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Beyond Black-Scholes option pricing: the role of volatility in local and stochastic volatility models

Master degree in Corporate Finance

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A mio padre, Antonio;
A mia mamma, Michela,
grazie di cuore.

If you want to make a fast buck,
just go in and tell them you want to make a fast buck!

from Fact and fantasy in the use of options, Fischer Black, 1975

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Abstract

Contrary to what one might think, derivatives, beyond their financial definition, have existed for a long time. History has provided information on the first derivatives already in ancient Greece, in the 6th century BC.

The reason for their wide spread use is, in part, due to the guarantees and benefits that they afford to those who buy them. In this sense, derivatives can be used for multiple purposes, such as to protect against risk, to block a price to be paid or received, to speculate on the price of a share of stock.

For this reason it is necessary to know how to provide a fair price of derivatives, for each of their strike price and for each of their maturity. In this sense, the research community have become increasingly interested in the world of derivatives and their modeling.

In effect, volatility modeling and option pricing is now a large and diverse area in finance mathematics.

This thesis provides a review of the most significant models for option pricing. It will be explained how the evaluation methods have implemented over the years, and how all of them are able to determine the fair initial price, avoiding mis-pricing between the option's price and the underlying.

As already mentioned, a fundamental part of these models is played by volatility. It will be shown how different assumptions about volatility dynamics will differentiate the models, starting from the assumption that it is constant, up to the models that assume it as a stochastic process.

The thesis starts introducing briefly the Asset Pricing theory, the Geometric Brownian Motion model for stock price, and then investigates the analytic solution for Black-Scholes differential equation for European options. We will see why this model suffers from several disadvantages, among which that it is not consistent with the implied volatility, which varies according to the maturity and the strike, and it is not constant as instead assumed by Black-Scholes. Other models will therefore be analyzed.

These models include Local Volatilty methods, based on the Fokker-Planck equation for the density of the underlying process, that takes into account the volatility smile and overcomes the limits of the Black-Scholes model with the so-called local implied volatility. However, this approach presents some limits: even though it's perfectly calibrated by construction, it is too poor in order to get the right market dynamics.

Then we focus on stochastic volatility models. The Heston Model considers the leverage effect and the clustering effect, which allows the volatility itself to be random and also allows it to take the non-normally distributed stock return into account. Nevertheless, although this models solve many problems of local volatility, they are not flexible enough to capture the right behavior of market dynamics.

For this reason we finally move to study the local-stochastic volatility models, since they have a new degree of freedom, called mixing factor, which can be used to impose the implied volatility dynamics, capturing the main advantages of the local volatility model and the stocastic volatility model.

Some definitions of derivatives, types and main option rules will be taken for granted.

This thesis was entirely written in LaTeX.

LaTeX

Chapter 1

Basic rules of mathematics for finance

Before starting the discussion of the methods used to evaluate options, we need to describe some basic mathematical results and notations.

The common thinking in finance is that since mathematicians started working in investment banks and large asset management funds at the end of the cold war, they have contributed greatly to change finance, allowing this *science-non-science* to evolve incredibly [17].

In effect, we can say more: the deeper branch of finance, the quantitative one, can be included in the area of applied mathematics. In this chapter it will be shown how phenomena discovered by quantum physics scientists and mathematicians have become the core of the finance world.

There are various steps to follow. First of all, the main results and mathematical notations used in asset pricing will be defined. These concepts are the basis for evaluating options, through methods that will be analyzed later.

The aim here is to generalize a model to describe the *random evolution* of stock prices. There are two important parameters concerning these models, through which the whole theory of option pricing will then develop. After that, it will be shown the tight connection between stocks prices and derivatives.

These first basic assumptions will generally be for all models. But when we move on to more ingenious models, we will require advanced mathematical tools and concepts for multidimensional stochastic processes, such as Cholesky's decomposition and Itô multidimensional lemma.

1.1 Fundamental concepts of mathematics and statistics

1.1.1 Stochastic calculus

The french mathematician Bachelier, at the beginning of the 20th century, included the Wiener process to incorporate the random nature of stock prices. This was the first step towards a stochastic approach to finance, which was however neglected for about 50 years. Indeed, it was only in 1969 that this model was revived, thanks to Robert Merton, who introduced stochastic calculus into the study of finance, trying to understand how prices are set in financial markets.
In this sense, it is easy to say that asset pricing lays its foundation in stochastic calculus. In effect, describing the process followed by a stock is like modeling a random system. There is therefore a need to learn how to govern this process.

Definition 1 (Stochastic process) *A **stochastic process**, $X(t)$, is a collection of random variables, indexed by a time variable t, and defined on some sample space Ω.*

Hence, a stochastic process is function of two variables: for a fixed instant of time t_i, it is a random variable; for a fixed outcome $w \in \Omega$, it is function of time [14].

The evolution followed by a stochastic process is called *realization*, a trajectory of the process X. Every path corresponds to a different realization of the random variable, for each instant of time t_m.

Mathematically, the first tool that allow us to describe the stochastic process is the *sigma-field* (or sigma-algebra). It is a collection of Ω, closed under complement and under countable-unions and countable intersections [2]. In this way, by combining events whose probability can be calculated, the result is an event that still belongs to the sigma-algebra.

Definition 2 (Sigma-field) *The **sigma-field**, $\mathcal{F}$, is collection of subsets of the sample space Ω where:*
(i) the empty set $\emptyset$ belongs to $\mathcal{F}$;

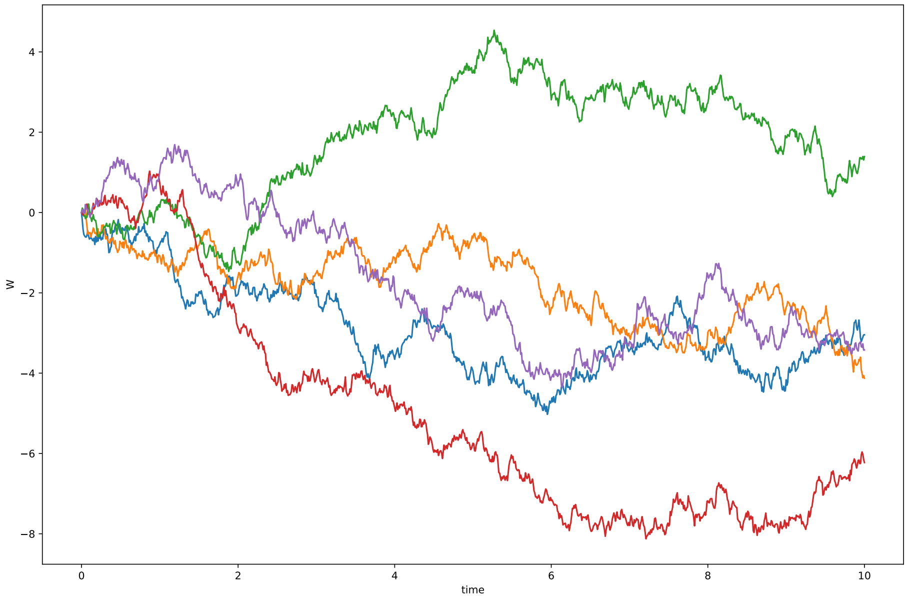

Figure 1.1: Simulation of the trajectory followed by a stochastic process, according to some asset price distribution.

(ii) whenever a set A belongs to $\mathcal{F}$, its complement A^c also belongs to $\mathcal{F}$;
(iii) whenever a sequence of sets $A_1, A_2, \ldots,$ belong to $\mathcal{F}$, their union $U_{i=1}^{\infty} A_i$ also belongs to $\mathcal{F}$.

The ordered sequence of sigma-field is called *filtration*, that can be interpreted as the set of all events occurring within time t. It represents the information available until a certain time T_i, providing the information generated by X in the interval [5]. Mathematically speaking, for each set in $\mathcal{F}(t)$, the distribution of random outcomes is known.

Definition 3 (Filtration) *Let $\mathcal{F}$ be the sigma-algebra of the probability space where the process is defined. A* ***Filtration*** *is a family of sub-sigma algebras of $\mathcal{F}$ increasing in t, i.e. $\mathcal{F}_s$ is contained in $\mathcal{F}_t$ ($\mathcal{F}_s \subset \mathcal{F}_t$) if $s < t$.*

If the filtration generated by process X, $\sigma(X_t)$, is contained in F_t for each t, the process is said to be **adapted** [2].

1.1.2 Wiener process

The Wiener process, or Brownian motion, is the basic building block to model uncertainty. In finance, it provides a model for the realistic asset realization, describing asset prices movements over time.

Definition 4 (Wiener process) *A stochastic process $W(t)$ is a* ***Wiener process*** *if the following properties hold:*
(i) $W(t_0) = 0$, i.e. it starts at zero,
(ii) $W(t)$ is a continuous function with no jumps,
(iii) $W(t)$ has stationary, independent increments,
(iv) each increments is normally distributed, $W(t) - W(s) \sim N(0, t-s)$, with mean equal to 0 and variance rate of $t - s$.

Remark 1 *Let $X(t)$ be a stochastic process. $X(t)$ is a* ***martingale*** *with respect to the filtration $\mathcal{F}_t$ if:*
(i) $X(t)$ is $\mathcal{F}$-adapted;
(ii) $\mathbb{E}[|X(t)|] < \infty$, i.e. the expected value of $X(t)$, for every t, exists and is finite.
(ii) $\mathbb{E}[X(t)|\mathcal{F}(s)] = X(s)$, for $t > s$, i.e. the expected value of a future state of the process, note the evolution of the same in previous times, depends only on the last observation made (Markov property).

The Wiener process, $W(t), t \in [0, t]$, is a martingale, i.e. the conditioned expectation of a future value is equal to its present value [14]:

$$\begin{aligned}\mathbb{E}[W(t+\Delta t) - W(t)|\mathcal{F}(t)] &= \mathbb{E}[W(t+\Delta t)|\mathcal{F}(t)] - \mathbb{E}[W(t)|\mathcal{F}(t)] \\ &= W(t) - W(t) = 0\end{aligned}$$

1.1.3 Stochastic differentiation

For a stochastic process, the Riemann-Stieltjes integral cannot be used, as the Wiener process is nowhere differentiable. It is therefore necessary to use the integral of Itô, that is a stochastic generalization of the Riemann-Stieltjes integral, where integrands and integrators are now stochastic processes [2].
To integrate a stochastic process $X(t)$ it is necessary to know the conditions that guarantee the existence of the stochastic integral.

Remark 2 *A stochastic process $X(t)$ is Itô integrable on the interval $[0,T]$ if these two conditions hold:*
(i) $X(t)$ is adapted for $t \in [0,T]$, i.e. $X(t)$ is $\mathcal{F}(t)$-measurable.
(ii) $X(t)$ is locally square-integrable process, $\int_0^T \mathbb{E}[X^2(t)dt] < \infty$.

Definition 5 (Itô Integral) *For any square-integrable process $X(t)$, adapted to the filtration generated by $W(t)$, with continuous sample path, the* ***Itô integral*** *is:*

$$I(T) = \int_0^T X(t)dW(t)$$

1.2 Stochastic model for asset and option pricing

So far it has been described how a Brownian motion can be able to describe the randomness of a stock prices. However, this is not enough to define the entire process followed by the evolution of a share price, which also depends on other variables, which will now be defined.

1.2.1 Geometric Brownian Motion

The Wiener process, $W(t)$, defined so far has a drift rate equal to 0 and a variance rate equal to 1. It is therefore necessary to generalize this process in such a way to have arbitrary drift rates and variances, where:
(i) *drift rate*, μ, is the mean change per unit time for a stochastic process;
(ii) *variance rate*, σ, is the variance per unit time for a stochastic process.
In effect, the Wiener process as defined involves some problems if used to describe the evolution of assets, because drift rate equal to zero means that the expected value of W at any given time t is equal to its current value, while in real life stocks trend upward or downward [4]. To capture this aspect of stocks is necessary to take a step further the Wiener process.

Definition 6 *A Generalized Wiener process is stochastic process where the change in a variable in time t has a normal distribution with constant mean and variance, both proportional to t:*

$$dS(t) = \mu dt + \sigma dW(t)$$

It is therefore assumed that the stock will grow, moving at a constant rate μ, where the $dW(t)$ term adds noise.

However, this model fails to capture a key aspect of stock prices: the expected percentage return required by investors from a stock is independent of the stock's price. In other words, with the assumption of constant expected drift rate, this model allows us to drift the stock either up or down in a given direction.
This, however, is inappropriate, since a stock may rise at a constant rate but crash the next day. It is therefore necessary to assume that the *expected return* (i.e. expected drift divided by the stock price) is constant, and not the expected drift rate[3].

The Geometric Brownian Motion (GBM) process is then introduced. It is the exponential transformation of the linear Brownian Motion, where the logarithm of the asset price follows an arithmetic Brownian motion, driven by a Wiener process $W(t)$.

Definition 7 (Geometric Brownian Motion) *The asset price $S(t)$ is said to follow a **Geometric Brownian Motion** process if it satisfies the following SDE:*

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{1}$$

(i) the Brownian motion, $W(t)$, is under the real-world measure P;
(ii) $\mu = \mu$ is the drift parameter, a constant growth rate of the stock;
(iii) σ is the constant percentage volatility parameter, the amount by which an asset price differs from its expected value.

