Optimal Monetary Policy with Liquidity Shock

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Abstract

This paper studies Ramsey-optimal monetary policy under commitment in a small-open economy with uninsurable idiosyncratic income and liquidity shocks, long-term nominal bonds and costly inflation. While my theoretical characterization of the optimal inflation plan is consistent with that in Nuno and Thomas (2020), the inclusion of liquidity shock has two significant implications for the equilibrium distributions of wealth and consumption, which consequently affect optimal inflation. First, debtors, especially those that are liquidity-constrained, consume much less, so redistributing resources to them is more important. Second, households in my model save more to insure for the liquidity shock, causing economy-aggregate net asset position to improve and cross-border redistribution loss to emerge.

1 Introduction

In recent years there have been renewed academic and policy interests in the redistributive effects of monetary policy, among other implications of household heterogeneity for the macro-economy. On the methodological side, advancements in computational techniques have enabled economics researchers to solve for and analyze monetary policy in macroeconomic models featuring complex settings of incomplete market and heterogeneous agents.

I build on Nuno and Thomas (2020)'s continuous-time model to analyze optimal monetary policy under commitment in an economy with uninsurable idiosyncratic income and *liquidity shocks*. As in their paper, I start with a Huggett (1993) model of incomplete market, i.e. uninsurable idiosyncratic income shock. In addition, households once hit with an income shock also have a probability of being hit by a simultaneous liquidity shock, which leaves them constrained at their respective current net asset position. Being unconstrained, the households can trade non-contingent claims, i.e. long-term nominal bonds, subject to an exogenous borrowing limit, in order to smooth consumption in the face of the idiosyncratic shocks. Importantly, I consider a small open economy, where assets are priced by risk-neutral foreign investors and the asset market does not have to clear domestically.

The novelty of this paper lies in the inclusion of an idiosyncratic and exogenous liquidity shock, in the fashion of Bilbiie (2020). This new shock has two implication for the model. First, the shock generates a proportion of households who cannot borrow or dissave to smooth their consumption in the face of a bad income shock. As opposed to Bilbiie (2020) and other heterogeneous-agents models (e.g. Kaplan et al. (2018)) where liquidity-constrained or hand-to-mouth households simply consume their labor income, liquidity-constrained households in my model still have to pay nominal interest on their debt (or receive nominal interest from their lending). Second, the unconstrained households are aware of the liquidity shock, so they in choosing their consumption and saving also try to partly insure for such event. As consumption in the constrained state is proportionate to their asset position, households have incentive to increase savings compared to a setting where there is no liquidity shock. This generally shifts the wealth distribution to the left and improves the economy's aggregate net asset position.

By introducing the liquidity shock, I attempt to feature in my model the liquidity problem faced by many in an economy. With respect to the asset-poor workers, once they lose their job, they might have difficulty accessing consumption credit, as most lending institution typically requires not only collateral to borrow against but also proof of income. In addition, unsecured credit is often notoriously expensive (Herkenhoff, 2019). On the other hand, even households with significant wealth might face difficulty in disposing some of their properties in exchange for cash, i.e. consumption. Kaplan et al. (2014) documents that the rich holds a large proportion of their wealth in illiquid assets.

In such set up, I solve for the Ramsey optimal monetary policy. The central bank commits at time-0 to an optimal path of inflation. I apply the methodology introduced in Nuño and Moll (2018) and Nuno and Thomas (2020) to address the issue of having a infinite-dimensional object, i.e. the wealth distribution, as a state variable. In particular, I employ Gateaux derivatives, which extend the concept of classical derivatives to infinite-dimensional spaces, to derive analytical first-order conditions for the Ramsey problem. The theoretical result is consistent with that of Nuno and Thomas (2020): the central bank must balance between the effects of several transmission channels of inflation. Firstly, non-zero inflation (deflation) directly hurts households' welfare as a source of dis-utility. This detrimental effect can be micro-founded on the basis of costly price adjustment. Secondly, unexpected inflation decreases time-0 market price of the long-term nominal bond, which implies a redistribution of real wealth from creditors to debtors. This amount to a gain, i.e. domestic redistribution gain, as poor households (debtors) have higher marginal utility from consumption (MUC). At the same time, if the country is a net debtor (creditor), wealth is transferred to (from) foreign creditors, implying a cross-border redistribution gain (loss). The latter effect stems from our setting of an open economy. Finally, increasing future inflation disturbs both current and future bond prices, i.e. decreasing bond prices, which hurts asset-poor and high-MUC households that need to sell new bonds. Furthermore, this tightens the debt limit of constrained households, who are very in need of extra consumption. In the long run, the central bank therefore must commit to undo its initial inflation stance, resulting in optimal inflation ”front-loading” as in Nuno and Thomas (2020). Using the same parameter values as Nuno and Thomas (2020) for my numerical exercise, however, I obtain much lower optimal initial and thereafter inflation levels. This comes from the fact that households in my model also save to insure for the liquidity shock, causing economy-aggregate net asset position to improve and cross-border redistribution loss to emerge.

Related literature. This paper relates to two strands of literature. The first is the body of literature on small-open economies. In this aspect, my model is most closely related to that of De Ferra et al. (2020), which builds on the open-economy framework from Gali and Monacelli (2005) and the heterogeneous-agent set up from Kaplan et al. (2018). De Ferra et al. (2020), however, focuses on analyzing positive response of the economy to exogenous aggregate shock, such as a sudden stop of foreign credit supply. Furthermore, the main transmission channel in De Ferra et al. (2020) is via exchange rate depreciation. Second, my paper features households with heterogeneous asset holding and limited market participation in an incomplete-market setting, which are characteristic of the emerging HANK literature. Most papers in this literature address positive questions. Recent works include Gornemann et al. (2016), McKay et al. (2016), Kaplan et al. (2018), and Luetticke (2021), among many others. Nuño and Moll (2018) provide the methodological framework for analyzing optimal monetary policy in a general equilibrium model with uninsurable idiosyncratic risk and infinite-dimensional wealth distribution. Nuno and Thomas (2020) and Bigio et al. (2019) apply this framework to normative research of monetary policy. My work builds on the former, rather than the latter's closed-economy and bank-credit setting. Like Nuno and Thomas (2020), I focus on the redistribution channel of monetary policy, in particular the Fisher channel, which was introduced by Fisher (1933) and re-identified in the HANK framework by Auclert (2019). My model adds to Nuno and Thomas (2020) by inclusion of liquidity shock, which facilitates limited asset market participation by a fraction of the population. Agents with limited asset market participation, or hand-to-mouth agents have been well-documented in the empirical literature. With respect to modeling, my liquidity shock is closest to Bilbiie (2020), which also let consumer be hit by exogenous shock and lose access to asset market. Other works generate liquidity-constrained agents by modeling an addition illiquid asset (e.g. Kaplan et al. (2018)).

2 Model

2.1 The economy

This model is a modified version of the one in Nuno and Thomas (2020) - an extension of a Hugget economy. The economy is small and open, in the sense that assets are priced by risk-neutral foreign investors and the asset market does not have to clear domestically. Assets (bonds) are nominal, non-contingent, and long-term. The model is in continuous time. In the domestic economy, there is a measure-one continuum of households $k \in [0,1]$, who trade the assets and one single consumption good. I normalize the World price of this consumption good to 1. The domestic price P_t has dynamics

$$\frac{dP_t}{dt} = \pi_t P_t$$

where π_t is the domestic inflation rate. Alternatively, one can interpret P_t as the nominal exchange rate and π_t as the rate of nominal exchange rate depreciation.

