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Pricing American-Bermudan Options through Least Square Methods

Porchia Paolo

SUPERVISOR

Pirra Marco

CO-SUPERVISOR

Morisi Manfredi

CANDIDATE

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Contents

Abstract

The popularity in academic research of the area of option pricing techniques improved since Black, Scholes, and Merton (1973) developed the first option pricing method, following the fast development of derivatives market of the last decades. The more an option has particular characteristics, the more it will be possible to find difficulties in selecting the right method, among the many existing, to evaluate it.

This thesis introduces the Least Squares Monte Carlo approach to option pricing applied by Longstaff and Schwartz (2001) on different types of American-style options. Specifically, this thesis refers to the valuation of an American-Bermudan put Option through this method, comparing the numerical results of this valuation with the ones obtained evaluating the prices of the Options using a dense Binomial Trees model as a benchmark. The application of these pricing methods will be executed through a program developed in Python language. Finally, the thesis analyzes the Numerical Results to discuss the advantages and disadvantages of the LSM approach for this specific

type of option when fronting various scenarios. It will show how the LSM method outperforms the traditional numerical method considering the trade-off between the amount of possible scenarios to test, the computational time required and the valuation precision, when a certain size of paths simulated, degrees of polynomials and exercise time points is exceeded.

List of Tables and Figures

Chapter 1

Introduction

In the last half of century, options, derivatives including options, and any other derivatives with similar features, became popular financial instruments traded, in both the Over-the-Counter (OTC) and the Exchange markets.

The price of an option depends on several factors: time to maturity, strike price, volatility and price of the underlying asset, interest rate and, eventually, dividends.

This thesis will concentrate on those types of options having the characteristic, typical of the so-called American options, of giving the possibility to the holder to exercise the option in many time points before the maturity. The prediction of the theoretical price of certain types of options is now more than a challenge for traders, derivatives issuers, and academic researchers. The importance of doing it quickly and precisely led to the development of various option pricing techniques, each of these to be chosen and applied based on the way it fits with the features of the option to be priced.

The Black-Scholes model for option pricing is one of the first and most celebrated works in this field. For the contribution that this model gave to the research, in 1997 Myron Scholes and Robert Merton received the Nobel prize for Economics. This model is going to be defined more in depth in the 2^{nd} chapter of the thesis.

Even if there have been found some analytical expressions for the price of American Options in many elementary forms (MCkean(1965); Roll (1977); Geske (1979); and Whaley (1981)), differently from what happens for European options, the Black-Scholes model for American options doesn’t provide the analytical solutions to determine the price of the option. This represents an issue for the traders willing to use this method, as most of the options issued and traded in the CBOE (Chicago Board of Options Exchange) are of the American type.

Eventually, a way to price American Options and similar, is by applying numerical solutions. The binomial model exposed by Cox, Ross and Rubinstein (1979) is probably the most popular technique giving a fast numerical solution determining American options' price. The binomial model is very intuitive, so it is commonly approached academically and in financial industry. In chapter 2 it will be described more in depth, as this thesis will use this model as the numerical benchmark to analyze the robustness and convergence precision of the prices evaluated through LSM.

One of the main issues with this and many other numerical methods is that the only stochastic factor considered in the valuation of the option is the price of the underlying asset, while the other relevant variables are assumed to be constants. It comes natural to consider that a method assuming factors like dividends, multiplicity and weights of underlying assets, volatility, and interest rate as constants could miss in flexibility, precision, and reliability compared to others. although it has great academic importance. In facts, the problem of using this model importing more factors as stochastic, is the so-called curse of dimensionality: the number of binomial nodes grows exponentially, making the model computationally unaffordable.

The necessity of finding numerical solutions considering multiple stochastic factors gave room to the Monte Carlo simulation. The first introducing this simulation for pricing options was Phelim Boyle in the article “Options: a montecarlo approach” (1977). The possibility to apply this technique on options with multiple stochastic factors led the evaluations of European Options’ price through this method to become popular by the years, while it kept being hard to implement it on American Options, because of computational issues. In facts, it’s required to calculate the optimal early exercise price, at a certain time, recursively, when processing the price of American Options. Since it’s just one single future path for any time spot, this calculation would show biased results.

From that moment, many studies came over trying to propose a way to price American Options through simulation. James Tilley suggested to define the optimal stopping time point along the paths through a simulation algorithm imitating the standard lattice. With a similar approach, Martineau and Barraquand (1995) proposed their method, the so-called Stratified State Aggregation along the Payoff. To price American options through Monte Carlo simulation, a model essentially related to the binomial model, instead, has been proposed by Broadie and Glasserman (1997).

Jacques Carriere (1996) defined a way to determine the optimal early exercise strategy associating the calculation of the numbers of conditional expectations through a backword induction system to the evaluation of American Options. To elaborate a precise approximation of the conditional expectation he combined advanced regression methods with the simulation. From this idea, Francis A. Longstaff and Eduardo S. Schwartz (2001) took the inspiration to elaborate a method based on simulation, called Least Squares Monte Carlo (LSM) algorithm. In their article they use a simple least squares cross-sectional regression to estimate the conditional expectation function for every exercise date and determine the optimal exercise strategy for each path. This work, that will be deepened more in the 4th chapter of this thesis, also applies this method to other specific types of options that are path-dependent, such as cancellable index amortizing swaps, and different forms of American-Bermudan options, (which is the type of option that will be analyzed in this thesis).

This thesis will focus on the application of the Least Squares method of Longstaff and Schwartz on pricing American-Bermudan Options. This brief introduction on the literature regarding analytical and numerical methods for option pricing, paying more attention on the issue of pricing American Options, has been given to show to the reader a basic idea of the area of research this thesis aims to contribute to, and why and how this research area developed in this direction in the last 50 years.

In chapter 2, it will be initially defined the financial background necessary to ensure to be acknowledged of every specific financial term and every basic financial process present in this thesis. Subsequently the main types of options will be described, paying particular attention to the types linked to the one evaluated in this thesis.

The attention will then be driven to the description of the methods of Option Pricing concerning this thesis: in the last two paragraphs of the second chapter there will be, at first, the illustration of the oldest within the most famous Black-Scholes model, defining its basic assumptions and obtaining its pricing formula for European Options. Then, there will be the illustration of the binomial tree model, the logic behind this method and some comparisons with other numerical methods, focusing on the advantages of the use of the former, which will be used later in this thesis as a benchmark to show the consistency of the LSM method successively approached to evaluate the American option.

In the third chapter the Monte Carlo simulation will be more specifically introduced and explained in its logics. As for the Black-Scholes method, this numerical method will be defined in its basic assumptions. Although the normal distribution N(0,1) is provided as a black-box function in almost every software, in the third chapter we will briefly illustrate how to generate pseudo-random numbers, proving how it's possible to convert random numbers sampled from U[0,1] in numbers sampled from the normal distribution N[0,1].

Successively, this method will be applied running the pricing formula of a European Call Option as for example, and the numerical results obtained through this simulation will be compared to the theoretical result coming from the former method, all of it involving a program to be run on python. We will prove how the Monte Carlo simulation results to be a valid instrument for pricing European Options. In facts, we will notice that the accuracy of the evaluations, in terms of robustness and convergence to the true value evaluated using the

Black-Scholes method, will be satisfying when a certain number of paths (10,000) are simulated.

Then, it will follow the definition and evaluation of the Least Square Method proposed by Longstaff and Schwarz to estimate the price of American-style Options. At first, there will be explained the reasons of the failure in pricing more complicated options than the European-style ones using the numerical methods explained before, and the reasons why it became so necessary to find a way to evaluate the American-style options. After the illustration of the LSM method, the biggest issue will be to prove how LSM could price these options successfully, managing their most challenging characteristics such as the path-dependency and the presence of multiple stochastic factors. Additionally, this thesis will provide the mathematical foundation of this simulation method in terms of convergence and robustness. Moreover, in this chapter it will follow the individuation of the proper basis function considered for this option type, the definition of the dynamic programming algorithm used to evaluate the American-Bermudan options, and its application on a put option.

In the last part of the fourth chapter, numerical results will be illustrated and analyzed, and then, in order to check the accuracy and the feasibility of the LSM simulation method, these will be compared to the results obtained applying the Binomial Tree model. This chapter ends up analyzing the performance of this method in terms of the trade-off between precision of the price evaluation and computational time, taking consideration of different amounts of possible exercise points in the time, different amounts of paths in simulation and different degrees of the basic functions used in the regression process.

We will realize how this method could represent a powerful tool to use in pricing American-Style option. When we impose a certain number of paths N simulated (10,000), of time-steps M introduced (100), and of degree of the regressive function implemented (2), the accuracy of our evaluations will result to be satisfying in terms of convergence to the average price of the simulations and to a benchmark price evaluated by Binomial Tree method, and in terms of robustness when changing the degree of the regressive function. Moreover, we'll notice how the computational effort needed will not be unaffordable, as it takes less than 19 seconds to get a sufficiently accurate evaluation.

Finally, this thesis will summarize its major findings and contribution, and some possible inspirations for future research on its related field.

CHAPTER 2

Foundation

2.1 Financial Background

In this chapter this thesis aims at deepening the most important financial terms and processes necessary to approach the illustrations and the analysis coming after.

A financial contract having the value deriving from (or depending on) the value of one or many other underlying assets, is called *derivative*. Popular examples of derivatives are swaps, options, futures and forwards. Even if the biggest part of the methods described in this thesis can be applied to many other types of derivatives, this thesis will concentrate its attention to the options.

The owner of an option contract holds the right to buy or sell the underlying asset at a predefined price, called Strike Price or Exercise price, at the expiration date (or maturity), and eventually in many predetermined points in the time before (this depends on the type of the option contract, which will characterize the subject of the next paragraph). Options giving the possibility to the holder to buy the underlying asset at a certain time and at a predefined price are called Call Options, while options giving the possibility to the holder to sell the underlying asset at a certain time and at a predefined price are called Put Options.

The parameters commonly considered to influence the most the value of an option contract are: the volatility of the underlying asset and its price; the dividends paid already and the one still to be paid on it; the time to maturity; the amount of future points in time available to exercise the option before the maturity (and their collocation in the timeline); and the interest rate.

It's logical that the relationship between the value of an option and its Strike Price, keeping anything else constant, works as such: the value of a call (put) option increases (decreases) when its strike price decreases, and vice versa. Coherently, the value of a call (put) option decreases (increases) when the price of its underlying asset decreases, and vice versa. Regarding its relationship with the volatility of the underlying asset, instead, the value of any type of option increases as the former increases, since it is a measure of the possibility for the underlying asset (and so for the option) to make big changes in its price development (making the asset derived riskier when volatility is higher), which comport the possibility to increase up to the infinite (hypothetically) when the value of the option increases, while in the other sense the option could never be worth less then limit value of 0, which is a value quite common to result, even in low volatility options. When the risk-free interest rate increases, the price of a call (put) option increases (decreases), as it generally happens with most of the financial instruments. The increase in the time missing to maturity generally influences the value of both the types of option positively, and so would do an increase in the amount of time points available to exercise the option before the maturity, especially when uniformly distributed.

A clever investor would often take advantage of *arbitrage* situations, if the possibility appeared. This is a trading strategy consisting in identifying securities priced differently in different markets, basically striking a combination of transactions, simultaneously purchasing where the price is lower and selling where it’ higher. This makes the investor gaining from entering a riskless investment. A trading strategy approached with the aim of secure the investment position on one asset is the so-called *hedging* strategy: meaning covering, this strategy can be implemented in multiple ways, and basically consists in opening other positions on other assets, typically having strictly decorrelated returns with the first position hold, in order to minimize the volatility of the investment in its entirety. An option is often recognized as a valid instrument to use in a hedging strategy. One last concept necessary to start describing the various methods considered in this thesis is the development process of the price of an asset compounded continuously: 1 unit invested today, will be worth e^{rt} in future, being “r” the compounded interest rate, and “t” the time expressed in years.

2.2 Definition of different types of options

As we have seen in the Financial Background’s chapter, the option contract gives to the holder the right to buy, or to sell, the underlying asset at a predetermined price in a predetermined time, depending on the contract referring to a call option or to a put option.

The right of the holder to exercise the option in a moment rather than in one or many others, is determined before in the option contract, and represents big part of the differences between some types of options and others.

The two basic types of option are the European style option and American style option, also defined as plain vanilla options. The denominations are not actually related to geographic location. In facts, these two types of option differ in the sense that the latter allows the holder to exercise it any time before expiration date, while the former allows to exercise it only at maturity. Obviously, the possibility of exercising the option before maturity represents an advantage for the holder, which is usually denoted in the price of the American option, that is a bit higher than the price of a European option having all other characteristics equal.

Other aspects influencing the categorization of the options can be the expiration cycle, the method used to trade them, and the underlying asset the options are related to. Generally, “Exotic” options are intended to be any of a broad category of options including more specific and complex financial structures than the typical vanilla options.

Some types of options giving less common approach to the rights of early exercise are:

- Bermudan style option: A more specific type of option which is quite popular in financial markets. In this case the holder has the right to exercise the option in a predefined amount of time, generally discretely spaced. This is a very important type of option for the academic research in the field of option pricing, especially for the methods considered in this thesis: in facts, Longstaff and Schwartz proposed their work evaluating, with the others, a Bermudan option having a big amount of time steps where to exercise the option, in order to better replicate the condition of an American-style Option.
- Canary option: This style of option is in a middle-way between European and Bermudan options, just like the geographical location of its name. The holder of a Canary option can typically exercise it at quarterly dates, but only after a predefined period (generally one year).

- Capped-style option: The profit of this option is predefined in the contract. It gets immediately exercised as the underlying reaches the predetermined price.
- Compound option: It’s an option deriving another option as underlying asset.
- Double option: Doesn’t get traded on exchanges, but only on commodities, it gives the holder the possibility to either buy or sell the underlying at the strike price K.
- Evergreen option: It allows the holder to exercise the right to buy or sell by providing the notice in a pre-determined period of time.
- Shout option: It allows the holder to lock-in in a price reached by the underlying in a moment before maturity, keeping such a price when exercising the option at maturity.
- Swing option: Typically adopted in energy trading, allows the option holder to purchase a predetermined amount of underlying at a predetermined price while maintaining a certain degree of flexibility in the amount purchased and the price paid. The holder can also exercise a predetermined amount of options for every time step precedingly specified.

The exotic options having standard exercise styles (as vanillas), but different methods in calculating the payoff values are:

- Basket option: It's an option deriving an asset composed by many assets (a "basket" of assets). A particular type of Basket option is the "Rainbow option", having the weights depending on the performance up to maturity of the assets composing the basket.
- Boston option: This option works as an American option, but the payoff can be collected only at maturity.
- Composite option: Also named "cross option" has the strike price denominated in a defined currency, and the underlying asset denominated in another one. So, it's suggested to consider the exchange rate volatility, and the correlations between the two currencies and the underlying asset, when trading these options. A type of composite option is the "quanto option", which is hedged by the exchange rate risk keeping it stable from the issuance of the contract.
- Exchange option: It gives the holder the right to exchange one asset for another, if convenient.

