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Applied Portfolio Management: an empirical analysis of Portfolio Optimization based on the FTSE MIB IT40 index

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List of Abbreviations

MPT	Modern Portfolio Theory
CML	Capital Market Line
MVP	Minimum Variance Portfolio
TP	Tangency Portfolio
CAPM	Capital Asset Pricing Model
PMPT	Post-Modern Portfolio Theory
SR	Sharpe Ratio
ER	Expected Return
Var	Variance
St. Dev	Standard Deviation
IT40	FTSE MIB IT 40 Index
MIB40	FTSE MIB IT 40 Index
FTSE MIB	FTSE MIB IT 40 Index
F.O.C.	First Order Conditions

EXECUTIVE SUMMARY

The primary objective of this thesis is to contribute to the comprehensive understanding of Modern Portfolio Theory (MPT) through a detailed analysis of FTSE MIB IT 40 index's stocks. The exploration aims to identify the Minimum Variance Portfolio (MVP) and the Tangency Portfolio (TP), key components of the efficient frontier, a critical construct in MPT. By constructing the efficient frontier, this work strives to delineate the set of optimal portfolios that provide the highest expected return for a given level of risk.
The thesis will also extend beyond the efficient frontier by drawing the Capital Market Line (CML), another seminal concept in MPT. The CML, which is tangent to the efficient frontier at the TP, represents portfolios that optimally combine risk-free and risky assets.
The primary instrument used in this analysis is Python, a programming language known for its statistical and financial libraries. With its robust capabilities in handling large datasets and conducting complex statistical analysis, Python provides an ideal platform for examining FTSE MIB IT 40 index's stocks.
It is essential to note that this research is built upon the foundational theories of MPT as first presented by Harry Markowitz in his 1952 paper, "Portfolio Selection," and his subsequent book, "Portfolio Selection: Efficient Diversification of Investments" [20][21]. The Capital Asset Pricing Model (CAPM) that underlies the construction of the CML was primarily inspired by the work of William Sharpe [28].
In spite of the criticisms and limitations of MPT highlighted by scholars such as Daniel Kahneman, Amos Tversky [16], and Benoit Mandelbrot [19], its principles still play a pivotal role in investment management and decision-making. Alternative approaches such as Behavioral Finance [31] and Post-Modern Portfolio Theory (PMPT) [26] further emphasize the need for ongoing analysis and refinement of portfolio optimization methodologies.
This thesis seeks to provide valuable insights into the practical implementation of MPT, bridging the gap between theory and practice. Finally, the main focus of this thesis lies in precisely applying the mathematical formulas found in the relevant literature, without relying on Python packages such as SciPy and its optimization function, scipy.optimize.minimize. This approach ensures compatibility with the principles of MPT, as different optimization methods available in the package, such as SLSQP (Sequential Least Squares Programming), may yield results that are not consistent with the theory.
The relevant code may be found in the Appendix.
In the following sections, we will introduce the variance minimization problem and its associated constraints, demonstrating the coherence of our approach.

Chapter One

INTRODUCTION – THE EARLY FORMS OF INVESTMENT THEORY

The world of investing is an intricate labyrinth of opportunities, challenges, and trade-offs. For a long time, investors have sought strategies and models to optimize their investment decisions, balancing the inherent trade-off between risk and return.

Before the advent of sophisticated data analytics and quantitative methods, in the mid-20th century investment decisions were largely qualitative in nature, guided by the principle 'buy low, sell high'.
This maxim served as a rudimentary yet widely accepted principle guiding investment decisions prior to the advent of quantitative methods, notably before the introduction of Modern Portfolio Theory (MPT) by Harry Markowitz in 1952 [20].
The principle revolves around capitalizing on market volatility by purchasing assets when their prices are low (typically during economic downturns) and selling when the prices are high (typically during economic booms).
Indeed, the 'buy low, sell high' maxim indeed encapsulates a deep understanding of market dynamics, particularly the concept of economic cycles.
These cycles, characterized by alternating periods of expansion (growth) and contraction (recession), have profound implications for the timing of investment decisions.
Irving Fisher, in "The Purchasing Power of Money" (1911), expounded on the influence of economic dynamics on investment, focusing on the impact of monetary policy and interest rates [9].
When economies are in expansion phases, central banks typically increase interest rates to contain inflation, which in turn can depress stock prices.
Conversely, during contraction phases, interest rates are usually lowered to stimulate economic activity, which can create attractive buying opportunities for investors. Thus, investors adhering to the 'buy low, sell high' strategy aim to capitalize on these cyclical fluctuations.
Hyman Minsky, in "Stabilizing an Unstable Economy" (1986), put forth the Financial Instability Hypothesis, where he suggested that periods of economic stability could unwittingly lead to speculative bubbles, and subsequent instability [24].
Investors following the 'buy low, sell high' principle need to be cognizant of such periods of exuberance, as they can lead to assets being overvalued. This forms the 'sell high' part of the maxim, where investors look to exit their positions before the bubble bursts.
However, it's worth noting that the principle also requires a keen understanding of individual businesses and their intrinsic value, a concept popularized by Benjamin Graham and David Dodd in "Security Analysis" (1934).

