Department of Economics and Finance

Teaching: Mathematical Methods for Economics and Finance

Introduction to the Schrödinger equation and an application to the stock market

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Introduction

Econophysics is the application of the principles of physics to the study of financial markets. It was first introduced by the theoretical physicist Eugene Stanley in 1995.

Physicists are currently contributing to the modeling of "complex systems" by using tools and methodologies developed in statistical mechanics and theoretical physics. Financial markets are remarkably well-defined complex systems, which are continuously monitored - down to time scales of seconds. Further, virtually every economic transaction is recorded, and an increasing fraction of the total number of recorded economic data is becoming accessible to interested researchers. Facts such as these make financial markets extremely attractive for researchers interested in developing a deeper understanding of modeling of complex systems.

In this thesis we will study an application of the Schrödinger equation to the stock market. In quantum mechanics the Schrödinger equation is a fundamental equation that determines the temporal evolution of the state of a system, for example of a particle, an atom or a molecule. Formulated by Erwin Schrödinger in 1925 and published in 1926, is a partial differential equation, linear, complex and non-relativistic with the wave function Ψ, and hase the form

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi(x,t)$$

It was formulated relying on de Broglie's hypothesis of the wave particle duality. According to the Copenhagen interpretation, the square modulus of the wave function is related to the probability of finding a particle in a given spatial region.
The application of this equation to the stock market is in the parallelism between the quantum world and the stock market. If we consider a stock as a micro system, defining a proper potential -forces that act on the stock- we can describe it by the wave function Ψ.

Through this thesis we will introduce the mathematical and physics tools to solve the Schrödinger equation for the specific potential defined in Chapter 3.

In the first Chapter we will introduce some elements of Functional Analysis, the concepts of distance, norm and inner product with the aim to define what a

Hilbert space is, since it is the space were wave functions lie. We will define what an operator is, with the further definition of linear operators, adjoint operator, and eventually self-adjoint operators. This is due to the fact that quantities of interest such as position or momentum are represented by "observables", which are self-adjoint linear operators acting on the Hilbert space. The position operator of quantum mechanics will be the price operator of our financial model. We will face some definition from spectral theory and the concepts of projections and bases.

When an observable corresponding to an operator A with discrete spectrum is measured in a system, the measured result will be one of the eigenvalues λ of A, and the probability of measuring a given eigenvalue λ_i will equal $\langle\Psi|P_i|\Psi\rangle$, where P_i is the projection onto the eigenspace of A corresponding to the λ_i. The last two sections of Chapter 1 are elements that will play a role when solving the Schrödinger equation of Chapter 3.

The second chapter leads, with the help of the mathematical tools explained in the first chapter, to an overview of the Schrödinger equation. After a brief introduction of quantum mechanics, the equation is presented. Further we expose the statistical interpretation, i.e. that the probability density of finding a particle at a given point, when measured, is proportional to the square of the magnitude of the particle's wave function at that point.

Momentum and position operators are then presented, they will be used in the third chapter when measuring the average rate of return and the rate of price change, which corresponds to the trend of the price in the stock market. The Heisenberg uncertainty principle is illustrated. Section 2.3 presents the formalism used in quantum mechanics. After a brief introduction to the Dirac's bra-ket notation, all the mathematical tools provided in the first chapter are used to describe the Schrödinger equation's components.

In section 2.4 three classical cases are presented and analytically solved: the infinite square well, the harmonic oscillator and the free particle. Those are three types of potential $V(x)$ well known in quantum physics. This will be helpful to see how the equation is actually solved for the simpler cases.

The third chapter illustrates the model. After an introduction to econophysics, the quantum model for the stock market is presented. The parallelism between quantum system and a stock is described. A proper Hamiltonian describing market's perturbations is built, to then face an initial value problem to be solved.

We set a change of variables, and guess a solution composed by two functions. After some algebraic computation, the problem is simplified to a free particle equation, whose solution is known from Chapter 2, and where the coefficients are found approximated to the second order. The distribution of the rate of return and its average evolution are found numerically, and the results of the model are presented.

Chapter 1

Elements of Functional Analysis

1.1 Metric Spaces

Definition 1.1.1. *Let X be a set. A function is said to be a* metric *or a* distance *if for each couple $x, y \in X$ associates a number $d(x, y)$ with the following properties:*

1. $d(x, y) \geq 0$*;* $d(x, y) = 0$ *iff* $x = y$ (positive definiteness)
2. $d(x, y) = d(y, x)$ (symmetry)
3. $d(x, y) \leq d(x, z) + d(z, y)$ (triangular inequality)

The couple (X, d) is said metric space.

Thanks to the concept of distance we can define a neighbourhood I of a point x_0 as

$$I(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

And from this we can define open sets, closed sets, accumulation points, boundary points, limit of a sequence of points in X, and anything that depends on distance[1].

1.2 Hilbert Spaces

1.2.1 Norm and Banach Space

Let us start this theory section by recalling the notion of *normed vector space*. Let X be a vector space on a real or complex field. A vector space is said to be

[1] See Herman, Strang, Calculus 1.

normed if you can define a *norm* in it:

$$\|\cdot\| : X \to \mathbb{R}$$

Such that, $\forall\lambda$ and $\forall\, x, y \in X$ the following properties are valid:

1. $\|x\| \geq 0$; $\|x\| = 0$ iff $x = 0$ *(positive definiteness)*
2. $\|\lambda x\| = |\lambda|\, \|x\|$ *(homogeneity)*
3. $\|x + y\| \leq \|x\| + \|y\|$ *(triangular inequality)*

A normed vector space is also *metric*, with the distance induced by:

$$d(x, y) = \|x - y\|$$

Definition 1.2.1. *Let p and q be two norms on a vector space X. Then p and q are called* equivalent *if there exist two real constants c and C with $c > 0$ such that $\forall x \in X$*

$$cq(x) \leq p(x) \leq Cq(x)$$

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements in X is said to be a *Cauchy sequence* if its terms become arbitrarily close to each other as the sequence progresses, i.e.

$$d(x_m, x_n) = \|x_m - x_n\| \to 0 \quad \text{per } m, n \to \infty$$

While it is said to be *convergent* to $x \in X$ if

$$d(x_n, x) = \|x_n - x\| \to 0 \quad \text{per } n \to \infty$$

Theorem 1.2.1. *Every sequence $\{x_n\}_{n\in\mathbb{N}}$ that converges, is a Cauchy sequence.*

Proof.
From the triangular inequality:

$$\|x_m - x_n\| \leq \|x_m - x\| + \|x_n - x\|$$

If the series converges, for n, m going to ∞ the right terms go to 0, hence also $\|x_m - x_n\| \to 0$, hence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. □
The inverse is not necessarily true, as we can see in the sequence following in the metric space $\mathbb{Q}$

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

It is a Cauchy sequence but has as limit the irrational number e that does not belong to $\mathbb{Q}$. If the inverse holds in a vector space, such vector space is said to be *complete*.

Definition 1.2.2. *A complete normed vector space is called* ***Banach space****.*

Let X, Y be two metric spaces and let $F : X \to Y$ be a function from X to Y. F is said to be *continuous* at $x \in X$ if $\forall \{x_n\}_{n\in\mathbb{N}} \subset X$,

$$\|x_n - x\|_X \to 0 \quad \text{implies} \quad \|F(x_n) - F(x)\|_Y \to 0$$

or, equivalently, if

$$\|F(y) - F(x)\|_Y \to 0 \quad \text{if} \quad \|y - x\|_X \to 0$$

Theorem 1.2.2. *Every norm in a space X is continuous in X.*

Proof.
Let $\|\cdot\|$ be a norm in X. From the triangular inequality,

$$\|y\| \leq \|y - x\| + \|x\| \quad \text{and} \quad \|x\| \leq \|y - x\| + \|y\|$$

Hence

$$|\|y\| - \|x\|| \leq \|y - x\|$$

So if $\|y - x\| \to 0$, also $|\|y\| - \|x\|| \to 0$, hence the norm is continuous. □

Given two equivalent norms p and q, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy with respect to the norm p if and only if is Cauchy with respect to the norm q. In particular, the space X is complete w.r.t. the norm p iff it is complete w.r.t. the norm q. Some examples are in order.

Space of continuous functions.

Let A be a compact subset on $\mathbb{R}^n$, the symbol $C^0(A)$, or simply $C(A)$ indicates the vector space of (real or complex) continuous functions in A. Let $X = C(A)$, endowed with the norm (called *maximum norm*)

$$\|f\|_{C(A)} = \max_A |f|$$

A sequence $\{f_m\}$ converges to f in $C(A)$ if

$$\max_A |f_m - f| \to 0$$

That is, if f_m converges *uniformly* to f in A. Since a uniform limit of continuous functions is continuous[2], $C(A)$ is a Banach space.

Note that other norms may be introduced in $C(A)$, for instance the *least squares* or $L^2(A)$ *norm*

$$\|f\|_2 = \left(\int_A |f|^2 \right)^{\frac{1}{2}}$$

[2]See the statement of the uniform limit theorem in James Munkres, Topology, 1999

Equipped with this norm $C(A)$ is not *complete*. Let, for example $A = [-1, 1] \subset \mathbb{R}$. The sequence

$$f_m(t) = \begin{cases} 0 & t \leq 0 \\ mt & 0 < t \leq \frac{1}{m} \\ 1 & t > \frac{1}{m} \end{cases} \quad (m \geq 1)$$

contained in $C([-1, 1])$, is a Cauchy sequence with respect to the L^2 norm. In fact (letting $m > k$)

$$\|f_m - f_k\|^2_{L^2(A)} = \int_{-1}^{1} |f_m(t) - f_k(t)|^2 dt = (m-k)^2 \int_0^{\frac{1}{m}} t^2 dt + \int_0^{\frac{1}{k}} (1-kt)^2 dt$$
$$= \frac{(m-k)^2}{3m^3} + \frac{1}{3k} < \frac{1}{3}\left(\frac{1}{m} + \frac{1}{k}\right) \to 0 \qquad \text{as } m, k \to \infty$$

However, f_n converges in $L^2(-1, 1)$ norm (and pointwise) to the Heaviside function

$$\mathcal{H}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Which is discontinuous at $t = 0$ and therefore does not belong to $C([-1, 1])$.

Summable and bounded functions.

Let Ω be an open set in $\mathbb{R}^n$ and $p \geq 1$ a real number. Let $X = L^p(\Omega)$ be the set of functions f such that $|f|^p$ is Lebesgue integrable in Ω. Identifying two functions f and g when they are *equal almost everywhere (a.e.)*, hence at all points in Ω, but for a subset of measure zero, $L^p(\Omega)$ becomes a Banach space when equipped with the norm *(integral norm of order p)*

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p\right)^{\frac{1}{p}}$$

The identification of two functions equal a.e. amounts to saying that an element of $L^p(\Omega)$ is not a single function but, actually, an equivalence class of functions, different from one another only on subsets of measure zero.

A function $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) is *essentially bounded* if there exists M such that

$$|f(x)| \leq M \qquad \text{a.e. in } \Omega \tag{1.1}$$

The infimum of all numbers M with property (1.1) is called *essential supremum of f*, and denoted by

$$\|f\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_\Omega |f|$$

If we identify two functions when they are equal a.e., $\|f\|_{L^\infty(\Omega)}$ is a norm in $L^\infty(\Omega)$ and $L^\infty(\Omega)$ becomes a Banach space.
From this definition we can state the Hölder inequality:

$$\left|\int_\Omega fg\right| \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} \tag{1.2}$$

Where p and q are *conjugate exponents*, i.e.

$$\frac{1}{p} + \frac{1}{q} = 1$$

The case $p = 1$ and $q = \infty$ is allowed.

Proof.

We first state Young inequality.
Given $a, b \in \mathbb{R}$, $a > 0, b > 0$ and given p and q conjugate exponents, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

This means that

$$\frac{|f(x)|}{\|f\|_p} \cdot \frac{|g(x)|}{\|g\|_q} \leq \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q$$

Integrating both sides

$$\frac{1}{\|f\|_p\|g\|_q}\int_\Omega |fg| = \frac{\|fg\|_1}{\|f\|_p\|g\|_q} \leq \frac{\|f\|_p}{p\|f\|_p} + \frac{\|g\|_q}{q\|g\|_q} = \frac{1}{p} + \frac{1}{q} = 1$$

□

1.2.2 Inner product and Hilbert Space

In order to define a Hilbert space we first need to recall the concept of *inner product*.

Definition 1.2.3. *Let X be a linear space over $\mathbb{R}$. An* inner or scalar product *in X is a function*

$$(\cdot, \cdot) : X \times X \to \mathbb{R}$$

With the following properties. For every x, y, $z \in X$ and scalars $\lambda, \mu \in \mathbb{R}$

1. $(x, x) \geq 0$ *and* $(x, x) = 0$ *if and only if* $x = 0$ (positivity)
2. $(x, y) = (y, x)$ (symmetry)
3. $(\mu x + \lambda y, z) = \mu(x, y) + \lambda(y, z)$ (bilinearity)

If the scalar field is $\mathbb{C}$, the inner product is defined as

$$(\cdot, \cdot) : X \times X \to \mathbb{C}$$

With the properties

1. $(x, x) \geq 0$ and $(x, x) = 0$ if and only if $x = 0$

2. $(x, y) = \overline{(y, x)}$

3. $(z, \mu x + \lambda y) = \overline{\mu}(z, x) + \overline{\lambda}(z, y)$

Where $\overline{(y, x)}$ indicates the *complex conjugate*. In the $\mathbb{C}$ the inner product is said to be *antilinear* with respect to its second argument or that it is a *sesquilinear form* in X.
The inner product induces the norm

$$\|x\| = \sqrt{(x, x)}$$

With respect to the above norm we state the following theorem.

Theorem 1.2.3. *Let x,y ∈ X. Then:*

1. *Schwarz's inequality:*

$$|(x, y)| \leq \|x\|\|y\|$$

   Equality holds if and only if x and y are linearly dependent

2. *Parallelogram law:*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

The Schwarz's inequality implies that the inner product is continuous. A linear space endowed with an inner product is called an *inner product space*.

Definition 1.2.4. *Let H be an inner product space. We say that H is a* ***Hilbert space*** *if it is complete with respect to the norm induced by the inner product.*

$L^2(\Omega)$ is one of the most important Hilbert spaces, with respect to the inner product

$$(u, v)_{L^2(\Omega)} = \int_\Omega uv$$

Two Hilbert spaces H_1 and H_2 are *isomorphic* if there exists a linear map $L : H_1 \to H_2$ which preserves the inner product, i.e.:

$$(x, y)_{H_1} = (L_x, L_y)_{H_2} \qquad \forall x, y \in H_1$$

1.3 Projections and Bases

1.3.1 Projections

As in finite-dimensional linear spaces, two elements x, y belonging to an inner product space are called *orthogonal* if $(x, y) = 0$, and we write $x \perp y$.

Now, if we consider a subspace V of $\mathbb{R}^n$, e.g. a hyperplane through the origin, every $\mathbf{x} \in \mathbb{R}^n$ has a unique orthogonal projection on V. In fact, if $\dim V = k$

and the unit vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ constitute an *orthonormal basis* in V, we may always find an orthonormal basis in $\mathbb{R}^n$, given by

$$\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k, \mathbf{w}_{k+1}, ..., \mathbf{w}_n$$

Where $\mathbf{w}_{k+1}, ..., \mathbf{w}_n$ are suitable unit vectors. Thus, if

$$\mathbf{x} = \sum_{j=1}^{k} x_j \mathbf{v}_j + \sum_{j=k+1}^{n} x_j \mathbf{w}_j$$

The projection of $\mathbf{x}$ on V is given by

$$P_V \mathbf{x} = \sum_{j=1}^{k} x_j \mathbf{v}_j$$

$P_V \mathbf{x}$ can also be defined without involving a basis in $\mathbb{R}^n$ as the point in V which minimizes the distance from $\mathbf{x}$.

Theorem 1.3.1. (Projection Theorem). *Let V be a closed subspace of a Hilbert space H. Then, for every $x \in H$, there exists a unique element $P_V x \in V$ such that*

$$\|P_V x - x\| = \inf_{v \in V} \|v - x\|$$

Moreover, the following properties hold:

1. *$P_V x = x$ if and only if $x \in V$*
2. *Let $Q_V x = x - P_V x$. Then $Q_V x \in V^\perp$ and*

$$\|x\|^2 = \|P_V x\|^2 + \|Q_V x\|^2$$

The elements $P_V x$, $Q_V x$ are called *orthogonal projections* of x on V and $V^\perp$, respectively. Even if V is not a closed subspace of H, the subspace $V^\perp$ is always closed. In fact, if $y_n \to y$ and $\{y_n\} \subset V^\perp$, we have, for every $x \in V$

$$(y, x) = \lim_{y_n \to y} (y_n, x) = 0$$

Whence $y \in V^\perp$

1.3.2 Bases

If $A \subset H$, we say A is *dense* in H if its closure[3], $\overline{A} = H$. A Hilbert space is said to be *separable* when there exists a countable dense subset of H. An *orthonormal basis* in a separable Hilbert space H is a sequence $\{w_k\}_{k \geq 1} \subset H$ such that

$$\begin{cases} (w_k, w_j) = \delta_{kj} & k, j \geq 1 \\ \|w_k\| = 1 & k \geq 1 \end{cases}$$

[3]The closure of a subset S consists of all points of the set plus its limit points.

Where δ_{kj} is the Kronecker delta:

$$\delta_{jk} := \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Every $x \in H$ may be expanded in the form:

$$x = \sum_{k=1}^{\infty} (x, w_k) w_k \tag{1.3}$$

The series (1.3) is called *generalized Fourier series*[4] and the numbers $c_k = (x, w_k)$ are the *Fourier coefficients* of x with respect to the basis $\{w_k\}$. Moreover:

$$\|x\|^2 = \sum_{k=1}^{\infty} (x, w_k)^2$$

An example of separable Hilbert space is $L^2(\Omega)$, $\Omega \subseteq \mathbb{R}^n$. In particular, the set of functions

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, ..., \frac{\cos mx}{\sqrt{\pi}}, \frac{\sin mx}{\sqrt{\pi}}$$

Constitutes an orthonormal basis in $L^2(0, 2\pi)$

Proposition 1.3.1. *Every separable Hilbert space H admits an orthonormal basis.*

1.4 Operators

1.4.1 Linear Operators

Let H be a Hilbert space. A *linear operator* from $D(A) \subset H$ into H is a function

$$A : D(A) \subset H \to H$$

such that, $\forall \alpha, \beta \in \mathbb{C}$ and $\forall f, g \in D(A) \subset H$

$$A(\alpha f + \beta g) = \alpha A f + \beta A g$$

For every linear operator we define its *Kernel*, $\mathcal{N}(A)$ and *Range*, $\mathcal{R}(A)$

Definition 1.4.1. *The* Kernel *of A, is the pre-image of the null vector in H:*

$$\mathcal{N}(A) = \{f \in D(A) : Af = 0\}$$

The Range *of A is the set of all outputs from points in $D(A)$:*

$$\mathcal{R}(L) = \{g \in H : \exists f \in D(A), Af = g\}$$

[4] More details in section 1.6

$\mathcal{N}(L)$ and $\mathcal{R}(L)$ are linear subspaces of $D(A)$ and H, respectively.

Definition 1.4.2. *A linear operator $A : H_1 \to H_2$ is* bounded *if there exists a number L such that $\forall f \in H_1$*

$$\|Af\|_{H_2} \leq L\|f\|_{H_1} \quad \forall f \in H_1$$

The number L controls the expansion rate operated by A on the elements of H_1. In particular, if $L < 1$, A contracts the sizes of the vectors in H_1. The infimum of the set of all constants L that satisfy the above property is the *norm of the operator*, indicated with $\|A\|$

$$\|A\| = \inf \{L \geq 0 : \|Af\| \leq L\|f\| \quad \forall f \in H_1\} \tag{1.4}$$

Or otherwise written as

$$\|A\| = \sup_{f \in H_1, f \neq 0} \left\{ \frac{\|Af\|}{\|f\|} \right\} \tag{1.5}$$

We indicate the set of all bounded operator from H_1 into H_2 as $\mathcal{B}(H_1, H_2)$. If $H_1 = H_2 = H$ the former becomes $\mathcal{B}(H)$.

Proposition 1.4.1. *A linear operator $A : H_1 \to H_2$ is bounded if and only if it is continuous.*

Proof.
Let A be bounded, $\forall f, f_0 \in X$ we have

$$\|A(f - f_0)\|_{H_2} \leq L\|(f - f_0)\|_{H_1}$$

So that, if $\|(f - f_0)\|_{H_1} \to 0$ also $\|A(f) - A(f_0)\|_{H_2} = \|A(f - f_0)\|_{H_2} \to 0$, hence A is continuous.
Let A be continuous. For $f = 0$ by definition of continuity, $\forall \varepsilon \ \exists \delta$ such that

$$\|Af\|_{H_2} \leq \varepsilon \quad \text{if} \quad \|f\|_{H_1} \leq \delta$$

Take $g \in H_1$ such that $\|g\|_{H_1} = 1$, take $\varepsilon = 1$ and let $h = \delta g$, so $\|z\|_{H_1} = \delta$ which implies

$$\delta\|Ag\|_{H_2} \leq \|Ah\|_{H_2} \leq 1$$

Or

$$\|Ag\|_{H_2} \leq \frac{1}{\delta}$$

□

Equipped with norm (1.4), $\mathcal{B}(H_1, H_2)$ is a Banach space.

