Department of Economics and Finance

Chair of Gambling: Probability and Decision

Optimal Betting with the Kelly Criterion: Applications to Sports Betting and Stock Market

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Ringrazio chi
durante il mio percorso
ha sempre scommesso su di me.

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Introduction

Gambling carries a great deal of risk, mostly due to the fact that gamblers can bet on different aspects of each game. This is done by casinos on purpose as most gamblers are not aware of the numerical and mathematical aspects (i.e. probabilities) behind every type of game leading them to make wrong choices in good faith. However, if some of these decisions have been taken a priori, using well constructed strategies, the probability of having a profit will be significantly higher. For example, if the gambler chooses too much capital to wager in a high risk bet, he could potentially lose all of his capital after a few number of rounds. On the other hand, if he chooses to wager in a low-risk bet using too little capital, he could potentially make his capital grow at a lower rate than it would if he had bet a larger amount of his capital. Therefore, the gambler needs a strategy to better manage his capital and determine the right amount of capital to wage in each bet. This amount, as well as having to be large enough to make his or her capital increase substantially overtime, cannot be too large in order to avoid ruining in a few number of rounds.

The *Kelly Criterion* is a key instrument to construct a suitable strategy when the game appears favorable for the gambler. Such a strategy was proposed for the first time by the scientist John Larry Kelly Jr. who worked for Bell Labs in 1956 (see [3]). Kelly developed a method for betting in different types of games, such as sports games, as shown in Chapter 3, and investing in stock market, as shown in

Chapter 4. However, this strategy cannot be used in every game as it is valid for positive expectation betting games only, or, in the analogous versions in finance, for investments with excess risk-adjusted expected rates of return.

First we need to identify the underlying mathematical aspects of all types of games, in order to be able to discern between fair and unfair games, Section 1.1, and understand how such frameworks behaves when such games are repeated n times.

Once these situations are identified, the gambler or the investor can manage to exploit such features by deciding the right amount of his capital to bet: this is the problem that we consider and try to solve here.

The aim of the Kelly Criterion works is to maximize what is called a *utility function*, that we will refer to as $g(f)$. Such a function is defined as the long term capital growth rate that the gambler gets if he bets a specific fraction f of its capital in each repetition of the game. By maximizing the utility function over the possible fractions f, the gambler can find the optimal fraction of capital to wager in the bet, that we will refer to as f^*. In Section 2.1.1 we show that the value of f that maximizes the utility function $g(f)$ in a bet with $V:1$ odds is $f^* = \frac{p(V+1)-1}{V}$.

The most important consequences of using the Kelly Criterion are that this strategy will allow the gambler to better manage his money, and that he will significantly lower the risk of catastrophic failure, since he does not bet all the capital he has at each and every round. These consequences are shown in Section 2.3 through simulations: we show that the use of a betting fraction $f = 1$, implies a catastrophic failure in a few rounds, while the profit increases by playing the optimal fraction f^* more than any other case with fraction $f \neq f^*$. In particular we show also the difference between betting a fraction f_1 with $g(f_1) > 0$ and betting a fraction f_2 with $g(f_2) < 0$. In the first case we observe an exponential increasing behaviour of our fortune in the long run, while in the second case our capital decreases exponentially fast in the long run.

In Chapter 3 we will apply this strategy to two different types of sports bet: horse racing and football. We will discuss how the optimal fraction is applied to these different environments and how we have to integrate our strategy when there are more than two outcomes in a game, and the gambler can make several bets in the same round, as in Section 3.1. In addition we will also give a brief study, made by Thorpe in [12] on the effects of correlation to our general framework and an experiment he carried out. In Section 3.3 instead we show how the optimal fraction predicted by the Kelly Criterion, rewritten in terms of quotes, depends on the difference between the quotation of a bookmaker and our estimate on the winning probability.
In Chapter 4 we will go over the application of the Kelly Criterion to the stock market. It is possible in fact to implement this strategy also in investments after having adapted our utility function to the new framework. For example a stock does not have fixed outcome as sports betting. On the other hand, the former has a certain range, or ranges, where the price may rise or decrease. For this reason in Section 4.3 we generalize the previous results given in Section 4.3 to a continuous framework. Finally, we will produce evidence of our above-mentioned findings, with the example in Section 4.3.1.

Chapter 1

Preliminaries

1.1 Fair and unfair games

In this chapter we give the preliminary definitions and results that we will use in the future chapters. Let us consider the following three different games:

- We throw a dice many times, we win 1 euro if the outcome is even or loose 2 euros if the outcome is odd.
- We have an unfair coin, with probability $p = 0.53$ of getting head and probability $q = 0.47$ of getting tail. We win 2 euros if the outcome of the coin toss is head or loose 2 euros if the result is tail.
- We have a deck of 40 cards; we win 5 euro if the suit of the card picked is red, or we loose 5 euros if the suit of the card drawn is black.

We define X_t as our capital at the t-th repetition of the game. After the first round of the first game we have:

$$X_1 = \begin{cases} 1 \cdot \frac{1}{2}, & \text{if we win,} \\ (-2) \cdot \frac{1}{2} & \text{if we loose.} \end{cases}$$

So the expectation of our capital after one round is

$$\mathbb{E}[X_1] = 1 \cdot \frac{1}{2} + (-2) \cdot \frac{1}{2} = \frac{1}{2} - 1 = -\frac{1}{2}, \tag{1.1}$$

Since the expectation of our capital at round one (1.1) is negative, the first game is unfavorable for us. In general, games with negative expectation of the game are called *subfair games.*

Let us now consider the capital in the first round of the second game. We have:

$$X_1 = \begin{cases} 0.53 \cdot 2, & \text{if we win,} \\ 0.47 \cdot (-2) & \text{if we loose,} \end{cases}$$

And so its expectation is:

$$\mathbb{E}[X_1] = 2 \cdot 0.53 + (-2) \cdot 0.47 = 0.12 \tag{1.2}$$

Since the expectation of our capital at round one (1.2) is positive, the second game is favorable for us. In general, games with positive expectations are called *superfair games.*

Let us now consider the capital in the first round of the third game.

$$X_1 = \begin{cases} \frac{20}{40} \cdot 5, & \text{if we win} \\ \frac{20}{40} \cdot (-5) & \text{if we loose} \end{cases}$$

In this category of games the expectations for the capital at the n-th round are zero, as shown by (1.3):

$$\mathbb{E}[X_1] = 5 \cdot \frac{20}{40} + (-5) \cdot \frac{20}{40} = 0 \tag{1.3}$$

Since the expectation of our capital at round one (1.3) is zero, the third game is either in favour nor against us.

In general, games with expectations equal to zero are called *fair games.*

1.2 Convergence of sequences of random variable

Suppose that we have an infinite sequence of random variables $X_1, X_2, ..., X_n, ...$ and we also define X to be a random variable.
We define, as $n \to \infty$, the *almost sure* limit as the random variable towards which, with probability 1, the sequence $\{X_n\}_{n\in\mathbb{N}}$ converges to:

$$\mathbb{P}\left(\lim_{n\to\infty} X_n = X\right) = 1.$$

We will write in this case $X_n \xrightarrow{a.p.} X$.
It could happen that the sequence $\{X_n\}_{n\in\mathbb{N}}$ has no almost sure limit. In such cases it is possible to consider weaker types of covergences that is the convergence *in probability* and *in distribution.*
We say that, as $n \to \infty$, the sequence $\{X_n\}_{n\in\mathbb{N}}$ *converges to* X *'in probability'* if

$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mathbb{P}\left(|X_n - X| > \varepsilon\right) = 0$$

Let F_{Xn} and F_X be the distribution functions of X_n and X, respectively. We say that, as $n \to \infty$, the sequence $\{X_n\}_{n\in\mathbb{N}}$ *converges to* X *'in distribution'* if, for any point $t \in \mathbb{R}$ in which F_X is continuous, we have

$$\lim_{n\to\infty} F_{Xn}(t) = F_X(t).$$

The following result, known as *Law of Large Numbers*, gives relevant information on the limit of proper sequences of random variables.

Law of Large Numbers : *Let* $\{X_n\}_{n\in\mathbb{N}}$*, be a sequence of independent identically distributed random variables, and suppose that* $\mathbb{E}[X_n] = \mu < \infty$ *and* $Var(X_n) =$

$\sigma^2 < \infty$ *for any* $n \in \mathbb{N}$*. For any* $n \in \mathbb{N}$ *we define* $S_n := \sum_{i=1}^{n} X_i$*. Then* $\frac{S_n}{n} \xrightarrow{a.s.} \mu$*.*

The above result can be generalized also to the case in which $\mu = \infty$.

1.3 Betting System

Suppose to repeat a game many times and denote by X_n the gambler's profit per unit bet at the n-th repetition of the game. Assume also that the outcome of each repetition is independent on the other ones. A betting system consists of a sequence of bet sizes $\{B_n\}_{n\in\mathbb{N}}$, where B_n represents the amount bet at the n-th repetition of the game. Moreover B_n depends only on the results of the previous rounds, that is $X_1, X_2, ..., X_{N-1}$.

More precisely

$$B_1 = b_1 > 0,$$

$$B_n = b_n \cdot (X_1, X_2, ..., X_{n-1}) \geq 0 \text{ for } n \geq 2,$$

$$B_n = b_n(X_1, X_2, ..., X_{n-1}) \geq 0 \text{ for } n \geq 2,$$

where b_1 is a constant and b_n is a nonrandom function of $n-1$ variables for each $n \geq 2$.

We denote by F_n the gambler's fortune after n rounds in the betting system previously defined. Then

$$F_n = F_{n-1} + B_n \cdot X_n,$$

for $n \geq 1$, and hence

$$F_n = F_0 + \sum_{i=1}^{n} B_i \cdot X_i,$$

for $n \geq 1$, where F_0 is a positive constant representing the gambler's initial fortune.

We assume that the gambler cannot bet more than he has, that is $B_n \leq F_{n-1}$ for $n \geq 1$.