The intrinsic value refers to the perceived true value of a company, considering all aspects including its financial performance, industry position, and future growth prospects [9].
Furthermore, to understand the context of this traditional approach, it's crucial to delve into some of the key influencers and their works that shaped the investment field during this era.
One of the pivotal figures in investment history, Benjamin Graham, promoted a philosophy of 'value investing'.
His seminal book "The Intelligent Investor" (1949) discusses the concept of a "margin of safety", essentially buying securities when they are undervalued by the market and selling them when their true value is realized [10].
Another important concept integral to this early investment philosophy was the idea of 'market timing'.
The notion here is that investors, by observing economic trends and indicators, could make educated guesses about future market movements and accordingly decide when to buy or sell securities.
William Peter Hamilton, in his book "The Stock Market Barometer" (1922), proposed Dow Theory, which became a foundation for technical analysis in investment decision making [13]. Named after Charles Dow, the founder of Wall Street Journal and creator of the Dow Jones Industrial Average, Dow Theory is a method used to interpret and predict trends in the stock market.
Dow Theory proposes that the stock market moves in identifiable trends and cycles, with three types of movements: primary (long-term), secondary (medium-term), and minor (short-term). The primary trends, which can last a year or more, are considered the main movements of the market, whereas the secondary and minor trends are seen as corrections and fluctuations within the primary trend.
This theory revolves around six core principles:

1. The market has three movements: As mentioned above, Dow identified three types of movements that stocks undergo, i.e., primary, secondary, and minor trends.
2. Market trends have three phases: Dow postulated that primary trends have three phases—accumulation, public participation, and distribution. The accumulation phase is characterized by knowledgeable investors entering the market, the public participation phase is when the masses begin to invest, and the distribution phase is when knowledgeable investors begin to sell.
3. The stock market discounts all news: Dow Theory asserts that all news - current and even future events - is reflected in the market price of stocks. This reflects the theory's foundational belief in the efficiency of markets.
4. Stock market averages must confirm each other: Here, the theory suggests that for a valid market trend to exist, it must be confirmed by more than one index. For instance, in the case of the US stock market, a bull or bear market signal is confirmed when both the Dow Jones Industrial and Transportation averages reach new highs or lows.

5. Trends are confirmed by volume: According to Dow Theory, trading volume should increase when the price movement aligns with the primary trend and decrease when it's contrary to the primary trend.
6. Trends exist until definitive signals prove they have ended: the final principle holds that a trend remains in effect until there's a clear signal that it has reversed.

Hamilton's work not only provided a structured way to interpret market trends but also laid the groundwork for the study of price charts and technical analysis, where the focus is more on price action rather than the underlying financial health of a company.

While Dow Theory provides valuable insights, it is not without limitations. It doesn't provide specific rules for buying and selling securities, nor does it account for sudden market events or changes in business cycles. Despite these limitations, its central tenets remain influential and are still considered by many investors and traders today.

On the other hand, while this era was dominated by qualitative decision-making approaches, there were attempts to incorporate quantitative elements.

John Burr Williams, in "The Theory of Investment Value" (1938), introduced the Dividend Discount Model (DDM) where the value of a stock was calculated based on the present value of its future dividends [33]. This was one of the first models that utilized a quantitative approach in investment decision making.

However, these methods had their limitations. The lack of robust statistical tools and methodologies made it challenging to make precise investment decisions. There was also a heavy reliance on individual judgement, which is inherently subjective and prone to cognitive biases.

In conclusion, the 'buy low, sell high' principle embodied the spirit of early investment decision making. Although rudimentary, this maxim encompassed several important concepts such as value investing, market timing, and economic cycles. Despite its limitations, it set the stage for the future evolution of investment practices and theories, paving the way for a more data-driven and systematic approach to investing.

Consequently, the advent of Modern Portfolio Theory (MPT) in 1952 by Harry Markowitz revolutionized the investing world by providing a quantitative, structured, and theoretically robust approach to investment management [20].

MPT, at its core, is a mathematical formulation used to optimize the construction of a portfolio of assets such that for a given level of risk, the expected return is maximized. The revolutionary aspect of MPT was its shift in focus from individual asset evaluation to the overall portfolio context. This marked a departure from the traditional security analysis, which was primarily rooted in the work of Graham and Dodd [11], and focused primarily on finding undervalued securities based on fundamental analysis.

Before the introduction of MPT, the general understanding was that risk was inseparable from reward, and to earn higher returns, investors needed to undertake higher risk. However, Markowitz demonstrated that it was possible to reduce the risk of a portfolio and simultaneously improve its expected return through the concept of diversification.

