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Acknowledgments

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Abstract

Investment portfolios are an essential variable in an individual's finances, with every consumer knowing the meaning of portfolio and diversification. Furthermore, globalization and digitalization transformed the nature of capital markets, improving the flow of information and the accessibility to investment opportunities. As a result, the daily volume of trade and the studies related to investment portfolios have constantly increased.

The first studies inherent to investment portfolios were introduced in 1938 by John Burr Williams with his book "*The Theory of Investment Value*".

The most significant contribution on which Modern Portfolio Theory (MPT) builds arrived in 1952 from the economist Harry Max Markowitz's publication *"Portfolio Selection"* in the Journal of Finance. MPT drove investors' focus from expected returns to risk tolerance and how the trade-off between these two variables can improve through diversification; it redefined the investment decision-making.

Later on, William F. Sharpe, John Lintner, and Jan Mossin relying on Markowitz's article introduced the Capital Asset Pricing Model (CAPM), an easy to interpret model to measure the risk-return trade-off.

Nowadays, most investment decisions are still carried out following the MPT ideas and implementing the CAPM.

The elaborate objective is to provide a keen understanding of the relationship between risk and return and how the two variables influence investment decisions in capital markets from both a theoretical and mathematical point of view.

The research is broken into three chapters.

Chapter 1 is an introduction to risk and returns notions on which portfolio theory relies. General concepts about risk, its measurement, and how it affects a portfolio and investment choices are covered. A strong emphasis is placed on the concept of diversification and its benefits.

Chapter 2 tests and provides empirical results to the theory covered in chapter 1, constructing a portfolio of stocks randomly selected and elaborating on it. Different portfolio strategies are utilized, and each result obtained is commented addressing the topics introduced in the first chapter.

Finally, chapter 3 concludes the study by providing further insight on how investments are carried out in practice and why financial models as the CAPM are not applicable to their full extent in the real market.

Keywords:
CAPM, Diversification, Portfolio, Return, Volatility

Table of Contents

Chapter 1

Risk Evaluation in Capital Markets

1.1 The Concept of Risk and Return

Whenever undertaking an investment, the main concerns are the risk involved and the return on the investment. Portfolios of securities are constructed fundamentally on these two determinants.

Risk in financial terms is to be intended as the probability that the realized return on an investment deviates from the expectations of return.
The risk-return trade-off is the building-block variable affecting investment decision. All sorts of investments involve a certain degree of risk. The two variables are directly related; investors expect their returns to increase as they undertake more risk. However, this is only a general assumption.

Generally, investors need to consider the time horizon and liquidity of an investment to determine the level of risk. Investors may consider long-term investments with greater risk when they do not need funds readily. Contrary, investors will consider short-term investments when they need funds readily. Ordinarily, the greater the time horizon, the greater the return on the initial investment.
With the term "liquid", we indicate the possibility to obtain cash at demand. Thus, liquid investments can be easily exchanged into cash, therefore bear less risk.

A riskless investment is one that grants a certain level of return with very low or null risk. It is customarily used as a means to compare the risk of an asset. Riskless assets are bought by investors who want to preserve their savings and access funds with ease.
Financial assets can be classified based on their level of risk:

1. Treasury bills with a maturity of 30 days are considered to be a riskless investment (i.e. a risk-free asset);
2. Long-term government bonds which price fluctuates with the market interest rate;
3. Common stock shares which returns are affected by the performance of the issuing firm.

In finance, risk is classified into two broad categories: market risk and specific risk.
Market risk, also known as systematic risk, refers to the fluctuations caused by events that affect the entire market and can not be eliminated through diversification.
Examples of market risks are financial crisis as stock market crashes, political tensions, fluctuations in the interest rates and natural catastrophes.
Specific risk, also known as unsystematic or idiosyncratic risk, is company or sector specific and can be mitigated by choosing uncorrelated investment opportunities (i.e., it can be diversified away).

1.2 Measures of Risk and Return

Risk is measured by implementing statistical measures to make investment decisions. Returns can be estimated by analyzing historical data.

Probability Distribution
Risk and return change with horizon and time. However, how they do change is unpredictable. Therefore, in order to estimate successive asset returns, we assume that returns are random variables Identically and Independently Distributed (IID) when successive returns are IID.

Probability distribution aims at describing all the possible values returns can assume; the probability that returns go up or down.

Implications of the IID assumption:
- Returns are serially uncorrelated. They do not co-vary over time with themselves, and there is no existence in trends in asset prices.
- Trends, cycles, or patterns are not predictable in returns; stocks follow a random walk.
- Returns and risk accumulate linearly over time

Assuming stock returns are IID simplifies the evaluation of risk and return; however, this assumption does not hold in most cases. It is often observed that they happen to be serially correlated. On top of that, risk and return do not accumulate linearly. The distribution of stock returns is log-normal therefore skewed in a direction rather than normal and bell-shape skewed.

Expected Return

$$E[R] = \sum_{i=1}^{n} p_i \times R_i$$

The expected return corresponds to a weighted average of the possible returns, using the probabilities of returns as weights.
Changes in the return of the stock can be assigned a probability of occurrence, which can be summarized with a probability distribution. Once obtained, they can be used as weights in the expected return formula.

Variance

$$Var(R) = \sigma^2 = \sum_{i=1}^{n} p_i \, (R_i - E[R])^2$$

The variance of the return corresponds to the expected square deviation from the mean: the distribution spread.
Variance is equal to zero whenever the return is risk-free; in other words, it does not deviate from the mean.

Standard Deviation

$$SD\ (R) = \sigma = \sqrt{Var(R)}$$

The standard deviation of the return is equal to the square root of the variance.
With the standard deviation, we refer to the volatility of the return. Moreover, it is easier to interpret as it brings back the risk measure to the original unit measure of the returns. The higher the standard deviation, the higher the volatility, therefore the greater the risk involved.

Realized Returns
The realized return from a stock that does not pay dividend is calculated as:

$$R_{t+1} = \frac{P_{t+1}}{P_t}$$

While, if the stock pays dividend:

$$R_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t}$$

In other words, is equal to the sum between the dividend yield and the capital gain rate. The latter formula can be used to compute the realized return on any security that provides regular cash flows (e.g., bonds and their coupon payments).

Realized Annual Returns
If we suppose dividends are paid quarterly and reinvested immediately into the company, the realized annual return is found as:

$$1 + R = (1 + R_1)(1 + R_2)(1 + R_3)(1 + R_4)$$

Beta

$$\beta_i = \frac{Cov(R_i, R_m)}{Var(R_m)}$$

It is equal to the ratio between the covariance of the return on stock *i* and the return on the market index and the variance of the market index return.

Alternatively:

$$\beta_i = \frac{\sigma_i * \rho_{i,m}}{\sigma_m}$$

Where:
- σ_i: standard deviation of stock *i*
- $\rho_{i,m}$: correlation between the stock *i* and the market index
- σ_m: standard deviation of the market index

Beta is among the most common measures of stock or portfolio volatility.

It expresses the systematic risk (non-diversifiable) that characterizes a stock compared to a benchmark in the same market segment. In general, the benchmark adopted in security analysis is the market index or market portfolio—for instance, the S&P 500.

A negative Beta indicates that the stock expected return is below that of the risk-free. The stock is inversely correlated with the market index; therefore, it will perform whenever the market is in a downturn. Including a stock with a negative Beta in a portfolio is an insurance against systematic risk.
Assets with Beta equal to 0 are infrequent but if close to 0 they contribute to reduce the risk of the portfolio. Positive Betas suggest positive correlation with the market.
The market index used as a benchmark in portfolio analysis has Beta equal to 1.
A stock with Beta equal to 1 if added to the portfolio does not provide additional risk nor extra expected return; the stock movement is directly related to the market index movements.
A beta greater than 1 indicates that the stock is more volatile than the market.

R-squared

$$R^2 = 1 - \frac{Unexpected\ Variation}{Total\ Variation}$$

R-squared is used to understand the percentage variance when the security variance is determined by a benchmark. In other terms, the percentage variation of a security with respect to changes in the benchmark or market index.
What is measured is not the performance of the portfolio, rather the correlation between the portfolio and the benchmark.

It ranges from 0 to 100%, where 0 means that the security movements are uncorrelated with the market index movements. Instead, 100% indicates that all movements of the security are explained by the market index movements. By convention, an R-squared above 70% indicates a strong correlation to the market index movements.

When evaluating a security, R-squared can be used in conjunction with Beta to understand its risk and returns better. A portfolio with a high R-squared and Beta close to 1 may outperform the market index utilized as a benchmark.
Moreover, a high R-squared can be adopted to verify Beta reliability.

Sharpe Ratio

$$S = \frac{R_p - r_f}{\sigma_p}$$

Sharpe Ratio gives a clear understanding of the excess obtained by investing in risky activities. In particular, it defines the trade-off obtained by combining a risky asset with a risk-free asset in a portfolio.

It corresponds to the ratio between the risk premium and the standard deviation of the excess return on the portfolio.
Generally, the greater the value of the Sharpe ratio, the more attractive the risk-adjusted return. Thus, the highest Sharpe ratio will offer the best trade-off between risk and return.

It is easy to calculate, and it allows an immediate comparison. However, it considers all risk, not just systematic risk (non-diversifiable), which is a valid measure of performance for an investor not interested in diversifying but who cares about investing all-in-one.

A Sharpe ratio above 1 indicates an acceptable investment. A portfolio with a Sharpe ratio above 2 is deemed very good and above 3 excellent.

Sortino Ratio
The Sortino ratio differs from the Sharpe ratio in that it only considers the standard deviation of the downside risk rather than that of the entire (upside and downside) risk; at the denominator, we will have downside risk. Hence, the numerator corresponds to the difference between the return and the targeted return.

$$SR = \frac{R_p - R_{TR}}{DD}$$

Because the Sortino ratio focuses only on the negative deviation of a portfolio's returns from the mean, it is thought to give a better view of its risk-adjusted performance since positive volatility is a benefit.

Tracking Error Volatility (TEV)
The Tracking Error Volatility aims to measure the spread from the targeted portfolio return, represented by the benchmark being the market index.

$$TEV = \sqrt{\sum_{i=1}^{n} (R_p - R_m)^2}$$

A TEV with a value close to 0 is efficiently replicating the performance of the benchmark. When the TEV reaches its tolerance level, the portfolio is rebalanced. As it indicates how much a portfolio deviates from the benchmark, it is also used to measure a fund's performance.

Information Ratio

$$IR_p = \frac{R_p - R_m}{TEV}$$

The Information Ratio is the difference between the expected return of the fund's portfolio *p* and the benchmark return *m* on the corresponding Tracking Error Volatility TEV.

The Information ratio allows testing whether a portfolio managed by a fund provides a return significantly larger than the benchmark.
Since the benchmark portfolio is supposedly efficient, the Information ratio is useful in evaluating evaluating the fund manager's skills; the greater, the better the manager ability to select profitable investment opportunities.

Treynor Ratio
Contrary to the Sharpe Ratio, Treynor Ratio only considers systematic risk and expresses the expected excess return over a risk-free asset per unit of systematic risk.

$$TR = \frac{R_p - R_f}{\beta_p}$$

One of the main limitations of the Treynor ratio is that Beta is challenging to estimate and may not represent all sources of systematic risk.

***M^2* measure (Modigliani squared)**
Like the Sharpe ratio, Modigliani squared focuses on total volatility as a measure of risk, but its risk adjustment leads to an easy-to-interpret differential return relative to the benchmark index.

Let $p*$ be a portfolio made of fund p and the risk-free asset with overall volatility of the market portfolio.
Because the market index and portfolio $p*$ have the same standard deviation, we can compare their performance by comparing returns:

M^2 = expected returns of portfolio $p*$ - expected returns market portfolio p

If the managed portfolio had a lower standard deviation than the index, it would be leveraged by borrowing money and investing the proceeds in the portfolio.

Value at Risk (VaR)
The Value at Risk statistical indicator measures the amount of potential loss that could occur in an investment or a portfolio over a given period.
It expresses the extent of the risk subject to a specific investment project.
Its calculation is not standardized.
To be computed it requires: a timeframe, a potential loss, and the probability of occurrence of the loss.

Jensen's Alpha
Jensen's Alpha is the intercept in the linear regression of the excess fund return on the market portfolio.

Represented in figure 1, Alpha corresponds the average return above or below the return predicted by the capital asset pricing model (CAPM), given the portfolio's Beta and the average market return. In other words, it indicates whether the fund is located above, or below, the security market line.

A positive value for Jensen's Alpha indicated that the portfolio managed by the fund manager has "beat the market" with their stock-picking skills and vice-versa.

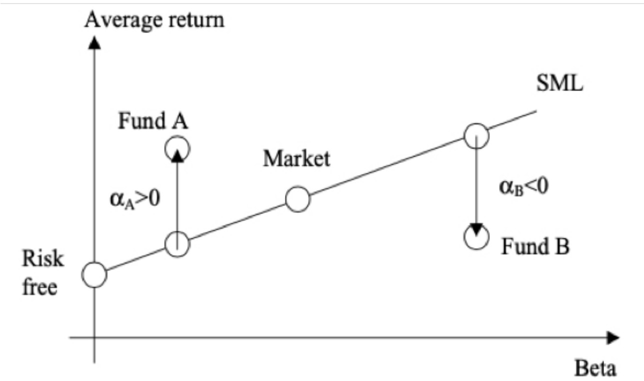

Figure 1: Graphical representation of Jensen's Alpha

1.3 Diversification and its Role

Diversification risk management strategy aims at constructing portfolios of uncorrelated securities. While this strategy does not provide the greatest returns for investors, it allows, for a given level of risk, to maximize the portfolio expected return.
Diversification is adopted to have constant and safe returns in the long-term rather than large but risky returns in the short-term.

In order to understand the fundamental role played by diversification in portfolio composition we highlight two types of risk: common and independent.

Common risk may affect a whole group of securities if they are perfectly correlated.
Think for instance to the petroleum shortage occurred May 2021 in the United States.
CNBC reports: *Oil prices rose on Tuesday, as lingering fears of gasoline shortages due to the outage at the largest U.S. fuel pipeline system after a cyber attack brought futures back from an early drop of more than 1%. Brent crude futures rose 35 cents, or 0.5%, to $68.67 a barrel. U.S. West Texas Intermediate (WTI) crude futures rose 49 cents, or 0.8%, to $65.41. Benchmark gasoline futures prices rose 1 cent to $2.14 a gallon.*[1]
This type of risk is broadly defined as market, systematic or undiversifiable risk as such fluctuations are market-wide and usually affect the entire global economy.
Conversely, independent risk concerns securities that do not share any correlation and it is generally defined as idiosyncratic, unsystematic or diversifiable risk. Fluctuations are firm-specific and do not affect the global economy but only firms individually.
While common risk may not be eliminated or reduced through diversification, independent risk can. By creating a portfolio of securities well diversified we reduce the impact of this risk.