- Low Exercise Price option: It works as a European style option with a predefined exercise price equal to $0.01.

  Other relevant exotic options differing both in the calculation of payoffs and in the style of exercise from vanillas, are:

- Asian options: This type of option, particularly relevant for this thesis, as we'll find out, has its payoff determined by the average price of the underlying asset in the final period having predetermined size, instead of being determined by the asset's price at maturity. In this case, the name is coherent with the geographical location in which it first has been modelled and traded, which is Tokyo. An example of payoff for an Asian put option could be the difference between the strike price and the daily average of the last 3 months, if positive, or zero if such a difference is negative. Traders use this option in order to reduce the risk related to unexpected bounces of the stock price, especially when getting closer to maturity.
- Barrier option: It involves the existence of a limit price, which, once it gets reached, activates the option to be exercised by the holder to exercise or, instead, deactivates it, imposing him not to exercise it anymore.

- Binary option: has a fixed payoff, which could be a predetermined amount or zero, depending on the level reached by the underlying asset's price at maturity.
- Chooser option: This contract gives the holder the right to choose to make this derivative a call or a put before a predefined date.
- Forward start option: This type of options has the strike price determined after a predefined amount of time. A series of forward start options is called "Cliquet option".
- Game option: Also named "Israeli option", it allows the writer to liquidate the option to the holder, paying the payoff at that moment plus a predefined penalty fee.
- Lookback option: This path dependent option gives the holder the right to buy (sell) the underlying asset at its lowest (highest) price reached in a predefined precedent period.
- Parisian option: The standard version of the Parisian option makes the payoff to depend by the maximum amount of consecutive time spent by the underlying instrument above or below the strike price. The cumulative version of this option depends on the total amount of time spent by the asset above or below the strike price.

- Parisian barrier option: The standard version of this type of option works as if the value of the underlying asset stays above or below a "price limit" for more than a certain amount of consecutive time, the option can be exercised, or it can no longer be exercised. The cumulative version of this option doesn't require the amount of time defining the threshold to be consecutive.
- Reoption: This type of derivative contract gives the holder the right to activate again an option expired without being exercised.

In this thesis, we will propose the application of Binomial tree and LSM pricing methods on an American-Bermudan Put Option, that's one of those proposed by Longstaff and Schwartz themselves in their masterpiece. The main features of this type of option are its exercisability any time before expiration (that's why American) but in a predefined big amount of exercise dates equally distant within each other (making it discrete, as a Bermudan option).

The LSM approach to American-style options' evaluation is based on imposing a very big number of exercise dates, as it's not actually possible to give a continuous possibility of exercise (which should technically be available for American options). This aspect makes the discrete American-style options evaluated through Longstaff-Schwarz method often termed as Bermudan-style options. The biggest is the amount of exercise dates made available and the closest the option gets to a pure American-style one.

2.3 Black-Scholes model

Fisher Black, Myron Scholes and Robert Merton are considered as pioneers in the field of the option pricing research. Thanks to the massive breakthroughs proposed through the model developed and published in the early 1970s, Scholes and Merton achieved to win the Nobel Prize for economic science in 1997, and Black got the mention for his contribution by the Swedish academy, as he died in 1995, resulting ineligible for the prize. The influence of Black-Scholes model on the way traders approach pricing and hedging options is evident and has its reflection in numerous subsequent research, as well as it gave several inputs to the growth in popularity and quality of financial engineering.

To follow, the framework of the Black-Scholes model will be defined, and the way the model can be implemented on the evaluation of a European call and put option (having the underlying stock paying zero dividends) will be illustrated.

In order to derive the Black-Scholes model, there are several explicit assumptions necessary to be made:

- There are no arbitrage opportunities.
- The underlying asset pays zero dividends.
- Securities are continuously available and possible to trade.
- It's always possible to buy any fraction of a share (Securities are always divisible).
- Short selling never undergoes restrictions.
- Transaction costs are not considered.
- The risk-free interest rate is constant and it's always possible to lend and borrow money instantly at that rate.
- The underlying asset's price follows a geometric Brownian motion with constant volatility and drift.

Successively some of these assumptions have been relaxed to extend the Black-Scholes model and make it more realistic.

Mainly, the model is based on the idea that the structure of the return on an option can be precisely replicated by a continuous rebalancing of a hedged portfolio composed by a risk-free asset gaining interests at a continuously compounded rate (which could be, for example, a government bond) and by a variable number of shares of the underlying asset. The return of such a portfolio depends exclusively on some known constant variables and on the time, as it is independent from the price movement of the stock. This return is deterministic, so, as it doesn't allow arbitrage opportunities, it cannot have an expected return, at the start, higher than the one on the initial investment compounded at the risk-free rate of interest (so the initial investment itself). In facts, the eventual arbitrage possibility would emerge by borrowing at a risk-free rate, moving the amount on the (eventually higher yielded) hedged portfolio, forcing the combination of the two assets to reach the risk-free condition, with a higher yield than the one to pay for the borrowing.

Before deriving the Black-Scholes pricing formulas, some notations used in this section will be defined:

- "S", the stock price.

- "f", the price of a derivative as a function of the stock price and of the time.
- "c", the European call option's price.
- "p", the European put option's price.
- "K", the strike price of an option.
- "r", the continuously compounded zero-risk interest rate annualized.
- "μ", the average growth rate of the stock annualized, that is the drift rate of S.
- "σ", the square root of the quadratic variation of the stock's price process, that is the volatility of the stock.
- "t", the time expressed in years; we generally adopt now = 0, at maturity = T.
- Π, the portfolio value.
- R, the accumulated profit or loss following a delta-hedging trading strategy.
- $N(x)$, the function of standard normal cumulative distribution, $\mathrm{N(x)} = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-\frac{x^2}{2}} dz$.

- "$N'(z)$", the derivate of the previous function, that represents the function of standard normal probability density, $\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$.

A Wiener process, also defined as Brownian motion, is a specific kind of Markov stochastic process having variance rate per year equal to 1, and a mean equal to zero. To be following a Wiener process, a variable z respects the next two properties:

First property: Given a little interval of time Δt, the equation for the change in Δz is:

$$\Delta z = \epsilon \sqrt{\Delta t} \tag{2.1}$$

being ϵ a standardized normal distribution $\phi(0, 1)$.

Second property: For any two different little interval of time Δt, the value of Δz is independent.

It is possible to define a generalized Wiener process in terms of dz as:

$$dx = adt + bdz \tag{2.2}$$

being a and b constant values.

A generalized Wiener process in which the constants a and b consist in being functions of the variables x and t, is called Itò process. The equation of this further kind of stochastic process is, then:

$$dx = a(x, t)dt + b(x, t)dz \qquad (2.3)$$

More specifically, the Itò process supposed to be followed by a variable x, is represented in the Itò's Lemma (K. Itò, 1951) involving a function G of x and t following the process:

$$dG = (\frac{\partial G}{\partial t} + \frac{\partial G}{\partial x}a + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\,)dt + \frac{\partial G}{\partial x}bdz \qquad (2.4)$$

The model assumes that the process of the stock price goes with the following geometric Brownian motion:

$$dS = \mu Sdt + \sigma Sdz \qquad (2.5)$$

Assuming f representing the price of a derivative (such a call or put option) having the stock priced by S as underlying asset, it must formally consist in a function of the price S and the time t. Applying the Itò's lemma from (2.4), the equation obtained is:

$$df = (\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\,)dt + \frac{\partial f}{\partial S}\,\sigma dz \qquad (2.6)$$

Converting the equations (2.5) and (2.6) in their discrete versions, we obtain:

$$\Delta S = \mu S\ \Delta t + \sigma S\ \Delta z \tag{2.7}$$

And

$$\Delta f = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\ \sigma\ \Delta z \tag{2.8}$$

Being ΔS and Δf the differences in values of S and f after a small time Δt. As the underlying of S is the same underlying of f, both these functions should follow the same Wiener process. So, formally, the Δz of (2.7) and the Δz of (2.8) should have the same value ($\epsilon\ \sqrt{\Delta t}$). This sets up the possibility of eliminating algebraically the Wiener process by composing a portfolio with the stock S and the derivative f.

The portfolio allowing this possibility is composed as follow:

$$\begin{cases} \dfrac{\partial f}{\partial S}, & \text{shares} \\ -1, & \text{derivative} \end{cases}$$

Meaning that the portfolio is made selling 1 unit of derivative for every $\frac{\partial f}{\partial S}$ of shares bought. The expression defining the price of this portfolio is, then:

$$\Pi = -f + \frac{\partial f}{\partial S} S \qquad (2.9)$$

The value's change of this portfolio $\Delta\Pi$ in the interval of time Δt results:

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S \qquad (2.10)$$

From the substitution of the equations (2.8) and (2.9) in the equation (2.10) we obtain:

$$\Delta\Pi = \left(- \frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2} \sigma^2 S^2\right)\Delta t \qquad (2.11)$$

This last expression evidences how the change in value $\Delta\Pi$ in the fraction of time Δt is not influenced by the stochastic process Δz, reducing to zero the level of risk for that amount of time. The portfolio, then, must gain immediately the same rate of return of the risk-free securities, for the assumption of zero arbitrage to hold, so:

$$\Delta\Pi = r\,\Pi\,\Delta t \qquad (2.12)$$

Being r the zero-risk interest rate. From combining the expressions (2.8), (2.9) and (2.12) it comes:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} r\,S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 = r\,f \qquad (2.13)$$

Next, the Black-Scholes PDE (partial differential equations) for a specific valuation of an option will be illustrated. The conditions determining the European call option's value c, are:

- $c(S, T) = \max(S - K, 0)$
- $c(S, t) \to S$ as $S \to \infty$
- $c(0, t) = 0$ for all t

The conditions determining the European put option's value p are similar to the call option's ones. First, the Black-Scholes PDE will be transformed in a diffusion equation, and consequently, the equations will be solved adopting the standard methods. So, being

$d1 = [\log_e(S_0/K) + (r + \sigma^2/2)Tx] / (\sigma \sqrt{T})$

$d2 = [\log_e(S_0/K) + (r - \sigma^2/2)Tx] / (\sigma \sqrt{T}) = d1 - \sigma \sqrt{T}$,

the values of the call and the put option will result to be:

$$c = S_0 \, N(d1) - Ke^{-rT} N(d2) \qquad (2.14)$$

$$p = Ke^{-rT} N(-d2) - S_0 \, N(-d1) \qquad (2.15)$$

In the article "The Pricing of Options and Corporate Liabilities", Black and Scholes (1973) produced a model that represents a milestone for the research in the field of option pricing, which is still widely used. The formula adopted to price European options, requires accepting several conditions. One assumption necessary to be made is that markets don’t produce arbitrage opportunities, as they denote how the return of a portfolio composed by a risk-free asset gaining interests at a continuously compounded rate and by a variable number of shares of the underlying asset, can perfectly replicate the return of the option itself. Merton, in one of the most relevant extensions of the Black-Scholes model, defined how there are no differences between the values of American options and European options having the same underlying assets, the same expiration date, and the same strike price, when the underlying asset doesn’t pay dividend. This because he found it never optimal to exercise the American option before the maturity. That’s one of the few cases in which, adopting this method, there has been a representation of a closed form solution for American options’ evaluation.

The fact of the Black-Scholes model imposing very strict assumptions to make it possible to rely on closed form solutions, often makes this method not appropriate for dealing with real market's features, bringing practisers to approximate the values through numerical methods instead of looking for theoretical values. In facts, in the financial industry, the implementation of numerical techniques for pricing options increased exponentially by the time, as these became more popular and sophisticated in academic research. In the section 2.4 it will be introduced the numerical method approached in this thesis: the binomial tree method.

2.4 Binomial Tree Model

As it has precedingly been illustrated, the Black-Scholes model requires numerous and strict assumptions to be imposed, making it often not completely adequate for the real market conditions. There have been many attempts to reduce or change many of these assumptions in many extensions of the Black-Scholes model, but none of them brought to available closed form solutions. As a solution, many practisers started looking for the approximated values involving numerical methods, rather than keeping looking for the theoretical values. In this Paragraph, the numerical method of the Binomial Tree will be defined, since it will be applied, then, as a benchmark method for the evaluation of American-Bermudan options' prices successively approached through the LSM method. This method, as well as the Finite Difference method, has been typically approached to solve the issue of giving a closed form solution for the evaluation of American Options and many other derivatives giving the holder the possibility to exercise before maturity, which couldn't be directly derived using the Black-Scholes method. Basically, most of the derivative types can be priced through this method, even though sometimes this has some issues to deal with can come out.

Cox, Rubinstein, and Ross were the first formalizing the Binomial pricing model for derivatives, in their paper "Option Pricing: A Simplified Approach", in 1979, after William Sharpe first proposed it as a concept in 1978.

This model consists in the representation in a diagram of the possible different paths available for the asset price over the time to maturity. The assumption that the underlying asset's price follows a random walk holds. For every time step, there is a defined probability that the price goes up by a certain percentage amount and a defined probability that the price goes down by a certain percentage amount. Making the time steps the smallest possible, the limit of the binomial tree brings to the lognormal assumption for the price of the stock, that's the same assumed for the Black-Scholes Model.

To follow the pricing mechanism related to the probability for any step:

Defining S_t the value of the stock at time t, it's price at next time step $t+1$, is going to be $S_{t+1} = uS_t$ with probability p, and $S_{t+1} = dS_t$ with probability $1-p$, being $u > 1 > d$.

Considering a call option on the stock expiring at the end of the actual time step, let f be the function representing its current value, S_0 be its actual price, and K be its Strike Price. From before, we obtain $f_u = \max(S_0 u - K, 0)$ and $f_d = \max(S_0 d - K, 0)$.

Let's consider a portfolio composed by a long position in a Δ number of shares for each unit of options in a short position. We'll extrapolate the number of shares Δ necessary to make a zero-risk portfolio.

