ and $\phi'(t) > 0$. The firm therefore pays a cost equal to $\phi(T)$ for the time it takes to produce F.
Suppose also the firm sustains, at the end of the time period, a cost which depends the production-rate strategy through the function $g : U \to [0, +\infty)$, $g'(u) > 0$.

$$\int_0^T g(u(t))\, dt$$

and a cost depending on the production x, over the time period, through the function $f : \mathbb{R} \to \mathbb{R}$, $f'(x) > 0$:

$$\int_0^T f(x(t))\, dt$$

The cost functional of the firm is given by:

$$J(u) = \phi(T) + \int_0^T f(x(t)) + g(u(t)) \tag{3.55}$$

We have an optimal control problem with $n = 1$, $k = 1$, admissible set $\mathbb{C}$ of piecewise continuous functions $u : [0, +\infty) \to U = [0, \alpha] \cap \mathbb{R}$.

The problem of the firm can be written as

$$\begin{aligned}
&\min_{u \in \mathcal{C}} J(u) = \phi(T) + \int_0^T g(u(t)) + f(x(t))\, dt \\
&\dot{x}(t) = u(t) \\
&x(0) = 0 \\
&x(T) = F \\
&U = [0, \alpha]
\end{aligned}$$

We can start by writing down the Hamiltonian

$$H(x(t), u(t), \lambda_0, \lambda(t)) = \lambda_0(f(x(t)) + g(u(t))) + \lambda(t)u(t)$$

Suppose $\lambda_0 = 1$. We have

$$H(x(t), u(t), 1, \lambda(t)) = f(x(t)) + g(u(t)) + \lambda(t)u(t)$$

The first condition of the PMP gives the followingive

$$\begin{aligned} u(t) &= 0 & \qquad g'(u(t)) &> -\lambda(t) \\ u(t) &= \alpha & \qquad g'(u(t)) &< -\lambda(t) \end{aligned}$$

The adjoint equation gives

$$\dot{\lambda}(t) = -f'(x(t))$$

and the free-end time condition gives

$$f(x(t)) + g(u(t)) + \lambda(t) + \frac{\partial}{\partial t}\phi(T) = 0 \quad \forall\ t \in [0, T]$$

For the case $\lambda_0 =$, the Hamiltonian is

$$H(x(t), u(t), 0, \lambda(t)) = \lambda(t)u(t)$$

The minimization condition gives

$$\begin{aligned} u(t) &= 0 & \qquad \lambda(t) &> 0 \\ u(t) &= \alpha & \qquad \lambda(t) &< 0 \end{aligned}$$

From the adjoint equation we get

$$\lambda(\dot{t}) = 0$$

so that $\lambda(t) = c \in \mathbb{R}$.

The free-end time condition becomes

$$\lambda(t)u(t) + \frac{\partial}{\partial t}\phi(t_1) = 0 \quad \forall\ t \in [0, T]$$

which gives the extremal control $u(t) \equiv \alpha$ for $\frac{\partial}{\partial t}\phi(T) > 0$ (as assumed before) and gives no solutions otherwise.

3.4.1 Particular case

We now consider the particular case with $f(x) = x$, $g(u) = u^2$ and $\phi(t) = t$.

The cost functional is

$$J(u) = T + \int_0^T x(t) + u(t)^2 \, dt$$

Suppose $\lambda_0 = 1$ The Hamiltonian is

$$f(x(t)) + g(u(t)) + \lambda(t)u(t) = x(t) + u(t)^2 + \lambda(t)u(t)$$

The minimizing condition on $u(t)$ gives

$$u(t) = 0 \qquad -\frac{\lambda(t)}{2} < 0, \ t \in [0, T] \tag{3.56}$$

$$u(t) = -\frac{\lambda(t)}{2} \quad -\frac{\lambda(t)}{2} \in [0, \alpha], \ t \in [0, T] u(t) = \alpha \quad -\frac{\lambda(t)}{2} > \alpha, \ t \in [0, T] \tag{3.57}$$

From the adjoint equation we get

$$\dot{\lambda}(t) = -1$$

that implies $\lambda(t) = c - t$, $c \in \mathbb{R}$.

By substituting this result in (3.56) and (3.57) we have

$$u(t) = 0 \qquad t < c, \ t \in [0, T]$$

$$u(t) = -\frac{t - c}{2} \quad t \in [c, c + 2\alpha], \ t \in [0, T] u(t) = \alpha \quad t > c + 2\alpha, \ t \in [0, T]\alpha$$

From the free-final time condition

$$H(x(t), u(t), \lambda(t)) + \frac{\partial}{\partial t}\phi(T) = 0 \quad \forall \ t \in [0, T]$$

we obtain

$$x(t) + u(t)^2 + \lambda(t)u(t) + 1 = 0 \quad \forall \ t \in [0, T] \tag{3.58}$$