[1] CNBC, Oil prices rise on nagging fears of fuel shortages, May 2021
https://www.cnbc.com/2021/05/11/oil-market-us-gulf-coast-pipeline.html

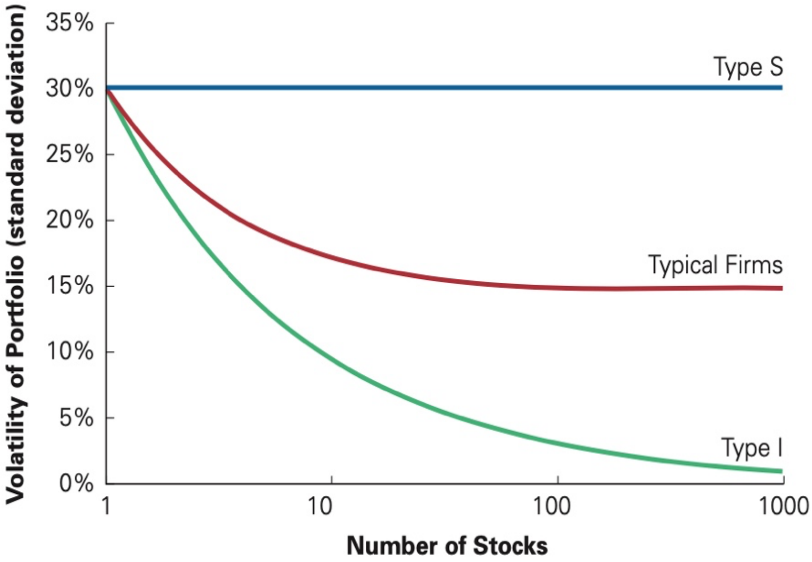

Figure 1.1: Volatility of Portfolios

The figure enlightens the variation in volatility for three types of firms after the increase in the number of stocks in the portfolio.
Firm type S bears only systematic risk and portfolio diversification does not have an impact on the volatility. Contrary, firm type I has only idiosyncratic risk and as the number of stocks in the portfolio increases volatility decreases until eliminated[2].
A typical firm however is subject to both type of risks. Diversification reduces portfolio volatility until only market risk remains; all firms are subject to a certain degree of market risk.

1.4 Equity Risk Premium and its Estimation

Investors are risk-averse and despite the existence of safe investment opportunities they occasionally decide to invest in risky activities.
For holding risky investments, they therefore require an extra return, such return is defined as the equity risk premium.

It is crucial to point out that the rewarded premium is not determined on the entire risk, rather only on systematic risk that cannot be diversified away. Investors do not earn a premium on idiosyncratic risk as such risk is diversifiable; portfolios able to eliminate completely risk must have same expected return of risk-free assets. This reasoning is in accordance with the Law of One Price: an economic theory that states that the price of identical goods in different markets must be the same after taking the currency exchange into consideration (i.e., if the prices are expressed in the same currency).[3]
If investors were to earn premiums also on diversifiable risk an arbitrage opportunity would arise: investors could purchase this securities in order to earn a premium and then diversify their portfolio to eliminate the risk.

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[2] Jonathan Berck, Peter DeMarzo, Corporate Finance 4thedition, chapter 4, 10.6 Diversification in Stock Portfolios
[3] Corporate Finance Institute, What is the Law of One Price (LOOP)?, https://corporatefinanceinstitute.com/resources/knowledge/economics/law-of-one-price-loop/

The premium earned by investors is therefore determined only on the amount of systematic risk carried by an asset and the stock volatility (or standard deviation) is not an appropriate indicator to measure the amount of the premium.

In order to assess the risk premium we must first determine the stock sensitivity to firm-specific risk. This is done by implementing an efficient portfolio: a portfolio that through diversification eliminates idiosyncratic risk, leaving only market risk. Such portfolio can be identified as the market portfolio which we assume to contain all securities traded in capital markets. However, in practice a portfolio containing all securities traded at a global level does not exist; for simplicity, the market portfolio is usually identified as a broad market index available on the market.

The S&P 500 is recognized worldwide as one of the premier benchmarks for the U.S. stock market's performance. The S&P 500 does not simply contain the 500 largest stocks; rather, it covers leading companies from leading industries. It represents a broad cross-section of the U.S. equity market, including common stocks traded on U.S. exchanges.
The S&P 500 represents the largest US companies based on market capitalization and it is constructed with weights based on firms' capitalization. Therefore, companies as Amazon.com, Apple Inc, and Microsoft Corp have the greatest influence on the index price performance[4].

Once determined a benchmark, the market portfolio, we can estimate Beta (β) of the security. Beta expresses in percentage the change in the security return after a change in the return in benchmark.

The risk premium required by investors can then be simply computed as the market excess return on the risk-free asset required for holding a risky investment.

$$Risk\ premium = E[R_m] - r_f$$

With the risk-free asset, beta of the investment, and the risk premium analysts and managers can evaluate an investment by computing the capital cost.
The assessment of the cost of capital corresponds to the expected return of an investment.

$$r_i = r_f + \beta_i \times (E[R_m] - r_f)$$

The above equation is known as the Capital Asset Pricing Model (CAPM). It is a widely used model to estimate the cost of capital and make investment decisions. The following sections will rely on the implication of the CAPM and a practical use of it.
Risk premium can be derived from the CAPM equation:

$$r_i - r_f = \beta_i \times (E[R_m] - r_f)$$

[4] Spglobal.com, Additional Info, S&P 500® The Gauge of the Market Economy
https://www.spglobal.com/spdji/en/documents/additional-material/sp-500-brochure.pdf

1.5 Portfolio Fundamentals

The economist Harry Max Markowitz in 1952 pioneered the Modern Portfolio Theory (MPT) for which he was awarded the Nobel Memorial Prize in Economic Sciences.
Also referred to as the Mean-Variance model, MPT consists in a mathematical framework aimed at optimizing a portfolio.
An optimized portfolio is not limited only to stock-picking, such stocks have to provide a safe return given a certain level of risk. The goal of an optimized portfolio is to combine stocks that do not move together, therefore the portfolio provides returns even during an economic downturn and grows during a boom.
Investors ultimately care about two variables: risk and return. A diversified portfolio is not for a speculator, is for an investor who wants the highest possible return for a given level of risk.

The stock analysis and the portfolio selection can be divided into two broad phases.
In the first stage the investor gathers information about potential securities which may satisfy his objective. Information is usually gathered through historical observations to build expectations about future returns.
The second stage starts from the end of the first. After gathering enough information to build expectations of the future we construct a portfolio on the basis of our expectations.[5]

The mean-variance model developed by Markowitz relies on several assumptions:

1. The total amount of risk of the portfolio depends on the volatility. Volatility is measured through statistical indicators as variance and standard deviation.
2. Investors are risk averse: they do not undertake risk willingly and take decisions on expected return variance.
3. Investors prefer to increase the number of stocks. Diversification is obtained by including in the portfolio stocks with low levels of co-movement.
4. Investors are rational and aim at maximizing utility: if two portfolios offer the same return for different level of risk, investors will choose the portfolio with the best trade-off.
5. Market information is fair and freely accessible to all investors.
6. Information is reflected immediately in the markets which are assumed to be efficient.

The technical analysis of a portfolio is carried out with the measures of risk and return described in section 1.2.

Income allocation in the portfolio depends on the weight assigned to an individual investment:

$$w_i = \frac{Value\ of\ investment\ i}{Total\ portfolio\ value}$$

Where w_i represents the percentage of wealth invested in an individual stock and the sum of the individual weights adds up to 1 ($\sum w_i = 1$).
Expected return on the portfolio can then be computed as the sum of the weighted individual returns:

[5] Markowitz, Harry Max. Portfolio Selection, *The Journal of Finance*, March 1952.

$$E(R_p) = \sum_{i=1}^{n} w_i E(R_i)$$

The variance of the returns on the portfolio corresponds to:

$$\sigma_p^2 = \sum_i w_i^2 \, \sigma_i^2 + \sum_i \sum_{j \neq i} w_i w_j \sigma_i \sigma_j \rho_{i,j}$$

The portfolio standard deviation is once again simply the square root of the variance:

$$\sigma_p = \sqrt{\sigma_p^2}$$

Therefore, for a portfolio of two stocks, expected returns and volatility are found:

$$E(R_p) = w_1 E(R_1) + \; w_2 E(R_2) = w_1 E(R_1) + (1 - w_1) E(R_2)$$

$$\sigma_p^2 = \; w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_1 \sigma_2 \rho_{1,2}$$

The total risk of the portfolio cannot be evaluated only through variance and standard deviation, it is necessary to assess how stock returns respond to shocks. The degree of co-movement is assessed with two statistical indicators: covariance and correlation.
Covariance measures the extent to which two random variables vary together. It corresponds to the product of the spread between the return and the expected return of both assets:

$$Cov(R_i, R_j) = E[(R_i - E(R_i)(R_j - E(R_j)]$$

A positive covariance means that the two stocks in question will exhibit increasing and decreasing returns at the same time. On the other hand, a negative covariance indicates that while a stock exhibits increasing returns, the other will exhibit decreasing returns.
As variance, covariance is not easy to interpret. The correlation is a statistical indicator that corresponds to a standardized and easy to interpret covariance. It is computed as the ratio between the covariance of the returns and the product of the standard deviations of the returns:

$$Corr\,(R_i, R_j) = \frac{Cov(R_i, R_j)}{\sigma(R_i)\sigma(R_j)}$$

As the covariance, correlation is either positive or negative, but the value ranges from -1 to +1 making interpretation straightforward.
As the correlation increases also the degree of co-movement intensifies. Stocks with a correlation of 1 are perfectly correlated and follow the same trends. Stocks with a correlation of 0, therefore also covariance, are independent and returns do not follow any predictable pattern. Finally, stocks with a correlation of -1 are perfectly negatively correlated and their returns move in exactly opposite direction.

Notice that the covariance of a stock with itself is always equal to the stock variance, while the correlation is equal to +1. Analytically it is proven as:

$$Cov(R_i, R_i) = E[(R_i - E(R_i)(R_i - E(R_i)] = E[(R_i - E(R_i)] = Var(R_i)$$

$$Corr\ (R_i, R_i) = \frac{Cov(R_i, R_i)}{\sigma(R_i)\sigma(R_i)} = \frac{Var(R_i)}{\sigma(R_i)^2} = 1$$

Estimates of a stock's mean return, variance and covariance can be determined through historical data. By observing returns on asset *i* over a total of periods from *t* to *T,* we find sample mean and sample variance as:

$$\widehat{R_\iota} = \frac{1}{T}\sum_{t=1}^{T} \widetilde{R_{\iota,t}}$$

Where $\widetilde{R_{\iota,t}}$ corresponds to the returns obtained on asset *i* in period *t.*

$$\widehat{\sigma_\iota^2} = \frac{1}{T-1}\sum_{t=1}^{T} \left(\widetilde{R_{\iota,t}} - \widehat{R_\iota}\right)^2$$

Computing the sample mean on the returns of another asset *j* we can define the sample covariance of asset *i* and *j* as:

$$\widehat{\sigma_{\iota,J}} = \frac{1}{T-1}\sum_{t=1}^{T} \left(\widetilde{R_{\iota,t}} - \widehat{R_\iota}\right)\left(\widetilde{R_{J,t}} - \widehat{R_J}\right)$$

1.6 The Efficient Portfolio

As mentioned throughout the antecedent sections, including in a portfolio stocks that are not perfectly correlated reduces the portfolio idiosyncratic risk until a certain point where only specific risk is left. The diversification process aims at constructing an efficient portfolio.

Firstly, we define a portfolio as "inefficient" if it is possible to obtain the same, or higher, expected return with another portfolio for the same, or lower, degree of volatility, and vice-versa. Investors being rational and risk-averse will aim at maximizing expected return, while minimizing volatility.
Markowitz Mean-Variance optimization can be represented on the xy-plane for every stock individually, where the y-axis represents the return and x-axis the standard deviation. This representation on the plane is practical in determining the efficient frontier. By deriving and then plotting the expected return and variance of returns of securities contained in the portfolio we obtain a hyperbola which corresponds to the efficient frontier. Portfolios outside this frontier are impossible to obtain through diversification, while portfolios in between the frontier are considered inefficient as, for a given risk level, there are portfolios with greater expected returns.
Figure 1.2 represents the efficient frontier obtained representing several portfolios.

Investors are rational and risk averse, therefore portfolios A and B will not be chosen because for lower levels of risk portfolios C and D offer greater expected returns. Portfolios E and F offer the highest returns as well as the highest standard deviation. Investors aim at moving up and left in order to maximize return and lower volatility. Notice also the point denoted as Minimum Variance Portfolio, the portfolio that minimizes risk. It differs from the Mean-Variance optimized portfolio because it does not provide the greatest return for a given level of risk. It can be interpreted as the turning point: portfolios laying on the efficient frontier past the minimum variance portfolio will be preferred to those laying beneath it.

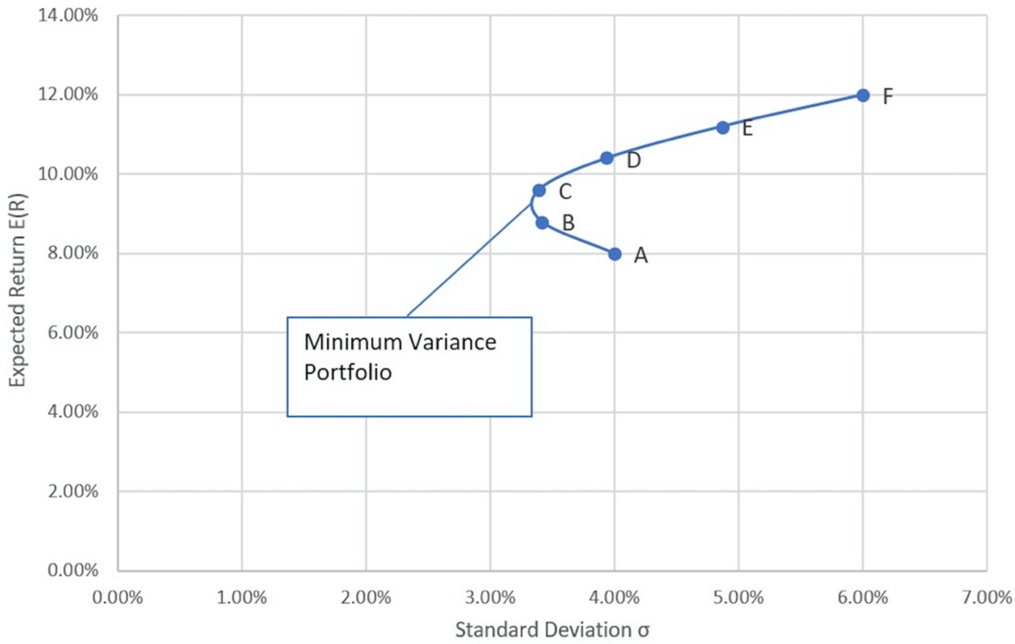

Figure 1.2 Efficient Frontier

Mathematically such portfolio is found solving a minimization problem which provides the weights necessary to minimize the portfolio variance, while keeping the sum of weights equal to one.

Expected return and variance of a portfolio with *n* assets uncorrelated:

$$E(R_p) = \sum_{i=1}^{n} w_i E(R_i)$$

$$\sigma_p^2 = \sum_j w_j^2 \, \sigma_j^2$$

The minimization and the constraint:

$$min_{w_N} \; \sigma_p^2 = w' \sum w \quad \text{s.t.} \;\; w_N{}'1 = 1$$

Introduce the Lagrangian function with λ as multiplier:

$$\mathcal{L} \equiv \frac{1}{2}\sum_{j=1}^{n} w_j^2 \sigma_j^2 - \lambda(\sum_{j=1}^{n} w_j - 1)$$

Variance is non-negative and we can apply first order condition:

$$\frac{\partial \mathcal{L}}{\partial w_j} = w_j \sigma_j^2 - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{j=1}^{n} w_j - 1 = 0$$

For $j = 1,2 \ldots n$ we have w_j:

$$w_j = \frac{\lambda}{\sigma_j^2}$$

$$\sum_{i=1}^{n} w_i = \lambda \sum_{i=1}^{n} \frac{1}{\sigma_i^2} = 1$$

$$\lambda = (\sum_{i=1}^{n} \frac{1}{\sigma_i^2})^{-1}$$

Weights of the Minimum Variance Portfolio are:

$$w_j^{mvp} = \frac{\frac{1}{\sigma_j^2}}{\sum_{i=1}^{n} \frac{1}{\sigma_j^2}}$$

Weights attached to asset *j* are proportional to the inverse of its variance.