Definition 1.4.3. *An operator $A : H_1 \to H_2$ is* closed *if $\forall \{f_n\}_{n \in \mathbb{N}} \in H_1$ and $\{Af_n\}_{n \in \mathbb{N}}$ such that*

$$f_n \to f \ \text{ and } \ Af_n \to g$$

Then

$$f \in H_1 \quad \text{and} \quad g = Af$$

Theorem 1.4.1. *Let $A : H_1 \to H_2$ be a closed operator. If $\exists A^{-1}$, then A^{-1} is closed.*

Definition 1.4.4. *An operator $A : H_1 \to H_2$ is* symmetric *if $\forall x, y \in H_1$*

$$(Ax, y) = (x, Ay)$$

1.4.2 Functionals and dual space

Definition 1.4.5. *We define* functional *a linear operator $A : H_1 \to H_2$ in which the arrival space H_2 is $\mathbb{R}$ (or $\mathbb{C}$, for complex Hilbert spaces).*

Definition 1.4.6. *The collection of all bounded linear functionals on a Hilbert space H is called* dual space *of H and denoted by H^*.*

From the definition of dual space we can state the Riesz's representation theorem which states that every Hilbert space is isomorphic to its dual space.

Theorem 1.4.2. (Riesz's Representation Theorem). *Let H be a Hilbert space, $\forall A \in H^*$ $\exists! g_A \in H$ such that*

$$Af = (f, g_A) \quad \forall f \in H \tag{1.6}$$

And

$$\|L\| = \|g_A\| \tag{1.7}$$

The Riesz's map $R : H^* \to H$ given by

$$A \longmapsto g_A$$

Is a canonical isometry, since it preserves the norm:

$$\|A\| = \|g_A\|$$

Indeed, an isometry is a distance-preserving transformation between metric spaces: it maps elements to a metric space such that the distance between the image elements in the new metric space is equal to the distance between the elements in the original metric space. Examples of isometries are rotation and translation, or a combination of the two.

We say that g_A is the *Riesz's element associated with A* with respect to the scalar product $(\cdot, \cdot)$. Moreover, H^* endowed with the inner product

$$(A_1, A_2)_{H^*} = (g_{A_1}, g_{A_2})$$

Is clearly a Hilbert space. This means that the Representation Theorem allows the identification of a Hilbert space with its dual.

1.4.3 Adjoint operator

The concept of *adjoint operator* extends the notion of transpose of an $m \times n$ matrix $\mathbf{A}$ and plays a crucial role in determining compatibility conditions for the solvability of several problems. The transpose $\mathbf{A}^T$ is characterized by the identity

$$(\mathbf{Ax},\ \mathbf{y})_{\mathbb{R}^m} = (\mathbf{x},\ \mathbf{A}^T\mathbf{y})_{\mathbb{R}^n} \qquad \forall \mathbf{x} \in \mathbb{R}^n, \forall \mathbf{y} \in \mathbb{R}^m$$

We use this definition to define the adjoint of a bounded linear operator.
Let $A : H_1 \to H_2$ be a bounded operator. If $y \in H_2$ is fixed, the real map

$$T_y : x \longmapsto (Ax, y)_{H_2}$$

Defines an element of H_1^*. From Riesz's Theorem $\exists! w \in H_1$ depending on y denoted by $w = A^\dagger y$ such that

$$T_y x = (x, w) = (x, A^\dagger y) \qquad \forall x \in H_1, \forall y \in H_2$$

This defines $A^\dagger$ as an operator from H_2 into H_1, which is called the *adjoint* of A. Precisely:

Definition 1.4.7. *The operator $A^\dagger : H_2 \to H_1$ defined by the identity*

$$(Ax, y)_{H_2} = (x, A^\dagger y)_{H_1} \qquad \forall x \in H_1, \forall y \in H_2$$

is called the adjoint *of A*

Symmetric matrices correspond to self-adjoint operators. Indeed we have the following definition

Definition 1.4.8. *Let $A : H_1 \to H_2$ be an operator. We say that A is* self-adjoint *if:*

1. $A = A^\dagger$
2. $H_1 = H_2$

Hence

$$(Ax, y) = (x, Ay) \tag{1.8}$$

In the finite-dimensional space, such operators are called *Hermitian*. An example of a self-adjoint operator in a Hilbert space H is the projection P_V on a closed subspace of H. In fact, recalling the Projection Theorem:

$$(P_V x, y) = (P_V x, P_V y + Q_V y) = (P_V x, P_V y) = (P_V x + Q_V x, P_V y) = (x, P_V y)$$

Proposition 1.4.2. *Let $A, A_1 \in \mathcal{B}(H_1, H_2)$ and $A_2 \in \mathcal{B}(H_2, H_3)$. Then:*

1. *$A^\dagger \in \mathcal{B}(H_1, H_2)$. Moreover $(A^\dagger)^\dagger = A$ and*

$$\|A^\dagger\|_{\mathcal{B}(H_2,H_2)} = \|A\|_{\mathcal{B}(H_2,H_2)}$$

2. *$(A_2, A_1)^\dagger = A_1^\dagger A_2^\dagger$. In particular, if A is an isomorphism, then*

$$(A^{-1})^\dagger = (A^\dagger)^{-1}$$

1.5 Spectral Theory

1.5.1 Spectrum of a matrix

Let $\mathbf{A}$ be a $n \times n$ matrix, and $\lambda \in \mathbb{C}$. Then, either the equation

$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{b}$$

has a unique solution for every $\mathbf{b}$ or there exists $\mathbf{u} \neq \mathbf{0}$ such that

$$\mathbf{Au} = \lambda\mathbf{u}$$

In the last case we say that λ,$\mathbf{u}$ constitutes an *eigenvalue-eigenvector pair*. The set of eigenvalues of $\mathbf{A}$ is called *spectrum of* $\mathbf{A}$, denoted by $\sigma(\mathbf{A})$. If $\lambda \notin \sigma(\mathbf{A})$, the *resolvent matrix* $(\mathbf{A} - \lambda\mathbf{I})^{-1}$ is well defined. The set

$$\rho(\mathbf{A}) = \mathbb{C} \setminus \sigma(\mathbf{A})$$

is called the *resolvent* of $\mathbf{A}$. If $\lambda \in \sigma(\mathbf{A})$, the kernel $\mathcal{N}(\mathbf{A} - \lambda\mathbf{I})$ is the subspace spanned by the eigenvectors corresponding to λ and it is called the *eigenspace* of λ. Note that $\sigma(\mathbf{A}) = \sigma(\mathbf{A}^T)$.
Symmetric matrices are particularly important: all eigenvalues $\lambda_1, ..., \lambda_n$ are real (possibly of multiplicity greater than 1) and there exists in $\mathbb{R}^n$ an orthogonal basis of eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_n$. The extension of those concepts in the Hilbert space setting is useful for the method of separation of variables.

1.5.2 Separation of variables

Suppose we have to solve the problem

$$\begin{cases} u_t = u_{xx} & x \in \Omega, t > 0 \\ u(x,0) = g(x) & x \in \Omega \\ u(x,t) = 0 & x \in \partial\Omega, t > 0 \end{cases}$$

Where Ω is a bounded one dimensional domain. Let us look for solutions of the form

$$u(x,t) = v(x)w(t)$$

And

$$u_t = v(x)w'(t)$$
$$u_{xx} = v''(x)w(t)$$

Substituting into the differential equation, we obtain

$$v(x)w'(t) = v''(t)w(t)$$

Separating variables

$$\frac{w'(t)}{w(t)} = \frac{v''(x)}{v(x)} = -\lambda$$

Since the left hand side of the equation only depends on the t variable while the right hand side of the equation only depend on the x variable, they must be equal to a constant, indeed $-\lambda$. This leads to two problems

$$w' + \lambda w = 0 \tag{1.9}$$

And

$$\begin{cases} -v'' = \lambda v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.10}$$

A number λ such that there exists a non trivial solution v for (1.10) is an eigenvalue of the operator $-\partial^2/\partial x^2$ in Ω and v is the corresponding eigenfunction. The problem can be solved if the following two properties hold:

1. There exists a sequence of real eigenvalues λ_k with corresponding eigenvectors u_k. Solving (1.9) for $\lambda = \lambda_k$ yields

$$w_k(t) = ce^{-\lambda_k t}$$

2. The initial data g can be expanded in series of eigenfunctions:

$$u(x) = \sum g_k u_k(x)$$

Then the solution is given by

$$u(x,t) = \sum g_k e^{-\lambda_k t} u_k(x)$$

The second condition requires that the set of eigenfunctions of $-\partial^2/\partial x^2$ constitutes a basis in the space of initial data. This leads to the problem of determining the *spectrum* of a linear operator in a Hilbert space, and in particular, of self-adjoint operators.

1.5.3 Spectrum of an operator

Definition 1.5.1. *Let $A : H_1 \to H_2$ be a bounded linear operator. A complex number λ is said to be in the* specrtum *of A, $\sigma(A)$, if $(\lambda I - A)$ is not invertible, where I is the identity operator.*

Definition 1.5.2. *Let H_1, H_2 be Hilbert spaces, and I the identity in H. Let $A : H_1 \to H_2$ be a bounded operator. A complex number λ is said to be in the* resolvent set *of A, $\rho(A)$, if the operator $(\lambda I - A)$ is one-to-one and onto*

$$\rho(A) = \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is one-to-one and onto}\}$$

Notice that $\sigma(A) \cup \rho(A) = \mathbb{C}$.
We define the operator $(\lambda I - A)^{-1}$ the *resolvent* of A, $R(\lambda, A)$. As a consequence of the closed graph theorem, that states that if the graph of a linear operator $A : H_1 \to H_2$ is closed in $H_1 \times H_2$, then A is bounded, we can deduce the fact

that if $\lambda \in \rho(A)$, the resolvent is bounded.

If H has finite dimension, any linear operator is represented by a matrix, so that its spectrum is given by the set of its eigenvalues. In infinitely many dimensions the spectrum may be divided in three subsets.

If $A - \lambda I$ is not one-to-one, $(A - \lambda I)^{-1}$ does not exists. This means that $\mathcal{N}(A - \lambda I) \neq \emptyset$, i.e. that the equation

$$Af = \lambda f \tag{1.11}$$

has non trivial solution. Then, we say that λ is an *eigenvalue* of A and that the non zero solutions of (1.11) are the *eigenvectors* corresponding to λ. The linear space spanned by these eigenvectors is called the *eigenspace* of λ and denoted by $\mathcal{N}(A - \lambda I)$. The set $\sigma_P(A)$ of the eigenvalues of A is called the point spectrum of A.

Definition 1.5.3. Point spectrum.
If $(\lambda I - A)$ is not injective, hence there exist two distinct elements $x, y \in X$ such that $(\lambda I - A)(x) = (\lambda I - A)(y)$, λ is said to be in the point spectrum *of A, denoted $\sigma_P(A)$.*

If $A - \lambda I$ is one-to-one, $\mathcal{R}(A - \lambda I)$ is dense in H, but $R(\lambda, A)$ is unbounded. Then, we say that λ belongs to the *continuous spectrum* of A, $\sigma_C(A)$.

Definition 1.5.4. Continuous spectrum.
If $(\lambda I - A)$ is injective, and its range is a dense subset R of X, λ is said to be in the continuous spectrum *of A, denoted $\sigma_C(A)$*

Finally, $A - \lambda I$ is one-to-one but $\mathcal{R}(A - \lambda I)$ is not dense in H. This defines the *residual spectrum* of A.

Definition 1.5.5. Residual spectrum.
If $(\lambda I - A)$ is injective, but its range is not dense in X, λ is said to be in the residual spectrum *of A, denoted $\sigma_R(A)$.*

So the spectrum is the disjoint union of these three sets:

$$\sigma(A) = \sigma_P(A) \cup \sigma_C(A) \cup \sigma_R(A)$$

Example
Let l^2 be the set of sequences $\mathbf{x}$=$\{x_m\}$ such that

$$\sum_{i=1}^{\infty} |x_m|^2 < \infty$$

Let $A : l^2 \to l^2$ be the shift operator which maps $\mathbf{x} = \{x_1, x_2, ...\} \in l^2$ into $\mathbf{y} = \{0, x_1, x_2, ...\}$. We have

$$(A - \lambda I)x = \{-\lambda x_1, x_1 - \lambda x_2, x_2 - \lambda x_3, ...\}$$

If $\lambda \neq 0$, then $\lambda \in \rho(A)$. In fact for every $\mathbf{z} = \{z_1, z_2, ...\} \in l^2$,

$$(A - \lambda I)^{-1}\mathbf{z} = \left\{-\frac{z_1}{\lambda}, -\frac{z_2}{\lambda} + \frac{z_1}{\lambda^2}, ...\right\}$$

Since $\mathcal{R}(A)$ contains only sequences whose first element is zero, $\mathcal{R}(A)$ is not dense in l^2, therefore $0 \in \sigma_R(A) = \sigma(A)$. For the purpose of this thesis, we are mainly interested in the spectrum of self-adjoint operators, where the following theorem is fundamental.

Theorem 1.5.1. *Let $A : H_1 \to H_2$ be a compact[5], self-adjoint operator on a separable Hilbert space. Then:*

a) $0 \in \sigma(A)$ *and* $\sigma(A) \setminus \{0\} = \sigma_P(A) \setminus \{0\}$

b) H_1 *has an orthonormal basis* $\{u_m\}$ *consisting of eigenvectors fo* A

c) If $\dim H_1 = \infty$*, the corresponding eigenvalues different from zero* $\{\lambda_m\}$ *can be arranged in a decreasing sequence* $|\lambda_1| \geq |\lambda_2| \geq ...$*, with* $\lambda_m \to 0$ *as* $m \to \infty$

Thus, the spectrum of a compact self-adjoint operator always contains $\lambda = 0$, which is not necessarily an eigenvalue. The other elements in $\sigma(A)$ are eigenvalues, arranged in a sequence converging to zero if H is infinite dimensional.

1.6 Fourier Analysis

Fourier analysis is the study of the way general functions may be approximated by the sums of simple trigonometric functions.

There are two types of Fourier expansions: Fourier series and Fourier transforms.

1.6.1 Fourier Series

Fourier's theorem states that any reasonably well-behaved function can be written as a discrete sum of trigonometric functions.

Theorem 1.6.1. *Consider a function $f(x)$ that is periodic on the interval $0 \leq x \leq L$, then $f(x)$ can be written as*

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nx}{L}\right) + b_n \sin\left(\frac{2\pi nx}{L}\right)\right]$$

[5]See Conway, A course in functional analysis, for the definition of compact operator.

a_n and b_n are specific coefficients that can be determined exploiting the orthogonality properties of trigonometric functions, i.e.

$$\int_0^L \sin\left(\frac{2\pi nx}{L}\right)\cos\left(\frac{2\pi mx}{L}\right)dx = 0$$

$$\int_0^L \cos\left(\frac{2\pi nx}{L}\right)\cos\left(\frac{2\pi mx}{L}\right)dx = \frac{L}{2}\delta_{nm} \tag{1.12}$$

$$\int_0^L \sin\left(\frac{2\pi nx}{L}\right)\sin\left(\frac{2\pi mx}{L}\right)dx = \frac{L}{2}\delta_{nm}$$

With δ_{mn} is the Kronecker delta.

Given these properties, we can calculate the coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)dx \tag{1.13}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi nx}{L}\right)dx \tag{1.14}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi nx}{L}\right)dx \tag{1.15}$$

Example (Sawtooth function)

Consider the function $f(x) = Ax$ for $-L/2 < x < L/2$ and period L shown in Figure 1.1.

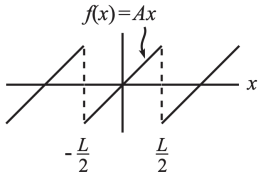

Figure 1.1: Sawtooth function

Given $f(x)$ is an odd function of x, a_n coefficients are zero, while the b_n ones are given by

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\left(\frac{2\pi nx}{L}\right)dx = \frac{2}{L}\int_{-L/2}^{L/2} Ax\sin\left(\frac{2\pi nx}{L}\right)dx$$

Integrating by parts

$$\begin{aligned}\int x\sin(rx)dx &= x\left(-\frac{1}{r}\cos(rx)\right)dx - \int -\frac{1}{r}\cos(rx)dx \\ &= -\frac{x}{r}\cos(rx) + \frac{1}{r^2}\sin(rx)\end{aligned}$$

Given $r \equiv 2\pi n/L$

$$\begin{aligned}b_n = \frac{2A}{L}\left[-x\left(\frac{L}{2\pi n}\right)\cos\left(\frac{2\pi nx}{L}\right)\Bigg|_{-L/2}^{L/2} + \left(\frac{L}{2\pi n}\right)^2\sin\left(\frac{2\pi nx}{L}\right)\Bigg|_{-L/2}^{L/2}\right] \\ = \left(-\frac{AL}{2\pi n}\cos(\pi n) - \frac{AL}{2\pi n}\cos(-\pi n)\right) + 0 \\ = -\frac{AL}{\pi n}\cos(\pi n) \\ = (-1)^{n+1}\frac{AL}{\pi n}\end{aligned}$$

So the Fourier trigonometric series of the Sawtooth function becomes

$$f(x) = \frac{AL}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin\left(\frac{2\pi nx}{L}\right)$$

The larger the number of terms included the better the approximation is. The partial series plots for 1, 3, 10, 50 terms are shown in 1.2

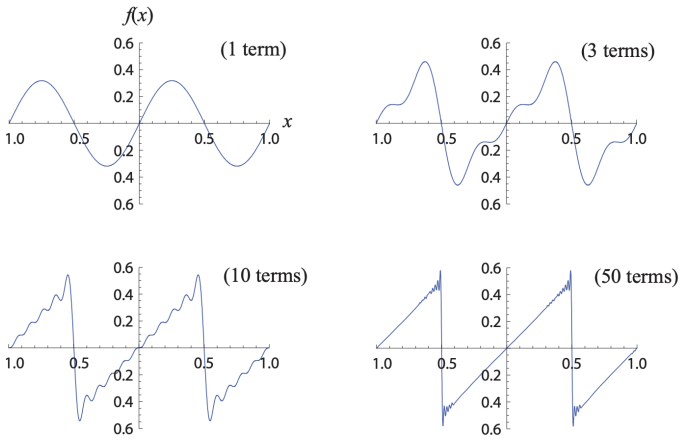

Figure 1.2: Partial series plots for the first 1, 3, 10 and 50 terms of the Sawtooth function

1.6.2 Fourier Exponential Series

A function that can be written in terms of sines and cosines can also be written in terms of exponentials, i.e.

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi nx/L} \tag{1.16}$$

Where the C_n coefficients are given by

$$C_n = \frac{1}{L}\int_0^L f(x)e^{-i2\pi nx/L}dx \tag{1.17}$$

Equation (1.17) exploits again the orthogonality of the exponential functions.
Example (Sawtooth function)

We consider another time the Sawtooth function

$$C_n = \frac{1}{L}\int_{-L/2}^{L/2} Axe^{-i2\pi nx/L}dx$$

Integrating by parts

$$\int xe^{-rx}dx = -\frac{x}{r}e^{-rx} - \frac{1}{r^2}e^{-rx}$$

This is true for $r \neq 0$. Given $r \equiv i2\pi n/L$ (hence $n \neq 0$) we obtain

$$C_n = -\frac{A}{L}\left(\frac{xL}{i2\pi n}e^{-i2\pi nx/L}\Big|_{-L/2}^{L/2} + \left(\frac{L}{i2\pi n}\right)^2 e^{-i2\pi nx/L}\Big|_{-L/2}^{L/2}\right)$$

The second of these terms yields zero because the limits produce equal terms. The first term is instead

$$C_n = -\frac{A}{L}\frac{(L/2)L}{i2\pi n}\left(e^{-i\pi n} + e^{i\pi n}\right)$$

Given the sum of exponential equal to $2(-1)^n$. So we are left

$$C_n = (-1)^n \frac{iAL}{2\pi n}$$

For $n \neq 0$, while if $n = 0$ the integral yields $C_0 = (A/L)(x^2/2)\big|_{-L/2}^{L/2} = 0$. We could expect this since Ax is an odd function.

We can then write the Fourier exponential series

$$f(x) = \sum_{n\neq 0}(-1)^n \frac{iAL}{2\pi n}e^{i2\pi nx/L}$$

Writing the exponential in trigonometric form

$$f(x) = \sum_{n\neq 0}(-1)^n \frac{iAL}{2\pi n}\left[\cos\left(\frac{2\pi nx}{L}\right) + i\sin\left(\frac{2\pi nx}{L}\right)\right]$$

Since $(-1)^n/n$ is an odd function of n and since $\cos(2\pi nx/L)$ is an even function of n the cosine terms sum to zero. Moreover, the function of sine is an odd function of n, hence we can restrict to the sum of just the positive integers and then double the result. Using $i^2(-1)^n = (-1)^{n+1}$ we can rewrite the solution as

$$f(x) = \frac{AL}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin\left(\frac{2\pi nx}{L}\right)$$

As obtained in the previous example.