Since the game is superfair, the gambler can maximize the expected profit by making the largest bet possible at each round, that is $B_n = F_{n-1}$ for $n \geq 1$. But then

$$F_n = F_{n-1} + B_n \cdot X_n = F_{n-1} + F_{n-1} \cdot X_n = F_{n-1} \cdot (1 + X_n) \tag{1.4}$$

and hence, by iteration we get for $n \geq 1$

$$\begin{aligned} F_n = F_{n-1} \cdot (1 + X_n) = F_{n-2} \cdot (1 + X_{n-1}) \cdot (1 + X_n) = \\ = F_{n-2} \cdot \prod_{i=n-1}^{n} (1 + X_i) = \\ = F_0 \cdot \prod_{i=1}^{n} (1 + X_i). \end{aligned}$$

So if there exists an index $j \in 1, ..., n$ such that $X_j = -1$, then $1 + X_j = 0$ and hence $F_n = 0$. This means that if at any of the first n rounds the gambler loses the amount of his bet, then he is ruined. So

$$\begin{aligned} \mathbb{P}(F_n = 0) = \mathbb{P}(\exists\ j \in \{1, ..., n\} \text{ such that } X_j = -1) = \\ = 1 - \mathbb{P}(\forall\ j \in \{1, ..., n\} \text{ we have } X_j \neq -1) \overset{indep.}{=} \\ \overset{indep.}{=} 1 - \prod_{i=1}^{n} \mathbb{P}(X_i \neq -1) \overset{id.distrib.}{=} 1 - \mathbb{P}(X \neq -1)^n = \\ = 1 - [1 - \mathbb{P}(X = -1)]^n \underset{n \to +\infty}{\longrightarrow} = 1, \end{aligned}$$

where X is a random variable distributed as X_i. Hence maximizing the expected profit may result in a disadvantageous betting system. We will analyze again this case in Section 2.1.1.

Chapter 2

Kelly Criterion in Superfair Games

Gambling includes many problems, and one of the most crucial one is finding superfair games, together with knowing how to perfectly manage its own portfolio. Since all casinos' games are subfair, superfair games has to be found in alternative environments, as sport betting or financial investment. For this type of games, an optimization of the portfolio has been analyzed by Kelly in 1956[1].

Let's analyze one of the first scenarios in which we could find ourselves: imagine that we face an infinitely wealthy opponent, in a superfair game with 1:1 odds[2]. Since we know that this is a superfair game, we could decide to bet all our fortune in each round: this decision is based on the assumption that the fortune you have at the t-th round, in case of winning, depends on the quantity you bet (k_t) times a factor that is how much the opponent will pay you in the case your bet is winning (a).

[1]The Bell System Technical Journal, July 1956, 'A new interpretation of information rate'

[2]The same reasoning used to optimize the portfolio can be extended also to the case in which we find ourselves in a bet with K:1 odds

If we take this assumption and we maximize the product

$$k_t \cdot a,$$

we know that, since a is a fixed number, we are left with maximizing k_t that is our capital: maximizing this function implies betting all of our capital.
As we bet all the capital in the first round, our function X_t, that describes our capital at round t-th, will satisfy

$$X_1 = \begin{cases} X_0 + a \cdot X_0, & \text{if we win,} \\ 0, & \text{if we loose.} \end{cases}$$

This strategy, although realistic, is not the best strategy for the gambler, since in case of lost his capital is zero after the first round, as we can observe in the above function[3].

A different strategy could be betting just a fraction of our capital round by round: we define f as the fraction of the capital that we at each round, and we define X_t, again as the function that represents our capital at round t [4]:

$$X_1 = \begin{cases} X_0 + f \cdot X_0 = X_0(1+f), & \text{if we win} \\ X_0 - f \cdot X_0 = X_0\ (1-f), & \text{if we loose} \end{cases} \tag{2.1}$$

Let's consider now the case where the gambler is allowed to make more than one bets in the same round and that the odds never change during the series of bets played, this type of betting system is called fixed odds betting.
Suppose that the gambler plays a series of $N \in \mathbb{N}$ bets and denote by $W_N \in \mathbb{N}$

[3]See Section 1.3

[4]We can observe that if $f = 1$, and so we bet the whole capital in each round, we find the same result above mentioned.

the number of wins and $L_N \in \mathbb{N}$ the number of losses, so that $W_N + L_N = \mathbb{N}$. By iteration, the equation for X_1 found before (2.1), we can see in a similar manner that, if the gambler won W_N bets and lost L_N bets, then his current capital at the end of the N bets is

$$X_N = X_0 \cdot (1+f)^{W_N} \cdot (1-f)^{L_N} \tag{2.2}$$

We can observe in (2.2) that the only variable under the control of the gambler is f, since X_0 refers to the capital at time zero and W_N and L_N are the number of times the player respectively wins and looses the bets.
Our aim is to find the value of f, that we will call f^*, that maximizes X_N in the long run.

2.1 Finding f^*

2.1.1 The expected value approach

Let us consider equation (2.2): W_N and L_N are binomial random variables, since we have defined the, as the number of times we win or loose a bet that has no push. In particular, $W_N \sim Bin(N,p)$, while $L_N \sim Bin(N,q)$, where $q = 1-p$[5].
Let us compute the expectation of X_N recalling that $W_N + L_N = N$

$$\mathbb{E}[X_N] = \sum_{w=0}^{N} X_0 \cdot \mathbb{P}(W_N = w) \cdot (1+f)^W \cdot (1-f)^{N-w}$$

Since $W_N \sim Bin(N,p)$, we get

$$\mathbb{E}[X_N] = X_0 \sum_{w=0}^{N} \cdot \binom{N}{w} p^w \cdot q^{N-w} \cdot (1+f)^w \cdot (1-f)^{N-w}$$

[5]The formal definition of the probability density of a Binomial random variable with N trials and p probability $X \sim \mathrm{Bin}(N,p)$ is reported here $\mathbb{P}(X=k) = \binom{N}{k} \cdot p^k \cdot (1-p)^{N-k}$.

$$= X_0 \sum_{w=0}^{N} \cdot \binom{N}{w} \cdot (p \cdot (1+f))^w \cdot (q \cdot (1-f))^{N-w}. \tag{2.3}$$

Recall that

$$(a+b)^N = \sum_{m=0}^{N} \binom{N}{m} \cdot a^m \cdot b^{N-m}$$

Hence, we can rewrite (2.3) as

$$\mathbb{E}[X_N] = X_0 \cdot (p \cdot (1+f) + q \cdot (1-f))^N$$

$$= X_0 \cdot (f \cdot (2 \cdot p - 1) + 1)^N. \tag{2.4}$$

We are left to maximize this function:

- we could maximize X_0, this implies that the more we bet, the more we win; as discussed above it is a possible solution but it is not very interesting in our line of research;
- since the game is superfair, $p > \frac{1}{2}$ and hence $f \cdot (2p-1) + 1 > 1$. So the function $f \cdot (2p-1)+1$ increases with N, p, f.

Then the only way to maximize X_N is to choose $f = f^* = 1$.
We already found this solution before and we know that this is not what a rationale gambler would adopt as a strategy, since it will lead to ruin in the long run with probability $p = 1$.

2.1.2 The log utility approach

If we want to determine the value of f^*, we firstly have to find a function $g(f)$ that we could maximize.
In particular, this utility function should however satisfy some properties:

- the function should be non-linear, since it must have some stationary points in some $f \in (0,1)$,

- the function must be continuous in the interval $(0,1)$, since the gambler. cannot bet either less or more than his capital.

First we define α_N as the geometric capital growth rate of α for an N series of bets, that is:

$$\alpha_N = \sqrt[N]{\frac{X_N}{X_0}}\,.$$

Applying the natural logarithm to both sides, we have

$$\ln(\alpha_N) = \frac{1}{N} \cdot \ln\left(\frac{X_N}{X_0}\right) \tag{2.5}$$

Using (2.2), the right hand side of equation (2.5) can be rewritten as

$$\frac{X_N}{X_0} = (1+f)^{W_N} \cdot (1-f)^{L_N} \iff$$

$$\ln\left(\frac{X_N}{X_0}\right) = W_N \cdot \ln(1+f) + L_N \cdot \ln(1-f) \iff$$

$$\ln(\alpha_N) = \frac{1}{N}\ln\left(\frac{X_N}{X_0}\right) = \frac{W_N}{N} \cdot \ln(1+f) + \frac{L_N}{N} \cdot \ln(1-f)\,. \tag{2.6}$$

and, with a very large N, by the Law of Large Numbers [6] we get

$$\begin{aligned} g(f) = \ln(\alpha) = \lim_{N\to\infty} \ln(\alpha_N) = \lim_{N\to\infty} \frac{1}{N} \cdot \ln\left(\frac{X_N}{X_0}\right) = \\ = \lim_{N\to\infty} \frac{W_N}{N} \cdot \ln(1+f) + \lim_{N\to\infty} \frac{L_N}{N} \cdot \ln(1-f) = \\ = p \cdot \ln(1+f) + q \cdot \ln(1-f). \end{aligned} \tag{2.7}$$

Note that maximizing $g(f)$, we maximize X_n in the long run. Let us look for stationary points of $g(f)$

$$g'(f) = \frac{p}{1+f} - \frac{q}{1-f} = 0 \iff$$

$$p - q = f(p+q) \iff p - q = f(p + (1-p)) \iff$$

[6] See Section 1.2

$$f = f^* = p - q = 2p - 1. \tag{2.8}$$

The next step is to find and analyze the second derivative of $g(f)$ and, more specifically, to prove that f^* is the maximum of $g(f)$.
The second derivative of $g(f)$ is

$$g''(f) = \frac{-p}{(f+1)^2} - \frac{q}{(f-1)^2}$$

and is negative for all f. Since $g(f)$ is continuous in $[0,1)$, $g(0) = 0$ and $\lim_{f\to 1^-} g(f) = -\infty$ then f^* is a local maximum of the function $g(f)$.
The value of the function in the local maximum is

$$g(f^*) = \ln(2) + p \cdot \ln(p) + q \cdot \ln(q).$$

Therefore

- if $p < \frac{1}{2}$, then the global maximum is $f = 0$, since we know that $f^* = 2p - 1 < 0$,[7]

- if $p > \frac{1}{2}$, then the global maximum is $f = f^* = 2p - 1$ with $(f^* \in (0,1))$.

So we have seen that, when playing a series of bets where the odds are in the gambler's favor, the gambler can either bet everything he has on each single coin toss risking the lost of all his capital, if he loses just one round, or he can use the so called *Kelly Criterion* and get the optimal fraction of capital to bet at every bet in the series, avoiding the risk of ruin.
Note that, by using the *Kelly Criterion* and not maximizing the expected value, gambler's short term gain is lowered while his long term gain is increased.

[7]As stated also before in our analysis this case is not taken into consideration in this dissertation since we are analyzing only superfair games.