As explained before, stocks correlation explains, in a standardized manner, the degree of co-movement of stocks. It affects a portfolio standard deviation and therefore also the shape of the efficient frontier.

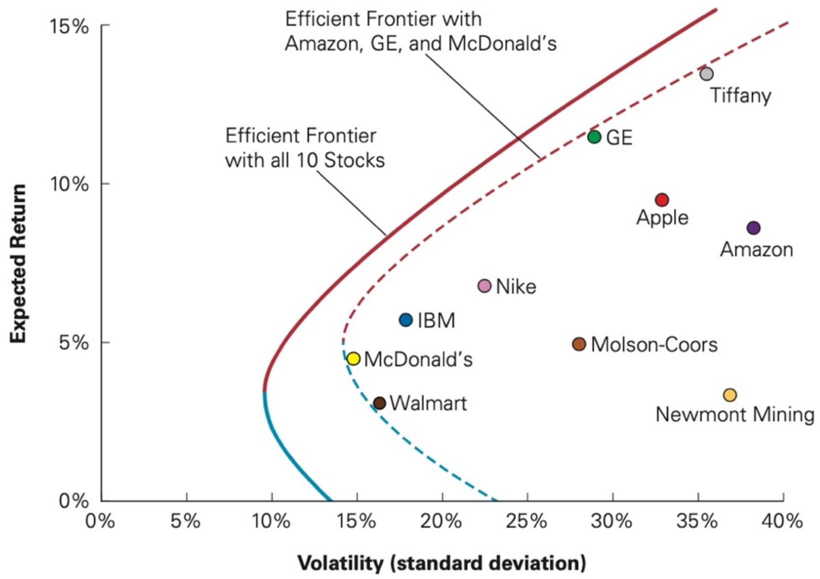

Figure 1.3 Efficient frontier with three stocks versus ten stocks[6]

The goal of moving left in the efficient frontier, in other words increase expected return and minimize volatility, can be achieved increasing the size of the portfolio.
Figure 1.4 demonstrates that moving from a portfolio of three stocks to a portfolio of ten stocks an investor can bend left the efficient frontier. The portfolio improves through diversification.

If we include the possibility to short-sell securities the efficient frontier shape changes.
Short-selling consists in borrowing from an investor and selling on the market an asset we do not own. Such practice is carried out when we predict a decrease in the asset price and we want to speculate on it. The outcome is positive when the asset price declines, and we are able to purchase and restitute the number of assets borrowed. However, this is a highly risky practice.
With short-selling, investors can keep negative weights in a portfolio and still have $\sum_{i=1}^{n} w_i = 1$. The possibility of achieving greater returns comes together with a higher volatility.
In figure 1.4 the efficient frontier without the possibility of short sales corresponds to the black curve. If short sales are included the frontier expands with no limit. Investors can choose to short sell an asset to purchase another and earn greater expected return for higher given level of risk.

[6] Jonathan Berck, Peter DeMarzo, Corporate Finance 4th edition, chapter 4, 11.4 Risk Versus Return: Choosing an Efficient Portfolio

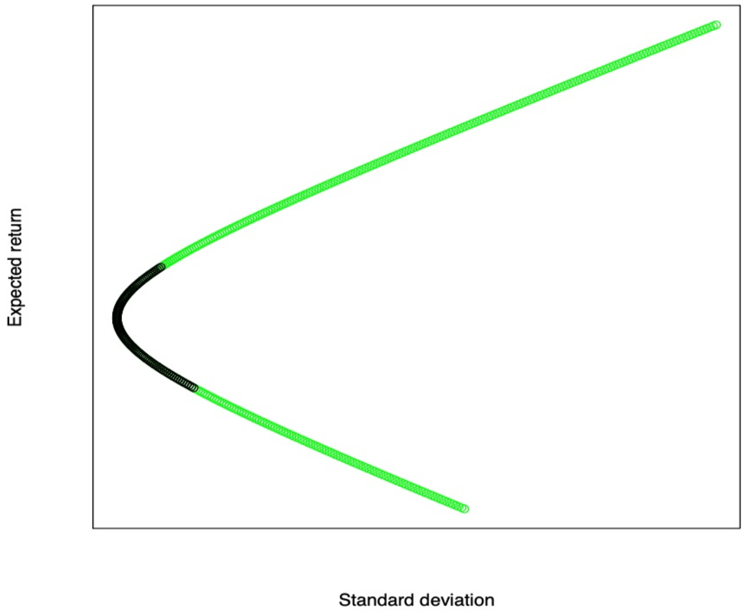

Figure 1.4 Efficient frontier with and without short sales[7]

Figure 1.5 sketches the shape of the efficient frontier when correlation between two stocks changes. If correlation is perfectly positive, the portfolio volatility is equal to the weighted average of volatility of stocks A and B and the frontier corresponds to a straight line. As correlation decreases the frontier bends left because of diversification. When equal to 0 stocks are independent. If correlation is perfectly negative it is possible to achieve a portfolio with no volatility.

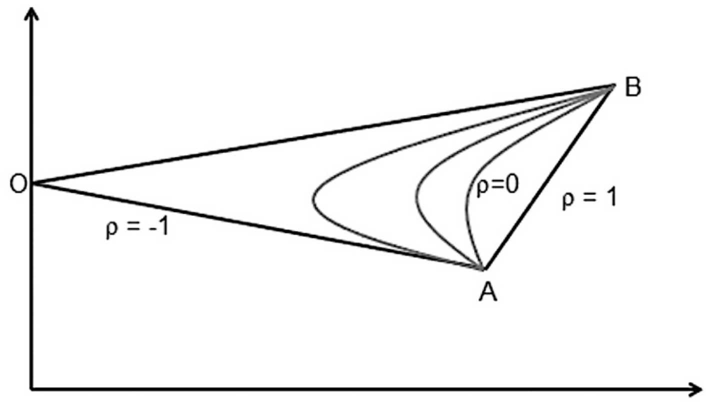

Figure 1.5 Effect of changes in correlation

Taking into consideration a risk-free asset it is possible to determine a single efficient portfolio that satisfies the investor's objective. A risk-free asset is an asset that provides a fixed return with little or no risk of default, therefore it is uncorrelated with other assets. Government securities can be utilized as risk-free assets.
The graph of the risk-free asset as pictured in figure 1.6 is linearly increasing as changes in expected return are always positive as the degree of volatility increases. By definition, the risk-free asset has a given level of expected safe return for no standard deviation.

[7] The efficient frontier with short sales allowed, University of California, Los Angeles Department of Statistics, Nicolas Christou

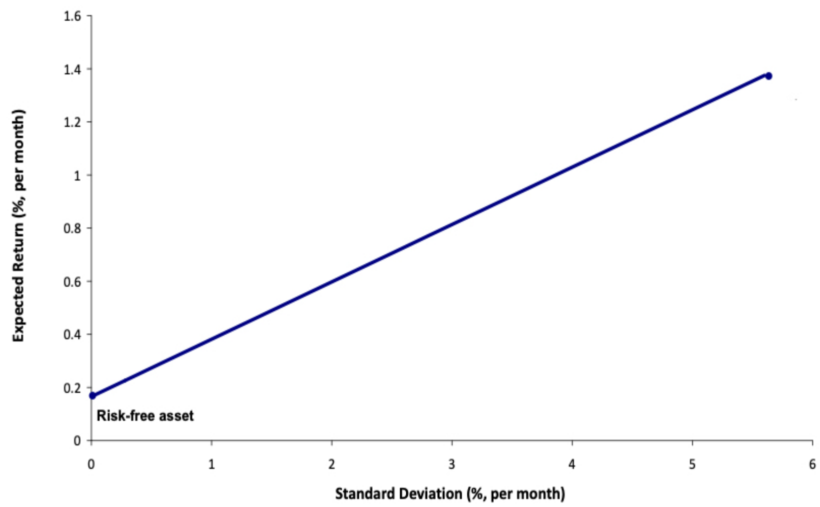

Figure 1.6 Portfolio frontier with a risk-free asset

The line obtained from the possible combinations of the risk-free asset with risky securities is defined as the Capital Allocation Line (CAL). The CAL equation is:

$$E(R_C) = r_f + \sigma_C \frac{E(R_p) - r_f}{\sigma_p}$$

Where, the risk-free rate is the y-intercept, the slope corresponds to the Sharpe ratio and *C* is the portfolio obtained from the combination of the risk-free asset with a portfolio *p* consisting of risky securities. Figure 1.7 depicts such combinations.

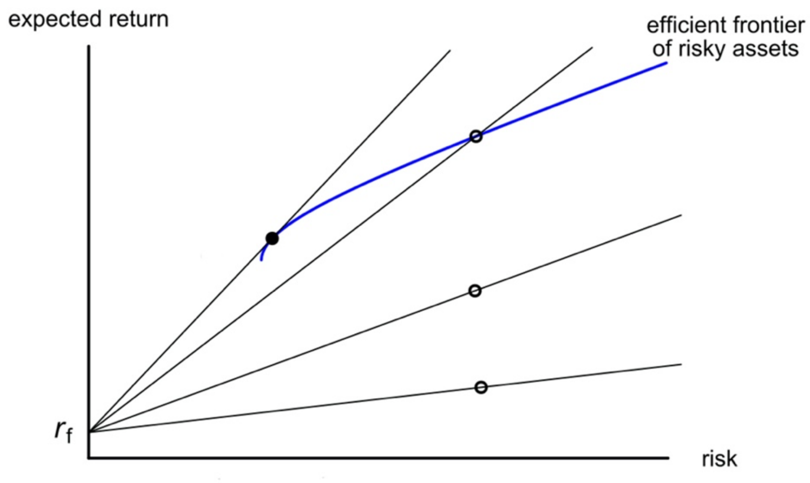

Figure 1.7 Capital Allocation Lines

Capital Allocation Line can be analytically derived constructing a portfolio *C* invested in portfolio *p* and in the risk-free asset:

$$E(R_C) = w_1 r_f + (1 - w_1)E(R_p) = r_f + (1 - w_1)(E(R_p) - r_f)$$

$$\sigma_C = (1 - w_1)\sigma_p$$

The equation obtained solving for $(1 - w_1)$ is the CAL equation.

The risk-free asset can be sold short in order to increase the wealth invested in a portfolio. This action is known as buying on margin as we say the portfolio is levered.
Buying on margin consists in borrowing funds in order to purchase securities. The money is borrowed at a rate and the investor provides a collateral for the loan which is usually the total of securities purchased. In addition, the investor is required to keep a maintenance margin at a pre-determined level. If the price of the securities drops and the maintenance margin goes below the pre-determined level, he is required to cover with money or sell the securities. In this case, the money borrowed comes from short-selling the risk-free asset at its interest rate. The possibility of obtaining greater returns comes together with higher volatility.

With the inclusion of a risk-free asset, investors construct efficient portfolios combining the risk-free asset with a portfolio of risky securities. The Two Fund Separation theorem states that the portfolio resulting from the combination of the two can be viewed as a portfolio of two sub-portfolios: one only of the risk-free asset and one only of risky assets. *An important implication is that an investor can separate her asset allocation decision into two steps: First, find the portfolio of risky assets that maximizes the Sharpe ratio; then, decide on the mix of the optimal risky portfolio and the risk-free asset, depending on her attitude toward risk.*[8]
The portfolio that maximizes the Sharpe ratio is found by the tangency between the CAL and the efficient frontier. In this case, the line is defined as the Capital Market Line (CML), a special case of the CAL where the efficient portfolio corresponds to the market portfolio (i.e. in this case to the tangent portfolio).
The Sharpe ratio being a measure of comparison of portfolio performance can be implemented to compare other portfolios to the tangent portfolio.
The tangent portfolio provides the best optimization to the risk-return tradeoff. Depending on the risk profile, investors will choose how much to allocate to the risk-free asset.
Figure 1.8 graphically represents the efficient frontier, now with the tangency point.

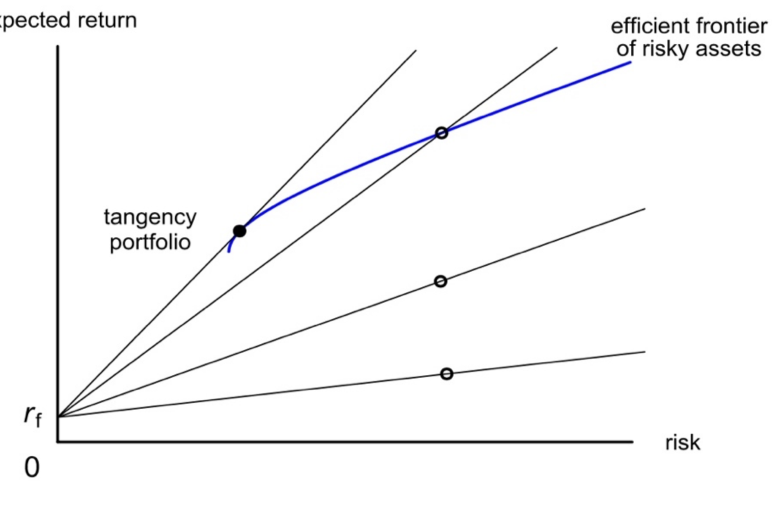

Figure 1.8 Tangent Portfolio

[8] Seung-Jean Kim and Stephen Boyd. Two-Fund Separation under Model Mis-Specification, Stanford University, January 2008.

When adding a new stock to a portfolio the goal is to optimize the risk-return tradeoff, therefore improve the portfolio Sharpe ratio. This could be done by buying on margin a stock *i.* The stock will improve the portfolio performance if the additional return earned from investment i is greater than the additional return earned from investing more funds in the current portfolio;

$$E[R_i] - r_f > SD(R_i) \times Corr(R_i, R_p) \times \frac{E(R_p) - r_f}{SD(R_p)}$$

Alternatively,

$$\frac{E(R_i) - r_f}{SD(R_i)} > Corr(R_i, R_p) \times \frac{E(R_p) - r_f}{SD(R_p)}$$

Beta of stock i can be defined as:

$$\beta_i = \frac{Cov(R_i, R_p)}{Var(R_p)} = \frac{\sigma_i * \rho_{i,p}}{\sigma_p}$$

Therefore:

$$E[R_i] > r_f + \beta_i \times (E[R_p] - r_f)$$

The portfolio Sharpe ratio improves when the expected return of investment i is greater than its required return.

$$r_i \equiv r_f + \beta_i \times (E[R_p] - r_f)$$

As the quantity of stock i in our portfolio increases, also does its correlation (and beta) with the portfolio. If stock *i* expected return is greater than its required return it improves the portfolio performance, therefore will be purchased and added to the stock until $E[R_i] = r_i$. Conversely, if its required return is lower it will be sold until $E[R_i] = r_i$.The following evaluation should be applied to each stock individually in order to obtain the portfolio with the highest Sharpe ratio. Following such reasoning, the portfolio that maximizes the Sharpe ratio is the efficient portfolio or tangent portfolio.

1.7 The Capital Asset Pricing Model

The Capital Asset Pricing Model (CAPM) relies on the assumptions introduced by Harry Markowitz. It was developed by William F. Sharpe[9], John Lintner[10] and Jan Mossin[11], and then further elaborated by Fischer Black[12].