By combining assets that are not perfectly positively correlated, Markowitz showed that it is possible to design a portfolio with a risk level lower than the weighted average risk of its constituent securities.
The critical building blocks of MPT include the ideas of portfolio return, portfolio variance, the correlation between securities, diversification, the efficient frontier, and the Capital Market Line (CML). The efficient frontier is a graphical representation of all optimal risk-return trade-offs, i.e., portfolios that offer the maximum expected return for a given level of risk. The CML, built upon the efficient frontier, introduces the concept of a risk-free asset and delineates the optimal trade-off between the risk-free asset and the risky portfolio.
Markowitz's pioneering work paved the way for subsequent advancements in financial theory, such as the Capital Asset Pricing Model (CAPM) by Sharpe [28] and the Black-Scholes-Merton option pricing model. However, these models, which are built on MPT, introduced additional assumptions and complexities, making Markowitz's MPT even more critical because of its foundational role.
The importance of MPT extends beyond academic finance. It is integral to the working of the financial industry, shaping practices in portfolio management, risk management, asset pricing, and financial regulation. Its principles guide the design of index funds, the asset allocation of pension funds, and the portfolio optimization algorithms of robo-advisors.
Despite its foundational role, MPT has also been the subject of criticisms and revisions, leading to the development of new theories and models. However, these new developments do not diminish the importance of MPT; they build upon it and refine it.
Especially, this thesis aims to revisit MPT, examine its principles and implications, and apply it to a contemporary context, thereby enhancing the understanding of this timeless theory.

Chapter Two

LITERATURE REVIEW – A FOCUS ON POST-MODERN PORTFOLIO THEORY (PMPT)

As already mentioned, while the foundation of Modern Portfolio Theory (MPT) was laid by Harry Markowitz in the 1950s, the field of financial portfolio management has witnessed significant evolution and criticism since then. Markowitz's mean-variance framework was instrumental in steering the conversation towards the significance of portfolio diversification and risk-return tradeoff [20].

However, the MPT's reliance on mean-variance optimization has been challenged and expanded over the years, particularly with the emergence of the Post-Modern Portfolio Theory (PMPT).

The Capital Asset Pricing Model (CAPM) formulated by Sharpe (1964) [28] expanded on MPT by introducing the concept of systematic risk, represented by beta.

The CAPM is a model used to determine a theoretically appropriate required rate of return on an asset, given that asset's systematic (non-diversifiable) risk. The model does this by introducing the concept of beta (β), a measure of an asset's sensitivity to market movements.

According to the CAPM, the expected return of a financial asset ($E(R_i)$) is equal to the risk-free rate (R_f) plus the asset's beta multiplied by the difference between the expected return of the market ($E(R_m)$) and the risk-free rate. In the CAPM formula, beta (β_i) measures the asset's risk in relation to the market. A beta of 1 indicates that the security's price moves with the market. A beta less than 1 suggests that the security is theoretically less volatile than the market, while a beta greater than 1 indicates that the security's price is theoretically more volatile than the market.

The risk-free rate (R_f) is the return on a risk-free asset, typically a government bond.

The expected return of the market ($E(R_m)$) is the average return on the capital market.

The difference between $E(R_m)$ and R_f is known as the market risk premium, which represents the excess return required by investors for taking on the additional risk of investing in the market portfolio as opposed to a risk-free asset.

The CAPM assumes that investors are risk-averse and, therefore, require higher expected returns for taking on higher risk. This concept is central to the model. It also assumes that the only risk that cannot be diversified away (hence, investors are compensated for) is systematic risk, which is measured by beta. Unsystematic risk, on the other hand, can be eliminated through diversification and hence is not rewarded with higher expected returns.

Despite its usefulness in estimating asset pricing and investment risk, CAPM is not without criticism. The model rests on several assumptions, such as the existence of a risk-free rate, the ability of investors to borrow and lend at that rate, and investors holding the same expectations about the market's future. In reality, these assumptions may not hold. Moreover, while beta is a helpful measure, it is based on past data and may not accurately predict future risk.

In fact, Fama and French (1992) [8] pointed out that the expected return as predicted by CAPM using beta was not completely accurate, indicating the presence of other influential factors.
Their investigation led to the development of the Fama-French Three-Factor Model, a significant contribution to financial economics.
Fama and French argued that the size of firms (market capitalization) and their book-to-market ratios were critical variables, which were often more predictive of stock returns than market beta alone. Their observations were based on empirical data that indicated a trend of small-cap companies and those with high book-to-market ratios outperforming the market, in contradiction with the predictions of the CAPM.
The size factor, often referred to as the "Small Minus Big" (SMB) factor, encapsulates the historical trend of smaller companies providing higher returns than larger companies. Small-cap stocks, they argued, often come with higher risk and potential growth opportunities, leading to higher expected returns as a compensation for the increased risk.
The value factor, or "High Minus Low" (HML), captures the excess returns of value stocks over growth stocks. Value stocks are typically companies that are perceived as undervalued by the market, and hence, offer higher returns when the market eventually corrects this undervaluation. Conversely, growth stocks, characterized by low book-to-market ratios, tend to have higher valuations with anticipated future growth priced in, and thus, lower expected returns.
The Fama-French model's introduction of these two additional factors, SMB and HML, provided a more robust and nuanced approach to understanding asset pricing and market behavior. However, as with any model, it is not without its constraints. It assumes that the relevance of the size and value factors is constant across time, which may not always be the case given the dynamic nature of financial markets.
Furthermore, subsequent research has uncovered additional factors, such as momentum and investment, that can contribute to explaining stock returns, leading to enhancements of the model such as the Fama-French Five-Factor Model.
Taking the critique further, behavioral finance scholars have argued that the assumptions of rationality and market efficiency foundational to MPT often don't reflect real-world scenarios [30][29]. The role of investor psychology and cognitive biases in financial decision-making and market outcomes is emphasized, challenging the theory's fundamental assumptions [17].
In fact, MPT assumes that investors are always rational, making optimal decisions based on available information to maximize their utility.
However, numerous psychological studies suggest that this is not always the case.
People are subject to a range of cognitive biases that can lead to sub-optimal decision making.
For instance, the overconfidence bias may cause investors to overestimate their ability to predict market movements, while loss aversion may make investors reluctant to sell losing investments in the hope of a rebound.