2.2 Bets with V:1 odds

Until now our focus has always been towards games with $1 : 1$ odds, that is with games that basically double your initial bet. Of course these games are not the only possible games that one could encounter in his betting experience and reality can be much more complex than the situations presented here.
Now let us generalize to the case of the payoff V to an entry price of 1 that is bets with $V : 1$ odds.
We rewrite equation (2.2) with the following equation:

$$X_N = X_0 \cdot (1 + V \cdot f)^{W_N} \cdot (1 - V \cdot f)^{L_N} \tag{2.9}$$

We can also rewrite equation (2.4) as

$$\mathbb{E}[X_N] = X_0 \cdot (p \cdot (1 + V \cdot f) + q \cdot (1 - V \cdot f))^N \tag{2.10}$$

Note that, in particular

$$\mathbb{E}[X_1] = X_0 \cdot (p \cdot (1 + V \cdot f) + q \cdot (1 - f)) > X_0$$

$$\iff X_0 \cdot (p + p \cdot V \cdot f + q - q \cdot f) > X_0$$

$$\iff X_0 \cdot (p + q) + X_0 \cdot f \cdot (p \cdot V - q) > X_0$$

$$\iff X_0 + X_0 \cdot f \cdot (p \cdot V - q) > X_0$$

$$\iff 1 + f \cdot (p \cdot V - q) > 1$$

$$\iff f \cdot (p \cdot V - q) > 0 \iff f \in (0, 1)$$

$$(p \cdot V - q) > 0 \iff$$

$$\iff p > \frac{1}{1 + V} \iff \mathbb{E}[X_1] = p \cdot V - (1 - p) > 0. \tag{2.11}$$

So for all $p > \frac{1}{1+V}$, the expected capital at time 1 is bigger than the initial capital and the game is superfair.
If $0 < p < \frac{1}{1+V}$ the expected value is negative meaning that the gambler is actually losing capital and therefore he should not place a bet in this particular game. This generalizes the assumption $p > 0.5$ made in the game of $1:1$ odds.
As shown in (2.7), we define the utility function $g(f)$ by

$$\begin{aligned} g(f) &= \lim_{N\to\infty} \ln(\alpha) = \lim_{N\to\infty} \frac{1}{N} \cdot \ln\left(\frac{X_N}{X_0}\right) = \\ &= p \cdot \ln(1 + V \cdot f) + q \cdot \ln(1 - f). \end{aligned} \tag{2.12}$$

To compute f^* we use the first derivative of $g(f)$

$$g'(f) = \frac{Vp}{1 + V \cdot f} - \frac{q}{1 - f} = 0\,,$$

that implies

$$f^* = \frac{p \cdot (V + 1) - 1}{V}\,. \tag{2.13}$$

As before, the next step is to show that f^* is the maximum of $g(f)$ in the interval $[0, 1)$. The second derivative of $g(f)$ is

$$g''(f) = \frac{-q}{(f-1)^2} + \frac{-V^2 \cdot p}{(V \cdot f + 1)^2} < 0.$$

Again, since the second derivative is negative f^* must be a maximum of the function. In particular, since $g(f)$ is continuous in $[0, 1)$ and $g''(f) < 0$, f^* is a global maximum of the function. Moreover by the conditions $p > \frac{1}{V+1}$ and $p < 1$ we get

$$f^* > \frac{\frac{1}{V+1} \cdot (V+1) - 1}{V} = 0,$$

$$f^* < \frac{1 \cdot (V+1) - 1}{V} = 1,$$

and hence $f^* \in (0,1)$.

Now it is possible to get the maximum geometric capital growth rate when using f^* in a game of odds $V:1$. Recall that

$$\alpha = \exp(p \cdot (\ln(1 + V \cdot f) + q \cdot (\ln(1 - f)) = (1 + V \cdot f)^p + (1 - g)^q$$

and hence by substituting f^* we get

$$\alpha^* = (p \cdot (V+1))^p \cdot \left(\frac{V - p \cdot (V+1) + 1}{V}\right)^q ,$$

that can be further simplified to

$$\alpha^* = (p \cdot (V+1))^p \cdot \left(\frac{q \cdot (V+1)}{V}\right)^q . \tag{2.14}$$

A natural question that arises is what does it happen if the gambler chooses to use a fraction of the capital $f_C < f^*$ or $f_C > f^*$? Since $g(0) = 0$, $g(f)$ is continuous and concave with maximum in f^* and $\lim_{f \to 1^-} g(f) = -\infty$, then there exists $f(0) \in (f^*, 1)$ such that $g(f_0) = 0$. In particular $g(f) > 0$ if and only if $0 < f < f_0$.

So, for a fraction $f \in (0, f_0)$, we have $g(f) > 0$ and hence X_N will converge to $+\infty$ almost surely. However, since $g(f^*) > g(f)$, the rate of growth of X_N will be slower then the one obtained with $f = f^*$.

2.3 Simulations

Consider the following bet: at each round if we bet b we win bV with probability p and we lose b with probability $1 - p$. Note that this game is superfair if the expected gain is positive, that is if

$$bV \cdot p - b \cdot (1 - p) > 0 \Rightarrow (V + 1)p > 1 .$$

Moreover, as shown in (2.13), the optimal fraction to bet is given by

$$f^* = \frac{p(V+1)-1}{V}\,.$$

We have realized two programs with MATLAB to compare the same bet changing the value of the parameters. More precisely, in the first program we compare our fortune, for different values of p, after $n = 50$ repetition of the same bet. In the second program instead, fixed the values of V and p, we compare our fortune changing the fraction to bet. In both the programs we fix $V = 2$ and the initial fortune $X_0 = 1$. Moreover we repeat the first bet for $n = 50$ and the second one for $n = 300$ times.

2.3.1 Comparison changing p

Below we have written the code in MATLAB of the first program. Once fixed the value of $p \in \{0.5, 0.65, 0.80, 0.95\}$, we realize two dynamics: in the first one we bet each time the optimal fraction `fstar`, while in the second one we bet everything we have in each bet, as these are the optimal values found in Sections 2.1.1 and 2.1.2. The first dynamic is represented by the vector `G`, while the second one by the vector `T`.

```
V=2; %V is the possible gain for one unit bet
X0=1; %X0 is the initial capital
n=50; %n is the number of repetitions of the bet

A=linspace(0,n,n+1);
k=0;

for p=0.5:0.15:0.95 %p is the probability of winning a bet
    k=k+1;
```

```
    G=zeros(n+1);
    T=zeros(n+1);
    G(1)=X0;
    T(1)=X0;
    fstar=((V+1)*p-1)/V; %ftsar is the optimal fraction
    for i=1:n
        if rand(1)<=p
            G(i+1)=G(i)*(1+fstar*V);
            T(i+1)=T(i)*(1+V);
        else
            G(i+1)=G(i)*(1-fstar);
            T(i+1)=0;
        end
    end

    subplot(2,2,k)
    plot(A,G,'b',A,T,'m')
    hold on
end
```

The plot in Figure 2.1 represents the output of the described program. The blue curve is the evolution of our fortune betting each time the optimal fraction, while the magenta curve is the evolution of our fortune betting each time everything we have. The left-top and the right -top plots are realized for $p = 0.5$ and $p = 0.65$, respectively, while the left-bottom and the right-bottom plots are realized for $p = 0.8$ and $p = 0.95$, respectively. As we have already said in the entire chapter and as we can see from the plots, betting everything at each round is not a good strategy since we lose everything in a few number of rounds (even if the winning probability at each round is 0.95).

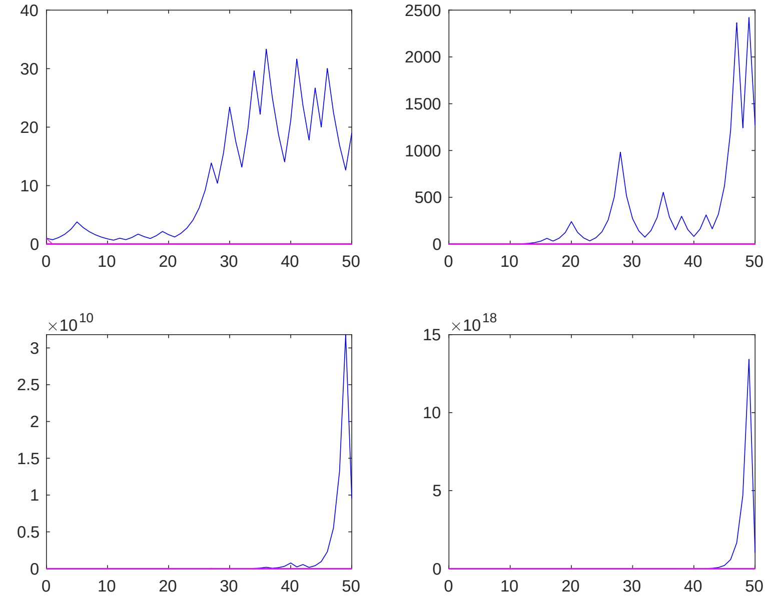

Fig. 2.1

2.3.2 Comparison changing the fraction to bet

Below we have written the code in MATLAB of the second program. Even in this case we have fixed $p = 0.5$. With such a choice, we can easily find the value $f_0 \in (0, 1)$ such that $g(f_0) = 0$, where g is the rate function given in (2.12). Indeed in this case we have

$$\frac{1}{2}\ln(1 + 2f) + \frac{1}{2}\ln(1 - f) = 0 \Rightarrow (1 + 2f)(1 - f) = 1 \Rightarrow f_0 = 0.5\,. \tag{2.15}$$

Moreover, by (2.13), in this case $f^* = 0.25$. In the program below we realize four dynamics:

- `F` represents the evolution of our fortune betting each time a fraction $f = 0.35 > f^* = 0.25$. Note also that $f = 0.30 < f_0 = 0.5$;
- `H` represents the evolution of our fortune betting each time a fraction $f = 0.15 < f^* = 0.25$;

- `G` represents the evolution of our fortune betting each time a fraction $f^* = 0.25$;
- `R` represents the evolution of our fortune betting each time a fraction $f = 0.6 > f_0 = 0.5 > f^* = 0.25$.

```
p=0.5; %p is the probability of winning a bet
V=2; %V is the possible gain for one unit bet
X0=1; % X0 is the initial capital
n=300; %n is the number of repetitions of the bet

fstar=((V+1)*p-1)/V; %fstar is the optimal fraction
f=0.35;
h=0.15;
r=0.6;

F=zeros(n+1);
H=zeros(n+1);
G=zeros(n+1);
R=zeros(n+1);
F(1)=X0;
H(1)=X0;
G(1)=X0;
R(1)=X0;

A=linspace(0,n,n+1);

for i=1:n
    if rand(1)<=p
```

```
        F(i+1)=F(i)*(1+f*V);
        H(i+1)=H(i)*(1+h*V);
        G(i+1)=G(i)*(1+fstar*V);
        R(i+1)=R(i)*(1+r*V);
    else
        F(i+1)=F(i)*(1-f);
        H(i+1)=H(i)*(1-h);
        G(i+1)=G(i)*(1-fstar);
        R(i+1)=R(i)*(1-r);
    end
end

figure (1) %referred to as figure 2.2
plot(A,F,'r',A,G,'b',A,H,'g')

figure(2) %referred to as figure 2.3
plot(A,R,'k')
```

The plot in Figure 2.2 and in Figure 2.3 represents the output of the described program:

- the blue curve in Figure 2.2 represents the dynamic `G` (with fraction f^*),
- the green curve in Figure 2.2 represents the dynamic `H` (with fraction $f < f^*$),
- the red curve in Figure 2.2 represents the dynamic `F` (with fraction $f \in (f^*, f_0)$),
- the black curve in Figure 2.3 represents the dynamic `R` (with fraction $f > f_0 > f^*$).