By substituting $u(t)$ with $\dot{x}(t)$ and $\lambda(t)$ with $c - t$ (**??**) becomes

$$x(t) + \dot{x}(t)^2 + (c - t)\dot{x}(t) + 1 = 0$$

Further computations (shown in the Appendix relatively to this exercise) lead to the following result: if $\alpha \leq 1$, the only extremal strategy is $u(t) \equiv \alpha$, while if $\alpha > 1$, there are two extremal strategies u_1 and u_2:

$$\begin{aligned}
u_1(t) &= \begin{cases} \frac{t^2}{4} + t & t \in [0, 2(\alpha - 1)] \\ u(t) = \alpha & t > 2(\alpha - 1) \end{cases} \\
u_2(t) &\equiv \alpha \qquad\qquad\qquad\qquad\qquad\qquad t > 0
\end{aligned}$$

The optimal among the two is the one for which the functional (3.55) is smaller.

Conclusions

I have applied Pontryagin's Maximum Principle to three different cases.

In the first one, *Cost minimization with a general cost function*, I analized the problem of a firm which has to supply a given quantity of a good at a specific date. The firm pays a cost, linear in the production (rate), which varies in time according to a known function. The solution provided by the theorem is to alternate periods of maximum production intensity with periods of minimum intensity, with the switching times between one mode and the other corresponding to the same level of cost. Such level is to be determined so to reach the given quanity at the end of the period.

In the second case, *A moving car* and its variant *A moving car 2*, I look at the problem of a car which has to cover a given space, in the minimum possible time. In the first variant, there are no costs associated with accelerating and braking, so the optimal solution is to accelerate at the maximum rate for a first segment of the time interval, and then to brake at maximum force in the remaining time. In the second variant, a cost is associated with accelerating and braking, and the solution becomes a three-segmented strategy, with an additional central period of constant speed. Pontryagin's Maximum Principle gives us the ratio of this segment over the whole interval as a function of the cost parameters, expliciting their effect on the solution.

In the third problem, I examined a firm which, having to supply a given quantity, faces the following conditions: a linear time cost, a quadratic production rate cost and a linear warehouse cost. In this case, two possible strategies arise from the application of the Maximum Principle: which of the two is optimal depends on parameter α, that expresses the maximum possible production rate.

From these results, we understand that the Maximum principle is able to tell us which parameters are relevant for the solution of a problem, and how each of them affects the final result.

Appendix A

Appendix

A.1 Exercises computations

A.1.1 Cost minimization with general price function

$\lambda_0 = 1$

Suppose $\lambda_0 = 0$. We would have

$$H(t, x, u, 0, \lambda) = \lambda u \tag{A.1}$$

For the Maximum Principle equation (2.1) in 2.1, we have

$$\begin{aligned} u(t) = 0 \quad \lambda(t) > 0 \\ u(t) = \alpha \quad \lambda(t) < 0 \end{aligned}$$

For the *adjoint equation* we have

$$\dot{\lambda}(t) \equiv 0$$

so $\lambda(t) \equiv c \in \mathbb{R}$.

If $c > 0$, than $u(t) \equiv 0$, so $x(t) \equiv 0$ and, in particular, $x(T) = 0 \neq F$.
If $c < 0$, than $u(t) \equiv \alpha$, so $x(T) = \alpha T > F$.
If $c = 0$, the non triviality condition ($(\lambda_0, \lambda(t)) \neq 0$ for all $t \in [0, T]$) is violated.

A.1.2 A moving car

Proof that $\lambda_0 = 1$

The Hamiltonian is:

$$H(t, x(t), u(t), 0, \lambda_(t) =< \lambda(t), g(t, x(t), u(t)) >= \lambda_1(t) x_2(t) + \lambda_2(t) u_1(t)$$

The minimum respect to $u_1(t)$ and $u_1(t)$ of the Hamiltonian is reached for

$$\begin{cases} u_1(t) = \alpha \\ u_2(t) = 0 \end{cases} \quad \text{for } \lambda_2(t) < 0 \begin{cases} u_1(t) = 0 \\ u_2(t) = \beta \end{cases} \quad \text{for } \lambda_2(t) > 0$$

Morover, we have

$$\nabla_{x(t)} H(t, x(t), u(t), 0, \lambda_(t) = (0, \lambda_1(t))$$

so we have

$$\dot{\lambda} = (0, -\lambda_1(t)) \tag{A.2}$$

(A.2) implies that $\lambda_1(t)$ is a constant, which we call d.
Then we have $\lambda_2(t) = -dt + f$ with $f \in \mathbb{R}$. Fix $\tau = \frac{f}{d}$ if $d \neq 0$.
We must necessairly have $d < 0$ and $f < 0$.
In fact, suppose $d = 0$; this implies $\lambda_2(t) = f$. Now

1. If $f > 0$, than $\lambda_2(t) > 0$ for every $t \in [0, T]$. By (A.5), this would imply $(u_1, u_2)(t) = (0, \beta)$for every $t \in [0, T]$; now, for (A.5), this would imply $x_2(t) < 0$ for every $t \in (0, T]$, violating condition $x_2(T) = 0$.
2. If $f = 0$, than we would have $\lambda_0 = 0$, $\lambda_1(t) \equiv 0$ and $\lambda_2(t) \equiv 0$, violating the non-triviality condition.
3. If $f < 0$, $\lambda_2(t) < 0$, implying $(u_1, u_2)(t) = (\alpha, 0)$, so that $x_2(t) > 0$, violating $x_2(T) = 0$.