2.2 Households

2.2.1 Assets, income, and borrowing constraint

Households trade nominal, domestic-currency denominated, non-contingent, long-term bonds among themselves and with foreign investors. For tractability, I follow Nuno and Thomas (2020) and assume that bonds pay exponentially declining nominal coupons. One bond bought (issued) at time t will generate a stream of nominal payments $\{\delta e^{-\delta(s-t)}\}_{s\in(t,\infty)}$. These payments add up to 1 unit of domestic currency over the infinite lifetime of the bond. From time-t point of view, one bond bought at time $t' < t$ is equivalent to $e^{\delta(t-t')}$ units of bonds bought at time t. Thus, the entire bond portfolio of household k at time t can be summarized by their current total nominal coupon payment, denoted as δA_{kt}. Then A_{kt} can be interpreted as the nominal face value of the bond portfolio, while δ is the amortization rate. A negative A_{kt} represents the face value of net outstanding debt. A_{kt} follows the law of motion

$$\frac{dA_{kt}}{dt} = A_{kt}^{new} - \delta A_{kt}$$

where A_{kt}^{new} is the net flow of new bonds bought at time t, in face value. Let Q_t denote the nominal price of bonds issued at time t. Household k is subject to the following budget constraint

$$Q_t A_{kt}^{new} = P_t(y_{kt} - c_{kt}) + \delta A_{kt}$$

where y_{kt} is the household's endowed income, and c_{kt} is their consumption at time t. Using this budget constraint, the law of motion for A_{kt} can also be expressed as

$$\frac{dA_{kt}}{dt} = \frac{P_t(y_{kt} - c_{kt}) + \delta A_{kt}}{Q_t} - \delta A_{kt}$$

Further define $a_{kt} \equiv A_{kt}/P_t$ as the real face value of the bond portfolio (one's net wealth). Its dynamics is then given,

$$\frac{da_{kt}}{dt} = \frac{\delta a_{kt} + y_{kt} - c_{kt}}{Q_t} - (\delta + \pi_t)a_{kt} \tag{1}$$

from combining the dynamics of A_{kt} and P_t. Here $\frac{\delta a_{kt}+y_{kt}-c_{kt}}{Q_t}$ is the real face value new bonds purchased at time t.

Household k lives in one of the three states. In state 1 and state 2, the household is endowed with y_1 and y_2 units of the good, respectively. Let $y_2 < y_1$. In these two states, they can trade the bonds freely subject to a exogenous borrowing limit,

$$a_{kt} \geq \phi,\ \phi < 0 \tag{2}$$

In state 3, the household is endowed with y_2 and cannot adjust their real face value of net wealth (net asset position), that is, they must choose the consumption level such that

$$\frac{da_{kt}}{dt} = 0 \tag{3}$$

The households move among these states following a three-state Poisson process. The process jumps from state 1 to state 2 and state 3 with probabilities $\lambda_1 p$ and $\lambda_1(1-p)$, respectively. A household jumps to state 1 from either of the two other states with probability λ_2.

2.2.2 Preferences

Household k has preference for consumption and inflation paths c_{kt} and π_t

$$\mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \left[u(c_{kt}) - x(\pi_t) \right] dt \right\}$$

where ρ is the positive discount rate. The consumption utility function $u(.)$ is bounded, continuous, strictly increasing and strictly concave for $c > 0$. The inflation (dis-)utility function $x(.)$ satisfies $x' > 0$ for $\pi > 0$ and $x' < 0$ for $\pi < 0$, $x'' > 0$ for all π, $x(0) = x'(0) = 0$. Dis-utility from inflation (or deflation) can be attributed to e.g. costly price adjustment.

The household chooses consumption at each t to maximize its welfare. The value function at time t is

$$v_t(a, S) = \max_{\{c_s\}_{s=t}^\infty} \mathbb{E}_t \left\{ \int_t^\infty e^{-\rho(s-t)} \left[u(c_s) - x(\pi_s) \right] ds \right\}$$

subject to the law of motion of asset position (1) and a borrowing limit that depends on the income-liquidity state S,

$$
\begin{aligned}
&a_{kt} \leq \phi, \text{ if } S \in \{S_1, S_2\} \\
&\frac{da_{kt}}{dt} = 0, \text{ if } S = S_3
\end{aligned}
$$

From now on, I use the notation $v_{it}(a) \equiv v_t(a, S_i)$ for the value function when the household is in state $i \in \{1, 2, 3\}$. The household-identifier subscript k is also dropped for brevity. So, $v_{1t}(a)$ is the value function at time t of the household in state 1 (high income, not liquidity constrained) with real face value of net wealth a. The Hamilton-Jacobi-Bellman (HJB) equation for this household's problem is

$$
\begin{aligned}
\rho v_{1t}(a) =& \frac{\partial v_{1t}}{\partial t} + \max_c \left\{ u(c) - x(\pi_t) + s_{1t}(a,c) \frac{\partial v_{1t}}{\partial a} \right\} \\
&+ \lambda_1 \left[(p) v_{2t}(a) + (1-p) v_{3t}(a) - v_{1t}(a) \right]
\end{aligned} \tag{4}
$$

where $s_{1t}(a,c)$ is the drift of asset position

$$
s_{1t}(a,c) = \frac{\delta a + y_1 - c}{Q_t} - (\delta + \pi_t) a
$$

Similarly, the HJB equation for the household in state 2 (low income, not liquidity constrained) is

$$
\begin{aligned}
\rho v_{2t}(a) =& \frac{\partial v_{2t}}{\partial t} + \max_c \left\{ u(c) - x(\pi_t) + s_{2t}(a,c) \frac{\partial v_{2t}}{\partial a} \right\} \\
&+ \lambda_2 \left[v_{1t}(a) - v_{2t}(a) \right]
\end{aligned} \tag{5}
$$

with the drift function

$$s_{2t}(a,c) = \frac{\delta a + y_2 - c}{Q_t} - (\delta + \pi_t)a$$

Since the households in state 1 and state 2 can freely adjust their asset position above the exogenous borrowing limit ϕ, their consumption satisfies first order conditions

$$u'(c_{it}(a^m)) = \frac{\partial v_{it}(a)}{\partial a} \tag{6}$$

where $c_{it}(a) \equiv c_t(a, S_i)$ is the optimal consumption policy function, $i \in \{1,2\}$. Consumption increases with nominal bond prices, because higher bond price (lower yield) makes it less attractive to buy bonds, i.e. save. Consumption decreases with the slope of the value function, because a steeper value function makes increasing bond holdings (improving net asset position) and thus saving more attractive.

The HJB equation for the household in state 3 (low income, liquidity constrained) is

$$\begin{aligned} \rho v_{3t}(a) =& \frac{\partial v_{3t}}{\partial t} + u(c_{3t}(a)) - x(\pi_t) + s_{3t}(a)\frac{\partial v_{3t}}{\partial a} \\ &+ \lambda_2 \left[v_{1t}(a) - v_{3t}(a) \right] \end{aligned} \tag{7}$$

where $s_{3t}(a) = 0$ since the household in this state cannot adjust their real face value of net wealth (net asset position). Consumption is such that this holds,

$$c_{3t}(a) = \delta a + y_2 - (\delta + \pi_t) Q_t a \tag{8}$$

The liquidity-constrained household can only consume their (low) labor income minus the minimum interest payment that keep their net wealth (asset position) from declining.

2.3 Foreign investors

Risk-neutral foreign investors trade the bonds previously introduced with domestic households. Foreign investors also have the outside option of investing elsewhere at (international) risk-free real rate $\bar{r}$. The nominal bonds issued at time t are priced by foreign investors

$$Q_t = \int_t^\infty \delta e^{-(\bar{r}+\delta)(s-t)-\int_t^s \pi_u du} ds$$

where δ is the amortization rate. The price also accounts for the (foreseen) path of domestic inflation. Foreign investors discount future nominal payments with inflation accumulated between the moment the bond is purchased and the moment each payment is made. As bonds have infinitive lives, foreign investors must consider the whole path of inflation $\{\pi_s\}_{s=t}^\infty$.

The dynamics of bond prices is given by taking derivative with respect to time

$$\frac{dQ_t}{dt} = (\bar{r} + \delta + \pi_t)Q_t - \delta \tag{9}$$

Equation (8), together with the boundary condition that the discounted value of Q_∞ is zero, complete the risk-neutral pricing of the bonds. In the asymptotic steady state, bond price is $Q_\infty = \frac{\delta}{\bar{r}+\delta+\pi_\infty}$ where π_∞ is the steady-state inflation level.

2.4 Central bank

At time-0, the central bank directly chooses and commits to a plan of inflation $\{\pi_t\}_{t=0}^\infty$ to maximize aggregate welfare (to be discussed in detail in Section 3). An equivalent setting (as in Nuno and Thomas (2020)) would be that the central bank trades with foreign investors a short-term security with instantaneous nominal interest rate R_t, where R_t is set by the central bank. Then, no-arbitrage implies that $\pi_t = R_t - \bar{r}$ and the central bank still effectively sets the inflation path.