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[9] Sharpe, William F. 1964. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." Journal of Finance. 19:3, pp. 425– 42.

[10] Lintner, John. 1965. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." Review of Economics and Statistics. Vol. 47, No. 1, pp. 13–37.

[11] Mossin, Jan. 1966. "Equilibrium in a Capital Asset Market", *Econometrica*, 34, 1966, pp. 768–783

[12] Fischer Black, Myron Scholes, & Michael Jensen, "The Capital-Asset Pricing Model: Some empirical tests", in Jensen, editor, Studies in the Theory of Capital Markets (1972)

The CAPM is widely adopted in the financial sector to evaluate the required rate of return of an asset. It is used in the decisional process by investors and can be applied to individual securities and portfolios to measure the risk-return trade-off.
Although, the empirical support to the CAPM is not much, it offers a straightforward way to predict expected returns. Following from the notions about the efficient portfolio, the CAPM allows an investor to predict mathematically the relation between risk and expected return, and therefore identify the efficient portfolio without knowing expected returns.

It builds on three main assumptions:

1. Complete agreement.
   Investors agree that asset returns over periods $t-1$ and t follow a normal distribution and investors can borrow and lend at the risk-free rate which does not depend on the amount.
2. Investors are risk-averse and hold efficient portfolios.
   Investors do not undertake risk willingly and take decisions on expected return and variance, therefore they will choose portfolios that maximize their expected return for a given level of risk. The wealth allocation of an investor depends individually on his aversion to risk.
   As a result of this assumption, the market portfolio is a portfolio located on the efficient frontier, the latter consisting of the market portfolio and the risk-free asset.
3. Homogeneity of expectations.
   Investors build expectations on historical data and information publicly and equally accessible. If investors have access to the same sources and are rational then it is possible to assume that, sharing common beliefs about the joint probability distributions of future returns, investing strategies will be similar.
   This assumption leads to the CAPM equilibrium which is obtained when all investors hold the tangent portfolio, therefore such portfolios as a whole will correspond to the market portfolio.

The analytical expression of the CAPM is:

$$E(R_i) = r_f + \beta_i(E(R_m) - r_f)$$

Where:

I. $E(R_i)$ is the expected return on asset *i*
II. r_f is the interest arising from the risk-free asset. No asset is identified to be free of risk, however the different government securities can be used as proxy depending on the duration of security *i.*
III. β_i measures in percentage the change in security *i* return after a change in the return in the market benchmark. It is obtained as: $\beta_i = \frac{Cov(R_i,R_m)}{Var(R_m)} = \frac{\sigma_i * \rho_{i,m}}{\sigma_m}$
   It measures the systematic risk (or market risk) characterizing asset *i.* Furthermore, it identifies the amount of risk we will add to a portfolio that cannot be diversified away.
IV. $E(R_m)$ is the expected market return which can be determined selecting a market benchmark which can be identified through historical returns of a market portfolio, for instance the market index S&P500.

V. $E(R_m) - r_f$ is the market premium; the difference between the market expected return and the risk-free asset. It identifies the excess return provided by the market.

VI. $\beta_i(E(R_m) - r_f)$ is the risk premium of asset *i.*

Alternatively, the CAPM equation can be rearranged:

$$E(R_i) - r_f = \beta_i(E(R_m) - r_f)$$

The risk premium on security *i,* $E(R_i) - r_f$, is equal to the market risk premium multiplied by beta, $\beta_i(E(R_m) - r_f)$.

A third way of expressing the CAPM analytically starts by considering beta:

$$\beta_i = \frac{\sigma_i * \rho_{i,m}}{\sigma_m}$$

Then, plugged into the CAPM equation and rearranged:

$$\frac{(E(R_i) - r_f)}{\sigma_i} = \rho_{i,m} \frac{(E(R_m) - r_f)}{\sigma_m}$$

Sharpe ratio of asset *i* = $Corr\left(R_i, R_j\right)$ × Sharpe ratio of the Market portfolio.
With $\rho_{i,m} \leq 1$ the Sharpe ratio of asset *i* will at its best be equal to the Sharpe ratio of the market portfolio, and assets having the same correlation with the market portfolio will thus have the same Sharpe ratio[13].

The following is a proof to the CAPM equation.
Consider investing in portfolio *p* consisting of two sub-portfolios, one of asset *i* and the other of the market portfolio *m.* Weights are respectively (α, 1 − α) where $\sum$ α = 1 and α ∈ [0, 1], meaning we cannot short-sell. Asset *i* is not efficient; lies on the feasible region (σ, r) but not on the efficient frontier. As α changes, *p* expected return and standard deviation vary in the feasible region $\left(\sigma(\alpha), \tilde{R}(\alpha)\right)$ parametrized by α.
Portfolio *p* expected return and variance are:

$$\widetilde{R_p} = \alpha\tilde{R}_i + (1 - \alpha)\tilde{R}_m$$

$$= \alpha\left(\tilde{R}_i - \tilde{R}_m\right) + \tilde{R}_m$$

$$\sigma_p = \sqrt{\alpha^2\sigma_i^2 + (1 - \alpha)^2\sigma_m^2 + 2\alpha(1 - \alpha)\sigma_{i,m}}$$

$$= \sqrt{\alpha^2\left(\sigma_i^2 + \sigma_m^2 - 2\sigma_{i,m}\right) + 2\alpha\left(2\sigma_{i,m} - \sigma_m^2\right) + \sigma_m^2}$$

[13] Perold, André F. "The Capital Asset Pricing Model", Journal of Economic Perspectives—Volume 18, Number 3—Summer 2004—Pages 3–24

For α = 0, $\left(\sigma(0), \tilde{R}(0)\right)$ = $(\sigma_m, \widetilde{R_m})$ while for α = 1, $\left(\sigma(0), \tilde{R}(0)\right)$ = $(\sigma_i, \widetilde{R_i})$.
As depicted in figure 1.8, the tangency between the curve of the efficient frontier of risky assets and the CML happens therefore at point $(\sigma_m, \widetilde{R_m})$ for α=0 and the curve's derivative

$$\frac{d\widetilde{R_p}}{d\sigma_{p_{\alpha=0}}}$$

is equal to the slope of the of the capital asset line at point *m,* where the slope of the line is given by

$$\frac{\tilde{R}_m - r_f}{\sigma_m}$$

Therefore, for α = 0

$$\frac{d\widetilde{R_p}}{d\sigma_p} = \frac{\tilde{R}_m - r_f}{\sigma_m}$$

Following calculus chain rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ we get

$$\frac{d\widetilde{R_p}}{d\sigma_p} = \frac{d\widetilde{R_p}/d\alpha}{d\sigma_p/d\alpha}$$

The derivative of portfolio *p* expected return $\widetilde{R_p}$ and standard deviation σ_p for α = 0 yields

$$\frac{d\widetilde{R_p}/d\alpha}{d\sigma_p/d\alpha} = \frac{\tilde{R}_i - \tilde{R}_m}{\left(\sigma_{i,m} - \sigma_m^2\right)/\sigma_m}$$

Keeping in mind that for α = 0, $\frac{d\widetilde{R_p}}{d\sigma_p} = \frac{\tilde{R}_m - r_f}{\sigma_m}$ then

$$\frac{\tilde{R}_i - \tilde{R}_m}{\left(\sigma_{i,m} - \sigma_m^2\right)/\sigma_m} = \frac{\tilde{R}_m - r_f}{\sigma_m}$$

Solving for $\tilde{R}_i$ we obtain the CAPM equation for asset *i,* $\tilde{R}_i - r_f = \beta_i(\tilde{R}_m - r_f)$.

In general, for a portfolio

$$\begin{aligned}\tilde{R}_p - r_f &= -r_f + \sum_{i=1}^{n} \alpha_i \tilde{R}_i \\ &= \sum_{i=1}^{n} \alpha_i\left(\tilde{R}_i - r_f\right) \\ &= \sum_{i=1}^{n} \alpha_i\, \beta_i\left(\tilde{R}_m - r_f\right) \;\; (CAPM\ equation\ asset\ i)\end{aligned}$$

$$= (\tilde{R}_m - r_f) \sum_{i=1}^{n} \alpha_i \beta_i$$

For $\beta_p = 1$, portfolio *p* expected rate of return will be equal to the market portfolio return; $\tilde{R}_p = \tilde{R}_m$. For $\beta_p > 1$, $\tilde{R}_p > \tilde{R}_m$ and for $\beta_p < 1$, $\tilde{R}_p < \tilde{R}_m$.
An asset *i* negatively correlated will have a negative beta, $\beta_i < 0$, and its expected rate of return will be lower than the risk-free rate, $\tilde{R}_i < r_f$. Assets negatively correlated are included in well-diversified portfolios as insurances.[14]

An alternative proof to the CAPM is obtained implementing the constrained maximization method.[15]
We construct a portfolio invested in asset *i*, in the market portfolio *m*, and in the risk-free asset r_f. The weights are respectively α_i, α_m, and $(1 - \alpha_i - \alpha_m)$; we are short-selling the risk-free asset. The constrained maximization problem to solve is:

$$\max\ \alpha_i \tilde{R}_i + \alpha_m \tilde{R}_m + (1 - \alpha_i - \alpha_m) r_f$$

while:

$$\alpha_i^2 \sigma_i^2 + \alpha_m^2 \sigma_m^2 + 2\alpha_i \alpha_m \sigma_{i,m} = \sigma_m^2$$

We introduce the Lagrangian function

$$\mathcal{L} = \alpha_i \tilde{R}_i + \alpha_m \tilde{R}_m + (1 - \alpha_i - \alpha_m) r_f + \lambda(\sigma_m^2 - \alpha_i^2 \sigma_i^2 - \alpha_m^2 \sigma_m^2 - 2\alpha_i \alpha_m \sigma_{i,m})$$

Differentiating with respect to *i* and *m* and setting the derivatives to zero, we obtain:

$$\mathcal{L}_{\alpha_i} = \tilde{R}_i - r_f - \lambda(2\alpha_i \sigma_i^2 + 2\alpha_m \sigma_{i,m}) = 0$$

$$\mathcal{L}_{\alpha_m} = \tilde{R}_m - r_f - \lambda(2\alpha_m \sigma_m^2 + 2\alpha_i \sigma_{i,m}) = 0$$

The market portfolio is efficient, therefore the solution to problem is obtained with $\alpha_m = 1$ and $\alpha_i = 0$. This is true as no other portfolio, in equilibrium, offers the best risk-return trade-off. In addition, notice that $\mathcal{L}_{\alpha_i}$ then implies:

$$\tilde{R}_i - r_f - \lambda(2\alpha_i \sigma_i^2 + 2\alpha_m \sigma_{i,m}) = 0$$

$$2\lambda = \frac{\tilde{R}_i - r_f}{\sigma_{i,m}}$$

which plugged into $\mathcal{L}_{\alpha_m}$ provides the CAPM equation

$$\tilde{R}_i - r_f = \frac{\sigma_{i,m}}{\sigma_m^2}(\tilde{R}_m - r_f)$$

[14] Sigman, Karl. "1 Capital Asset Pricing Model (CAPM)", 2005, Columbia University, pp. 1-6
[15] Quintin, Erwan. "CAPM: a formal proof", Wisconsin School of Business at UW Madison

The graphical representation on the xy-plane of the CAPM equation

$$E(R_i) = r_f + \beta_i(E(R_m) - r_f)$$

is defined as the Security Market Line (SML). The x-axis corresponds to the beta of the individual stocks and the y-axis to the individual expected returns.

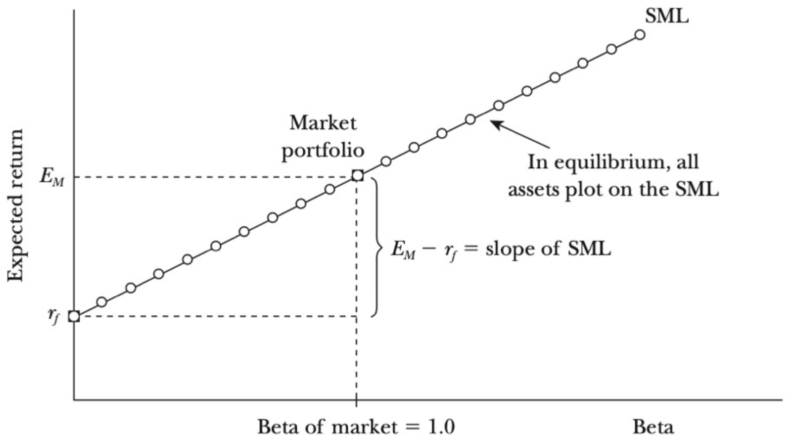

Figure 1.9 The Security Market Line

Figure 1.9 represents the linear relationship between the beta and the expected return of a stock. The y-axis intercept corresponds to the risk-free asset which has a beta equal to zero and the slope corresponds to the market premium $E(R_m) - r_f$ which is proportional to beta β. The market portfolio has beta equal to 1 as it represents the market risk. In equilibrium the market portfolio is efficient; all assets required return will be equal to the expected return and they will lie on the SML. Assets lying above the SML will offer greater expected returns than those predicted by the CAPM, thus are regarded as underpriced. In equilibrium, the increasing demand for such assets will bring prices up and their expected returns will lower until lying on the SML. Conversely, assets lying beneath the SML will offer lower returns than those predicted by the CAPM, thus are regarded as overpriced. In equilibrium, the decreasing demand will bring prices down and their expected returns will increase until lying on the SML.

Chapter 2

Portfolio Selection and Empirical Results

Chapter 2 lays out numerical examples and empirical results to the theory covered in chapter 1.

The analysis is carried out on a general portfolio, and the risk and return measures implemented are those covered throughout chapter 1.
The portfolio is constructed through a random selection of stocks among the eleven distinct market sectors enlisted in the Global Industry Classification Standard (GICS), narrowed down by S&P Dow Jones and MSCI[16]. The sectors are: information technology, health care, financials, consumer discretionary, communication services, industrials, consumer staples, energy, utilities, real estate, and materials.
Each stock contained in the portfolio is randomly selected and represents one of the eleven different sectors. Investing in different sector provides an immediate degree of diversification, essential to mitigate the idiosyncratic risk of companies within a specific market segment.

The analysis and comparison of the portfolios are carried out using four different methods:

1. Equally Weighted Portfolio
2. Minimum Variance Portfolio
3. Portfolio Optimization
4. Capital Asset Pricing Model

The technical data is extrapolated from the Yahoo! Finance website and the selected time horizon is four years, starting 1/January/2016 until 1/January/2020. The horizon is selected to avoid the effects of the global financial crisis and the latest COVID-19 global pandemic. Such a decision provides reliable estimates of returns and volatility.
Returns and volatility are computed using historical data with monthly frequency.

The technical tool implemented in the analysis is Excel and its built-in functions for data analysis.

2.1 Stocks Selection and Portfolio Size

The stocks used for the analysis are randomly selected among those with the greatest market capitalization per sector, assuming all stocks are equally valuable.

The first issue raised during the selection is the number of stocks to include in the portfolio in order to benefit from diversification. A general premise is that as the number of stocks in a portfolio increases, the total volatility decreases. However, investors and professionals do not agree on a precise number of stocks.
Investors have to take into consideration many factors that may influence the size of their portfolio, among these the most crucials being: wealth, transaction costs, accessibility,

[16] The Global Industry Classification Standard (GICS) - https://www.msci.com/gics

skill, time horizon, and time allocation. Keeping track of a portfolio requires time, especially if the number of stocks it contains is significant. As a result, professional investors stand better chances of constructing and managing an extensive portfolio, whereas retail investors have to dedicate more time and effort.