When the price increases, the portfolio is worth

$$S_0 \, u \, \Delta - f_u$$

When the price decreases, the portfolio is worth, instead

$$S_0 \, d \, \Delta - f_d$$

To make a riskless portfolio, the last equations must be equal

$$S_0 \, u \, \Delta - f_u = S_0 \, d \, \Delta - f_d$$

Consequently

$$\Delta = \frac{fu - fd}{S0u - S0d} \qquad (2.16)$$

This situation shows a riskless portfolio, since it keeps the same value independently by the direction of the stock price. Because of the Assumption of No-Arbitrage, the growth of the portfolio's value must immediately follow the risk-free rate r. So

$$S_0 \;\; \Delta - f = (S_0 \, u \, \Delta - f_u) \, e^{-rT}$$

Isolating the option price f

$$f = S_0 \, u \, \Delta(1 - u \, e^{-rT}) + f_u \, e^{-rT} \qquad (2.17)$$

Importing the value of Δ from (2.16) and evaluating, the price becomes

$$f = e^{-rT}[\ p\ f_u + (1 - p)\ f_d\ ] \tag{2.18}$$

being

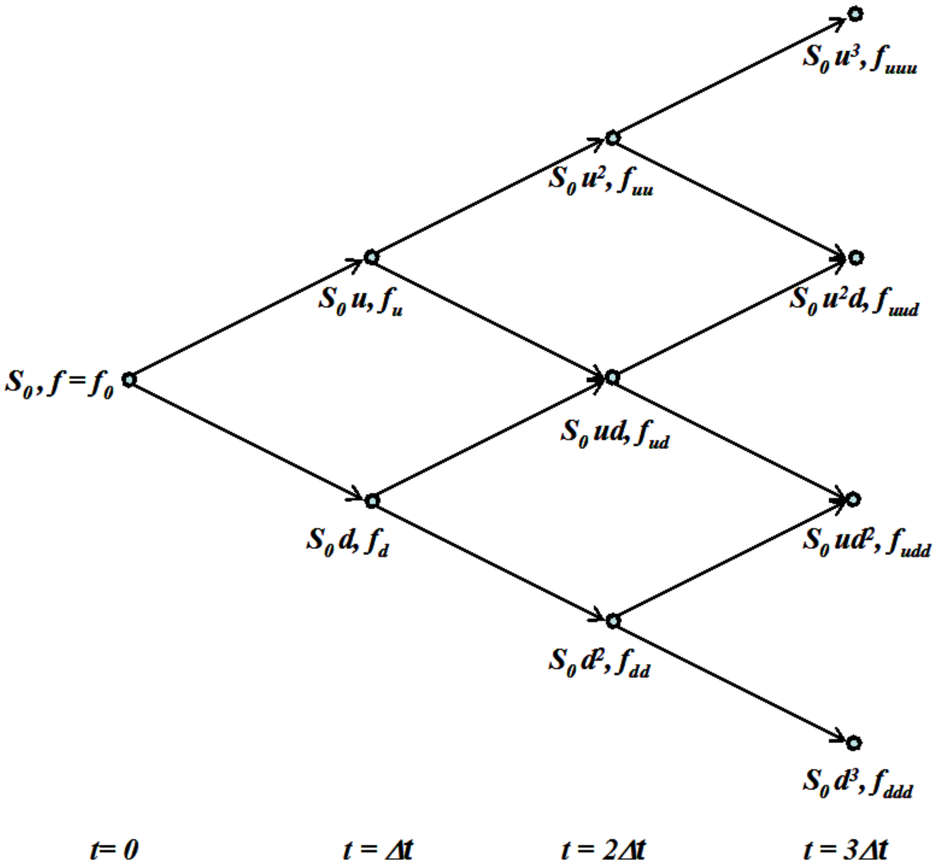
$$p = \frac{e^{-rT} - d}{u - d} \tag{2.19}$$

At this point, it’s noticeable how the probability of the stock price to move up or down is not involved in the pricing formula for option in (2.18). This aspect is also maintained when the amount of time steps in considered in the development of the price paths increases. The more time nodes there are in a binomial model, the more accurate the approximation to the real stock price moving can result.

Typically, a satisfying approximation can be obtained by developing the binomial tree on 30 or more timesteps. In facts, imposing 30 time-steps propose 31 final stock prices, and 2^{30} possible price paths to follow for the asset: more than one billion.

To follow, an exemplificative representation of a binomial tree developed on three time-nodes will be illustrated.

Figure 2.1: Binomial tree on three time-nodes



The graph above shows how the price increases or decreases for each time step.

For the evaluation of the option's price, starting from the values obtained at the end of each path, the same pricing formula of the single-node binomial model (2.18) needs to be applied in each node of the last binomial tree (fig. 4.1). It's required a backward induction from the last options' prices.

$$f = e^{-r3\Delta T}[\, p^3 f_{uuu} + p^2 (1 - p) f_{uud} + p (1 - p)^2 f_{udd} + (1 - p)^3 f_{ddd}]$$

We can then formalize the value for any model at n multiple nodes

$$f = e^{-rn\Delta T}[\, p^n f_{uu\ldots uu} + p^{n-1} (1 - p) f_{uu\ldots ud} + \ldots + p (1 - p)^{n-1} f_{ud\ldots dd} + (1 - p)^n f_{dd\ldots dd}] \quad (2.20)$$

Being ΔT the time running from a node to another.

The formula above holds for European Options, as it doesn't consider the influence on the price of the possibility to early exercise the option.

In facts, to adapt the model to American Option pricing it's required to check at each node whether it's profitable to exercise, and to adjust the price of the option accordingly.

For an American call, the value of the option at a node is given by

$$f_{xx} = = \max (K - S_{xx,}\ e^{-r\Delta T} [p\, f_{xxu} + (1\text{-}p)\, f_{xxd}] \quad (2.21)$$

So, if the value of the payoff gainable exercising the option is greater than the value of the option itself, we assign that price for the

American option at that node. Otherwise, we assign the value the option would have when unexercised.

When a node shows an option price higher than it would be without exercising, it influences the prices of the nodes following back through the tree. This would make an American option to be more valuable than a European option with the same characteristics.

This basic model has been successively updated with many extensions due to its ductility and simplicity. One of the most famous updates has been proposed by White and Hull (1990) importing in the model the dividends payment and some multivariate evaluation problems.

CHAPTER 3

Simulative methods for option pricing

3.1 Monte Carlo Simulation.

Monte Carlo simulation is usually applied for derivatives where the payoff is dependent on the history of the underlying variable, or for derivatives with multiple factors of uncertainty or other complicated features. In facts, as we precedingly stated, one of the main issues with numerical methods like binomial trees is that they consider the price of the underlying asset as the only stochastic factor in the valuation of the option, while the other relevant variables are assumed to be constants. In order not to have computational issues related to the curse of dimensionality, the development of an accurate simulative approach seemed to be necessary.

Regarding some historical references, Stanislaw Ulam coined the term “Monte Carlo method” in the late 40s. Then, Phelim Boyle resulted to be the pioneer of the application of this simulative approach for pricing options in his article “Options: a Monte Carlo approach” (1977). The possibility to apply this technique on options with multiple stochastic factors led the evaluations of European Options’ price through this method to become popular by the years, while it kept being hard to implement it on American Options, because of computational issues. In facts, it’s required to calculate the optimal early exercise price at a certain time, recursively, in the process of pricing American Options. Since it’s just one single future path for any time spot, this calculation would show biased results. From that moment, many studies came over trying to propose a way to price American Options through simulation. James Tilley suggested to define the optimal stopping time point along the paths through a simulation algorithm imitating the standard lattice.

With a similar approach, Martineau and Barraquand (1995) proposed their method, the so-called Stratified State Aggregation along the Payoff. In the fourth chapter of this thesis, we will introduce a development for Monte Carlo simulation, useful in pricing American-style options.

Turning our attention on the illustration of the method, we'll take in exam the calculation of the following integral to get the fundamental idea of the Monte Carlo simulation

$$\int_A g(x)f(x)dx = \bar{g} \tag{3.1}$$

Identifying *g(x)* as an arbitrary function, *f(x)* as a density function of probability, and *A* as the range of integration. An estimation of $\bar{g}$ through Monte Carlo can be obtained generating an independent and identical distributed sample $\{x_1, x_2, \ldots, x_{n-1}, x_n\}$ from $f(x)$ and developing

$$\hat{g} = \frac{1}{n}\sum_{i=0}^{n} g(x_i) \tag{3.2}$$

The value of the variance $\hat{s}^2$ of $\hat{g}$ is

$$\hat{s}^2 = \frac{1}{n}\sum_{i=0}^{n} (g(x_i) - \hat{g})^2 \tag{3.3}$$

Being

$$\frac{\hat{g}-\bar{g}}{\sqrt{\hat{s}^2/n}} \to N(0, 1) \tag{3.4}$$

When $n \to \infty$, it's possible to obtain the confidence intervals.

This approach has been used by Boyle (1977) for pricing a European option having the underlying asset following a geometric Brownian motion. It appears necessary to re-propose from the equation (2.5) the assumption that, being dz a Wiener process,

$$dS = \mu S dt + \sigma S dz \tag{3.5}$$

For the path of S to be simulated, the life of the option needs to be divided in N intervals of time Δt with very short length. Then, the equation (3.5) must be approximated by

$$S(t + \Delta t) - S(t) = \mu S(t) + \sigma S(t)\epsilon\sqrt{\Delta t} \tag{3.6}$$

Being ϵ a random sample from a normal distribution $N(0,1)$.

It's then possible to construct a path for the stock price S repeating this process using N random samples from $N(0,1)$.

Anyway, calculating $\ln S$ rather than S would generally bring to a more accurate approximation. Differently from S, the stochastic process by Itò's lemma followed by $\ln S$ results as

$$d \ln S = (\mu - \frac{\sigma^2}{2})dt + \sigma dz \tag{3.7}$$

For short time intervals Δt, as for dS in the equation (3.6), d ln S can be discretely approximated as follows

$$\ln S(t + \Delta t) - \ln S(t) = (\mu - \frac{\sigma^2}{2})\Delta t + \sigma\epsilon\sqrt{\Delta t} \qquad (3.8)$$

Automatically, for all T, the equation (3.8) results as follows

$$\ln S(T) - \ln S(0) = (\mu - \frac{\sigma^2}{2})T + \sigma\epsilon\sqrt{T} \qquad (3.9)$$

Developing it to obtain a function of S in T, we get

$$S(T) = S(0) \exp[(\mu - \frac{\sigma^2}{2})T + \sigma\epsilon\sqrt{T}] \qquad (3.10)$$

The estimations of the values of the stock price S at any time T provided by the equation (3.10) constructs the path for the stock price.

An aspect of this type of estimation is that $\sqrt{n}$ has inverse proportionality with the standard error. Aiming at improving the accuracy of the simulation reducing the size of the standard error, Boyle (1977) introduces antithetic variates and control variates. The possibility of Monte Carlo simulation to handle the problem of the payoff depending on both the paths followed by the underlying asset S represents the key advantage on numerical methods like Binomial Tree, which could show more problems in the application on such an environment. Even though another major drawbacks of Monte Carlo simulation consist in this being computationally time consuming sometimes, making it hard to apply on American-style options (issue that this thesis aims to solve in Chapter 4, through the evaluation by Least Squares method), there are many examples of the application of the former advantage showed in successive researches: one of the most relevant for this thesis comes from Paul Glasserman and Mark Broadie (1996), who explained how to price Asian Options through Monte Carlo simulation.

3.2 Application on European option

As defined in the previous section (3.1), there are

$$d \ln S = (\mu - \frac{\sigma^2}{2})dt + \sigma dz \tag{3.11}$$

and

$$S(T) = S(0) \exp[(\mu - \frac{\sigma^2}{2})T + \sigma\epsilon\sqrt{T}] \tag{3.12}$$

Let's make the function $f(S,t,K,T)$ represent the value of an option at time t having underlying asset worth S, strike price K and expiration at time T. Naturally, the value of such a function at time 0 will have to be equal to its expected value at maturity discounted at the interest rate r. So

$$f(S,0,K,T) = e^{-rT}\,\mathbb{E}g(S(0)\exp[(\mu - \frac{\sigma^2}{2})T + \sigma\epsilon\sqrt{T}] \quad (3.13)$$

g(x) represents the payoff function. The value of g(x) for a European call option is

$$g(x) = \max(S(T) - K, 0)$$

Denoting an amount M of independent and identically distributed samples generated from N(0,1) as $\{y_1, y_2, \ldots, y_{M-1}, y_M\}$, the value of the option f will be, by the Law of Large Numbers

$$f(S,0,K,T) = \frac{1}{M} e^{-rT} \sum_{i=0}^{M} g(S\,(0)\exp[(\mu - \frac{\sigma^2}{2})T + \sigma\, y_i \sqrt{T}]) \quad (3.14)$$

Before proceeding with the execution of the simulation on a European call option, although the normal distribution N(0,1) is provided as a black-box function in almost every software, we will now briefly illustrate how to generate pseudo-random numbers, proving how it's possible to convert random numbers sampled distributed between [0,1] in numbers sampled from the normal distribution N[0,1].

For a random sample N(0,1) to be generated, we could start from generating a random sample distributed between [0,1]. In facts, statistical randomness doesn't imply necessarily "real" randomness, that is objective unpredictability. In many cases it's sufficient to use *Pseudorandomness*. The function generating a pseudorandom number is provided by most of the programs, and it's typically based on the linear congruential generator that, in order to generate these numbers, uses the recurrence

$$Xn+1 = (aXn + b) \bmod \mathrm{m}$$

Using the theorem below, it is possible to transfer a series of samples supposed to be independent and uniformly distributed between 0 and 1 into a series of samples extracting from the normal distribution N(0,1)

Theorem: Imposing U_1 and U_2 independent and uniformly distributed between 0 and 1, defining

$$X1 = \sqrt{-2 \ln \mathrm{U}_1} \cos(2\pi U_2), Y1 = \sqrt{-2 \ln \mathrm{U}_2} \sin(2\pi U_2)$$

Automatically, X_1 and Y_1 are independent and the distribution of both is of $N(0, 1)$.

Demonstration: first, we need to express and prove the following two lemmas are necessary

Lemma 1: Assuming (X_1, X_2) and (Y_1, Y_2) to have the same distribution, and $g(x_1, x_2)$ and $h(x_1, x_2)$ to be the 2-dimension real functions, define:

$$\begin{cases} Z1 = g(X1, X2)\,, \\ Z2 = h(X1, X2)\,, \end{cases}$$

$$\begin{cases} W1 = g(Y1, Y2)\,, \\ W2 = h(Y1, Y2)\,, \end{cases}$$

Automatically, (Z_1, Z_2) and (W_1, W_2) will have the same distribution as well.

Demonstration: This lemma can be proved assuming (X_1, X_2) and (Y_1, Y_2) to have a joint distribution density function $f(x, y)$.

$$P(Z1 \le z, Z2 \le w) = P(g(X1, X2) \le z, h(X1, X2) \le w)$$

$$= \int R2\ I\{g(x, y) \le z, h(x, y) \le w\} f(x, y) dxdy$$

$$= P(g(Y1, Y2) \le z, h(Y1, Y2) \le w)$$

$$= P(W1 \leq z, W2 \leq w) \tag{3.15}$$

Thus, (Z_1, Z_2) and (W_1, W_2) have the same distribution.

Lemma 2: Supposing X and Y to be independent and to be distributed in $\mathrm{N}(0,1)$, and supposing (R, Θ) to be determined by the following polar coordinates transformation:

$$\Delta\colon \begin{cases} \mathrm{X} = \mathrm{R}\cos(\Theta), \\ \mathrm{Y} = \mathrm{R}\sin(\Theta), \end{cases} \tag{3.16}$$

In such a case, R has the Rayleigh Distribution, and Θ is uniformly distributed between $[0, 2\pi]$.