Remark 3 Companies usually pay dividends during the year, but this aspect is not taken into account in the basic model of Brownian Geometric Motion. To fix this, it is assumed a continuous and constant dividend yield of size $qS(t)$, i.e. a proportional dividend of size $qS(t)dt$.

$$dS(t) = (\mu - q)S(t)dt + \sigma S(t)dW(t)$$

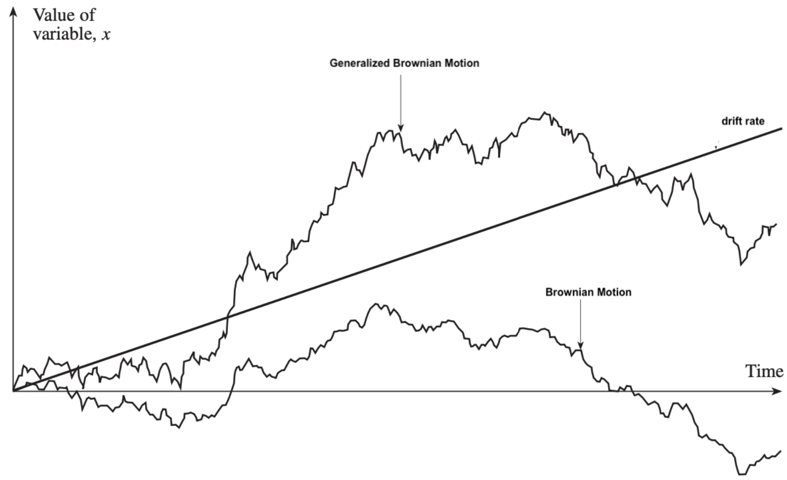

Figure 1.2: Differences between a Wiener process and a Geometric Brownian Motion, with drift rate μdt

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1.2.2 Markov property

An important rule must be defined: past history of a stock is irrelevant on today's valuation, as it is assumed that today's price incorporates all past information (memoryless).
This is the so-called Markov process, that is a stochastic process in which only the current value of the variable is relevant to predict the future. In mathematical terms, the conditional probability distribution of future states depends only on the present state, and not on the history [14].

Definition 8 (Markov property) *The adapted stock price process $S(t)$ on a filtered probability space has the* ***Markov property****, if for each bounded and measurable function $g : R^N \to R$,*

$$E[g(S(t))|\mathcal{F}(s)] = E[g(S(t))|S(s)],\ s \leq t$$

Brownian Motion is a particular type of Markov stochastic process with a mean change of *zero* and a variance rate of 1.0 per year.

1.2.3 Itô process

Definition 9 (Itô process) *An **Itô process** is a type of Geometric Brownian Motion process where the constants μ and σ are dependent on the value of the underlying variable x and time t. In mathematical terms [5]:*

$$dX(t) = \mu(t, X(t))dt + (t, X(t))dW(t) \tag{2}$$

All option pricing models define the process followed by the stocks as an Itô process. But what about the option pricing?
Since an option is nothing more than a function of the price of the underlying asset and time, we need a link between the stochastic process of the stock (underlying) and that of the derivative.

In this sense, it is fundamental to introduce the Itô lemma, which is the tool that allows us to derive the stochastic process followed by the function of a variable based on the process followed by the variable itself [2], i.e. it finds the stochastic process followed by a differentiable function $G(x, t)$.

Definition 10 (Itô's lemma) *Let $g(t, X)$ be a function of $X = X(t)$ and time t, with continous partial derivatives, $\frac{\partial g}{\partial X}, \frac{\partial g^2}{\partial X^2}, \frac{\partial g}{\partial t}$. A stochastic variable $Y(t) := g(t, X)$ then also follow an Itô process, governed by the same Wiener process $W(t)$, that is:*

$$\mathrm{d}Y(t) = \left(\frac{\partial g}{\partial t} + \mu(t, X)\frac{\partial g}{\partial X} + \frac{1}{2}\frac{\partial^2 g}{\partial X^2}\sigma^2(t, X)\right)\mathrm{d}t + \frac{\partial g}{\partial X}\sigma(t, X)\mathrm{d}W(t)$$

Itô lemma is a calculation tool for the processes described by stochastic differential equations (SDEs). It is a way of calculating the stochastic process followed by a function of a variable from the stochastic process followed by the variable itself [3].

Remark 4 Itô lemma shows that the random variable S of the Geometric Brownian Motion is log-normally distributed. Using $dX(t) = logS$:

$$dX(t) = (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dW(t),$$

1.3 Multidimensionality

When more complex option pricing models will be analyzed, we will face with multi-dimensional stochastic processes. In this case it is necessary to move to SDE system, with uncorrelated (independent) Brownian motions, $\bar{\mathbf{W}}(t)$.
The mathematical tool needed to change the formulation of multidimensional processes from $\mathbf{W}(\mathbf{t})$ dependent Brownian motions to $\bar{\mathbf{W}}(t)$ independent motions is the *Cholesky decomposition* [5].

1.3.1 Cholesky decomposition

Definition 11 (Cholesky decomposition) *Each symmetric positive definite matrix C has a unique factorization, the so-called* ***Cholesky decomposition****, of the form,*

$$\mathbf{C} = \mathbf{L}\mathbf{L}^T$$

with L is a lower triangular matrix with positive diagonal entries.

1.3.2 Itô multidimensional lemma

Let $X(t)$ be a stochastic process based on independent Brownian motions.

$$d\mathbf{X}(t) = \bar{\mu}(t, \mathbf{X}(t))dt + \bar{\sigma}(t, \mathbf{X}(t))d\bar{\mathbf{W}}(t) \tag{3}$$

It can be expressed in function of drift $\bar{\mu}$ and volatility terms $\bar{\sigma}$ as:

$$\begin{bmatrix} dX_1 \\ \dots \\ dX_n \end{bmatrix} = \begin{bmatrix} \bar{\mu_1} \\ \dots \\ \bar{\mu_n} \end{bmatrix} dt + \begin{bmatrix} \bar{\sigma}_{1,1} & \dots & \bar{\sigma}_{1,n} \\ \dots & \dots & \dots \\ \bar{\sigma}_{n,1} & \dots & \bar{\sigma}_{n,n} \end{bmatrix} \begin{bmatrix} d\bar{W}_1 \\ \dots \\ d\bar{W}_n \end{bmatrix}$$

Let $g \equiv g(t, \mathbf{X}(t))$ be differentiable on $\mathbb{R}$ x $\mathbb{R}^n$. The increment $dg(t, \mathbf{X}(t))$ is governed by the SDE, named as the Itô lemma for processes with independent Brownian Motion:

$$dg(t, \mathbf{X}(t)) = \frac{\partial g}{\partial t}dt + \sum_{j=1}^{n} \frac{\partial g}{\partial x_j} dX_j(t) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j} dX_i(t) dX_j(t)$$

$$= (\frac{\partial g}{\partial t} + \sum_{i=1}^{n} \bar{\mu}_i(t, \mathbf{X}(t)) \frac{\partial g}{\partial x_i} + \frac{1}{2} \sum_{i,j,k=1}^{n} \bar{\sigma}_{i,k}(t, \mathbf{X}(t)) \bar{\sigma}_{j,k}(t, \mathbf{X}(t)) \frac{\partial^2 g}{\partial x_i \partial x_j}) dt$$
$$+ \sum_{i,j=1}^{n} \bar{\sigma}_{i,j}(t, \mathbf{X}(t)) \frac{\partial g}{\partial x_i} d\bar{W}_j(t)$$

1.4 Quick remarks of derivatives definitions

A **derivative** can be defined as a financial instrument whose value depends on the values of an underlying variables. A stock option, for example, is a derivative whose value is dependent on the price of a stock [3].

An **option** is a contract that gives the buyer the right, but not the obligation, to buy or sell an underlying asset at a specific price (the so-called **strike price,** K) on or before a certain date.

A **call** option is a contract that gives the buyer the right, but not the obligation, to buy an underlying asset at a specific price on or before a certain date (the so-called **expiry date**)[3]. The payoff is given by:

$$\max(S - K, 0)$$

where S is the value of the underlying at the maturity date.

A **put** option is a contract that gives the buyer the right, but not the obligation, to sell an underlying asset at a specific price on or before a certain date (the so-called **expiry date**)[3]. The payoff is given by:

$$\max(K - S, 0)$$

where S is the value of the underlying at the maturity date.

There are many different variations of puts and calls, the most popular being *European* and *American* option types. The main difference between European and American options is the exercise date: while the European can be exercised only at the date of expiration, the American options may be exercised at any time before, or on, the date of expiration.

As we will see in later chapters, it is possible to obtain an analytical formula for the value of the derivative under the assumptions of no arbitrage opportunities (Black, Merton and Scholes Theory [7]). In this approach the dynamical model for the underlying of the derivative plays a fundamental role and in particular its volatility.

Chapter 2

Characteristics of Volatility

The concept of volatility is of paramount importance. Pricing models vary around its definition and use. It will be shown how it can be calculated and its central role in option pricing.

This chapter starts first of all with volatility various definitions. After that, it will be shown a deep analysis of volatility and its relationship with the different strike prices and the different maturities of stocks, the so-called *volatility surfaces*, and the implications that exist with the strike price, the so-called *volatility smiles*.

Implied volatility, which is the basis of all these assumptions, plays a crucial role in the valuation models: although this is calculated by the Black-Scholes equation (as the only variable that is not observable) it plays a fundamental role in the other models that will be shown later.

In essence, the ability to model volatility is crucial to option pricing, since in the stochastic differential equation it governs the asset price. All models strongly depend on it.

2.1 Definitions of volatility

Among the definitions, volatility, σ, is the measure of uncertainty about the stock returns, measured as the degree of variation of the price of financial instruments over time.

2.1.1 Historical volatility

Historical volatility estimates the fluctuations of a financial securities by measuring price changes over predetermined periods of time, i.e. using past empirical price data [3].
The σ is provided by using empirical asset price data and is calculated as:

$$\hat{\sigma} = \frac{\sqrt{\hat{V}}}{\Delta t}$$

where $\hat{V}$ is the sample variance of the log-returns of the stock:

$$\hat{V} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

with x_i as the log-returns of the stock:

$$x_i = ln\left(\frac{X(t_i)}{X(t_{i-1})}\right)$$

and $\bar{x}$ as the sample mean:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

2.1.2 Diffusion term - volatility

In stochastic differential equations, $\sigma(t, S(t))$ can be defined as the diffusion term that measures the randomness in asset return $S(t + \Delta t) - S(t)$ [1].
In this sense, the volatility is related to the standard deviation of the logarithmic price increments, conditioned on price observations, that are usually described with the following stochastic process in the Black and Scholes Theory:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

2.1.3 Implied volatility

The implied volatility, σ_{imp}, is the volatility implied by option prices observed in the market. It is defined as the parameter of the Black-Scholes solution that provides the market option price $V_c^{mkt}(K,T)$ at time $t_0 = 0$.

Remark 5 *The implied volatility is derived from the Black-Scholes formula, which will be subsequently derived. However, it is necessary here to define the formula from which implied volatility is calculated.*
A call option value can be written as:

$$V_c(t,S) = S(t)F_{N(0,1)}(d1) - Ke^{-r(T-t)}F_{N(0,1)}(d2)$$

A put option value can be written as:

$$V_p(t,S) = Ke^{-r(T-t)}F_{N(0,1)}(d2) - S(t)F_{N(0,1)}(-d1)$$

where:

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$
$$d_2 = \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T-t}.$$

$F_{N(0,1)}$ = *cumulative distribution function of a standard normal variable.*

Matematically speaking, σ_{imp} is the σ-value, for which:

$$V_c(t_0, S; K, T, \sigma_{imp}, r) = V_c^{mkt}(K,T) \tag{4}$$

As opposed to historical volatility, which is backward looking, it is forward looking.

The implied volatility is relatively low for at-the-money options, while it becomes progressively higher for into-the-money and out-of-the money options [1].

Remark 6 (Option moneyness) The intrinsic value of an option is the difference between the strike price and underlying asset price. It is possible to have:

1. At the money (ATM), that are options whose strike price is at or very near to the current market price of the underlying.

2. In the money (ITM), that are options whose value in a strike price that is favorable in comparison to the market price of the underlying asset:

    (a) An in-of-the-money call option means that the underlying price is trading above the strike price of the call.

    (b) An in-of-the-money put option means that the underlying's price is below the put's strike price.

3. Out of the money (OTM), are options that has no intrinsic value.

    (a) An out-of-the-money call option means that the underlying price is trading below the strike price of the call.

    (b) An out-of-the-money put option means that the underlying's price is above the put's strike price.

2.2 Volatility surface

The plot of the implied volatility of an option as a function of its strike price K and time to maturity T, i.e. in a (t, S)-plane, is known as *volatility surface*. It defines the appropriate volatilities for the valuation of options with different strike prices and different maturities [1].

Since Black-Scholes assumes that the volatility of the underlying asset is constant, the volatility surface in Black-Scholes method is completely flat.
However, in real life this is not true: Black-Scholes assumptions are violated, and with one σ input, the Black-Scholes model can only match one market quote at a strike and maturity.
This is why other various models have been developed over the years; in the next chapters, it will be shown the local volatility model and stochastic volatility model.

1. Local volatility models take into account the volatility smile, in which the volatility in the $dS(t)/S(t)$ term is written as a function of $S(t)$.

2. In stochastic volatility models, volatility has its own dynamics described by a stochastic process.

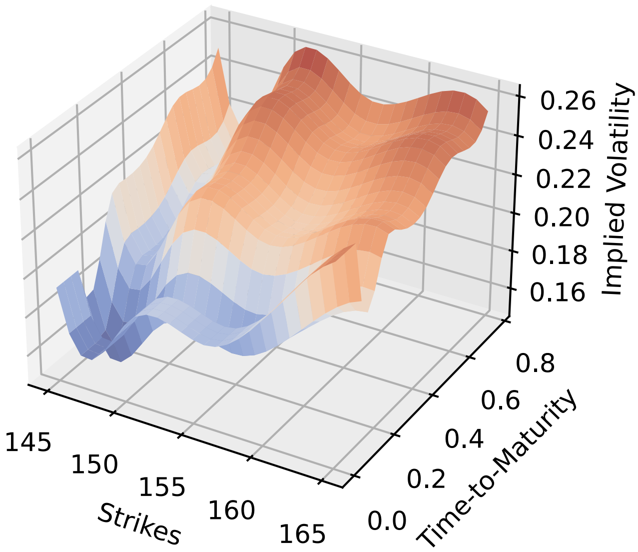

Figure 2.1: An example of implied volatility surface with a the three-dimensional surface that relates the implied volatilities of options with different strikes and different maturities.

As said, the Black-Scholes formula is used to find the volatility value. In doing so, however, we come up against the biggest inconsistency of the model itself: the value of the implied volatility in the prices is not constant but varies according to the maturity and the strike. This behavior is known as an implied volatility smile.

2.3 Volatility smile

The relationship between implied volatility and strike price for options with a certain maturity is known as a *volatility smile* (thanks to the *u-shaped* relationship [1]), thus relating the implied volatilities as a function of exercise prices.
Mathematically speaking, implied volatility is generally a convex function of the strike price, with lowest point on the plot is usually around the at the money point where $S(t) = K$.