2.5 Equilibrium

Given the central bank's policy $\{\pi_t\}_{t=0}^{\infty}$, I define an equilibrium of this model as paths for bond prices $\{Q_t\}_{t=0}^{\infty}$, households' value function $\{v_t(a,S_i)\}_{t=0}^{\infty}$ and consumption function $\{c_t(a,S_i)\}_{t=0}^{\infty}$, and the joint density of (net) wealth and income-liquidity state $\{f_t(a,S_i)\}_{t=0}^{\infty}$ for $i \in \{1,2,3\}$ such that, at every time t, (i) households maximize their welfare taking as given equilibrium prices, (ii) markets clear. We have in this economy the bond market and the goods (and labor) market. Since this is an endowment economy, the latter is of little interest. Markets clear in the sense that the current account identity holds,

$$\frac{d\bar{a}_t}{dt} = \frac{\delta\bar{a}_t + \bar{y}_t - \bar{c}_t}{Q_t} + (\delta + \pi_t)\bar{a}_t \tag{10}$$

where $\bar{a}_t$, $\bar{y}_t$, $\bar{c}_t$ are the economy-aggregate at time t of (real face value) net wealth, income, and consumption, respectively. These aggregates can be calculated as the cross-household average,

$$\bar{g}_t \equiv \sum_{i=1}^{3} \int_{\phi}^{\infty} g_t(a,S_i) f_t(a,S_i) da$$

where g_t is any function of individual variable, e.g. net wealth or consumption.

The state of an economy at each time t is summed up by the joint distribution of net wealth and income-liquidity state, which has the density function $f_t(a,S_i)$. Its dynamics is given by a set of Kolmogorov Forward (KF) equations

$$\frac{\partial f_{1t}(a)}{\partial t} = -\frac{\partial}{\partial a}\left[s_{1t}(a) f_{1t}(a)\right] - \lambda_2 f_{1t}(a) + \lambda_1 \left[f_{3t}(a) + f_{2t}(a)\right] \tag{11}$$

$$\frac{\partial f_{2t}(a)}{\partial t} = -\frac{\partial}{\partial a}\left[s_{2t}(a) f_{2t}(a)\right] - \lambda_1 f_{2t}(a) + \lambda_2(p) f_{1t}(a) \tag{12}$$

$$\frac{\partial f_{3t}(a)}{\partial t} = -\lambda_1 f_{3t}(a) + \lambda_2(1-p) f_{1t}(a) \tag{13}$$

where $s_{it}(a)$, $i \in \{1,2\}$ are the drifts of individual net wealth evaluated at optimal consumption. It is worth noting that the consumption policy of those in state 3 is not optimal, but rather one such that the drift s_{3t} is zero. Hence, households in state 3 do not move from one asset position to another, and the change in point density $f_{3t}(a)$ only comes from those jumping into and out of state 3.

2.6 Transmission channels of monetary policy

To conclude this section, I will briefly review the transmission channels of inflation that are discussed in Nuno and Thomas (2020) and further discuss how inflation affects liquidity-constrained households in particular.

2.6.1 Three main channels

First, positive inflation (or deflation) directly shows up in households' preference as a source of dis-utility $x(\pi_t)$. This effect, by construction, is asymmetric across households. Furthermore, the dis-utility from inflation is exogenous to the households' consumption and saving decisions, as inflation levels are set by the central bank and taken as given by the households. This dis-utility cost can be attributed to costly price rigidities, which typically show up in representative-agent New-Keynesian model. I follow Nuno and Thomas (2020) to abstract from the modeling of other New-Keynesian transmission channels, e.g. labor supply. In this endowment economy where aggregate income (output) is given, labor supply can be considered fixed.

Second, inflation causes changes in bond prices, which in turn change the real market value (not face value) of households' bond portfolios. Define real market value of net wealth $a_t^m \equiv Q_t a_t$. Since bonds are competitively priced by foreign investors, the real returns on a_t^m must be identical to the international real return $\bar{r}$. This can be seen from the dynamics of a_t^m, derived from combining the dynamics of Q_t and a_t from equations (1) and (7)

$$\frac{da_t^m}{dt} = \bar{r}a_t^m + y_t - c_t$$

with initial condition $a_0^m = Q_0 a_0$.

The central bank, therefore, cannot use inflation to change the real returns on a_t^m, which is fixed at international level $\bar{r}$. However, inflation can affect households' initial real market value of net wealth a_0^m via Q_0. Recall from Section 2.3 that in pricing Q_0, foreign investors must take into account the whole future inflation path, as accumulated inflation erodes the value of future coupon payments

$$Q_0 = \int_0^\infty \delta e^{-(\bar{r}+\delta)t - \int_0^t \pi_u du} dt$$

Therefore, unexpected inflation along the path $\{\pi_t\}_{t=0}^\infty$ will decrease time-0 bond price Q_0, while initial face-value asset position a_0 is predetermined. Then, (unexpected) inflation increases a_0^m for those who have $a_0 < 0$ (debtors) and decreases a_0^m for those who have $a_0 > 0$ (creditors). This amounts to redistribution of wealth, in market value terms, from creditors to debtors by the central bank. This is the Fischer channel.

The third and final channel discussed in Nuno and Thomas (2020), denoted the "liquidity channel", is via changing the borrowing limit in real market value terms,

$$a_t^m \geq Q_t \phi, \ \phi < 0$$

This channel also operates through changing the bond price Q_t. By unexpectedly rising the future inflation path $\{\pi_s\}_{s=t}^\infty$, the central bank puts downward pressure on bond price Q_t and thus tightens the real market value borrowing limit $Q_t\phi$. This limits the ability to borrow, in real market value terms, of households with net asset position a_t nearing the exogenous borrowing limit ϕ. Deflationary policies, on the other hand, will relax the borrowing limit.

2.6.2 Effects on liquidity-constrained households

It is worthwhile to further discuss how the channels operate on liquidity-constrained households, i.e. those are in state 3, which is the novelty of this paper. Consider a debtor (one with $a_0 < 0$) in state 3 at time 0. I can write their contemporaneous consumption as a function of their debt (net asset position) in real market value

$$c_3(a_0^m) = y_2 + \left[\frac{\delta}{Q_0} - (\pi_0 + \delta)\right] a_0^m$$

where I use equation (8) and the identity $a_0^m = Q_t a_0$. Denote $\hat{r}(Q_0, \pi_0) \equiv \frac{\delta}{Q_0} - (\pi_0 + \delta)$,

$$c_3(a_0^m) = y_2 + \hat{r}(Q_0, \pi_0) a_0^m$$

Then $\hat{r}$ is the effective payment rate the household must make on its net debt a_0^m to meet its liability, i.e. keeping a_t constant. Notice that in the asymptotic steady state, $\hat{r} = \bar{r}$.

The Fischer channel affects the household normally: any positive inflation shock along the path $\{\pi_t\}_{t=0}^{\infty}$ will decreases the market value of their debt (improves their asset position a_0^m) via decreasing the time-0 bond price Q_0. Holding the rate $\hat{r}$ constant, the household has to pay for less debt and therefore get to consume more instantly (a_0^m goes up, so c_3 goes up). This typically improves their welfare, as the household is in a state with low income and cannot borrow, so their contemporaneous consumption is below the optimal level.

However, inflation also affects $\hat{r}$. Suppose the central bank announces at time-0 a positive shock to π_t with any $t > 0$. Then Q_0 will go down and make $\hat{r}$ go up. The in-debt household must pay more on their debt and consume less. I interpret this as the "liquidity channel": one can think that a household in state 3 is at their personal borrowing limit $Q_0 a_0$ (a_0 predetermined), and inflation tightens this personal limit, forcing them to consume less contemporaneously. Thus, this liquidity channel of inflation affects liquidity-constrained households across the whole distribution, tightening the constraint for debtors but relaxing that of creditors. Notice that a positive shock to π_0, however, has ambiguous effect on liquidity-constrained households' time-0 consumption.