If $d > 0$

1. If $f = 0$ we have $(u_1, u_2)(t) = (0, \beta)$, so $x_2 > 0$ violating $x_2(T) = 0$.
2. If $f < 0$ we have $u(t) = \beta$, violating $x_2(T) = 0$.

So we have $f > 0$.
This implies $(u_1, u_2)(t) = (0, \beta)$ for $[0, \frac{f}{d}]$, so that $x_2(\tau) = -\beta\tau$, $x_2(\tau\frac{\alpha+\beta}{\alpha}) = 0$ and $x_1(t) < [0, \tau\frac{\alpha+\beta}{\alpha}]$.
Since $u(t) = \alpha$ on $[\tau\frac{\alpha+\beta}{\alpha}, T]$, we have $x_2(t) > 0$ on $[0, +\infty)$, violating $x_2(T) = 0$.

If $d < 0$, we must have $f < 0$.
In fact if $f \geq 0$ we have $(u_1, u_2)(t) = (\alpha, 0)$ on $(0, T]$, so $x_2(t) < 0$ on $[0, T]$, violating $x_2(T) = 0$.

If $d, f < 0$, we have

$$\begin{cases} u_1(t) = 0 \\ u_2(t) = \beta \end{cases} \quad \text{for } t \in [0, \tau] \begin{cases} u_1(t) = \alpha \\ u_2(t) = 0 \end{cases} \quad \text{for } t \in (\tau, T] \tag{A.3}$$

and

$$x_1(t) = -\frac{\beta}{2}t^2 \quad \text{on } [0,\tau)$$
$$x_1(t) = -\frac{\beta}{2}\tau^2 - \tau(\alpha+\beta)(t-\tau) + \frac{\alpha}{2}(t^2-\tau^2)$$

Now, for the free-final time autonomous problem condition

$$H(t, x(t), u(t), \lambda_0, \lambda(t)) =$$

we have

$$\lambda_1(t)x_2(t)d + \lambda_2(t)(u_1(t) - u_2(t))$$
$$d\beta t + (-dt + f)\beta \quad \text{on } [0,\tau]$$

that is, $\beta f = 0$, which contradicts $\beta > 0$, $f < 0$.

Proof that $d < 0$ for $\lambda_0 = 1$

If $d = 0$, we have

1. For $f > 0$ we have $(u_1, u_2)(t) \equiv (0, \beta)$ (which violates $x_2(T) = 0$).
2. For $f < 0$ we have$(u_1, u_2)(t) = (\alpha, 0)$, which violates $x_2(T) = 0$. For $f = 0$, than, for $H(t, x(t), u(t), 1, \lambda_(t)) \equiv 0$ we have $1 = 0$, that is false.

If $d > 0$, we have $f > 0$, since for $f \leq 0$ we have $(u_1, u_2)(t) \equiv (\alpha, 0)$ so $x_2(t) > 0$ violates $x_2(T) = 0$.

For $d, f > 0$, we have

$$\begin{cases} u_1(t) = 0 \\ u_2(t) = \beta \end{cases} \quad \text{for } t \in [0,\tau] \begin{cases} u_1(t) = 0 \\ u_2(t) = \beta \end{cases} \quad \text{for}\, t \in (\tau, T] \tag{A.4}$$

We would have $x_2(t) < 0$ on $(0, \tau\frac{\alpha+\beta}{\beta})$, so $x_1 < 0$ on $[0, \tau\frac{\alpha+\beta}{\beta}]$, so that $x_1(t) = F$ is never verified on this interval.
Then, since $u(t) = \beta$ on $[\tau\frac{\alpha+\beta}{\beta}, T]$, we have $x_2(t) > 0$ on $[\tau\frac{\alpha+\beta}{\beta}, T]$, violating $x_2(T) = 0$.

Proof $f = -\frac{1}{\alpha}$

We have

$$\begin{cases} u_1(t) = \alpha \\ u_2(t) = 0 \end{cases} \quad \text{for } t \in [0,\tau] \begin{cases} u_1(t) = 0 \\ u_2(t) = \beta \end{cases} \quad \text{for}\, t \in (\tau, T] \tag{A.5}$$

So the condition

$$H(t, x(t), u(t), 1, \lambda_(t)) \equiv 0 \tag{A.6}$$

implies

$$1 + d\alpha t + (-dt + f)\alpha = 0 \quad \text{for } t \in [0, \tau) \tag{A.7}$$

gives $f = -\frac{1}{\alpha}$.