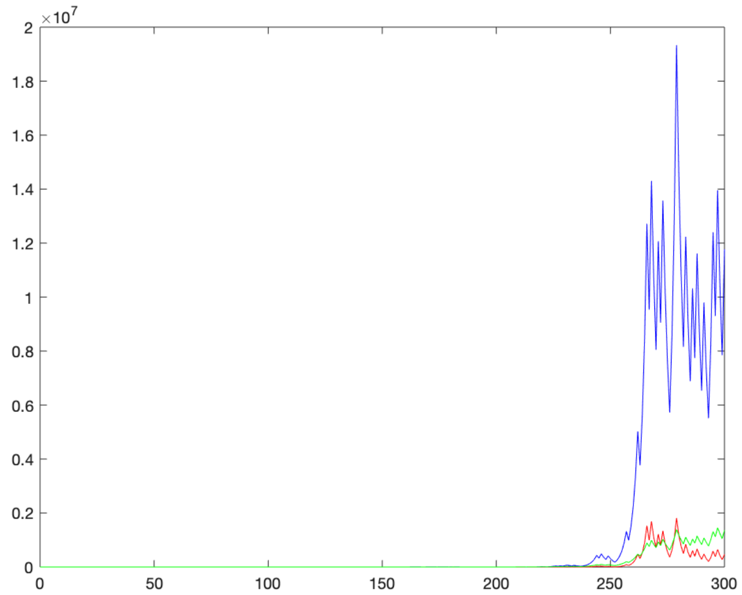
×10^7
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
50
100
150
200
250
300

Fig. 2.2

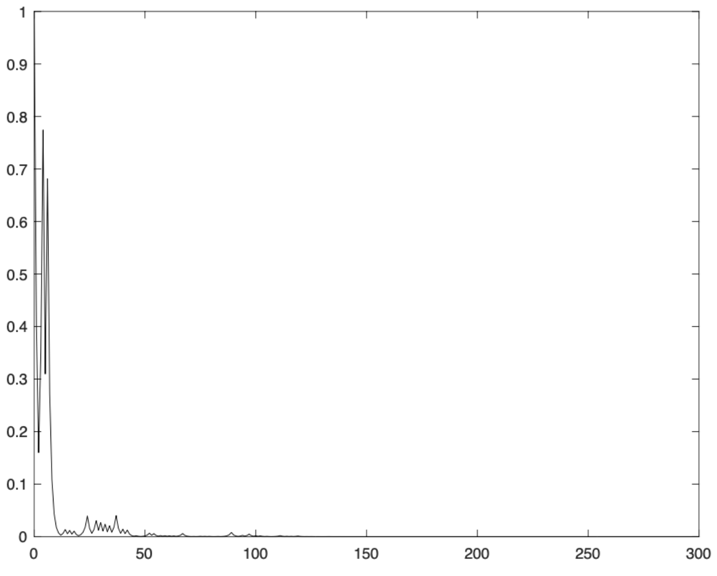
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
50
100
150
200
250
300

Fig. 2.3

As we have studied in this chapter and as the plot shows, betting the optimal fraction maximize our fortune in the long run. It is relevant to observe the difference between betting a fraction f such that the rate $g(f)$ is positive (that is the curves blue, green and red) and betting a fraction f such that the rate $g(f)$ is negative (that is the black curve). Indeed in this last case, the evolution of our fortune is dramatically worst than the ones in the other cases. This is a consequence of the fact that a negative rate means an exponential decrease of the fortune in the long run.

Chapter 3

Kelly Criterion in Sports Betting Strategies

Horse racing is just one of the activities among sports betting in which we can apply the Kelly Criterion. What's more, it can be used also in card games like Blackjack.
In the first section of this chapter we will mainly focus on the Horse Racing Game, but the same techniques can be applied to other sports bets with similar structure. In the second section we will instead focus on the general analysis made by Thorp in [12] on sports bets. Finally we will discuss an application of the Kelly criterion in football.

3.1 Horse Racing

In Horse Racing a gambler can make several different types of bets: one can decide to bet on which horse may win the race, or finish second or third, as well as simultaneous bets of the above described types. Furthermore, all these bets have in common that the gambler has to choose on which horse/s to place his bet/s

where each horse has a different probability of finishing in a certain place.
In this section we will analyze the case of a gambler betting simultaneously on all the horses in a race. The gambler will divide all his capital on bets on all the horses, and a bet on each horse is a bet that the horse will win the race.

Suppose a gambler wants to divide all his capital on three bets on different horses called, for the sake of this example, A, B and C. The bets will be on the winning horse, and the gambler will place all the three different bets on the three different horses winning the race.
Each horse has a different probability of winning the race $p_A, p_B, p_C \in [0,1]$, where $p_A + p_B + p_C = 1$, and the gambler knows the probabilities of success of each horse. Also, the fraction of capital he bets on each horse is $f_A, f_B, f_C \in [0,1]$, where $f_A + f_B + f_C = 1$.
The probabilities of each horse winning do not change between races, and the odds of the game are 1 to 1.
For example, if the gambler plays a single race and horse B wins the race, then of course horses A and C lose and the gambler's capital at time 1 [1] at time one becomes

$$X_1 = X_0 \cdot (1 + f_B - f_A - f_C)$$

The gambler gains the amount of capital he bet on horse B and loses the amount of capital he bet on horses A and C.
Next, assume he bets on two consecutive races, in the first race horse A won and in the second race horse C won, capital is now

$$X_2 = X_1 \cdot (1 - f_A - f_B + f_C) = X_0 \cdot (1 + f_A - f_B - f_C) \cdot (1 - f_A - f_B + f_C).$$

Suppose now that the gambler is playing a series of $N \in \mathbb{N}$ races and define

- $W_N^A \in \mathbb{N}$ as the amount of wins for horse A,

[1] Described by the function X_t previously defined, see Section2.1.1

- $W_N^B \in \mathbb{N}$ as the amount of wins for horse B,
- $W_N^C \in \mathbb{N}$ as the amount of wins for horse C;

hence $W_N^A + W_N^B + W_N^C = N$. Using the equation for the capital at time 2, we can easily manipulate it to get that the gambler's capital after N bets is given by

$$X_N = X_0 \cdot (1+f_A-f_B-f_C)^{W_N^A} \cdot (1-f_A+f_B-f_C)^{W_N^B} \cdot (1-f_A-f_B+f_C)^{W_N^C}. \quad (3.1)$$

Now is up to the gambler to decide how to optimally divide his capital in three parts for the three bets. One way he could accomplish this is under the Kelly principle, but first we have to determine a utility function to which we will apply the natural logarithm.

We define $G(f)$ as the logarithmic capital growth function of a series of N bets, in particular

$$\begin{aligned} G(f_A, f_B, f_C) = \ln\left(\frac{X_N}{X_0}\right) = W_N^A \cdot \ln(1 + f_A - f_B - f_C) + \\ + W_N^B \cdot \ln(1 - f_A + f_B - f_C) + W_N^C \cdot \ln(1 - f_A - f_B + f_C). \end{aligned}$$

Let us divide now $G(f)$ by N and use equation (2.5)

$$\begin{aligned} \ln \alpha_N = \frac{1}{N} G(f) = \frac{1}{N} \cdot \ln\left(\frac{X_N}{X_0}\right) = \frac{W_N^A}{N} \cdot \ln(1 + f_A - f_B - f_C) + \\ + \frac{W_N^B}{N} \cdot \ln(1 - f_A + f_B - f_C) + \frac{W_N^C}{N} \cdot \ln(1 - f_A - f_B + f_C). \end{aligned}$$

Taking N large and applying the Law of Large Numbers (see Chapter1.2) we get

$$\begin{aligned} g(f_A, f_B, f_C) = \ln(\alpha_N) = p_A \cdot \ln(1 + f_A - f_B - f_C) + \\ + p_B \cdot \ln(1 - f_A + f_B - f_C) + p_C \cdot \ln(1 - f_A - f_B + f_C). \end{aligned}$$

This equation represents the utility function of the capital in a horse race after a very large number of bets. The aim now is to calculate the maximum of this function: this maximization problem can be defined as a constrained nonlinear multi-variable optimization problem. More precisely we have to maximize the function

$$g(f_A, f_B, f_C) = \ln(\alpha_N) = p_A \cdot \ln(1 + f_A - f_B - f_C) + \\ + p_B \cdot \ln(1 - f_A + f_B - f_C) + p_C \cdot \ln(1 - f_A - f_B + f_C),$$

subject to the constraints

$$h(f_A, f_B, f_C) = f_A + f_B + f_C - 1 = 0$$
$$f_A \geq 0,\ f_B \geq 0,\ f_C \geq 0\,.$$

It is possible to get the optimal betting ratios (f_A^*, f_B^*, f_C^*) analytically by inserting the equality constraint $f_A = 1 - f_B - f_C$ into the function g. Hence we have to maximize

$$g(f_B, f_C) = p_A \cdot \ln(2 \cdot (1 - f_B - f_C)) + p_B \cdot \ln(2 \cdot f_B) + p_C \cdot \ln(2 \cdot f_C)$$

over the domain

$$\Delta = \{f_B, f_C \in \mathbb{R}^2 \,:\, f_B + f_C \leq 1,\ 0 \leq f_B, 0 \leq f_C\}\,.$$

A picture of the domain Δ is given in Figure 3.1.

To find the critical point of this function we calculate the partial derivatives and we put them equal to zero. So we have

$$\frac{\partial}{\partial f_B} g(f_B, f_C) = \frac{2 \cdot p_A}{2 \cdot f_B + 2 \cdot f_C - 2} + \frac{p_B}{f_B}$$
$$\frac{\partial}{\partial f_C} g(f_B, f_C) = \frac{2 \cdot p_A}{2 \cdot f_B + 2 \cdot f_C - 2} + \frac{p_C}{f_C}$$

and hence the critical point is

$$(f_B^*, f_C^*) = (p_B, p_C)\,, \tag{3.2}$$

that implies $f_A^* = p_A$. So the optimal betting ratios are the probabilities of each horse winning the race. Now we are left to prove that the optimal betting ratios

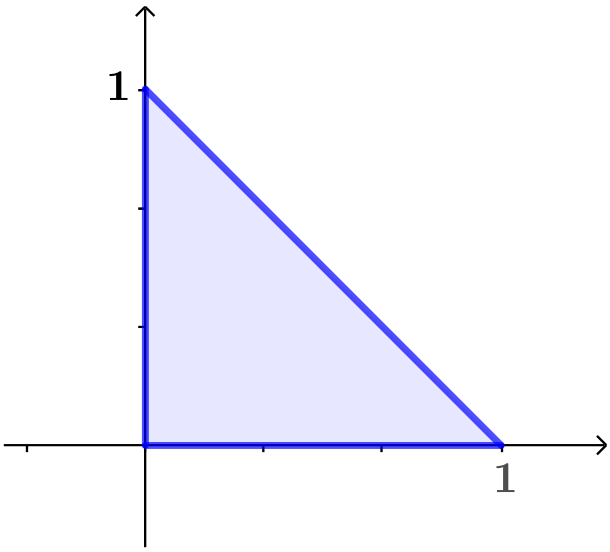

Fig. 3.1: The domain Δ, where in the coordinate axes there are f_B and f_C.