Demonstration: The joint distribution density function of (X,Y) is $f(x, y) = \frac{1}{2\pi}\exp(-\frac{x^2+y^2}{2})$. Also, for (R, Θ) the range is

$$\{(r, \theta) | r \geq 0, \theta \in [0, 2\pi]\}$$

Being α the angle of amplitude for (x, y), we define the set $D = \{(x, y) | \sqrt{x^2 + y^2} \leq r, \alpha \in [0, \theta)\}$. Under the transformation Δ, $\{R \leq r, \Theta \leq \theta\} = \{(X, Y) \in D\}$. So, it's noticeable how the joint distribution function for (R, Θ) can be

$G(r, \theta) = P(R \leq r, \Theta \leq \theta)$

$= P((X, Y\ ) \in D)$

$= \int_D \frac{1}{2\pi} \exp(-\frac{x^2+y^2}{2})\mathrm{dxdy}$ (let $x = t \cos \alpha, y = t\sin \alpha$)

$= \frac{1}{2\pi} \int_0^{\theta} \mathrm{d\alpha} \int_0^{r} \exp(-\frac{t^2}{2})\mathrm{dt}$

$= \frac{\theta}{2\pi} \int_0^{r} \exp\left(-\frac{t^2}{2}\right) \mathrm{t\ dt}$

Since $G(r, \theta)$ is continuous, and it's derivable whereas its derivate is defined (always except for limit linear lines), using the following derivative we will obtain the joint distribution density function for (R, Θ):

$g(r, \theta) = \left(-\frac{\partial^2}{\partial r\, \partial \theta}\right) \mathrm{G}(r, \theta) = \frac{1}{2\pi}\exp(-\frac{r^2}{2})$

Being $r \geq 0, \theta \in [0, 2\pi)$.

In $g(r, \theta)$ the variables are divided. Consequently, R and Θ are independent and have respectively the density function below:

$$gR(r) = \exp(-\frac{r^2}{2})I_{[0,\infty)} \tag{3.17}$$

$$g\Theta(\theta) = 1\ 2\pi\ I_{[0,2\pi)} \tag{3.18}$$

These last two equations show R to have the Rayleigh distribution and Θ to have the $[0, 2\pi)$ uniform distribution.

Imposing $R_1 = \sqrt{-2 \ln \mathrm{U}_1} = 2\pi U_2$, its distribution function results to be

$$\mathrm{F}(r) = P(R_1 \leq r)$$

$$= P(\sqrt{-2 \ln \mathrm{u}_1} \leq r)$$

$$= P(u1 \geq \exp(-\frac{r^2}{2})) \tag{3.19}$$

$$= \int_{\exp\left(-\frac{r^2}{2}\right)}^{1} \mathrm{du}_1$$

$$= 1 - exp\left(-\frac{r^2}{2}\right) (r \geq 0)$$

The density function of R_1

$$f(r) = F'(r) = r\ exp\left(-\frac{r^2}{2}\right) \tag{3.20}$$

From the last equation we can notice R_1 to have the Rayleigh distribution too. Consequently, (R_1, Θ_1) has the same distribution of (R, Θ) in equation (3.16). (X_1, Y_1) and (X, Y) in (3.16) will have the same distribution too, according to the Lemma. This means that X_1

and Y_1 are proved to be independent and to have the normal distribution $N(0, 1)$.

After this illustration of the framework of Monte Carlo simulation for pricing European options, an example of its application will be provided.

Successively the accuracy and efficiency of this method will be discussed. We will use the Black-Scholes model as the benchmark method, and we'll compare the true value coming from it with the results from Monte Carlo simulation.

3.3 Numerical Results

In the following example, we will give to each variable a predefined value, and from that we will simulate the payoff value of a European call option.

For simplicity, the expiration date of the option will be at T=1 year. The payoff of the option will be evaluated from 5 different starting prices of the underlying stock, from 26 to 34. The strike price will be

30, the volatility of returns σ of the underlying will be equal to 0.4, and the risk-free interest rate will be $r=0.06$.

As already illustrated, there will be 5 different values of S. For every S, we will run different amounts of paths for 5 different times. The first execution will simulate 10 paths for each of the 5 prices, the second will simulate 100 paths, the third 1000 paths, and the last 10000 paths. The aim of simulating different sizes of samples is to prove how the accuracy of this method increases proportionally with the amounts of paths. In facts, the payoff values will be then compared to the ones obtained through Black-Scholes pricing model, chosen as the benchmark method for the evaluation of European options because of its simplicity and accuracy. The code for the program executing the simulation and the benchmark method will be presented at the bottom of this thesis. The results are in the table that follows.

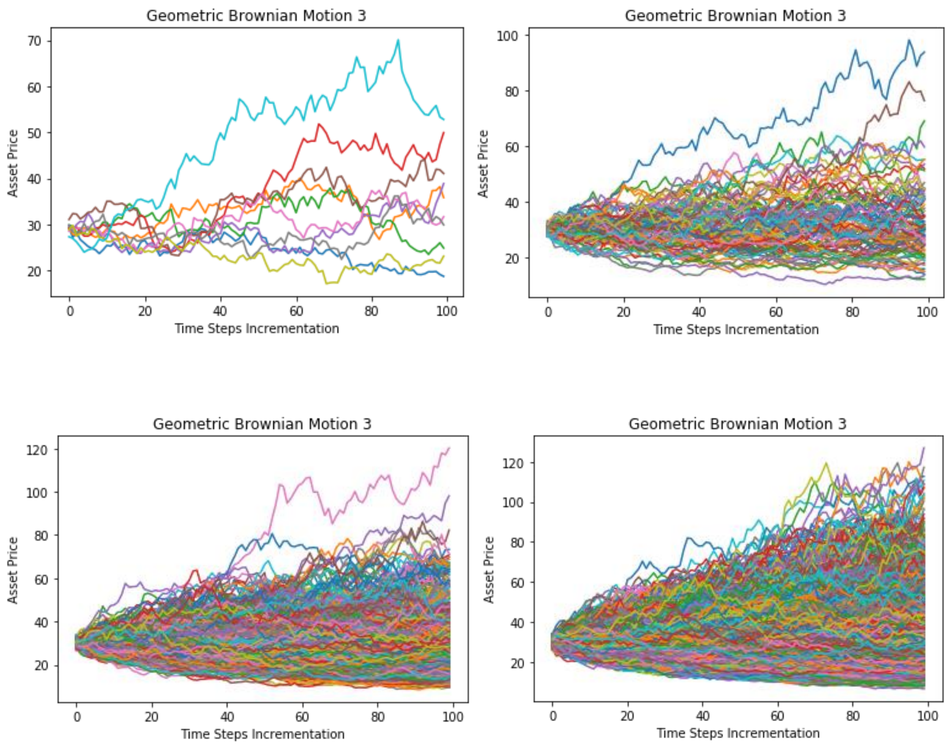
Table 3.1: European Call option with Monte Carlo Simulation

S	26	28	30	32	34	s.e.	av. bias	%bias
B-S	3.265	4.333	5.542	6.875	8.318			
N								
10	4.059	5.188	6.064	5.851	9.835	0.126	0.942	3,33%
100	3.730	5.274	5.730	6.801	7.162	0.040	0,565	1,99%
1000	3.217	4.599	4.741	7.135	8.800	0.013	0,371	1,31%
10000	3.306	4.419	5.473	6.976	8.464	0.004	0,089	0,31%

The table above shows how the Monte Carlo simulation method results to be a valid option in pricing European options, unless the number of paths simulated are too low. In facts, it's noticeable how the prices of the Option evaluated with Monte Carlo gets closer to the *true* value of the option, evaluated with Black-Scholes, as the number of paths simulated increases. For example, for the simulations with 10 paths, we see how the difference between the two values is always bigger than 10% for each of the 5 starting prices S_0, which is unacceptable. For the simulations with 100 paths, we can notice how the difference between a simulated price and a true one can abundantly exceed 10% (as it does for the evaluation of the payoff when $S_0=26$), but for two starting prices S_0 out of five ($S_0=30$ and $S_0=32$) is lower, indicating an improving trend. Not by chance, when executing the evaluation simulating 1,000 different paths for each starting price S_0, only for two starting prices S_0 out of five ($S_0 = 28$ and $S_0 = 30$) the differences between the true expected payoff and the simulated one exceeds 5%. Finally, 10,000 paths simulated seem to be enough to obtain an acceptable approximation. In facts, in this case, for all the 5 evaluations the differences between the simulated option price and the true one is less than 2%.

In the figure below we represent the change of the geometric Brownian Motions as the number of paths simulated increases. We take in example the simulation behaviour when evaluating the option price for $S_0 = 30$. As the number of paths simulated increases, we can notice how the possible area covered by the asset price gets more clear and foreseeable.

Figure 3.1: 4 Brownian motions development increasing N



The illustration of Monte Carlo above regards the simplest of the option types, as the aim was to stress the mechanism of this method in the clearest way possible. It's also possible to apply this method for more complex tasks. In facts, it can be used when the number of random variables increase (e.g., it can consider variables correlated with various sources of risk as inflation or volatility of commodities), and generally when compounding in the uncertainty. It can also be applied when the option shows more complex features, like having the payoff related to the average of the underlying price (Asian options), or having the payoff depending on several underlying assets (Basket options, Rainbow options, etc.) when the simulation considers the correlation.

However, although many techniques for reducing the computations can be applied (e.g., variance reduction techniques), Monte Carlo method is quite time consuming, which makes numerical methods to be preferable alternatives in some cases.

In the next chapter we will illustrate and apply a development of Monte Carlo simulation method useful for pricing Options granting the right of early exercise. More specifically, the framework of the LSM will be exposed approaching an American-Bermudan option, benchmarking its values with the ones obtained using binomial tree method.

CHAPTER 4

Least Squares Method for options pricing

In chapter 2 and chapter 3 we defined the basis necessary to introduce the simulative method for pricing options of different types, which is what this chapter is going to face. In facts we've seen an example of Monte Carlo simulation algorithm for pricing European options, and we discussed its utility comparing it to the Black-Scholes method precedingly defined. In this chapter we will develop this work implementing and discussing the application of the simulative method on American-Bermudan options. Regarding the benchmark method for the discussion of the simulation, we will apply the Binomial tree model defined in chapter 2, as it's well suited for valuing options with early exercise, although it's often considered difficult to implement when the elaboration of multiple variable factors is required.

We've noticed how Monte Carlo simulation results to be strong in handling the evaluation of path-dependent factors and multiple factors, and regarding the uncertainties related to its application on options granting early exercise, we will see in this chapter how the Least Square Monte Carlo algorithm (LSM) developed by Longstaff and Schwartz in 2001 could represent a valid instrument for this task.

The main difference to consider between the evaluations of European and American options, for Monte Carlo simulation to be used, is that the European option requires only the last value of the simulated path, while the American one needs all of it.

Since the advantages of many numerical methods are combined in LSM, many types of derivatives under a universal class of price dynamics can be approached through this method. Another advantage of this method is that its implementation is simple and efficient. All these aspects explain how the popularity of this technique among traders and academicals increased over the time.

4.1- The LSM Algorithm

To follow, the framework of the Least Square Monte Carlo method will be defined and implemented on an American option discretized in its dates of exercise, so to give the clearest idea of the dynamics of the model.

As previously said, American options typically grant to the holder the possibility to exercise any time before maturity, while European options don't. For every exercise point, in order to define a fair value of the option, it's required to decide whether to exercise it or to keep it. The determination of the exercise value at every time-step is intuitive: for American Put options it's max (K-S, 0), while for American Call Options it's max (S-K, 0). The computation of the continuation value, instead, represents the key issue for the determination of the option value. Between the various ways to determine the continuation values proposed by academicals and researchers, the Least Square Monte Carlo Algorithm (LSM) exposed by Longstaff and Schwartz (2001) results to be one of the most successful.

In order to determine the best-fit relationship between the value of exercise and the value of continuing for each time step, the use of the least square analysis is involved in the LSM algorithm. The nature of this algorithm is iterative, and the construction it executes of the estimated expected value of the American-style option develops time-step by time-step, with the condition of the option not to have been exercised before each time-step of the evaluation. The estimation of such a conditional expectation is obtained through linear regression.

Getting more in depth, the first step consists in generating, under adequate price dynamics, a group of sample paths. Successively, as we said before, the interval of possible exercise times, which should be continuous for all the life of the option as by definition of American options, will be instead approximated with a discrete set of time points. Then, all the discounted future payoffs realized (by exercising the option) will be regressed on functions of the state which can variate at each of the time steps. Applying the dynamic programming principle, we can obtain a complete estimation of the strategy of optimal early exercise. This principle implies that the option should be exercised as soon as both it's in-the-money (having payoff higher than 0) and it has an estimated value of conditional expectation of continuation lower than the value it would have when exercising immediately. The option's estimated value results to be, then, the estimated expected payoff discounted at time 0.

In order to explain efficiently the mechanism behind the LSM algorithm, it will follow the illustration of a simple numerical example. In table 4.1 (see below) we simulate 10 paths for a stock price running on three different time steps and having current Stock price $S_0 = 30$ at time 0. The aim is to evaluate the price of an American Call option having such a stock as underlying, exercisable at time t=1 and at time t=2, at a strike price K = 30. We suppose in facts, for simplicity, that the stock price has the same characteristics of the one we considered in pricing the European Call option in Chapter 3, recycling that code to obtain the values needed in this example. So, the volatility is $\sigma = 0.4$ and the risk-free interest rate is $r=0.06$. After developing the 10 paths by printing the values at time 1 and at time 2, we obtain the following table:

Table 4.1: Simulations of 10 Paths running through 2 Exercise dates.

Path	t=0	t=1	t=2
1	30	29.02	28.70
2	30	29.76	31.79
3	30	30.74	31.20
4	30	30.07	31.86
5	30	30.59	31.08
6	30	29.51	31.03
7	30	29.88	28.59
8	30	29.57	29.07
9	30	31.74	31.03
10	30	28.18	31.26

At time $t=2$, to identify the optimal strategy it's simply necessary to exercise when $S_2 > 30$, so that the option is in-the money. The cash flow realized approaching the optimal strategy at time $t=2$, conditional on the fact that the option doesn't get exercised before time 2, is represented in the following table.

Table 4.2: Payoff for call option from table 4.1 in t=2

Path	t=0	t=1	t=2
1	-	-	0
2	-	-	1.79
3	-	-	1.20
4	-	-	1.86
5	-	-	1.08
6	-	-	1.03
7	-	-	0
8	-	-	0
9	-	-	1.03
10	-	-	1.26

Now we will get the values for time t=1. Here the option holder needs to decide whether to exercise the option or to keep it until the next time spot. That is if the path is in-the-money, otherwise it becomes not relevant to choose, since exercising would be useless as the option's payoff is zero at that time (which is the minimum value possible in any case). From this simulation, there result to be 4 paths out of 10 in-the-money at time t=1 (paths 3;4;5;9). In the table below we define as $\tilde{s}(1)$ the stock price at time t=1 when in-the-money, and as $\tilde{y}(1)$ the discounted value (at r=0,05 and Δt=0,01, since in the code we compute 100 time-steps per year) of the payoff at time t=2 for the corresponding path if the option is not exercised.