3 Optimal monetary policy

3.1 The central bank's problem

The central bank chooses and commits to its monetary policy at time zero, i.e. a plan of inflation $\{\pi_t\}_{t=0}^{\infty}$, to maximize aggregate welfare. I only consider the case where the central bank can credibly commits to its plan, so the problem is a Ramsey problem. I assume the central bank is utilitarian, that is it attribute the same Pareto weight to each household. The central bank's objective is then to maximize

$$\begin{aligned} W_0 &\equiv \mathbb{E}_{f_0(a,S)}\left[v(a,S)\right] \\ &= \int_0^{\infty} e^{-\rho t}\left\{\sum_{i=1}^{3}\int_{\phi}^{\infty}\left[u(c_t(a,S_i)) - x(\pi_t)\right] f_t(a,S_i)da\right\}dt \end{aligned}$$

The value functional of the central bank at time zero is given by

$$W[f_0(a,S)] = \max_{\{X_t\}_{t=0}^{\infty}} W_0 \tag{14}$$

where $X_t \equiv [\pi_t, Q_t, v_t(a,S), c_t(a,S), f_t(a,S)]$; subject to the KF (distribution dynamics) equations (11)-(13), the households' HJB equations (4), (5) and (7), the optimal/constrained consumption policies (6) and (8), and the bond pricing equation (9).

Given an initial distribution $f_0(a,S)$, a Ramsey allocation is composed of paths for inflation $\{\pi_t\}_{t=0}^{\infty}$ and bond price $\{Q_t\}_{t=0}^{\infty}$; and sequences of functions for household value $\{v_t(a,S)\}_{t=0}^{\infty}$, consumption policy $\{c_t(a,S)\}_{t=0}^{\infty}$, and distribution $\{f_t(a,S)\}_{t=0}^{\infty}$ that solve the central bank's problem (14).

3.2 Solution to the central bank's problem

The Lagrangian for the Ramsey problem $\mathcal{L}[\pi, Q, v, c, f] \equiv \mathcal{L}_0$ is given by

$$
\begin{aligned}
\mathcal{L}_0 = \int_0^{\infty} e^{-\rho t} \Bigg\{ & \sum_{i=1}^{3} \int_{\phi}^{\infty} [u(c_{it}(a)) - x(\pi_t)] f_{it}(a) \\
& + \zeta_{1t}(a) \left[-\frac{\partial f_{1t}(a)}{\partial t} - \frac{\partial}{\partial a}[s_{1t}(a) f_{1t}(a)] - \lambda_2 f_{1t}(a) + \lambda_1 [f_{3t}(a) + f_{2t}(a)] \right] \\
& + \zeta_{2t}(a) \left[-\frac{\partial f_{2t}(a)}{\partial t} - \frac{\partial}{\partial a}[s_{2t}(a) f_{2t}(a)] - \lambda_1 f_{2t}(a) + \lambda_2(p) f_{1t}(a) \right] \\
& + \zeta_{3t}(a) \left[-\frac{\partial f_{3t}(a)}{\partial t} - \lambda_1 f_{3t}(a) + \lambda_2(1-p) f_{1t}(a) \right] \\
& + \theta_{1t}(a) \left[-\rho v_{1t}(a) + \frac{\partial v_{1t}}{\partial t} + u(c_{1t}(a)) - x(\pi_t) + s_{1t}(a) \frac{\partial v_{1t}}{\partial a} + \lambda_1 [p v_{2t}(a) + (1-p) v_{3t}(a) - v_{1t}(a)] \right] \\
& + \theta_{2t}(a) \left[-\rho v_{2t}(a) + \frac{\partial v_{2t}}{\partial t} + u(c_{2t}(a)) - x(\pi_t) + s_{2t}(a) \frac{\partial v_{2t}}{\partial a} + \lambda_2 [v_{1t}(a) - v_{2t}(a)] \right] \\
& + \theta_{3t}(a) \left[-\rho v_{3t}(a) + \frac{\partial v_{3t}}{\partial t} + u(c_{3t}(a)) - x(\pi_t) + \lambda_2 [v_{1t}(a) - v_{3t}(a)] \right] \\
& + \eta_{3t} [u(c_{3t}(a)) - u(\delta a + y - (\pi_t + \delta) Q_t a)] + \sum_{i=1}^{2} \eta_{it} \left[u'(c_{it}(a)) - \frac{1}{Q_t} \frac{\partial v_{it}}{\partial a} \right] \Bigg\} da\, dt \\
& + \int_0^{\infty} e^{-\rho t} \mu_t \left[Q_t(\bar{r} + \pi_t + \delta - \delta - \frac{dQ}{dt}) \right] dt
\end{aligned}
\tag{15}
$$

where ζ, θ, η, and μ are the Lagrangian multipliers for the KFEs, HJBs, consumption policies, and bond pricing equation, respectively.

This is a optimal control problem. The central bank maximizes welfare taking into account the dynamics of net wealth-state distribution (reflected in the constraints derived from KFEs) and of households' value functions (HJBs). It also understands the pricing behavior of foreign investors and the optimization behavior of unconstrained households. Notice that the consumption policy constraint of households in state 3, corresponding to multiplier η_{3t}, is different from the optimization conditions of those in states 1 and 2. It instead reflects their constrained consumption level as given in equation (8).

In order to obtain the first-order conditions for the problem, I use the variational approach introduced by Nuno and Thomas (2020), by taking Gâteaux derivatives of the Lagrangian (15) with respects to functions π, Q, v, c, f. The Gâteaux derivative is a generalization of the standard derivative to the infinite-dimensional spaces. It is necessary to employ this approach because the suitable function space for the central bank's problem (14) is infinite-dimensional. For instance, the optimal policy π is not an ordinary function but a functional that map the infinite-dimensional object f_0 (the initial distribution) and t (time) into $\mathbb{R}$.

To illustrate, let h be an arbitrary function in the same function space as π, the Gâteaux derivative with respect to π is

$$\frac{d}{d\alpha}\mathcal{L}_0[\pi + \alpha h, Q, v, c, f]|_{\alpha=0} \equiv \lim_{\alpha \to 0} \frac{\mathcal{L}_0[\pi + \alpha h, Q, v, c, f] - \mathcal{L}_0[\pi, Q, v, c, f]}{\alpha} \tag{16}$$

The first-order conditions are such that Gâteaux derivatives are equal to zero for any function h.

In the appendix, I solve for these first-order conditions and derive the following observations. The Lagrangian multipliers ζ corresponding to the KFEs must be equal to the value function v. Intuitively, in equilibrium, the social (shadow) value of an individual household, represented by ζ, must be equal to its private value v. Furthermore, the Lagrange multipliers θ_i and η_i, $i = 1, 2$, corresponding to the *unconstrained* households' HJB equations and first-order conditions are all zero. These constraints are slack, which means the central would choose the same consumption policy as these optimizing households. This is not the case for the liquidity-constrained households in state 3, however. Recall that liquidity-constrained households have low income and are forbidden to borrow, so by construction, their consumption c_3 is set lower than the optimal level. Therefore, the multiplier η_3, representing the social value of relaxing this consumption (liquidity) constraint, has a positive value. Some algebraic manipulation shows that the value of η_3 coincides with f_3. Intuitively, the central bank assigns the same Pareto weight to each household, so the social value from marginally improving welfare of those in asset position a is equal to their mass f.

Using the fact that $\zeta = v$, $\theta = 0$, $\eta_i = 0$ for $i = 1, 2$ and $\eta_3 = f_3$, taking Gâteaux derivative with respect to π gives the optimality condition

$$\mu_t Q_t = x'(\pi_t) + \int_{\phi}^{\infty} a Q_t u'(c_{3t}(a)) f_{3t}(a) da + \sum_{i=1}^{2} \int_{\phi}^{\infty} a \frac{\partial v_{it}}{\partial a} f_{it}(a) da \tag{17}$$

where μ_t is the multiplier associated with the bond pricing equation, of which dynamics is given by taking Gâteaux derivative with respect to Q,

$$-(\pi_t + \delta) \int_{\phi}^{\infty} a f_{3t}(a) da + \sum_{i=1}^{2} \int_{\phi}^{\infty} -\frac{\delta a + y - c}{Q_t^2} \frac{\partial v_{it}}{\partial a} f_{it}(a) da + \mu_t(\bar{r} + \pi_t + \delta - \rho) + \frac{d\mu_t}{dt} = 0 \tag{18}$$

with $\mu_0 = 0$. Equations (17) and (18) provide the optimal inflation path, which together with equations (4)-(13) forms the complete solution to the Ramsey problem.