(f_B^*, f_C^*) are the global maximum of the function $g(f_B, f_C)$.
The domain Δ is closed and bounded and $g(f_B, f_C)$ is continuous over the interior of Δ. Since

$$\lim_{f_B \to 1, f_C = 0} (g(f_B, f_C)) = -\infty \,, \qquad \lim_{f_C \to 1, f_B = 0} (g(f_B, f_C)) = -\infty \,,$$

$$\lim_{f_C \to 1, f_B = 0} (g(f_B, 1 - f_B)) = -\infty \,,$$

we get that the maximum of $g(f_B, f_C)$ must be in the interior of Δ.
We need to apply the second derivative test on the critical point found to know if the optimal betting ratios is relative maximum, relative minimum or saddle point.
Since

$$\frac{\partial}{\partial^2 f_B} g(p_B, p_C) = -\frac{4 \cdot p_A}{(2 \cdot p_B + 2 \cdot p_C - 2)^2} - \frac{1}{p_B} \,,$$

$$\frac{\partial}{\partial^2 f_C} g(p_B, p_C) = -\frac{4 \cdot p_A}{(2 \cdot p_B + 2 \cdot p_C - 2)^2} - \frac{1}{p_C} \,,$$

$$\frac{\partial}{\partial f_B \partial f_C} g(p_B, p_C) = -\frac{4 \cdot p_A}{(2 \cdot p_B + 2 \cdot p_C - 2)^2} \,,$$

then

$$D = \frac{\partial}{\partial^2 f_B} g(p_B, p_C) \cdot \frac{\partial}{\partial^2 f_C} g(p_B, p_C) - \left(\frac{\partial}{\partial f_B \partial f_C} g(p_B, p_C) \right)^2 =$$
$$= \frac{4 \cdot p_A}{(2 \cdot p_B + 2 \cdot p_C)^2 \cdot p_B} + \frac{4 \cdot p_A}{(2 \cdot p_B + 2 \cdot p_C)^2 \cdot p_C} + \frac{1}{p_B \cdot p_C} .$$

So $D > 0$ and $\frac{\partial}{\partial^2 f_B} g(f_B, f_C) > 0$. Therefore the critical point in (3.2) is a relative maximum. Note also that, since $p_B, p_C \in (0, 1)$, then the critical point is in the interior of the domain Δ.

The final step is to prove the relative maximum is the global maximum.

First we will find the value of the function $g(f_B, f_C)$ on the the boundary of the domain Δ and at the relative maximum.

Note that

$$\begin{aligned}
\exp\left(g(f_B, f_C)\right) &= \exp\left(p_A \cdot \ln\left(2 \cdot (1 - f_B - f_C)\right) + p_B \cdot \ln\left(2 \cdot f_B\right) + p_C \cdot \ln\left(2 \cdot f_C\right)\right) = \\
&= \exp\left(p_A \cdot \ln\left(2 \cdot f_A\right)\right) \cdot \exp\left(p_B \cdot \ln\left(2 \cdot f_B\right)\right) \exp\left(p_C \cdot \ln\left(2 \cdot f_C\right)\right) = \\
&= \exp\left(\ln\left(2 \cdot f_A\right)^{p_A}\right) \cdot \exp\left(\ln\left(2 \cdot f_B\right)^{p_B}\right) \cdot \exp\left(\ln\left(2 \cdot f_C\right)^{p_C}\right) \\
&= (2 \cdot f_A)^{p_A} \cdot (2 \cdot f_B)^{p_B} \cdot (2 \cdot f_C)^{p_C} = 2 \cdot f_A^{p_A} \cdot f_B^{p_B} \cdot f_C^{p_C} .
\end{aligned}$$

and (recalling that $f_A = 1 - f_B - f_C$)

- when $f_B = 0$, we get $\exp\left(g(0, f_C)\right) = 0$,
- when $f_C = 0$, we get $\exp(g(f_B, 0)) = 0$,
- when $f_C = 1 - f_B$, we have $f_A = 0$ and hence we get $\exp\left(g(f_B, f_C = 1 - f_B)\right) = 0$.

The value of the utility function at the relative maximum is $\exp\left(g(f_B, f_C)\right) = 2 \cdot p_A^{pA} \cdot p_B^{p_B} \cdot p_C^{p_C} > 0$. So the global maximum cannot be on the boundary of .

Moreover using Weierstrass's Extreme Value Theorem, [2] for the function e^g, the global maximum value of the function $g(f_B, f_C)$ is on the relative maximum point. This means that (f_B^*, f_C^*) is the point of maximum of the function $g(f_B, f_C)$.
In conclusion, by using (3.2), the optimal betting ratios can easily be calculated for this type of game.
A final point of interest is for what values of (p_A, p_B, p_C) should the gambler play the game. By inserting the critical point in (3.1) into the function g, we get

$$p_A \cdot \ln(2 \cdot p_A) + p_B \cdot \ln(2 \cdot p_B) + p_C \cdot \ln(2 \cdot p_C) > 0\,. \tag{3.3}$$

We say that the gambler should play the game if (3.3) is satisfied. Indeed, if such a constraint holds for a selected (p_A, p_B, p_C), then the gambler's fortune geometric growth rate will be positive and he should bet in the Horse Racing game.

3.2 Thorp's analysis on sports betting

Edward O. Thorp in [12] exhibits an application of the Kelly criterion system to bets on different sports during realized in 1993. The result hereby presented in Figure 3.2 is of significant interest and confirms the hypothesis made throughout this dissertation: "After 101 days of bets, our $50,000 bankroll had a profit of $123,000, about $68,000 from Type 1 sports and about $55,000 from Type 2 sport. The expected returns are shown as about $62,000 for Type 1 and about $27,000 for Type 2. One might assign the additional $34,000 actually won to luck, but this is likely to be at most partly true because our expectation estimates from the model were deliberately chosen to be conservative"[3].

Being a risk averse bettor, made him under-bet the f^* he found for the bets.

[2]**Weierstrass's Extreme Value Theorem**: *A continuous function in a closed bounded domain has a global maximum and a global minimum.*

[3]Sixth section, *Sports Betting*, in [12], page 18, lines 36-39.

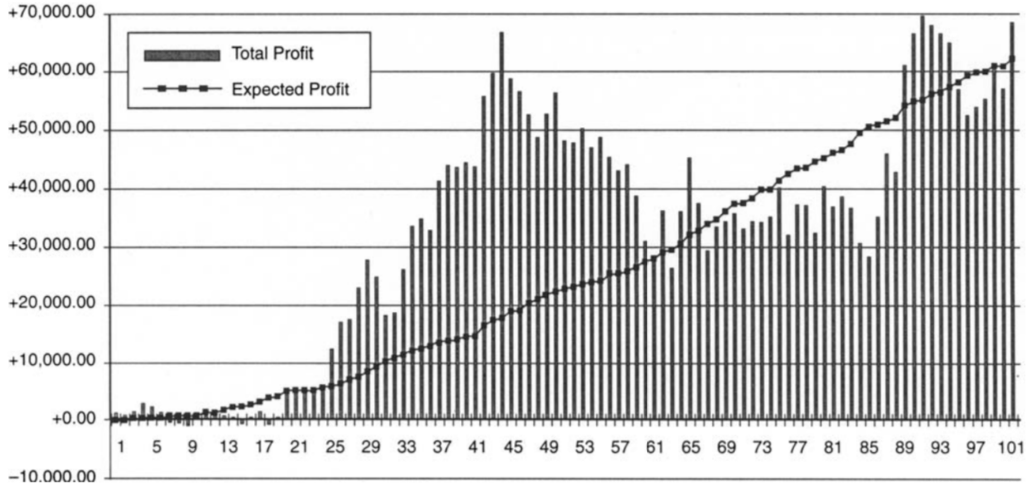

Fig. 3.2: This is 101 days of bets during the end of 1993 and the beginning of 1994

He explains that "The reason [of the under-betting] is that using too large an f^* and over-betting is much more severely penalized than using too small an f^* and under-betting"[4]. Since they have looked for bets where $p-q=0.06$, where p is the probability of winning and $q=1-p$ is the probability of loosing, Thorp says: "Our typical expectation was about 6% so our total bets[...] were about \$2,000,000 or about \$20,000 per day. We typically placed from five to fifteen bets a day and bets ranged from a few hundred dollars to several thousand each, increasing as our bankroll grew."[5].

Thorp then goes on explaining two examples, that I here present, used to discuss the difference between multiple bets on the same event, when these bets are independent, Example 1[6], and when they are dependent, Example 2[7].

[4]Sixth section, *Sports Betting*, in [12], page 18, lines 41-43.
[5]Sixth section, *Sports Betting*, in [12], page 19, lines 5-6.
[6]Subsection *Example 6.1* in [12], page 19.
[7]Subsection *Example 6.2* in [12], page 20.

Example 1

Suppose we bet simultaneously on two independent favorable coins with betting fractions f_1 and f_2 and with success probabilities p_1 and p_2, respectively. Then the expected growth rate is given by

$$\begin{aligned} g(f_1, f_2) = p_1 p_2 \ln(1 + f_1 + f_2) + p_1 q_2 \ln(1 + f_1 - f_2) \\ + q_1 p_2 \ln(1 - f_1 + f_2) + q_1 q_2 \ln(1 - f_1 - f_2)\,. \end{aligned} \tag{3.4}$$

To find the optimal f_1^* and f_2^* we solve the system of equations $\frac{\partial}{\partial f_1} g(f_1, f_2) = 0$ and $\frac{\partial}{\partial f_2} g(f_1, f_2) = 0$. This leads to the conditions

$$f_1 + f_2 = \frac{p_1 p_2 - q_1 q_2}{p_1 p_2 + q_1 q_2} := c\,, \qquad f_1 - f_2 = \frac{p_1 q_2 - q_1 p_2}{p_1 q_2 + q_1 p_2} := d\,, \tag{3.5}$$

that imply

$$f_1^* = (c + d)/2\,, \qquad f_2^* = (c - d)/2\,.$$

These equations pass the symmetry check: interchanging the indexes 1 and 2 throughout maps the equation set into itself.