Table 4.3: Exercise payoff vs Continuation value in t=1

Path	$\tilde{s}(1)$	$\tilde{y}(1)$
1	-	-
2	-	-
3	30.74	$1.20e^{-0.0005}$
4	30.07	$1.86e^{-0.0005}$
5	30.59	$1.08e^{-0.0005}$
6	-	-
7	-	-
8	-	-
9	31.74	$1.03e^{-0.0005}$
10	-	-

Regressing the discounted cash flow $\tilde{y}(1)$ to a function of the current stock price $\tilde{s}(1)$ and to a constant, it is possible to estimate the conditional expected value of the continuation to time 2. So,

$E[Y\ |S(1) = s(1)] =$ -0,0145782235 s(1)+ 0,490923969 .

When $\tilde{s}(1) \geq 1{,}33$ the payoff from immediate exercise is higher than the conditional expectation of continuing, making it convenient, in our case, to exercise the option at time 1 only for path 9.

Table 4.4 will show the cash flow matrix correspondent to this case.

Table 4.4: Cash flows

Path	t=1	t=2
1	-	-
2	-	1.79
3	-	1,20
4	-	1.86
5	-	1.08
6	-	1.03
7	-	-
8	-	-
9	1.74	-
10	-	-

The average of all values discounted at their current value is the price of the option. So

$$c = \frac{(1.74)e^{-0.0005}+(1.79+1.20+1.86+1.08+1.03+1.03)e^{-0.001}}{10} = 0.973$$

Let's analyze the mechanism of this method technically deeper.

Assuming a horizon of time [0, T] and an underlying probability space $(\Omega, \mathcal{F}, P)$. All the possible developments of the stochastic economy on [0,T] are set in Ω, that has ω as its typical element. P is the measure of the probability defined on the sets (Ω) in $\mathcal{F} = \mathcal{F}T$, that is the σ-field of all the different events possible in the time horizon [0,T]. The associated price dynamic generates the filtration represented by $\mathbb{F} = (\mathcal{F}t\,; t \in [0, T])$. It's possible to assume that a measure Q of the risk neutral probability exists, if the conditions of the no-arbitrage paradigm hold.

The aim of this definitions is to evaluate the price of an American-style option generating random cash-flows in [0,T]. The fair price of the option is evaluated maximizing the expected value of all the discounted cash flows, being the research of the maximum extended over all the different time steps with respect to $\mathbb{F}$ (as cash flows can happen every time when dealing with American options). The pricing paradigm of no-arbitrage holding would also imply that expected value of the cash flows that remains with respect to Q is equal to the value of continuation, when following the rule of optimal stopping. Let's impose using L discrete times $0 < t1 \leq t2 \leq ... \leq tL = T$ to approximate the continuous time interval of possible spots of early exercise. The function of the expected value can be expressed as follows

$$G(\omega;tl) = E_Q[\sum_{j=l+1}^{L} \exp(-\int_{t1}^{tj} r(\omega, u)du)\ CF(\omega, t_j ;t_l , T)|\mathcal{F}t_l] \quad (4.1)$$

The option generates the path of cash flows denoted as $CF(\omega, t_j ;t_l , T)|\mathcal{F}t_l]$, which is conditional on the fact that the option doesn't get exercised at t or before. Moreover, $r(\omega, t)$ is the zero-risk interest rate, and the holder of the option is assumed to be following at all time s, $tl \leq s \leq T$ the optimal exercise strategy.

As suggested by the name, in order to estimate the function of the conditional expectation at each of the possible exercise time spots, LSM adopts least squares regression. Going more in depth, we'll assume, at time t_1, that the functional form of $G(\omega;t_l)$, which is unknown, could be expressed as a countable linear combination of $\mathcal{F}_{tl}$-measurable basis functions. Since the type of derivatives we're focusing on have random payoffs with finite-variance, the space of square integrable functions will include the conditional expectation function $G(\omega;t_l)$, under the appropriate measure. $G(\omega;t_l)$ can be expressed as a countable linear combination of elements of the countable orthonormal basis present in this Hilbert space.

Substantially, $M < \infty$ elements from such a basis are used by LSM for the approximation. We'll define it $G_M(\omega;t_l)$. When the number of sample paths gets closer to infinity, the conform value from the regression $\hat{G}_M(\omega;t_l)$ converge to $G_M(\omega;tl)$ in mean square and in probability. That's implied by the fact that there are too weak assumptions regarding the existence of moments.

The LSM algorithm, as previously mentioned, holds the idea that least-square regression approximates the conditional expectation for each exercise date. It is possible to express the conditional expectation function at time $t= t_{L-1}$ $G_M(\omega;t_{L-1})$ as a linear combination of various orthonormal basis functions ($p_j(X)$) like the Hermite polynomial, the

Legendre's, the Laguerre's, the Chebyshev's, the Jacobi's, or the Genbauer's one. The function would be, then

$$G(\omega;t_{L-1}) = \sum_{j=0}^{\infty} a_j p_j(X),\ a_j \in \mathbb{R} \tag{4.2}$$

That is, after the approximation previously defined

$$G_M(\omega;t_{L-1}) = \sum_{j=0}^{M} a_j p_j(X),\ a_j \in \mathbb{R} \tag{4.3}$$

We repeat this procedure going backwards through every time step until we reach the first exercise date.

4.2 – LSM Convergence and Robustness

The approach of the LSM method, as defined before, involves using discrete exercise dates and regressing on some basic functions selected for the approximation of the real value of American-style option. Now, the thesis will concentrate on the analysis of the robustness of the basic function's selection and of the convergence of the algorithm with discrete time N, and the differences in these evaluations from type of option to type of option. In their paper, Longstaff and Schwartz provided many results for the convergence aspect. At first, they analyzed the correlation between the true value of an option being continuously exercisable and the value obtained from the evaluation through LSM with discrete exercise dates.

Proposition 1. Assuming V(X) to express the American-style option's true value and $CF(w_i; D, M)$ to express the discounted cash flow obtained through the application of the LSM algorithm, the inequality exposed below will hold almost for sure.

$$V(X) \geq \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} CF(w_i; D, M)$$

From the expression above it is defined how any value obtained through the application of the stopping rule implied by the LSM algorithm can't be bigger than the American-style option's true value, since the former is based on the stopping rule maximizing its value.

Generally, when there is a certain amount of discrete dates of exercise $\sum t_1, t_2, .., t_n = M$, basis functions D, and price paths N, limits are necessary to be considered when dealing with a result of general convergence of LSM. In that sense, Longstaff and Schwartz proposed what follows

Proposition 2. Let the option value V(X) depend on a single state variable X that follows a Markov process having support on $(0, \infty)$. Moreover, let the only possible exercise dates be t_1 and t_2. (so to make it easier generalizing to the case of M dates of exercise) and let to be absolutely continuous the conditional expectation function $G(\omega;t_1)$, that also respects the following

$$\int_0^{\infty} e^{-X} G^2(\omega;t_1)dX < \infty$$

$$\int_0^{\infty} e^{-X} G^2{}_X(\omega;t_1)dX < \infty$$

Then, there exist an $M < \infty$ such that, for any $\varepsilon > 0$

$$\lim_{N\to\infty} P[|V(X) - \frac{1}{N}\sum_{i=1}^{N} CF(w_i;D,M))| > \varepsilon] = 0$$

Such a result denotes how by choosing a linear combination of enough basis functions D and having the amount of paths $N \to \infty$, the LSM result converge to the probability's true value. This result also provides another important implication, that is, that in order to get the true value V accurately estimated, it's not necessary to select an infinite number of basis functions D in the regression.

Extending the work of Longstaff and Schwartz, other LSM results of convergence have been established by Clement, Lamberton, and Protter, in their paper “An analysis of a least squares regression method for American option pricing” (2002). They managed to prove that, under certain general conditions, the LSM algorithm almost surely converges. Moreover, their research provides the determination of the rate of convergence and demonstrated the normalized error to be asymptotically Gaussian.

It comes natural to raise the doubt that whether the choice of different basic functions could automatically provide different results or not, since the regression involves such a choice. In that sense, the literature contains many papers related to this LSM’s robustness issue.

Javier Navas and Manuel Moreno (2003) developed an analysis of the various basis functions’ impact on the price of the option. In their research they price an American put option, a Bermuda call option, and an American-Bermuda-Asian option on a maximum of 5 assets, applying the LSM algorithm. The analysis consisted in the evaluation of in-sample and out of- sample prices of option, and the determination of the standard errors for each of the different kinds and amounts of basis function used in the regression.

It comes out how LSM shows to have different degrees of robustness when pricing different types of options. In facts, the research demonstrates how the LSM technique results to be very robust when pricing the American put option. Standard errors are very small in value and the resulting option values are very similar within each other when using different polynomial basis, also variating in the degree (from 3 to 20) of the various polynomials.

For the other two types of options though, as they are more complex in their features, there is no guarantee for the robustness of the method, and there is not a clear indication on the proper basic function to be chosen. In facts, the option values resulted for those two types of option have consistent differences within each other when changing the features of the basic function adopted for the regression.

Another way to denote the accuracy of the LSM method in pricing the value of an American-style option is by implementing the Hedging Policy and estimating the Hedging Error.

As for the case of the Binomial Tree model described in paragraph 2.4, the logic behind the definition of the hedging strategy for the LSM consists in holding, for every option written (that we will arbitrarily consider of a put type), a portfolio that replicates precisely the value of the option, composed by an amount Δ of the option's underlying stock shares and by some of the risk-free financial asset "B" (which could be a government bond).

In the case of the American-Bermudan option, the equality must hold in all the M time-steps in which the option is exercisable. The resulting equation at each exercise date t will be

$$V_t\ (S_t) = B_t + \Delta t\ (St)\ S_t$$

Being $V_t(S_t)$ the option's value at time t.

Since the stock price S_t changes its value every time for every path, this holding should be adjusted in a self-financing way, in order to replicate the option accurately. On that purpose, we would need to go forward in calculating the delta and the value of the option step-by-step, starting from time t = 0. The value invested in the risk-free asset at time 0 should satisfy, then, the equation

$$B_0 = V_0(S_0) - \Delta_0(S_0)\, S_0$$

For every successive exercise date t, in order to find the new Δ_t, we would have to adequate the weights in the portfolio by changing our exposition on the risk-free asset B_t. Being r the risk-free interest rate, the amount B_t invested on the risk-free asset will then be evaluated by noting that the portfolio evolves, from a time-step to the next, as follows

$$B_{t-1} + \Delta_{t-1}(S_{t-1})\, S_{t-1} \Rightarrow B_{t-1}\, e^{r\Delta t} + \Delta_{t-1}(S_{t-1})\, S_t$$

That will define our replicating strategy by setting

$$B_t = B_{t-1}\, e^{r\Delta t} + (\Delta_{t-1} - \Delta_t)\, S_t$$

The holder of the American-Bermudan put option has the right to sell a stock at the strike price K at each exercise date. The counterpart trying to replicate the option and hedge from this dynamic, compensate by receiving the (negative) amount $(S_t - K)$ coming from such a sale of the option at time t, making the replicating portfolio react by liquidating its Δ_t in stockholding. What is left for the hedger would then be

$$HedERR_t = S_t - K + B_t + \Delta_t\, S_t$$

That defines the Hedging Error.

The assumption required to have a perfect replication strategy of the option is that the values of Δ amd B at the optimal exercise time must be respectively $\Delta = -1$ and $B = K$, so that the Hedging Error will be $HedERR = 0$.

In the following sections, we will apply this pricing technique on an American-Bermudan put option. We will evaluate the robustness of the LSM method for this type of option, developing the analyses with basic functions of various degrees, looking for confirming the results obtained by Navas and Moreno (2003) by applying our personal code to different scenarios (variating also the amount of exercise dates and paths). Moreover, we will evaluate the convergence to a benchmark value of the option, which will be evaluated through binomial tree method.

4.3 Set-up of LSM model and Numerical Results

In this section, we will use LSM to evaluate the price of the American-Bermudan Put option similar to the one used by Longstaff and Schwarz in their masterpiece, comparing the result with a Binomial Tree with equal density of time-steps.

As for the Monte Carlo simulation executed in Chapter 3 on the European option, we will test out if the evaluation of the price of the American-style options with LSM simulation will result more convergent to a benchmark price and more robust, as the number of paths sampled and as the size of the regression function's degree increase.

To be consistent with the evaluations previously made in this thesis, and not to be too distant from the paper of Longstaff and Schwartz, we will keep using the same parameters used before for the underlying asset of the options that we are going to evaluate, but differently from the evaluation in chapter 3, the option will be of a put-type, rather than a call.

So, the fixed variables considered in the estimations are the current stock price $S_0 = 26$; the strike price $K = 30$; the continuously compounded annual interest rate $r = 0.06$; the returns' volatility $\sigma = 0.4$; the expiration date is $T = 1$ year; and the amount of simulations executed for every analysis $E = 5$.

In the first analysis the amount of paths simulated N will variate from 10 to 100000, in order to evaluate the convergence with the fixed value of the options resulted using a binomial tree having same density of time steps, and to evaluate its correlation with the size of N. In the same way, for the evaluation of the robustness of this method, we will compare its applications variating the degree D of the basis function used as regression function for the evaluation of the continuation value. The last analysis will consider a change in the number of time-steps M considered for each set of 5 evaluations. The computational issues (in the measure of the time) fronted while applying the LSM model will be evaluated in every analysis, and its correlation with the size of each variable factor will be discussed at the end of this paragraph. The code for all these evaluations is included in the Appendix.

The two tables below show the results for the evaluation of the American-Bermudan put option when variating the amount of paths simulated (as we did for the European call option in chapter 3) and simulating each combination for 5 times.

Table 4.5: Evaluating American put option variating N.

E	1	2	3	4	5	Average	S.E.	Av.Bias	%Bias	time
B-T	5.865	5.865	5.865	5.865	5.865	5.865				0.1 s
N										
10	7.240	7.506	5.126	4.663	6.289	6.164	0.502	0.299	5.10%	0.1 s
100	5.500	5.289	6.295	5.783	5.345	5.642	0.165	-0.223	-3.80%	0.4 s
1000	5.913	5.846	5.800	5.753	5.718	5.806	0.031	-0.059	-1,01%	1.1 s
10000	5.832	5.819	5.829	5.819	5.873	5.834	0.009	-0.031	-0.53%	18 s
100000	5.866	5.853	5.869	5.848	5.860	5.859	0.004	-0.006	-0.01%	234 s

The table above show the convergence results for the LSM evaluations to the value obtained with Binomial Tree's model, denoted as B-T, and the variations in the robustness levels of these evaluations, when variating N, keeping $r = 0.06$; $\sigma = 0.4$; $S_0 = 26$; $K = 30$; $M = 100$; $E = 5$; $D = 2$. Obviously, the number of time-steps considered in the evaluation through binomial tree is equal to the one used for LSM, i.e., $M = 100$.

For what concerns the American-Bermudan put option, we can notice how well LSM works. In facts, for all the $E = 5$ evaluations made, the value of the option obtained with LSM tends to converge to the value obtained with Binomial Tree method as N increases, reaching acceptable indicators of accuracy when the paths sampled are $N \geq 10{,}000$. Specifically, we notice this from the development of the module of the difference in percentage between the average of the 5 evaluations and the benchmark price, which is smaller and smaller as N increases, getting to a satisfying value smaller than 1% at $N = 10{,}000$ (%Bias $= 0.53\%$).