A.1.3 A moving car 2

$f = -\frac{s}{\alpha} - c$

For the free end time condition

$$H(x(t), u(t), \lambda(t)) = 0 \quad \forall t \in [0, T]$$

we have

$$s + \alpha c + d\alpha t + (-dt + f)\alpha \quad \forall t \in [0, \tau_1)$$

that gives the result.

Central interval proportion

Since $f = -\frac{s}{\alpha} - c$, $\tau_1 = \frac{f+c}{d}$ and $\tau_2 = \frac{f-c}{d}$, we have $\tau_1 = -\frac{s}{\alpha d}$ and $\tau_2 = -\frac{s}{\alpha d} - \frac{2c}{d}$, so $\tau_2 = \tau_1 - \frac{2c}{d}$.

Now, from $\beta T = \alpha\tau_1 + \beta\tau_2$, by we have

$$\beta T = -\frac{s}{d} - \beta\left(-\frac{s}{\alpha d} - \frac{2c}{d}\right)$$

from which we get

$$d = -\frac{\sigma}{T}$$

with

$$\sigma = s\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + 2c = s\delta + 2c$$

remembering $\delta = \frac{\alpha+\beta}{\alpha\beta}$.
This implies

$$\frac{\tau_2 - \tau_1}{T} = -\frac{2c}{d} = 2c\frac{T}{\sigma}\frac{1}{T} = \frac{2c}{\sigma} = \frac{2c}{s\delta + 2c}$$

A.1.4 General cost function on state and control, free final time

Case $\lambda_0 = 1$

We have

$$u(t) = 0 \qquad t < c,\ t \in [0,T] \tag{A.8}$$
$$u(t) = -\frac{t-c}{2} \qquad t \in [c, c+2\alpha],\ t \in [0,T] \tag{A.9}$$
$$u(t) = \alpha \qquad t > 2\alpha + c,\ t \in [0,T]\alpha \tag{A.10}$$

and

$$x(t) + \dot{x}^2 + (c-t)\dot{x}(t) + 1 = 0 \quad \forall\, t \in [0,T] \tag{A.11}$$

Deriving the LHS of (A.11) respect to t:

$$\dot{x} + 2\dot{x}\ddot{x} - \dot{x} + (c-t)\ddot{x} = 2\dot{x}\ddot{x} + (c-t)\ddot{x} = 0 \quad \text{for a.e. } t \in [0,T]$$

There are two solutions to (A.5): $\ddot{x} = 0$ and $\dot{x} = \frac{t-c}{2}$. These solutions must be intended in the sens that, for every sub-interval of $[0,T]$, it must be veriefied either one or the other.

If, $\ddot{x} = 0$, than $u(t)$ is constant; for (A.8), (A.9) and (A.10), this can be true only if $u(t) \equiv 0$ or $u(t) \equiv \alpha$.
Suppose $u(t) \equiv 0$; by substituting in (A.11), we get $x(t) = -1$, which is impossible since $x(0) = 0$ and $\dot{x} = u \geq 0$, so $x(t) \geq 0$ for all $t > 0$.

If $u(t) = \alpha$, by substituitng in (A.11) we get $x(t) = \alpha^2 - \alpha c + \alpha t - 1$; moreover, for (A.10), we also need $t \geq 2\alpha + c$.

If $u(t) = \frac{t-c}{2}$; by substituiting in (A.11), we get $c^2 = 4$, so $c = \pm 2$; for (A.9), we also need $t \in [c, c+2\alpha]$.
If $c = 2 > 0$, for (A.8) we'd have $u(t) = 0$ in $[0,c]$, which has already been excluded; therefore $c = -2$.

Now, we can divide the possible cases in two: $c = -2$ and $c \neq -2$.

If $c \neq 2$, the case $u(t) = \frac{t-c}{2}$ is exluded, so we can only have $u(t) \equiv \alpha$ on $[0,T]$; for (A.10), this requires $2\alpha + c \leq 0$, so $c \leq -2\alpha$.
By substituting in $u(t) \equiv \alpha$ in (A.11) we have, thanks to the initial condition $x(0) = 0$:

$$x(0) = 0 = -\alpha^2 - \alpha c - 1$$

that gives $c = -\frac{\alpha^2+1}{\alpha}$.

The condition required before, $2\alpha + c \leq 0$, gives $\alpha \leq 1$ (and, in particular,

$\alpha < 1$ since $\alpha = 1$ implies $c = -2$).

To resume, if $c \neq -2$, than $c = -\frac{\alpha^2+1}{\alpha}$ and $\alpha < 1$.

Now, suppose $c = -2$; by substituting it in (A.8), (A.9) and (A.10), and excluding the case $u(t) = 0$, we have

$$u(t) = \frac{t}{2} + 1 \qquad t \in [0, 2(\alpha - 1)] \tag{A.12}$$

$$u(t) = \alpha \qquad t > 2(\alpha - 1) \tag{A.13}$$

for $\alpha > 1$ and

$$u(t) = \alpha \qquad t \in [0, T]$$

if $\alpha \leq 1$.