A first answer to the question regarding "how many stocks make a diversified portfolio?" was provided by John L. Evans and Stephen H. Archer. Their study asserted that a portfolio of 10 stocks is sufficient to capture the stock market return.[17]
Later on, Meir Statman added that *"a well-diversified portfolio of randomly chosen stocks must include at least 30 stocks for a borrowing investor and 40 stocks for a lending investor".*[18]
Lawrence Fisher and James H. Lorie, in their study "Some Studies of Variability of Returns on Investments in Common Stocks", published in 1970 in The Journal Of Business, came up with an approximate ideal number of stocks. They concluded that a portfolio of 32 randomly selected stocks grants almost entirely the benefits of diversification.[19]
In 2001 a study was advancing that the older studies were no more reliable and that at least a portfolio of 50 stocks was necessary.[20] Following the latest reasoning, in 2007, a research posed that even 100 stocks are not sufficient.[21]

No consensus has been reached about the size of a well-diversified portfolio; however, investors in the US generally assume that the appropriate range is between 20 and 30 stocks. For simplicity, the analysis is conducted on a portfolio of 11 stocks traded on the NYSE and NASDAQ. The stocks are randomly selected and represent a different sector individually amid the eleven GICS sectors[22]:

I. Microsoft (Information Technology)
II. Johnson & Johnson (Health care)
III. JP Morgan Chase and Co (Financials)
IV. Amazon (Consumer Discretionary)
V. Facebook (Communication Services)
VI. American Airlines (Industrials)
VII. Procter & Gamble (Consumer Staples)
VIII. Chevron Corp (Energy)
IX. American Water Works (Utilities)
X. American Tower Corp (Real Estate)
XI. Cemex SAB de CV ADR (Materials)

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[17] John L. Evans and Stephen H. Archer. "Diversification and the Reduction of Dispersion: An Empirical Analysis". The Journal of Finance, Vol. 23, No. 5, Dec. 1968, pp. 761-767
[18] Meir Statman. "How Many Stocks Make a Diversified Portfolio?". The Journal of Financial and Quantitative Analysis, Vol. 22, No. 5, Dec. 1968, pp. 353-363
[19] Lawrence Fisher and James H. Lorie. "Some Studies of Variability of Returns on Investments In Common Stocks". The Journal of Business, Vol. 43, No. 2, Apr. 1970, pp. 99-134
[20] John Y. Campbell, Martin Lettau, Burton G. Malkiel, and Yexiao Xu. "Have Individual Stocks Become More Volatile? An Empirical Exploration of Idiosyncratic Risk". The Journal of Financial, Vol. 56, No. 1, Feb. 2001, pp. 1-43
[21] Domian, Dale L. and Louton, David A. and Racine, Marie D. "Diversification in Portfolios of Individual Stocks: 100 Stocks are Not Enough". The Financial Review, Vol. 42, No. 4, Nov. 2007, pp. 577-570
[22] Spglobal.com, S&P Dow Jones Indices, Research & Insights, Investment Themes – Sectors https://www.spglobal.com/spdji/en/landing/investment-themes/sectors/

Microsoft Corp	MSFT
Johnson & Johnson	JNJ
JPMorgan	JPM
Amazon	AMZN
Facebook	FB
American Airlines	AAL
Procter & Gamble	PG
Chevron Corp	CVX
America Water Works	AWK
American Tower Corp	AMT
Cemex SAB de CV	CX

Table 1.1 Stocks and their Tickers

Data relative to the stocks is extrapolated from Yahoo! finance website starting 1/January/2016 until 1/January/2020. Historical returns are computed with monthly frequency on the adjusted closing price to consider possible corporate actions that may affect prices, such as dividends, stock splits, or rights offerings.

Date	MSFT	JNJ	JPM	AMZN	FB	AAL	PG	CVX	AWK	AMT	CX
01/01/16	-7,64%	0,74%	-4,74%	-5,87%	-4,71%	5,16%	-0,86%	-3,50%	-0,14%	-2,27%	22,30%
01/02/16	9,33%	3,59%	5,19%	7,44%	6,72%	0,30%	2,52%	15,78%	6,89%	11,03%	31,41%
01/03/16	-9,70%	3,59%	6,72%	11,11%	3,05%	-15,41%	-2,66%	7,11%	5,56%	2,45%	6,43%
01/04/16	6,28%	0,54%	4,04%	9,58%	1,05%	-8,01%	1,98%	-1,15%	1,84%	1,35%	-14,63%
01/05/16	-2,78%	8,41%	-4,80%	-0,99%	-3,81%	-11,03%	4,48%	4,89%	14,63%	7,40%	-2,99%
01/06/16	10,77%	3,24%	2,94%	6,04%	8,45%	25,40%	1,09%	-2,24%	-2,28%	2,41%	23,99%
01/07/16	1,38%	-4,70%	6,34%	1,36%	1,76%	2,25%	2,81%	-1,85%	-10,40%	-2,06%	8,37%
01/08/16	0,87%	-0,35%	-1,35%	8,86%	1,70%	1,16%	2,79%	3,40%	1,62%	-0,04%	-4,22%
01/09/16	4,03%	-1,81%	4,01%	-5,67%	2,12%	10,90%	-3,29%	1,78%	-1,07%	3,91%	9,32%
01/10/16	0,57%	-4,04%	16,59%	-4,97%	-9,60%	14,38%	-4,27%	6,50%	-2,12%	-12,73%	-10,02%
01/11/16	3,82%	4,23%	7,63%	-0,09%	-2,85%	0,79%	1,96%	6,56%	0,38%	3,33%	2,82%
01/12/16	4,04%	-1,70%	-1,92%	9,82%	13,27%	-5,23%	4,19%	-5,40%	1,49%	-1,53%	15,32%
01/01/17	-1,04%	7,91%	7,67%	2,62%	4,01%	4,77%	4,79%	1,03%	6,21%	10,91%	-8,64%
01/02/17	3,56%	2,59%	-3,07%	4,91%	4,80%	-8,55%	-1,34%	-3,65%	0,22%	5,88%	7,21%
01/03/17	3,95%	-0,87%	-0,96%	4,34%	5,77%	0,76%	-2,80%	-0,62%	2,56%	3,62%	1,65%
01/04/17	2,02%	3,87%	-5,03%	7,53%	0,81%	13,59%	1,64%	-3,02%	-1,98%	4,70%	-6,72%
01/05/17	-0,74%	3,83%	11,26%	-2,68%	-0,32%	4,17%	-1,07%	1,85%	0,23%	0,86%	13,91%
01/06/17	5,47%	0,33%	0,44%	2,04%	12,10%	0,24%	4,21%	4,66%	4,04%	3,54%	3,08%
01/07/17	2,85%	-0,26%	-0,45%	-0,73%	1,61%	-11,30%	2,39%	-1,44%	-0,25%	8,60%	-4,12%
01/08/17	0,16%	-1,16%	5,08%	-1,96%	-0,64%	6,36%	-1,40%	10,28%	0,53%	-7,68%	-2,47%
01/09/17	11,67%	7,23%	5,34%	14,97%	5,38%	-1,41%	-5,10%	-1,37%	8,47%	5,63%	-10,68%
01/10/17	1,19%	-0,06%	4,49%	6,47%	-1,60%	7,84%	5,01%	2,67%	4,33%	0,18%	-6,41%
01/11/17	2,14%	0,89%	2,32%	-0,62%	-0,41%	3,28%	2,10%	6,20%	0,39%	-0,88%	-1,19%
01/12/17	11,07%	-1,10%	8,16%	24,06%	5,91%	4,40%	-6,03%	0,13%	-9,09%	4,04%	10,53%
01/01/18	-1,31%	-6,01%	0,37%	4,24%	-4,59%	-0,13%	-8,36%	-10,71%	-4,58%	-5,67%	-20,99%
01/02/18	-2,21%	-0,70%	-4,79%	-4,30%	-10,39%	-4,04%	0,97%	2,91%	4,04%	4,31%	1,07%

01/03/18	2,47%	-1,30%	-1,08%	8,21%	7,64%	-17,38%	-8,75%	9,71%	5,42%	-6,18%	-6,19%
01/04/18	5,69%	-5,43%	-1,13%	4,05%	11,50%	1,42%	2,08%	-0,65%	-3,97%	2,01%	-4,03%
01/05/18	0,20%	2,19%	-2,63%	4,31%	1,32%	-12,61%	6,68%	2,60%	3,26%	4,19%	10,07%
01/06/18	7,58%	9,21%	10,32%	4,57%	-11,19%	4,16%	3,61%	-0,13%	3,36%	3,40%	13,57%
01/07/18	5,89%	1,64%	0,22%	13,24%	1,83%	2,38%	3,49%	-6,19%	-0,82%	0,59%	-4,83%
01/08/18	2,21%	3,27%	-1,52%	-0,48%	-6,41%	2,37%	0,34%	4,21%	1,02%	-2,56%	-0,71%
01/09/18	-6,61%	1,32%	-3,39%	-20,22%	-7,70%	-15,12%	6,55%	-8,69%	0,64%	7,83%	-28,41%
01/10/18	3,82%	4,94%	2,70%	5,77%	-7,37%	14,48%	7,52%	6,53%	7,77%	5,57%	1,98%
01/11/18	-8,01%	-11,59%	-12,20%	-11,13%	-6,77%	-19,82%	-2,74%	-7,64%	-4,38%	-3,83%	-6,23%
01/12/18	2,82%	3,12%	6,02%	14,43%	27,16%	11,40%	4,95%	5,39%	5,40%	9,86%	12,86%
01/01/19	7,28%	2,68%	1,65%	-4,59%	-3,14%	-0,39%	2,96%	4,30%	6,22%	1,92%	-10,85%
01/02/19	5,72%	2,98%	-3,00%	8,59%	3,25%	-10,62%	5,58%	4,05%	3,09%	11,87%	-4,33%
01/03/19	10,73%	1,01%	14,64%	8,19%	16,02%	7,62%	2,34%	-2,53%	3,77%	-0,89%	-0,86%
01/04/19	-5,30%	-7,12%	-8,00%	-7,86%	-8,24%	-20,33%	-2,67%	-5,17%	4,46%	7,39%	-10,43%
01/05/19	8,71%	6,93%	5,51%	6,68%	8,75%	20,10%	6,55%	10,38%	3,12%	-2,07%	2,91%
01/06/19	1,72%	-6,50%	3,76%	-1,42%	0,64%	-6,44%	7,65%	-1,07%	-1,05%	3,95%	-16,04%
01/07/19	1,17%	-1,43%	-4,62%	-4,85%	-4,41%	-13,77%	2,51%	-4,38%	10,93%	8,78%	5,34%
01/08/19	1,18%	1,55%	7,13%	-2,27%	-4,09%	2,87%	3,45%	1,78%	-2,01%	-3,94%	4,53%
01/09/19	3,12%	2,06%	6,14%	2,35%	7,62%	11,46%	0,10%	-2,07%	-0,77%	-0,96%	-3,83%
01/10/19	5,59%	4,13%	6,32%	1,36%	5,21%	-4,39%	-1,34%	0,85%	-1,82%	-1,86%	-1,06%
01/11/19	4,53%	6,83%	5,80%	2,61%	1,79%	0,11%	2,33%	3,90%	1,94%	7,38%	1,34%

Table 1.2 Historical Returns

Historical returns are then used to compute their average, variance, and standard deviation on a monthly and then annual basis:

MONTHLY											
Average	2,56%	1,01%	2,22%	2,75%	1,55%	-0,04%	1,25%	1,19%	1,64%	2,21%	0,41%
Variance	0,00239657	0,00180208	0,00327341	0,00555797	0,00544869	0,01034627	0,00152728	0,00280416	0,00206790	0,00265828	0,01255357
Std dev	0,04948404	0,04290987	0,05783228	0,07535775	0,07461329	0,10281629	0,03950288	0,05352678	0,04596584	0,05211591	0,11325403
ANNUAL											
Average	30,70%	12,17%	26,59%	32,94%	18,64%	-0,48%	15,05%	14,29%	19,68%	26,49%	4,88%
Variance	0,02875885	0,02162498	0,03928094	0,06669559	0,06538431	0,12415525	0,01832731	0,03364987	0,02481485	0,03189936	0,15064285
Std dev	0,17141774	0,14864415	0,20033691	0,26104692	0,25846801	0,35616608	0,13684198	0,18542220	0,15923035	0,18053482	0,39232345

Table 1.3 Monthly and Annual Securities Statistics

2.2 Method 1: Equally Weighted Portfolio

With the first method, a portfolio with equal allocation per stock is constructed.

Ideally, the portfolio obtained with equal allocation will provide a certain degree of return for a lower level of volatility, or standard deviation, compared to the individual stocks.

The monthly returns on the equally weighted portfolio of the 11 stocks selected are:

Dates	Returns
01/01/16	-0,14%
01/02/16	9,11%
01/03/16	1,66%
01/04/16	0,26%
01/05/16	1,22%
01/06/16	7,25%
01/07/16	0,48%
01/08/16	1,31%
01/09/16	2,20%
01/10/16	-0,88%
01/11/16	2,60%
01/12/16	2,94%
01/01/17	3,66%
01/02/17	1,14%
01/03/17	1,58%
01/04/17	1,58%
01/05/17	2,85%
01/06/17	3,65%
01/07/17	-0,28%
01/08/17	0,65%
01/09/17	3,65%
01/10/17	2,19%
01/11/17	1,29%
01/12/17	4,74%
01/01/18	-5,25%
01/02/18	-1,19%
01/03/18	-0,68%
01/04/18	1,05%
01/05/18	1,78%
01/06/18	4,41%
01/07/18	1,59%
01/08/18	0,16%
01/09/18	-6,71%
01/10/18	4,88%
01/11/18	-8,58%

01/12/18	9,40%
01/01/19	0,73%
01/02/19	2,47%
01/03/19	5,46%
01/04/19	-5,75%
01/05/19	7,05%
01/06/19	-1,35%
01/07/19	-0,43%
01/08/19	0,93%
01/09/19	2,29%
01/10/19	1,18%
01/11/19	3,50%
01/12/19	

Table 1.4 Equally Weighted Portfolio Monthly Returns

The monthly and annual statistics are:

MONTHLY	
Average	1,52%
Variance	0,00120209
Std dev	0,03504593
ANNUAL	
Average	18,29%
Variance	0,01442502
Std dev	0,12140268

Table 1.5 Equally Weighted Portfolio Monthly and Annual Statistics

A comparison between the individual stocks and the equally weighted portfolio monthly and annual statistics provides insight into the results.
In this basic scenario, the statistics obtained on the portfolio are simply an equally weighted average of the individual statistics. Nonetheless, by just investing the same wealth into these 11 stocks, we can reduce diversifiable or idiosyncratic risk to a level lower than any individual stock on a monthly and annual basis. The portfolio's monthly and annual standard deviation (approximately 3,50% and 12,14%) are less than any stock considered individually.

The covariance and correlation statistical indicators explain the degree of co-movement of the stocks in the portfolio.