It is actually possible to have different results from an empirical observation:

if implied volatility is a concave function of the strike price, it will be defined the *volatilty frown* [5].
Moreover, when the plot is downward sloping, it is defined as a *volatility skew*, i.e. implied volatilities are higher for low strikes than for high strikes [5].

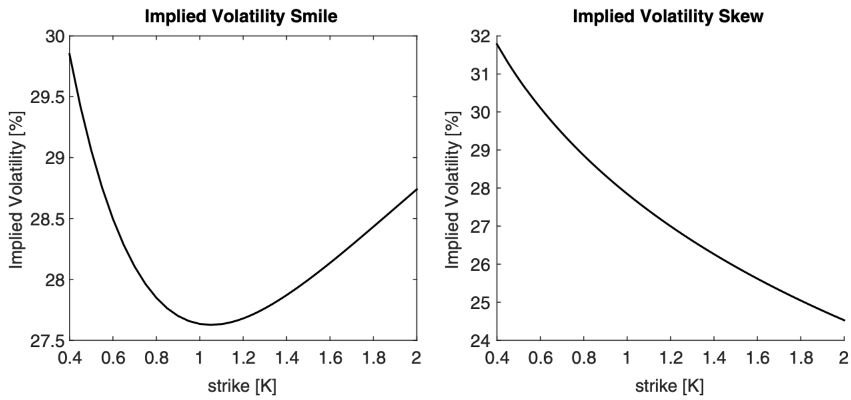

Figure 2.2: Possible implied volatility shapes. In the left figure we see a so-called volatilty smile. In the right figure a volatilty skew.

Both Local volatility and Stochastic volatility models take the volatility smile into account.

For a given strike price and maturity, the correct volatility to price an option derivative is the same, whether it is a call option or a put option. In this sense, it is possible to say that volatility smile is the same, as well as volatility surface, is the same for (European) calls and puts, thanks to the Put-Call parity.

Remark 7 (Put-Call parity)

Put-call parity shows the relationship that has to exist between european put and call options that have the same underlying asset, maturity, and strike prices.

$$V_c(t, S) = V_p(t, S) + S(t) - Ke^{-r(T-t)}$$

Chapter 3

Constant volatility model

The Black-Scholes model is the basic building block of all option pricing models. Proposed in 1973, it was the first model to model the stock price as a Geometric Brownian Motion, with the assumption of constant volatility.

In this chapter, the structure of the model and the details of the derivation of the pricing formula will be presented.

Black-Scholes analysis of European options provides a closed-form solution to option pricing, which requires only observable variables, except volatility.

The two main assumptions of the model are that volatility is constant and that the underlying follows a geometric Brownian motion.
Its derivation will be shown, as well as its relationship with the dimension to the risk, the so-called Greeks.

Finally, a list of the pros and cons of this model will be made. We will see in the next chapters how other models try to overcome them.

3.1 Black-Scholes method

Black and Scholes derived their famous partial differential equation (PDE) for the valuation of option derivatives in 1973. With their equation, it is possible to compute the fair value for the option price at any time $t \in [0, T]$, and at any future stock price $S(t) \in [0, S\max]$.
The key hypothesis is that the interest rate r and volatility σ are constants or known functions of time t. Other assumptions includes:
(1) There are no arbitrage opportunities
(2) The underlying follows a geometric Brownian motion
(3) The market is completely liquid
(4) Short selling is possible
(5) There are no transaction costs
(6) The underlying and the derivative are traded at any instant of time and can have any positive real value
(7) There are no dividends

3.1.1 Derivation of the partial differential equation

The derivation of the pricing PDE is based on the concept of a replicating portfolio, i.e. a risk-free portfolio containing shares and options. In theory, in the absence of arbitrage opportunities the rate of return of this portfolio must be equal to the risk-free interest rate, as both the stock and the option are subject to the same source of uncertainty, i.e. the change in the stock price.

The replicating portfolio, $\Pi(t, S)$, consists of:
(i) one long position in the option $V(t, S)$;
(ii) a short position of size Δ in the underlying $S(t)$.

$$\Pi(t, S) = V(t, S) - \Delta S(t)$$

Where the underlying for the financial derivative contract is a stock price process, $S \equiv S(t)$, that is assumed to be a Geometric Brownian Motion, with the dynamics under the real-world measure P:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

The derivative contract, $V \equiv V(t,S)$, represents the value of a European option. It is a function of the stochastic process $S(t)$ and of time t. It is thank to Itô's lemma that it is possible to build a model that describes the dynamics of the derivatives:

$$dV(t,S) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2$$

$$= \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dW$$

Applying Itô's lemma it is possibile to show the evolution for an infinitesimal change in portfolio value:

$$d\Pi = dV - \Delta dS$$

$$= \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dW - \Delta[\mu S dt + \sigma S dW]$$

$$= \left[\frac{\partial V}{\partial t} + \mu S(\frac{\partial V}{\partial S} - \Delta) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right]dt + \sigma S(\frac{\partial V}{\partial S} - \Delta)dW$$

This portfolio contains some randomness, which depends on the Brownian Motion W. Setting up a risk-less portfolio, i.e. setting up a delta-neutral position in the replicating portfolio, it is possible to eliminate the risk:

Remark 8 *The Delta, Δ, is the sensitivity of the option with respect to the stock, i.e. the rate of change in the option value with respect to a change in the stock value $\Delta = \frac{\partial V}{\partial S}$.*

In this sense, the dW-terms cancel out, and the infinitesimal change of portfolio $\Pi(t,S)$, in time instance dt, is given by:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt$$

Since it is a risk-less portfolio, it must do not contain the drift parameter μ, which drives the stock $S(t)$ under the real-world measure P. The only value the portfolio depends on is volatility, σ, which is the representation of the uncertainty about the future behavior of the stock prices.

The return provided by the portfolio is the risk-free rate, as a money-savings account, $M(t) = M(t0)\epsilon^{r(t-t_0)}$, that for an amount $\Pi \equiv \Pi(t, S)$ can be expressed as:

$$d\Pi = r\Pi dt$$

where r is to the constant interest rate on a money-savings account.

The change in portfolio value is thus:

$$d\Pi = r\left(V - S\frac{\partial V}{\partial S}\right)dt$$

Equating the change in this risk-less portfolio and replicating portfolio, the final Black-Scholes partial differential equation for the value of the option $V(t, S)$ is revealed:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{5}$$

It is a parabolic PDE, that must be satisfied by the price of any derivative dependent on a non-dividend-paying stock, holding for both calls and puts [1]. Moreover, since risk factors do not enter the equation, it is said that Black-Scholes equation is independent of risk preferences, i.e. the assumption that all investors are risk neutral.

3.1.2 Black-Scholes solution

The natural condition for the Black-Scholes PDE is a final condition, where:

$$V(T, S) = H(T, S)$$

where $H(T,S)$ is the payoff function:

$$V_c(T,S) = H(T,S) = max(S(T) - K, 0), \text{for a call option}$$

$$V_p(T,S(T)) = H(T,S(T)) = max(K - S(T), 0), \text{for a put option}$$

Using Feynman-Kac theorem [2], that form the basis for a closed-form expression for the option value, it is possible to compute the correct expectation of a discounted payoff function.
It provides a link between the partial differential equation of a diffusion process and its expectation.

Given the money-savings account, modeled by $dM(t) = rM(t)dt$, with constant interest rate r, let $V(t,S)$ be a sufficiently differentiable function of time t and stock price $S = S(t)$. Suppose that $V(t,S)$ satisfies the following partial differential equation, with general drift term, $\mu(t,S)$, and volatility term, $\sigma(t,S)$:

$$\frac{\partial V}{\partial t} + \mu(t,S)\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2(t,S)\frac{\partial^2 V}{\partial S^2} - rV = 0$$

The solution $V(t,S)$ at any time $t < T$ is then given by:

$$V(t,S) = e^{-r(T-t)}\mathbb{E}[H(T,S)|\mathcal{F}(t)]$$

The Feynman-Kac theorem also holds true in logarithmic coordinates, i.e., when $X(t) = logS(t)$. With this transformation of variables, the resulting log-transformed Black-Scholes PDE reads:

$$\frac{\partial V}{\partial t} + r\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2\left(\frac{\partial V}{\partial X} + \frac{\partial^2 V}{\partial X^2}\right) - rV = 0$$

The final solution $V(t,X)$ gives the representation of option value:

$$V(t,X) = e^{-r(T-t)}\mathbb{E}[H(T,X)|\mathcal{F}(t)]$$

A closed-form solution of the Black-Scholes PDE for a European options with a constant strike price K can be finally derived, with $H_c(T,S) = max(S(T) - K, 0)$.
A call option value can be written as:

$$V_c(t,S) = S(t)F_{N(0,1)}(d1) - Ke^{-r(T-t)}F_{N(0,1)}(d2) \tag{6}$$

A put option value can be written as:

$$V_p(t,S) = Ke^{-r(T-t)}F_{N(0,1)}(d2) - S(t)F_{N(0,1)}(-d1) \tag{7}$$

where:

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$
$$d_2 = \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T-t}.$$

$F_{N(0,1)}$ = cumulative distribution function of a standard normal variable.

3.2 Hedge parameters - Greek letters

Important information are provided with the so-called *hedge parameters*, i.e. the option Greeks, where each Greeks measures a different dimension to the risk in an option position.

3.2.1 Delta, Δ

As seen, Delta, Δ, is the sensitivity of the option value V with respect to small changes in stock S, or, as usually defined, it is the rate of change of the option price with respect to the price of the underlying asset [3].

$$\Delta = \frac{\partial V}{\partial S} \tag{8}$$

Practically, it is the number of stocks that one must buy or sell to hedge against the risk, i.e. to be delta-neutral: if $\Delta = 0.1$, then it is necessary to buy 0.1 of a stock to be delta neutral. A negative number implies that short-selling of stocks should take place.

As seen, Delta is crucial in the derivation of the Black-Scholes differential equation, as it allows to set up a risk-less portfolio consisting of a position in an option on a stock and a position in the stock, i.e. to build a portfolio delta-neutral. It is therefore derived from this hedging hypothesis, keeping the value of the portfolio

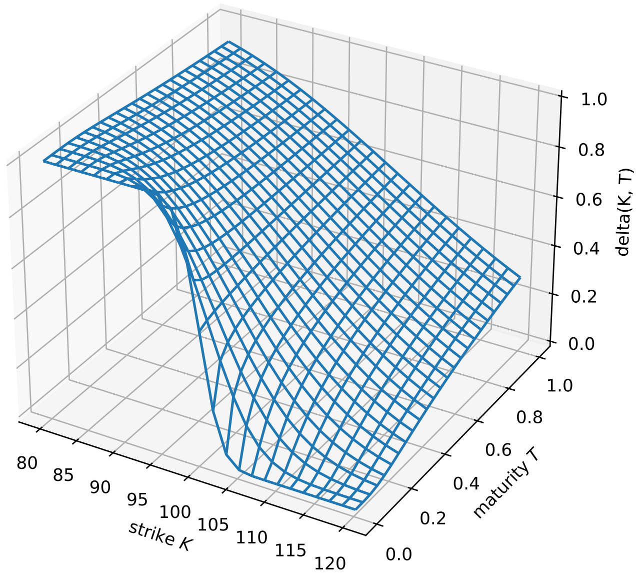

Figure 3.1: An example of Delta of the call option. It varies both with moneyness and maturity date T.

stable, not making it change when the underlying asset moves. In this sense, the portfolio derivative must be equal to 0.

$$\frac{\partial \Pi(t,S)}{\partial S} = \Delta = \frac{\partial V}{\partial S}$$

Moreover, we can also express Delta in function of a call and a put options. Respectively:

$$\Delta = \frac{\partial}{\partial S} V_c(t,S) = F_{N(0,1)}(d_1)$$

$$\Delta = \frac{\partial}{\partial S} V_p(t,S) = F_{N(0,1)}(d_1) - 1$$

3.2.2 Gamma, Γ

Gamma, Γ, is a measure of the rate of change of Δ with respect to the price of the underlying asset. It could be thought of as the Δ of Delta, measuring the sensitivity of the option delta [3].

Mathematically, it is the second partial derivative of the portfolio with respect to asset price.

$$\Gamma = \frac{\partial \Delta}{\partial S}$$

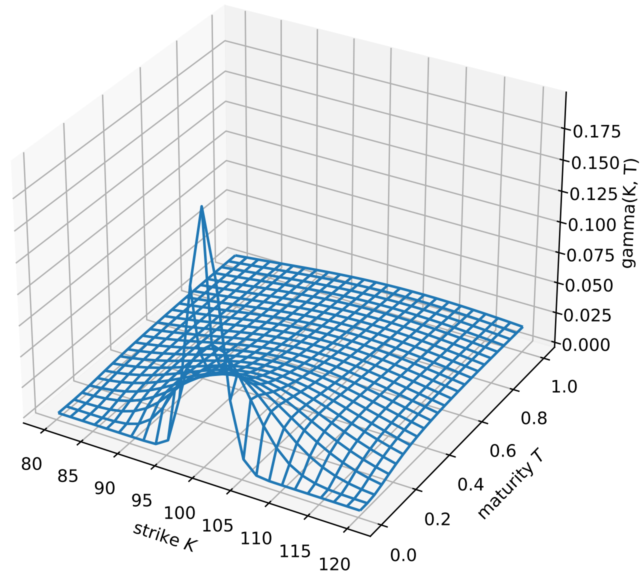

Figure 3.2: An example of Gamma. It has the highest values around the ATM level for short maturities.

When gamma is extremely positive or negative, the delta is very sensitive to the price of the underlying asset, thus changing rapidly. A frequent adjustment is therefore necessary in this case.

3.2.3 Theta, Θ

Theta, Θ, is a measure of V with respect to the passage of time, t. It is the rate of change of the value of the portfolio with respect to the passage of time with all else remaining the same [3].
This is sometimes referred to as the *time decay* of the value of the option: mathematically, the Θ of a derivative is the rate of change of the value with respect to the passage of time (partial derivative with respect to small t).

$$\Theta = -\frac{\partial V}{\partial t}.$$

3.2.4 Relationship between Greeks and Black-Scholes

Note that every derivative is a Greek in the Black-Scholes differential equation:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

Thus by direct substitution we have,

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV$$

According to Black-Scholes method, it can be created a delta-neutral portfolio building up a portfolio where $\Delta = 0$.