Another crucial assumption of MPT is the efficient market hypothesis (EMH), which posits that all relevant information is fully and immediately reflected in asset prices. However, behavioral finance scholars have noted that markets often exhibit anomalies that are inconsistent with the EMH, such as short-term momentum and long-term reversals in stock returns. These phenomena, which are difficult to explain within the EMH framework, suggest that markets are not always fully efficient and that investor behavior can influence asset prices.

Behavioral finance scholars also highlight the influence of herd behavior and investor sentiment on market dynamics. During periods of market euphoria or panic, collective investor behavior can drive asset prices away from their fundamental values, leading to potential bubbles and crashes. This behavioral aspect is largely absent in MPT, which assumes that markets are always in equilibrium.

In summary, behavioral finance challenges the assumptions of rationality and market efficiency in MPT, asserting that psychological biases and investor sentiment play significant roles in financial decision-making and market outcomes.

In response to these criticisms and the perceived limitations of MPT, the Post-Modern Portfolio Theory (PMPT) was proposed in the 1980s. PMPT deviates from MPT by prioritizing downside risk, arguing that investors are more concerned with potential losses than overall volatility [32].

The PMPT introduced Sortino Ratio, which measures the risk-adjusted return of an investment, asset, or portfolio, factoring in the downside risk [32]. This allowed investors to better gauge the performance of their portfolios during market downturns.

$$\text{Sortino Ratio} = \frac{R_p - r_f}{\sigma_d}$$

where:

R_p = Actual or expected portfolio return

r_f = Risk-free rate

σ_d = Standard deviation of the downside

Formula 1: The Sortino Ratio

The numerator of the Sortino Ratio consists of the difference between the actual or expected return of the investment or portfolio and the risk-free rate. The risk-free rate typically represents the return on a hypothetical investment with zero risk, often proxied by government bonds.

By subtracting the risk-free rate, the ratio accounts for the opportunity cost of capital, or the minimum return an investor would require in order to justify the risk taken on by the investment.

The denominator of the Sortino Ratio, the standard deviation of the downside, is what distinguishes this ratio from other risk-adjusted measures.

The standard deviation of the downside, also known as downside deviation, specifically measures the volatility of negative returns. This focus on negative returns aligns with the intuition that investors are primarily concerned with downside volatility that might lead to financial losses, rather than total volatility that includes upward price movements.
In essence, the Sortino Ratio provides a measure of the "excess return" (return above the risk-free rate) per unit of bad risk taken (downside volatility). A higher Sortino Ratio indicates that the investment provides more return for each unit of downside risk, implying better risk-adjusted performance.
It's worth noting, however, that the Sortino Ratio, while insightful, is dependent on several assumptions. Firstly, it assumes that the investor is risk-averse and cares only about downside risk. Secondly, the ratio assumes that returns are normally distributed, which may not always be the case in financial markets. Finally, different assumptions about the risk-free rate or the expected return can lead to varying Sortino Ratios for the same investment or portfolio.
The PMPT approach gained momentum as it addressed investors' practical concerns more directly than MPT.
Nonetheless, it has not escaped criticism. PMPT has been critiqued for its computational complexity and for the subjectivity involved in defining the downside risk [27].

Both MPT and PMPT have made significant contributions to portfolio management theory and practice. Despite their limitations, they provide essential frameworks for understanding portfolio diversification, risk management, and the interplay of risk and return.

Chapter Three

ANALYSIS – MPT AND APPLIED MPT TO THE FTSE MIB IT 40 INDEX

INTRODUCTION

In this chapter we will illustrate the theory and mathematical foundations behind MPT. Such concepts and formulas will be then applied to the FTSE MIB index in Chapter 2 using Python (the coding script is reported in the Appendix, divided by its correspondent section).

Chapter 3.1: The Assumption of Perfect Capital Markets

In order to begin our analysis of the MPT, we start by introducing the assumption that capital markets are *perfect*. This requires that:

- Markets are *frictionless*: namely, there are no *transaction costs* or any other constraints to trading;
- Investors can purchase or *short sell* unlimited amounts of any asset. Notice that by *short selling* we mean the possibility of borrowing a certain security and selling it with the future obligation to return such an asset, inclusive of any income that this may generate during the holding period;
- Assets are *infinitely divisible*, that is, it possible to acquire or short sell any portion of a certain traded security;
- All investors possess *homogenous information* on asset returns.

Chapter 3.2: The Covariance matrix, expected return and weights' vectors

Now, suppose there exist $N \geq 2$ risky assets, but not a risk-free one. Assume these N assets are *linearly independent*. This means that no asset return can be expressed as a linear combination of the returns of the other assets.