Suppose $\alpha > 1$; than, by integrating (A.12) and (A.13):

$$x(t) = \frac{t^2}{4} + t \qquad t \in [0, 2(\alpha - 1)] x(t) = -(\alpha - 1)^2 + \alpha t \qquad t > 2(\alpha - 1)$$

Now, if the firm reaches the production level F for $t \leq 2(\alpha - 1)$, we have $T \in [0, 2(\alpha - 1)]$; this implies

$$x(T) = \frac{T^2}{4} + T = F$$

that gives $T = 2[\sqrt{F+1} - 1]$.

From $T \leq 2(\alpha - 1)$, we obtain $\sqrt{F+1} \leq \alpha$, which requires (as has already been assumed for the case under the exam) $\alpha > 1$ (since $F > 0$).
If,instead, we had $\sqrt{F+1} > \alpha$; than $T > 2(\alpha - 1)$, which gives $T = \frac{F}{\alpha}$. If $\alpha \leq 1$, we have $u(t) = \alpha$ on $[0, T]$ and $T = \frac{F}{\alpha}$.

Case $\lambda_0 = 0$

The Hamiltonian, in this case, is

$$\lambda(t) u(t)$$

For the Minimum principle, we have

$$u(t) = \alpha \qquad \lambda(t) < 0 \tag{A.14}$$

$$u(t) = 0 \qquad \lambda(t) > 0 \tag{A.15}$$

For the adjoint equation we have $\dot{\lambda}(t) \equiv 0$, so $\lambda = c \in \mathbb{R}$.

Exluding $c = 0$ for non-triviality, we have

$$u(t) = \alpha \qquad c < 0 \tag{A.16}$$
$$u(t) = 0 \qquad c > 0 \tag{A.17}$$

The free final time condition gives

$$H(x(t), u(t), \lambda(t)) + \frac{\partial}{\partial t}\phi(T) = \lambda(t)u(t) + 1 = 0 \quad \forall\ t \in [0, T] \tag{A.18}$$

If $c > 0$, than $u(t) = 0$, so (A.27) gives $1 = 0$.
If $c < 0$, than $u(t) \equiv \alpha$, so we get $c = -\frac{1}{\alpha}$.

A.2 Optimal Control Theory: from the origins to nowadays

Optimal control theory can be conceived as the last step of a long journey, which started with geometrical problems of Ancient Greek philosphers, went throught the 17^{th}, 18^{th} and 19^{th} centuries when the Caclulus of Variations (the "ancessor" of optimal control theory) whas developed, ending with the fundamental contributions of Lev Pontryagin and Richard Bellman in the 1950s'.

As a premise to the series of events that led to maturity of Calculus of Variation, and eventually, of Optimal control Theory, here is the standard form of a Calculus of Variations problem:

$$\min_{x \in C^1} \max \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, dt \tag{A.19}$$

$$x(t_0) = x_0, \qquad x(t_1) \in X_f \subseteq \mathbb{R}^n \tag{A.20}$$

where $n \in \mathbb{N}$.
The problem asks to find the function $x : [t_0, t_1] \to \mathbb{R}^n$ that respects (A.19) and (A.20). Calculus of Variations is the subject that studies how to find the funciton $x(t)$ that minimizes/maximizes intergral (A.19), while respecting the initial and final conditions (A.20).

Now, one of the most antique mathematical problems that can be arranged in this form was studied for the first time, around 300 B.C., by the greek philosopher Euclid of Alexandria. He was trying to find the shortest curve that connects two points (A and B) on a plane (*Figure A.1*).The solution may seem obvious: the straight line is the shortest. However, proving this mathematically is not as trivial. Indeed, Euclid did not leave the burden to others (some centuries after). If the two points A and B have coordinates (t_0, a) and (t_1, b), the problem can be written in a Calculus of Variations structure as this: :

$$\begin{aligned} &\min_{x \in C^1} \int_{t_0}^{t_1} \sqrt{1 + x'(t)^2}\, dt \\ &x(t_0) = a \\ &x(t_1) = b \end{aligned}$$

Another example of a Calculus of Variations problem is contained in the epic of queen Dido of Virgil's Aeneid. Dido was the daughter of the Phoenician king in the 9th century B.C., and was forced to a long exile by her brother, Pigmalion, who assassinated her husband. At the end of a long journey in the Mediterrean see, Queen Dido ended up in Tunis. There, king Iarba allowed her to delimit the perimeter of a piece of land which would the have become her own city (Carthage). The queen was allowed to sign the border using only a