There is relation among coefficients obtained in the exponential series and in the trigonometric series, given the formulas

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2} \quad \text{and} \quad \sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$$

We can get these relations among the coefficients

$$C_n = \frac{a_n - ib_n}{2} \quad \text{and} \quad C_{-n} = \frac{a_n + ib_n}{2}$$

And $C_0 = a_0$

1.6.3 Fourier Transform

The Fourier trigonometric or exponential series require the function to be periodic. A non periodic function can still be written as a series if considering the period of infinite length, passing from sums to integrals. Starting from equations (1.16) and (1.17), we define $k_n = 2\pi n/L$ with difference among k_n and k_{n-1} of $2\pi/L$ since $n \in \mathbb{Z}$. This difference goes to zero as $L \to \infty$, letting us consider k a continuous variable. Equation (1.16) becomes

$$f(x) = \int_{-\infty}^{\infty} C(k_n)e^{ik_n x}dk_n \tag{1.18}$$

With $C(k_n) \equiv (L/2\pi)C_n$. Recalling equation (1.17)

$$C(k_n) \equiv \frac{L}{2\pi}C_n = \frac{L}{2\pi}\cdot\frac{1}{L}\int_{-\infty}^{\infty} f(x)e^{-ik_n x}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ik_n x}dx \tag{1.19}$$

Dropping the n variable, the previous equations can be summed up in

$$f(x) = \int_{-\infty}^{\infty} C(k)e^{ikx}dk \qquad C(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx \tag{1.20}$$

$C(k)$ is known as the *Fourier transform* of $f(x)$.

1.7 Bessel funciton

Bessel functions are canonical solutions of the Bessel's equation, an ordinary differential equation of the form

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{1.21}$$

Where ν is a constant and defines the order of the Bessel's equation.

Bessel equation of order zero

In this case $\nu = 0$, so equation (1.21) reduces to

$$L[y] = x^2y'' + xy' + x^2y = 0 \tag{1.22}$$

We seek for a solution in series form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

We find the first derivative

$$y' = \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1}$$

And the second derivative

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-2}$$

Substituting in the Bessel's equation

$$\begin{aligned} L[y] &= x^2 \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-2} + x \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} + x^2 \sum_{n=0}^{\infty} a_n x^{r+n} \\ &= \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} \end{aligned} \tag{1.23}$$

We change the index of the third term to have a function of the same power of x.

Let $m = n + 2$, equation (1.23) becomes

$$L[y] = \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} + \sum_{m=2}^{\infty} a_n x^{r+m} \tag{1.24}$$

That can be reduced as

$$L[y] = a_0[r(r-1)+r]x^r + a_1[(r+1)r+(r+1)]x^{r+1} + \sum_{n=2}^{\infty} \{a_n[(r+n)(r+n-1)+(r+n)]+a_{n-2}\}x^{r+n} = 0 \tag{1.25}$$

Since this equation must be true for all x, coefficients must sum up to zero, building up a *recurrence relation*

$$a_n(r) = -\frac{a_{n-2}(r)}{(r+n)(r+n-1)+(r+n)} = -\frac{a_{n-2}(r)}{(r+n)^2} \quad n \geq 2 \tag{1.26}$$

When r is equal to zero, hence when we seek for the $y_1(x)$ solution, the first term of equation (1.25) goes to zero, while the second does not, this means that $a_1 = 0$, and hence, from equation (1.26), $a_3 = a_5 = \cdots = 0$. So only coefficients with even index are nonzero, starting from $r = 0$

$$a_n(0) = -\frac{a_{n-2}(0)}{n^2} \quad n = 2, 4, 6, \ldots$$

We set $n = 2m$ to indicate it is an even index, obtaining

$$a_{2m}(0) = -\frac{a_{2m-2}(0)}{(2m)^2} \quad m = 1, 2, 3, \ldots$$

Thus

$$a_2(0) = -\frac{a_0}{2^2} \quad a_4(0) = \frac{a_0}{2^4 2^2} \quad a_6(0) = -\frac{a_0}{2^6(3\cdot 2)^2}$$

Leading to the general case

$$a_{2m}(0) = \frac{(-1)^m a_0}{2^{2m}(m!)^2} \quad m = 1, 2, 3, \ldots \tag{1.27}$$

Inserting in the y function

$$y_1(x) = a_0 \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right] \quad x > 0 \tag{1.28}$$

Function in brackets is known as the *Bessel function of the first kind of order zero* and is denoted by $J_0(x)$.

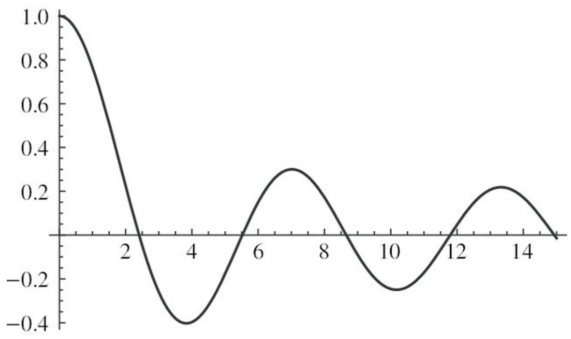

Figure 1.3: Bessel function of the first kind of order zero, denoted by $J_0(x)$

Bessel functions have plenty of properties, for the purpose of this thesis we present the recurrence formulae

$$2J'_n(z) = J_{n-1}(z) - J_{n+1}(z) \tag{1.29}$$

$$\frac{2n}{z}J_n(z) = J_{n-1}(z) + J_{n+1}(z) \tag{1.30}$$

Chapter 2

The Schrödinger equation

2.1 Brief Introduction to Quantum Mechanics

Quantum Mechanics is a fundamental theory in physics that provides a description of the physical properties of nature at the scale of atoms and subatomic particles, since classical physics has been resulted not sufficient to represent the microscopic world. The first main difference is the concept that energy in the quantum world can only assume certain values of energy and not a continuum of energies. Max Planck in 1900 assumed that any energy-radiating atomic system can theoretically be divided into a number of *discrete* energy levels, with each level of energy described by

$$E_n = nhv \qquad n = 1, 2, 3, \ldots \tag{2.1}$$

Where v is the frequency of radiation of energy and h is a numerical value called Planck's constant

$$h \approx 6.63 \times 10^{-34} J \cdot s \tag{2.2}$$

Equation (2.1) means that energy cannot assume any value $(-\infty, +\infty)$, but rather an integer multiple of the elementary quantity hv, that for this reason is called *quantum of energy*.

In 1923, the french physicist Louis de Broglie theorized a new cinematic, based on the idea that matter had both a corpuscular and wave nature: the now well known *wave-particle duality*. It is from this new way to see matter that Erwin Schrödinger postulated in 1925 his equation for describing a quantum-mechanical system. It was then published in 1926, forming the basis for the work that resulted in his Nobel Prize in Physics in 1933. From Planck's hypothesis of quantized energy, Schrödinger supposed that a particle with energy E and momentum q can be represented as a wave packet with frequency $v = E/h$ and wave length $\lambda = h/p$, where h is Planck's constant and p is the momentum.

From those ideas Schrödinger wrote his equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V(x)\Psi \tag{2.3}$$

t is time.

x is the position of the particle.

m is the mass of the particle

i is the imaginary unit.

$V(x)$ is the potential the particle is subject to.

$\hbar = h/2\pi$ is the reduced Planck constant.

Δ is the Laplace operator (or Laplacian) and is the sum of all unmixed second partial derivatives

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

Note For the purpose of this elaborate, the equation is written in the one dimensional case.

Equation (2.3) is a linear partial differential equation depending on time and position.

2.2 The wave function

2.2.1 Statistical Interpretation

The solution of the Schrödinger equation, the wave function $\Psi(x,t)$, describes the quantum state of a system. The manner Ψ represents the state of a particle is explained by Born's statistical interpretation of the wave function, formulated by Max Born in 1926.

It states that the probability density of finding a particle at a given point, when measured, is proportional to the square of the magnitude of the wave function $|\Psi(x,t)|^2$.
Indeed, the integral

$$\int_a^b |\Psi(x,t)|^2 dx$$

gives the probability of finding the particle between a and b at time t. Hence the probability is represented by the area under the graph of $|\Psi(x,t)|^2$ from a to b. It follows that the integral of $|\Psi|^2$ must be 1, i.e. the probability of the particle to be somewhere in the space is 100%. Hence here we find the first property of the wave function

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = 1 \tag{2.4}$$

Equation (2.4) is know as *normalization condition*. Any wave function for which the integral of the square modulus converges can be normalized, if multiplied for an opportune constant.

One might claim that even if we have normalized the wave function for a certain time, say $t = 0$, we cannot be certain that that Ψ will *stay* normalized as time goes on and Ψ evolves.

Here comes a crucial property of the wave function, that is the preservation of the normalization. Here is the proof.

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2dx = \int_{-\infty}^{+\infty}\frac{\partial}{\partial t}|\Psi(x,t)|^2dx \tag{2.5}$$

By the product rule,

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi^*\Psi) = \Psi^*\frac{\partial\Psi}{\partial t} + \frac{\partial\Psi^*}{\partial t}\Psi \tag{2.6}$$

Now Schrödinger equation says that

$$\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi \tag{2.7}$$

and hence, taking the complex conjugate of equation (2.7)

$$\frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^* \tag{2.8}$$

So

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right) = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right] \tag{2.9}$$

The integral in Equation (2.5) can now be evaluated explicitly:

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2dx = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{+\infty} \tag{2.10}$$

But $\Psi(x,t)$ must go to zero as x goes to $\pm\infty$, otherwise the wave function would not be normalizable. It follows that

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2dx = 0 \tag{2.11}$$

And hence that the integral is *constant*, independent of time. If Ψ is normalized at $t = 0$ it *stays* normalized for all future time □

2.2.2 Momentum

Any observable, i.e. any quantity which can be measured, is associated with a self-adjoint linear operator. The operators must yield real eigenvalues, since

they are values which may come up as the result of a measurement. As we saw in section 1.4 this means that the operator must be Hermitian. The probability of each eigenvalue is related to the projection of the physical state on the subspace related to that eigenvalue.

Back to the position operator, for a particle in state Ψ, the expectation value of the position operator, x, is:

$$\langle x \rangle = \int_{-\infty}^{+\infty} x|\Psi(x,t)|^2 dx \tag{2.12}$$

The value $\langle x \rangle$ is the average of repeated measurements on an ensemble of identically prepared systems. As time goes on, $\langle x \rangle$ will change, due to the time dependence of Ψ, and we might be interested in how fast it moves. Referring to equations (2.9) and (2.12), we see that

$$\frac{d\langle x \rangle}{dt} = \int x \frac{\partial}{\partial t}|\Psi|^2 dx = \frac{i\hbar}{2m} \int x \frac{\partial}{\partial x}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right) dx \tag{2.13}$$

This expression can be simplified, using integration by parts and the fact that Ψ goes to zero at $\pm\infty$, into

$$\frac{d\langle x \rangle}{dt} = -\frac{i\hbar}{2m} \int \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right) dx \tag{2.14}$$

Performing another integration by parts

$$\begin{aligned} \int \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right) dx = \int \Psi^* \frac{\partial \Psi}{\partial x} - \int \frac{\partial \Psi^*}{\partial x}\Psi = \\ \Psi^*\Psi|_{-\infty}^{+\infty} - \int \frac{\partial \Psi^*}{\partial x}\Psi - \int \frac{\partial \Psi^*}{\partial x}\Psi = \\ -2\int \frac{\partial \Psi^*}{\partial x}\Psi \end{aligned} \tag{2.15}$$

Hence equation (2.14) becomes

$$\frac{d\langle x \rangle}{dt} = -\frac{i\hbar}{m} \int \Psi^* \frac{\partial \Psi}{\partial x} dx \tag{2.16}$$

This result is the *velocity* of the expectation value of x, denoted as $\langle v \rangle$.

Actually, it is customary to work with *momentum* ($p = mv$), rather that velocity:

$$\langle p \rangle = m\frac{d\langle x \rangle}{dt} = -i\hbar \int \Psi^* \frac{\partial \Psi}{\partial x} dx \tag{2.17}$$

Hence we have found the expression for two fundamental operators:

$$\langle x \rangle = \int \Psi^*(x)\Psi dx \tag{2.18}$$

$$\langle p \rangle = \int \Psi^* \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi dx \tag{2.19}$$

That represents respectively, position and momentum in quantum mechanics.

Other quantities can be expressed in terms of position and momentum. Kinetic energy is

$$K = \frac{1}{2} m v^2 = \frac{p^2}{2m} \tag{2.20}$$

With expectation

$$\langle K \rangle = -\frac{\hbar^2}{2m} \int \Psi^* \frac{\partial^2 \Psi}{\partial x^2} dx \tag{2.21}$$

2.2.3 Uncertainty principle

The wavelenght of Ψ, λ is related to the momentum of the particle by the de Broglie formula:

$$p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda} \tag{2.22}$$

Thus a spread in wavelenght corresponds to a spread in momentum, and our general observation now says that the more precisely determined a particle's position is, the less precisely is its momentum. Quantitatively,

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \tag{2.23}$$

Where σ_x is the standard deviation in x, and σ_p is the standard deviation in p. This is Heisenberg's famous *uncertainty principle*.

2.3 Formalism

2.3.1 Dirac's bra and ket notation

Bra-ket notation was introduced by Paul Dirac to describe a state of a system. In such notation, the vectors in the space are called *ket vectors*, or simply *kets* and are denoted as $|\alpha\rangle$.

A ket $|\alpha\rangle$ can be expressed in a simpler way by the n-tuple of its components, $\{a_n\}$, with respect to a specified orthonormal basis:

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{pmatrix}$$

Two kets can be summed

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle$$

And the sum $|\gamma\rangle$ is also a ket. The product between a ket $|\alpha\rangle$ and a number c is still a ket. The constant c can stay on the right or left side of the ket

$$c\,|\alpha\rangle = |\alpha\rangle\,c$$

If $c = 0$ the resulting ket is the *null* ket. An operator Q acts on a ket from the left:

$$A(|\alpha\rangle) = A\,|\alpha\rangle$$

and the result is another ket.
We define *autoket* of an operator, indicated with the notation

$$|a'\rangle\,, |a''\rangle\,, |a'''\rangle\,, ...$$

A ket with the property

$$A\,|a'\rangle = a'\,|a'\rangle\,, A\,|a''\rangle = a''\,|a'\rangle\,, A\,|a'''\rangle = a'''\,|a'''\rangle$$

Hence the action of A over an autoket produces the same ket times a factor. The set of numbers $\{a', a'', a''', ...\}$ represents the set of the eigenvalues of the A operator.

We then define the *bra vectors*, or simply *bras*, denoted with the symbol $\langle\alpha|$, the mirror image of the symbol for a ket vector.

Thanks to this notation we can introduce the inner product in the bra-ket notation as the complex number:

$$\langle\alpha|\beta\rangle = a_1^* b_1 + a_2^* b_2 + ... + a_n^* b_n$$

Let us enunciate some properties of the inner product.

$$\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^* \tag{2.24}$$

Hence, $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$ are complex conjugates, that is the second property of the inner product in $\mathbb{C}$ stated in the first Chapter. From equation (2.24) we deduce that $\langle\alpha|\alpha\rangle$ is real (the only way a number can be equal to its conjugate is that the imaginary part is zero, hence that the number belongs to $\mathbb{R}$).

But since $\langle\cdot|\cdot\rangle$ is an inner product we remind the property:

$$\langle\alpha|\alpha\rangle \geq 0 \tag{2.25}$$

From which we can define the norm

$$\|\alpha\| = \sqrt{\langle\alpha|\alpha\rangle}$$

An operator Q acts on a bra from the right

$$(\langle\alpha|)Q = \langle\alpha|\,Q$$

And the result is another bra.

We can then build the product $|\alpha\rangle\langle\beta|$ in this order, known as *outer product*. The result is an operator:

$$|\alpha\rangle\langle\beta| := \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix} \begin{pmatrix} b_1^* & b_2^* & \dots & b_n^* \end{pmatrix} = \begin{pmatrix} a_1 b_1^* & a_1 b_2^* & \dots & a_1 b_n^* \\ a_2 b_1^* & a_2 b_2^* & \dots & a_2 b_n^* \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1^* & a_n b_2^* & \dots & a_n b_n^* \end{pmatrix}$$

Another property is that if the operator Q is defined as

$$Q = |\beta\rangle\langle\alpha|$$

Then

$$Q^\dagger = |\alpha\rangle\langle\beta|$$

2.3.2 Hilbert Space

In the previous Chapter we stated various elements of functional analysis that will be useful to solve the Schrödinger equation. Quantum theory is based on two constructs: *wave functions* and *operators*. The state of a system is represented by its wave function, observables are represented by operators. Mathematically, wave functions satisfy the defining conditions for abstract vectors and are represented by kets, while operators act on them as linear transformations.

Linear transformations, A, are represented by matrices (with respect to the specified basis), which act on vectors (to produce new vectors) by the ordinary rules of matrix multiplication:

$$|\beta\rangle = A|\alpha\rangle = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

In quantum mechanics we encounter functions instead of vectors, in *infinite*-dimensional spaces.

To represent a possible physical state, the wave function Ψ must be normalized:

$$\int |\Psi|^2 dx = 1$$

The set of all square-integrable functions on a specified interval is a space we have already met before, i.e. the $L^2(\Omega)$ space, which is a Hilbert space when embedded with the proper inner product

$$\langle f|g\rangle = \int_\Omega f(x)^* g(x) dx$$

If f and g are both square-integrable, their inner product is guaranteed to exists thanks to the Schwarz's inequality.

Moreover, the inner product of $f(x)$ with itself,

$$\langle f|f\rangle = \int_\Omega |f(x)|^2 dx$$

is real and non-negative, and equals *zero* only when $f(x) = 0$ a.e.

Finally, a set of functions is *complete* if any other function in the space can be expressed as a linear combination of them

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x) \tag{2.26}$$

If the functions $\{f_n(x)\}$ are orthonormal, the coefficients are given by the so called *Fourier's trick*.

We exploits the othonormality of f_n multypling both sides of equation (2.26) by $f_m(x)^*$ and integrate

$$\int f_m(x)^* f(x) dx = \sum_{n=1}^{\infty} \int f_m(x)^* f_n(x) dx = \sum_{n=1}^{\infty} c_m \delta_{mn} = c_m$$

So the general way to find coefficients is

$$c_n = \langle f_n|f\rangle$$

2.3.3 Observables

An observable is a physical quantity that can be measured. Examples include position and momentum. The expectation value of an observable $Q(x, p)$ can be expressed in inner-product notation:

$$\langle Q\rangle = \int \Psi^* \hat{Q} \Psi dx = \langle \Psi|\hat{Q}\Psi\rangle \tag{2.27}$$

Where $\hat{Q}$ is the operator constructed from Q by the replacement

$$p \to \hat{p} \equiv \frac{\hbar}{i}\frac{d}{dx}$$

These operators are linear, in the sense that

$$\hat{Q}[af(x) + bg(x)] = a\hat{Q}f(x) + b\hat{Q}g(x)$$

For any function f and g and any complex number a and b.
The outcome of a measurement has to be *real*, and so, a fortiori, is the average of many measurements:

$$\langle Q\rangle = \langle Q\rangle^* \tag{2.28}$$

But the complex conjugate of an inner product reverses the order, so

$$\langle \Psi | \hat{Q}\Psi \rangle = \langle \hat{Q}\Psi | \Psi \rangle \tag{2.29}$$

And this must hold true for any wave function Ψ. Thus operators representing observables have the very special property that

$$\langle f | \hat{Q} f \rangle = \langle \hat{Q} f | f \rangle \qquad \forall f(x) \tag{2.30}$$

But can be shown that they satisfy the ostensibly stronger condition:

$$\langle f | \hat{Q} g \rangle = \langle \hat{Q} f | g \rangle \qquad \forall f(x), g(x) \tag{2.31}$$

From equation (1.8) we know that such operators are Hermitian operators.

Suppose you want to prepare a state such that every measurement of Q is certain to return the same value, q. An example are the stationary states of the Hamiltonian: when you make a measurement of the total energy on a particle in the stationary state Ψ_n is certain to yield the corresponding allowed energy E_n.