The *covariance matrix* is defined as an $N \times N$ matrix of the returns on the N risky assets. This is a matrix of elements $\{\sigma_{ij}\}^{N}_{i,j=1}$, where $\sigma_{ij} \equiv \mathrm{Cov}(\tilde{r}_i, \tilde{r}_j)$ and $\tilde{r}_i$ is the return on asset i during the holding period, while $\sigma_{ii} = \sigma_i^2 \equiv \mathrm{Var}(\tilde{r}_i)$. The matrix **V** is symmetric and, because the N assets are linearly independent, positive definite. It also possesses *full rank*, so that it is *invertible*. Hence, let **e** be a $N \times 1$ vector of the expected returns on the N assets. This is a vector of elements $\{e_i\}^{N}_{i=1}$, where $e_i \equiv E[\tilde{r}_i]$.

Now we intrude the *weights vector*, that is, the proportion of the investor's wealth invested in the N assets. Given that short-selling assets is permitted, the weights in the vector **w** can be negative. Nevertheless, individuals cannot invest more than their wealth, so these weights must add up to 1:

Formula 2: The first Constraint (the Portfolio Weights must add up to 1)

$$\sum_{i=1}^{N} w_i = 1'w = 1$$

This restriction qualifies the meaning of the vector **w**. In fact, this vector characterizes the *composition* of the portfolio, not its overall value. Now, suppose the investor chooses the portfolio's composition characterized by the vector **w**, then the return on portfolio *p*, $\tilde{r}_p$, will be:

Formula 3: The Return on the Portfolio

$$\tilde{r}_p = \sum_{i=1}^{N} w_i \tilde{r}_i = w'\tilde{r}$$

Formula 3 states that the return on the portfolio corresponds to the weighted average, with weights given by the proportion of wealth invested in the individual assets, of the assets' returns.

Furthermore, Thanks to the linearity of the expectation operator and the properties of the variance, we find that, given the weights in the vector **w**, the expected return of our portfolio *p* and its variance are as follows:

Formula 4: The Expected Return on the Portfolio

$$E[\tilde{r}_p] = \sum_{i=1}^{N} w_i e_i = w'e$$

Formula 5: The Portfolio Variance

$$\sigma_p{}^2 = \sum_{j=1}^{N} * \sum_{j=1}^{N} w_i w_j \sigma_{ij} = w'Vw$$

the expected return on portfolio *p* corresponds to the weighted average of the expected returns on the *N* assets with weights given by the proportion of wealth invested in the single assets.

Chapter 3.3: The Constrained Minimization Problem

Now, we would like to choose, among all possible portfolios made of all possible combinations of our stocks, the one that carries *minimum variance*, i.e. the minimum risk.

Such portfolio is the MVP and it corresponds to the solution of the following constrained minimization problem:

Formula 6: The Constrained Variance Minimization Problem

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}'\mathbf{1} = 1\,, \\ & \mathbf{w}'\mathbf{e} \;=\; E_p\,. \end{aligned}$$

Notice that we want to minimize a *quadratic form* and since the covariance matrix is positive definite, we can assume this problem has a unique solution, obtained by solving the system of its *first order conditions*. This system is derived as follows.

Chapter 3.3(A): The Associated Lagrangian system

Define the *Lagrangian* associated with the optimization problem as:

Formula 7: The Associated Lagrangian System

$$\min_{\mathbf{w},\lambda,\gamma} \mathcal{L} \;\equiv\; \frac{1}{2}\mathbf{w}'\,\mathbf{V}\,\mathbf{w} \;+\; \lambda\,(E_p \;-\; \mathbf{w}'\mathbf{e}) \;+\; \gamma\,(1 \;-\; \mathbf{w}'\mathbf{1})\,.$$

Its F.O.C. are defined as follows:

Formula 8: The First Order Conditions of the Associated Lagrangian System

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \mathbf{w}} &\equiv \mathbf{V}\,\mathbf{w} \;-\; \lambda\,\mathbf{e} \;-\; \gamma\,\mathbf{1} \;=\; 0\,,\\
\frac{\partial \mathcal{L}}{\partial \lambda} &\equiv E_p \;-\; \mathbf{w}'\mathbf{e} \;=\; 0\,,\\
\frac{\partial \mathcal{L}}{\partial \gamma} &\equiv 1 \;-\; \mathbf{w}'\mathbf{1} \;=\; 0\,.
\end{aligned}$$

By pre-multiplying the first equation of the F.O.C. by the inverse of the covariance matrix, $\mathbf{V}^{-1}$, we obtain the following:

Formula 9: The Weights in terms of *λ* and *γ*

$$\mathbf{w} \;=\; \lambda\,(\mathbf{V}^{-1}\,\mathbf{e}) \;+\; \gamma\,(\mathbf{V}^{-1}\,\mathbf{1})\,.$$

Now, it is possible to do some little algebra and obtain the following system:

Formula 10: Reordering the F.O.C. of the Langrangian System

$$\begin{aligned}
B\,\lambda \;+\; A\,\gamma &\;=\; E_p\,,\\
A\,\lambda \;+\; C\,\gamma &\;=\; 1\,,
\end{aligned}$$

Where:

Formula 11: Defining A, B and C

$$A \;=\; \mathbf{1}'\,\mathbf{V}^{-1}\,\mathbf{e}\,, \quad B \;=\; \mathbf{e}'\,\mathbf{V}^{-1}\,\mathbf{e} \quad \text{and} \quad C \;=\; \mathbf{1}'\,\mathbf{V}^{-1}\,\mathbf{1}\,.$$

Solving this system of equations in γ and λ we find that:

Formula 12: The Lagrangian System's λ and γ

$$\lambda = \frac{1}{D}(CE_p - A) \quad \text{and} \quad \gamma = \frac{1}{D}(B - A\,E_p)$$

Where $D = BC - A^2$.