Another relevant factor resulting from the table above regards the robustness of the values obtained, which increases consistently as N increases. In facts, we can see how the values of the five simulations made for every different size of N go closer and closer to their average value, as N increases. Specifically, the value highlighting this behaviour is the Standard Error (S.E.), which is the variance of the E $= 5$ evaluations for each set of Ns, divided by the square root of E itself ($\frac{\sigma}{\sqrt{E}}$). This value develops becoming smaller and smaller as N increases, granting higher and higher results' accuracy.

The following two tables illustrate the impact on convergence and robustness of the application of the LSM model when variating the degree of the regression function D.

Table 4.6: Evaluating American put option variating D.

E	1	2	3	4	5	Average	S.E.	Av.Bias	%Bias	time
B-T	5.865	5.865	5.865	5.865	5.865	5.865				0.1 s
D										
1	5.920	5.800	5.809	5.850	5.791	5.832	0.021	-0.033	-0.56%	17s
2	5.832	5.819	5.829	5.819	5.873	5.834	0.009	-0.031	-0.53%	18 s
4	5.837	5.851	5.846	5.887	5.844	5.853	0.008	-0.012	-0.20%	19 s
6	5.844	5.844	5.830	5.860	5.817	5.839	0.006	-0.026	-0.44%	19 s
8	5.851	5.827	5.820	5.859	5.838	5.839	0.006	-0.026	-0.44%	20 s
10	5.856	5.863	5.842	5.866	5.858	5.857	0.004	-0.008	-0.14%	23 s

This table above show the results for the evaluation of the American-Bermudan put option when variating the degree of the regression function used to determine the continuation value at every time-step, keeping r $=0.06$; $\sigma=0.4$; $S_0=26$; K $=30$; M $=100$; E $=5$; N $=10{,}000$. We can notice how low the impact of the change in the degree of such a function is on the accuracy of the evaluations, with a little exception for the cases in which the degree is equal to $\mathrm{D}=1$, because the regression would be too simple as $y\sim$ x. In facts, for American-Bermudan put option, when $\mathrm{D}=1$ it's the only case in which the Standard Error is bigger than 0.01 (0.021), while for all the other evaluations it orbits between 0.009 and 0.004.

Moreover, for what concerns the convergence to the benchmark value, the tables above show acceptable scores for all the values of D (%Bias $<1\%$), and very small positive correlation between the size of D and the accuracy of the evaluations (Bias tends to decrease a little).

Since the effect of the changes in the degree of the regressive function seems to be so irrelevant on the robustness of the evaluations, we decided to use the regression function with $\mathrm{D}=2$ for the analysis on the effects of the changes in size of the other variables (although, as we will see, it wouldn't influence the computational effort that much).

The last analysis will evaluate the impact on the convergence and on the robustness of the LSM model for pricing options of the change in the amount of time-steps M used to simulate the continuity of exercisability. The following table illustrate the results of this analysis.

Table 4.7: Evaluating American put option variating M

	E	1	2	3	4	5	Average	S.E.	Av.Bias	%Bias	time
T-steps											
10	LSM	5.888	5.927	5.873	5.860	5.863	5.834	0.011	-0.096	-1.62%	2 s
	B-T			5.930							0.1s
50	LSM	5.802	5.817	5.852	5.820	5.845	5.827	0.008	-0.047	-0.80%	8 s
	B-T			5.874							0.1s
100	LSM	5.832	5.819	5.829	5.819	5.873	5.834	0.009	-0.031	-0.53%	18 s
	B-T			5.865							0.1s
500	LSM	5.852	5.837	5.834	5.852	5.868	5.849	0.005	-0.012	-0.20%	54 s
	B-T			5.861							5.4s
1000	LSM	5.851	5.858	5.848	5.859	5.845	5.852	0.005	-0.011	-0.19%	123 s
	B-T			5.863							20s

As anticipated before, Table 4.7 show the effect on the accuracy of the LSM evaluations when a change in the number of Exercise dates M occurs, keeping $r = 0.06$; $\sigma = 0.4$; $S_0 = 26$; $K = 30$; $E = 5$; $N = 10{,}000$; $D = 2$.

For the American-Bermudan put option, it is noticeable that in order to have an acceptable degree of robustness (S.E. < 0.01 is the threshold we took in all our analysis), it's necessary to take 50 time-steps or more, even though the differences when variating this factor is much less consistent than the difference when variating N. In facts, when $M = 10$ the S.E. is already only 0.011, which is only 0.003 bigger than the S.E. when $M = 50$, and 0.006 bigger (the double) than the one of $M = 1000$.

Regarding the Convergence to the benchmark value, in order to obtain an acceptable value of the inherent indicator (%Bias $< 1\%$), again the number of time-steps to impose must be $M \geq 50$, although again there is lower consistency of correlation (but still remarkable) than the one for the analysis in which N variates, as the % Bias when $M = 10$ is only 0.82 % higher than when $M = 50$ and 1.43% higher than when $M = 1000$.

Then, it's important to highlight how developing the evaluation of the Binomial Tree increasing the number of exercise dates (which is the only factor able to improve the accuracy of the model, as it indirectly brings to increase also the amount of price paths, when increasing), the price has a little tendency to decrease. That's actually a good sign for the convergence theory of the LSM methods, as in all the analysis developed with a minimum degree of accuracy, the option's value resulted from LSM is often a little lower than the value resulted from Binomial Tree, but tended to increase as the size of variables sampled (i.e. N, D in part, and M) increases. This suggests that, for this type of option, as more N, M, and eventually D get close to ∞, the two values converge to the same price.

In all the three analyses just executed, a factor we still miss to determine is the effect on the computational effort required to make each analysis when changing the size of the variables. A necessary premise to be made before, is that the measures of the computational time evaluated in this thesis are to be referred to the execution of the code, present in the Appendix, using a processor Intel(R) Celeron(R) N4020 CPU @ 1.10GHz and a RAM memory of 4,00 GB (3,83 GB usable). From the tables above we can notice how, for all the three factors we decided to increment (N, D, and M), there is a consequent increment in the computational time required for the evaluation of the option price, which is an expectable result, as the amount of processes to be developed increases. Anyway, there are much different developments of this tendency for each of the three variables analyzed. In facts, the increase in the computational effort required when increasing the degree D of the regression function used to determine the continuation value is very low (only 20% more of computational time required when passing from $\mathrm{D} = 2$ to $\mathrm{D} = 10$). That would make it theoretically easy to use a high degree D of the regression function, if convenient (although we've seen it doesn't make an appreciable difference in terms of convergence and robustness, making us decide to save some computational power for the other analyses, keeping $\mathrm{D} = 2$).

The increase in the computational time required when increasing the amount of exercise dates M, instead, is quite significative. In facts from table 4.7 we can see how it's generally almost 4 times slower when increasing by 5 times the amount of time-steps M to develop. For this reason, we decided to operate in the other two analyses keeping $\mathrm{M} = 100$, which gives sufficiently accurate results, without costing too much more of computational time, and without approximating too much the possibility to exercise the option continuously, as $\mathrm{M} = 50$ when $\mathrm{T} = 1$ year would be too far away from a discrete American option (it would mean that the option could be exercised once a week, which is too much more like a Bermudan's feature than an American's one).

Moreover, we can notice how the binomial tree also increases consistently in its computational time as M increases, since for every single time-step added, it doubles the paths generated (in facts, it's determined how, maintaining all the other variables fixed, when M gets bigger than a value around 5000 the LSM method becomes faster than the binomial tree method to compute).

Finally, the analysis showing a much higher correlation between a change in a variable factor and a change in computational time needed, is when the variable changed is the number of paths N. In facts, we can see how reaching the minimum accuracy threshold of N $= 10{,}000$, maintaining all the other variables fixed at the predefined values, the analysis already requires 18 seconds to be computed, and developing it with $\mathrm{N} = 100{,}000$ paths the time required becomes 234 seconds. As in this case the accuracy of the computation doesn't improve too much (S.E. $= 0.004$ rather than 0.009), we decided to execute the other two analyses with N $= 10{,}000$.

To demonstrate the sufficient goodness of the standard values we took in all the analyses (N $= 10{,}000$, D $= 2$, M $= 100$), it will follow a comparison between the option evaluation with these canonical values and with the maximum values taken combined (N $= 100{,}000$, D $= 10$, M $= 1000$).

TABLE 4.8: Comparison between evaluation MAX (N $= 100{,}000$, D $= 10$, M $= 1000$) and evaluation STA (N $= 10{,}000$, D $= 2$, M $= 100$)

	B-T	1	2	3	4	5	Average	S.E.	Av.Bias	%Bias	time
MAX	5.863	5.854	5.861	5.864	5.858	5.855	5.858	0.002	-0.005	-0.05%	1230s
STA	5.865	5.832	5.819	5.829	5.819	5.873	5.834	0.009	-0.031	-0.53%	18s

From the table above we can notice how the method gets more accurate when maximizing all the variables correlated to an improvement of convergence and robustness of the evaluation (standard error decreases by 4 times, convergence with the benchmark value improves by 6 times), although the computational time required increases exponentially (taking 20 minutes and 30 seconds, it increases by more than 65 times from the values obtained maintaining standard values of the three variables). That makes the effort probably too big to make it convenient to increase the variables sizes that much, if imagining to adopt this simulative method broadly, since the standard variables' values considered already gives us acceptable results in terms of convergency and robustness level.

CHAPTER 5

Conclusions

This thesis stressed out the work of the research on analytical and numerical methods for pricing options, developing a comparison between the results coming from simulative methods applied on both European-style and American-style options, and respectively the Black-Scholes model and the Binomial Tree model, which are the most popular ones adopted in the Academical approaches. One of the aims of this thesis, in facts, was to develop the analyses in a manner that was the most comprehensible possible for students and practisers, applying the models to which the readers are supposed to be more used. Not by chance, we decided to define all the basic elements, and to illustrate all the basic notions necessary to better understand the different processes of the various models for option pricing.

After the illustration of the Monte Carlo simulation and its application on option pricing, we proved its efficiency in pricing European-style options, since the resulting option value became closer and closer to the value obtained with Black-Scholes as the number of paths simulated increased.

Afterwards, the biggest issue to overcome was to readapt our program for Monte Carlo simulation to make it work for pricing American-style options, which differ from the European-style ones by their feature of continuous exercisability. We decided to evaluate them through the Least Squares method proposed by Longstaff and Schwartz (2001), as it was proved to be efficient for many other types of options. In facts, another aim of this thesis is to represent a basis for future developments of this method on the various types of options (as the ones we described in paragraph 2.2) or derivatives in general.

In order to confirm the impressions from the literature on the efficiency of the LSM on pricing American-style options, we developed many simulations with our updated code (integrating the linear regression for determining the continuation value of the option at each time-step). We arranged these simulations variating the size of the different variables influencing the accuracy of the evaluations (possibility that is not contemplated while adopting numerical methods as Binomial Tree model, which allows to variate only the number of exercise dates).

The results obtained applying this development of the Monte Carlo simulation confirmed that the LSM method represents e credible tool for pricing American-style options with a certain accuracy and elasticity, without necessarily employ excessive computational effort. In facts, we showed how the program developed simulations of prices of a certain level of convergence and robustness (<1% of %Bias with the benchmark price, and <0.01 of Standard Error), in an acceptable amount of time (18 seconds using average level of processor and RAM memory).

Moreover, in this research we obtained important indications on the level of correlation between the accuracy of the option price evaluation and the size of each of the three different variables of the simulation supposed to influence the simulation accuracy (the amount of exercise dates M, the number of paths simulated N, and the degree of the regression function D). In facts, we denoted how increasing M and N can substantially improve the accuracy of the evaluation, while increasing D has a much lower impact on that (although it’s also the less expensive improvement in terms of computational costs), confirming another conclusion of the previous research, regarding the satisfying robustness level of the price simulated on selecting different polynomials of different degrees $D \geq 2$ for the regression.

As for future research, this thesis aims to represent a basis for the development of the LSM model into a more accurate and sophisticate tool for pricing option types with more variable or complex features, such as options having payoff depending on more underlying assets (Rainbow or Basket options), or options depending on the average of the price of the underlying in predetermined amount of time (Asian options), or compounding in the uncertainty (variable risk-free interest rate).

After demonstrating its efficiency on American-style options, this work aims to contribute to the idea that using simulations it's possible to obtain many advantages in the evaluation and in the risk management of financial instruments, making it much more common and promising to insist on the research on this type of methods. Moreover, as the size of the derivatives' market is increasing and many programming tools are improving and at the same time getting more accessible, it will become much easier to implement more complex simulative methods (or integrations of the existing methods, like LSM), and to apply them on more complex and realistic financial instruments.

Another aspect we may want to improve in future is to provide more detailed convergence and robustness results of the LSM when challenging particular scenarios, eventually using different types and combinations of basic functions, determining further mathematical foundation for this method.

Considering that in many cases the residuals of the regression could be heteroskedastic, as LSM always consider the ordinary least squares to estimate the conditional expectation, another development of this work could be that of identifying the most adapt type of regression techniques for each scenario. We could try testing, for example, weighted least squares or generalized least squares, and discuss the efficiency and computational costs for every test.

Finally, another possible development of this method for future research could be that of combining it with other pricing models (more advanced than Black-Scholes model or Binomial tree model) very popular in the financial industry, as these eventually reflect more accurately the real market (e.g., Stochastic Volatility models, or Variance-Gamma models, or Jump-Diffusion models).

Appendix

Related python codes.