	MSFT	JNJ	JPM	AMZN	FB	AAL	PG	CVX	AWK	AMT	CX
MSFT	0,00239657	0,00068678	0,00118781	0,00212221	0,00168601	0,0022936	0,00015459	0,00052658	-5,453E-05	0,00028793	0,00124512
JNJ	0,00068678	0,00180208	0,00080222	0,00093301	0,00034834	0,00130971	0,00054085	0,00087003	0,00095731	0,00082408	0,00121164
JPM	0,00118781	0,00080222	0,00327341	0,00126962	0,00092578	0,00311346	-6,158E-05	0,00107901	-0,00032	-0,0006809	0,00099531
AMZN	0,00212221	0,00093301	0,00126962	0,00555797	0,00322625	0,0019651	-0,0002467	0,00076147	3,3446E-05	0,00040802	0,00225058
FB	0,00168601	0,00034834	0,00092578	0,00322625	0,00544869	0,00175717	0,00021152	0,00054179	1,367E-06	0,00060361	0,00242213
AAL	0,0022936	0,00130971	0,00311346	0,0019651	0,00175717	0,01034627	0,0004795	0,00120965	-0,0009953	-0,0011849	0,00350232
PG	0,00015459	0,00054085	-6,158E-05	-0,0002467	0,00021152	0,0004795	0,00152728	0,00025402	0,00048449	0,00084883	0,0002294
CVX	0,00052658	0,00087003	0,00107901	0,00076147	0,00054179	0,00120965	0,00025402	0,00280416	0,00086964	-2,503E-05	0,00186688
AWK	-5,453E-05	0,00095731	-0,00032	3,3446E-05	1,367E-06	-0,0009953	0,00048449	0,00086964	0,0020679	0,00104021	2,2633E-05
AMT	0,00028793	0,00082408	-0,0006809	0,00040802	0,00060361	-0,0011849	0,00084883	-2,503E-05	0,00104021	0,00265828	0,00071603
CX	0,00124512	0,00121164	0,00099531	0,00225058	0,00242213	0,00350232	0,0002294	0,00186688	2,2633E-05	0,00071603	0,01255357

Table 1.6 Covariance Matrix

	MSFT	JNJ	JPM	AMZN	FB	AAL	PG	CVX	AWK	AMT	CX
MSFT	1	0,33047154	0,42408437	0,58147934	0,46657124	0,46060619	0,08080427	0,20312741	-0,0244951	0,11407589	0,22700344
JNJ	0,33047154	1	0,33029987	0,2948081	0,11116591	0,30331725	0,32600752	0,38703083	0,49590805	0,37651297	0,25474366
JPM	0,42408437	0,33029987	1	0,29765652	0,21921076	0,53499593	-0,0275432	0,3561442	-0,1229778	-0,2308396	0,15526559
AMZN	0,58147934	0,2948081	0,29765652	1	0,58626523	0,25914074	-0,0846847	0,19288335	0,00986546	0,1061517	0,26943462
FB	0,46657124	0,11116591	0,21921076	0,58626523	1	0,23403171	0,07332433	0,13860565	0,00040723	0,15860213	0,29286474
AAL	0,46060619	0,30331725	0,53499593	0,25914074	0,23403171	1	0,1206249	0,2245773	-0,2151882	-0,2259327	0,30731281
PG	0,08080427	0,32600752	-0,0275432	-0,0846847	0,07332433	0,1206249	1	0,12274772	0,27262332	0,42127126	0,05239134
CVX	0,20312741	0,38703083	0,3561442	0,19288335	0,13860565	0,2245773	0,12274772	1	0,36113598	-0,009167	0,31465266
AWK	-0,0244951	0,49590805	-0,1229778	0,00986546	0,00040723	-0,2151882	0,27262332	0,36113598	1	0,44366531	0,00444209
AMT	0,11407589	0,37651297	-0,2308396	0,1061517	0,15860213	-0,2259327	0,42127126	-0,009167	0,44366531	1	0,12395098
CX	0,22700344	0,25474366	0,15526559	0,26943462	0,29286474	0,30731281	0,05239134	0,31465266	0,00444209	0,12395098	1

Table 1.7 Correlation Matrix

Covariance and correlation matrix are functional and easy-to-interpret means to understand the degree of stocks co-movement of returns. They are symmetric matrices with *n* entries corresponding to the number of variables and their transpose.
The diagonal of the correlation matrix will have values equal to 1 as each variable comoves with itself.
Most stocks exhibit a weak correlation; this is possibly explained by the fact that they refer to different market sectors. Few stocks exhibit moderate correlation: MSFT and AMAZN and FB, AAL and JPM.

Range	Correlation Strength
0	No correlation
0 to ±0.25	Negligible correlation
±0.25 to ±0.50	Weak Correlation
±0.50 to ±0.75	Moderate correlation
±0.75 to ±1	Very strong correlation
±1	Perfect correlation

Table 1.8 Range and Strength of Correlation

Annual volatility and returns are plotted on xy-space with the y-axis representing the return and the x-axis the standard deviation:

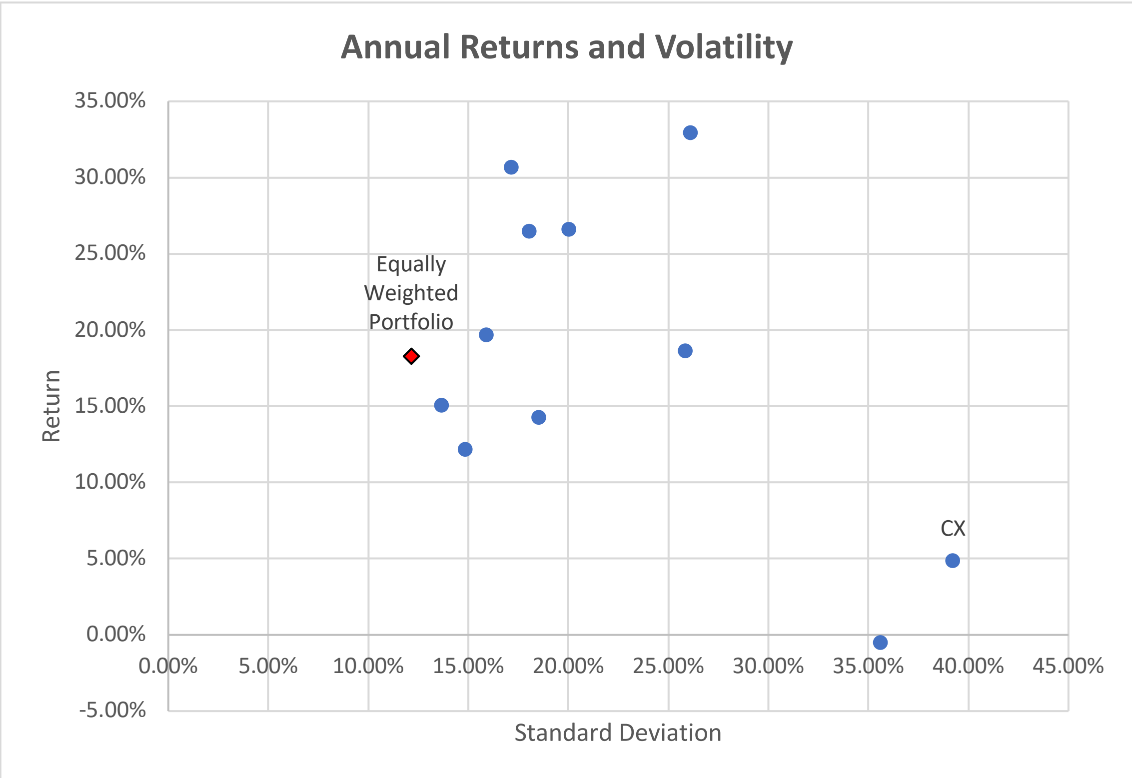

Figure 2 Annual Returns and Volatility

2.3 Method 2: Minimum Variance Portfolio

With method number 2, the objective is to construct a portfolio with the eleven stocks in question that minimizes the level of risk. The Minimum Variance Portfolio (MVP) is a portfolio that diversifies away all the degree of idiosyncratic risk.
The MVP obtained includes the option to short sell securities.
Statistics are computed with historical adjusted returns. Excel Solver is then implemented to find feasible solutions by trial and error.

The first step consists in building a variance-covariance matrix of the stocks' historical returns (refer to *Table 1.6*). Then, individual annual average returns are computed:

MSFT	0,30697112
JNJ	0,12431952
JPM	0,26592271
AMZN	0,32941737
FB	0,18640343
AAL	-0,00481403
PG	0,15049019
CVX	0,14288596
AWK	0,19676582
AMT	0,26490097
CX	0,04883802

Table 1.9 Annual Individual Returns

Weights of the portfolio are then assigned randomly to the stocks' returns to minimize volatility. For ease of purpose, the weights initially assigned are that of the equally weighted portfolio constructed in section 2.2.

MSFT	0,09091	0,30697112
JNJ	0,09091	0,12431952
JPM	0,09091	0,26592271
AMZN	0,09091	0,32941737
FB	0,09091	0,18640343
AAL	0,09091	-0,00481403
PG	0,09091	0,15049019
CVX	0,09091	0,14288596
AWK	0,09091	0,19676582
AMT	0,09091	0,26490097
CX	0,09091	0,04883802

Table 2 Individual Weights and Annual Returns

Where the annual statistics are:

ANNUAL	
Average	18,29%
Variance	0,01442502
Std dev	0,12140268

Table 2.1 Equally Weighted Portfolio Annual Statistics

The goal is to minimize the current variance level. First, the minimization problem is resolved by implementing Excel Solver and assigning a target level of average annual return. Then, running the software, we obtain combinations of weights that provide the targeted return. A necessary condition is that the sum of the weights is always equal to one.

The following weights are obtained taking into consideration the possibility to short sell the stocks:

MSFT	-0,23490	-0,18087	-0,09082	-0,00077	0,08928	0,13430	0,17933	0,26938	0,35943	0,44948	0,53952
JNJ	0,56987	0,47509	0,31714	0,15919	0,00124	-0,07774	-0,15671	-0,31467	-0,47262	-0,63057	-0,78852
JPM	-0,15097	-0,09950	-0,01370	0,07209	0,15789	0,20078	0,24368	0,32947	0,41527	0,50106	0,58686
AMZN	-0,14896	-0,12264	-0,07879	-0,03493	0,00893	0,03086	0,05279	0,09664	0,14050	0,18436	0,22821
FB	0,24007	0,20550	0,14789	0,09028	0,03267	0,00386	-0,02495	-0,08256	-0,14017	-0,19778	-0,25539
AAL	0,13933	0,11733	0,08067	0,04400	0,00734	-0,01099	-0,02933	-0,06599	-0,10266	-0,13932	-0,17599
PG	0,47728	0,45517	0,41831	0,38146	0,34460	0,32617	0,30775	0,27089	0,23404	0,19718	0,16033
CVX	0,17421	0,15526	0,12366	0,09207	0,06047	0,04468	0,02888	-0,00272	-0,03431	-0,06591	-0,09750
AWK	0,08811	0,11046	0,14770	0,18495	0,22220	0,24082	0,25945	0,29670	0,33394	0,37119	0,40844
AMT	-0,18575	-0,14433	-0,07531	-0,00628	0,06274	0,09726	0,13177	0,20079	0,26982	0,33884	0,40787
CX	0,03171	0,02853	0,02324	0,01794	0,01265	0,01000	0,00735	0,00206	-0,00324	-0,00853	-0,01383
Sum	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00

Table 2.2 Excel Solver Weights for given Portfolio Return (with short sales)

The portfolio annual statistics are then computed for each set of returns:

Variance	0,00184 9039	0,00152 9975	0,00110 8768	0,00082 577	0,00068 0979	0,00066 0412	0,00067 4397	0,00080 6023	0,00107 5857	0,00148 3899	0,00203 0149
Std dev	0,04300 0454	0,03911 4896	0,03329 8174	0,02873 6210	0,02609 5579	0,02569 8484	0,02596 9153	0,02839 0538	0,03280 0252	0,03852 1404	0,04505 7172
Average	0,02000 0001	0,05000 0001	0,10000 0000	0,14999 9999	0,20000 0000	0,22499 9998	0,25000 0000	0,30000 0000	0,35000 0000	0,40000 0001	0,45000 0001

Table 2.3 Portfolio Statistics for each Set of Weights

Variance is computed with excel functions implementing the transpose of the weights multiplied by the variance-covariance matrix:

$$\sigma_p^2 = \sqrt{w_t \sum w} = MMULT(TRANSPOSE(w); MMULT(\sum, w))$$

Standard deviation corresponds to the square root of the variance:

$$\sigma_p = \sqrt{\sigma_p^2}$$

The average portfolio return is instead computed as the weighted average of the returns multiplied by the individual weights:

$$Average\ Annual\ Return = w^T \mu = SUMPRODUCT(w, \mu)$$

The portfolios obtained are then represented on the xy-space and the efficient frontier is graphed:

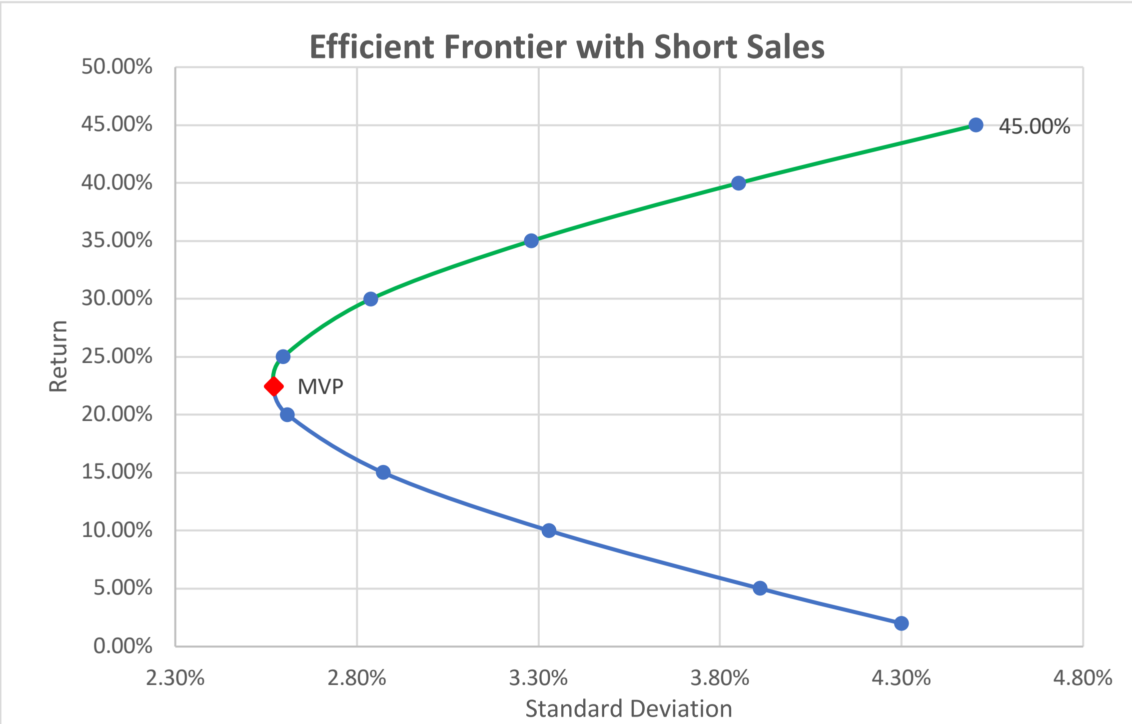

Figure 2.1 Efficient Frontier with Short Sales

The point denoted by the red dot with coordinates (0,025698484; 0,224999998) corresponds to the MVP obtained with the set of weights computed by Excel Solver. At this point, return and standard deviation are approximately 22.5% and 2,57%, respectively. Portfolios lying on the blue line beneath the MVP are inefficient portfolios. On the other hand, portfolios lying on the green line, or efficient frontier, are efficient portfolios.

If weights can only assume positive values, in other words, short sales are not allowed, Excel Solver provides different sets of weights.