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV \tag{9}$$

This shows the inverse relationship between Θ and Γ: when the former is extremly positive, the latter tends to be large and negative, and viceversa.

3.2.5 Vega, V

The Vega of an option is the partial derivative of its value with respect to volatility σ [3].
It therefore represents the risk factor derived from the volatility itself, thus going to the assumptions of Black-Scholes method, where volatility is constant. In effect, as it will be shown next, in real life there is evidence that volatility changes randomly.

$$V = \frac{\partial V}{\partial \sigma}$$

When Vega is highly positive or highly negative, the portfolio's value is very sensitive to small changes in volatility.

3.3 Limits of Black-Scholes model

The assumptions made to derive the Black-Scholes model do not reflect the reality of the financial markets.

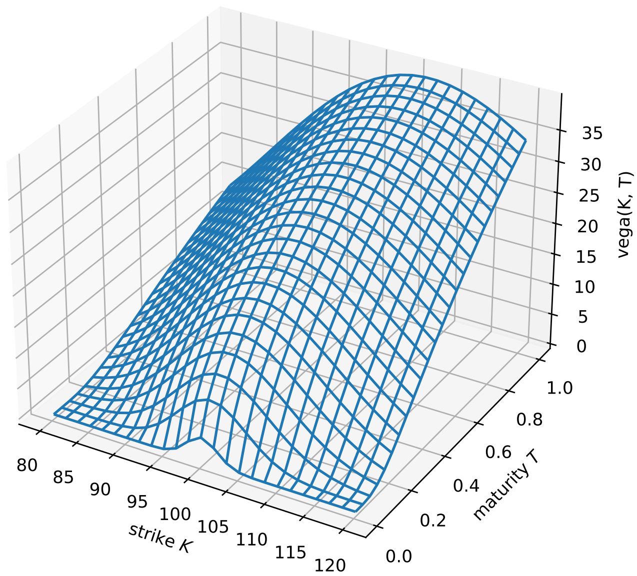

Figure 3.3: An example of Vega. It increases with T and decreases from the ATM level in both directions, i.e. OTM and ITM

3.3.1 Log-returns

Geometric Brownian motion predicts that log returns are distributed like a normal one. Empirically, however, we observe a distribution whose tails (especially the left) are much fatter than those expected, i.e. the extreme realizations in the real world occur with a greater frequency than the theoretical one.

3.3.2 Leverage effect

Leverage effect ca be defined as the negative correlation between stock prices and volatility.

$$leverage = \frac{v}{MKT}$$

Where v is the the total debt of a firm and MKT is the market capitalisation. In effect, it is empirically proven that falls in underlying correspond to a peak in the volatility values and vice-versa.

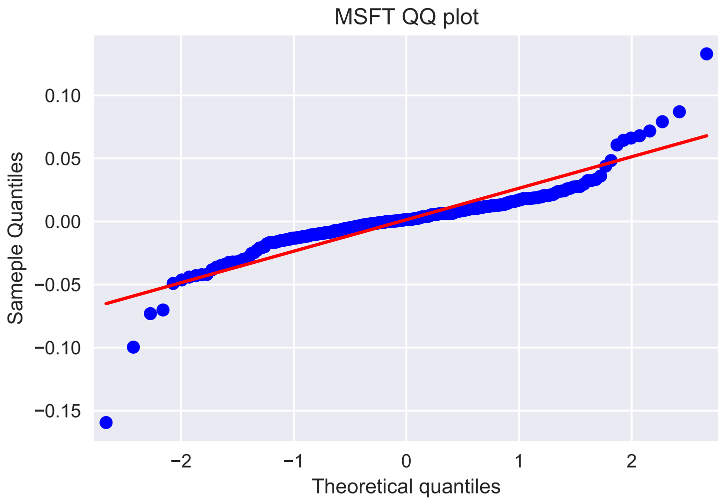

Figure 3.4: Log-return. MSFT's QQ plot of log-return distributions: many observations fall outside the red line, i.e. extreme observations are not in line with the normal distributed quantiles.

3.3.3 Volatility clustering

Volatility clustering can be defined as a positive auto-correlation of the absolute log-returns over few days. In other words, it refers to the phenomenon whereby volatility tends to crystallize on the most recent values: therefore periods of high volatility are more likely to be followed by periods with high volatility and vice-versa *(see Figure 3.6)*.

3.3.4 Implied volatilty hypothesis

As said, the key assumptions of the Black-Scholes theory is that volatility is constant. This, however, is not consistent with the observations of the financial market: proceeding with the calculation of the implied volatility through the Black-Scholes equation, it shows a so-called implied volatility smile.

3.4 Alternative models

Models that take these disadvantages into account have thus been developed.

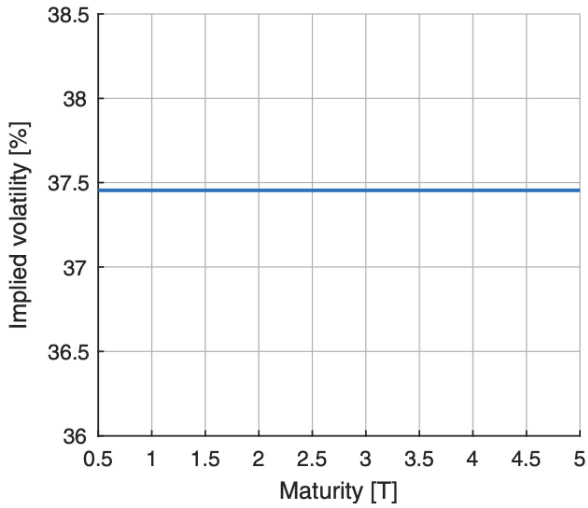

Figure 3.5: The volatility term structure for Black-Scholes model with constant volatility σ.

1. In Local Volatility models, volatility is a function of both the current asset and time. These models are based on the implied volatility observed by the market, and then automatically-calibrates, reproducing furthermore exactly the smiles and inclinations of the market volatility.

2. In Stochastic volatility models, volatility of the asset follows a random process. They needs to be re-calibrated frequently, to determine the open parameters of the underlying stock process so that model and market option prices fit.

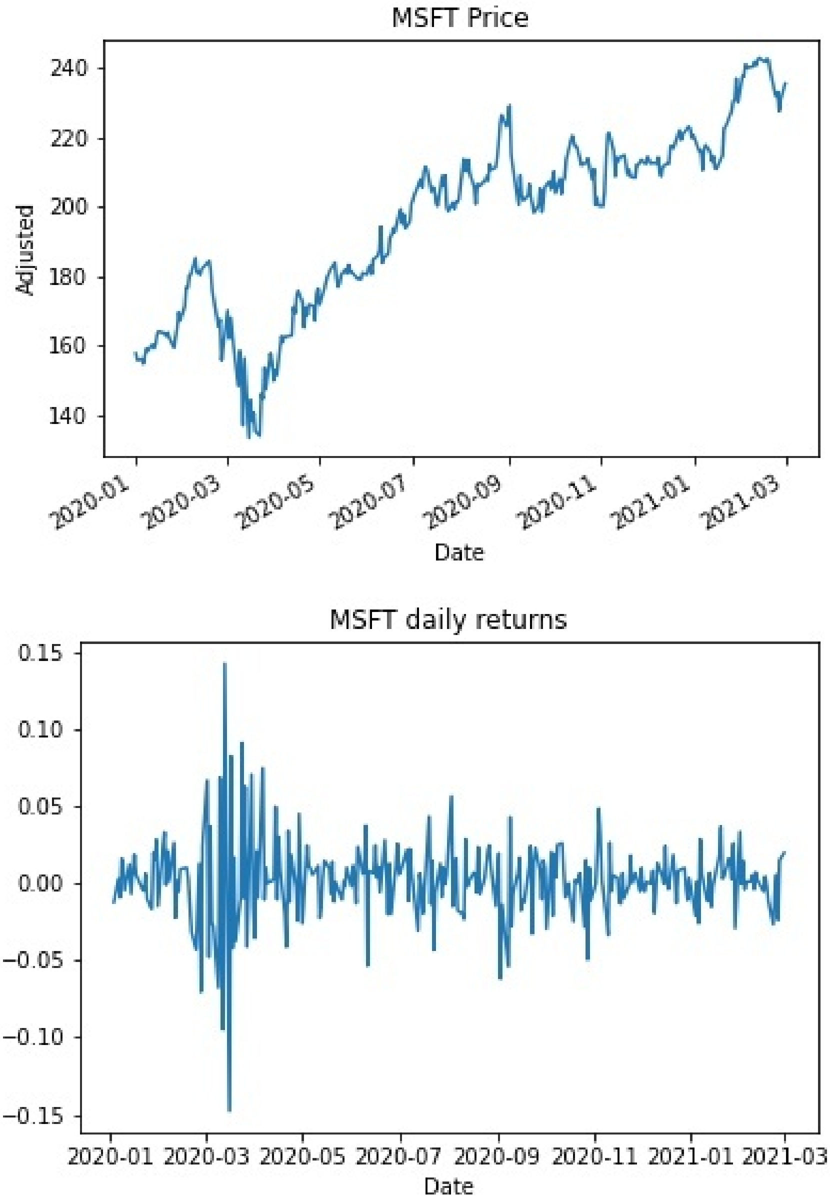

Figure 3.6: Volatility Clustering. MSFT's log-returns: the series seems to randomly fluctuate around zero, meaning there is little autocorrelation.

Chapter 4

Local volatility models

As seen, the Black-Scholes model is based on the assumption that the stochastic process of the underlying evolves according to a geometric Brownian motion, whose diffusion coefficient, i.e. volatility, is constant.

However, in real life the value of this implied volatility is not constant but varies according to the maturity and the strike, generating the so-called implied volatility smile.

In essence, Black-Scholes model is that it is not able to recognize the implied volatility smile.

The so-called *Local-Volatility* models have therefore been developed, in which volatility is expressed as a deterministic function of time and of the underlying.

These models have a great advantage: it is consistent with the market. Dupire's volatility, as will be shown, is exact by construction. So, no additional source of randomness is introduced into the model. In this case it is said that the model is complete.

4.1 Local Volatility methods

In local volatility models, the volatility of the process is a function of the stock $S(t)$.
The approach is to take into account the volatility smile, overcoming the limits of the Black-Scholes model, where the volatility is constant, which is not consistent with the empirical observation that implied volatility varies with the option's expiration date.
In essence, local volatility models aim to be consistent with the volatility smiles observed on the market, setting the diffusive parameter of the underlying, the so-called *local volatility*, in order to fit market prices [1].
The structure of local volatility models is made by:

$$dS(t) = rS(t)dt + \sigma_{LV}(t, S)S(t)dW(t)$$

With the diffusive parameter σ_{LV} as a deterministic function dependent on time and on the value of the underlying.

There are two main models supporting the local-volatility:

1. Parametric Local Volatility, that consists in assuming a parametric shape dependent on several parameters, which will be calibrated on the market surface.

2. Dupire Local Volatility, that will be shown in this thesis, where the local volatility is calculated directly from the market prices by exploiting the partial differential equation that governs asset dynamics.

4.1.1 Local Volatility derivation

Local volatility modeling is based on the financial version of the Fokker-Planck equation, which describes the evolution over time of the probability density function (PDF) of the position of a particle [5].
In this case, the *implied stock density*, is provided from option prices via differentiation of the call prices.

$$f_{S(t)}(y)) = e^{-r(T-t_0)} \frac{\partial^2}{\partial y^2} V_c(t_0, S_0; K, T)$$

4.1.2 Fokker-Planck Equation

The transition density $f_{S(t)}(y) = f_S(t, y; t0, S0)$, i.e. the marginal density of spot at maturity T, associated to the general SDE for $S(t)$,

$$dS(t) = \mu(t, S)dt + \sigma(t, S)dW(t), \tag{10}$$

evolves following the Fokker-Planck equation.

Definition 12 (Fokker-Planck PDE) *For the diffusion process as defined, the* ***Fokker-Planck PDE*** *for the probability density $f_{S(t)}(y)$ of the random variable $S(t)$ is:*

$$\frac{\partial}{\partial t} f_{S(t)}(y) + \frac{\partial}{\partial y}[\mu(t, y) f_{S(t)}(y)] - \frac{1}{2}\frac{\partial^2}{\partial y^2}[\sigma^2(t, y) f_{S(t)}(y)] = 0$$

$$f_{S(t_0)}(y) = \delta(y = S_0)$$

4.1.3 Dupire Equation

The risk neutral price of a call with strike K and maturity T is given by:

$$\begin{aligned}\frac{\partial V_c(t_0, S_0; K, T)}{\partial T} &= \frac{\partial}{\partial T}(e^{-r(T-t_0)} \int_K^{+\infty} (y - K) f S(t)(y) dy \\ &= -rV_c(t_0, S_0; K, T) + e^{-r(T-t_0)} \int_K^{+\infty} \frac{\partial f_{S(t)}(y)}{\partial T} dy\end{aligned}$$

Since it was seen that $f_{S(t)}(y)$ satisfies the the Fokker-Planck PDE, integral can be written as

$$\int_K^{+\infty} (y - K) \frac{\partial f_{S(T)}(y)}{\partial T} dy =$$

$$-r \int_K^{+\infty} (y - K) \frac{\partial (y f_{S(T)}(y))}{\partial y} dy + \frac{1}{2} \int_K^{+\infty} (y - K) \frac{\partial^2 \sigma_{LV}(T, y) y^2 f_{S(T)}(y))}{\partial y^2} dy$$

with $\mu(t, S(t)) = rS$ and $\sigma(t, S) = \sigma_{LV}(t, S)S$.

Remark 9 The partial derivative of a call option with respect to the strike is given by the following expression.

Let $V_c(t_0, S_0; K, T)$ be the call price, with payoff max $(S - K, 0)$. The partial derivative of a call option with respect to the strike is given by the following expression:

$$\begin{aligned}\frac{\partial V_c(t_0, S_0; K, T)}{\partial K} &= -e^{\hat{a}r(T-t_0)} \int_K^{\infty} f_{S(T)}(y)dy \\ &= -e^{-r(T-t_0)}(1 - F_{S(T)}(K))\end{aligned}$$

where $F_{S(T)}$ is the CDF of stock $S(T)$ at time T.