Binomial Tree for American put option.

```
#This is the code for the evaluation of an American Put option price
with the Binomial Tree method.

# Importing the libriaries necessary.

import numpy as np

import matplotlib.pyplot as plt

np.set_printoptions(formatter={'float':lambda x: '%6.5f' % x}) #
printing untill the fifth digit after comma.

%matplotlib inline

S0 = 26

T = 1.0

r = 0.06

sigma = 0.4
```

```
#Sm = 30

#Tree_list = []

def Tree_Creation(M):

        dt = T / M

        u = np.exp(sigma * np.sqrt(dt))

        d = 1 / u

        S = np.zeros((M + 1, M + 1))

        S[0, 0] = S0

        y = 1

        for t in np.arange(1, M + 1):

            for i in np.arange(y):

                S[i, t] = S[i, t-1] * u

                S[i+1, t] = S[i, t-1] * d

            y += 1

        return S

M=100
```

```
TREE=Tree_Creation(M)

dt = T / M  # Redefining dt outside the function

# Importing the risk neutral probability of a price moving up

u = np.exp(sigma * np.sqrt(dt))

d = 1 / u

p_rn=(np.exp(r*dt)-d)/(u-d)

def Pay_Off_Call_Option(S,K):

    return np.maximum(K-S,0)

def
American_Call_Price_Binomial_Tree_Definition(tree_price,pay_off,s
trike):

    #Developing the tree for the option price initiating the matrix where
to store it, with the same dimension of the tree for the underlying

    option_tree=np.zeros((M+1,M+1))

    exercise_option_tree=np.zeros((M+1,M+1)).astype(int) #
developing the matrix suggesting where to exercise the option and
where to not
```

```
    #the last date value is the pay-off
    option_tree[:,-1]=pay_off(tree_price[:,-1],strike)
    exercise_option_tree[:,-1]=1*(tree_price[:,-1]<strike)
    # Developing the price going backwards, applying at each node the expression above
    for i in np.arange(M-1,-1,-1):
        for j in np.arange(i+1):
            option_tree[j,i]=np.maximum(np.exp(-r*dt)*(p_rn*option_tree[j,i+1]+(1-p_rn)*option_tree[j+1,i+1]),pay_off(tree_price[j,i],strike))
            exercise_option_tree[j,i]=1*(np.exp(-r*dt)*(p_rn*option_tree[j,i+1]+(1-p_rn)*option_tree[j+1,i+1])<pay_off(tree_price[j,i],strike))


    return [option_tree,exercise_option_tree]
K=30 # strike
```

```
call_price_tree,exercise_option_tree=American_Call_Price_Binomial
_Tree_Definition(TREE,Pay_Off_Call_Option,K)
print(call_price_tree)
print(exercise_option_tree)
```

Monte Carlo simulation and Black-Scholes model for European call option.

```
#This is the code for Monte Carlo simulation in pricing European Call
option and the black-scholes evaluation as a benchmark
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm
def Stock_price_paths(S, T, r, q, sigma, steps, N):
    """
    To follow, the inputs definitions:
    #S Current Stock Price
```

```
    #K = Strike Price

    #r = risk-free interest rate

    #q = annual dividend yield

    #sigma =annual volatility

    #T = Time to maturity: 1 year = 1; 1 months = 1/12



    The output definition:

    # [steps,N] Matrix of paths of the Stock price

    """

    dt = T/steps

                                    #S_{T} = ln(S_{0})+\int_{0}^T(\mu-
\frac{\sigma^2}{2})dt+\int_{0}^T \sigma dW(t)

    S_T = np.log(S) +  np.cumsum(((r - q - sigma**2/2)*dt +\

                      sigma*np.sqrt(dt) * \

                      np.random.normal(size=(steps,N))),axis=0)
#application of the formula above using numpy library
```

```
    return np.exp(S_T)

S0 = 26 #lower stock price  S_{0}

Sm = 30 #median stock price  S_{2}

K = 30 #strike price

T = 1 #time to maturity in years

r = 0.06 #risk-free rate per year in %

q = 0.00 #dividend rate per year in %

sigma = 0.4 #volatility per year in %

steps = 10 #time steps

N = 100 #number of trials for every S

v = 5 #number of simulations for each S

paths_list=[]

for i in range(v):

    paths = Stock_price_paths(S0,T,r, q,sigma,steps,N)

    paths_list.append(paths)
```

```
    S0=S0+ Sm*(1/15)

    plt.plot(paths);

    plt.xlabel("Time Steps Incrementation")

    plt.ylabel("Asset Price")

    plt.title("Geometric Brownian Motion " + str(i+1))

    plt.show()


def black_scholes_call_payoff(S,K,T,r,q,sigma):

    """

    Inputs

    #S Current Stock Price

    #K = Strike Price

    #r = risk-free interest rate

    #q = annual dividend yield

    #sigma =annual volatility

    #T = Time to maturity: 1 year = 1; 1 months = 1/12
```

```
    Output
    # call_price = option price
    """
    d1 = (np.log(S/K) + (r - q + sigma**2/2)*T) / sigma*np.sqrt(T)
    d2 = d1 - sigma* np.sqrt(T)

    call_price = S * np.exp(-q*T)* norm.cdf(d1) - K * np.exp(-
r*T)*norm.cdf(d2)
    return call_price
S0=26
v=5
for i in range(v):
    paths_payoffs = np.maximum(paths_list[i][-1]-K, 0)
    option_present_price = np.mean(paths_payoffs)*np.exp(-r*T)
#final value discounted back to present value
```

```
    black_scholes_price =
black_scholes_call_payoff(S0,K,T,r,q,sigma)
    S0 = S0+ Sm*(1/15)
    print(f"Black Scholes Price for S" + str(i) + " is " +
str(black_scholes_price) )
    print(f" Monte Carlo Simulation price for S" + str(i) + " is " +
str(option_present_price) )
```

LSM for American put option.

```
#This is the code for the evaluation of an American Put option
Through LSM. At the end of it, it's illustrated the integration of the
cod to adapt it to the Moving Window Asian option by changing the
payoff function.
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm
```

```
import math

def Stock_price_paths(S, T, r, q, sigma, steps, N):

    """

    To follow, the inputs definitions:

    #S Current Stock Price

    #K = Strike Price

    #r = risk-free interest rate

    #q = annual dividend yield

    #sigma =annual volatility

    #T = Time to maturity: 1 year = 1; 1 months = 1/12

    The output definition:

    # [steps,N] Matrix of paths of the Stock price

    """

    dt = T/steps

                                    #S_{T} = ln(S_{0})+\int_{0}^T(\mu-
\frac{\sigma^2}{2})dt+\int_{0}^T \sigma dW(t)
```

```
    S_T = np.log(S) + np.cumsum(((r - q - sigma**2/2)*dt +\
                          sigma*np.sqrt(dt) * \
                          np.random.normal(size=(steps,N))),axis=0)
#application of the formula above using numpy library
    return np.exp(S_T)
def estimate_coef(x, y): #definition of the least square regression
function
    model = np.polyfit(x, y, 2)
    #print(model)
    return (model[0], model[1], model[2])


S0 = 26 #lower stock price S_{0}
Sm = 30 #median stock price S_{2}
K = 30 #strike price
T = 1 #time to maturity in years
r = 0.06 #risk-free rate per year in %
```

```
q = 0.00 #dividend rate per year in %

sigma = 0.4 #volatility per year in %

steps = 100 #time steps

N = 100000 #number of trials for every S

v = 1 #number of simulations for each S

dt = T/steps

paths = Stock_price_paths(S0,T,r, q,sigma,steps,N) #simulating the paths


    #plt.plot(paths);                        #plotting the paths

    #plt.xlabel("Time Steps Incrementation")

    #plt.ylabel("Asset Price")

    #plt.title("Geometric Brownian Motion " + str(i+1))

    #plt.show()

path_payoff_US = np.zeros((steps,N))   #creating the zero-matrix for all the matrix we will need in computation of the payoff
```

```
path_exercise_US = np.zeros((steps-1,N))

coefficient_regression = np.zeros((steps-1,3))

discounted_cash_flow = np.zeros((steps,N))

stopping_rule = np.zeros((steps,N))

discounted_cash_flow_final =np.zeros(N)

cash_flow =np.zeros((steps,N))


paths = Stock_price_paths(S0,T,r, q,sigma,steps,N)    #creating the
payoff matrix

for j in range(steps):

    path_payoff_US[-j,:] = np.maximum(-paths[-j,:]+K, 0)

#printing (path_payoff_US)


for j in range(steps-1):

    path_exercise_US[-j,:] = np.exp(-r*dt)*path_payoff_US[-j,:]

#creating the matrix with the payoff discounted by un time-step
```

```
for j in range(steps-1):   #creating the coefficients of the regression for each time step
    coefficient_regression[-j,:] = estimate_coef(paths[j-1,:],path_exercise_US[-j,:])


for j in range(steps):      #iterating the LS regression in every time step to evaluate the continuation value, then executing the option every time (from 0) the payoff is higher than the continuation value
    for y in range(N):
        if path_payoff_US[-j-1,y] > 0 :
            continuation_value = coefficient_regression[-j,0]*pow(paths[-j-1,y], 2) + coefficient_regression[-j, 1]*paths[-j-1, y] + coefficient_regression[-j, 2]
            if continuation_value  < path_payoff_US[-j-1,y]:
                stopping_rule[-j-1,y] = 1
        if path_payoff_US[-j-1,y] == 0 :
```

```
            stopping_rule[-j-1,y] = 0


for j in range(steps):      #imposing the exercise of the option to
happen at the first convenient time
    for y in range(N):
        if stopping_rule[-j,y] ==1 :
            stopping_rule[-j+1:,y] =0
for j in range(N):
    if np.sum(stopping_rule[:,j]) == 0:
        stopping_rule[-1,j] = 1
for j in range(steps):
    for y in range(N):
        if stopping_rule[j,y] ==1:


            cash_flow[j,y] = path_payoff_US[j,y]    #creating the cash
flow matrix
```

```
        else:
            cash_flow[j,y] = 0
#print(path_payoff_US)
#print(cash_flow)
#print(stopping_rule)
for j in range(steps):      #discounting the cash flow matrix
    for y in range(N):
            discounted_cash_flow[-j,y] = np.exp(-r*dt*(steps-
j))*cash_flow[-j,y]
for j in range(N):      #defining the option value by taking an average
of all the discounted cash flows
    discounted_cash_flow_final[j]  =
np.sum(discounted_cash_flow[:,j])
PriceUS_list = np.average(discounted_cash_flow_final)
print(PriceUS_list)
```

Summary Report

Abstract

Title: Pricing American-Bermudan options through Least Square Methods

Academic year: 2020-2021

Course: Corporate Finance – Chair in Asset Pricing

Author: Manfredi Morisi – Matr. 718881

Supervisor: Prof. Paolo Porchia

Co-Supervisor: Prof. Marco Pirra

Key Words: Option evaluation, dynamic programming, simulative methods, regressive methohds, computation accuracy.

Purpose: This thesis presents an analysis of the accuracy of simulative methods for pricing option contracts.

Methodology: The methodology is based on a simulative approach to define an optimal value of different option's

	types, so to have the possibility to consider multiple stochastic factors.
Software:	Python
Literature review:	This work is based on theories of pricing models, simulative methods and regressive methods with dynamic programming, as well as on previous findings about Monte Carlo Simulations for option pricing.
Empirical framework:	The analysis is based on various simulations of the price of two types of having respectively one and three variating factors.
Findings:	The thesis concludes that the adoption of simulative methods represents a successful tool for pricing options, as it allows to evaluate derivatives with multiple stochastic factors without impacting negatively on the accuracy the evaluation and on the computational effort required.

1 Introduction

This thesis concentrates on developing the work of Francis A. Longstaff and Eduardo S. Schwartz presenting a strategy to evaluate American-style options through Monte Carlo simulation, solving the issue of the path-dependency using a variable regressive function.

In order to get to the method developed in the final analysis of this thesis, we define all the basic knowledges necessary to understand the various aspects of this research. After the definition of every specific financial term and every basic financial process present in this thesis, we describe the main types of options, paying particular attention to the types linked to the one evaluated in this thesis.

Successively, we illustrate the most representative methods for pricing European-style and American-style options in the academical field: Black-Scholes model and Binomial Tree model. These two methods will be adopted as benchmark methods for the evaluations with simulative methods of respectively European and American-Bermudan option.

After the illustration of the foundation of this research, the Monte Carlo simulation method is introduced and explained in its logics. Before the application of this method on the evaluation of a European Call option's price, this thesis provides a brief illustration of the generation of pseudo-random numbers, proving how it's possible to convert random numbers sampled from U[0,1] in numbers sampled from the normal distribution N[0,1].

The positive results obtained on these evaluations explain why the application of the Monte Carlo method became much more popular over time in this specific research field. In facts, we notice that the accuracy of the evaluations, in terms of robustness and convergence to the true value evaluated using the Black-Scholes method, reach a satisfying level when a certain number of paths (10,000) are simulated.

The possibility of reaching accurate evaluations using a method that enables the analysis to consider multiple stochastic factors (which can't happen with basic numerical methods), led to the development of the research on the application of simulative methods on other types of derivatives. Specifically, the most popular and challenging case regards the evaluation of path-dependent derivatives such as American-style options. This thesis aims to take up such a challenge by applying the Least Square Monte Carlo algorithm (LSM) proposed by Longstaff and Schwartz on an American-Bermudan Put option. This method determines the optimal early exercise strategy using a simple least squares cross-sectional regression to estimate the conditional expectation function for every exercise date and determine the optimal exercise strategy for each path, starting from the expiration date, iterating backwords until the first exercise date. The LSM algorithm is implemented in a Python code attached at the top of the Appendix.

Net of an affordable computational effort required (19 seconds), the evaluations obtained using LSM applying a certain number of paths N simulated (10,000), of time-steps M introduced (100), and of the degree of the regressive function implemented (2), result very accurate in terms of convergence to the average price of the simulations and to a benchmark price evaluated by Binomial Tree method, and in terms of robustness when changing the degree of the regressive function. Finally, this thesis summarizes its major findings and contribution, and some possible inspirations for future research on its related field.

2 Foundation

This thesis develops defining the most important financial terms and processes necessary to approach the illustrations and the analysis coming after. We provide the definitions of derivative, option, put and call type, strike price, arbitrage, and hedging. Then, we describe the correlation of an option’s price with its underlying’s volatility and with the risk-free interest rate, and the development process of an asset compounded continuously.

Moreover, this thesis provides the definition of the various kinds of option contracts, starting from the definition of the plain vanilla options, which are European-style and American-style ones, following with the definition of Exotic option, and its categorization in three sets of types, linked to the expiration cycle, to the method used to trade them, and to the underlying asset the options are related to: some types of options giving less common approach to the rights of early exercise (Bermudan option; Canary option; Evergreen option; and so on), other options having standard exercise styles (as vanillas), but different methods in calculating the payoff values (Basket option; Rainbow option; Exchange option; and so on) and other options differing both in the calculation of payoffs and in the style of exercise from vanillas (Asian option; Barrier option; Binary option; and so on).

Specifically, this thesis proposes the application of the pricing methods described next on a typical European Call option and on an American-Bermudan Put option, which has exercisability at any time before expiration (that's why American) but in a predefined big amount of exercise dates equally distant within each other (making it discrete, as a Bermudan option). However, we illustrated all these different types of options because we think that the results obtained in this thesis show that it could be worth to try to develop this analysis on more types of derivatives in future research.

In the third paragraph of the second Chapter, this thesis provides the illustration of the masterpiece developed by Black, Scholes and Merton, that represents the first and most important model for pricing and hedging options.

First, we defined the various conditions required when applying Black-Scholes, as its precision is strictly related to how close the referring market is to a market with perfect competition. Successively, we define how the model is based on the idea that the structure of the return on an option can be precisely replicated by a continuous rebalancing of a hedged portfolio composed by a risk-free asset gaining interests at a continuously compounded rate (which could be, for example, a government bond) and by a variable number of shares of the underlying asset. The return of such a portfolio depends exclusively on some known constant variables and on the time, as it is independent from the price movement of the stock. Then, we defined a generalized Wiener process

$$dx = adt + bdz \tag{2.2}$$

And the Itò's Lemma

$$dG = \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial x}a + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}bdz \tag{2.4}$$

So, we derive the Black-Scholes model

$$\Delta f = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\,\sigma\,\Delta z \tag{2.8}$$

And the Black-Scholes partial differential equations (PDE)

$$c = S_0 N(d1) - Ke^{-rT}N(d2) \tag{2.14}$$

$$p = Ke^{-rT}N(-d2) - S_0 N(-d1) \tag{2.15}$$

Before going on illustrating the Binomial Tree model, we denote the limits of the Black-Scholes model in creating accurate estimations of the option value, as it imposes very strict assumptions, and it doesn't provide closed form solutions for path-dependent derivatives.