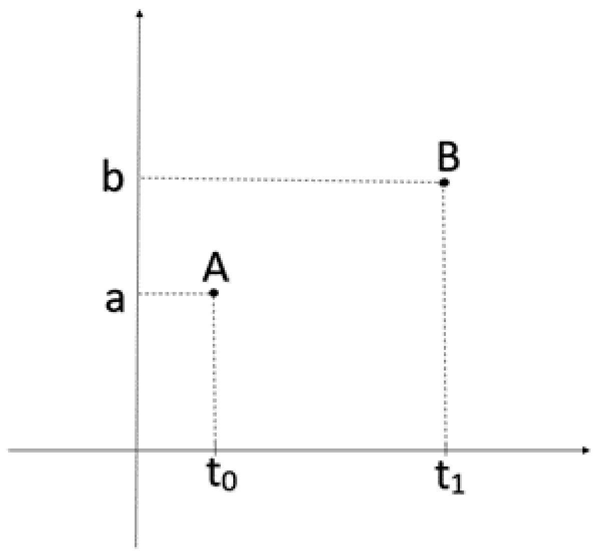

Figure A.1:

bull's hide. Now, moving to a mathematical formulation, if we hypothesize that the animal's skin had a given length B, this problem can be written as

$$\max \int_{t_0}^{t_1} x(t)\,dt$$
$$x(t_0) = x(t_1) = 0$$
$$\int_{t_0}^{t_1} \sqrt{1 + x'(t)^2} = B$$

with t_1 a free variable (*Figure A.2*).
This problem, which falls in the cathegory of *Isoperimetric problems*, is also present in the collected in the mathematical book of Pappus of Alexandria (Mathamatical Collection, Book 5) where are inserted several discoveries from characters such as Eulcid, Archimedes and Zeonodrus.

For several centuries, nothing relevant happened in the research of Calculus of Variations, and we move directly to the 17^{th} century. At that time Pierre De Fermat was studying the problem of finding the path traversed by a ray of light while being refracted through a medium with varying density, such air and water, or reflected by a surface, such as a flat or spherical mirror.
Fermat stated that "nature works in those ways which are easier and faster". Fermat's Principle, that is, that light minimizes the time it takes to go from a point A to a point B, equals affirming that light chooses the trajectory

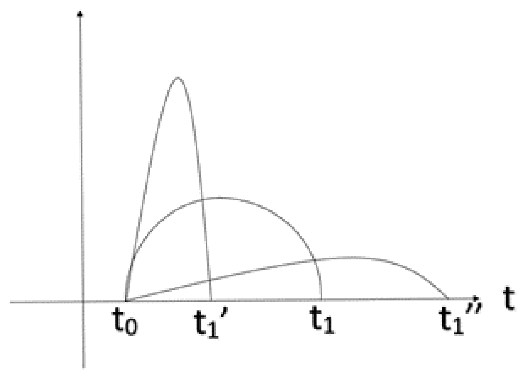

Figure A.2:

$(y(x), z(x))$ that solves the following Calculus of Variations problem:

$$\min_{x \in C^1} \int_{x_0}^{x_1} \frac{\sqrt{z'(x)^2 + y'(x)^2 + 1}}{v(x, y, z)} \, dt$$
$$(z, y)(x_0) = (z_0, y_0) \qquad\qquad (z, y)(x_1) = (z_1, y_1)$$

where $(z, y) : x \mapsto (z, y)$ is the function that defines the position is space of the ray of light, $v(x, y, z)$ is the velocity of light that must be expressed as a function of the coordinate (x, y, z), and (z_0, y_0, x_0) and (z_1, y_1, x_1) are the two points the light passes throught.

Towards the end of the same century, two other the important mathematicians of the time, Isaac Newton and G.W. von Leibniz, had developed, working completely independently from each other, the infinitesimal calculus. The modern definition of "derivative" was, for the first time, published by Newton in 1704 in the *Acta eruditorum*, one of the most influential scientific journals.

Despite the fundamental contribution of Newton and Leibnitz, it is nowadays believed that the event which really lighted the fuse of research in the Calculus of Variations happened in 1696. At the time Johann Bernoulli, who was the youngest member of one of the most renowned families of mathematicians (the Bernoullis), published, again on the *Acta eruditorum*, a problem known as the "Brachistochrone" (form ancient greek *brachistos*, shorter, and *chronos*, time).

The problem proposed by Bernoulli was about finding the curve that an object, falling under gravity force, would follow to cover the distance between two points, the origin $(0, 0)$ and $B = (x_B, y_B)$ (*Figure A.3*) in the minimum possible time

$$\min y \in C^1 \frac{1}{\sqrt{2g}} \int_0^{x_B} \frac{\sqrt{1+y'(x)}}{\sqrt{y(x)}}\, dx$$
$$y(0) = 0$$
$$y(x_B) = y_B$$

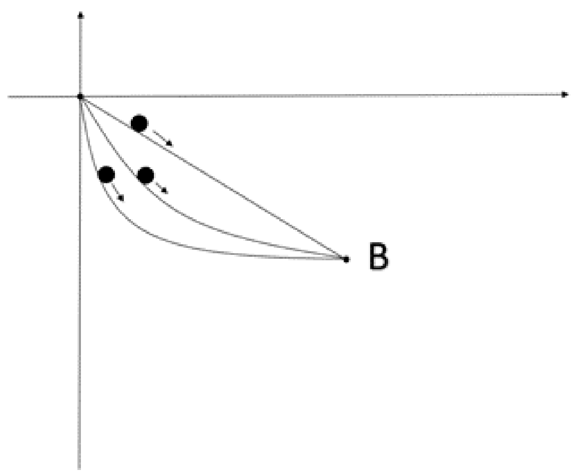

Figure A.3:

Among the great minds that solved this problem were Leibniz, de l'Hopital, Newton and Johann's younger brother, Jacob (with whom he had a quite thorny relationship). The solution of the problem corresponds to the *cycloid* generated by a point on a circumference, such that this point is at the origin and then, by making the circumference rotate on the x axis, we should have that A equals to the arch of circumference AB (*Figure A.4*).