Remark 10 *The second derivative with respect to K:*

$$\frac{\partial^2 V_c(t_0, S_0; K, T)}{\partial K^2} = -e^{-r(T-t_0)} F_{S(T)}(K)$$

Both integrals in the equation can be expressed in terms of call option values:

$$-\int_K^{+\infty} \frac{\partial (y f_{S(t)}(y))}{\partial y} dy = e^{-r(T-t_0)}[K\frac{\partial V_c(t_0, S_0; K, T)}{\partial K} - V_c(t_0, S_0; K, T)]$$

and

$$\int_K^{+\infty} \frac{\partial^2 \sigma_{LV}(T, y) y^2 f_{S(t)}(y))}{\partial y^2} dy = -e^{-r(T-t_0)} \sigma_{LV}^2(T, K) \frac{\partial^2 V_c(t_0, S_0; K, T)}{\partial K^2}$$

Collecting all terms obtained, we find the Dupire equation.

Definition 13 (Dupire equation) *Let $V_c(T, K)$ be the price of a European call with strike price equal to K and maturity T. Then it satisfies the following equation:*

$$\frac{\partial}{\partial T} V_c = -rK \frac{\partial V_c}{\partial K} + \frac{1}{2}\sigma_{LV}^2(T, K) K^2 \frac{\partial^2 V_c}{\partial K^2} \tag{11}$$

4.1.4 Dupire Formula

From the inverse formula, we can find the parameter of the volatility $\sigma_{LV}(T, K)$. It is important to note, as mentioned, that the volatility $\sigma_{LV}(T, K)$ is provided from market quotes of option values, so it perfectly fit to the market option quotes.

$$\sigma_{LV}^2(T, K) = \frac{\frac{\partial V_c(t_0, S_0; K, T)}{\partial T} + rK \frac{\partial V_c(t_0, S_0; K, T}{\partial K}}{\frac{1}{2} K^2 \frac{\partial^2 V_c(t_0, S_0; K, T)}{\partial K^2}} \tag{12}$$

4.2 Implied local volatility

The local volatility function relies on the available implied volatility surface, $\sigma_{imp}(T, K)$, for each expiry date T and strike price K [6]. In effect, the easiest way to calculate local volatility $\sigma_{LV}(T, K)$ is to express it in terms of implied volatilities $\sigma_{imp}(T, K)$.

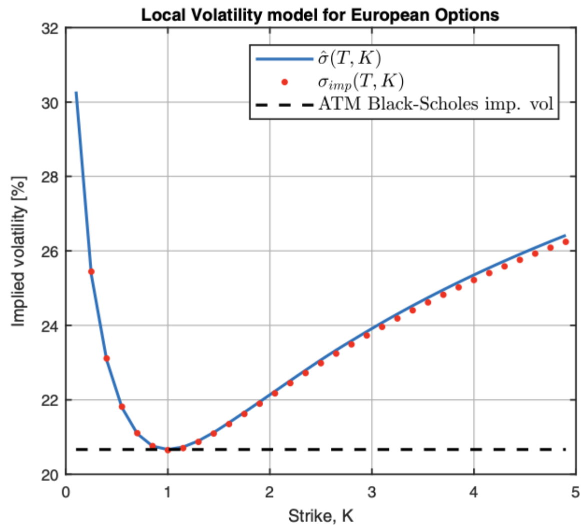

Figure 4.1: Consistency of the volatility calculated by the local volatility model versus the volatility calculated by the Black-Scholes, in comparison to the true implied volatility.

To relate implied (Black-Scholes) volatilities, $\sigma_{imp}(T, K)$ with Dupire's local volatilty $\sigma_{LV}(T, K)$, we use the generic arbitrage-free formula European call option prices, expressed in functions of two variables:

(i) $w = \sigma^2_{imp}(T, K)(T - t_0)$

(ii) $y = log\frac{K}{S_0 e^{r(T-t_0)}} = log\frac{K}{S_0} - r(T - t_0)$

For the variables w and y, we define the call price, $c(y, w)$, as:

$$V_c(t_0, S_0; K, T) = S_0[F_{\mathcal{N}(0,1)}(d_1)] - e^y F_{\mathcal{N}(0,1)}(d_2)] = c(y, w)$$

where:

$$d_1 = \frac{1}{2}\sqrt{w} - \frac{y}{\sqrt{w}} and d_2 = d_1 - \sqrt{w}$$

With the partial derivative with respect to T and with respect to K, it is possible to find the relationship between the σ_{LV} and σ_{imp}. In the final result, beyond the calculations, the local Dupire variance is obtained:

$$\sigma_{LV}^2(T,K) = \frac{\frac{\partial w}{\partial T} + rK\frac{\partial w}{\partial K}}{1 + K\frac{\partial w}{\partial K}(\frac{1}{2} - \frac{y}{w}) + \frac{1}{2}K^2\frac{\partial^2 w}{\partial K^2} + \frac{1}{2}K^2(\frac{\partial w}{\partial K})^2(-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w^2})} \tag{13}$$

Where:

(i) $\frac{\partial w}{\partial T} = \sigma_{imp}^2 + 2(T - t_0)\sigma_{imp}^2\frac{\partial \sigma_{imp}^2}{\partial T}$

(ii) $\frac{\partial w}{\partial K} = 2(T - t_0)\sigma_{imp}^2\frac{\partial \sigma_{imp}^2}{\partial K}$

(iii) $\frac{\partial^2 w}{\partial K^2} = 2(T - t_0)(\frac{\partial \sigma_{imp}}{\partial K})^2 + 2(T - t_0)\sigma_{imp}^2\frac{\partial^2 \sigma_{imp}^2}{\partial K^2}$

Local volatility function $\sigma_{LV}^2(T,K)$ can thus be expressed in terms of the implied volatilities σ_{imp}.

4.3 Limits of Local volatility models

Local volatility models overcome some imperfections of the Black-Scholes model.

In fact, Local volatility models use the observed implied volatilities market option quotes as input to the model, to then generate the local volatility value for each strike price K and for each expiry date T.

Therefore, calibration to these input European options is thus highly accurate.

However, there are drawbacks in these models: first of all, it is unthinkable that all the source of uncertainty is due solely to the underlying and that local volatility is conditioned only by the value of the spot is too strong an assumption to describe the behavior of the market.

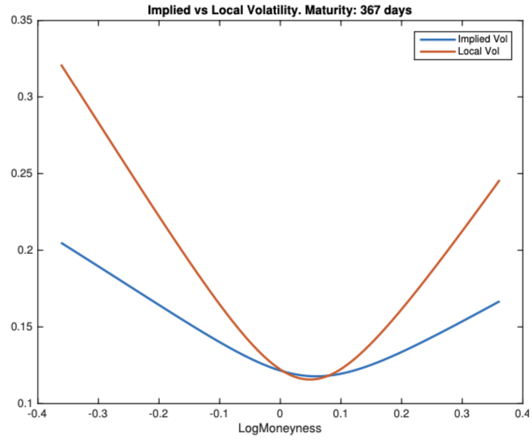

Figure 4.2: Implied Volatility versus Local Volatility.

Moreover, there are no derivatives (options) for so many maturities and for all types of shares, and it is therefore difficult to generate a clear stock density for these shares. In this sense, Local volatility models may suffer from significant mis-pricing inaccuracy when dealing with financial derivatives products that depend on the volatility paths and, generally, on density functions.

The last problem in using local volatility models is the low robustness they offer as the data changes: the local volatility surface tends to move a lot in the face of small shifts in the spot value. This is an indication of scarce consistency with the dynamics observed on the market. One of the purposes of the stochastic volatility models is precisely to overcome this problem.

Chapter 5

Stochastic Volatility models

In stochastic volatility models volatility has its own dynamics described by a stochastic process.

Some new mathematical and stastical conditions must be included: some econometric behaviors observed in the historical series of volatility are introduced, such as mean reversion, i.e. the assumption that an asset's price will tend to converge to the average price over time.

An important aspect is that for the stochastic volatility, the discounted characteristic function can be derived, which forms the basis of the Fourier option pricing techniques. It allows a quick calibration process, i.e. an easy choice of parameters that allow to better replicate market data.

It will be also introduced the techniques to correlate independent Brownian motions, dealing with a two-dimensional option pricing PDE.

The Heston stochastic volatility model is the most studied of the stochastic volatility models, where the asset's variance follows a Cox-Ingersoll-Ross process. The pros and cons of this method will be analyzed, as well as its calibration process.

5.1 Mathematics underlying the model

Stochastic volatility models describes the evolution of the volatility of an underlying asset assuming that the volatility of the asset is described by an additional stochastic process, in a different way than the corresponding asset prices.

In this case, the assumption of constant volatility in asset pricing is modeling the volatility as a diffusion process, with two-dimensional option pricing PDE (system of SDEs), which is correlated to the asset price process $S(t)$ [5].

Moreover, important assumptions concern the implied volatility smile and skew, which is present in the market, and can be accurately recovered by stochastic volatility models, especially for options with a medium to long time to the maturity date.

5.1.1 Stochastic volatility - CIR process

One of the most famous stochastic process describing volatility stems from the process square-root of Cox, Ingersoll and Ross (COS), that models the dynamics of variance $dv(t)$:

$$dv(t) = k(\bar{v} - v(t))dt + \gamma\sqrt{v(t)}dW_v^{\mathbf{Q}}(t) \tag{14}$$

This stochastic process is a mean reverting square-root process, composed of a deterministic term, $k(\bar{v} - v)dt$, to which a stochastic term $\gamma\sqrt{v(t)}dW_v^{\mathbf{Q}}$ is added.

In the process the volatility therefore oscillates around the average value v (mean-reverting process), where the parameter $\bar{v}$ is called long-term volatility. The value of k indicates the speed of this oscillation and determines its frequency (reversion speed), while γ (the so-called volatility of volatility) indicates the intensity of the disturbance generated by the Brownian motion, i.e. controls the volatility of the variance process.

As mentioned, the process models a mean reversion feature for the volatility: if the volatility exceeds or goes below its mean, it is driven back to the mean with the speed k of mean reversion [1].

5.1.2 Characteristic function

A general asset price stochastic volatility model can be defined by a two-dimension system of stochastic differential equation SDEs, with a diffusive volatility structure:

$$dS(t) = rS(t)dt + a(t,v)S(t)dW_x(t)$$

$$dv(t) = b(t,v)dt + c(t,v)dW_v^Q(t)$$

Where the underlying Brownian motions are correlated:

$$dW_x^Q dW_v^Q = \rho_{x,v}dt \leq 1$$

Under some conditions on the drift b and volatlity c coefficients, this class of models can be included in the class of affine diffusions [5], which guarantees that the corresponding discounted characteristic function can be derived. It is can then possible to express the model in terms of the independent Brownian motions.

Remark 11 Stochastic models in the class of affine diffusion (AD) processes can be expressed by the following stochastic differential form.

$$dX(t) = \bar{\mu}(t, X(t))dt + \bar{\sigma}(t, X(t))d\tilde{W}(t)$$

where $\tilde{W}$ is a column vector of independent Brownian motion and the function $\bar{\mu}$ and $\bar{\sigma}$ is of the following form [2]:

$$\bar{\mu}(t, X(t)) = a_0 + a_1 X(t)$$

$$(\bar{\sigma}(t, X(t))\bar{\sigma}(t, X(t))^T)_{ij} = (c_0)_{ij} + (c_1)_{i,j}^T X_j(t)$$

For this class of processes, the characteristic function, $\phi_X(u; t, T)$, can be determined, often in closed form.

$$\phi_X(u; t, T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r(s)ds + iu^T X(T)} | \mathcal{F}(t)] = e^{\bar{A}(u,\tau) + \bar{B}^T(t,\tau)X(t)}$$

Where the coefficients $\bar{A}(u, \tau)$ and $\bar{B}^T(t, \tau)$ satisfy the ordinary differential equations (ODEs):

$$\frac{d\bar{A}}{d\tau} = -r_0 + \bar{B}^T a_0 + \frac{1}{2}\bar{B}^T c_0 \bar{B}$$

$$\frac{d\bar{B}}{d\tau} = -r_1 + a_1^T\bar{B} + \frac{1}{2}\bar{B}^T c_1 \bar{B}$$

These ODEs can be derived by adding the general solution for the characteristic function, in the pricing PDE, which originates from the asset price process.

With the log-transformation $X(t) = logS(t)$, for $X(t) = [X(t), v(t)]^T$:

$$\begin{bmatrix} dX(t) \\ xv(t) \end{bmatrix} = \begin{bmatrix} \bar{\mu}_1(t, X(t)) \\ \bar{\mu}_2(t, X(t)) \end{bmatrix} dt + \begin{bmatrix} \bar{\sigma}_1(t, X(t)) & \bar{\sigma}_{1,2}(t, X(t) \\ \bar{\sigma}_2(t, X(t)) & \bar{\sigma}_{2,2}(t, X(t) \end{bmatrix} \begin{bmatrix} dW_x(t) \\ dW_v(t) \end{bmatrix}$$

With the drift terms as:

$$\bar{\mu}(t, X(t)) = \begin{bmatrix} r - \frac{1}{2}a^2(t,v) \\ b(t,v) \end{bmatrix} = \begin{bmatrix} a_0 + a_1 v \\ b_0 + b_1 v \end{bmatrix}$$

With the diffusion term as:

$$\bar{\sigma}(t, X(t))\bar{\sigma}(t, X(t))^T = \begin{bmatrix} a^2(t,v) & \rho_{x,v} a(t,v)c(t,v) \\ \rho_{x,v} a(t,v)c(t,v) & c^2(t,v) \end{bmatrix} =$$

$$\begin{bmatrix} c_{0,1,1} + c_{1,1,1}v & c_{0,1,2}. + c_{1,1,2}v \\ c_{0,2,1} + c_{1,2,1} & c_{0,2,2} + c_{1,2,2}v \end{bmatrix}$$

It follows that:

$$a^2(t,v) = (\sqrt{v})^2 = c_{0,1,1} + c_{1,1,1}v$$

$$\rho_{x,v} a(t,v)c(t,v) = \rho_{x,v}\sqrt{v}c(t,v) = c_{0,1,2} + c_{1,1,2}v$$

$$c^2(t,v) = c_{0,2,2} + c_{1,2,2v}$$

The generator of the stochastic volatility model, for $X = [X_1(t), X_2(t)]^T =: [X(t), v(t)]^T$, is given by,

$$\mathcal{A} = \sum_{i=1}^{2} \bar{\mu}_i(t, X)\frac{\partial}{\partial X_i} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}(\bar{\sigma}(t,X)\bar{\sigma}(t,X)^T)_{i,j}\frac{\partial^2}{\partial X_i \partial X_j} \tag{15}$$

By the Feynman-Kac theorem, the corresponding PDE for $V(t, X)$ is then known to be:

$$\frac{\partial}{\partial t}V + \mathcal{A}V - rV = 0$$

with the corresponding solution

$$V(t, X) = e^{-r(T-t)}\mathrm{E}^{\mathrm{Q}}[H(T, X)|\mathcal{F}(t)] \tag{16}$$

where H is the boundary condition of the PDE.