In the situation described the standard deviation of Q, in a determinate state, would be zero

$$\sigma^2 = \langle (\hat{Q} - \langle Q \rangle)^2 \rangle = \langle \Psi | (\hat{Q} - q)^2 \Psi \rangle = \langle (\hat{Q} - q)\Psi | (\hat{Q} - q)\Psi \rangle = 0 \tag{2.32}$$

This means that

$$\hat{Q}\Psi = q\Psi \tag{2.33}$$

This is the eigenvalue equation for the operator $\hat{Q}$; Ψ is an eigenfunction of $\hat{Q}$, and q is the corresponding eigenvalue. So determinate states are eigenfunctions of $\hat{Q}$. This means that determinate states of the total energy are eigenfunctions of the Hamiltonian:

$$\hat{H}\psi = E\psi \tag{2.34}$$

2.3.4 Eigenfunctions of a Hermitian Operator

Our attention is thus directed to the eigenfunctions of Hermitian operators, that represents, as stated before, determinate states of observables. These fall into two categories: If the spectrum is *discrete*, i.e. the eigenvalues are separated from one another, then the eigenfunctions lie in Hilbert space and they constitute physically realizable states. If the spectrum is *continuous* then the eigenfunctions are not normalizable, and they do not represent possible wave functions (though linear combinations of them may be normalizable). Some operators have a discrete spectrum only, like for example, the Hamiltonian for the harmonic oscillator, some have only a continuous spectrum, like the free particle Hamiltonian, and some have both a discrete part and a continuous part, like the Hamiltonian for a finite square well.

Discrete Spectra

Mathematically, the normalizable eigenfunctions of a Hermitian operator have two important properties

Theorem 2.3.1. *The eigenvalues of Hermitian operator are real*

Proof.
Suppose

$$\hat{Q}f = qf$$

Hence $f(x)$ is an eigenfunction of $\hat{Q}$ with eigenvalue q and

$$\langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle$$

Since $\hat{Q}$ is Hermitian

$$q\langle f|f\rangle = q^*\langle f|f\rangle$$

q is a number, so it comes outside the integral, and because the first function is the inner product is complex conjugated so too is the q on the right.

Since $\langle f|f\rangle$ cannot be zero, due to the fact that the zero function is not a legal eigenfunction, $q = q^*$, and hence q is real. □

Theorem 2.3.2. *Eigenfunctions of Hermitian operators belonging to distinct eigenvalues are orthogonal*

Proof.
Suppose

$$\hat{Q}f = qf$$

And

$$\hat{Q}g = q'g$$

And $\hat{Q}$ is Hermitian. Then

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$$

So

$$q'\langle f|g\rangle = q^*\langle f|g\rangle$$

Where the inner products exist because the eigenfunctions are in Hilbert space by assumption. But q is real, so if $q' \neq q$ it must be that $\langle f|g\rangle = 0$ □

Moreover, the eigenfunctions of an observable operator are complete, i.e. any function in Hilbert space can be expressed as a linear combination of them.

Continuous Spectra

If the spectrum of a Hermitian operator is *continuous*, the eigenfunctions are not normalizable, hence the proofs of the two theorems stated when facing the discrete spectrum fail, because the inner products may not exist. Nevertheless, there is a sense in which the three essential properties of reality, orthogonality and completeness still hold.

For example, let $f_p(x)$ be the eigenfunction and p the eigenvalue of the momentum operator

$$\frac{\hbar}{i}\frac{d}{dx}f_p(x) = pf_p(x) \tag{2.35}$$

The general solution is

$$f_p(x) = Ae^{ipx/\hbar}$$

This is not square-integrable, for any (complex) value of p, hence the momentum han no eigenfuncitons in Hilbert space. However, restricting the spectrum to real eigenvalues, we recover a kind of orthonormality

$$\int_{-\infty}^{+\infty} f^*_{p'}(x)f_p(x)dx = |A|^2\int_{-\infty}^{+\infty} e^{i(p-p')x/\hbar}dx = |A|^2 2\pi\hbar\delta(p-p')$$

Choosing $A = 1/\sqrt{2\pi\hbar}$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$$

Then

$$\langle f_{p'}|f_p\rangle = \delta(p-p') \tag{2.36}$$

Where δ is the Dirac delta:

$$\delta(x) = \begin{cases} +\infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

Which is also constrained to satisfy the identity

$$\int_{-\infty}^{+\infty}\delta(x)dx = 1$$

Equation (2.36) is called *Dirac orthonormality.*

About completeness, we take the example of the position operator. Let $g_y(x)$ be the eigenfunction and y the eigenvalue:

$$xg_y(x) = yg_y(x) \tag{2.37}$$

Here y is a fixed number (for any given eigenfunction), but x is a continuous variable. The only function of x with the property to be the same whether you multiply it by x or by the constant y is the zero function, except in the one point where $x = y$. Hence we find again the Dirac function

$$g_y(x) = A\delta(x-y)$$

These eigenfunctions are also complete:

$$f(x) = \int_{-\infty}^{+\infty} c(y) g_y(x) dy = \int_{-\infty}^{+\infty} c(y) \delta(x - y) dy$$

With the trivial solution in this case for $c(y)$ that if equal to $f(y)$.

2.3.5 Generalized Statistical Interpretation

If you measure an observable $Q(x, p)$ on a particle in the state $\Psi(x, t)$ you are certain to get *one of the eigenvalues* of the Hermitian operator $\hat{Q}(x, -i\hbar\frac{d}{dx})$. If the spectrum of $\hat{Q}$ is discrete, the probability of getting the paticular eigenvalue q_n associated with the orthonormalized eigenfunction $f_n(x)$ is

$$|c_n|^2$$

Where

$$c_n = \langle f_n | \Psi \rangle \tag{2.38}$$

If the spectrum is continuous, with real eigenvalues $q(z)$ and associated Dirac-orthonormalized eigenfunctions $f_z(x)$, the probability of getting a result in the range dz is

$$|c(z)|^2 dz$$

Where

$$c(z) = \langle f_z | \Psi \rangle \tag{2.39}$$

When making a measurement, the wave function *collapses* to the corresponding eigenstate.

As stated before, the eigenfunctions of an observable operator are complete, to the wave function can be written as a linear combination of them:

$$\Psi(x, t) = \sum_n c_n \phi_n(x) \tag{2.40}$$

Given the orthonormality of the eigenfunctions we can compute coefficients with Fourier's trick

$$c_n = \langle \phi_n | \Psi \rangle = \int \phi_n(x)^* \Psi(x, t) dx \tag{2.41}$$

The generalized statistical interpretation tells us that $|c_n|^2$ is the probability that a measurement of Q would yield the value q_n, i.e. the eigenvalue of ϕ_n.

Since the total probability must be one

$$\sum_n |c_n|^2 = 1 \tag{2.42}$$

This follows from the normalization of the wave function

$$\begin{aligned} 1 = \langle\Psi|\Psi\rangle &= \langle(\sum_{n'} c_{n'} f_{n'}| \sum_n c_n \phi_n)\rangle \\ &= \sum_{n'}\sum_n c^*_{n'} c_n \langle\phi_{n'}|\phi_n\rangle \\ &= \sum_{n'}\sum_n c^*_{n'} c_n \delta_{n'n} \\ &= \sum_n c^*_n c_n \\ &= \sum_n |c_n|^2 \end{aligned}$$

Similarly, the expectation value of Q should be the sum over all possible outcomes of the eigenvalue times the probability of getting that eigenvalue

$$\langle Q\rangle = \sum_n q_n |c_n|^2 \tag{2.43}$$

Indeed

$$\langle Q\rangle = \langle\Psi|\hat{Q}\Psi\rangle = \langle(\sum_{n'} c_{n'}\phi_{n'}|\hat{Q}\sum_n c_n\phi_n)\rangle$$

But $\hat{Q}\phi_n = q_n\phi_n$, so

$$\langle Q\rangle = \sum_{n'}\sum_n c^*_{n'} c_n q_n \langle f_{n'}|f_n\rangle = \sum_{n'}\sum_n c^*_{n'} c_n q_n \delta_{n'n} = \sum_n q_n |c_n|^2$$

2.4 Solution of the Schrödinger equation

In the first chapter we introduced some elements of functional analysis that are used in quantum mechanics as we have seen in Section 2.3. We also gave an explanation of the statistical meaning of the wave function and of the operators used to compute quantities of interest.

We can now face the real problem of the equation, that is, how to actually find $\Psi(x,t)$ in the first place. Hence we need to solve the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

for a specified potential. The final aim is to be able to solve the equation for a potential that depends both on time and position, but first we start with the easier case with a potential that is independent of time.

Whenever the Schrödinger equation presents a time-independent potential, it can be solved with the method of *separation of variables*, i.e. the hypothesis

that our solution takes the form of a *product* of a function of t and a function of x,

$$\Psi(x,t) = \psi(x)\phi(t)$$

Where we make a distinction between the lower case ψ that indicates a function of x alone, in contrast with the upper case Ψ that is the function of both x and t.

Given our guess solution, we compute the derivatives appearing in the equation:

$$\frac{\partial \Psi}{\partial t} = \psi \frac{d\phi}{dt}$$
$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{d^2\psi}{dx^2}\phi$$

Replacing the found derivatives in the equation we obtain

$$i\hbar\psi\frac{d\phi}{dt} = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\phi + V\psi\phi$$

We now divide each side by $\psi\phi$

$$i\hbar\frac{1}{\phi}\frac{d\phi}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} + V$$

Following the same reasoning of Section 1.5.2, since the left side depends only on t and the right side depends only on x, the only way this can possibly be true is if both sides are *constant*.

$$i\hbar\frac{1}{\phi}\frac{d\phi}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} + V = E$$

Then

$$\phi' = -i\frac{E}{\hbar}\phi \tag{2.44}$$

And

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right)\psi = E\psi \tag{2.45}$$

The method of separation of variables has turned a partial differential equation into two ordinary differential equations.

Let us now solve each of these.
About the first one, it is easy to solve:

$$\frac{\phi'}{\phi} = -i\frac{E}{\hbar}$$

Integrating both sides we obtain

$$\ln|\phi| = -i\frac{E}{\hbar}t + c$$

Elevating e to each side we found the solution

$$\phi(t) = C_1 e^{-iEt/\hbar}$$

About the second equation, this is the so called *time-independent Schrödinger equation* and can be solved only when the potential $V(x)$ is specified.

One may claim that the hypothesis that the solution has the form $\psi(x)\phi(t)$ is way too strong, and that in this way we can find only a tiny subset of all possible solutions. However, the solutions we find with separation of variables are indeed of great interest for some reasons.

First of all, they are states of a definite total energy. We know from classical physics that the total energy of a system is the kinetic energy plus the potential energy, i.e.

$$E = \frac{1}{2}mv^2 + V$$

Where V is the potential energy and the kinetic one is expressed by its formula. Observing the left hand side of the time-independent equation, the operator acting on ψ is given by

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V$$

Given the momentum operator we have previous defined in equation (2.19) and the kinetic energy defined in equation (2.20), we recognize the total energy of the system in what in classical physics is called the *Hamiltonian*:

$$H(x,p) = \frac{p^2}{2m} + V(x) \tag{2.46}$$

From what we can derive the corresponding *Hamiltonian operator* by the substitution

$$p \to \frac{\hbar}{i}\frac{\partial}{\partial x}$$

That is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \tag{2.47}$$

We can then rewrite the time-independent Schrödinger equation in the form

$$\hat{H}\psi = E\psi$$

The expectation value of the Hamiltonian, hence the expectation value of the total energy is

$$\langle H \rangle = \int \psi^* \hat{H} \psi dx = E \int |\psi|^2 dx \tag{2.48}$$

Since the normalization of Ψ entails the normalization of ψ, last equation can be rewritten in

$$E \int |\Psi|^2 dx = E$$

Moreover

$$\hat{H}^2\psi = \hat{H}(\hat{H}\psi) = \hat{H}(E\psi) = E(\hat{H}\psi) = E^2\psi$$

And hence

$$\langle H^2 \rangle = \int \psi^* \hat{H}^2 \psi dx = E^2 \int |\psi|^2 dx = E^2$$

So the variance of H is

$$\sigma_H^2 = \langle H \rangle^2 - \langle H^2 \rangle = E^2 - E^2 = 0 \tag{2.49}$$

Hence $\sigma_H = 0$, meaning that every member of the sample must assume the same value E, in order to have zero spread among measurements.

So we have just shown that a separable solution has the property to have a value of the total energy always equal to E.

Another important property of the solutions obtained with the method of separation o variables is that they are *stationary states*. A stationary state is a quantum state with all observables independent of time. This means that the system remains in the same state as time elapses, in every observable way. For the Hamiltonian of a single particle, this means that the particle has a constant probability distribution for its position, momentum, etc. In fact, the probability density

$$|\Psi(x,t)|^2 = \Psi^*\Psi = \psi^* e^{+iEt/\hbar} \psi e^{-iEt/\hbar} = |\psi(x)|^2 \tag{2.50}$$

Due to the fact that the time dependent factors cancel out, does not depend on time. Analogously, the computation of the expectation value of any operator can be reduced to

$$\langle Q(x,p) \rangle = \int \psi^* Q\left(x, \frac{\hbar}{i}\frac{d}{dx}\right) \psi dx$$

This means, as stated in the definition of stationary state, that the expectation value of every observable of such a solution is independent of time. In particular, $\langle x \rangle$ is constant, and hence $\langle p \rangle = 0$. Nothing ever *happens* in a stationary state.

This does not mean that the wave function itself is stationary. We know that the solution with separation of variable has the component $\phi(t)$ that indeed depends on time. Given the expression of $\phi(t) = e^{-iEt/\hbar}$ the wave function continually changes to form a standing wave, i.e. a wave that oscillates over time but whose peak amplitude profile does not move in space.

Finally, solutions found with the method of separation of variable have a last important property: the general solution is a linear combination of separable solutions.

Given the total energy of system property, we know that the time-independent Schrödinger equation yields an infinite collection of solutions $\psi_1(x), \psi_2(x), ...$ each of them associated with the total value of energy $E_1, E_2, ...$ We have then a different wave function for each allowed energy:

$$\Psi_n(x,t) = e^{-iE_n t/\hbar} \psi_n(x)$$

The time-dependent Schrödinger equation has the property that any linear combination of solutions is itself a solution. This is the so called *superposition principle*.

Quantum superposition is a fundamental principle of quantum mechanics that states that the linear combination of two or more quantum states is another valid quantum state, or conversely, that every quantum state can be written as a linear combination of two or more distinct states. This is due to the mathematical property of linear equation: since the Schrödinger equation is linear, any linear combination of solution is also a solution. Such a property is so important that is the first described by Paul Dirac in its book *Principles of Quantum Mechanics*, where it describes the phenomenon with these words:

The general principle of superposition of quantum mechanics applies to the states, [that are theoretically possible without mutual interference or contradiction], of any one dynamical System. It requires us to assume that between these states there exist peculiar relationships such that whenever the System is definitely in one state we tan consider it as being partly in each of two or more other states. The original state must be regarded as the result of a kind of superposition of the two or more new states, in a way that cannot be conceived on classical ideas. Any state may be considered as the result of a superposition of two or more other states, and indeed in an infinite number of ways. Conversely any two or more states may be superposed to give a new state.

This means that once we have found the separable solution, we can construct a general one of the form

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar} \tag{2.51}$$

Hence, finding the right constants $c_1, c_2, ...$ every solution of the time dependent Schrödinger equation can be written as this infinite sum.

So given the above information, we can face again the problem of solving the equation given an initial condition $\Psi(x,0)$. Using the separation of variables you find $\phi(t)$ and an infinite set of $\psi_n(x)$ for any possible energy value E_n. To fit the initial condition, you can rewrite $\Psi(x,0)$ as a linear combination of the solutions found

$$\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

And you can always do this thanks to the superposition principle.

In this way, you can construct $\Psi(x,t)$ by multiplying the time independent components of the initial state by their time dependent component

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar} = \sum_{n=1}^{\infty} c_n \Psi_n(x,t) \tag{2.52}$$

The separable solutions themselves

$$\Psi_n(x,t) = c_n \psi_n(x) e^{-iE_n t/\hbar} \tag{2.53}$$

are stationary states, hence all probabilities and expectation values are independent of time, but this property is not shared by the general solution of equation (2.52) since exponentials do not cancel out when computing $|\Psi|^2$ as energies are different for different stationary states.

What we need know is to actually find the $\psi(x)$ solution in the first place. This is possible only knowing the potential $V(x)$ and in the following sections we will analyze the solutions for three well known potential examples.

2.4.1 The infinite square well

Let's take the potential $V(x)$ of the infinite square well, that takes the form

$$V(x) = \begin{cases} 0 & -a \leq x \leq a \\ \infty & \text{otherwise} \end{cases} \tag{2.54}$$

A particle in this potential is free but bounded to stay between the two ends $-a$ and a since out of this region an infinite force prevents.

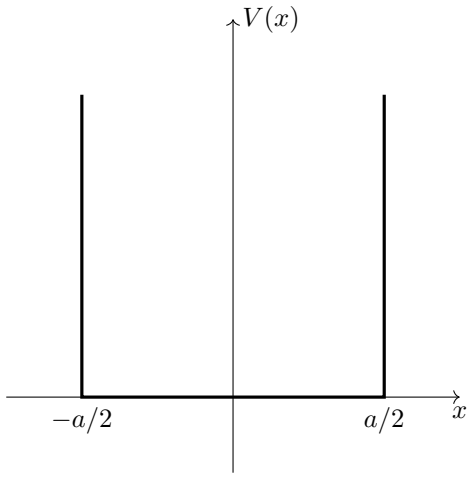

Figure 2.1: Infinite quantum well potential.

This means that outside the well the probability of finding the particle is zero, i.e. $\psi(x) = 0$. On the other side, inside the well the equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

Introducing the variable k

$$k \equiv \frac{\sqrt{2mE}}{\hbar} \tag{2.55}$$

The equation becomes

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

That is a second order differential equation with constant coefficients.

In order to solve it we write the characteristic polynomial $P(\lambda)$ associated to the equation

$$P(\lambda) = \lambda^2 + k^2$$

And we find its roots

$$\lambda = \pm ik$$

Since the roots are complex conjugates, the solution is with sine and cosine

$$\psi(x) = A\sin kx + B\cos kx \tag{2.56}$$

A and B are constants fixed by the boundary conditions of the problem. Good boundary conditions are that ψ and $d\psi/dx$ are both continuous. The first requires

$$\psi(-a/2) = \psi(a/2) = 0$$

Since outside the well $\psi(x) = 0$. Substituting in the solution

$$\psi(-a/2) = A\sin(-\frac{ka}{2}) + B\cos(-\frac{ka}{2}) = 0 \tag{2.57}$$

Since $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$ we can rewrite (2.57) as

$$\psi(-a/2) = B\cos(\frac{ka}{2}) - A\sin(\frac{ka}{2}) = 0$$

The other boundary condition is that

$$\psi(a/2) = B\cos(\frac{ka}{2}) + A\sin(\frac{ka}{2}) = 0$$

Summing the two we have

$$2B\cos(\frac{ka}{2}) = 0$$

From this system either $B = 0$ or $\cos(\frac{ka}{2}) = 0$. $B = 0$ gives the null trivial solution which we are not interested in, hence $\cos(\frac{ka}{2})$ must be zero, so this leads to

$$\frac{ka}{2} = \frac{n\pi}{2} \tag{2.58}$$

Since cosine function is pair, $\cos(-\theta) = \cos(\theta)$ we can absorb the minus into a and find the distinct values of k

$$k_n = \frac{n\pi}{a} \quad \text{with } n = 1, 2, 3, \ldots \tag{2.59}$$

Recalling equation (2.55) and (2.58) we infer that the boundary conditions influence values of the energy E:

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2} \tag{2.60}$$

This is a peculiar and fundamental aspect of quantum mechanics: the system can have just some allowed values of energy.

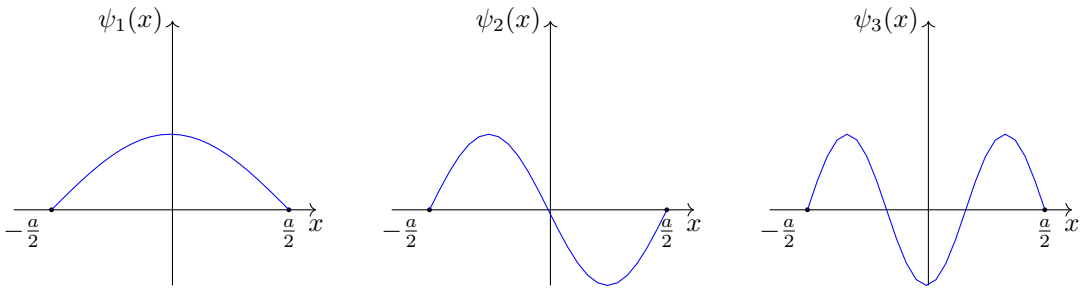

Figure 2.2: The first three stationary states of the infinite square well.

Eventually we find B by normalizing the function:

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} |2B|^2 \cos^2(kx)dx$$

The integral can be solved by the change of variable $z = kx$, so $\frac{dz}{k} = dx$

$$\int_{-\frac{ka}{2}}^{\frac{ka}{2}} \frac{|2B|^2}{k} \cos^2(z)dz = \frac{4|B|^2}{2k}(z + \sin z \cos z)\Big|_{-\frac{ka}{2}}^{\frac{ka}{2}} = |B|^2\frac{a}{2} = 1$$

So

$$B = \sqrt{\frac{2}{a}}$$

Where we took a positive value of B for simplicity.