3.3 Optimal inflation rule

Denote the real market value of a household's net asset position (Net Nominal Position) as $NNP_t(a) \equiv Q_t a$ and the marginal utility of consumption $MUC_t(a, S) \equiv u'(c(a, S))$. Equation (17) is equivalent to

$$\begin{aligned} x'(\pi_t) = & \underbrace{cov_{f_t(a,S)}[-NNP_t(a), MUC_t(a,S)]}_{\text{Domestic redistribution gain}} \\ & \underbrace{+ \mathbb{E}_{f_t(a,S)}[-NNP_t(a)] * \mathbb{E}_{f_t(a,S)}[MUC_t(a,S)]}_{\text{Cross-border redistribution gain/loss}} + \mu_t Q_t \end{aligned} \tag{19}$$

where I use the first-order condition for consumption (6) and the identity $cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$.

This expression clearly represents the central bank's trade-offs in choosing the optimal inflation path. Optimality requires that the marginal direct cost of dis-utility $x'(\pi)$ must be equal to the marginal gain, which is divided into three terms. The first term $cov_{f_t(a,S)}[-NNP_t(a), MUC_t(a,S)]$ reflects the effect of redistributing wealth from creditors to debtors domestically. The covariance is generally positive: the poorer households, i.e. those with more debt, have lower consumption levels and thus higher marginal utility from consumption. On the other hand, wealthy households, i.e. those with positive asset position, have relatively higher consumption and lower marginal utility from consumption. Therefore, the central bank is biased towards redistributing wealth, and effectively consumption from creditors to debtors using positive inflation. Furthermore, there are constrained debtors in our economy, who have even lower consumption levels due to the fact that they are forbidden to borrow for consumption. Inflation would also channel resources to these households and provide them with badly needed consumption. The existence of liquidity shocks, in this aspect, would strengthen the central bank's inflationary bias.

The second term, $\mathbb{E}_{f_t(a,S)}[-NNP_t(a)] * \mathbb{E}_{f_t(a,S)}[MUC_t(a,S)]$, reflects the effect of redistributing wealth cross-border, in aggregate term. Recall that we defined the aggregate net wealth,

$$\bar{a}_t \equiv \sum_{i=1}^{3} \int_{\phi}^{\infty} a_t f_t(a, S_i) da$$

If the country has net debt in aggregate, that is $\mathbb{E}_{f_t(a,S)}[-NNP_t(a)] = -\bar{a}_t > 0$, we say the country is a net debtor. When this is the case, the term $\mathbb{E}_{f_t(a,S)}[-NNP_t(a)] * \mathbb{E}_{f_t(a,S)}[MUC_t(a,S)]$ is positive: inflation redistributes wealth from foreigners to domestic households, who have in average marginal utility $\mathbb{E}_{f_t(a,S)}[MUC_t(a,S)]$. This formulation comes from the fact that foreigners' welfare does not enter the central bank objective function. Conversely, if the country is a net creditor, inflation transfers wealth abroad and harm domestic households in aggregate. This is more likely the case in our economy with liquidity shocks.

Households in state 1, who have high income, would try to save, not only for the income shock, but also for the liquidity shock. Saving is ever more important because once the household is liquidity-constrained, their consumption will be directly proportionate to their wealth (debt) level. Households in state 2, even though they have low income, also refrain from taking up too much debt, as they know this might subject them to forced interest payment and drastically low consumption if they enter state 3 in the future. Economy-wide aggregate net asset position, therefore, tend to be higher once we allow for liquidity shocks, to such an extent that the country may become a net creditor. This effect goes against the inflationary bias and might undo the effect on domestic redistribution gain.

Finally, the third term $\mu_t Q_t$ reflects the cost of defecting from previous commitment about time-t inflation. The central bank effectively make promises about time-t inflation to foreign investors, who price the bond taking as given the promised inflation path. The term equals zero at time 0, since the central bank has not been subject to previous commitments. After that, the multiplier μ_t is shown to become increasingly negative over time in our numerical exercise. Intuitively, increasing future inflation disturbs both current and future bond prices, i.e. decreasing bond prices, which hurts asset-poor and high-MUC households that need to sell new bonds. This effect ensures that inflation is tempered in the long-run. We will see from the numerical solution in Section 4, as well as in Nuno and Thomas (2020)'s theoretical analysis, that optimal long-run inflation converges to 0. With respect to liquidity-constrained households, equation (18) shows that they contribute to the dynamics of the multiplier $d\mu_t/dt$ with the term

$$(\pi_t + \delta)\int_{\phi}^{\infty} a f_{3t}(a)da = (\pi_t + \delta)\,\mathbb{E}_{f_{3t}(a)}\left[NNP_t(a)\right]$$

If households in state 3 constitute a net debtor in aggregate ($\mathbb{E}_{f_{3t}(a)}[NNP_t(a)] < 0$), the term is negative and thus further pushes for decreasing inflation, as inflation tightens the debt limit of these liquidity-constrained debtors in real term, as discussed in section 2.6.2.

4 Numerical analysis

4.1 Algorithm

Equations (17) and (18) provides the complete solution, i.e. an optimal inflation path, to our Ramsey problem. However, there is no close-form solution and we must proceed to find a numerical solution. The algorithm is as follows. We start by guessing an inflation path, for instance, $\pi_t = 0$ for all t. Given the inflation path, we can compute the path of bond prices. Then we can solve for the households' HJB equation, the KF equation and their respective dynamics taking inflation and prices as given. We do this by applying the scheme of finite difference method provided by Achdou et al. (2017) and Nuño and Moll (2018). Using the functions and moments obtained, we calculate the whole path of multiplier μ_t under equation (18) and the implied optimal inflation path under equation (17). If this implied optimal inflation path is close enough to our initial guess, we have found the solution. Otherwise, we update our guess and iterate until the two coincide. Nuno and Thomas (2020) highlights that solving numerically the continuous-time model is significantly more efficient than doing so in discrete time, because solving the HBJ and KF equation makes use of Matlab's (or any other common software package) fast routines to handle sparse matrices.

4.2 Calibration

Before initiating the algorithm described above, we need to pin down some parameters. My calibration bases largely on that of Nuno and Thomas (2020), for the sake of comparability. The calibration is supposed to be mainly illustrative, as I want to highlight the effects of inclusion of liquidity shocks. Table 1 summarizes the calibration.

Time frequency is 1 year. For preferences, I assume functional forms $u(c) = log(c)$ and

Parameter	Value	Description	Source/Target
ψ	5.5	coefficient inflation dis-utility	Nuno and Thomas (2020)
λ_2	0.72	transition rate unemployed-to-employed	job finding rate =0.72
λ_1	0.8	transition rate employed-to-unemployed	unemployment rate = 0.1
y_1	1.03	employment income	Hall and Milgrom (2008)
y_2	1.03	unemployment income	Hall and Milgrom (2008)
ρ	0.0302	subjective discount rate	Nuno and Thomas (2020)
ϕ	-3.6	debt limit	Nuno and Thomas (2020)
$\bar{r}$	0.03	international real interest rate	standard
δ	0.19	amortization rate	bond duration = 4.5 years
p	0.98	probability unconstrained given unemployment	illustrative

Table 1: Calibration

$x(\pi) = \psi\frac{\pi^2}{2}$. The algorithm and general results are robust for other CRRA utility functions, i.e. those with coefficient of relative risk aversion different from 1. The quadratic form of inflation dis-utility cost can be micro-founded by modeling firms explicitly and allowing them to set prices subject to standard quadratic price adjustment costs à la Rotemberg (1982).

The parameters for income process are set to feature European labor market following Blanchard and Galí (2010). Let income states y_1 and y_2 be "employment" and "unemployment" as in Huggett (1993). The Poisson rates λ_1 and λ_2 of jumping between unemployment and employment are set such that the unemployment rate is $\frac{\lambda_2}{\lambda_1+\lambda_2} = 0.1$, and the annual job finding rate is $\lambda_2 = 0.72$. I normalize average income to 1 and set $y_2/y_1 = 0.71$ per Hall and Milgrom (2008).