Finally, by plugging γ and λ into Formula 9 we obtain the solution to the variance minimization system:

Formula 13: The Solution to the Minimization Problem

$$\mathbf{w}_p = \mathbf{g} + \mathbf{h}\,E_p$$

Where the vectors **g** and **h** are defined as:

Formula 14: The g and h vectors

$$\mathbf{g} = \frac{1}{D}[B(\mathbf{V}^{-1}\mathbf{1}) - A(\mathbf{V}^{-1}\mathbf{e})],$$

$$\mathbf{h} = \frac{1}{D}[C(\mathbf{V}^{-1}\mathbf{e}) - A(\mathbf{V}^{-1}\mathbf{1})].$$

Indeed, given the choice of the Expected Return E_p, it is possible to derive the optimal composition of the MVP as a combination of the **g** and **h** vectors.

Chapter 3.4: The Mathematics of the Portfolio Frontier

Our next step is to understand the mathematics behind the graphical representation of the Portfolio Frontier. Firstly, we must define what is a Frontier Portfolio: *the portfolio p, characterized by the vector of weights* ***w**$_p$, which solves the optimal choice problem (mentioned above) is a **frontier portfolio***.

Therefore, the **Portfolio Frontier** is defined as *the set of frontier portfolios*.

The portfolio *variance* associated with the optimal portfolio *p* is obtained by inserting this expression for $\mathbf{w}_p$ in $\mathbf{w}'\mathbf{V}\mathbf{w}$. Therefore, to the expected return E_p corresponds the following standard deviation (defined as the square root of the portfolio variance):

Formula 15: The Portfolio Standard Deviation

$$\sigma_p = \frac{1}{\sqrt{D}}\left[CE_p^2 - 2AE_p + B\right]^{1/2}$$

This expression identifies an *hyperbola* in the standard-deviation-mean (σ, E) space. Furthermore, varying E_p returns different values for σ_p.

Indeed, solving for E_p we obtain the following:

Formula 16: The Hyperbola Equation

$$E_p = \frac{A}{C} \pm \frac{\sqrt{D(C\sigma_p^2 - 1)}}{C}$$

Also, we have the following equation for the asymptotes:

Formula 17: The Hyperbola's Asymptotes Equation

$$\hat{E}_{1,2} = \frac{A}{C} \pm \sqrt{\frac{D}{C}}\sigma_p$$

The Portfolio Frontier will finally look like this:

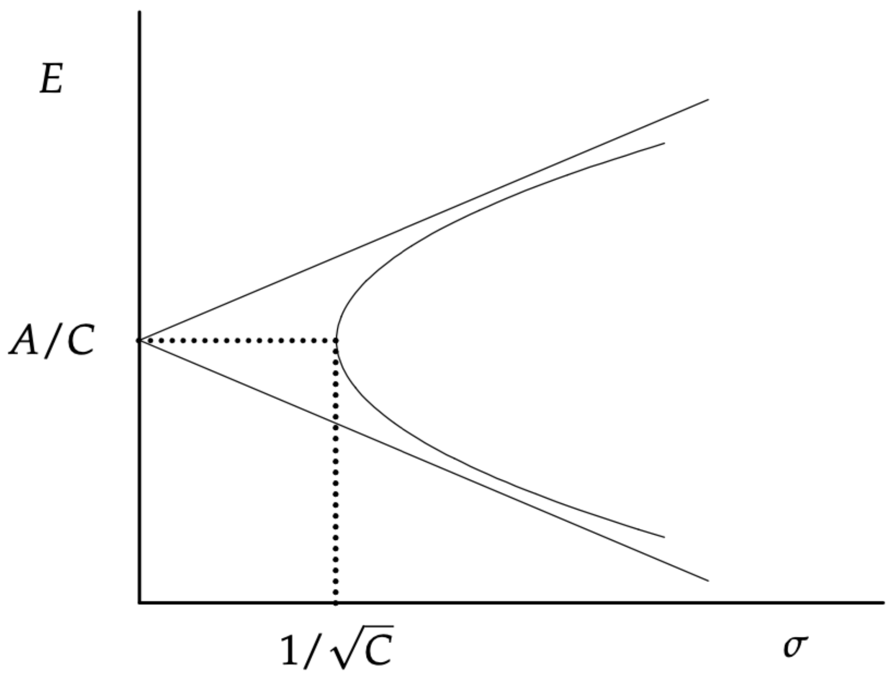

Figure 1: A Graphical Representation of the Portfolio Frontier

We are going to derive a similar graph for the Portfolio frontier in Chapter 2.6.