An alternate form is instructive. Let $m_i = p_i - q_i$, for $i = 1, 2$, so that $p_i = (1+m_i)/2$ and $q_i = (1-m_i)/2$. Substituting these relations in (3.5) and simplifying, we get

$$\begin{aligned} c &= \frac{m_1 + m_2}{1 + m_1 m_2}\,, & d &= \frac{m_1 - m_2}{1 - m_1 m_2}\,, \\ f_1^* &= \frac{m_1 \cdot (1 - m_2^2)}{1 - m_1^2 m_2^2}\,, & f_2^* &= \frac{m_2 \cdot (1 - m_2^2)}{1 - m_1^2 m_2^2}\,, \end{aligned}$$

which shows clearly the factors by which the f_i^* are each reduced from m_i^*. Since m_1, m_2 are typically small, the reduction factors are typically very close to 1.

In the special case $p_1 = p_2 = p$, we have $d = 0$ and $m = p - q$. Then we get

$$f^* = f_1^* = f_2^* = c/2 = \frac{(p - q)}{(2(p^2 + q^2))} = m/(1 + m)$$

as the optimal fraction to bet on each coin simultaneously. Note that if we do the single bet sequentially, the optimal fraction to bet is $f^* = m$. So playing simultaneously reduces the optimal fraction to bet.

Generally simultaneous sports bets are on different games and typically not numerous so they were approximately independent and the appropriate fractions were only moderately less than the corresponding single bet fractions. However one could ask himself if is this always true for independent simultaneous bets? Simultaneous bets on blackjack hands at different tables are independent but at the same table they have a pairwise correlation that has been estimated at 0.5 (see [2]). This should substantially reduce the Kelly fraction per hand. The blackjack literature discusses approximations to these problems. On the other hand, correlations between the returns on securities can range from nearly -1 to nearly 1. An extreme correlation often can be exploited to great advantage through the techniques of "hedging". The risk averse investor may be able to acquire combinations of securities where the expectations add and the risks tend to cancel. The optimal betting fraction may be very large.

Example 2

We have two favorable coins as in the previous example but now their outcomes are not necessarily independent. For simplicity, assume the special case where the two bets have the same payoff distributions, but with the following joint density

$$\begin{aligned} &\mathbb{P}((X_1, X_2) = (1,1)) = c + m\,, \qquad \mathbb{P}((X_1, X_2) = (1,-1)) = b\,, \\ &\mathbb{P}((X_1, X_2) = (-1,1)) = b\,, \qquad \mathbb{P}((X_1, X_2) = (-1,-1)) = c\,, \end{aligned} \tag{3.6}$$

where $c, m, b \geq 0$ are such that $2c + m + 2b = 1$. Note that the condition $2c + m + 2b = 1$ implies

$$b = (1-m)/2 - c \tag{3.7}$$

and therefore $0 \le c \le (1-m)/2$.

Note that, since the joint density is symmetric, we have that X_1 and X_2 are identically distributed and in particular

$$\mathbb{P}(X_1 = 1) = \mathbb{P}((X_1, X_2) = (1,1)) + \mathbb{P}((X_1, X_2) = (1,-1)) = c + m + b = \frac{1+m}{2}\,,$$
$$\mathbb{P}(X_1 = -1) = \mathbb{P}((X_1, X_2) = (-1,1)) + \mathbb{P}((X_1, X_2) = (-1,-1)) = b + c = \frac{1-m}{2}\,.$$

So

$$\mathbb{E}[X_1] = \mathbb{E}[X_2] = 1 \cdot \frac{1+m}{2} - 1 \cdot \frac{1-m}{2} = m\,,$$
$$\mathbb{E}[X_1^2] = \mathbb{E}[X_2^2] = 1 \cdot \frac{1+m}{2} + 1 \cdot \frac{1-m}{2} = 1\,,$$

and hence

$$\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \mathbb{E}[X_2^2] - \mathbb{E}[X_2]^2 = 1 - m^2\,.$$

We can also compute the covariance of X_1 and X_2. Note that

$$\mathbb{P}(X_1 X_2 = 1) = \mathbb{P}((X_1, X_2) = (1,1)) + \mathbb{P}((X_1, X_2) = (-1,-1)) = 2c + m\,,$$
$$\mathbb{P}(X_1 X_2 = -1) = \mathbb{P}((X_1, X_2) = (-1,1)) + \mathbb{P}((X_1, X_2) = (1,-1)) = 2b\,,$$

and so

$$\mathbb{E}[X_1 X_2] = 1 \cdot \mathbb{P}(X_1 X_2 = 1) - 1 \cdot \mathbb{P}(X_1 X_2 = -1) = 2c + m - 2b \underset{(3.7)}{=} 4c + 2m - 1\,.$$

So we have that the covariance of X_1 and X_2 is

$$\mathrm{Cov}(X_1, X_2) = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] = 4c + 2m - 1 - m^2 = 4c - (1-m)^2$$

and hence the correlation coefficient of X_1 and X_2 is

$$\rho(X_1, X_2) = \frac{\mathrm{Cov}(X_1, X_2)}{\sqrt{\mathrm{Var}(X_1)\mathrm{Var}(X_2)}} = \frac{4c - (1-m)^2}{1 - m^2}\,. \tag{3.8}$$

In this case the expected growth rate is

$$
\begin{aligned}
g(f_1, f_2) = \mathbb{P}((X_1, X_2) = (1,1)) \cdot \ln(1 + f_1 + f_2) + \\
+ \mathbb{P}((X_1, X_2) = (1,-1)) \cdot \ln(1 + f_1 - f_2) \\
+ \mathbb{P}((X_1, X_2) = (-1,1)) \cdot \ln(1 - f_1 + f_2) + \\
+ \mathbb{P}((X_1, X_2) = (-1,-1)) \cdot \ln(1 - f_1 - f_2)\,,
\end{aligned}
$$

that is

$$
\begin{aligned}
g(f_1, f_2) = (c+m)\ln(1 + f_1 + f_2) + b\ln(1 + f_1 - f_2) + \\
+ b\ln(1 - f_1 + f_2) + c\ln(1 - f_1 - f_2)\,.
\end{aligned}
\tag{3.9}
$$

The symmetry of the function in (3.9) shows that its point of maximum, which we know to be unique, must have the form (f^*, f^*), for some value of f^*. Hence we can reduce the maximization problem to a one dimensional problem in which the function to maximize is obtained from the function $g(f_1, f_2)$ in (3.9) fixing $f_1 = f_2 = f$. So we have to maximize the function

$$
g(f) = (c+m) \cdot \ln(1 + 2f) + c \cdot \ln(1 - 2f)\,. \tag{3.10}
$$

By easy computations we obtain that the point of maximum of $g(f)$ is

$$
f^* = \frac{m}{2(2c+m)}\,.
$$

Note that f^* is decreasing in c and, recalling that $0 \le c \le (1-m)/2$ and (3.8), we have in particular that

- if $c = (1-m)/2$, then $\rho(X_1, X_2) = 1$ and $f^* = m/2$;
- if $c = 0$, then $\rho(X_1, X_2) = -\frac{1-m}{1+m}$ and $f^* = 0.5$.

Moreover, when X_1 and X_2 are uncorrelated, that is $\rho(X_1, X_2) = 0$, we have

$$
\rho(X_1, X_2) = \frac{4c - (1-m)^2}{1 - m^2} = 0 \Rightarrow c = \frac{(1-m)^2}{4}
$$

and hence $f^* = \frac{m}{1+m^2}$.

3.3 Kelly criterion with bookmaker's quotes: application to football

In Section 2.2, we have seen that if we consider a game with $V : 1$ odds, then the optimal fraction to bet is

$$f^* = \frac{(V+1)p - 1}{V - 1}\,, \tag{3.11}$$

where p is the probability of winning the bet. Consider now a bet on a football match. In this case the relevant instrument that we use to bet are the bookmaker's quotes and our estimate of the winning probability. In terms of the notation introduced above, the quote of the bookmaker is represented by $V + 1$. Moreover such a quote is obtained as $1/\tilde{p}$, where $\tilde{p}$ is the probability of winning estimated by the bookmaker. Inserting the identity $V + 1 = 1/\tilde{p}$ in (3.11), we get a formula for f^* in terms of only p and $\tilde{p}$, that is

$$f^* = \frac{p - \tilde{p}}{1 - \tilde{p}}\,. \tag{3.12}$$

So the fraction f^* measures how bigger is our prediction of the winning probability with respect to the one of the bookmaker. In particular our betting trick is to find "errors" in bookmaker's quotes and use them to bet a fraction f^* of our budget. Obviously this is possible if only our estimate of the winning probability p is bigger than the one proposed by the bookmaker, that is $\tilde{p}$.

Let us now apply the above reasoning to the football match of the 2020/2021 Champions League Final between Manchester City and Chelsea. In this match, the bookmakers are leaning towards the Citizens, obviously underestimating the chances of the Pensioners. Such a situation would work in our favor.

According to the quotes, the bookmaker estimates the probability that Chelsea will not lose during the regular time at 51.8% (that is, the quote is 1.93). We

estimated this outcome at 58% ($> 51.8\%$).
So we can compute f^* through (3.12) with $p = 0.58$ and $\tilde{p} = 0.518$, or equivalently through (3.11) with $V+1 = 1.93$ and $p = 0.58$. In both the case we get $f^* = 0.1283$, which means that we will bet $12,83\%$ of our bankroll.

The success of this method is determined by the ability to calculate the correct probability of the event outcome. Kelly's strategy looks like a competition between a gambler and a bookmaker in the accuracy of determining the success rate of a particular event.
As stated before, this strategy is used when there is a deal with value-bets, that is when the bookmaker made a mistake in calculating the probability of a certain outcome and set wrong quotes.

Chapter 4

Kelly Criterion for Asset Allocation and Money Management

4.1 Stock Market

An other field in which the *Kelly Criterion* can be applied is the activity of investing in the stock market. Comparing to the Horse Racing setting discussed in the previous chapter, in this type of activity the gambler becomes an investor and the horses he could bet on becomes stocks and stock options. The investor can buy or sell stock and stock options on the market and each of them has a different probability of increasing or decreasing its value. For example, a stock worth 50 dollars in October, can increase of 10 percent to the price of 55 dollars or it can decrease over the same period by 5 percent to 45 dollars in November.
Suppose an investor has the possibility of buying two stocks A and B. Each stock has a different probability of changing value: stock A, respectively stock B, increases its value with probability p_A, respectively p_B, and decreases its value with

probability $q_A = 1 - p_A$, respectively $q_B = 1 - p_B$. Both stocks can rise or fall independently of each other.