We then found the solutions inside the well

$$\psi_n(x) = \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi}{a}x\right) \tag{2.61}$$

So we get an infinite set of solutions, one for each positive integer n. The ψ_n associated to lowest energy possible, hence ψ_1 is called *ground state*, while others are named *excited states*. We enunciate some properties of the ψ_n functions:

1. They are alternatively even and odd with respect to the center of the well (0)
2. Each new level of energy carries a function with one more node (zero crossing)
3. They are *orthonormal*
4. They are *complete*, any other function can be expressed as a linear combination of them:

$$f(x) = \sum_{n=1}^{\infty} c_n \psi_n(x) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{a}x\right)$$

Where c_n are found with Fourier's trick

$$c_n = \int \psi_n(x)^* f(x) dx \tag{2.62}$$

These properties are not peculiar to the infinite square well, the first is true whenever the potential itself is a symmetric function, the second is universal, regardless the shape of the potential.

The stationary states are

$$\Psi_n(x,t) = \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t} \tag{2.63}$$

And the general solution of equation (2.52) becomes:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t} \tag{2.64}$$

As stated in Section 2.3.5, $|c_n|^2$ is the probability that a measurement of the energy would yield the value E_n.

2.4.2 The Harmonic Oscillator

The quantum harmonic oscillator has the following potential

$$V(x) = \frac{1}{2} m\omega^2 x^2 \tag{2.65}$$

That leads to the time-independent Schrödinger equation

$$-\frac{\hbar}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi \tag{2.66}$$

We now present the analytic method to solve the equation.

It is customary to "clean" the equation by introducing the dimensionless variable ξ

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x \tag{2.67}$$

So we can rewrite the equation in terms of this new variable

$$\frac{d^2\psi}{d\xi^2} = (\xi^2 - K)\psi \tag{2.68}$$

Where K is the total energy of the system expressed in units of $(1/2)\hbar\omega$:

$$K \equiv \frac{2E}{\hbar\omega} \tag{2.69}$$

We will now attempt to solve equation (2.68). To begin with, notice that for very large value of ξ, ξ^2 dominates over K, and hence we can write

$$\frac{d^2\psi}{d\xi^2} \approx \xi^2\psi \tag{2.70}$$

The above equation can be traced back to the parabolic cylinder function to obtain the approximate solution

$$\psi(\xi) \approx Ae^{-\xi^2/2} + Be^{+\xi^2/2} \tag{2.71}$$

We can immediately see that the second term of the equation is not normalizable since it blows up as $|x| \to \infty$, so the physically acceptable solutions have the asymptotic form, for large value of ξ

$$\psi(\xi) \to h(\xi)e^{-\xi^2/2} \tag{2.72}$$

For the function $h(\xi)$ we look for a power series.

$$h(\xi) = a_0 + a_1\xi + a_2\xi^2 + \cdots = \sum_{k=0}^{\infty} a_k\xi^k \tag{2.73}$$

This is known as the Frobenius method for solving differential equations. According to Taylor's theorem, any function well-behaved function can be expressed as a power series without loss of generality. Differentiating equation (2.72), we have

$$\frac{d\psi}{d\xi} = \left(\frac{dh}{d\xi} - \xi h\right)e^{-\xi^2/2}$$

and

$$\frac{d^2\psi}{d\xi^2} = \left(\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + (\xi^2 - 1)h\right)e^{\xi^2/2}$$

We can then rewrite the Schrödinger equation in terms of h

$$\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + (K - 1)h = 0 \tag{2.74}$$

We compute now the first and second derivative of $h(\xi)$

$$\frac{dh}{d\xi} = a_1 + 2a_2\xi + 3a_3\xi^2 + \cdots = \sum_{k=0}^{\infty} k a_k \xi^{k-1}$$

And

$$\frac{d^2h}{d\xi^2} = 2a_2 + 2 \cdot 3a_3\xi + 3 \cdot 4a_4\xi^2 + \cdots = \sum_{k=0}^{\infty} (k+1)(k+2)a_{k+2}\xi^k$$

And put them in (2.74) to obtain

$$\sum_{k=0}^{\infty} [(k+1)(k+2)a_{k+2} - 2ka_k + (K-1)a_k]\xi^k = 0 \tag{2.75}$$

And so

$$(k+1)(k+2)a_{k+2} - 2ka_k + (K-1)a_k = 0$$

We can see then that the coefficients are codependent, i.e.

$$a_{k+2} = \frac{2k+1-K}{(k+1)(k+2)} a_k \tag{2.76}$$

So from a_0 we can recursively find all the even-numbered coefficients, while a_1 generate the odd-numbered coefficients.

However, in finding the solution we need to remember that we want normalizable solutions. For large values of k, we can approximate (2.76) into

$$a_{k+2} \approx \frac{2}{k} a_k$$

So a general a_k is approximately equal (here the example for even-numbered coefficients)

$$a_k \approx \prod_{k=0,2,4,\ldots}^{\infty} \frac{2}{k} = \frac{1}{(k/2)!}$$

So the equation for $h(\xi)$ becomes

$$h(\xi) \approx \sum \frac{1}{(k/2)!} \xi^k = \sum \frac{1}{j!} \xi^{2k} = e^{\xi^2}$$

This causes h to blow up as we go on with its coefficients a_k, with the consequence that ψ blows up to. This behaviour is in contrast with the constraint that the ψ must be normalizable.

To overcome this problem the power series must *terminate*. There has to be a maximum value n such that $a_{n+2} = 0$, in order to truncate either the odd or the even series, the other one will be truncated from the start with $a_1 = 0$ if n is

even and $a_0 = 0$ if n is odd. In order to terminate this series we recall equation (2.76)

$$K = 2k + 1$$

But from equation (2.69)

$$\frac{2E}{\hbar\omega} = 2n + 1$$

Meaning that

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{2.77}$$

The energy in quantized, as we expected from quantum theory.

So we found, again, n solutions ψ_n that depend on the grade of the polynomial $h_n(\xi)$. Those polynomials are called Hermite polynomials, $H_n(\xi)$, leading to the normalized stationary states of the harmonic oscillator:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2} \tag{2.78}$$

2.4.3 The free particle

We now consider the case where $V(x) = 0$. The time independent Schrödinger equation then reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \tag{2.79}$$

And as in the case of the infinite square well, introducing the variable

$$k \equiv \frac{\sqrt{2mE}}{\hbar}$$

The equation becomes

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

And the solution, in exponential form, is

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \tag{2.80}$$

But differently from the infinite square well, we do not face boundary conditions that can help us to find the value of k and, hence, of the energy of the system. Let us go back the general solution, reintroducing the time dependence, $\exp(-iEt/\hbar)$

$$\Psi(x,t) = Ae^{ik(x-\frac{\hbar k}{2m}t)} + Be^{ik(x+\frac{\hbar k}{2m}t)} \tag{2.81}$$

In this form, any function of x and t that depends on x and t in the combination $(x \pm vt)$, for some constant v represents the wave function of a particle travelling in the $\mp x$ direction, at speed v. A fixed point on the waveform, such a maximum or a minimum, corresponds to a fixed value of the argument, and hence to x and t such that

$$x \pm vt = c$$

Or

$$x = \mp vt + c$$

Where c is a constant. The first term in equation (2.81) represents a wave traveling to the right, while the second represents a wave going to the left. Since they only differ by the sign in front of k, we can write

$$\Psi_k(x,t) = Ae^{i(kx - \frac{\hbar k^2}{2m}t)} \tag{2.82}$$

And

$$k \equiv \pm\frac{\sqrt{2mE}}{\hbar} \quad \text{with} \quad \begin{cases} k > 0 & \text{travelling to the right} \\ k < 0 & \text{travelling to the left} \end{cases} \tag{2.83}$$

The problem of this solution is that it is not normalizable, since the integral

$$\int_{-\infty}^{+\infty} \Psi_k^* \Psi_k dx = \infty$$

This means that that in the case of the free particle the separable solutions do not represent physically realizable states. In the sense that there is no such a thing as a free particle with a definite energy.

Still separable solutions play a mathematical role in the general solution, that is a linear combination of them, as a integral over the continuous variable k, instead of a sum over the discrete values.

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{i(kx - \frac{\hbar^2}{2m}t)} dk \tag{2.84}$$

For appropriate value of $\phi(k)$, this wave function can be normalized.

In order to find $\phi(k)$, we start from the initial wave function $\Psi(x,0)$, since it has to satisfy the condition

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{ikx} dk \tag{2.85}$$

We here recognize equation (1.20), and get the relation

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(k) e^{ikx} dk \\ F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx \end{aligned} \tag{2.86}$$

The existence of those integrals is guaranteed by the fact that $\Psi(x,0)$ is normalizable by definition.

So in order to find $\phi(k)$ we need to compute the Fourier inverse

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x,0) e^{-ikx} dx \tag{2.87}$$

Chapter 3

A quantum model for the stock market

We now illustrate an application of the Schrödinger equation to the stock market, a model suggested by Chao Zang and Lu Huang in 2010, in the journal *Physica A: Statistical Mechanics and its Applications* [3].

3.1 Introduction to econophysics

Econophysics is a research field that applies physics to economics problems. It denotes the activities of physicists who are working on economics problems to test a variety of new conceptual approaches deriving from the physical sciences.

Physicists are currently contributing to the modeling of "complex systems" by using tools and methodologies developed in statistical mechanics and theoretical physics. Financial markets are remarkably well-defined complex systems, which are continuously monitored - down to time scales of seconds. Further, virtually every economic transaction is recorded, and an increasing fraction of the total number of recorded economic data is becoming accessible to interested researchers. Facts such as these make financial markets extremely attractive for researchers interested in developing a deeper understanding of modeling of complex systems.

3.2 The model

The model we are presenting is an application of quantum mechanics to the stock market. As quantum mechanics describes the micro world, it is reasonable a parallelism between a stock and a micro system.

We face a corpuscular property of a stock since it is always traded at a certain price, but at the same time a wave behavior is met in the fluctuation

of such a price. In this way, the wave particle duality, cornerstone of quantum mechanics, is satisfied.

In the model, the square modulus of the wave function $\Psi(x,t)$, represents the price probability distribution. In Dirac notation, following the superposition principle -equation (2.51)- we write

$$|\Psi(x,t)\rangle = \sum_n c_n |\psi_n\rangle \tag{3.1}$$

With c_n computed with the mentioned formula $c_n = \langle\phi_n|\Psi\rangle$ and $|\phi_n\rangle$ possible state of the stock system. The superposition principle of quantum mechanics has been studied in the stock market by Shi [15] and Piotrowski [16]. The trading process can be seen as a physical measurement, that forces the wave function to collapse in one of its possible states, with a certain price associated.

As explained in Section (2.3.5), $|c_n|^2$ represents the probability to observe state n, and

$$P(t) = \int_a^b |\Psi(x,t)|^2 dx \tag{3.2}$$

Is the probability to find the stock between price a and b and time t.

The energy of the stock, hence the Hamiltonian, can be viewed as the intensity of the price movement.

On the other side, we introduce the variable p for the momentum

$$p = m\frac{d}{dt}x \tag{3.3}$$

That denotes the rate of price change, i.e. its trend.

The variable m is the equivalent of the mass in quantum mechanics. In our model, the mass of the stock can be seen as a measure of its inertia. Stocks with bigger mass will have prices difficult to move. Analogously to reality, stocks with larger market capitalization move slower than the smaller market capitalization ones.

We can then enunciate the uncertainty principle of quantum mechanics applied in our stock market model

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \tag{3.4}$$

With σ_x and σ_p standard deviation of price and trend respectively.

An example of the uncertainty principle in finance is the knowledge at a certain time of nothing but the exact price of the stock. As a result, one cannot know the rate and direction of price change in future times, meaning that the uncertainty of the trend seems to be infinite (right as the uncertainty principle states).

In real stock market, one can get information also on how many buyers and sellers there are near the current price (e.g. investors in China are able to see

five or ten bid and ask prices and their volumes on the screen via stock trading software). Hence what we get is more a distribution on the price, where we can evaluate the standard deviation, and from (3.4), an estimate of the stock trend.

For example, if a trader may see a number of buyers far greater than the number of sellers near the current price, she may predict that the price will rise the next time. In finance, the standard deviation of the asset price is usually an indicator of the financial risks. Introducing the uncertainty principle of quantum theory may be helpful in the study of risk management theory.

As unit of price we will take the Chinese Yuan, with an estimation of its standard deviation, according to [5] of $\sigma_x = 10^{-3}$ Yuan.

On the other side, the standard deviation of the trend can be approximated, given the small value of σ_x, to

$$\sigma_p = \sigma\left(\frac{d}{dt}x\right) = \sqrt{\left\langle\left(\frac{d}{dt}x\right)^2\right\rangle - \left\langle\frac{d}{dt}x\right\rangle^2} \approx \sqrt{\left\langle\left(\frac{d}{dt}x\right)^2\right\rangle} \tag{3.5}$$

The average stock price change is estimated as 10^{-2} Yuan per ten seconds in Chinese stock market. We can then estimate the mass of the stock thanks to the property of the standard deviation that

$$\sigma(cX) = |c|\sigma(X)$$

From equation (3.4), given the value of $\hbar$ in equation (2.2)

$$\sigma_x\sigma_p = 10^{-3}|m|10^{-2} \geq 6.63 \times 10^{-34} \approx 10^{-33}$$

So

$$m \approx 10^{-28}$$

From the parallelism mentioned, we present the Schrödinger equation describing the evolution of a stock micro system.

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = H\Psi(x,t) \tag{3.6}$$

Where, as in quantum mechanics, H is the Hamiltonian, an operator depending on price and time.

The difficulty is to find the right Hamiltonian for the stock system, an operator able to involve the broad range of factors influencing the price, from the economic environment to the psychology of investors. We will construct an Hamiltonian simulating the fluctuation of the stock price in Chinese market under an ideal periodic impact of external factors.

3.3 The infinite square well

In Chinese stock market there is the price limit rule, applied to most stocks in China, according to which the rate of return in a trading day cannot be

more that $\pm 10\%$[1] of the previous day's value. From this assumption we can build our Hamiltonian as a one dimensional infinite square well, with a width of $d_0 = x_0 \times 20\%$, with x_0 previous day's closing price of a stock.

We introduce a transformation of coordinate

$$x' = x - x_0 \tag{3.7}$$

To obtain a symmetric infinite square well with width d_0, with the stock price transformed in its absolute return. Let us introduce another variable, r,

$$r = \frac{x'}{x_0} \tag{3.8}$$

That is the rate of return, a dimensionless variable. The width of the well becomes $d_0 = 20\%$. This leads us to evaluate the mass of the stock again, with a dimensionless value of 10^{-30}.

We will choose a cosine squared function to describe the return distribution, due to the fact that return distribution can be described approximately by the Gaussian [6], and the shapes of the two are close. This decision is made because the ground state of the symmetric infinite square well is a cosine function, as we know from equation (2.61), that we can rewrite according to the new variables as

$$\psi_1(r) = \sqrt{\frac{2}{d}} \cos\left(\frac{\pi r}{d}\right) \tag{3.9}$$

With the corresponding energy from equation (2.60) of

$$E_1 = \frac{\hbar^2 \pi^2}{2md^2}$$

We know from Chapter 2 that the square modulus of (3.9) is the probability distribution of the rate of return. We can infer the similarity of this distribution and the Gaussian one by the fact that the maximal value of the density is reached at zero return, and both decrease symmetrically and gradually on right and left sides. The main difference between the two is that the cosine square distribution does not have fat tails and the sharp peak. But besides that, the cosine squared distribution is a good approximation for the Gaussian distribution with a large variance, as shown in Figure 3.3.

[1] https://www.szse.cn/English/rules/siteRule/

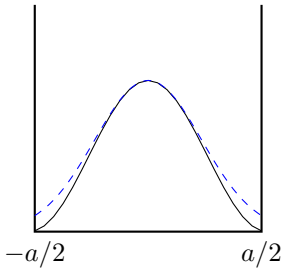

Figure 3.1: Cosine squared function (black line) and Gaussian distribution (dashed blue line) compared.

We want to introduce in the equation something we can compare with the role of information on prices. New information has the consequence to make the stock's price either raise or decline, we can then contemplate this oscillating behavior, assuming a periodic appearance of news, by adding the function $\cos(\omega t)$. The variable ω can be seen as the frequency of appearance of different kind of information, and is assumed to be $\omega = 10^{-4}/s$. The value is reasonable, as it means that the information fluctuates in a single cycle of about four trading days.

The stock here is similar to a charged particle moving in the electromagnetic field, where the external field of stock market is constructed by the information. The stock price may be influenced by such a field.

The potential energy of the stock can be similarly expressed as $eFr\cos(\omega t)$, with e a constant and where F denotes the magnitude of the external field, with a value of $F = 10^{-9}$. Thanks to the addition of such a field, we can rewrite the Hamiltonian as

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + eFr\cos(\omega t) \tag{3.10}$$

The first term of the Hamiltonian, i.e. the kinetic energy, represents some intrinsic properties of the stock. On the other side, the second term of the Hamiltonian, i.e. the potential energy of the stock, reflects the cyclical impact the stock feels due to the information field.

The multiplication of $eFcos(\omega t)$ for the rate of return r has as a result a line whose slope changes over time depending on the value of the cosine factor. So our potential $V(x,t)$ becomes

$$V(x,t) = \begin{cases} eFr\cos(\omega t) & -d/2 < x < d/2 \\ \infty & \text{otherwise} \end{cases} \tag{3.11}$$

As shown in Figure 3.2

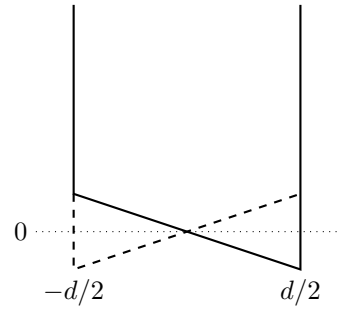

Figure 3.2: The addition of the periodic field into the Hamiltonian causes the slope of the bottom of the quantum well to change periodically.

With the constructed Hamiltonian, we face the following initial value problem (IVP)

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\Psi(r,t) = \left[-\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial r^2} + eFr\cos(\omega t)\right]\Psi(r,t) \\ \\ \Psi(d/2,t) = \Psi(-d/2,t) = 0 \end{cases} \tag{3.12}$$

Definition 3.3.1. *Let $H_1, \mathcal{T}, H_2$ be Hilbert spaces. A function $\Psi : H_1 \times \mathcal{T} \to H_2$ is a global classical solution of the PDE*

$$F(x_1,\dots,x_n,t,\Psi_{x_1},\dots,\Psi_{x_1,x_1},\Psi_{x_1,x_2},\dots,\Psi_{x_1,x_1,x_1},\dots) = 0 \quad \forall \boldsymbol{x}, t \in H_1 \times \mathcal{T} \tag{3.13}$$

If it is differentiable at every $\boldsymbol{x}, t \in H_1 \times \mathcal{T}$ and it satisfies (3.13).

Where the unknown $\Psi = \Psi(x_1,\dots,x_n,t)$ is a function of $n+1$ variables and $\Psi_{x_j},\dots,\Psi_{x_ix_j},\dots$ are its partial derivatives. The highest order of differentiation occurring in the equation is the *order of the equation.*

The case we are studying with equation (3.12) is a second order partial differential equation depending on two variables, r and t.

Proposition 3.3.1. *The function $\Psi(r,t)$ defined as*

$$\Psi(r,t) = \exp\left[-\frac{iE_ct}{\hbar} - \frac{ieFr\sin(\omega t)}{\hbar\omega} - \frac{ie^2F^2(2\omega t - \sin(2\omega t)}{8\hbar m\omega^3}\right] \times$$
$$\sum_{l=-\infty}^{+\infty} A_l \exp(-il\omega t)\left\{\exp\left[ik_l\left(r - \frac{eF\cos(\omega t)}{m\omega^2}\right)\right] + (-1)^l\exp\left[-ik_l\left(r - \frac{eF\cos(\omega t)}{m\omega^2}\right)\right]\right\} \tag{3.14}$$

Is a solution to the IVP of equation (3.12).