For values of subjective discount rate ρ and debt limit ϕ, I also set those equal to the values in Nuno and Thomas (2020) to highlight the changes from including liquidity shocks. Notice that Nuno and Thomas (2020) set ρ and ϕ jointly to target UK, Sweden, and Baltic countries' steady-state net international investment position (NIIP) over GDP ($\bar{a}/\bar{y}$) and gross credit to household over GDP ($\bar{b}/\bar{y}$). For the model with liquidity shock, we would need different values of ρ and debt limit ϕ to match these features. I will also consider such calibration in the Appendix. In addition, I set the world real interest rate $\bar{r} = 0.03$ as in standard practice. Finally, amortization rate $\delta = 0.19$ gives average bond duration of 4.5 years, which is consistent with Auclert (2019). For illustrative purpose, I choose the probability of a liquidity shock given an income shock $(1-p) = 0.02$. With the parameters chosen by Nuno and Thomas (2020), arbitrarily large probability of liquidity shock would make the wealth distribution degenerate.

4.3 Stationary distribution under exogenous zero inflation rate

Recall from section 3 that the optimal policy is a functional of the time-0 distribution $f_0(a,S)$, which is an infinitive-dimensional object. For the calibration, I consider the steady state distribution under a zero inflation policy as the initial distribution. Figure 1 shows the this initial distribution together with other equilibrium objects un the zero inflation policy.

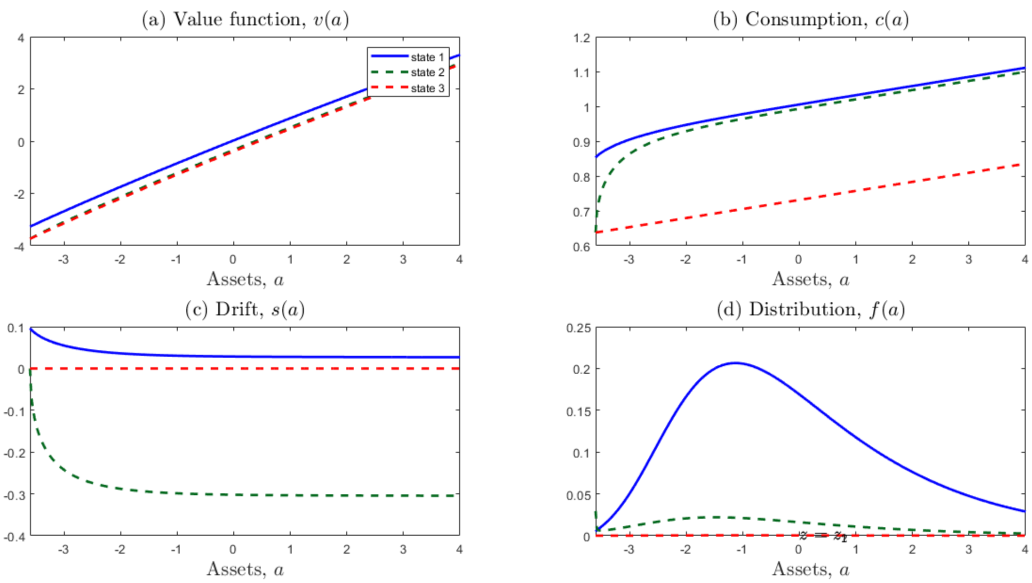

Figure 1: Time-0 equilibrium objects ($p = 0.02$)

It can be seen from panel (b) that given the same net asset position and income, households which are met with the liquidity shock, i.e. those in state 3, have consumption levels much lower than unconstrained households. Panel (d) shows the initial distribution of wealth. There is a significant mass of households in state 2 at the debt limit ϕ.

4.4 Optimal inflation and transition dynamics

Given the initial distribution, we can proceed to solve for optimal transition dynamics. Figure 2 shows the optimal path of inflation and the transition dynamics of relevant variables.

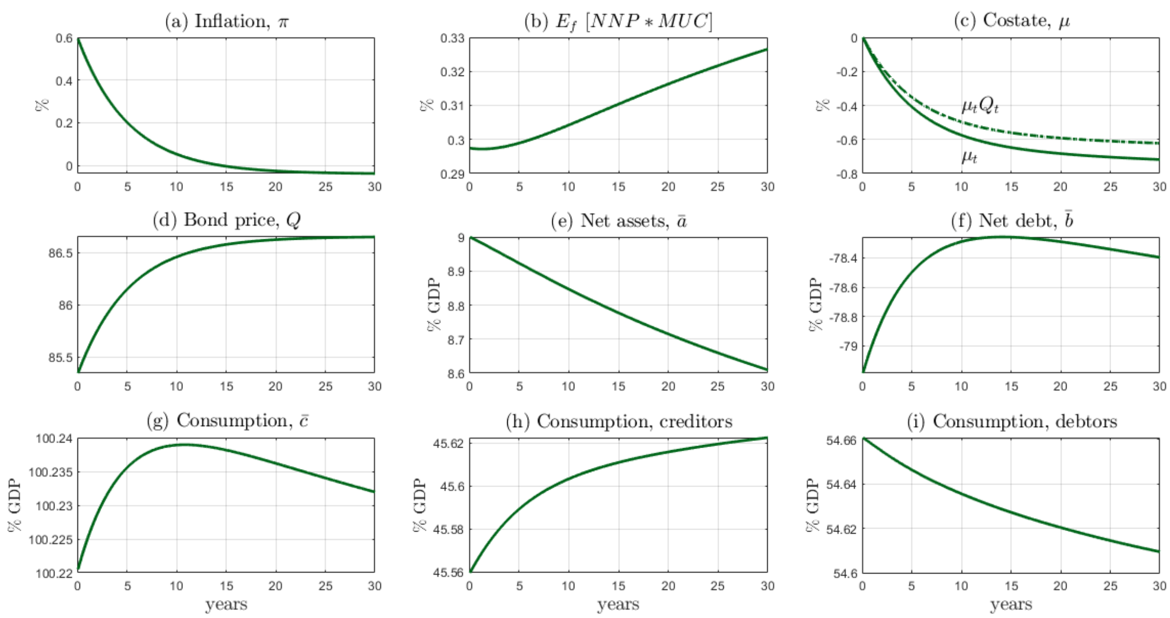

Figure 2: Optimal inflation path and transitional dynamics

At time-0, we can see that given $\mu_0 = 0$ and our functional form for inflation dis-utility, the optimal inflation level depends only on domestic and cross-border redistribution motive,

$$\pi_0 = \frac{1}{\psi}\{\underbrace{cov_{f_0(a,S)}[-NNP_0(a), MUC_0(a,S)]}_{\text{Domestic redistribution gain}} + \underbrace{\mathbb{E}_{f_0(a,S)}[-NNP_0(a)] * \mathbb{E}_{f_0(a,S)}[MUC_0(a,S)]}_{\text{Cross-border redistribution gain/loss}}\}$$

With our calibration, the second term is negative, as the country is a net creditor with aggregate net asset position at 9 percent of GDP. So positive inflation comes at the cost of transferring some wealth from domestic households to foreigners, while the central bank do not care about the latter. This can be seen from panel (e), where the country aggregate net asset position deteriorates over time. However, there still exists some inflationary bias: optimal policy recommends positive inflation of around 0.6 percent at time-0. This means the domestic redistribution gain still somewhat dominate the cross-border redistribution loss. The recommended inflation level is however much lower than that in Nuno and Thomas (2020) (above 4%), where the country is a net debtor with aggregate net asset position at minus 25 percent of GPD.