Let us assume that, when the stock increases in value, the percentage by which it increases is constant, that it does not change between months, and that at the end of the month the investor has a capital increase of what he invested into buying the stock multiplied by a factor of how much the stock's value increased. Using the previous example, the investor would have 55 dollars in November because the stock increased ten percent. In the same way, if the stock increases again the next month (by 10 percent because the percentage it increases is constant) then he would have $60,5$ dollars. The same applies if the stock lowers in price. If the stock rose in value then the investor's capital is the amount of capital he invested in the stock multiplied by $V \in (1, \infty)$ and if the stock fell in value then the decrease in capital for the investor is the amount of capital he invested multiplied by the fraction $Z \in (0, 1)$.

This case strongly resembles the case of the general coin toss but in this case it is defined as a coin toss game of V to Z odds for each stock option. The fraction of capital he invests on each stocks A and B are, respectively, $f_A, f_B \in [0, 1]$, where $f_A + f_B = 1$.

For example, if the investor were to invest in the two stocks in October and in November the value of stock A rose and the value of stock B fell, then his current capital would be

$$X_1 = X_0 \cdot (V_A \cdot f_A + Z_B \cdot f_B)$$

If in December stock B rose and stock A fell, we have

$$X_2 = X_1 \cdot (Z_A \cdot f_A + V_B \cdot f_B) = X_0 \cdot (V_A \cdot f_A + Z_B \cdot f_B) \cdot (Z_A \cdot f_A + V_B \cdot f_B).$$

Suppose now that the investor is investing in stock options for a series of $N \in \mathbb{N}$ months and define

- $WL \in \mathbb{N}$ as the number of months in which stock A rose in value and B fell in value;

- $LW \in \mathbb{N}$ as the number of months in which stock B rose in value and A fell in value;

- $WW \in \mathbb{N}$ as the number of months in which stocks A and B rose in value together;

- $LL \in \mathbb{N}$ as the number of months in which stocks A and B fell in value together;

with $WL + LW + WW + LL = N$.

Then, the equation describing the current capital of the investor at the N-th month is

$$
\begin{aligned}
X_N =& X_0 \cdot (V_A \cdot f_A + Z_B \cdot f_B)^{WL} \cdot (Z_A \cdot f_A + V_B \cdot f_B)^{LW} \cdot \\
& \cdot (V_A \cdot f_A + V_B \cdot f_B)^{WW} \cdot (Z_A \cdot f_A + Z_B \cdot f_B)^{LL}.
\end{aligned}
$$

Then we can define the utility function as follows

$$
\begin{aligned}
G(f_A, f_B) = \ln\left(\frac{X_N}{X_0}\right) &= WL \cdot \ln(V_A \cdot f_A + Z_B \cdot f_B) + LW \cdot \ln(Z_A \cdot f_A + V_B \cdot f_B) + \\
&+ WW \cdot \ln(V_A \cdot f_A + V_B \cdot f_B) + LL \cdot \ln(Z_A \cdot f_A + Z_B \cdot f_B).
\end{aligned}
$$

As seen Section 3.1 for the Horse Race, the above function represents the logarithmic capital growth function of N discrete series of investments. Dividing by N, we get

$$
\begin{aligned}
g(f_A, f_B) = \frac{1}{N} \cdot G(f_A, f_B) = \frac{1}{N} \cdot \ln\left(\frac{X_N}{X_0}\right) &= \frac{WL}{N} \cdot \ln(V_A \cdot f_A + Z_B \cdot f_B) + \\
&+ \frac{LW}{N} \cdot \ln(Z_A \cdot f_A + V_B \cdot f_B) + \frac{WW}{N} \cdot \ln(V_A \cdot f_A + V_B \cdot f_B) + \\
&+ \frac{LL}{N} \cdot \ln(Z_A \cdot f_A + Z_B \cdot f_B)
\end{aligned}
$$

and, by the Law of Large Numbers (see Chapter 1.2), for N large we have

$$\begin{aligned} g(f_A, f_B) = &\, p_A \cdot q_B \cdot \ln(V_A \cdot f_A + Z_B \cdot f_B) + q_A \cdot p_B \cdot \ln(Z_A \cdot f_A + V_B \cdot f_B) + \\ &+ p_A \cdot p_B \cdot \ln(V_A \cdot f_A + V_B \cdot f_B) + q_A \cdot q_B \cdot \ln(Z_A \cdot f_A + Z_B \cdot f_B) \,. \end{aligned} \tag{4.1}$$

The next step is to find the maximum of the utility function to get the optimal betting ratios. This problem can be defined as a nonlinear constrained optimization problem and it can be solved by MATLAB. The optimization problem is defined as the maximization of equation (4.3) subject to the equality constraint $h(f_A, f_B) := f_A + f_B - 1 = 0$ and the bound constraints $f_A, f_B \in [0, 1]$. It is also possible to insert the constraint $h(f_A, f_B) = 0$ into the objective equation and attempt to solve it analytically. Equation (4.3) becomes

$$\begin{aligned} g(f_A) = &\, p_A \cdot q_B \cdot \ln(V_A \cdot f_A + Z_B \cdot (1 - f_A)) + \\ &+ q_A \cdot p_B \cdot \ln(Z_A \cdot f_A + V_B \cdot (1 - f_A)) + \\ &+ p_A \cdot p_B \cdot \ln(V_A \cdot f_A + V_B \cdot (1 - f_A)) + \\ &+ q_A \cdot q_B \cdot \ln(Z_A \cdot f_A + Z_B \cdot (1 - f_A)) \,. \end{aligned}$$

and the domain of g is

$$\Delta = \{f_A \in \mathbb{R} \,:\, 0 \leq f_A \leq 1\} \,.$$

Now we have to find the critical point. The first derivative of the function $g(f_A)$ is

$$\begin{aligned} g'(f_A) = &\, \frac{p_A \cdot p_B \cdot (V_A - V_B)}{V_A \cdot f_A + V_B \cdot (1 - f_A)} + \frac{p_A \cdot q_B \cdot (V_A - Z_B)}{V_A \cdot f_A + Z_B \cdot (1 - f_A)} + \\ &- \frac{p_B \cdot q_A \cdot (V_B - Z_A)}{Z_A \cdot f_A + V_B \cdot (1 - f_A)} + \frac{q_A \cdot q_B \cdot (Z_A - Z_B)}{Z_A \cdot f_A + Z_B \cdot (1 - f_A)} \,. \end{aligned}$$

The equation $g'(f_A) = 0$ is difficult to solve and therefore we will not continue attempting to find the solution analytically, but rather we will use the MATLAB function `fmincon` whose aim is to find a constrained minimum of a function of several variables. Note that, since we are interested to the maximum of g, we need to apply `fmincon` to the function $-g$.

4.2 Betting on correlated stocks

To illustrate both the Kelly criterion and the size of the securities markets, we return to the study of the effects of correlation as in Example 2 in Section 3.2. Consider the pair of bets U_1 and U_2, with joint distribution given by

$$\begin{aligned} &\mathbb{P}((U_1,U_2)=(m_1+1,m_2+1))=a\,, \qquad \mathbb{P}((U_1,U_2)=(m_1-1,m_2+1))=\frac{1}{2}-a\,,\\ &\mathbb{P}((U_1,U_2)=(m_1+1,m_2-1))=\frac{1}{2}-a\,, \qquad \mathbb{P}((U_1,U_2)=(m_1-1,m_2-1))=a\,, \end{aligned} \tag{4.2}$$

where $0 \le a \le \frac{1}{2}$ and $m_1, m_2 \ge 0$. Since the joint distribution is symmetric, then $U_1 - m_1$ and $U_2 - m_2$ are identically distributed. Moreover

$$\mathbb{P}(U_1 = m_1+1) = \mathbb{P}((U_1,U_2)=(m_1+1,m_2+1)) + \mathbb{P}((U_1,U_2)=(m_1+1,m_2-1)) = \frac{1}{2}$$

and hence $\mathbb{P}(U_1 = m_1-1) = 1-\mathbb{P}(U_1 = m_1+1) = \frac{1}{2}$. So $\mathbb{E}[U_1] = m_1$ and similarly $\mathbb{E}[U_2] = m_2$.

To measure the correlation between U_1 and U_2 we need to compute $\mathrm{Cov}(U_1, U_2)$ and $\mathrm{Var}(U_1), \mathrm{Var}(U_2)$. Note that

$$\mathbb{E}[U_1^2] = (m_1-1)^2 \cdot \frac{1}{2} + (m_1+1)^2 \cdot \frac{1}{2} = m_1^2 + 1$$

and similarly $\mathbb{E}[U_2^2] = m_2^2 + 1$. Hence

$$\mathrm{Var}(U_1) = \mathbb{E}[U_1^2] - \mathbb{E}[U_1]^2 = m_1^2 + 1 - m_1^2 = 1$$

and similarly $\mathrm{Var}(U_2) = 1$. Moreover

$$\begin{aligned}\mathbb{E}[U_1U_2] &= a[(m_1+1)(m_2+1) + (m_1-1)(m_2-1)]+ \\ &+ \left(\frac{1}{2} - a\right)[(m_1-1)(m_2+1) + (m_1+1)(m_2-1)] = \\ &= 2a(m_1m_2+1) + (1-2a)(m_1m_2-1) = m_1m_2 + 4a - 1\,.\end{aligned}$$

So

$$\mathrm{Cov}(U_1, U_2) = \mathbb{E}[U_1U_2] - \mathbb{E}[U_1]\mathbb{E}[U_2] = m_1m_2 + 4a - 1 - m_1m_2 = 4a - 1$$

and hence the correlation coefficient of U_1, U_2 is

$$\rho(U_1, U_2) = \frac{\mathrm{Cov}(U_1, U_2)}{\sqrt{\mathrm{Var}(U_1)\mathrm{Var}(U_2)}} = 4a - 1$$

that increases from -1 to 1 as a increases from 0 to $\frac{1}{2}$.

Finding a general solution for (f_1^*, f_2^*) to the above bet appears algebraically complicated, but specific solutions are easy to find numerically. Even with reduction to the special case $m_1 = m_2 = m$ and the use of symmetry to reduce the problem to finding $f^* = f_1^* = f_2^*$, a general solution is still much less simple. By the way, if we instead consider the instance when $a = 0$ and $m_1 =_2= m$ (so that $\mathrm{Cov}(U_1, U_2) = -1$), then $g(f) = \ln(1 + 2mf)$ which is increasing in f. This pair of bets is a *sure bet* and one should bet as much as possible.

This is a simplified version of the classic arbitrage of securities markets: find a pair of securities which are identical or equivalent and trade at disparate prices [1]. Buy the relatively underpriced security and sell short the relatively overpriced security, achieving a correlation of -1 and locking in a risk-less profit.

An example occurred in 1983, as reported by Thorp in [12], when his investment partner bought \$330 million worth of old AT&T and sold short \$332.5 million

[1]For the *Law of One Price* two securities yielding the same dividends should have the same price.

worth of issued new AT&T plus the new seven sisters regional telephone companies. Much of this was done in a single trade as part of what was then the largest dollar value block trade ever done on the New York Stock Exchange (December 1, 1983).