5.2 Heston model

The most used stochastic volatility model is the so-called Heston stochastic volatility models (Heston SVM), defined by two stochastic differential equations (2D system of SDEs), one for the underlying asset price $S(t)$, and one for the variance process $v(t)$.

$$dS(t) = rS(t)dt + \sqrt{v(t)}S(t)dW_x^Q(t), S(t_0) = S_0 > 0$$

$$dv(t) = k(\bar{v} - v(t))dt + \gamma\sqrt{v(t)}dW_v^Q(t), v(t_0) = v_0 > 0.$$

Where the underlying Brownian motions are correlated:

$$dW_x^Q dW_v^Q = \rho_{x,v}dt$$

5.2.1 Heston option pricing PDE derivation

To define the Heston option pricing PDE, it is possible to carry on the same method used to derive Black-Scholes equation, that is building up a replicating portfolio with value $\Pi(t, S, v)$.

The hedging portfolio in the case of stochastic volatility consists of:

1. the option sold, with value $V(t, S, v)$
2. $-\Delta$ units of the underlying asset $S(t)$,
3. $-\Delta_1$ units of another option, which is bought with value $V_1(t, S, v; K_1, T)$, with the goal to hedge the risk associated with the random volatility.

The final replicating portfolio is made by:

$$\Pi(t, S, v) = V(t, S, v; K, T) - \Delta S - \Delta_1 V_1(t, S, v; K_1, T) \tag{17}$$

Where V_1 is an option with the same maturity of V, but with a different strike K_1. As did with Black-Scholes PDE derivation, with Itô lemma it is possible to compute the stochastic differential and derive the process for an infinitesimal change of the portfolio $\Pi(t, S, v)$, eliminating the randomness of the portfolio with a proper hedge, canceling the terms in dS and dv:

$$\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta = 0$$

$$\frac{\partial V}{\partial v} - \Delta_1 \frac{\partial V_1}{\partial v} - \Delta = 0$$

The final portfolio evolves in deterministic way, and must earn only the risk-free rate:

$$d\Pi = r(V - \Delta S - \Delta_1 V_1)dt$$

Both option value V and option value V_1 are a function of the independent variables t, S, v. They must be equal to a specific function, $g(t, S, v)$, which only depends on the independent variables S, v, t.

The final option pricing PDE is provided by:

$$\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \rho_{x,y}\gamma Sv\frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2}\gamma^2 v\frac{\partial^2 V}{\partial v^2} + rS\frac{\partial V}{\partial S} + k(\bar{v} - v)\frac{\partial V}{\partial v} - rV = 0$$

5.2.2 Heston characteristic function and solution

The Heston dynamics, with X(t) = logS(t), are thus given by the following system of SDEs:

$$dX(t) = (r - \frac{1}{2}v(t))dt + \sqrt{v(t)}dW_x(t)$$

$$dv(t) = k(\bar{v} - v(t))dt + \gamma\sqrt{v(t)}dW_v(t)$$

Where the underlying Brownian motions are correlated:

$$dW_x^Q dW_v^Q = \rho_{x,v}dt$$

This model is affine, and can be expressed in terms of two independent Brownian motions.

$$\begin{bmatrix} dX(t) \\ dv(t) \end{bmatrix} = \begin{bmatrix} r - \frac{1}{2}v(t) \\ k(\bar{v} - v(t)) \end{bmatrix} dt + \begin{bmatrix} \sqrt{v(t)} & 0 \\ \rho_{x,v}\gamma\sqrt{v(t)} & \gamma\sqrt{(1 - \rho_{x,y}^2)vt} \end{bmatrix} \begin{bmatrix} dW_x(t) \\ dW_v(t) \end{bmatrix}$$

with:

$$\bar{\sigma}(t, X(t))\bar{\sigma}(t, X(t))^T = \begin{bmatrix} v(t) & \rho_{x,v}\gamma v(t) \\ \rho_{x,v}\gamma v(t)) & \gamma^2 v(t) \end{bmatrix}$$

which is affine in its state variables $X(t) = [X(t), v(t)]^T$ for constant γ and $\rho_{x,v}$.

It is possible to define as the so-called *Heston ODEs* the system of:

$$\frac{d\bar{A}}{d\tau} = k\bar{v}\bar{C} + r(\bar{B} - 1)$$

$$\frac{d\bar{B}}{d\tau} = 0$$

$$\frac{d\bar{C}}{d\tau} = \bar{B}(\bar{B} - 1)/2 - (k - \gamma\rho_{x,v}\bar{B})\bar{C} + \gamma^2\bar{C}^2/2$$

The solution of this system is given by:

$$\bar{A}(u, \tau) = r(iu - 1)\tau + \frac{k\bar{v}\tau}{\gamma^2}(k - \gamma\rho_{x,v}iu - D_1) - \frac{2k\bar{v}}{\gamma^2}log(\frac{1 - ge^{-D_1\tau}}{1 - g})$$

$$\bar{B}(u, \tau) = iu$$

$$\bar{C}(u, \tau) = \frac{1 - e^{-D_1\tau}}{\gamma^2(1 - ge^{-D_1\tau}}(k - \gamma\rho_{x,v}iu - D_1)$$

where

$$D_1 = \sqrt{(k - \gamma\rho_{x,v)iu)^2 + (u^2 + iu)\gamma^2}}$$

$$g = \frac{k - \gamma\rho_{x,v}iu - D_1}{k - \gamma\rho_{x,v}iu + D_1}$$

5.2.3 Calibration

Calibration is the process that makes a model consistent with the market, trying to minimize the difference between market plain vanilla option prices and the model prices, such that $V_{mkt} = V_H$ [1].
In particular, the calibration problem therefore consists in finding Ω such that:

$$\sigma^{mkt} \approx \sigma^{hest}$$

It is important to underline that, however, the existence and uniqueness of the Ω solution cannot be guaranteed in any way. For this reason the calibration process becomes a numerical *optimization problem*. In this sense, a target function is defined, where parameters that vary are $\Omega = (\rho_{x,v}, v, v_0, \bar{v}, k, \gamma)$

$$\min_{\Omega} \sum_i \sum_j w_{i,j} (V_c^{mkt}(t_0, S_0; K_i, T_j) - V_c^{hest}(t_0, S_0; K_i, T_j, \Omega))^2$$

and

$$\min_{\Omega} \sum_i \sum_j w_{i,j} (\sigma_{imp}^{mkt}(t_0, S_0; K_i, T_j) - \sigma_{imp}(t_0, S_0; K_i, T_j, \Omega))^2$$

1. $V_c^{mkt}(t_0, S_0; K_i, T_j)$ is the call option price for strike K and maturity T in the market;

2. $V_c^{mkt}(t_0, S_0; K_i, T_j, \Omega)$ is the Heston call option value;

3. σ_{imp}^{mkt} is the implied volatilities from the market

4. σ_{imp} is the implied volatilities from the Heston model

5. $w_{i,j}$ is some weighting function.

The goal is therefore to set a $\mathcal{J}(\Omega)$ equal to the sum of the squared differences between the prices of the European model and market calls:

$$\mathcal{J}(\Omega) = \sum_i \sum_j \left(V_{ci}^{mkt}(T_i, K_{ij}, \Omega) - V_{ci}^{hest}(T_i, K_{ij})^2 \right) \tag{18}$$

This choice leads to greater weighting of the errors committed At-The-Money, as the Out and In ones depend little on volatility.

5.2.4 Heston Implied volatility

As seen, in the Heston model the variance, $v(t) = \sigma^2(t)$, is governed by the *mean reverting* CIR process, where the process does not move towards infinite values, but tends to oscillate around a determined level (long-term average) [5].

An in-depth analysis is due, as each parameter has a specific effect on the implied volatility curve generated by the following dynamics:

1. when the correlation between stock and variance process, $\rho_{x,v}$, gets increasingly negative, the slope of the skew in the implied volatility curve increases.

2. an increasing volatility-of-volatility, γ, increases the implied volatility curvature.

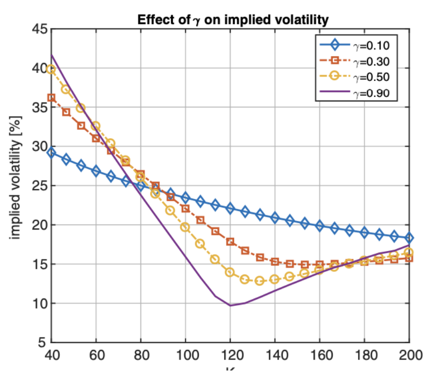

Figure 5.1: Relationship between the variation of the parameter γ (left) and the variation of the parameter $\rho_{x,v}$ (right), compared with the strike price K.

3. the speed of mean reversion, k, of $v(t)$ has a limited effect on the implied volatility smile or skew, but determines the speed at which the volatility converges to the long-term volatility .

5.3 Limits of Stochastic volatility models

While in local volatility models, volatility is a deterministic function of time and stock prices and is therefore perfectly correlated with stock price, in stochastic volatility models volatility follows a stochastic process of its own. This is the

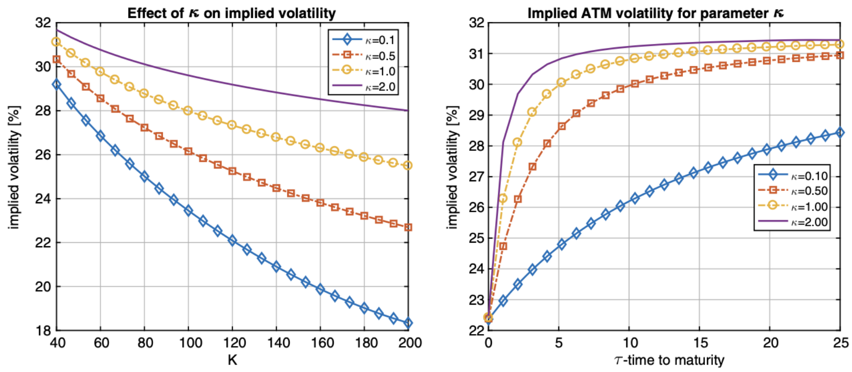

Figure 5.2: Relation between the Heston k parameter on the implied volatility as a function of strike K (left side) and its contribution on an ATM volatility (right).

rationale behind this new model, also based on market observations which clearly state that there is no perfect correlation empirically observed.

Moreover, Stochastic Volatility models are able to explain in a consistent way the smile observed on the market even if it is difficult to calibrate the prime maturities, also managing to provide a fairly realistic dynamics of the underlying.

The Heston model considers the leverage effect and the clustering effect, which allows the volatility itself to be random and also allows it to take the non-normally distributed stock return into account [6].

However, these models are not perfect: the extra randomness from the second stochastic process is one of the main problems.

A review of advantages and disadvantages is provided below:

1. The closed-form solution allows the calibration.

2. Heston model price dynamics allows for non-lognormal probability distribution.

3. The volatility is mean-reverting.

4. Heston models takes into account the leverage effect (negative correlation of stock returns and implied volatility), allowing to change the correlation

between the stock price and the volatility.

The disadvantages, as mentioned, concern the estimation of the parameters and the calibration:

1. Since volatility is not easily observable in the market, the parameters values in the Heston Model are not easily estimated.

2. The results depending greatly on the parameters used, thus and therefore the whole model depends on the calibration.

3. The Heston Model fails to produce decent results for short maturity. To perform well, the further extensions of the model are necessary, such as Local-Stochastic volatility model

5.4 Towards Local-Stochastic Volatility model

In this section we briefly describe the local-stochastic volatility model, which purpose is to unify the two previous categories of models while maintaining the positive aspects of one and eliminating the negative ones of the other.

The disadvantages and advantages of the local volatility model and the stochastic volatility model have already been explained. For example, calibrating the Heston model, however, is not always an easy task, and also not all types of smiles and implied volatility skews can be modeled using Heston dynamics. (aggiungi rif)

The Local-Stochastic Volatility model (SLV) was therefore developed. More precisely, it will be defined as the Heston-Stochastic Local Volatility (H-SLV) model, that is a model where the Heston stochastic volatility model where a non-parametric local volatility component is added.

This model can be transformed into the pure stochastic volatility model, if the local volatility component $\bar{\sigma}(t, S(t)) = 1$, or into the local volatility model, if the stochastic component of the variance, $b_v(t, v(t)) = 0$.

5.4.1 Local-Stochastic Volatility models

The stochastic-local volatility (SLV) model is governed by the following system of SDEs:

$$dS(t)/S(t) = rdt + \bar{\sigma}(t, S(t))\bar{\xi}(v(t))dW_x(t)$$

$$dv(t) = a_v(t, v(t))dt + b_v(t, v(t))dW_v(t)$$

$$dW_x(t)dW_v(t) = \rho_{x,v}dt$$

with:

1. correlation parameter $\rho_{x,v}$ between the corresponding Brownian motions, W_x, W_v, and constant interest rate r.
2. $\bar{\sigma}(t, S(t))$ governs local volatility.
3. $\bar{\xi}(v(t)$ governs the stochastic volatility.
4. The terms $a_v(t, v(t))$ and $b_v(t, v(t))$ represent the drift and diffusion of the variance process, respectively.

The question around which the model revolves is the following: the term $\bar{\sigma}(t, S(t))$, defined as the densities implied by the market and the model are equal, is not specified.
It is therefore necessary to define this local component part.