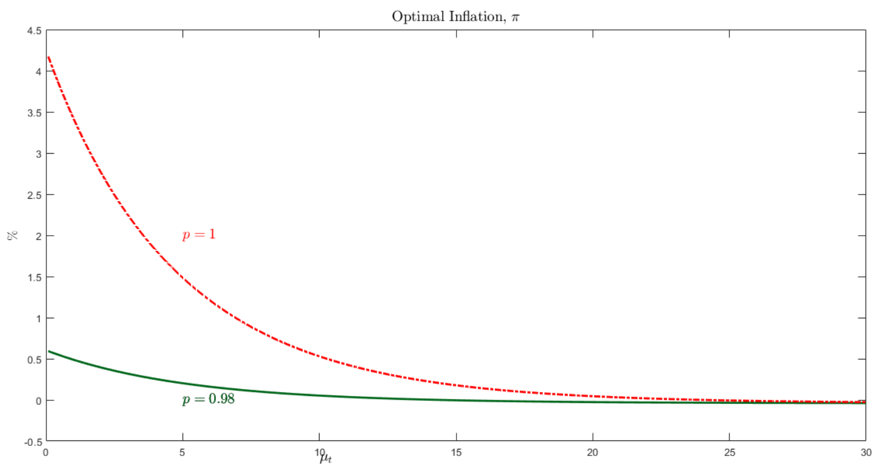

Figure 3: Optimal inflation path with and without liquidity shock

Over time, the optimal inflation path is dictated by the dynamics of μ_t, according to equations (17) and (18). The multiplier μ_t and consequently the term $\mu_t Q_t$ becomes more and more negative over time, implying the central bank must commit to undo initial inflation and bring bond price back to the steady-state level. Optimal long-run inflation converges to 0, due the fact that the subjective discount rate is very close to the international real interest rate. The last two panels show that optimal inflation favors debtors initially, by redistributing wealth from creditors to the latter. This effect fades as inflation converge to near-0 long-run optimal level.

5 Conclusion

Building on the seminal work of Huggett (1993) and the recent methodological contribution of Nuno and Thomas (2020), I model and analyze optimal monetary policy in a continuous-time, incomplete-market, small-open economy allowing for liquidity shock. I derive theoretical and normative results consistent with that in Nuno and Thomas (2020), while the numerical solution recommends a significantly lower inflation level. This resulted from the fact that using the same parameter values, our model gives rise to welfare loss from cross-border redistribution, as opposed to cross-border gain in their work.

This paper aims to characterize an important feature of many economies, that is not everyone can have access to consumption credit, especially during spells of unemployment. To this end, my model involves liquidity shock that hit simultaneously with the income shock, leaving some low-income households constrained even though they are far from the exogenous debt limit. The simultaneity of income and liquidity shock, however, poses a problem for calibration. As the unemployment rate is generally below 10 percent for most real economies, liquidity-constrained households cannot account for a larger proportion (up to 40-50 percent) of the population, as suggested by the empirical literature studying consumption behavior (see for example, Campbell and Mankiw, 1989). This is a point for future improvement. Another potential extension is a more detailed characterization of the income distribution, either by more using a more complicated income process or by endogenizing labor supply.

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Appendices

A Solution to the central bank's problem

First, we need to define some math notation. Given the states t and a, define the operater $\mathcal{A}$

$$\mathcal{A}v \equiv \begin{pmatrix} s_1(t,a)\frac{\partial v_1(t,a)}{\partial a} & +\lambda_1\left[pv_2(t,a)+(1-p)v_3(t,a)-v_1(t,a)\right] \\ s_2(t,a)\frac{\partial v_2(t,a)}{\partial a} & +\lambda_2\left[v_1(t,a)-v_2(t,a)\right] \\ 0 & +\lambda_2\left[v_1(t,a)-v_2(t,a)\right] \end{pmatrix}$$

where $v \equiv \begin{pmatrix} v_1(t,a) \\ v_2(t,a) \\ v_3(t,a) \end{pmatrix}$. Then the HJBs can be expressed as

$$\rho v = \frac{\partial v}{\partial t} + \max_c \{u(c) - x(\pi) + \mathcal{A}v\}$$

Let $\Phi \equiv \{1,2,3\} \times \mathbb{R}$. Define

$$\langle f, g\rangle_\Phi \equiv \sum_{i=1}^{3} \int_\Phi f_i g_i da$$

$$\langle f, g\rangle_{\Phi_{1,2}} \equiv \sum_{i=1}^{2} \int_{\Phi_{1,2}} f_i g_i da$$

$$\langle f, g\rangle_{\Phi_3} \equiv \int_{\Phi_3} f_3 g_3 da$$

where f, g are functions in the spaces of Lebesgue-integrable functions $L^2(\Phi)$. Given some operator $\mathcal{A}$, its adjoint is operator $\mathcal{A}^*$, such that $\langle f, \mathcal{A}v\rangle_\Phi = \langle \mathcal{A}^* f, v\rangle_\Phi$. The adjoint of our operator $\mathcal{A}$ is

$$\mathcal{A}^* f = \begin{pmatrix} \frac{\partial(s_1 f_1)}{\partial a} & - & \lambda f_1 + \lambda_2 f_2 \\ \frac{\partial(s_2 f_2)}{\partial a} & - & \lambda f_2 + \lambda_1 p f_1 \\ 0 & - & \lambda_2 f_3 + \lambda_1 (1-p) f_1 \end{pmatrix}$$

The Lagrangian to our problem can be written as

$$\begin{aligned}
\mathcal{L}(\pi, Q, f, v, c) = & \int_0^\infty e^{-\rho t} \langle u(c_t) - x(\pi_t) \rangle_\Phi dt \\
& + \int_0^\infty \langle e^{-\rho t} \zeta_t, \mathcal{A}^* f_t - \frac{\partial f_t}{\partial t} \rangle_\Phi dt \\
& + \int_0^\infty \langle e^{-\rho t} \theta_t, u(c_t) - x(\pi_t) + \mathcal{A} v_t + \frac{\partial v_t}{\partial t} - \rho v_t \rangle_\Phi dt \\
& + \int_0^\infty \left[\langle e^{-\rho t} \eta_t, u'(c_t) - \frac{1}{Q_t} \frac{\partial v}{\partial a} \rangle_{\Phi_{1,2}} + \langle e^{-\rho t} \eta_t, u(c_t) - u(\delta a + y_2 - (\delta + \pi_t) Q_t a) \rangle_{\Phi_3} \right] dt + \\
& + \int_0^\infty e^{-\rho t} \mu_t \left[Q_t (\bar{r} + \pi_t + \delta) - \delta - \dot{Q}_t \right] dt
\end{aligned}$$

The Gateaux derivative with respect to f is

$$\begin{aligned}
\frac{d}{d\alpha} \mathcal{L}(\pi, Q, f + \alpha h, v, c) = & - \int_0^\infty e^{-\rho t} \langle u(c_t) - x(\pi_t) + \frac{\partial \zeta}{\partial t}, h_t \rangle_\Phi dt \\
& + \langle \zeta_0, h_0 \rangle_\Phi - \lim_{T \to \infty} \langle e^{-\rho T} \zeta_T, h_T \rangle_\Phi
\end{aligned}$$

where I use defintion of the adjoint operator $\langle \zeta_t, \mathcal{A}^* f_t \rangle_\Phi = \langle A^* \zeta_t, f_t \rangle_\Phi$ and integrate by parts

$$\int_0^\infty \langle e^{-\rho t} \zeta_t, \frac{\partial f_t}{\partial t} \rangle_\Phi = \langle \zeta_0, f_0 \rangle_\Phi - \lim_{T \to \infty} \langle e^{-\rho T} \zeta_T, f_T \rangle_\Phi + \int_0^\infty e^{-\rho t} \langle \frac{\partial \zeta}{\partial t} - \rho_t, f_t \rangle_\Phi dt$$

before taking derivative. The derivative must be equal for any pertubation h in the defined space, such that $h_0 = 0$ since the initial distribution f_0 is given. Use transversality condition $\lim_{T \to \infty} e^{-\rho T} \zeta_T = 0$ (since $\zeta \in L^2(\Phi)$), we have the PDE for ζ

$$\rho \zeta = \frac{\partial \zeta}{\partial t} + u(c) - x(\pi) + \mathcal{A} \zeta$$

which looks exactly like that of the HJBs. The boundary conditions are also the same, so they must have the same solution, i.e. $\zeta = v$.