The coefficients A_l are approximated to the second order to

$$A_l = i^l \left\{ \begin{array}{l} J_l(q) + \dfrac{q(q^2-\pi^2)v^2}{64}[J_{l+1}(q) - J_{l-1}(q)] - \dfrac{3q^2v}{32}[J_{l+2}(q) - J_{l-2}(q)] \\ -\dfrac{q^2v^2}{32}[J_{l+2}(q) + J_{l-2}(q)] - \dfrac{q^3v^2}{64}[J_{l+3}(q) - J_{l-3}(q)] \\ +\dfrac{9q^4v^2}{2048}[J_{l+4}(q) + J_{l-4}(q)] \end{array} \right\} \tag{3.15}$$

And

$$v = \frac{\hbar\omega}{E_c} \quad q = \frac{k_0 eF}{m\omega^2} \tag{3.16}$$

Proof.
To find the exact solution we make a transformation of the coordinate system. The wave function of our system responds to a time varying external field, related to a time varying, spatially uniform force $f(t)$. We can eliminate the time-dependency of the field with a transformation of variable whenever we face an equation in the form

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + [U(x,t) - xf(t)]\Psi(x,t) \tag{3.17}$$

In our case, $U(x,t) = 0$ inside the well. We now transform equation (3.17) to a system with the coordinates ξ, t, where

$$\xi = x - q(t) \tag{3.18}$$

And

$$q(t) = \frac{1}{m}\int p(t)dt \tag{3.19}$$

With

$$p(t) = \int f(t)dt \tag{3.20}$$

Combining (3.12) and (3.17) we infer that

$$f(t) = -eF\cos(\omega t)$$

So we can compute $p(t)$

$$p(t) = \int -eF\cos(\omega t)dt = -\frac{1}{\omega}eF\sin(\omega t)$$

And then $q(t)$

$$q(t) = \frac{1}{m}\int -\frac{1}{\omega}eF\sin(\omega t)dt = \frac{1}{m\omega^2}eF\cos(\omega t)$$

So we can now introduce the coordinate ξ for our system

$$\xi = r - \frac{eF\cos(\omega t)}{m\omega^2} \tag{3.21}$$

We express $\Psi(r,t)$ in the following form

$$\Psi(r,t) = \phi(\xi,t)\chi(r,t) \tag{3.22}$$

With

$$\chi(r,t) = \exp\left[-\frac{iEt}{\hbar} + \frac{ixp(t)}{\hbar} - \int \frac{ip^2(t)}{2\hbar m}dt\right] \tag{3.23}$$

To compute the integral, we first compute $p^2(t)$

$$p^2(t) = \frac{1}{\omega^2}e^2F^2\sin^2(\omega t)$$

So the integral inside (3.23) becomes

$$\int \frac{ip^2(t)}{2\hbar m}dt = \int \frac{i}{2\hbar m}\frac{e^2F^2}{\omega^2}\sin^2(\omega t)dt = \frac{i}{2\hbar m}\frac{e^2F^2}{\omega^2}\int \sin^2(\omega t)dt$$

$$\int \sin^2(\omega t) = \frac{1}{2\omega}[\omega t - \sin(\omega t)\cos(\omega t)] = \frac{1}{4\omega}[2\omega t - \sin(2\omega t)]$$

Hence the $\chi(r,t)$ function becomes

$$\chi(r,t) = \exp\left[-\frac{iE_ct}{\hbar} - \frac{ieFr\sin(\omega t)}{\hbar\omega} - \frac{ie^2F^2(2\omega t - \sin(2\omega t)}{8\hbar m\omega^3}\right] \tag{3.24}$$

Where E_c denotes the energy of the driven system. In order to substitute $\Psi(r,t)$ with $\chi(r,t)\phi(\xi,t)$ in equation (3.12) we compute the first derivative with respect to t and the second derivative with respect to r of our new wave function.

Using the product rule of derivative, $\frac{\partial}{\partial t}\Psi$ becomes

$$\frac{\partial}{\partial t}[\chi\phi] = \chi_t\phi + \phi_t\chi$$

Where we omit the independent variables for ease. The first derivative with respect to t of ϕ is given by

$$\phi_t = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial t} \tag{3.25}$$

Given $\frac{\partial\phi}{\partial\xi} = 1$, equation (3.25) becomes

$$\phi_t = \frac{\partial\phi}{\partial t} + \frac{eF\sin(\omega t)}{m\omega} \tag{3.26}$$

$$\chi_t = \exp\left[-\frac{iE_c t}{\hbar} - \frac{ieFr\sin(\omega t)}{\hbar\omega} - \frac{ie^2F^2(2\omega t - \sin(2\omega t)}{8\hbar m\omega^3}\right]\cdot \\ \cdot\left[-\frac{iE_c}{\hbar} - \frac{ieFr\cos(\omega t)}{\hbar} - \frac{ie^2F^2}{4\hbar\omega^2 m} + \frac{ie^2F^2\cos(2\omega t)}{4\hbar\omega^2 m}\right] \tag{3.27}$$

To shorten, we put

$$\beta(r,t) = -\frac{iE_c}{\hbar} - \frac{ieFr\cos(\omega t)}{\hbar} - \frac{ie^2F^2}{4\hbar\omega^2 m} + \frac{ie^2F^2\cos(2\omega t)}{4\hbar\omega^2 m}$$

So that

$$\chi_t = \beta\chi$$

For the second derivative with respect to r, we first notice that

$$\phi(\xi,t)_r = \frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial r}$$

But $\xi = r - q(t)$, so its derivative with respect to r is 1.

$$\Psi_r = \chi_r\phi + \phi_\xi\chi$$

And

$$\Psi_{rr} = \chi_{rr}\phi + 2\chi_r\phi_\xi + \phi_{\xi\xi}\chi$$

We now compute the second derivative of χ with respect to r

$$\chi_r = \exp\left[-\frac{iE_c t}{\hbar} - \frac{ieFr\sin(\omega t)}{\hbar\omega} - \frac{ie^2F^2(2\omega t - \sin(2\omega t)}{8\hbar m\omega^3}\right]\cdot \\ \cdot\left[-\frac{ieF\sin(\omega t)}{\hbar\omega}\right] \tag{3.28}$$

$$\chi_{rr} = \exp\left[-\frac{iE_c t}{\hbar} - \frac{ieFr\sin(\omega t)}{\hbar\omega} - \frac{ie^2F^2(2\omega t - \sin(2\omega t)}{8\hbar m\omega^3}\right]\cdot \\ \cdot\left[-\frac{e^2F^2\sin^2(\omega t)}{\hbar^2\omega^2}\right] \tag{3.29}$$

To shorten, we put

$$\alpha(t) = -\frac{ieF\sin(\omega t)}{\hbar\omega}$$

So that

$$\chi_r = \alpha\chi$$

And

$$\chi_{rr} = \alpha^2\chi$$

Our equation can now pass from this form

$$i\hbar\left[\chi_t\phi+\chi\left(\phi_t+\frac{eF\sin(\omega t)}{m\omega}\phi_\xi\right)\right]=-\frac{\hbar^2}{2m}\left[\chi_{rr}\phi+2\chi_r\phi_\xi+\phi_{\xi\xi}\chi\right]+eFr\cos(\omega t)\chi\phi$$

To the one with the above specified variables:

$$i\hbar\left[\beta\chi\phi+\chi\left(\phi_t+\frac{eF\sin(\omega t)}{m\omega}\phi_\xi\right)\right]=-\frac{\hbar^2}{2m}\left[\alpha^2\chi\phi+2\alpha\chi\phi_r+\phi_{\xi\xi}\chi\right]+eFr\cos(\omega t)\phi\chi$$

In order to simplify the equation we divide both side by $\chi(r,t)$, so we obtain

$$i\hbar\left(\beta\phi+\phi_t+\frac{eF\sin(\omega t)}{m\omega}\phi_\xi\right)=-\frac{\hbar^2}{2m}\left[\alpha^2\phi+2\alpha\phi_\xi+\phi_{\xi\xi}\right]+eFr\cos(\omega t)\phi \tag{3.30}$$

After this division, we subtract from both sides

$$\left[rf(t)+\frac{p^2(t)}{2m}\right]\phi+\frac{i\hbar p(t)}{m}\phi_\xi$$

Let us compute the unsolved entities inside the brackets

$$rf(t)=-eFr\cos(\omega t)$$

$$\frac{p^2(t)}{2m}=\frac{e^2F^2\sin^2(\omega t)}{2m\omega^2}=\frac{2m}{\hbar}\alpha$$

Notice that

$$p(t)=-\frac{eF\sin(\omega t)}{\omega}=\frac{\hbar}{i}\alpha$$

So we subtract this quantity

$$\left[-eFr\cos(\omega t)+\frac{2m}{\hbar}\alpha\right]\phi+\frac{\hbar^2\alpha}{m}\phi_\xi$$

From equation (3.30)

$$i\hbar\beta\phi+i\hbar\phi_t+\frac{i\hbar eF\sin(\omega t)}{m\omega}\phi_\xi+eFr\cos(\omega t)\phi-\frac{2m}{\hbar}\alpha\phi-\frac{\hbar^2\alpha}{m}\phi_\xi=$$

$$=-\frac{\hbar^2\alpha^2}{2m}\phi-\frac{\hbar^2\alpha}{m}\phi_\xi-\frac{\hbar^2}{2m}\phi_{\xi\xi}+eFr\cos(\omega t)\phi+eFr\cos(\omega t)\phi-\frac{2m}{\hbar}\alpha\phi-\frac{\hbar^2\alpha}{m}\phi_\xi$$

With some simplification we are left with

$$\left[i\hbar\beta+\frac{\hbar^2\alpha^2}{2m}-eFr\cos(\omega t)+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}\right]\phi=-i\hbar\phi_t \tag{3.31}$$

Let us unwrap the previous simplification

$$i\hbar\beta=E_c+eFr\cos(\omega t)+\frac{e^2F^2}{4\omega^2m}-\frac{e^2F^2\cos(2\omega t)}{4\omega^2m} \tag{3.32}$$

Given the trigonometric rule

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

The former (3.32) becomes

$$i\hbar\beta = E_c + eFr\cos(\omega t) + \frac{e^2F^2\sin^2(\omega t)}{2\omega^2 m}$$

On the other side

$$\frac{\hbar^2}{2m}\alpha^2 = -\frac{e^2F^2\sin^2(\omega t)}{2\omega^2 m} \tag{3.33}$$

So equation (3.31) becomes

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} - E_c\right]\phi = i\hbar\phi_t \tag{3.34}$$

We here recognize the problem of the free particle from Section 2.4.3, and hence we know from equation (2.82) that the solution is in the form

$$\phi(\xi, t) = \sum_l A_l \exp\left[\pm ik_l\xi - il\omega t\right] \tag{3.35}$$

With the energy expressed as

$$E = E_c \pm l\hbar\omega$$

We can define k

$$k_l = \frac{\sqrt{2m(E_c + l\hbar\omega)}}{\hbar}$$

Given the above expression we can now write Ψ

$$\Psi(r,t) = \exp\left[-\frac{iE_ct}{\hbar} - \frac{ieFr\sin(\omega t)}{\hbar\omega} - \frac{ie^2F^2(2\omega t - \sin(2\omega t)}{8\hbar m\omega^3}\right] \times$$
$$\sum_{l=-\infty}^{+\infty} A_l\exp(-il\omega t)\left\{\exp\left[ik_l\left(r - \frac{eF\cos(\omega t)}{m\omega^2}\right)\right] + (-1)^l\exp\left[-ik_l\left(r - \frac{eF\cos(\omega t)}{m\omega^2}\right)\right]\right\} \tag{3.36}$$

$\Psi(r,t)$ is of the form $\exp[-iEt/\hbar]u(t)$, with $u(t) = (t + 2\pi/\omega)$ [17] from which we infer the magnitude of the period, i.e. $2\pi/\omega$. The coefficients A_l are to be determined combining the boundary conditions at $r = \pm d/2$ and the following Fourier expansion [12]

$$\exp\left(i\frac{k_leF\cos(\omega t)}{m\omega^2}\right) = \sum_{n=-\infty}^{\infty} i^nJ_n\left(\frac{k_leF}{m\omega^2}\right)\exp(in\omega t) \tag{3.37}$$

Where $J_n\left(\frac{k_l eF}{m\omega^2}\right)$ is the n-th Bessel function. We know need equation (3.36) satisfy the boundary condition of the infinite square well of our model, i.e.

$$\Psi(\frac{d}{2},t) = \Psi(-\frac{d}{2},t) = 0 \tag{3.38}$$

We can reduce [8] the summation of equation (3.37), and hence reduce equation (3.36) that at the boundary turns out to be

$$\sum_{l=-\infty}^{\infty} (-i)^l A_l \left[\exp\left(ik_l\frac{d}{2}\right) + (-1)^l \exp\left(-ik_l\frac{d}{2}\right)\right] J_{n+l}\left(\frac{k_l eF}{m\omega^2}\right) = 0 \qquad \forall n \tag{3.39}$$

So far, all equation have been exact. In order to find an approximate solution of equation (3.39), we define two dimensionless variables

$$v = \frac{\hbar\omega}{E_c} \tag{3.40}$$

$$q = \frac{k_0 eF}{m\omega^2} \tag{3.41}$$

And expanding the wave vector $k_l = k_0\sqrt{1+lv}$ to the second order of v. From the Euler's formula

$$\frac{e^{ix} + e^{-ix}}{2} = \cos(x)$$

We rewrite equation (3.39) as

$$\sum_{l=-\infty}^{\infty} (-i)^l A_l \cos\left(\frac{k_0 d\sqrt{1+lv}}{2}\right) J_{n+l}\left(\frac{k_l eF}{m\omega^2}\right) = 0 \qquad \forall n \tag{3.42}$$

The Taylor expansion to the second order of $\sqrt{1+x}$ is

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8} \tag{3.43}$$

Hence

$$k_l = k_0\sqrt{1+lv} \approx k_0\left(1 + \frac{1}{2}lv - \frac{1}{8}l^2v^2\right) \tag{3.44}$$

Substituting in the Bessel argument

$$J_l\left(\frac{k_l eF}{m\omega^2}\right) = J_l\left(\frac{k_0 eF}{m\omega^2}\sqrt{1+lv}\right) \approx J_l\left[q\left(1 + \frac{1}{2}lv - \frac{1}{8}l^2v^2\right)\right] \tag{3.45}$$

We introduce another variable for simplicity

$$\eta_l = q\left(\frac{1}{2}lv - \frac{1}{8}l^2v^2\right) \tag{3.46}$$

So the approximation term of equation (3.45) becomes

$$\begin{aligned} J_l(q+\eta_l) &\approx J_l(q) + J_l'(q)\eta_l + \frac{1}{2}J_l''(q)\eta_l^2 \\ &\approx J_l(q) + \frac{1}{2}J_l'(q)qlv\left(1-\frac{1}{4}lv\right) + \frac{1}{8}J_l''(q)q^2l^2v^2 \end{aligned} \tag{3.47}$$

Exploiting the recurrence formula, (1.29) we know that

$$J_l'(q) = \frac{1}{2}\left[J_{l-1}(q) - J_{l+1}(q)\right] \tag{3.48}$$

$$J_l''(q) = \frac{1}{4}\left[J_{l-2}(q) - 2J_l(q) + J_{l+2}(q)\right] \tag{3.49}$$

We substitute (3.48) and (3.49) in (3.47) and obtain

$$\begin{aligned} J_l + \frac{qlv}{4}J_{l-1} - \frac{qlv}{2}J_{l+1} - \frac{ql^2v^2}{16}J_{l-1} + \frac{ql^2v^2}{16}J_{l+1} \\ + \frac{q^2l^2v^2}{32}J_{l-2} - \frac{q^2l^2v^2}{16}J_l + \frac{q^2l^2v^2}{32}J_{l+2} \end{aligned} \tag{3.50}$$

We omit the argument of the Bessel function for simplicity. Now we exploit the second recurrence formula (1.30)

$$lJ_l(q) = \frac{q}{2}\left[J_{l-1} + J_{l+1}\right] \tag{3.51}$$

$$l^2J_l = \frac{q}{4}[q(J_{l+2} + J_l + J_{l-2}) + 2(J_{l-1} - J_{l+1})] \tag{3.52}$$

We can derive from (3.51), (3.52)

$$lJ_{l+1} = (l+1)J_{l+1} - J_{l+1} = \frac{q}{2}[J_l + J_{l+2}] - J_{l+1} \tag{3.53}$$

$$lJ_{l+2} = (l+2)J_{l+2} - 2J_{l+2} = \frac{q}{2}[J_{l+1} + J_{l+3}] - 2J_{l+2} \tag{3.54}$$

And analogously

$$\begin{aligned} l^2J_{l+1} &= (l+1)^2J_{l+1} - 2lJ_{l+1} - J_{l+1} \\ &= \frac{q}{4}[q(J_{l+3} + J_{l+1} + J_{l-1}) + 2(J_l - J_{l+2})] - 2lJ_{l+1} - J_{l+1} \end{aligned} \tag{3.55}$$

$$\begin{aligned} l^2J_{l+2} &= (l+2)^2J_{l+2} - 4lJ_{l+2} - 4J_{l+2} \\ &= \frac{q}{4}[q(J_{l+4} + J_{l+2} + J_l + 2(J_{l+1} - J_{l+3})] - 4lJ_{l+2} - 4J_{l+2} \end{aligned} \tag{3.56}$$

The same reasoning is for J_{l-1}, J_{l-2}. Now we substitute (3.56), (3.55), (3.54), (3.53), (3.52) in (3.50) and exploit the Taylor expansion of the cosine function

$$\begin{aligned} &\cos\left[\frac{k_0d}{2}\left(1 + \frac{1}{2}lv - \frac{1}{8}l^2v^2\right)\right] \\ &\approx -\sin\left(\frac{k_0d}{2}\right)\frac{k_0d}{4}\left[\frac{1}{2}lv - \frac{1}{8}l^2v^2\right] - \cos\left(\frac{k_0d}{2}\right)\left(\frac{k_0d}{2}\right)^2\frac{1}{4}l^2v^2 \end{aligned} \tag{3.57}$$

To obtain

$$A_l = i^l \left\{ \begin{array}{l} J_l(q) + \dfrac{q(q^2-\pi^2)v^2}{64}[J_{l+1}(q) - J_{l-1}(q)] - \dfrac{q^2 v}{32}[J_{l+2}(q) - J_{l-2}(q)] \\ -\dfrac{q^2v^2}{32}[J_{l+2}(q) + J_{l-2}(q)] - \dfrac{q^3v^2}{64}[J_{l+3}(q) - J_{l-3}(q)] \\ +\dfrac{9q^4v^2}{2048}[J_{l+4}(q) + J_{l-4}(q)] \end{array} \right\} \tag{3.58}$$

With the required condition that

$$k_0 d = \frac{\pi}{\sqrt{1+q^2v^2/8}} \tag{3.59}$$

□

The authors use numerical methods to simulate the wave function's distribution at different times, specifically at time $t = 0s$, $t = 1000s$ and $t = 2500s$, plotted in Figure 3.3 respectively with solid, dotted and dashed line.

At time zero the distribution of the rate of return is nearly symmetric with the previous day's closing price, which corresponds to the initial state of the stock, with a zero return being the most probable. By adding the external field of information in, the distribution of the rate starts its evolution over time. When $t = 1000s$ and $t = 2500s$ the probability density shift around the starting one maintaining the same peak value.

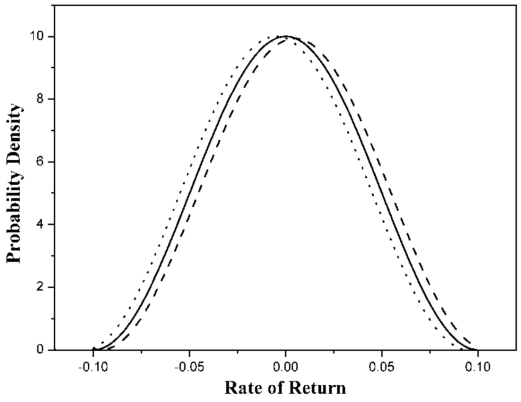

Figure 3.3: Numerical simulations of the probability density of the rate of return at $t = 0$ (solid line), $t = 1000s$ (dotted line) and $t = 2500s$ (dashed line), parameters are $e = 10^{-19}$, $m = 10^{-30}$, $\omega = 10^{-4}$, $F = 10^{-19}$ and $d = 0.2$. At $t = 0$ the distribution corresponds to the initial state of the system. The external field makes the distributions imbalance at $t = 1000s$ and $t = 2500s$.

The change of distribution reflects the imbalance of the market under the influence of external information.

We know that the average value of the rate of return can be written as

$$\langle r(t) \rangle = \int_{-d/2}^{d/2} \Psi^*(r,t) r \Psi(r,t) dr \tag{3.60}$$

The fluctuating average rate of return is shown in Figure 3.4. For the parameters selected, the average rate of return vibrates about 20 times within a period, that have been estimated being of $2\pi/\omega$, while the amplitude of the fluctuation achieves about $\pm 3\%$ as shown in Figure 3.4. This underlines the intrinsic volatility of the price, despite the market information.

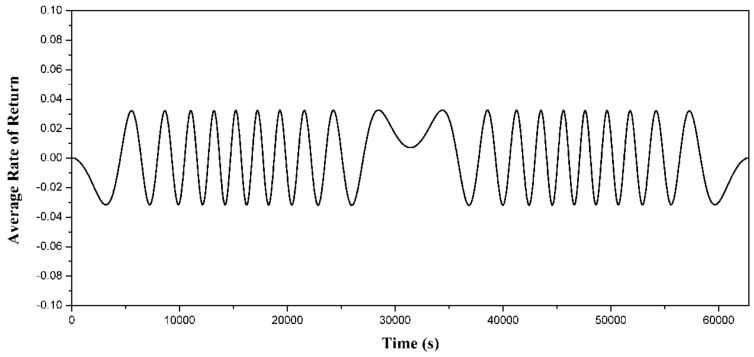

Figure 3.4: The average rate of return fluctuates in a cycle under the market information with the same parameters as in Figure 3.3. The fluctuation of a single cycle is given in the figure,in which the symmetry axis appears at $t = \pi/\omega$ and the amplitude of the fluctuation achieves $\pm 3\%$.