Numerical methods like binomial tree model represented an intuitive and functional alternative to Black-Scholes model as it gives a closed-form solution for pricing American-style options, although, as for many other numerical methods, it allows to consider only the price of the underlying asset as stochastic factor in the evaluation, which led to a successive increase in popularity of simulative methods.

This model consists in the representation in a diagram of the possible different paths available for the asset price over the time to maturity. The assumption that the underlying asset's price follows a random walk holds. For every time step, there is a defined probability that the price goes up by a certain percentage amount and a defined probability that the price goes down by a certain percentage amount. Defining S_t the value of the stock at time t, it's price at next time step $t+1$, is going to be $S_{t+1} = uS_t$ with probability p, and $S_{t+1} = dS_t$ with probability $1-p$, being $u > 1 > d$.

Then, $f_u = \max(S_0 u - K, 0)$ and $f_d = \max(S_0 d - K, 0)$.

Making the time steps the smallest possible, the limit of the binomial tree brings to the lognormal assumption for the price of the stock, that's the same assumed for the Black-Scholes Model.

Then we define the hedging strategy by determining the number of shares Δ necessary to make a zero-risk portfolio

$$\Delta = \frac{\mathrm{fu} - \mathrm{fd}}{\mathrm{S0u} - \mathrm{S0d}} \tag{2.16}$$

That, imported in

$$f = S_0 u\, \Delta(1 - u\, e^{-rT}) + f_u\, e^{-rT} \tag{2.17}$$

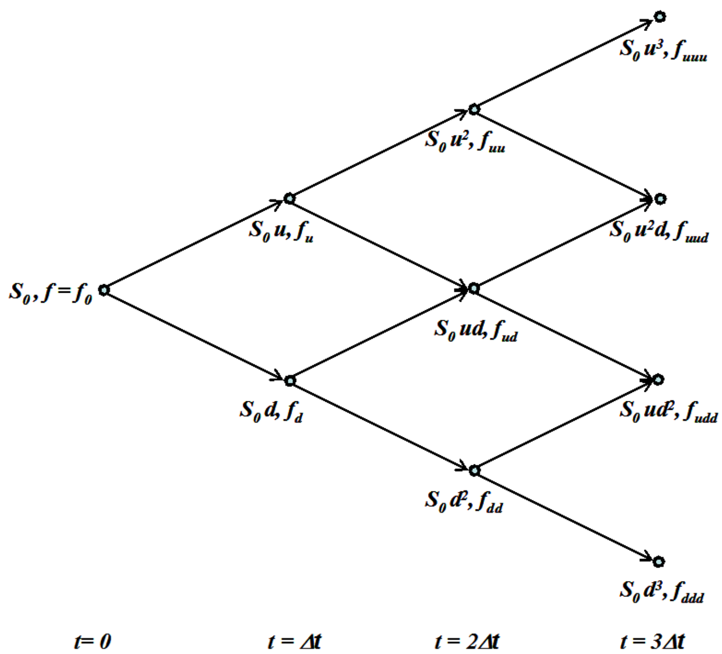
brings to

$$f = e^{-rT}[\, p\, f_u + (1 - p)\, f_d\,] \tag{2.18}$$

being

$$p = \frac{e^{-rT} - d}{u - d} \tag{2.19}$$

Figure 2.1: Binomial tree on three time-nodes



To adapt the model to American Option pricing it's required to check at each node whether it's profitable to exercise, and to adjust the price of the option accordingly.

For an American call, the value of the option at a node is given by

$$f_{xx} = = \max\left(K - S_{xx},\ e^{-r\Delta T}\left[p\ f_{xxu} + (1-p)\ f_{xxd}\right]\right) \quad (2.21)$$

When a node shows an option price higher than it would be without exercising, it influences the prices of the nodes following back through the tree making an American option to be more valuable than a European option with the same characteristics.

3 Simulative methods for option pricing

In the third chapter we develop the illustration of the Monte Carlo simulation taking in exam the calculation of the following

$$\int_A g(x)f(x)dx = \bar{g} \tag{3.1}$$

Identifying *g(x)* as an arbitrary function, *f(x)* as a density function of probability, and *A* as the range of integration.

Then, from the development of the equation (2.6)

$$S(T) = S(0)\ \exp[(\mu - \frac{\sigma^2}{2})T + \sigma\epsilon\sqrt{T}] \tag{3.10}$$

Which constructs the paths of the stock price S at any time T.

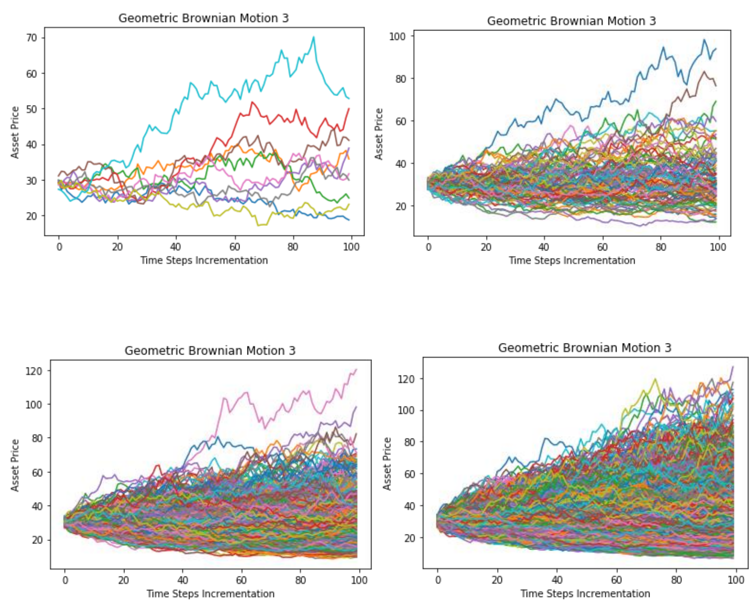
Applying it on a European option, we get the definition of its value

$$f(S, 0, K, T) = \frac{1}{M}\, e^{-rT} \sum_{i=0}^{M} g(S\ (0)\ \exp[(\mu - \frac{\sigma^2}{2})T + \sigma\, y_i \sqrt{T}]) \tag{3.14}$$

Successively, although our software provides the normal distribution N[0,1] as a black-box function, we illustrate how to generate pseudo-random numbers, proving that it's possible to convert random numbers sampled distributed between [0,1] in numbers sampled from the normal distribution N[0,1].

Then, we evaluate a European Call option having expiration at T=1 year. The payoff of the option is evaluated from 5 different starting prices of the underlying stock, from 26 to 34. The strike price is 30, the volatility of returns σ of the underlying is 0.4, and the risk-free interest rate is r=0.06. The paths sampled for each price will be at first 10, then 100, then 1000, and finally 10,000. This, in order to prove how the accuracy of this method increases proportionally with the amounts of paths. As previously said, we notice how sampling an amount of paths $N \geq 10{,}000$ we obtain a sufficiently accurate value of the option for every simulation (S.E. < 0.004; % Bias < 1%), when compared to the true values obtained with Black-Scholes.

Figure 3.1: 4 Brownian motions development increasing N



4 Least Squares Method for options pricing

In the fourth chapter, we illustrate the development of the Monte Carlo simulation proposed by Longstaff and Schwarz in their masterpiece, the LSM, which aims to evaluate American-style option integrating, at every time-step from maturity going backwards, a simple least squares cross-sectional regression to estimate the conditional expectation function.

We can obtain a complete estimation of the strategy of optimal early exercise through the application of the dynamic programming principle implying that the option should be exercised as soon as both it's in-the-money (having payoff higher than 0) and it has an estimated value of conditional expectation of continuation lower than the value it would have when exercising immediately. The option's estimated value is, then, the estimated expected payoff discounted at time 0.

In the illustration present at the bottom of the first paragraph of this thesis we propose an example on an American call option having two exercise dates, to which it follows a deeper functional analysis.

Then, it follows the definition of the dynamics determining the accuracy of the estimations in terms of convergence to a benchmark price, convergence to an average of the estimates, and robustness relatively to the degree of the regressive function, which are going to be the criteria defining the quality of the numerical results obtained in the last paragraph of the chapter. Then, it will follow the documentation of a series of results coming from the findings of the literature regarding the convergence and the robustness of the estimations with LSM method on different types of options.

Before introducing it to the analysis and the findings of this thesis itself, we illustrate the implementation of the Hedging policy with the Δ estimator on the LSM method, which consists in the fact that the following equality must hold for all of the M time-steps evaluated

$$V_t\,(S_t) = B_t + \Delta t\,(St)\,S_t$$

Resulting in the definition of the Hedging Error

$$HedERR_t = S_t - K + B_t + \Delta_t\,S_t$$

That should be worth 0 if the portfolio replicates perfectly the option.

In the last part of the chapter, we set up the LSM model, evaluating an American-Bermudan Put option having same characteristics of the European Call option from chapter 3, the degree of the regression function D = 2, and the amount of time-steps M = 100 in all the evaluations, except, of course, for the evaluations in which M or D variate. We premise that the benchmark values for the following analysis are obtained by applying a binomial tree model having the same size of M (its code is present in the appendix as well).

So, the analysis consists in the evaluation of the option by variating the amount of paths N at first, from N = 10 to N = 100,000 denoting how a sufficiently appreciable level of accuracy results to come when $N \geq 10{,}000$.

For the analysis in which we variate the degree of the regressive function adopted to evaluate the continuation value at each step, we notice how the accuracy of the evaluation is not particularly sensitive to the change in the degree of the function, except for the too simple case of D = 1.

Then, the analysis in which the amount of time-steps M variates, results to show acceptable accuracy when the value of M is $M \geq 50$, although we still prefer to use M ≥ 100, which better simulates the features of the American option (continuously exercisable).

Finally, we analyze the correlation between the size of the various variables with the computational effort required to make the estimations. As expectable, we see how the factor determining the most the increase in computational time required is the size of the sampled paths N. A little less than N, but still consistently, M influences the computational time too, while variating D doesn't bring to appreciable differences. The last analysis shows a comparison between the estimation with the factors N, M, and D having the minimum acceptable values to get enough accurate results, and the estimation with the maximum values of the three variables measured (N = 100,000 ; M = 1,000 ; D = 10). The latter requires 65 times more effort to be computed respect to the former, while the accuracy is only 6 times better (although still acceptable).

5 Conclusion

This thesis stressed out the work of the research on analytical and numerical methods for pricing options, developing a comparison between the results coming from simulative methods applied on both European-style and American-style options, and respectively the Black-Scholes model and the Binomial Tree model, which are the most popular ones adopted in the Academical approaches. In facts, we aimed to develop the analyses in a manner that was the most comprehensible possible for students and practisers, applying the models to which the readers are supposed to be more used.

After successfully illustrating and implementing the simulative methods for pricing European options, the biggest issue to overcome for this thesis was to readapt our program for Monte Carlo simulation to make it work for pricing American-style options, which differ from the European-style ones by their feature of continuous exercisability.

We can say that the results coming from our analysis strengthen the conclusions derived in the literature, as it showed how LSM methods represents an efficient tool for pricing path-dependent options with a certain accuracy level, without necessarily requiring massive computational efforts.

Moreover, differently from the traditional numerical methods, LSM allows the development of the evaluations imposing various amounts of stochastic factors, which makes it more adapt to the real market situations, and which pique our curiosity on an eventual implementation on more complex financial instruments. Also, as many programming tools are improving, it will become more and more accessible to integrate this method to more complex and computationally heavy scenarios and pricing models.

Bibliography

[1] A.-S. Chen and P.-F. Shen, (2003), Computational Complexity Analysis of Least-squares Monte Carlo (LSM) for Pricing US Derivatives. Applied Economics Letters, 10, 223-229

[2] Black, F. and M. Scholes, (1973) The Pricing of Options and Corporate Liabilities. Journal of Political Economy. 81, 673-654

[3] Boyle, Phelim P. (1977) Options: A Monte Carlo Approach. Journal of Financial Economics. 4, 323-338

[4] Broadie, M. and P. Glasserman, (1997) Pricing American-Style Securities Using Simulation. Journal of Economic Dynamics and Control. 21, 1323-1352

[5] Broadie, M. and Kaya, O. (2006) Exact Simulation of Stochastic Volatility and Other Affine Jump Diffusion Processes. Operations Research 54, 217-231.

[6] Carriere, J. (1996) Valuation of Early-Exercise Price for Options Using Simulations and Non-parametric Regression. Insurance: Mathematics and Economics. 19, 19-30

[7] Clement, Emmanuelle, Lamberton, Damien, and Protter, Philippe (2002) An Analysis of a Least Squares Regression Method for American Option Pricing. Finance and Stochastics, 6, 449-471

[8] Cox, John C. and Ross, Stephen A., and Rubenstein, Mark (1979) Option Pricing: A Simplified Approach. Journal of financial Economics 7, 229-263.

[9] Glasserman, P.(2004) Monte Carlo Methods in Financial Engineering (Springer)

[10] Hull, John C. (2006) Options, Futures, and Other Derivatives, 6th edition, Chapter 21, Pearson Education. 520-524

[11] Ito, K. (1951) On stochastic differential equations. Memoirs, American Mathematical Society, 4. 1-51

[12] Ibanez, Alfredo, and Zapatero, Fernando (1998), Monte Carlo Valuation of American Options through Computation of the Optimal Exercise Frontier. Working paper, University of Southern California.

[13] Stentoft, Lars, (2003), Convergence of the Least Squares Monte Carlo Approach to American Option Valuation, school of economics and management, University of Aarhus, working paper.

[14] Long, Yun (2010), Monte Carlo Simulation in Option Pricing, Master thesis in statistics and applied probability, National University of Singapore.

[15] Longstaff Francis. A. and Schwartz Eduardo S., (2001), Valuing American options by simulation: a simple least squares approach, The Review of Financial Studies vol. 14, no.1, 113-148

[16] Navas, J. and Moreno, M. (2003) On the Robustness of Least-Squares Monte Carlo (LSM) for Pricing American Derivatives. Review of Derivatives Research 6, 107-128.

[17] Schwartz, Eduardo. S. (1977) The Valuation of Warrants: Implementing a New Approach. Journal of Financial Economics 4, 79-93.

[18] Sgubbi, Elena (2017), Metodi di regressione per programmazione dinamica stocastica, Master thesis in mathematical science, Università di Bologna.

[19] Thom, Howard (2009), Longstaff Schwartz Pricing of Bermudan Options and their Greeks, University of Oxford.

[20] Tilley, J.A, (1993), Valuing American Options in a Path Simulation Model. Transactions of the Society of Actuaries. 45, 83-104

[21] Option (finance) - Wikipedia, and related Bibliography. www.wikipedia.org

[22] Python documentation centre, www.python.org