MSFT	0,00000	0,00000	0,00000	0,00000	0,09988	0,28524	0,54441	0,00000
JNJ	0,10325	0,31983	0,33611	0,15932	0,00000	0,00000	0,00000	0,00000
JPM	0,00000	0,00000	0,00000	0,03869	0,16514	0,23210	0,03617	0,00000
AMZN	0,00000	0,00000	0,00000	0,00000	0,01212	0,03876	0,18846	1,00000
FB	0,00000	0,00000	0,00000	0,06840	0,02802	0,00000	0,00000	0,00000
AAL	0,68276	0,48767	0,21310	0,05538	0,00101	0,00000	0,00000	0,00000
PG	0,00000	0,03134	0,32107	0,38895	0,33877	0,00000	0,00000	0,00000
CVX	0,00000	0,00000	0,04879	0,10268	0,05631	0,00000	0,00000	0,00000
AWK	0,00000	0,00000	0,00000	0,16971	0,21995	0,14147	0,00000	0,00000
AMT	0,00000	0,00000	0,00000	0,00000	0,06740	0,30243	0,23096	0,00000
CX	0,21400	0,16116	0,08094	0,01687	0,01142	0,00000	0,00000	0,00000
	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00

Table 2.4 Excel Solver Weights for given Portfolio Return (without short sales)

The annual statistics of the portfolios are computed as above.

Variance	0,006678695	0,004084289	0,001564781	0,000832256	0,000674901	0,000918862	0,001649894	0,005557966
Std dev	0,081723283	0,063908442	0,039557309	0,028848854	0,025978854	0,030312729	0,040618891	0,074551766
Average	0,020000000	0,050000000	0,100000000	0,149999999	0,204500002	0,270000001	0,300000001	0,329417371

Table 2.5 Portfolio Statistics for each Set of Weights

The weights obtained without short sales provide different portfolios. The MVP now has coordinates (0,025978854; 0,204500002). Therefore, the portfolio return and standard deviation are approximately 20,45% and 2,60%, respectively.
Now, the efficient frontier without short sales reaches its limit at point (0,074551766; 0,329417371). As explained in section 1.6, the efficient frontier with short sales has no limit; investors can decide to short sell an asset and invest in others.
By not including short sales, the efficient frontier eventually reaches its limit. The last portfolio, obtained by investing 100% in Amazon stock, determines the boundary with coordinates (0,074551766; 0,329417371). One of the main drawbacks of excel solver is that to achieve the target level of return, the sets of weights obtained often result in concentrated portfolios.

2.4 Method 3: Portfolio Optimization

Method number 3 regards the optimization of the portfolio. Theory about portfolio optimization is covered in section 1.6.

The optimization is accomplished through diversification. Each security in the portfolio will be assigned a positive weight (short sales are not allowed), with the sum of the weights equal to one. The efficient portfolio obtained corresponds to the portfolio with the highest Sharpe ratio, the tangent portfolio.

A risk-free asset is necessary to determine a single efficient portfolio that corresponds to the tangency between the Capital Market Line (CAL) and the efficient frontier. The risk-free asset implemented is the return on the 5 years United States Government Bond. Data regarding the returns of the risk-free asset is found on the official website of the US Department of Treasury[23]. The risk-free rate for the given timeframe is:

Risk-free rate	1.97%

Table 2.6 5 years United States Government Bond return

Combining the risk-free asset and the securities we obtain the CAL, with equation:

$$E(R_C) = r_f + \sigma_C \frac{E(R_p) - r_f}{\sigma_p}$$

The CAL intercept corresponds to the risk-free asset while its slope to the Sharpe ratio. Excel solver is implemented to find the optimal diversification that maximizes the Sharpe ratio. The weights obtained are:

MSFT	0.23257
JNJ	0.00000
JPM	0.22414
AMZN	0.03232
FB	0.00000
AAL	0.00000
PG	0.08751
CVX	0.00000
AWK	0.18880
AMT	0.23466
CX	0.00000
Sum	1.00000

Table 2.7 Weights Tangent Portfolio

[23] US Department of the Treasury, Daily Treasury Yield Curve Rates https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield

The statistics obtained from this allocation are:

Portfolio Return	25.41%
Portfolio Variance	0.94%
Portfolio Std Dev	9.71%
Risk free rate	1.97%
Sharpe Ratio	2.41425639

Table 2.8 Tangent Portfolio Statistics

The maximum value for the Sharpe ratio obtained from this portfolio is 2.41425639. In general, a portfolio with a Sharpe ratio above 2 is deemed very good by investors.

Figure 2.2 is a graphical representation of the tangent portfolio obtained from the tangency between the CAL and the efficient frontier. The y-axis intercept corresponds to the rate of return on the risk-free asset, the 5 years United States Government Bond.

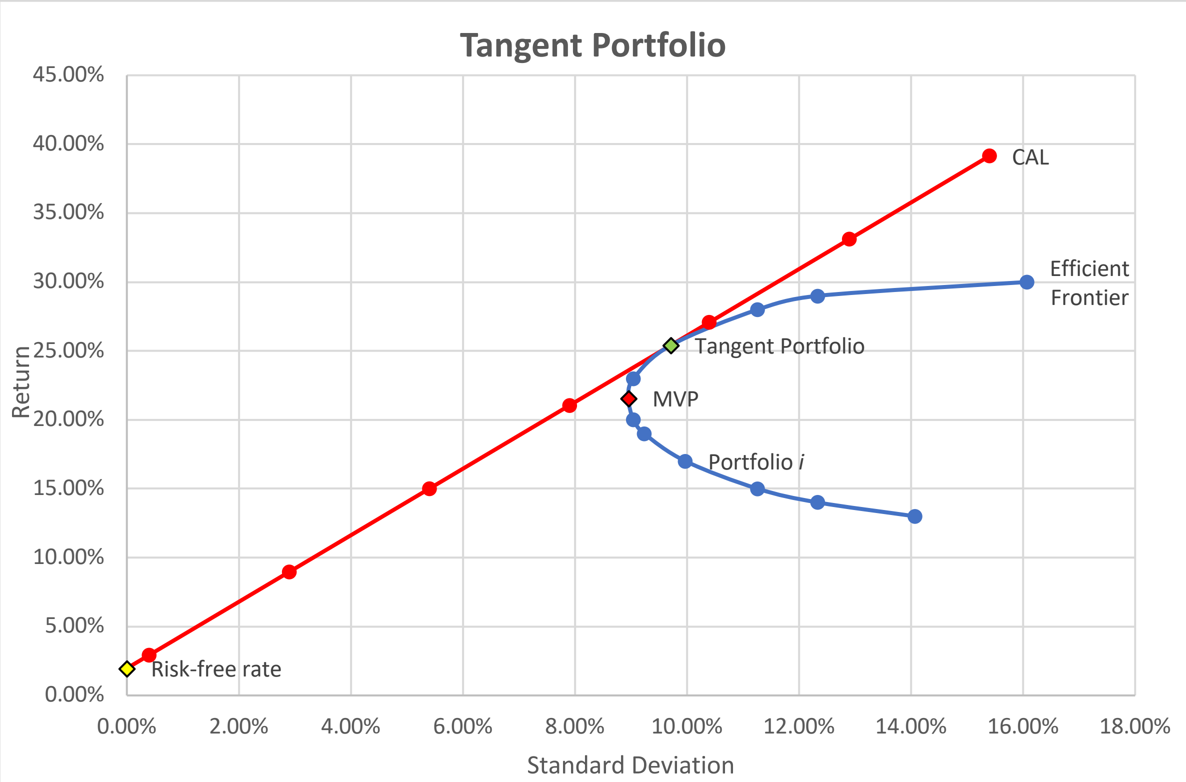

Figure 2.2 The Tangent Portfolio

2.5 Method 4: Capital Asset Pricing Model

The fourth and last method applies the Capital Asset Pricing Model to the portfolio. The objective is to identify the required rate of return of the stocks that form the portfolio implementing the notions introduced in section 1.7.

The securities' returns used for the computations are defined at the beginning of the analysis (recall Table 1.3).

The risk-free asset implemented in the CAPM is the 5 years United States Government Bond introduced with method 3 with a rate of 1.97%. The expected market return is computed with the historical returns of the S&P 500 broad market index in four years, starting 1/January/2016 until 1/January/2020. S&P 500 returns for the given time frame are:

S&P 500 Returns
-0.00412836
0.065991115
0.002699398
0.015324602
0.000910921
0.035609801
-0.001219243
-0.001234451
-0.019425679
0.034174522
0.018200762
0.017884358
0.03719816
-0.000389197
0.009091209
0.011576251
0.004813775
0.019348826
0.000546433
0.019302979
0.022188135
0.028082628
0.00983163
0.056178704
-0.038947372
-0.026884499
0.002718775
0.021608342
0.004842436
0.036021556

0.030263211
0.004294287
-0.069403356
0.017859357
-0.091776895
0.078684402
0.029728889
0.017924256
0.039313498
-0.065777731
0.068930164
0.013128152
-0.018091627
0.017181178
0.020431771
0.034047037
0.028589819

Table 2.9 S&P 500 returns

The monthly and annual statistics are:

MONTHLY	
Average	1.14%
Variance	0.10%
Std dev	3.25%
ANNUAL	
Average	13.7%
Variance	1.24%
Std dev	11.26%

Table 3 S&P 500 Annual Statistics

The market risk premium therefore is:

Market Risk Premium	11.75%

Table 3.1 Market Risk Premium

It is the premium investors receive for holding the market index. It is computed as the difference between the market return and the risk-free rate $(E(R_m) - r_f)$.

Beta of each stock is estimated through the built-in function in excel. It is computed as $\beta_i = \frac{\sigma_i * \rho_{i,m}}{\sigma_m}$ using stocks' individual returns for the given period.

MSFT	1.059169853
JNJ	0.72331686
JPM	1.154203949
AMZN	1.528702694
FB	1.287739146
AAL	1.917509267
PG	0.302824618
CVX	0.894423783
AWK	0.153031134
AMT	0.100609546
CX	1.413591696

Table 3.2 Individual Stocks' Beta

All stocks have positive betas, with the lowest being of AMT (0.100609546) and the greatest of AAL (1.917509267). All stocks exhibit positive betas due to the selection made at the beginning of the analysis: the stocks are randomly selected and represent individually a different sector among the eleven GICS sectors on which the S&P 500 is constructed; therefore, a positive well-distributed correlation is expected.

The stocks' required return can then be computed with the CAPM equation $E(R_i) = r_f + \beta_i(E(R_m) - r_f)$:

MSFT	14.41%
JNJ	10.47%
JPM	15.53%
AMZN	19.93%
FB	17.10%
AAL	24.50%
PG	5.53%
CVX	12.48%
AWK	3.77%
AMT	3.15%
CX	18.58%

Table 3.3 Stocks' Required Return predicted with CAPM

The required rates of return differ from the expected rates of return essentially because the former is the minimum rate an investment must have, given its riskiness, to be considered, while the latter is the potential rate provided by such investment.

Stocks' Alphas are again estimated through excel built-in functions:

MSFT	1.35%
JNJ	0.21%
JPM	0.90%
AMZN	1.00%
FB	0.08%
AAL	-2.23%
PG	0.91%
CVX	0.17%
AWK	1.46%
AMT	2.09%
CX	-1.21%

Table 3.4 Individual Stocks' Alpha

Positive Alphas indicate greater expected returns than those required by CAPM for a given level of risk. However, values do not deviate excessively. Securities too underpriced are targeted and exploited by arbitrageurs who profit from it.

Alternatively, a stock Beta and Alpha can be evaluated by running a regression on the returns. Regressions provide further statistical insights. For instance, if we run a regression on MSFT stock, we obtain:

MSFT Regression Statistics	
Multiple R	0.69560475
R Square	0.48386597
Adjusted R Square	0.47239632
Standard Error	0.03594339
Observations	47

Table 3.5 Summary of MSFT Regression Statistics

	Coefficients	***Standard Error***	***t Stat***	***P-value***	***Lower 95%***	***Upper 95%***	***Lower 95.0%***	***Upper 95.0%***
Intercept	0.01347387	0.005564386	2.421448257	0.01955278	0.00226662	0.02468112	0.00226662	0.02468112
X Variable	1.05916985	0.163071499	6.49512551	5.7325E-08	0.73072699	1.38761271	0.73072699	1.38761271

Table 3.6 Statistics Output

The R square result indicates that approximately half of the MSFT movements (48.39%) are correlated with the S&P 500. The values of Beta and Alpha are respectively the coefficients of the Intercept and the X variable.

On the xy-plane, we can plot the graphical representation of the CAPM. The Security Market Line (SML) plots the linear relationship between a stock's Beta and its required rate of return.

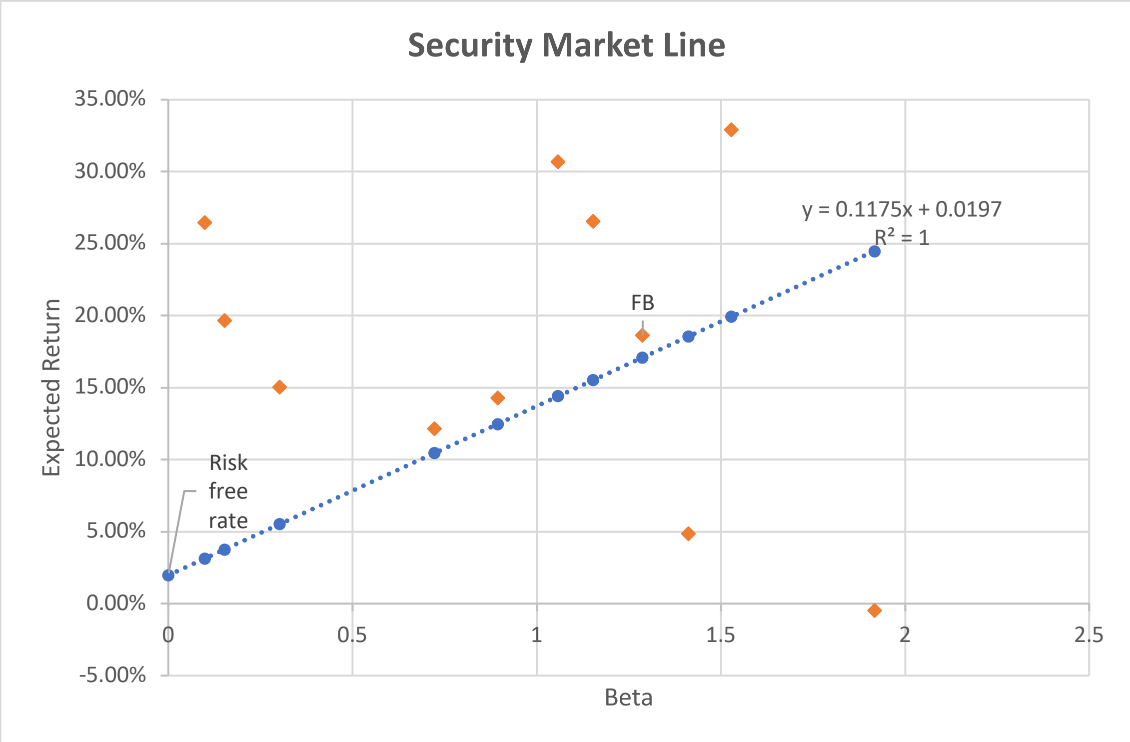

Figure 2.3 The Security Market Line

The intercept on the y-axis with the risk-free rate is the first point of the linear relationship. The equilibrium occurs when the assets' returns lie on the SML. Hence, their required rate of return is equal to their actual return. Stocks above the SML are underpriced and offer greater expected returns than their required rate, while stocks beneath are overpriced and offer lower expected returns. Take for instance FB, in equilibrium it would lay on the SML however the exhibited return is greater than that predicted by the CAPM. As it lays above the SML equilibrium we assert that through the analysis FB stock is underpriced. The risk-free asset has, by definition, Beta equal to 0 as its returns are safe and uncorrelated with the market.