The Gateaux derivative with respect to c is

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{L}(\pi, Q, f, v, c+\alpha h) = &\int_0^\infty e^{-\rho t}\langle\left(u'(c_t) - \frac{1}{Q_t}\frac{\partial \zeta}{\partial a}\right)h_t, f_t\rangle_{\Phi_{1,2}} dt + \int_0^\infty e^{-\rho t}\langle\left(u'(c_t) - \frac{1}{Q_t}\frac{\partial v}{\partial a}\right)h_t, \theta_t\rangle_{\Phi_{1,2}} dt \\ &+ \int_0^\infty e^{-\rho t}\langle u''(c_t)h_t, \eta_t\rangle_{\Phi_{1,2}} dt \\ &+ \int_0^\infty e^{-\rho t}\langle u'(c_t)h_t, \theta_t\rangle_{\Phi_3} dt + \int_0^\infty e^{-\rho t}\langle u'(c_t)h_t, f_t\rangle_{\Phi_3} dt - \int_0^\infty e^{-\rho t}\langle u'(c_t)h_t, \theta_t\rangle_{\Phi_3} dt\end{aligned}$$

which should be equals to zero for any functional h. Consider the first line, the FOC $u'(c_t) - \frac{1}{Q_t}\frac{\partial \zeta}{\partial a} = 0$ holds for households in state 1 and state 2, and $u''(c_t) > 0$ (u strictly concave). So $\eta_1 = \eta_2 = 0$.

Consider the second line and the case $\theta_3 = 0$, which we will check later. Then $\eta_3 = f_3$.

The Gateaux derivative with respect to c is

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{L}(\pi, Q, f, v+\alpha h, c) = &\int_0^\infty e^{-\rho t}\langle \mathcal{A}^*\theta_t - \frac{\partial \theta}{\partial t}, h_t\rangle_\Phi \\ &- \langle \theta_0, h_0\rangle_\Phi + \lim_{T\to\infty}\langle e^{-\rho T}\theta_T, h_T\rangle_\Phi\end{aligned}$$

which must be zero for any h. Since $\theta \in L^2(\Phi)$ we have transversality condition $\lim_{T\to\infty} e^{-\rho T}\theta_T = 0$. Also, h_0 can be positive so $\theta_0 = 0$. We finally have

$$\frac{\partial \theta}{\partial t} = \mathcal{A}^*\theta_t$$

$$\theta_0 = 0$$

This is a KF equation with initial zero mass everywhere $\theta_0 = 0$. Therefore, the distribution is 0 everywhere at all time $\theta = 0$.

The Gateaux derivative with respect to π gives equation (17) and with respect to Q gives equation (18).

B Numerical algorithm

Here I describe the numerical algorithm to compute the optimal Ramsey plan. The complete algorithm proceeds as follows

- **Step 0:** Compute the stationary steady-state distribution f_N, bond price Q_N and inflation π_N. Set initial guess for optimal inflation path $\pi^{(1)} \equiv \{\pi_t^{(1)}\}_{t=0}^{\infty} = \pi_N$, and iteration count $n := 1$.

- **Step 1:** Given $\pi^{(n)}$, compute the path of bond prices $Q^{(n)}$ using equation (9).

- **Step 2:** Given bond prices $Q^{(n)}$ and inflation path $\pi^{(n)}$, solve the household's transition problem (HJB) and find the path of distributions (KFE) using the method described by Achdou et al. (2017) (also used in Nuno and Thomas (2020) and Nuño and Moll (2018)). We obtain paths of consumption function $\mathbf{c}^{(n)}$, value function $\mathbf{v}^{(n)}$ and net wealth distribution $\mathbf{f}^{(n)}$.

- **Step 3:** Given $\mathbf{c}^{(n)}$, $\mathbf{v}^{(n)}$, and $\mathbf{f}^{(n)}$, compute Lagrange multiplier $\mu^{(n)}$ using equation (18)

$$\mu_{t+1}^{(n)} = \mu_{t+1}^{(n)}[1 + dt(\rho - \bar{r} - \pi_t^{(n)} - \delta)] + \frac{dt}{Q_t^{(n)}} \left[\sum_{i=1}^{3} \sum_{j=1}^{J} (\delta a_j + y_i - c_{t,i,j}^{(n)}) u' \left(c_{t,i,j}^{(n)} \right) f_{t,i,j}^{(n)} da \right]$$

  with $\mu_0^{(n)} = 0$.

- **Step 4:** . Given $\mathbf{c}^{(n)}$, $\mathbf{v}^{(n)}$, $\mathbf{f}^{(n)}$, and $\mu^{(n)}$ iterate steps 1-4 until $\pi^{(n)}$ satisfies equation (17)

$$\Delta_t^{(n)} \equiv \sum_{i=1}^{3} \sum_{j=1}^{J} a_j Q_t^{(n)} u' \left(c_{t,i,j}^{(n)} \right) f_{t,i,j}^{(n)} da + \psi \pi_t^{(n)} - Q_t^{(n)} \mu_t^{(n)} = 0$$

  If $\Delta_t^{(n)}$ is close enough to 0 for all t, we have found the Ramsey solution. Otherwise, we update the inflation path guess $\pi^(n+1) = \pi^(n) - \xi \Delta^{(n)}$ with coefficient $\xi = 0.05$.

C Alternative calibration

Here I jointly set households' discount rate ρ and ad-hoc borrowing limit ϕ such that the steady-state net international investment position (NIIP) over GDP, i.e. aggregate asset position in our model ($\bar{a}/\bar{y}$) is around -25 percent, and gross household debt to GDP ($\bar{a}/\bar{y}$) is around 90 percent. These numbers are features of Nuno and Thomas (2020) target economies, namely Sweden, the UK and Baltic countries. Under probability of liquidity shock $1 - p = 0.02$, such calibration give $\rho = 0.03019$ and $\phi = -3.5$, corresponding to aggregate asset position at -25.4 and total debt at 86.8 percent of GDP. Figure 4 illustrates the optimal transitional dynamics under such calibration.

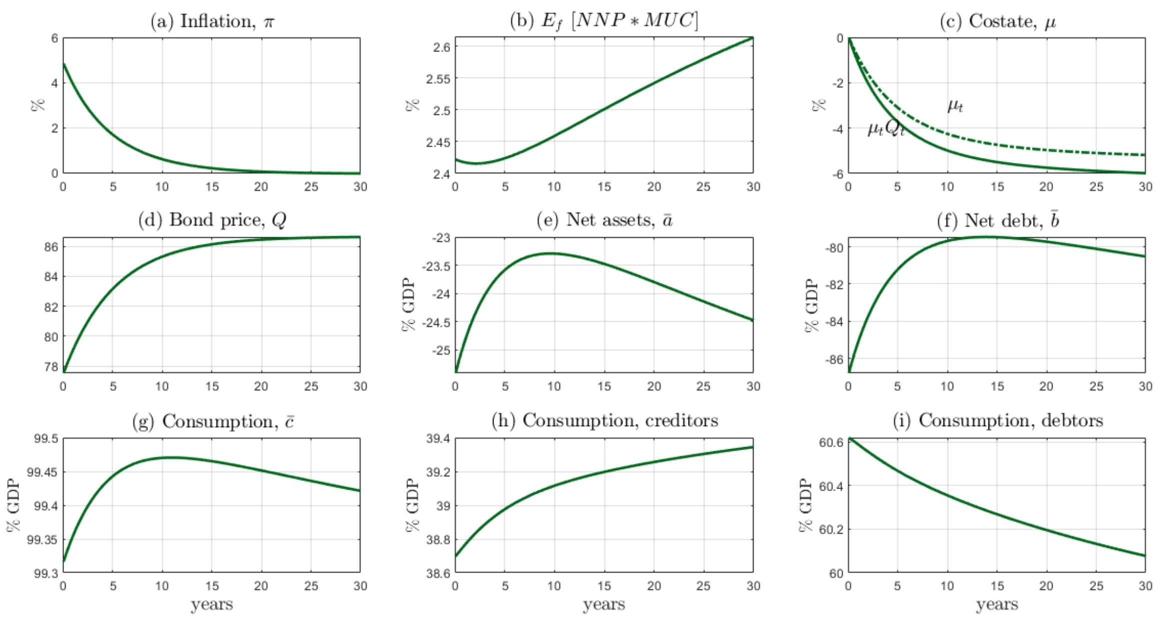

Figure 4: Optimal inflation path and transitional dynamics under alternative calibration

Optimal initial inflation is close and a bit higher than that in Nuno and Thomas (2020), which is expected given that here we calibrated such that the cross-border redistribution motive is roughly the same as theirs.