Since this problem had some similarities with *Fermat's* principle of the time minimizing light's path, Johann Bernoulli praiesed himself to "have with one blow solved two fundamental problems, one optical and the other mechanical".

A few years later, on of the biggest mathematicians of the 18^{th} century, Leonhard Euler, entered the scene in the Calculus of Variations Odyssey. His father, Paul Euler, was in a tight bound with the Bernoullis: they were both of Protestant faith and Paul, while studying theology at the university of Basel, was hosted by Jacob Bernoulli, who had a chair of Mathematics at the same university.

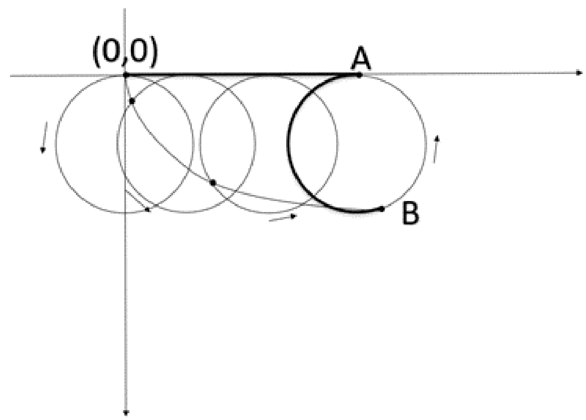

Figure A.4:

When Jacob died, Johanne took his chair, and some years later, the thirteen years old son of Paul, Leonhard, become his student. Instead of following in the footsepts of his father and become a protestant minister, he captured Johanne's attention for his extra-ordinary mathematical talent. The two formed a close friendship, and Johann encouraged the young Euler to study advanced textbooks in mathematics, while giving him also private lessons.
A few years later, Leonhard had already published his first mathematical textbook ("On finding the equation of geodesic curves") and was actively working on isoperimetric problems. In 1744 Euler published what would become the work that estabilished the Calculus of Variation a proper field of study, *Methodus Inveniendi Lineas Curva [...]*.
In the book, problems were scripted in the forms (A.19), (A.20) reported at the beginning of this section. Moreover, Euler proved the fundamental result of Calculus of Variations:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}(t)}L(t,x(t),\dot{x}(t)) = \frac{\partial}{\partial x(t)}L(t,x(t),\dot{x}(t)) \tag{A.21}$$

known as the *Euler Equation.*

Some years later, in 1755, a young teenager of Turin, Joseph Lagrange, sent a document to Euler, where he had reworked some of the results in *Methodus Inveniendi Lineas Curva [...]* purely analytically, instead of employing the geometric procedures of the author. Euler was enthusiastic of Lagrange's new treatment of the argument, since it was was arranged in a much more elegant and convenient way.

Euler himself re-defined the maximum-minimum problems he was working on, which were before denoted as “isoperimetric problems”, as “calculi variationum” problems, borrowing the term the “δ calculus” that Lagrange used in his work, where “δ” stood for “variation”.
From there on the Euler euqation (A.21) has become known to the world’s mathemathical community as the *Euler-Lagrange* equation.

Almost a century after, in 1830s’, an Irish mathematician, William Hamilton, was studying a subject today known as *Hamiltonian mechanics.* Hamilton employed the same integral form (A.19) to describe the law of motion of matter. In particular, the integrated function, $L(t, x(t), \dot{x}(t))$, expressed the difference bertween kinetic and potential energy of a particle at time t, while $x(t)$ was the space coordinate vector.

His *Principle of Least action* affirms that the law of motion is the function $x(t)$ that minimizes the integral, that is, the solution of the relative Caluclus of Variations problem.
However, instead of using the Euler-Lagrange Equation, he set up this system, known as*Hamiltonian system*:

$$\begin{aligned}
&H(t, x(t), \lambda(t)) =< \lambda(t), \dot{x}(t, x(t), \lambda(t) > -L(t, x, \dot{x}) \qquad (A.22)\\
&\dot{x} = \frac{\partial}{\partial \lambda(t)} H(t, x(t), \lambda(t))\\
&\dot{\lambda}(t) = -\frac{\partial}{\partial} H(t, x(t), \lambda(t))\\
&\lambda(t) = \frac{\partial}{\partial \dot{x}(t)} L(t, x(t), \dot{x}(t))
\end{aligned}$$

These equations can be proved to be equivalent to the Euler Lagrange equation.

Hamilton’s contribution was fundamental, since the Hamiltonian function (A.22) became a an essential component of Optimal Control Theory.