SML Summary Data				
Stocks	**Beta**	**CAPM return**	**Return**	**Alpha**
rf	0	1.97%		
MSFT	1.059169853	14.41%	30.70%	1.35%
JNJ	0.72331686	10.47%	12.17%	0.21%
JPM	1.154203949	15.53%	26.59%	0.90%
AMZN	1.528702694	19.93%	32.94%	1.00%
FB	1.287739146	17.10%	18.64%	0.08%
AAL	1.917509267	24.50%	-0.48%	-2.23%
PG	0.302824618	5.53%	15.05%	0.91%
CVX	0.894423783	12.48%	14.29%	0.17%
AWK	0.153031134	3.77%	19.68%	1.46%
AMT	0.100609546	3.15%	26.49%	2.09%
CX	1.413591696	18.58%	4.88%	-1.21%

Table 3.7 SML Summary Data

To obtain the statistics of an Equally Weighted portfolio following the results obtained from the CAPM we simply compute the averages:

Portfolio Averages	
BETA	0.95773841
ALPHA	0.00429557
Expected Return	13.22%

Table 3.8 Portfolio Statistics with CAPM

The MVP and the optimized portfolio with the returns predicted by the CAPM have the following statistics:

MVP	
Portfolio Return	8.61%
Portfolio Variance	0.80%
Portfolio Std Dev	8.96%
Sharpe Ratio	0.74224939

Table 3.9 MVP Statistics with CAPM Returns

The statistics for the optimized portfolio are:

Optimized Portfolio	
Portfolio Return	14.14%
Portfolio Variance	1.48%
Portfolio Std Dev	12.18%
Sharpe Ratio	0.998836898

Table 4.0 Optimized Portfolio Statistics with CAPM Returns

2.6 Evaluation of Results and Considerations

Through the four methods implemented to construct a portfolio, we provided practical examples of the notions introduced throughout chapter 1.
The differences between the data obtained with these evaluations are substantial, and they can be further analyzed and compared to obtain more information.

The eleven stocks selected to form a portfolio represent a different market segment individually. Therefore, following the GICS classification of market sectors, we provide an immediate significant degree of diversification.

The following table provides a quick comparison of the returns and volatility available for the portfolios developed with the four methods through excel built-in functions.
Although we should not rely only on return and volatility, table 4.1 points out the first differences between the portfolios.

Portfolios Averages				
	Equally Weighted	**MVP**	**Optimization**	**CAPM**
Expected Return	18.29%	22.50%	25.41%	13.22%
Variance	1.44%	0.07%	0.94%	
Std Dev	12.14%	2.57%	9.71%	

Table 4.1 Portfolios Averages

Beginning with the first method, the Equally Weighted Portfolio, the statistics are fair and we can diversify our portfolio by investing the same proportion into each stock. Figure 2 from section 2.2 clearly demonstrates how the portfolio significantly reduces volatility, with its standard deviation lower than each stock's standard deviation.
Equal weighted indices are highly adopted in practice. The most common are the S&P 500 Equal Weight Index, the NASDAQ-100 Equal Weight Index, and the Russell 1000® Equal Weight Index.
The Minimum Variance Portfolio MVP, for a given return of 22.50%, provides total volatility equal to 2.57%. At first sight, the statistics of the MVP seem to be better than those of the Equally Weighted portfolio as it provides a greater return for lower volatility. However, recalling tables 2.2 and 2.3 from section 2.3, such results are obtained considering the possibility to short-sell assets. Accordingly, while it may seem that the total risk of the MVP is lower compared to the other portfolios, we need to be aware that short selling is a hazardous activity, and in the worst-case scenario, the potential profits could result in severe losses.
The optimized portfolio provides the highest rate of return as its objective is to maximize return. However, once again, we notice one of the main drawbacks of excel solver: to obtain this level of return (without including short sales), some stocks are not included in the portfolio. Recalling table 2.7 from section 2.4, we see that the result is a concentrated portfolio formed by only six stocks, therefore limiting the benefits of diversification.
Lastly, the average expected return for the portfolio predicted by the CAPM differs significantly from the returns of the other methods. The difference that arose is not easily explained only through mathematical results; we need to take into consideration also the dynamics occurring in capital markets. Chapter 3 will provide explanations and examples for the difference between the returns predicted by the CAPM and those observed.

Regarding validity and reliability, the research is carried out using trusted sources such as university books and famous publications, while all computations and results are obtained by implementing Excel and its built-in functions.

For ease of computations, the analysis is carried out on a small portfolio using monthly observations on four years. The results obtained are easy to interpret and adequately demonstrate the concepts introduced in chapter 1. However, the analysis could be improved by including more stocks in the portfolio and considering a broader time frame using weekly or daily observations. As the time horizon and the number of observations increase, also does the reliability of the results. While the increase in the size of the portfolio ideally improves the diversification process.

Chapter 3

Capital Markets are not Efficient in Practice

In this conclusive chapter, the focus will be moved toward the investor behavior and the efficiency of capital markets; the latter regards the efficient market portfolio and why, in practice, capital markets are not efficient.

Section 1.7 clarifies the notions inherent to the CAPM. Despite having little empirical support, the CAPM is still widely adopted in the financial market to evaluate stocks' required rates of return. According to this model, the market portfolio is the only efficient portfolio available to investors who cannot consistently do better than the market without taking on additional risk. Investors will not be able to do better in the long term as the equilibrium described by the CAPM occurs when assets' returns correspond to the required rate of return predicted by the CAPM. In addition, investors who perform better than the market portfolio take on greater risk, which leads their portfolios to underperform in the long run.

3.1 Investors' Expectations

In order to understand and explain deviations from the returns predicted by the CAPM, we elaborate on the third assumption: the homogeneity of expectations.

Investors' expectations on the stock market come from historical data and information publicly and equally accessible. The core idea is that all investors are ration and have access indiscriminately to such information. As a result, they must all hold the tangent or efficient portfolio, or in other words, the market portfolio.
If stocks were to present significantly positive alphas, then investors, having access to the same information, should immediately rush to purchase the stock until alpha is zero and the opportunity of profit is inexistent. In practice, however, it is never the case. For investors to hold stocks with positive alphas, some must be holding stocks with negative alphas. Professionals investors might be able to purchase positive alphas stocks from inexpert investors because of differences in available information.
The equilibrium towards the CAPM aims, where all investors profit by holding the market portfolio, is a safe strategy for inexpert investors. As a result, the assumption on which the CAPM relies is not that of homogenous expectations, rather rational expectations where all investors correctly interpret the information available. The inefficiency of the market portfolio arises because of investors, particularly if:

- Investors incorrectly interpret the information available and end up holding stocks with negative alphas;
- Or are willingly holding inefficient portfolios for reasons different from return and volatility

3.2 Investment Styles

Investment styles or strategies are the approaches an individual or fund manager chooses when picking stocks to form a portfolio. As financial assets are classified, the same reasoning applies to investment styles.
The first degree of differentiation occurs depending on the level of risk a fund manager is willing to undertake. Conservative funds offer short time horizon portfolios with low levels of risk by investing mainly in fixed-income investments like government and corporate bonds and money market funds. Moderate or value funds' portfolios are an in-between; a significant percentage (more than 50%) of investments consists of stocks from large-capitalization companies while the rest of bonds and often a small percentage also of money market funds. The high returns offered are subject to a discrete level of volatility. Aggressive, growth, or hedge funds are more volatile and require a greater time horizon to exhibit positive returns. Aggressive funds' portfolios consist almost entirely of stocks and a small percentage of bonds; therefore, economic downturns severely affect the portfolio return.

Portfolio management is carried out following two broad styles: active or passive management. Portfolio active management requires individuals or funds to do in-depth research and invest with high frequency, therefore requiring commissions frequently. The objective is to achieve greater returns than those of a market benchmark through market forecasting. Nonetheless, actively managed portfolios are subject to greater degrees of market volatility. Mutual funds are actively managed funds.
Conversely, passive portfolio management objective is to mimic the return of a benchmark by weighting the stocks within it. Therefore, investment frequency and fees charged are lower compared to an actively managed portfolio. Exchange-Traded Funds are passively managed funds.

Stocks fall under two general categories: growth or value stocks.
Investors purchase growth stocks as they believe that the company is an innovator with potential growth in the upcoming years. Growth stocks generally do not offer dividends as earnings are reinvested and may involve greater risk if the company is not well established. Generally, growth stocks are found in companies with small to medium market capitalization. On the other hand, investors purchase value stocks to earn constant returns through dividends. Value stocks are typically associated with sound and well-established companies, as firms with large market capitalization. Therefore, they exhibit a lower level of risk.

3.3 Investors' Behavior

Investors fail to diversify properly when selecting stocks and constructing portfolios, and the outcome is a sub-optimal portfolio. In addition, investment decisions are often biased, and the result is a portfolio concentrated in one sector or in a particular geographical area impeding investors from fully benefiting from diversification.

Deviating from the equilibrium suggested by the CAPM implies a greater volume of daily trading. If all investors were to hold the market portfolio, the daily trade volume would

be low as the market portfolio is a passive portfolio, and investors are not required to rebalance it every day. In practice, however, most of the investors do not hold passive portfolios; as a consequence, the volume of daily trade is excessively high.
An explanation for the excessive volume of trade occurring in the financial markets is the overconfidence bias of investors. Generally, investors believe they are above average, failing to assess their skills appropriately and making wrong investment decisions. As a result, they take on more risk, and expected profits result in losses when considering trading commissions and bid-ask spread.
The volume of trade constantly increases and decreases as a response to economic downturns.
An example of overconfidence bias can be found in James Montier's study[24].

Another pattern of investors is to sell winning stocks and hold losers. It is commonly addressed as the disposition effect and consists of selling stocks whose value has risen and holding stocks whose value has declined. Individuals, not only in financial markets, tend to take on greater risk once facing a loss in order to cover for the mistake.

The average investor is generally not a full-time trader. The amount of time and resources available to the investor for trade is not that of an analyst, and investment decisions are often a product of a third party recommendation or the environment around him.
Recommendations significantly impact the stock prices when the aggregate number of investors purchasing the stock is significant. The effect is an excessive increase in prices. It has been demonstrated that the average investor often purchases stocks from big-cap companies, those covered by the news, and those who are "big trends" and are experiencing significant returns at the time of purchase. Often investors may also follow investing strategies of others with the belief that their strategy is less profitable because of wrong assumptions or limited information.
A takeover offer is the most common profitable opportunity that an investor can come aware of from news. When a takeover bid is announced, share prices usually spike up, and if the takeover is successful, they stay constant; if not, they adjust to the original level.

Professionals traders, particularly mutual funds managers, can better exploit the information available and turn it into a profit because of their advantageous position and their non-apparent skills. Jonathan B. Berk and Jules H. van Binsbergen in their study "Measuring Managerial Skill in the Mutual Fund Industry," provided a clear insight into the topic implementing a different analysis compared to the previous ones. The average mutual fund manager's strategy does not seem to outperform that of an individual investor. However, through their stock picking, the best mutual fund managers can cover the value destruction. As a result, stock picking or market timing talent do exist, and the most skilled fund managers are highly retributed to manage larger funds.[25]

[24] James Montier, "Behaving Badly", Dresdner Kleinwort Wasserstein - Global Equity Strategy, February 2006
[25] Jonathan B. Berk, Jules H. van Binsbergen, "Measuring Managerial Skill in the Mutual Fund Industry", Journal of Financial Economics, October 2015, pp 1-20

3.4 Is the Market Portfolio Efficient?

Several research and articles regarding the validity of the CAPM have been written, particularly the efficiency of the market portfolio used as benchmark in the evaluation. The general claim is that the market portfolio implemented in the CAPM as a key variable is inefficient.

One of the possible explanations for the inefficiency of the market portfolio is investors' behavior. Investors subject to behavioral biases fail to properly diversify their investments, thus holding inefficient portfolios. Furthermore, investors may be willingly holding inefficient portfolios for reasons other than risk and return.
A relevant consideration is that an efficient market portfolio may exist; it would need to include anything with marketable value. However, it is impossible to fully diversify, including all existing investments, as comparable data is not available or hard to find for determinate sectors. Following the latter assertion, among the critiques to CAPM, one of the most recognized is that of Richard Roll.[26]

Theory and empirical support of expected return are not yet fully reliable in the financial market. All the existing models are imprecise, and while multifactor models are deemed to better capture the required rate of return of a stock, they are not easy to implement.
Despite the controversy, the CAPM remains a valid one-factor model widely implemented in the financial sector to evaluate the cost of capital.

[26] Roll, Richard (March 1977), "A critique of the asset pricing theory's tests Part I: On past and potential testability of the theory", *Journal of Financial Economics*, **4** (2): 129–176

Conclusion

The elaborate is intended to be understood and commented on by individuals who have a discrete knowledge of capital markets and financial instruments and have developed an interest in portfolio risk management. The thesis relies on the Modern Portfolio Theory and aims at providing a solid understanding of how risk and return are evaluated in capital markets through statistical indicators and how these variables affect investment decisions. Furthermore, it suggests the implementation of technical tools and mathematical frameworks to conduct the evaluation.

The risk-return trade-off is the essential variable affecting investment decisions. Risk is defined as the probability that the realized return on an investment deviates from the expectations of return.
Despite the existence of safe investments, individuals approach financial markets in search of assets that provide greater return rates for given levels of risk. It is of a common belief that expected returns increase as we undertake more risk.
Risk is classified into two broad categories: market risk and specific risk. The former, also defined as systematic risk, affects the entire market and cannot be diversified away. The latter, also known as unsystematic or idiosyncratic risk, affects specific companies and can be diversified away by choosing uncorrelated investments.
The benefits of diversification are evident as it is the only 'tool' to mitigate specific risk. With investment diversification, we can significantly improve the risk profile of our portfolio, even to the extent where only market risk affects the portfolio while having more safe and constant returns.
Investors are rewarded for holding risky assets, however, only on the specific risk. The reward or extra return is defined as the risk premium. It is mathematically assessed as the difference between the return on the market portfolio and the return on the risk-free asset, an asset that grants a certain level of return null risk.
The market portfolio is the benchmark adopted to evaluate a security level of systematic risk and its risk premium. It is essential in the evaluation as it represents, in theory, all the securities globally traded.
The Beta statistical indicator expresses the amount of systematic risk characterizing a stock; in other terms, it is the percentage change in the security return after a change in the return in the benchmark.
With the risk-free asset, the market benchmark, and the stocks' Beta, we lay down the equation for the CAPM: $E(R_i) = r_f + \beta_i\big(E(R_m) - r_f\big)$ used to compute a stock required rate of return. Firms and individuals frequently adopt this model during the decisional process to predict stocks' expected returns.

Recommendations for further study:
The CAPM has been subject to extensive critiques regarding its assumptions, and its reliability is constantly questioned.
Multifactor models were developed following the CAPM, and they include other risk factors that may affect a security price. Despite the difficulty in implementing them, multifactor models can better capture a stock's required rate of return. Among these, one of the most adopted is the Fama-French three-factor model, which also considers the size of firms, book-to-market values, and excess returns on the market.

List of Figures

List of Tables

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