Conclusion

In this thesis we have explained some elements of Functional Analysis to have the tools to introduce the Schrödinger equation, a partial differential equation that describes a quantum-mechanical system. This equation has then been applied to the stock market, with the parallelism between quantum systems and stocks, and between the potential a particle is subject to and the forces that move a stock's price. To simulate how the fluctuation of information crashing the market influences the price return, we build our potential using a cosine function of time, and add an external field that simulate external factors affecting the price, resulting in the potential $eFr\cos\omega t$. We then find an exact solution with approximate parameters to the second order. The solution of the equation, the wave function $\Psi(r,t)$ has the statistical interpretation we explained in Chapter 2, i.e. that the square modulus of it is the return's probability distribution. With a numerical computation, we get the probability distribution of the rate of return under the action of the external field. The peak of the distribution of the rate of return oscillates around the one of the starting distribution. The average rate of return, vibrates about 20 times within a period of $2\pi/\omega$, with an amplitude of $\pm 3\%$ reflecting the intrinsic volatility of the stock price despite external factors, showing the wave behavior that confirms the analogy between particles and stocks. The model gives a new theory of quantum finance, that can be further implemented with the broad range of possible potentials we could build that can quantify the factors affecting the stock price.
There is a strong belief we can further work on the connection between quantum world and stocks from the market, building models able to predict the future average return, or exploiting the uncertainty principle in the study of risk management theory.

Appendix A

Elements of probability theory

A.1 Algebras and $\sigma-$algebras of sets

We denote by X a non empty set, by $\mathcal{P}(X)$ the set of all parts of X, and by $\emptyset$ the empty set.

Definition A.1.1. *Let $\mathcal{F}$ be a nonempty subset of $\mathcal{P}(X)$. $\mathcal{F}$ is said to be an algebra if*

1. $\emptyset, X \in \mathcal{F}$
2. $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$
3. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

Where A^c is the complement of A.

Definition A.1.2. *An algebra $\mathcal{F}$ in $\mathcal{P}(X)$ is said to be a $\sigma-$algebra if, for any sequence $\{A_n\}_n$ of elements of $\mathcal{F}$, we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.*

Definition A.1.3. *A set Ω together with an associated $\sigma-$algebra, $\mathcal{F}$, i.e., the pair $(\Omega, \mathcal{F})$, is called a measurable space.*

A.2 Probability Theory

We give a formal definition of the *probability space* or *probability triple*, $(\Omega, \mathcal{F}, P)$.

Definition A.2.1. *The triple $(\Omega, \mathcal{F}, P)$ is a probability (measure) space if*

- *Ω is the sample space, that is, some (possibility abstract) set.*
- *$\mathcal{F}$ is a $\sigma-$algebra of sets (events) - the measurable subsets of Ω. Its elements, $\{\omega\}$, of Ω, are called elementary events.*

- *P is a probability measure.*

Definition A.2.2. *P is said to be a probability measure if it satisfies the following Kolmogorov axioms:*

1. *For any $A \in \mathcal{F}$, there exists a number $P(A) \geq 0$; the probability of A.*
2. *$P(\Omega) = 1$.*
3. *Let $\{A_n, n \geq 1\}$ be disjoint. Then*

$$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$$

Definition A.2.3. *Let X be a real valued random variable. The distribution function of X is*

$$F(x) = P(X \leq x) \quad x \in \mathbb{R}$$

Definition A.2.4. *A distribution function F is*

- *Discrete iff for some countable set of numbers $\{x_j\}$ and point masses $\{p_j\}$*

$$F(x) = \sum_{x_j \leq x} p_j \quad \forall x \in \mathbb{R}$$

  The function p is called probability function.

- *Continuous iff it is continuous for all x.*

- *Absolutely continuous iff there exists a non-negative integrable function f, such that*

$$F(b) - F(a) = \int_a^b f(x)dx \quad \forall a < b$$

  The function f is called the density of F.

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LUISS

Department of Economics and Finance

Teaching: Mathematical Methods for Economics and Finance

Introduction to the Schrödinger equation and an application to the stock market
Summary

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Chapter 1

Elements of Functional Analysis

1.1 Metric Spaces

Definition 1.1.1. *Let X be a set. A function is said to be a* metric *or a* distance *if for each couple $x, y \in X$ associates a number $d(x, y)$ with the following properties:*

1. $d(x, y) \geq 0$*;* $d(x, y) = 0$ *iff* $x = y$ (positive definiteness)
2. $d(x, y) = d(y, x)$ (symmetry)
3. $d(x, y) \leq d(x, z) + d(z, y)$ (triangular inequality)

The couple (X, d) is said metric space.

1.2 Hilbert Spaces

1.2.1 Norm and Banach Space

Let us start this theory section by recalling the notion of *normed vector space*. Let X be a vector space on a real or complex field. A vector space is said to be normed if you can define a *norm* in it:

$$\|\cdot\| : X \to \mathbb{R}$$

Such that, $\forall\lambda$ and $\forall\, x, y \in X$ the following properties are valid:

1. $\|x\| \geq 0$; $\|x\| = 0$ iff $x = 0$ *(positive definiteness)*
2. $\|\lambda x\| = |\lambda|\, \|x\|$ *(homogeneity)*
3. $\|x + y\| \leq \|x\| + \|y\|$ *(triangular inequality)*

A normed vector space is also *metric*, with the distance induced by:

$$d(x, y) = \|x - y\|$$

Definition 1.2.1. *Let p and q be two norms on a vector space X. Then p and q are called* equivalent *if there exist two real constants c and C with $c > 0$ such that $\forall x \in X$*

$$cq(x) \leq p(x) \leq Cq(x)$$

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements in X is said to be a *Cauchy sequence* if its terms become arbitrarily close to each other as the sequence progresses, i.e.

$$d(x_m, x_n) = \|x_m - x_n\| \to 0 \quad \text{per } m, n \to \infty$$

While it is said to be *convergent* to $x \in X$ if

$$d(x_n, x) = \|x_n - x\| \to 0 \quad \text{per } n \to \infty$$

Theorem 1.2.1. *Every sequence $\{x_n\}_{n\in\mathbb{N}}$ that converges, is a Cauchy sequence.*

The inverse is not necessarily true, if the inverse holds in a vector space, such vector space is said to be *complete*.

Definition 1.2.2. *A complete normed vector space is called* ***Banach space****.*

Let X, Y be two metric spaces and let $F : X \to Y$ be a function from X to Y. F is said to be *continuous* at $x \in X$ if $\forall\{x_n\}_{n\in\mathbb{N}} \subset X$,

$$\|x_n - x\|_X \to 0 \quad \text{implies} \quad \|F(x_n) - F(x)\|_Y \to 0$$

or, equivalently, if

$$\|F(y) - F(x)\|_Y \to 0 \quad \text{if} \quad \|y - x\|_X \to 0$$

Theorem 1.2.2. *Every norm in a space X is continuous in X.*

Given two equivalent norms p and q, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy with respect to the norm p if and only if is Cauchy with respect to the norm q. In particular, the space X is complete w.r.t. the norm p iff it is complete w.r.t. the norm q.

Let Ω be an open set in $\mathbb{R}^n$ and $p \geq 1$ a real number. Let $X = L^p(\Omega)$ be the set of functions f such that $|f|^p$ is Lebesgue integrable in Ω. We introduce the *integral norm of order p*

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p\right)^{\frac{1}{p}}$$

A function $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) is *essentially bounded* if there exists M such that

$$|f(x)| \leq M \quad \text{a.e. in } \Omega \tag{1.1}$$

The infimum of all numbers M with property (1.1) is called *essential supremum of f*, and denoted by

$$\|f\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{\Omega} |f|$$

We further state the Hölder inequality:

$$\left| \int_\Omega fg \right| \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} \tag{1.2}$$

Where p and q are *conjugate exponents*, i.e.

$$\frac{1}{p} + \frac{1}{q} = 1$$

The case $p = 1$ and $q = \infty$ is allowed.

1.2.2 Inner product and Hilbert Space

In order to define a Hilbert space we first need to recall the concept of *inner product*.

Let X be a linear space over $\mathbb{R}$. An *inner o scalar product* in X is a function

$$(\cdot, \cdot) : X \times X \to \mathbb{R}$$

with the following properties. For every x, y, $z \in X$ and scalars $\lambda, \mu \in \mathbb{R}$

1. $(x, x) \geq 0$ and $(x, x) = 0$ if and only if $x = 0$ *(positivity)*
2. $(x, y) = (y, x)$ *(symmetry)*
3. $(\mu x + \lambda y, z) = \mu(x, y) + \lambda(y, z)$ *(bilinearity)*

If the scalar field is $\mathbb{C}$, the inner product is defined as

$$(\cdot, \cdot) : X \times X \to \mathbb{C}$$

With the properties

1. $(x, x) \geq 0$ and $(x, x) = 0$ if and only if $x = 0$
2. $(x, y) = \overline{(y, x)}$
3. $(z, \mu x + \lambda y) = \overline{\mu}(z, x) + \overline{\lambda}(z, y)$

Where $\overline{(y, x)}$ indicates the *complex conjugate*. In the $\mathbb{C}$ the inner product is said to be *antilinear* with respect to its second argument or that it is a *sesquilinear form* in X.
The inner product induces the norm

$$\|x\| = \sqrt{(x, x)}$$

With respect to the above norm we state the following theorem.

Theorem 1.2.3. *Let x,y ∈ X. Then:*

1. *Schwarz's inequality:*

$$|(x, y)| \leq \|x\|\|y\|$$

2. *Parallelogram law:*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

The Schwarz's inequality implies that the inner product is continuous. A linear space endowed with an inner product is called an *inner product space.*

Definition 1.2.3. *Let H be an inner product space. We say that H is a* ***Hilbert space*** *if it is complete with respect to the norm induced by the inner product.*

$L^2(\Omega)$ is one of the most important Hilbert spaces, with respect to the inner product

$$(u, v)_{L^2(\Omega)} = \int_\Omega uv$$

Two Hilbert spaces H_1 and H_2 are *isomorphic* if there exists a linear map $L : H_1 \to H_2$ which preserves the inner product, i.e.:

$$(x, y)_{H_1} = (L_x, L_y)_{H_2} \qquad \forall x, y \in H_1$$

1.3 Projections and Bases

1.3.1 Projections

As in finite-dimensional linear spaces, two elements x, y belonging to an inner product space are called *orthogonal* if $(x, y) = 0$, and we write $x \perp y$.

Now, if we consider a subspace V of $\mathbb{R}^n$, e.g. a hyperplane through the origin, every $\mathbf{x} \in \mathbb{R}^n$ has a unique orthogonal projection on V. In fact, if $\dim V = k$ and the unit vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ constitute an *orthonormal basis* in V, we may always find an orthonormal basis in $\mathbb{R}^n$, given by

$$\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k, \mathbf{w}_{k+1}, ..., \mathbf{w}_n$$

Where $\mathbf{w}_{k+1}, ..., \mathbf{w}_n$ are suitable unit vectors. Thus, if

$$\mathbf{x} = \sum_{j=1}^{k} x_j \mathbf{v}_j + \sum_{j=k+1}^{n} x_j \mathbf{w}_j$$

The projection of $\mathbf{x}$ on V is given by

$$P_V \mathbf{x} = \sum_{j=1}^{k} x_j \mathbf{v}_j$$

$P_V \mathbf{x}$ can also be defined without involving a basis in $\mathbb{R}^n$ as the point in V which minimizes the distance from $\mathbf{x}$.

Theorem 1.3.1. (Projection Theorem). *Let V be a closed subspace of a Hilbert space H. Then, for every $x \in H$, there exists a unique element $P_V x \in V$ such that*

$$\|P_V x - x\| = \inf_{v \in V} \|v - x\|$$

Moreover, the following properties hold:

1. *$P_V x = x$ if and only if $x \in V$*
2. *Let $Q_V x = x - P_V x$. Then $Q_V x \in V^\perp$ and*

$$\|x\|^2 = \|P_V x\|^2 + \|Q_V x\|^2$$

The elements $P_V x$, $Q_V x$ are called *orthogonal projections* of x on V and $V^\perp$, respectively.

Even if V is not a closed subspace of H, the subspace $V^\perp$ is always closed.

1.3.2 Bases

If $A \subset H$, we say A is *dense* in H if its closure, $\overline{A} = H$. A Hilbert space is said to be *separable* when there exists a countable dense subset of H. An *orthonormal basis* in a separable Hilbert space H is a sequence $\{w_k\}_{k \geq 1} \subset H$ such that

$$\begin{cases} (w_k, w_j) = \delta_{kj} & k, j \geq 1 \\ \|w_k\| = 1 & k \geq 1 \end{cases}$$

Where δ_{kj} is the Kronecker delta. Every $x \in H$ may be expanded in the form:

$$x = \sum_{k=1}^{\infty} (x, w_k) w_k \tag{1.3}$$

The series (1.3) is called *generalized Fourier series* and the numbers $c_k = (x, w_k)$ are the *Fourier coefficients* of x with respect to the basis $\{w_k\}$. Moreover:

$$\|x\|^2 = \sum_{k=1}^{\infty} (x, w_k)^2$$

Proposition 1.3.1. *Every separable Hilbert space H admits an orthonormal basis*

1.4 Operators

1.4.1 Linear Operators

Let H be a Hilbert space. A *linear operator* from $D(A) \subset H$ into H is a function

$$A : D(A) \subset H \to H$$

such that, $\forall \alpha, \beta \in \mathbb{C}$ and $\forall f, g \in D(A) \subset H$

$$A(\alpha f + \beta g) = \alpha A f + \beta A g$$

Definition 1.4.1. *A linear operator $A : H_1 \to H_2$ is* bounded *if there exists a number L such that $\forall f \in H_1$*

$$\|Af\|_{H_2} \leq L\|f\|_{H_1} \quad \forall f \in H_1$$

The infimum of the set of all constants L that satisfy the above property is the *norm of the operator*

$$\|A\| = \inf \{L \geq 0 : \|Af\| \leq L\|f\| \quad \forall f \in H_1\} \tag{1.4}$$

We indicate the set of all bounded operator from H_1 into H_2 as $\mathcal{B}(H_1, H_2)$. If $H_1 = H_2 = H$ the former becomes $\mathcal{B}(H)$.

Proposition 1.4.1. *A linear operator $A : H_1 \to H_2$ is bounded if and only if it is continuous.*

Equipped with norm (1.4), $\mathcal{B}(H_1, H_2)$ is a Banach space.

Definition 1.4.2. *An operator $A : H_1 \to H_2$ is* closed *if $\forall \{f_n\}_{n\in\mathbb{N}} \in H_1$ and $\{Af_n\}_{n\in\mathbb{N}}$ such that*

$$f_n \to f \text{ and } Af_n \to g$$

Then

$$f \in H_1 \quad \text{and} \quad g = Af$$

Theorem 1.4.1. *Let $A : H_1 \to H_2$ be a closed operator. If $\exists A^{-1}$, then A^{-1} is closed.*

Definition 1.4.3. *An operator $A : H_1 \to H_2$ is* symmetric *if $\forall x, y \in H_1$*

$$(Ax, y) = (x, Ay)$$

1.4.2 Functionals and dual space

Definition 1.4.4. *We define* functional *a linear operator $A : H_1 \to H_2$ in which the arrival space H_2 is $\mathbb{R}$ (or $\mathbb{C}$, for complex Hilbert spaces).*

Definition 1.4.5. *The collection of all bounded linear functionals on a Hilbert space H is called* dual space *of H and denoted by H^**

Theorem 1.4.2. (Riesz's Representation Theorem). *Let H be a Hilbert space, $\forall A \in H^*$ $\exists! g_A \in H$ such that*

$$Af = (f, g_A) \quad \forall f \in H \tag{1.5}$$

And

$$\|L\| = \|g_A\| \tag{1.6}$$

The Representation Theorem allows the identification of a Hilbert space with its dual.

1.4.3 Adjoint operator

The concept of *adjoint operator* extends the notion of transpose of an $m \times n$ matrix $\mathbf{A}$ and plays a crucial role in determining compatibility conditions for the solvability of several problems.

Definition 1.4.6. *The operator $A^\dagger : H_2 \to H_1$ defined by the identity*

$$(Ax, y)_{H_2} = (x, A^\dagger y)_{H_1} \qquad \forall x \in H_1, \forall y \in H_2$$

is called the adjoint *of A*

Definition 1.4.7. *Let $A : H_1 \to H_2$ be an operator. We say that A is* self-adjoint *if:*

1. $A = A^\dagger$
2. $H_1 = H_2$

Hence

$$(Ax, y) = (x, Ay) \tag{1.7}$$

In the finite-dimensional space, such operators are called *Hermitian*.

Proposition 1.4.2. *Let $A, A_1 \in \mathcal{B}(H_1, H_2)$ and $A_2 \in \mathcal{B}(H_2, H_3)$. Then:*

1. *$A^\dagger \in \mathcal{B}(H_1, H_2)$. Moreover $(A^\dagger)^\dagger = A$ and*

$$\|A^\dagger\|_{\mathcal{B}(H_2,H_2)} = \|A\|_{\mathcal{B}(H_2,H_2)}$$

2. *$(A_2, A_1)^\dagger = A_1^\dagger A_2^\dagger$. In particular, if A is an isomorphism, then*

$$(A^{-1})^\dagger = (A^\dagger)^{-1}$$

1.5 Spectral Theory

1.5.1 Spectrum of a matrix

Let $\mathbf{A}$ be a $n \times n$ matrix, and $\lambda \in \mathbb{C}$. Then, either the equation

$$\mathbf{A}\mathbf{x} - \lambda\mathbf{x} = \mathbf{b}$$

has a unique solution for every $\mathbf{b}$ or there exists $\mathbf{u} \neq \mathbf{0}$ such that

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$$

In the last case we say that λ,$\mathbf{u}$ constitutes an *eigenvalue-eigenvector pair*. The set of eigenvalues of $\mathbf{A}$ is called *spectrum of* $\mathbf{A}$, denoted by $\sigma(\mathbf{A})$. If $\lambda \notin \sigma(\mathbf{A})$, the *resolvent matrix* $(\mathbf{A} - \lambda\mathbf{I})^{-1}$ is well defined. The set

$$\rho(\mathbf{A}) = \mathbb{C} \setminus \sigma(\mathbf{A})$$

is called the *resolvent* of $\mathbf{A}$. If $\lambda \in \sigma(\mathbf{A})$, the kernel $\mathcal{N}(\mathbf{A} - \lambda\mathbf{I})$ is the subspace spanned by the eigenvectors corresponding to λ and it is called the *eigenspace* of λ. Note that $\sigma(\mathbf{A}) = \sigma(\mathbf{A}^T)$ Symmetric matrices are particularly important: all eigenvalues $\lambda_1, ..., \lambda_n$ are real (possibly of multiplicity greater than 1) and there exists in $\mathbb{R}^n$ an orthogonal basis of eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_n$. The extension of those concepts in the Hilbert space setting is useful for the method of separation of variables.

1.5.2 Separation of variables

Suppose we have to solve the problem

$$\begin{cases} u_t = u_{xx} & x \in \Omega, t > 0 \\ u(x,0) = g(x) & x \in \Omega \\ u(x,t) = 0 & x \in \partial\Omega, t > 0 \end{cases}$$

Where Ω is a bounded one dimensional domain. Let us look for solutions of the form

$$u(x,t) = v(x)w(t)$$

And

$$\begin{aligned} u_t &= v(x)w'(t) \\ u_{xx} &= v''(x)w(t) \end{aligned}$$

Substituting into the differential equation, we obtain

$$v(x)w'(t) = v''(t)w(t)$$

Separating variables

$$\frac{w'(t)}{w(t)} = \frac{v''(x)}{v(x)} = -\lambda$$

Since the left hand side of the equation only depends on the t variable while the right hand side of the equation only depend on the x variable, they must be equal to a constant, indeed $-\lambda$. This leads to two problems

$$w' + \lambda w = 0 \tag{1.8}$$

And

$$\begin{cases} -v'' = \lambda v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.9}$$

A number λ such that there exists a non trivial solution v for (1.9) is an eigenvalue of the operator $-\partial^2/\partial x^2$ in Ω and v is the corresponding eigenfunction. The problem can be solved if the following two properties hold:

1. There exists a sequence of real eigenvalues λ_k with corresponding eigenvectors u_k. Solving (1.8) for $\lambda = \lambda_k$ yields

$$w_k(t) = ce^{-\lambda_k t}$$

2. The initial data g can be expanded in series of eigenfunctions:

$$u(x) = \sum g_k u_k(x)$$

Then the solution is given by

$$u(x,t) = \sum g_k e^{-\lambda_k t} u_k(x)$$

The second condition requires that the set of eigenfunctions of $-\partial^2/\partial x^2$ constitutes a basis in the space of initial data. This leads to the problem of determining the *spectrum* of a linear operator in a Hilbert space, and in particular, of self-adjoint operators.