4.3 Continuous approximation

In applying the Kelly criterion to the securities markets, we meet new analytic problems. A bet on a security typically has many outcomes rather than just a few, as in most gambling situations. This leads to the use of continuous instead of discrete probability distributions. We are led to find f to maximize

$$g(f) = \mathbb{E}[\ln(1 + f \cdot X)] = \int \ln(1 + f \cdot x) dP(x),$$

where $P(x)$ is the probability distribution of the random variable X.
Frequently the problem is to find an optimum portfolio from among n securities, where n may be a large number. In this case x and f are n-dimension vectors and $f \cdot x$ is their scalar product. We also have constraints on $f = (f_1, \ldots, f_n)$, since we always need $1 + f \cdot x > 0$ (so that $\ln(\cdot)$ is defined) and $\sum_{i=1}^{n} f_i = 1$ (or some $c > 0$) to normalize to a unit (or to a $c > 0$) investment.
The maximization problem is generally solvable because $g(f)$ is concave.
There may be other constraints as well for some or all i such as $f_i \geq 0$, where no short selling is allowed, or $f_i \leq M_i$ or $f_i \leq m_i$ (limits amount invested in i-th security), or $\sum_{i=1}^{n} |f_i| \leq M$ (limits on total leverage to meet margin regulations or capital requirements).
Note that in some instances there is not enough of a good bet or investment to allow betting the full f^*, predicted by the model, so one is forced to under-bet, reducing somewhat both the overall growth rate and the risk.
Together with the advantages of understanding correlation, there are other tools

that can turn in favor of a gambler, or in the case of the securities market, an investor: continuous approximation.

Let X be a random variable with $P(X = m + s) = P(X = m - s) = 0.5$. Then

$$\mathbb{E}[X] = m\,, \qquad \mathrm{Var}(X) = s^2\,.$$

If the initial capital is V_0 and we bet a fraction f of V_0 with return per unit of X, the result is

$$V(f) = V_0(1 + (1-f)\cdot r + fX) = V_0(1 + r + f\cdot(X - r))$$

where r is the rate of return on the remaining capital, e.g., invested in Treasury bills. Then

$$\begin{aligned} g(f) = \mathbb{E}[G(f)] = \mathbb{E}\left[\ln\left(\frac{V(f)}{V_0}\right)\right] &= \mathbb{E}[\ln(1 + r + f(X - r))] = \\ &= 0.5\ln(1 + r + f(m - r + s)) + 0.5\ln(1 + r + f(m - r - s)). \end{aligned}$$

Now subdivide the time interval into n equal independent steps, keeping the same drift and the same total variance. Thus m, s^2 and r are replaced by m/n, s^2/n and r/n, respectively. We have n independent $X_i, i = 1, \ldots, n$, with

$$P(X_i = m/n + sn^{-1/2}) = P(X_i = m/n - sn^{-1/2}) = 0.5.$$

Then

$$\frac{V_n(f)}{V_0} = \prod_{i=1}^{n}(1 + (1-f)r + f\cdot X_i).$$

Taking $\mathbb{E}[\log(\cdot)]$ on both sides of the above equation gives $g(f)$. Moreover, expanding the result in a power series, leads to

$$g(f) = r + f(m - r) - s^2 f^2/2 + O(n^{-1/2}) \tag{4.3}$$

where $O(n^{-1/2})$ has the property $n^{1/2}O(n^{-1/2})$ is bounded as $n \to \infty$.
Letting $n \to \infty$ in (4.3) we have

$$g_\infty(f) = r + f(m-r) - s^2 f^2/2\,. \tag{4.4}$$

Note that the point of maximum and the maximum of $g_\infty(f)$ are given by

$$f^* = \frac{m-r}{s^2}\,, \qquad g_\infty(f^*) = \frac{(m-r)^2}{2s^2} + r\,. \tag{4.5}$$

The limit $V = V_\infty(f)$ of $V_n(f)$ as $n \to \infty$ corresponds to a log normal diffusion process, which is a well-known model for securities prices. The security here has instantaneous drift rate m, variance rate s^2, and the risk-less investment of cash earns at an instantaneous rate r. Then $g_\infty(f)$ in equation (4.4) is the (instantaneous) growth rate of capital with investing our betting fraction f.
There is nothing special about our choice of the random variable X in terms of restrictions: any bounded random variable with mean $E[X] = m$ and variance $Var(X) = s^2$ will lead to the same result.
Note that f no longer needs to be less than or equal to 1 and also the usual problems that arises from the logarithm function, that is being undefined for negative arguments, have disappeared. Moreover, $f < 0$ causes no problems. This simply corresponds to selling the security short.
If $m < r$ this could be advantageous. Note further that the investor who follows the policy f must now adjust is investment instantaneously, that means adjusting in tiny increments whenever there is a small change in V. This idealization appears in option theory.
Our previous growth functions for finite sized betting steps were approximately parabolic in a neighborhood of f^* and often in a range up to $0 \le f \le 2f^*$, where also often $2f^* = f_c$. Now with the limiting case (4.4), $g_\infty(f)$ is exactly parabolic and very easy to study.

Log normality of $\frac{V(f)}{V_0}$ means that $\log\left(\frac{V(f)}{V_0}\right)$ is $\mathcal{N}(M, S^2)$ distributed, with mean $M = g_\infty(f)\cdot t$ and variance $S^2 = Var(G_\infty(f))\cdot t$ for any time t, where

$$G_\infty(f) = \ln\left(\frac{V_\infty(f)}{V_0}\right).$$

This allows to determine, for instance, the expected capital growth and the time tk required for $V(f)$ to be at least k standard deviations above V_0.
First, we can show by our previous methods that $Var(G_\infty(f)) = s^2 f^2$, hence its standard deviation is $\mathrm{SDev}(G_\infty(f)) = sf$. Solving

$$t_k g_\infty(f) = k t_k^{1/2}\mathrm{SDev}(G_\infty(f))$$

we get

$$t_k g_\infty(f) = k t_k^{1/2}\mathrm{SDev}(G_\infty(f)) \Rightarrow t_k^{1/2} = \frac{ksf}{g_\infty(f)} \Rightarrow t_k = \frac{k^2 s^2 f^2}{g_\infty(f)^2}. \tag{4.6}$$

Note that both t_k and $t_k g_\infty(f)$ increase as f increases, for $0 \le f < f^+$ where f^+ is the positive root of $s^2 f^2/2 - (m-r)f - r = 0$ and $f^+ > 2f^*$.
From general portfolio theory we know that the capital asset pricing model (CAPM) says that the market portfolio lies on the Markowitz efficient frontier E in the (s, m) plane at a (generally) unique point $P = (s_0, m_0)$ such that the line determined by P and $(s = 0, m = r)$ is tangent to E at P. The slope of this line is the Sharpe ratio $S = (m_0 - r_0)/s_0$ and from before we know that $g_\infty(f^*) = S^2/2 + r$ so the maximum growth rate $g_\infty(f^*)$ depends, for any fixed r, only on the Sharpe ratio. Again, from before we can state that $f^* = 1$ when $m = r + s^2$, in which case the Kelly investor will select the market portfolio without borrowing or lending. If $m > r + s^2$ the Kelly investor will use leverage and if $m < r + s^2$ he will invest partly in Treasury bills and partly in the market portfolio. Thus the Kelly investor will dynamically reallocate as f^* changes over time because of fluctuations in the forecast m, r and s^2, as well as in the prices of the portfolio securities.

Since $g_\infty(1) = m - s^2/2$, we have that the portfolios in the (s, m) plane satisfying $m - s^2/2 = C$, where C is a constant, all have the same growth rate.
In the continuous approximation, the Kelly investor appears to have the utility function $U(s, m) = m - s^2/2$. Thus, for any (closed, bounded) set of portfolios, the best portfolios are exactly those in the subset that maximizes the one parameter family $m - s^2/2 = C$.

4.3.1 The S&P 500 Index Example

Using historical data of easy finding, we make the rough estimates $m = 0,11$, $s = 0,15$, $r = 0,06$.
The equations we need are the generalizations of (4.5) and (4.6) with $f = cf^*$, for some $c > 0$. We have

$$
\begin{aligned}
&cf^* = c(m-r)/s^2,\\
&g_\infty(cf^*) = ((m-r)^2 \cdot (c - c^2/2))/s^2 + r,\\
&\mathrm{SDev}(G_\infty(cf^*)) = c(m-r)/s,\\
&tg_\infty(cf^*) = k^2c^2/(c - c^2/2 + rs^2/(m-r)^2)\,,\\
&t(k, cf^*) = k^2c^2((m-r)^2/s^2)/((m-r)^2/s^2)(c - c^2/2) + r)^2\,,
\end{aligned}
\tag{4.7}
$$

and, if we define $\tilde{m} = m - r$, $\tilde{G}_\infty = G_\infty - r$, $\tilde{g}_\infty = g_\infty - r$, and we substitute them into equation (4.7), we get equation (4.6), showing the relation between the two sets.
From Equations (4.7) with $c = 1$, we find

$$
\begin{aligned}
&f^* = 2.22,\\
&g_\infty(f^*) = 0.115,\\
&\mathrm{SDev}(G_\infty(f^*)) = 0.33,\\
&tg_\infty(f^*) = 0.96k^2,\\
&t = 8.32k^2 \text{ years.}
\end{aligned}
$$

Thus, with $f^* = 2.22$, after 8.32 years we have the 84% of probability that $V_n > V_0$ at the $k = 1$ standard deviations level of significance. Moreover we have $\ln(V_n/V_0) = 0.96$ so that the median value of V_n/V_0 is around $e^{0.96} = 2.61$. With the usual unlevered $f = 1$, and $c = 0.45$, using (4.6) we find

$$
\begin{aligned}
&g_\infty(1) = m - s^2/2 = 0.09875,\\
&\mathrm{SDev}(G_\infty(f^*)) = 0.15,\\
&tg_\infty(1) = 0.23k^2,\\
&t(k, 0.45f^*) = 2.31k^2 \text{ years.}
\end{aligned}
$$

Writing $tg_\infty = h(c)$ in equation (4.6) as

$$h(c) = k^2/(1/c + rs^2/((m-r)^2c^2) - 1/2)\,,$$

we see that the measure of riskiness, $h(c)$, increases, at least up to the point $c = 2$, corresponding to $2f^*$. Writing $t(k, cf^*) = t(c)$ as

$$t(c) = k^2((m-r)^2/s^2)/((m-r)^2/s^2)(1-c/2) + r/c^2$$

shows that $t(c)$ also increases as c increases, at least up to the point $c = 2$. Thus for smaller, that is, for a more conservative, $f = cf^*$, $c \leq 2$, specified levels of $P(V_n > V_0)$ are reached earlier. For $c < 1$, this comes with a reduction in growth rate, which reduction is relatively small for f near f^*.

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