Notably, we observe that the point with coordinates ($1/\sqrt{C}$, A/C) represents the combination of the standard deviation and the expected value of the return on the *minimum variance portfolio*, ie. that portfolio which reaches the global minimum for the variance of its return.

Chapter 3.4(A): The Efficient Frontier

From the symmetry of the portfolio frontier in Figure 1 we see that this can be divided into two *branches* joined by the point corresponding to the minimum variance portfolio.

From this we see that, bar the minimum variance portfolio, we have two different portfolios on the portfolio frontier with the same variance. However, we can expect that for the same level of risk investors prefer portfolios with greater expected returns. Therefore, we define the upper branch of the hyperbola in Figure 1 the *efficient frontier*. It is possible to provide a more precise definition to the Efficient Frontier, however, we must start by the definition of Efficient Portfolio: *A portfolio p on the frontier portfolio with expected return, E_p, larger than or equal to the expected return of the minimum variance portfolio, $E_p \geq A/C$, is an* ***efficient portfolio****.* Therefore, the Efficient Frontier is defined as *the set of all Efficient Portfolios*.

Chapter 3.5: The Mathematics of the Capital Market Line

In this chapter we will understand the mathematics behind the graphical representation of the Portfolio Frontier.

To begin, suppose we augment the existing portfolio of assets with a risk-free asset, characterized by a certain return rate r_f. We will now reexamine the mean-variance optimization problem discussed earlier, taking into account N+1 assets, one of which offers a risk-free rate of return r_f, where $r_f < A/C$.

In this situation, we can still represent a portfolio by selecting a specific vector w that denotes the proportions of initial wealth allocated to the N risky assets. However, we are no longer required to impose the constraint that the sum of the elements of w equals 1. In fact, given the weight vector *w* for the risky assets, a portfolio is deemed appropriate as long as the proportion of the investor's wealth invested in the risk-free asset is equal to **1 - w'1**. By doing so, the investor fully utilizes all available resources, ensuring that all wealth is invested:

Formula 18: The new Restriction

$$w'1 + (1 - w'1) = 1$$

It should be noted that the proportion of wealth allocated to the risk-free asset, 1 - w'1, can take negative values, indicating that the investor engages in short-selling of the risk-free asset, effectively borrowing at the risk-free rate r_f. Conversely, if 1 - w'1 is positive, our investor lends at the risk-free rate r_f. In essence, the inclusion of the risk-free asset assumes that investors have the ability to freely borrow and lend at a specified interest rate.

The optimization problem defined before for the Portfolio Frontier is now defined as:

Formula 19: The new Constrained Variance Minimization Problem

$$\min_{\mathbf{w}} \quad \frac{1}{2}\mathbf{w}'\,\mathbf{V}\,\mathbf{w}$$
$$\text{s.t.} \quad \mathbf{w}'\mathbf{e} + (1 - \mathbf{w}'\mathbf{1})\, r_f = E_p$$

As before, we need to impose a Lagrangian system to be able to solve Formula 19.
The Lagrangian system is therefore the following:

Formula 20: The new Associated Lagrangian System

$$\min_{\mathbf{w},\lambda} \mathcal{L} \equiv \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w} + \lambda\left[E_p - \mathbf{w}'\mathbf{e} - (1-\mathbf{w}'\mathbf{1})\,r_f\right]$$

Here also the respective F.O.C.:

Formula 21: The new F.O.C. of the Associated Lagrangian System

$$\mathbf{w} = \lambda\mathbf{V}^{-1}(\mathbf{e} - \mathbf{1}r_f)$$

$$r_f + \mathbf{w}'(\mathbf{e} - \mathbf{1}r_f) = E_p$$

By substituting the first F.O.C. into the second we get that $\lambda = \frac{E_P - r_f}{H}$, where $H = (e - 1r_f)'V^{-1}(e - 1r_f)$, then:

Formula 22: The new Solution to the Minimization Problem

$$\mathbf{w}_p = \mathbf{V}^{-1}(\mathbf{e} - \mathbf{1}r_f)\frac{E_p - r_f}{H}$$

Now, by inserting the vector w_p into Formula 5, i.e. the expression for the variance of the portfolio return, and taking the square root, we obtain the following:

Formula 23: The Equation of the Capital Market Line

$$E_p = r_f \pm \sqrt{H}\,\sigma_p$$

Chapter 3.5(A): The Tangent Portfolio and the Sharpe Ratio

Evidently, the upper portion of the diagram represents the revised efficient frontier.
This linear segment, characterized by a slope of $\sqrt{H}$, is known as the capital market line (CML). The slope of the CML reflects the trade-off between risk and return along the efficient frontier.
It is defined as the ratio of the expected excess return of any efficient portfolio to its standard deviation, $(E_p - r_f) / \sigma_p$.
This ratio is referred to as the **Sharpe Ratio** of the efficient frontier and quantifies the benefit of obtaining a higher expected excess return for a given increase in risk exposure.
The trade-off between risk and return along the CML is determined by the characteristics of the N risky assets and the risk-free asset, as indicated by the expression for the scalar H.