1.5.3 Spectrum of an operator

Definition 1.5.1. *Let $A : H_1 \to H_2$ be a bounded linear operator. A complex number λ is said to be in the* specrtum *of A, $\sigma(A)$, if $(\lambda I - A)$ is not invertible, where I is the identity operator.*

Definition 1.5.2. *Let H_1, H_2 be Hilbert spaces, and I the identity in H. Let $A : H_1 \to H_2$ be a bounded operator. A complex number λ is said to be in the* resolvent set *of A, $\rho(A)$, if the operator $(\lambda I - A)$ is one-to-one and onto*

$$\rho(A) = \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is one-to-one and onto}\}$$

Notice that $\sigma(A) \cup \rho(A) = \mathbb{C}$.

We define the operator $(\lambda I - A)^{-1}$ the *resolvent* of A, $R(\lambda, A)$. In infinitely many dimensions the spectrum may be divided in three subsets.

Definition 1.5.3. Point spectrum.
If $(\lambda I - A)$ is not injective, hence there exist two distinct elements $x, y \in X$ such that $(\lambda I - A)(x) = (\lambda I - A)(y)$, λ is said to be in the point spectrum *of A, denoted $\sigma_P(A)$.*

Definition 1.5.4. Continuous spectrum.
If $(\lambda I - A)$ is injective, and its range is a dense subset R of X, λ is said to be in the continuous spectrum *of A, denoted $\sigma_C(A)$*

Definition 1.5.5. Residual spectrum.
If $(\lambda I - A)$ is injective, but its range is not dense in X, λ is sais to be in the residual spectrum *of A, denoted $\sigma_R(A)$*

So the spectrum is the disjoint union of these three sets:

$$\sigma(A) = \sigma_P(A) \cup \sigma_C(A) \cup \sigma_R(A)$$

Theorem 1.5.1. *Let $A : H_1 \to H_2$ be a compact*[1]*, self-adjoint operator on a separable Hilbert space. Then:*

a) $0 \in \sigma(A)$ *and* $\sigma(A) \setminus \{0\} = \sigma_P(A) \setminus \{0\}$

b) H_1 *has an orthonormal basis* $\{u_m\}$ *consisting of eigenvectors fo* A

c) If $\dim H_1 = \infty$*, the corresponding eigenvalues different from zero* $\{\lambda_m\}$ *can be arranged in a decreasing sequence* $|\lambda_1| \geq |\lambda_2| \geq ...$*, with* $\lambda_m \to 0$ *as* $m \to \infty$

Thus, the spectrum of a compact self-adjoint operator always contains $\lambda = 0$, which is not necessarily an eigenvalue. The other elements in $\sigma(A)$ are eigenvalues, arranged in a sequence converging to zero if H is infinite dimensional.

1.6 Fourier Analysis

Fourier analysis is the study of the way general functions may be approximated by the sums of simple trigonometric functions.

There are two types of Fourier expansions: Fourier series and Fourier transforms.

1.6.1 Fourier Series

Fourier's theorem states that any reasonably well-behaved function can be written as a discrete sum of trigonometric functions.

Theorem 1.6.1. *Consider a function $f(x)$ that is periodic on the interval $0 \leq x \leq L$, then $f(x)$ can be written as*

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \right]$$

a_n and b_n are specific coefficients that can be determined with the following formula:

$$a_0 = \frac{1}{L} \int_0^L f(x) dx \tag{1.10}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{2\pi n x}{L}\right) dx \tag{1.11}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{2\pi n x}{L}\right) dx \tag{1.12}$$

[1] See Conway, A course in functional analysis, for the definition of compact operator.

1.6.2 Fourier Exponential Series

A function that can be written in terms of sines and cosines can also be written in terms of exponentials, i.e.

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi nx/L} \tag{1.13}$$

here the C_n coefficients are given by

$$C_n = \frac{1}{L}\int_0^L f(x)e^{-i2\pi nx/L}dx \tag{1.14}$$

1.6.3 Fourier Transform

The Fourier trigonometric or exponential series require the function to be periodic. A non periodic function can still be written as a series if considering the period of infinite length, passing from sums to integrals. The Fourier transform of a function f is given by

$$f(x) = \int_{-\infty}^{\infty} C(k_n)e^{ik_n x}dk_n \tag{1.15}$$

With

$$C(k_n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ik_n x}dx \tag{1.16}$$

1.7 Bessel funciton

Bessel functions are canonical solutions of the Bessel's equation, an ordinary differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{1.17}$$

The solution can be written in the form

$$y_1(x) = a_0\left[1 + \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right] \quad x > 0 \tag{1.18}$$

Function in brackets is known as the *Bessel function of the first kind of order zero* and is denoted by $J_0(x)$. We present the recurrence formulae

$$2J_n'(z) = J_{n-1}(z) - J_{n+1}(z) \tag{1.19}$$

$$\frac{2n}{z}J_n(z) = J_{n-1}(z) + J_{n+1}(z) \tag{1.20}$$

Chapter 2

The Schrödinger equation

2.1 Brief Introduction to Quantum Mechanics

Quantum Mechanics is a fundamental theory in physics that provides a description of the physical properties of nature at the scale of atoms and subatomic particles. In the quantum world can only assume certain values of energy , i.e. $E_n = nhv$ with $n \in \mathbb{N}$, v is the frequency of radiation of energy and h is a numerical value called Planck's constant that has an approximate value of $6.63 \times 10^{-34} J \cdot s$.

In 1923, the french physicist Louis de Broglie theorized a new cinematic, based on the idea that matter had both a corpuscular and wave nature: the now well known *wave-particle duality*. It is from this new way to see matter that Erwin Schrödinger postulated in 1925 his equation for describing a quantum-mechanical system:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V(x)\Psi \tag{2.1}$$

t is time.

x is the position of the particle.

m is the mass of the particle

i is the imaginary unit.

$V(x)$ is the potential the particle is subject to.

$\hbar = h/2\pi$ is the reduced Planck costant.

Δ is the Laplace operator (or Laplacian) and is the sum of all unmixed second partial derivatives

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

2.2 The wave function

2.2.1 Statistical Interpretation

The manner Ψ represents the state of a particle is that the probability density of finding a particle at a given point, when measured, is proportional to the square of the magnitude of the wave function $|\Psi(x,t)|^2$.
Indeed, the integral $\int_a^b |\Psi(x,t)|^2 dx$ gives the probability of finding the particle between a and b at time t. It follows that the integral must be 1, so that

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = 1 \tag{2.2}$$

Equation (2.2) is know as *normalization condition*.

2.2.2 Observables

Any observable, i.e. any quantity which can be measured, is associated with a self adjoint linear operator. The operators must yield real eigenvalues, since they are values which may come up as the result of a measurement. As we saw in section 1.4 this means that the operator must be Hermitian. Here we present three operators:

$$\langle x \rangle = \int \Psi^*(x) \Psi dx \tag{2.3}$$

$$\langle p \rangle = \int \Psi^* \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi dx \tag{2.4}$$

$$\langle K \rangle = -\frac{\hbar^2}{2m} \int \Psi^* \frac{\partial^2 \Psi}{\partial x^2} dx \tag{2.5}$$

That represents respectively, position, momentum and kinetic energy.

2.2.3 Uncertainty principle

A spread in wavelenght corresponds to a spread in momentum, meaning that the more precisely determined a particle's position is, the less precisely is its momentum. Quantitatively,

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \tag{2.6}$$

Where σ_x is the standard deviation in x, and σ_p is the standard deviation in p. This is Heisenberg's famous *uncertainty principle*.

2.3 Formalism

2.3.1 Dirac's bra and ket notation

Bra-ket notation was introduced by Paul Dirac to describe a state of a system. In such notation, the vectors in the space are called *ket vectors* and are denoted

as $|\alpha\rangle$.
Two kets can be summed

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle$$

And the sum $|\gamma\rangle$ is also a ket. The product between a ket $|\alpha\rangle$ and a number c is still a ket. An operator Q acts on a ket from the left, $A(|\alpha\rangle) = A\,|\alpha\rangle$ and the result is another ket. An *autoket* of an operator is a ket with the property $A\,|a'\rangle = a'\,|a'\rangle$. The set of numbers $\{a'\}$ represents the set of the eigenvalues of the A operator.

We then define the *bra vectors*, denoted with the symbol $\langle\alpha|$, the mirror image of the symbol for a ket vector. The inner product in the bra-ket notation is the complex number:

$$\langle\alpha|\beta\rangle = a_1^* b_1 + a_2^* b_2 + ... + a_n^* b_n$$

With the property $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$ Hence, $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$ are complex conjugates, while $\langle\alpha|\alpha\rangle$ is real.

From this inner product we can define the norm

$$\|\alpha\| = \sqrt{\langle\alpha|\alpha\rangle}$$

An operator Q acts on a bra from the right

$$(\langle\alpha|)Q = \langle\alpha|\,Q$$

And the result is another bra.

2.3.2 Hilbert Space

Quantum theory is based on two constructs: *wave functions* and *operators*. The state of a system is represented by its wave function, a ket, while observables are operators that act on it.

2.3.3 Eigenfunctions of a Hermitian Operator

Our attention is thus directed to the *eigenfunctions of Hermitian operators*, that represents, as stated before, determinate states of observables. These fall into two categories: If the spectrum is *discrete*, i.e. the eigenvalues are separated from one another, then the eigenfunctions lie in Hilbert space and they constitute physically realizable states. If the spectrum is *continuous* then the eigenfunctions are not normalizable, and they do not represent possible wave functions (though linear combinations of them may be normalizable).

Discrete Spectra

Mathematically, the normalizable eigenfunctions of a Hermitian operator have two important properties

Theorem 2.3.1. *The eigenvalues of a Hermitian operator are real*

Theorem 2.3.2. *Eigenfunctions of Hermitian operators belonging to distinct eigenvalues are orthogonal*

Moreover, the eigenfunctions of an observable operator are *complete*: Any function in Hilbert space can be expressed as a linear combination of them.

Continuous Spectra

If the spectrum of a hermitian operator is *continuous*, the eigenfunctions are not normalizable. Nevertheless, there is a sense in which the three essential properties of reality, orthogonality and completeness still hold.

2.3.4 Generalized Statistical Interpretation

If you measure an observable $Q(x,p)$ on a particle in the state $\Psi(x,t)$ you are certain to get *one of the eigenvalues* of the hermitian operator $\hat{Q}(x,-i\hbar\frac{d}{dx})$. If the spectrum of $\hat{Q}$ is discrete, the probability of getting the paticular eigenvalue q_n associated with the orthonormalized eigenfunction $f_n(x)$ is $|c_n|^2$, with $c_n = \langle f_n|\Psi\rangle$.

2.4 Solution of the Schrödinger equation

We need to solve the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

for a specified potential.

Whenever the Schrödinger equation presents a time-independent potential, it can be solved with the method of *separation of variables*, $\Psi(x,t) = \psi(x)\phi(t)$, lower case ψ that indicates a function of x alone.

About the first one, the solution is given by: $\phi(t) = C_1 e^{-iEt/\hbar}$ About the second equation, this is the so called *time-independent Schrödinger equation* and can be solved only when the potential $V(x)$ is specified.

Some properties of the solution with separation of variables are presented.

1. They are states of a definite total energy.
2. Solutions are stationary states, i.e. with all observables independent of time.
3. The general solution is a linear combination of separable solutions.

2.5 The infinite square well

Let's take the potential $V(x)$ of the infinite square well, that takes the form

$$V(x) = \begin{cases} 0 & -a \leq x \leq a \\ \infty & \text{otherwise} \end{cases} \tag{2.7}$$

A particle in this potential is free but bounded to stay between the two ends $-a$ and a since out of this region an infinite force prevents.

The equation is a second order differential equation with complex eigenvalue, so the solution has the following form:

$$\psi(x) = A \sin kx + B \cos kx \tag{2.8}$$

With

$$k_n = \frac{n\pi}{a} \qquad \text{with } n = 1, 2, 3, \ldots \tag{2.9}$$

Boundary conditions influence values of the energy E, $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

Eventually we find B by normalizing the function, and get an infinite set of solutions depending on n. Due to the superposition principle, the general solution is:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t} \tag{2.10}$$

2.6 The Harmonic Oscillator

The quantum harmonic oscillator has the following potential

$$V(x) = \frac{1}{2}m\omega^2 x^2 \tag{2.11}$$

That leads to the time-independent Schrödinger equation

$$-\frac{\hbar}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi \tag{2.12}$$

The equation is solved with an analytic method that leads to the solution:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2} \tag{2.13}$$

Where $H_n(\xi)$ are the Hermite polynomials of the dimensionless variable $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x$.

2.7 The free particle

We now consider the case where $V(x) = 0$. The time independent Schrödinger equation then reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \tag{2.14}$$

We set $k = \pm\frac{\sqrt{2mE}}{\hbar}$, and the solution, in exponential form, is

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{i(kx - \frac{\hbar^2}{2m}t)} dk \tag{2.15}$$

With

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x,0) e^{-ikx} dx \tag{2.16}$$

Chapter 3

A quantum model for the stock market

I am now presenting an application of the Schrödinger equation to the stock market, a model suggested by Chao Zang and Lu Huang in 2010, in the journal *Physica A: Statistical Mechanics and its Applications*.

3.1 The model

The model we are presenting is an application of quantum mechanics to the stock market. As quantum mechanics describes the micro world, it is reasonable a parallelism between a stock and a micro system.

We face a corpuscular property of a stock since it is always traded at a certain price, but at the same time a wave behavior is met in the fluctuation of such a price. In this way, the wave particle duality, cornerstone of quantum mechanics, is satisfied.

In the model, the square modulus of the wave function $\Psi(x,t)$, represent the price probability distribution. The trading process can be seen as a physical measurement, that forces the wave function to collapse in one of its possible states, with a certain price associated.

$$P(t) = \int_a^b |\Psi(x,t)|^2 dx \tag{3.1}$$

Is the probability to find the stock between price a and b and time t. While

$$p = m\frac{d}{dt}x \tag{3.2}$$

Is the momentum of the stock, i.e. its rate of price change. The variable m is the equivalent of the mass in quantum mechanics, it can be seen as a measure of its inertia. The uncertainty principle is satisfied. As unit of price we will take

the Chinese Yuan, with an estimation of its standard deviation of $\sigma_x = 10^{-3}$ Yuan and an average price change is estimated as 10^{-2} Yuan per ten seconds, and a mass of $m \approx 10^{-30}$. The Schrödinger equation describing the evolution of a stock micro system is then

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = H\Psi(x,t) \tag{3.3}$$

Where, as in quantum mechanics, H is the Hamiltonian that needs to be defined.

3.2 The infinite square well

In Chinese stock market there is the price limit rule according to which the rate of return in a trading day cannot be more that $\pm 10\%$ of the previous day's value. From this assumption we can build our Hamiltonian as a one dimensional infinite square well, with a width of $d_0 = 20\%$, with p_0 previous day's closing price of a stock. We define the dimensionless variable for the rate of return, $r = \frac{x-x_0}{x_0}$. We will choose a cosine square function to describe the return distribution. We will build a Hamiltonian reflecting the oscillation of new information affecting the price, with the form

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + eFr\cos(\omega t) \tag{3.4}$$

With the constructed Hamiltonian, we face the following initial value problem (IVP)

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\Psi(r,t) = \left[-\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial r^2} + eFr\cos(\omega t)\right]\Psi(r,t) \\ \\ \Psi(d/2,t) = \Psi(-d/2,t) = 0 \end{cases} \tag{3.5}$$

That has the solution

$$\Psi(r,t) = \exp\left[-\frac{iE_c t}{\hbar} - \frac{ieFr\sin(\omega t)}{\hbar\omega} - \frac{ie^2F^2(2\omega t - \sin(2\omega t)}{8\hbar m\omega^3}\right]\times$$
$$\sum_{l=-\infty}^{+\infty} A_l \exp(-il\omega t)\left\{\exp\left[ik_l\left(r - \frac{eF\cos(\omega t)}{m\omega^2}\right)\right] + (-1)^l\exp\left[-ik_l\left(r - \frac{eF\cos(\omega t)}{m\omega^2}\right)\right]\right\} \tag{3.6}$$

The coefficients A_l are approximated to the second order to

$$A_l = i^l\left\{\begin{array}{l} J_l(q) + \dfrac{q(q^2-\pi^2)v^2}{64}[J_{l+1}(q) - J_{l-1}(q)] - \dfrac{3q^2v}{32}[J_{l+2}(q) - J_{l-2}(q)] \\ \\ -\dfrac{q^2v^2}{32}[J_{l+2}(q) + J_{l-2}(q)] - \dfrac{q^3v^2}{64}[J_{l+3}(q) - J_{l-3}(q)] \\ \\ +\dfrac{9q^4v^2}{2048}[J_{l+4}(q) + J_{l-4}(q)] \end{array}\right\} \tag{3.7}$$

With

$$v = \frac{\hbar\omega}{E_c} \quad q = \frac{k_0 eF}{m\omega^2} \tag{3.8}$$

The authors use numerical methods to simulate the wave function's distribution at different times, specifically at time $t = 0$, $t = 1000s$ and $t = 2500s$, plotted in 3.1 respectively with solid, dotted and dashed line.

At time zero the distribution of the rate of return is nearly symmetric with the previous day's closing price, which corresponds to the initial state of the stock, with a zero return being the most probable. By adding the external field of information in, the distribution of the rate starts its evolution over time. When $t = 1000s$ and $t = 2500s$ the the probability density shift around the starting one but the peak values seem not to change.

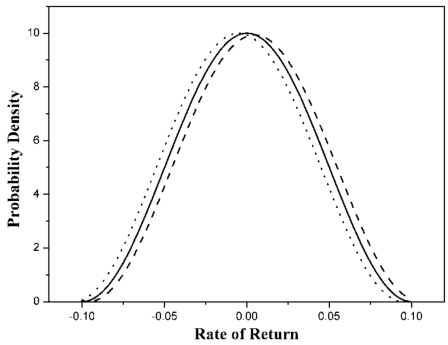

Figure 3.1: Numerical simulations of the probability density of the rate of return at $t = 0$ (solid line), $t = 1000s$ (dotted line) and $t = 2500s$ (dashed line), parameters are $e = 10^{-19}$, $m = 10^{-30}$, $\omega = 10^{-4}$, $F = 10^{-19}$ and $d = 0.2$. At $t = 0$ the distribution corresponds to the initial state of the system. The external field makes the distributions imbalance at $t = 1000s$ and $t = 2500s$.

The change of distribution reflects the imbalance of the market under the influence of external information.

We know that the average value of the rate of return can be written as

$$\langle r(t) \rangle = \int_{-d/2}^{d/2} \Psi^*(r,t) r \Psi(r,t) dr \tag{3.9}$$

The fluctuating average rate of return is shown in Figure 3.2. For the parameters selected, the average rate of return vibrates about 20 times within a period, that have been estimated being of $2\pi/\omega$, while the amplitude of the fluctuation achieves about $\pm 3\%$. This underlines the intrinsic volatility of the price, despite the market information.

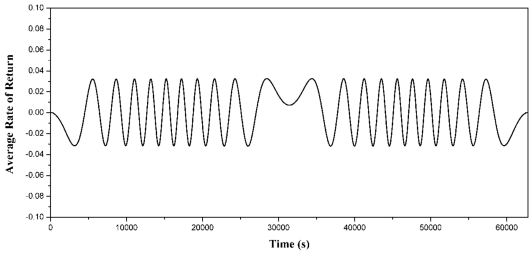

Figure 3.2: The average rate of return fluctuates in a cycle under the market information with the same parameters as in the 3.1. The fluctuation of a single cycle is given in the figure,in which the symmetry axis appears at $t = \pi/\omega$ andthe amplitude of the fluctuation achieves $\pm 3\%$.

Conclusion

In this thesis we have explained some elements of Functional Analysis to have the tools to introduce the Schrödinger equation, a partial differential equation that describes a quantum-mechanical system. This equation has then been applied to the stock market, with the parallelism between quantum systems and stocks, and between the potential a particle is subject to and the forces that move a stock's price. To simulate how the fluctuation of information crashing the market influences the price return, we build our potential using a cosine function of time, and add an external field that simulate external factors affecting the price, resulting in the potential $eFr\cos\omega t$. We then find an exact solution with approximate parameters to the second order. The solution of the equation, the wave function $\Psi(r,t)$ has the statistical interpretation we explained in Chapter 2, i.e. that the square modulus of it is the return's probability distribution. With a numerical computation, we get the probability distribution of the rate of return under the action of the external field. The peak of the distribution of the rate of return oscillates around the one of the starting distribution. The average rate of return, vibrates about 20 times within a period of $2\pi/\omega$, with an amplitude of $\pm 3\%$ reflecting the intrinsic volatility of the stock price despite external factors, showing the wave behavior that confirms the analogy between particles and stocks. The model gives a new theory of quantum finance, that can be further implemented with the broad range of possible potentials we could build that can quantify the factors affecting the stock price.
There is a strong belief we can further work on the connection between quantum world and stocks from the market, building models able to predict the future average return, or exploiting the uncertainty principle in the study of risk management theory.