5.4.2 Specify the local volatility

Although local volatility component represents the local part of the process, it does not coincide with Dupire volatility.
Intuitively, it is a function whose purpose is to correct the implicit volatility smileys obtained with the stochastic part by pushing them towards the market ones.

For the derivation of this component an extension will be made to the lemma of Itô, through the *Tanaka-Meyer formula*.

Definition 14 (Tanaka-Meyer formula) *Given a probability space* $(\Omega, \mathcal{F}, \mathbb{Q})$*, let, for* $X(t) = X(t_0) + X_I(t) + X_{II}(t)$ *be a semi-martingale, where* $X_I(t)$ *is a continuous local martingale, and* $X_{II}(t)$ *is a cadlag adapted process of locally bounded variation, i.e.* $X_{II}(t)$ *is defined on* $\mathbb{R}$ *and is right-continuous with left limits almost everywhere. For the function* $g(x) = (x-a)^+$*, with* $a \in \mathbb{R}$*, it follows that,*

$$g(X(t)) = g(X(t_0)) + \int_{t_0}^{t} 1_{X(z)>a} dX_I(z) + \int_{t_0}^{t} 1_{X(z)>a} dX_{II}(z) + \frac{1}{2}\int_{t_0}^{t} g''(X(z))(dX_I(z))^2$$

This formula is useful for determining the dynamics of option value.

For the derivation, we start from the price of a European call option, and study its dynamics by applying Itô's lemma.

$$V_c(t_0, S_0) = \frac{M(t_0)}{M(t)}\mathbb{E}^{\mathbb{Q}}[(S(t)-K)^+|\mathcal{F}(t_0)]$$

where $dM(t) = rM(t)dt$.

$$\begin{aligned} dV_c(t_0, S_0) &= (d\frac{1}{M(t)})\mathbb{E}[(S(t)-K)^+] + \frac{1}{M(t)}d\mathbb{E}[(S(t)-K)^+] \\ &= -\frac{r}{M(t)}\mathbb{E}[(S(t)-K)^+]dt + \frac{1}{M(t)}d\mathbb{E}[(S(t)-K)^+] \end{aligned}$$

where the Fubini theorem justifies the equality:

$$d(\mathbb{E}[(S(t)-K)^+]) = \mathbb{E}[d(S(t)-K)^+]$$

The Itô lemma is not suitable since the function $g(x) = (x-a)^+$ is not differentiable at the point $x = a$.
It is therefore necessary to apply the Tanaka-Meyer formula. The formula becomes:

$$(S(t)-K)^+ = (S(t_0)-K)^+ + \int_{t_0}^{t} 1_{X(z)>K} dS(z) + \frac{1}{2}\int_{t_0}^{t} \delta(S(z)-K)(dS(z))^2$$

where δ is the Dirac delta function.
In differentiable form:

$$d(S(t)-K)^+ = 1_{(S(t_0)>K)} dS(t) + \frac{1}{2}\delta(S(t)-K)(dS(t))^2$$

Substituting the SLV dynamic of $S(t)$, shown at the beginning, it is possible to derive the dynamics of the European call option price $V_c(t_0, S_0)$:

$$dV_c(t_0, S_0) = \frac{rK}{M(t)}\mathbb{E}[1_{S(t)>K}]dt + \frac{1}{2M(t)}\mathbb{E}[\delta(S(t) - K)\bar{\sigma}^2(t, S(t))\bar{\xi}^2(v(t))S^2(t)]dt$$

$$dV_c(t_0, S_0) = \frac{rK}{M(t)}\mathbb{E}[1_{S(t)>K}]dt + \frac{1}{2M(t)}\mathbb{E}[\delta(S(t) - K)\bar{\sigma}^2(t, S(t))\bar{\xi}^2(v(t))S^2(t)]dt$$

Where the expectations are conditioned on $\mathcal{F}(t_0)$

From this equation we can find the SLV local volatility term:

$$\bar{\sigma}^2(t, K) = \frac{\sigma_{LV}^2(t, K)}{\mathbb{E}[\bar{\xi}^2 v(t))|S(t) = K]}$$

The SLV local volatility component $\bar{\sigma}^2(t, K)$ thus consists of two components, a deterministic local volatility σ_{LV} and a conditional expectation $\mathbb{E}[\bar{\xi}^2(v(t))|S(t) = K]$.

Chapter 6

Application: Volatility estimates and Simulation of Heston model

In this final chapter we will put into practice what we have seen in the previous chapters.

In the first section we will do a simple study on implied volatility, taking historical data of Amazon.com, Inc. (ticker: AMZN) from Yahoo Finance website.

Thus, we will calculate the implied volatility, showing how to use the strike prices and implied volatility for different maturities in order to calculate any other volatilities for any given strike and expiration date. Here, we will see that the Black-Scholes constant volatility assumption does not hold.

Moreover, we will also make a comparison between the volatility surface (from Black-Scholes model) and local volatility surface (from Local Volatility model *a la Dupire*).

In the second part we will see how the Heston model works and its calibration, evaluating it with the market values. We will make a simple simulations, starting from Heston pricing up to calibration.

6.1 Volatilty estimates

European options on an equity underlying such as a stock trade for different combinations of strikes and maturities.

First of all, to build a volatility surface let's take some data from Amazon.com, Inc. (ticker: AMZN).
The reference data are:
(i) Initial date: Nov 06, 2015
(ii) Adj. Close price: 659.37
(iii) Eight different Strikes prices
(iv) Exipiration times that go forward from month to month for 24 months (up to 2 years).

It is important to underline that implied volatility is available from Yahoo Finance data, so we don't need to compute it from option prices.

In this way, we can thus build a [8x24] matrix where for each row there is a different exipiration time, and each column corresponds to various strikes as given in strikes, i.e. a sample matrix of volatility quote by expiration date and strike.

Interpolating these volatilities, it is possible to construct the implied volatility surface for any other combination of strike and time-to-maturity.

Remark 12 *Recall that the implied volatility of a European (Call or Put) option with market quote is the value of the volatility σ^{imp} which substituted in the Black-Scholes formula gives us this market quote.*

Now the Black-Scholes volatility surface can be constructed using the *BlackVarianceSurface* function (See Appendix) and the volatilities for any given strike and expiry pair can be easily obtained using *BlackVarSurface*.

As we know, it turns out that the Black-Scholes implied volatility for these options with different maturities and strikes is not the same, generating the so-called *volatility smile*.

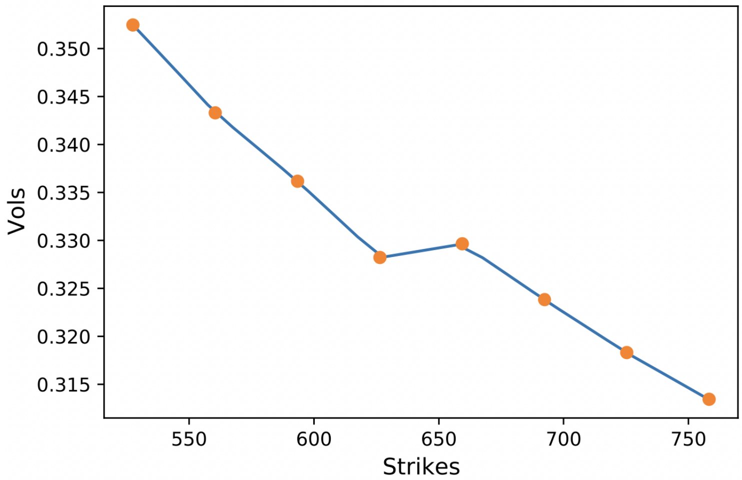

Figure 6.1: Volatility as a function of the strike prices (orange points) and interpolation of these points.

We can then plot the volatility surface; in the construction, we create a three-dimensional plot where the x-axis is the strike price, the z-axis is the implied volatility, and the y-axis is the time to maturity.

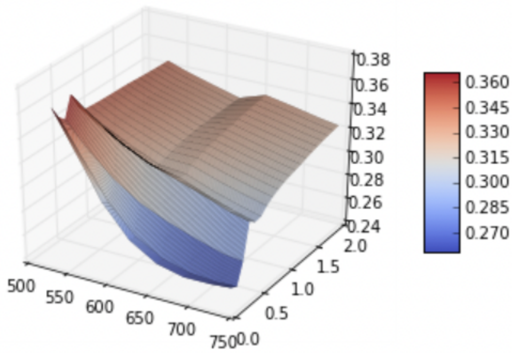

Figure 6.2: Volatility surface (plotting Strike, Expiration, and Implied Volatility)

We can highlight some features on the graph above: first of all, some volatility skews levels out as the option expiration date increases.

Furthermore, especially for short maturities, there is a great variability for the different strike prices. Hence, we see pronounced curvature for short maturities, and flatter surfaces for longer maturities.

The appearence of the market implied volatility surface shows that the Black-Scholes model is far from accurate. In effect, we can go further in the analysis: we know that in Dupire Local Volatility, the local volatility is calculated directly from the market prices by exploiting the partial differential equation that governs asset dynamics.

We know from the Dupire formula (12) that the volatility $\sigma_{LV}(T,K)$ is provided from market quotes of option values, so it perfectly fits to the market option quotes.

$$\sigma^2_{LV}(T,K) = \frac{\frac{\partial V_c(t_0,S_0;K,T)}{\partial T} + rK\frac{\partial V_c(t_0,S_0;K,T}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 V_c(t_0,S_0;K,T)}{\partial K^2}}$$

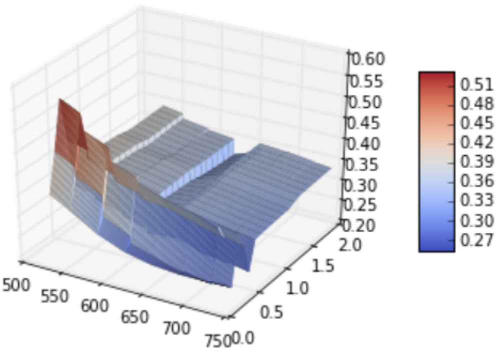

Figure 6.3: Local volatility surface (a la Dupire).

If we compare implied volatility surface with local volatility surface, we can see that the fit is pretty good. Actually, the Local Volatility models, unlike Black-Scholes model, are very good and even perfect at fitting arbitrage-free surface of implied volatilities (smile), via Dupire's formula.

6.2 Pricing of Heston model and Calibration

As seen in *Chapter 5*, an alternative to Local Volatility models are the Stochastic Volatility models, that can produce a more realistic volatility surface, where the smile is almost self similar, compared to local volatility models which flatten out the forward volatility curve and vanish the smile.

In the Stochastic Volatility model the asset price and its volatility are both assumed to be random processes and can change over time, hence it gives more realistic dynamics of the volatility smile, with a two-factor model that assumes separate dynamics for both the stock price and instantaneous volatility.

In this section we show how to price European options with the Heston stochastic volatility model.

The Heston model is defined a system of stochastic differential equations where the stock price follows a geometric Brownian motion and its variance follows a Cox-Ingersoll-Ross (CIR) process.
The basic Heston model assumes that S_t, the price of the asset, is determined by a stochastic process:

$$dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S,$$

where ν_t, the instantaneous variance, is given by CIR process,

$$d\nu_t = \kappa\left(\theta - \nu_t\right) dt + \xi\sqrt{\nu_t} dW_t^\nu$$

We already described volatility v as a mean reverting stochastic process with a constant volatility of volatility σ.
The two stochastic processes have a correlation ρ.

Let's define Heston parameters:
(i) $k = 4.1$
(ii) $\sigma = 0.3$
(iii) $\rho = -0.7$
(iv) $v_0 = 0.04$
(v) $\theta = 0.06$

(vi) $r = 0$
(vii) $S_0 = 1$
(viii) $T = 0.4$
(ix) $logMoneyness = 0.1$

Using these data and the characteristic function seen in *Chapter 5*, it is possible to derive both the correct price for a call and the Heston Implied volatility. In effect, we can use the model to deduce the implied volatility for a given option, using the given parameters to obtain a plot called Heston Volatility Smile.

To run the script below you will need the *HestonPValue* and *hestonEuropeanCall* (see Appendix).

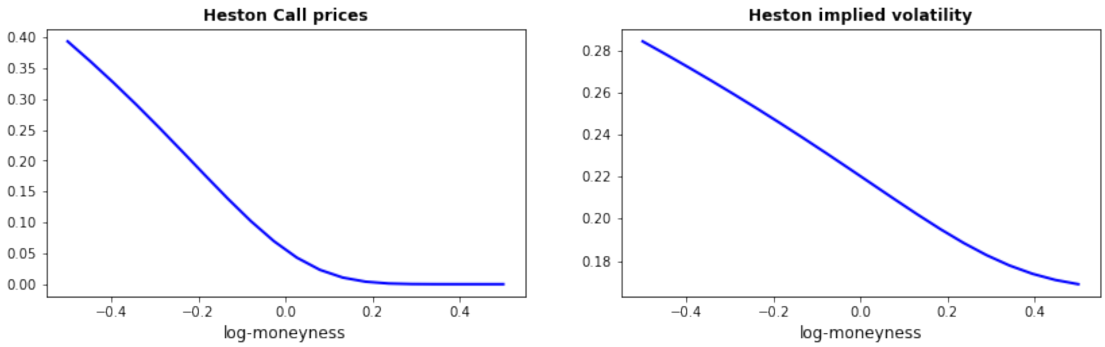

Figure 6.4: Heston call price and Heston implied volatility.

We assumed that the model parameters were given. Actually, when we have the Heston model and a pricing engine, we can pick the quotes with all strikes and maturities in order to calibrate the model. In this sense, we will *calibrate* the Heston model to fit to market volatility quotes with (for example) one year maturity.

In effect, in order to estimate option prices under the Heston Model, we need to find the five unknown input parameters, which are initial volatility, long-term volatility, volatility of the stochastic volatility process, volatility mean-reverting speed and correlation between stock price and volatility.

These five parameters are unknown because they cannot be easily observed in the market. The way to find these five parameters is to calibrate the Heston Model to the option market prices. In this way, we can obtain the five parameters that reflect the behaviors of the options that are traded in the real market.

However, in practice, it is not possible to match exactly the observed market prices. Thus the problem of calibrating the Heston Model is formulated as an optimization problem. Our objective is to minimize the pricing error between the model prices and the market prices for a set of data.