A few years later, Jacobi, one of the main contributors to Optimal Control Theory, reviewed Hamilton’s work and noticed that some results could be better expressed with this partial differential equation (now known as Hamilton-Jacobi equation):

$$V_t(t, x(t)) + H(t, p, x(t)) = 0 \qquad (A.23)$$

The function $V(t, x(t)$ was called “action function” (now *value function*), and gives the value of the integral (A.19) when $x(t)$ is a minimizing function, at the starting point $t_0 = t$ and $x_0 = x$. If $x(t)$ is an minimizing function, than $V(t, x)$ must verify (A.23) for all $(t, x(t))$, $t \in [t_0, t_1]$.

Optimal control theory’s origins date around 1950s’; the focus of mathematical engineering was on finding optimal driving inputs for controlled dynamical systems (of the kind displayed in *Figure* **??**); the transition from the Calculus of Variations structure to the Optimal Control's arrangement occurred by expliciting the derivative of the x function, $\dot{x}$, by constraining it to be a solution of a differential equation:

$$\dot{x}(t) = g(t, x(t), u(t)) \quad \text{for a.e. } t \in [t_0, t_1] \tag{A.24}$$

where $u(t)$ is the *control function*; the objective of minimizing (or maximizing) the integral (A.19) remains the same, even thought it's not achieved by direclty choosing a function $x(t)$ (or its derivative) as in Calculus of Variation, but by "controlling" its derivative.

During this period (1950's/60s'), he advent of computers allowed to process mathematical operations that were not even thinkable up to that time; moreover, the end of the Second World War worked as a powerful propellent for research in all fields of technology.

The main contributions to Optimal Control Theory were those of Richard Bellman and Lev Pontryagin's and his group.

Richard Bellman was an american mathematician employed at RAND (Research AND develpement), as institution founded in USA in 1948, after the War made clear how vital technological progress and scientific knowledge were in the struggle for a competitive advantage over other Countries.
The objective of this no-profit governmental organization was to reunite the brightest intellectuals of USA (mathematicians, engineers, physicists, chemists, etc.) to quicken the progress of military or aerospace technologies.
Bellman's work evolved from Calculus of Variations to a new and stricIty algorithm-related field: dynamic programming.

This is the "Bellman's equation":

$$-\frac{\partial}{\partial t}V(t, x(t)) = \max_{u(t)\in U} H(t, x(t), u(t), \frac{\partial}{\partial x(t)}V(t, x(t)))$$

known also as Hamilton-Jacobi-Bellman's. It's originated by inserting a *feedback control* function in the Hamilton Jacobi equation (A.23), where the classical Hamiltonian (A.22) was substituted by *Maximized Hamiltonian*, and$\dot{x}$ was substituted with it's derivative g of the Optimal Control structure (A.24).

Similarly as in the US, also in the USSR intellectuals, such as mathematicians and phisicists, were pushed towards development of military technologies. Lev Pontryaginwas one of the leading mathematicians of Russia and head of the Department of Differential equations at the Steklov Institute in Moscow, who devoted his deep knowledge of dynamical systems to advancements in optimal control theory.

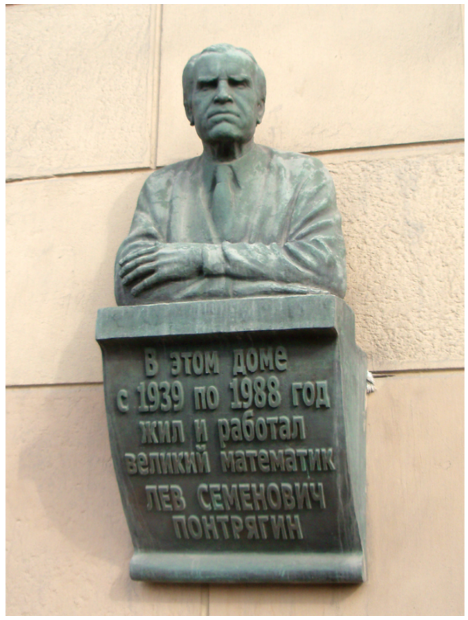

Figure A.5: monument to Lev Pontryagin in Moscow

Pontryagin's Maximum Principle (equations (A.25), (A.26) and (A.27)) was developed by Lev himself and some of his students: V.G. Boltianskii, R.V. Gamkrelidze and E.F. Mischelnko. In 1961, they published their results in "The Mathematical Theory of Optimal Processes", and were rewarded with the *Lenin Prize*

$$u(t) \in \arg\,\max_{u(t) \in U} H(t, x(t), u(t), \lambda(t)) \tag{A.25}$$

$$\dot{\lambda}(t) = -\frac{\partial}{\partial x(t)} H(t, x(t), u(t), \lambda(t)) \tag{A.26}$$

$$\lambda(t_0) = 0 \tag{A.27}$$

With respect to Bellman's equation, the PMP provides a less computation heavy approach, however with a more limited field of applicability.

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