Here we have defined the Levenberg-Marquardt (LM) algorithm, which finds local minima and is very sensitive to initial conditions. It involves an iterative procedure, with the aim of finding the right parameters of the model curve such that the sum of the squares of the deviations is minimized [21].

In this sense, we look at the quality of calibration by pricing the options used in the calibration using the model and lets get an estimate of the relative error.

Strikes	Market Value	Model Value	Relative Error (%)
527.50	184.23863	177.26114	-3.7872037
560.46	162.13295	156.75990	-3.3139776
593.43	141.96233	138.13403	-2.6967003
626.40	123.03554	121.33552	-1.3817289
659.37	108.50169	106.27965	-2.0479306
692.34	93.29771	92.86049	-0.4686261
725.31	79.64951	80.95876	1.6437604
758.28	67.62146	70.44848	4.1806550

Average Abs Error (%) : 2.440

Figure 6.5: Quality of calibration with estimate of the relative error.

Appendix A

Appendix: Python code

A.1 Volatility estimates

```
import QuantLib as ql
import math

day_count = ql.Actual365Fixed()
calendar = ql.UnitedStates()

calculation_date = ql.Date(6, 11, 2015)

spot = 659.37
ql.Settings.instance().evaluationDate = calculation_date

dividend_yield = ql.QuoteHandle(ql.SimpleQuote(0.0))
risk_free_rate = 0.01
dividend_rate = 0.0
flat_ts = ql.YieldTermStructureHandle(
    ql.FlatForward(calculation_date, risk_free_rate, day_count))
dividend_ts = ql.YieldTermStructureHandle(
    ql.FlatForward(calculation_date, dividend_rate, day_count))

expiration_dates = [ql.Date(6,12,2015), ql.Date(6,1,2016),
```

```
ql.Date(6,2,2016),
ql.Date(6,3,2016), ql.Date(6,4,2016), ql.Date(6,5,2016),
 ql.Date(6,6,2016), ql.Date(6,7,2016), ql.Date(6,8,2016),
ql.Date(6,9,2016), ql.Date(6,10,2016), ql.Date(6,11,2016),
ql.Date(6,12,2016), ql.Date(6,1,2017), ql.Date(6,2,2017),
ql.Date(6,3,2017), ql.Date(6,4,2017), ql.Date(6,5,2017),
ql.Date(6,6,2017), ql.Date(6,7,2017), ql.Date(6,8,2017),
ql.Date(6,9,2017), ql.Date(6,10,2017), ql.Date(6,11,2017)]
\\
strikes = [527.50, 560.46, 593.43, 626.40, 659.37, 692.34,
725.31, 758.28]
data = [
[0.37819, 0.34177, 0.30394, 0.27832, 0.26453, 0.25916, 0.25941,
0.26127], [0.3445, 0.31769, 0.2933, 0.27614, 0.26575, 0.25729,
0.25228, 0.25202], [0.37419, 0.35372, 0.33729, 0.32492, 0.31601,
0.30883, 0.30036, 0.29568], [0.37498, 0.35847, 0.34475, 0.33399,
0.32715, 0.31943, 0.31098, 0.30506], [0.35941, 0.34516, 0.33296,
0.32275, 0.31867, 0.30969, 0.30239, 0.29631], [0.35521, 0.34242,
0.33154, 0.3219, 0.31948, 0.31096, 0.30424, 0.2984], [0.35442,
0.34267, 0.33288, 0.32374, 0.32245, 0.31474, 0.30838, 0.30283],
[0.35384, 0.34286, 0.33386, 0.32507, 0.3246, 0.31745, 0.31135,
0.306], [0.35338, 0.343, 0.33464, 0.32614, 0.3263, 0.31961,
0.31371, 0.30852], [0.35301, 0.34312, 0.33526, 0.32698, 0.32766,
0.32132, 0.31558, 0.31052], [0.35272, 0.34322, 0.33574, 0.32765,
0.32873, 0.32267, 0.31705, 0.31209], [0.35246, 0.3433, 0.33617,
0.32822, 0.32965, 0.32383, 0.31831, 0.31344], [0.35226, 0.34336,
0.33651, 0.32869, 0.3304, 0.32477, 0.31934, 0.31453], [0.35207,
0.34342, 0.33681, 0.32911, 0.33106, 0.32561, 0.32025, 0.3155],
[0.35171, 0.34327, 0.33679, 0.32931, 0.3319, 0.32665, 0.32139,
0.31675], [0.35128, 0.343, 0.33658, 0.32937, 0.33276, 0.32769,
0.32255, 0.31802], [0.35086, 0.34274, 0.33637, 0.32943, 0.3336,
0.32872, 0.32368, 0.31927], [0.35049, 0.34252, 0.33618, 0.32948,
0.33432, 0.32959, 0.32465, 0.32034], [0.35016, 0.34231, 0.33602,
0.32953, 0.33498, 0.3304, 0.32554, 0.32132], [0.34986, 0.34213,
```

```
0.33587, 0.32957, 0.33556, 0.3311, 0.32631, 0.32217], [0.34959,
0.34196, 0.33573, 0.32961, 0.3361, 0.33176, 0.32704, 0.32296],
[0.34934, 0.34181, 0.33561, 0.32964, 0.33658, 0.33235, 0.32769,
0.32368], [0.34912, 0.34167, 0.3355, 0.32967, 0.33701, 0.33288,
0.32827, 0.32432], [0.34891, 0.34154, 0.33539, 0.3297, 0.33742,
0.33337, 0.32881, 0.32492]]


implied_vols = ql.Matrix(len(strikes), len(expiration_dates))
for i in range(implied_vols.rows()):
    for j in range(implied_vols.columns()):
        implied_vols[i][j] = data[j][i]

black_var_surface = ql.BlackVarianceSurface(
    calculation_date, calendar,
    expiration_dates, strikes,
    implied_vols, day_count)

strike = 600.0
expiry = 1.2 # years
black_var_surface.blackVol(expiry, strike)

import numpy as np
% matplotlib inline
from mpl_toolkits.mplot3d import Axes3D
import matplotlib.pyplot as plt
from matplotlib import cm

strikes_grid = np.arange(strikes[0], strikes[-1],10)
expiry = 1.0 # years
implied_vols = [black_var_surface.blackVol(expiry, s)
                for s in strikes_grid]
actual_data = data[11]
```

```
fig, ax = plt.subplots()
ax.plot(strikes_grid, implied_vols, label="Black Surface")
ax.plot(strikes, actual_data, "o", label="Actual")
ax.set_xlabel("Strikes", size=12)
ax.set_ylabel("Vols", size=12)
legend = ax.legend(loc="upper right")

plot_years = np.arange(0, 2, 0.1)
plot_strikes = np.arange(535, 750, 1)
fig = plt.figure()
ax = fig.gca(projection='3d')
X, Y = np.meshgrid(plot_strikes, plot_years)
Z = np.array([black_var_surface.blackVol(y, x)
              for xr, yr in zip(X, Y)
                  for x, y in zip(xr,yr) ]
             ).reshape(len(X), len(X[0]))

surf = ax.plot_surface(X,Y,Z, rstride=1, cstride=1,
cmap=cm.coolwarm, linewidth=0.1)
fig.colorbar(surf, shrink=0.5, aspect=5)

local_vol_surface = ql.LocalVolSurface(
    ql.BlackVolTermStructureHandle(black_var_surface),
    flat_ts,
    dividend_ts,
    spot)

plot_years = np.arange(0, 2, 0.1)
plot_strikes = np.arange(535, 750, 1)
fig = plt.figure()
ax = fig.gca(projection='3d')
X, Y = np.meshgrid(plot_strikes, plot_years)
Z = np.array([local_vol_surface.localVol(y, x)
              for xr, yr in zip(X, Y)
```

```
                    for x, y in zip(xr,yr) ]
                ).reshape(len(X), len(X[0]))

surf = ax.plot_surface(X, Y, Z, rstride=1, cstride=1,
cmap=cm.coolwarm, linewidth=0.1)
fig.colorbar(surf, shrink=0.5, aspect=5)
```

A.2 Pricing of Heston model

```
from scipy.integrate import quad
import cmath
import numpy as np
import matplotlib.pyplot as plt
from math import isnan
from scipy.optimize import bisect
from scipy.stats import norm

def Heston_P_Value(hestonParams,r,T,s0,K,typ):
    kappa, theta, sigma, rho, v0 = hestonParams
    return 0.5+(1./np.pi)*quad(lambda xi:
    Int_Function_1
    (xi,kappa,theta, sigma,rho,v0,r,T,s0,K,typ),0.,500.)[0]

def Int_Function_1
(xi,kappa,theta,sigma,rho,v0,r,T,s0,K,typ):
    return (cmath.e**(-1j*xi*np.log(K))*
    Int_Function_2
    (xi,kappa,theta,sigma,rho,v0,r,T,s0,typ)/(1j*xi)).real

def Int_Function_2(xi,kappa,theta,sigma,rho,v0,r,T,s0,typ):
    if typ == 1:
        w = 1.
        b = kappa - rho*sigma
```

```
    else:
        w = -1.
        b = kappa
    ixi = 1j*xi
    d = cmath.sqrt((rho*sigma*ixi-b)*(rho*sigma*ixi-b)
    sigma*sigma*(w*ixi-xi*xi))
    g = (b-rho*sigma*ixi-d) / (b-rho*sigma*ixi+d)
    ee = cmath.e**(-d*T)
    C = r*ixi*T + kappa*theta/(sigma*sigma)*((b-rho*sigma*ixi-d)
    *T - 2.*cmath.log((1.0-g*ee)/(1.-g)))
    D = ((b-rho*sigma*ixi-d)/(sigma*sigma))*(1.-ee)/(1.-g*ee)
    return cmath.e**(C + D*v0 + ixi*np.log(s0))

def heston_EuropeanCall(hestonParams,r,T,s0,K):
    a = s0*Heston_P_Value(hestonParams,r,T,s0,K,1)
    b = K*np.exp(-r*T)*Heston_P_Value(hestonParams,r,T,s0,K,2)
    return a-b

def heston_Impliedvol(hestonParams,r,T,s0,K):
    myPrice = heston_EuropeanCall(hestonParams,r,T,s0,K)
    ## Bisection algorithm when the Lee-Li algorithm breaks down
    def smileMin(vol, *args):
        K, s0, T, r, price = args
        return price - BlackScholes(True, s0, K, T, r, 0., vol)
    vMin = 0.000001
    vMax = 10.
    return bisect(smileMin, vMin, vMax,
    args=(K, s0, T, r, myPrice),
    rtol=1e-15, full_output=False, disp=True)

#Heston parameters
kappa = 4.1
sigma = .3
rho = -0.7
```

```
v0 = 0.04
theta = 0.06
r = 0.
s0 = 1.
T = 0.4
logMoneyness = 0.1

hestonParams = kappa, theta, sigma, rho, v0
cp = heston_EuropeanCall(hestonParams,r,T,s0,
s0*np.exp(logMoneyness))
iv = heston_Impliedvol(hestonParams,r,T,s0,
s0*np.exp(logMoneyness))
print("Heston Call Price: %.4f " %cp)
print("Heston Implied volatility: %.2f%%" %(100.*iv))

logMoneynesses = np.linspace(-.5, .5, 20)

calls = [heston_EuropeanCall(hestonParams,r,T,s0,s0*np.exp(x))
for x in logMoneynesses]
ivs = [heston_Impliedvol(hestonParams,r,T,s0,s0*np.exp(x))
for x in logMoneynesses]

fig = plt.figure(figsize=(14,8))
plt.subplot(2, 2, 1)
plt.plot(logMoneynesses, calls, 'b-', linewidth=2)
plt.title("Heston Call prices", fontsize=12,
fontweight='bold')
plt.xlabel(u'log-moneyness', fontsize=12)
plt.subplot(2, 2, 2)
plt.plot(logMoneynesses, ivs, 'b-', linewidth=2)
plt.title("Heston implied volatility", fontsize=12,
fontweight='bold')
plt.xlabel(u'log-moneyness', fontsize=12)
plt.show()
```

A.3 Heston Calibration

```
#dummy parameters
v0 = 0.01; kappa = 0.2; theta = 0.02; rho = -0.75; sigma = 0.5;

process = ql.HestonProcess(flat_ts, dividend_ts,
                           ql.QuoteHandle(ql.SimpleQuote(spot)),
                           v0, kappa, theta, sigma, rho)
model = ql.HestonModel(process)
engine = ql.AnalyticHestonEngine(model)
# engine = ql.FdHestonVanillaEngine(model)

heston_helpers = []
black_var_surface.setInterpolation("bicubic")
one_year_idx = 11 # 12th row in data is for 1 year expiry
date = expiration_dates[one_year_idx]
for j, s in enumerate(strikes):
    t = (date - calculation_date )
    p = ql.Period(t, ql.Days)
    sigma = data[one_year_idx][j]
    #sigma = black_var_surface.blackVol(t/365.25, s)
    helper = ql.HestonModelHelper(p, calendar, spot, s,
ql.QuoteHandle(ql.SimpleQuote(sigma)),
flat_ts,
dividend_ts)
    helper.setPricingEngine(engine)
    heston_helpers.append(helper)

lm = ql.LevenbergMarquardt(1e-8, 1e-8, 1e-8)
model.calibrate(heston_helpers, lm,
                ql.EndCriteria(500, 50, 1.0e-8,1.0e-8, 1.0e-8))
theta, kappa, sigma, rho, v0 = model.params()
```

```
print "theta = %f, kappa = %f, sigma = %f, rho = %f,
v0 = %f" % (theta, kappa, sigma, rho, v0)

avg = 0.0

print "%15s %15s %15s %20s" % (
    "Strikes", "Market Value",
     "Model Value", "Relative Error (%)")
print "="*70
for i, opt in enumerate(heston_helpers):
    err = (opt.modelValue()/opt.marketValue() - 1.0)
    print "%15.2f %14.5f %15.5f %20.7f " % (
        strikes[i], opt.marketValue(),
        opt.modelValue(),
        100.0*(opt.modelValue()/opt.marketValue() - 1.0))
    avg += abs(err)
avg = avg*100.0/len(heston_helpers)
print "-"*70
print "Average Abs Error (%%) : %5.3f" % (avg)
```

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