It may be now useful to visualize such CML in a graph:

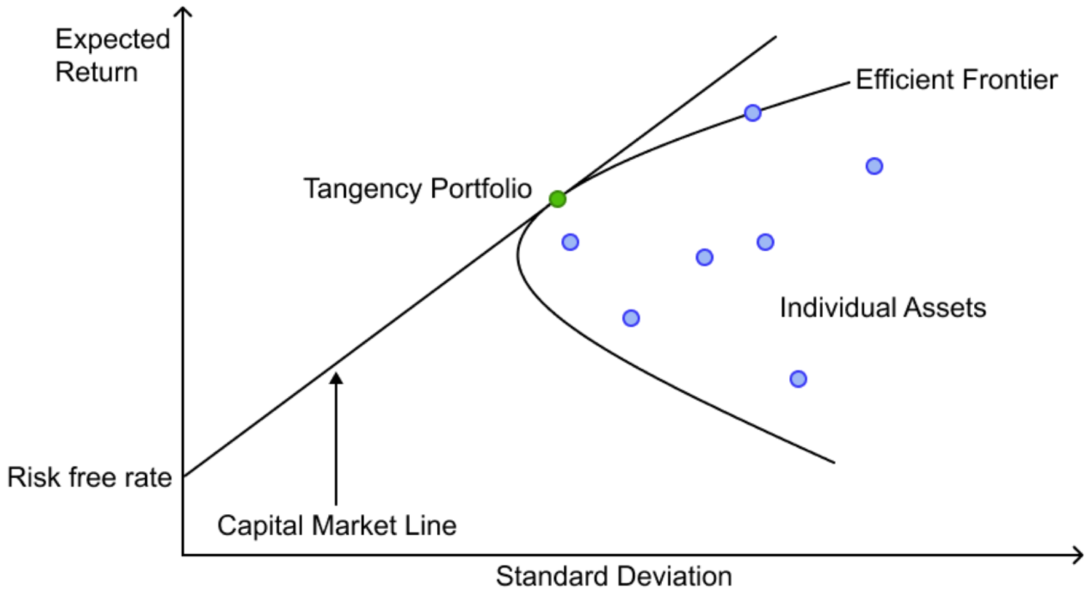

Figure 2: A Graphical Representation of the CML and Portfolio Frontier

We are going to derive a similar graph for the CML and the Portfolio frontier in Chapter 2.7.

A notable characteristic of the CML is its tangency to the frontier of risky asset portfolios, as depicted in Figure 2. The point of tangency corresponds to a portfolio of risky assets, the Tangency Portfolio, which lies within the efficient portion of the risky portfolio frontier.

This tangent portfolio, obtained by imposing the condition $\mathbf{1'w_p = 1}$ in the equation of Formula 21, possesses a key property: its Sharpe ratio, $(E_T - r_f) / \sigma_T$, is equivalent to the slope of the CML and thus equal to $\sqrt{H}$. This implies that the points on the CML represent portfolios formed by combining the risk-free asset, which yields a risk-free rate rf, with the tangent portfolio T, which generates a random return $\tilde{r}_T$. Furthermore, the Sharpe ratio of the tangent portfolio (TP) represents the maximum Sharpe ratio achievable.

INTRODUCTION – THE FTSE MIB IT40 INDEX

We will start our analysis by describing the selected index from the Milan bourse: the FTSE MIB IT40.

The FTSE MIB is the primary benchmark index for the Italian stock market. This index represents approximately 80% of the domestic market capitalization and consists of highly important and liquid companies across various ICB sectors in Italy. The FTSE MIB Index measures the performance of 40 Italian stocks and aims to replicate the sector weights of the broader Italian equity market. The Index is derived from the universe of traded securities on the main stock market of Borsa Italiana (BIt). Each stock is analyzed for size and liquidity, and the Index provides an overall accurate representation by sectors. The FTSE MIB Index is weighted by market capitalization, adjusted for the float of its components[1].

In particular, the 40 listed stocks are the following (in alphabetic order):

Table 1: The stocks of the FTSE MIB IT40 index

Company Name (Spa)	Ticker (.MI)
A2A	A2A
Amplifon	AMP
Assicurazioni Generali	G
Azimut Holding	AZM
Banca Generali	BGN
Banca Mediolanum	BMED
Banco BPM	BAMI
Banca Monte dei Paschi di Siena	BMPS
Bper Banca	BPE
Davide Campari Milano	CPR
CNH Industrial	CNHI
Diasolin	DIA
Enel	ENEL
Eni	ENI
Erg	ERG
Ferrari	RACE
FinecoBank	FBK
Hera	HER
Interpump Group	IP
Intesa Sanpaolo	ISP
Inwit	INWT
Italgas	IG

[1] Borsa Italiana Spa. “FTSE MIB”. Borsaitaliana.it. https://www.borsaitaliana.it/borsa/indici/indici-in-continua/dettaglio.html?indexCode=FTSEMIB

Iveco Group	IVG
Leonardo	LDO
Mediobanca	MB
Moncler	MONC
Nexi	NEXI
Pirelli & C.	PIRC
Poste Italiane	PST
Prysmian	PRY
Recordati	REC
Saipem	SPM
Snam	SRG
Stellantis	STLAM
STMicroelectronics	STMMI
TIM	TIT
Tenaris	TEN
Terna – Rete Elettrica Nazionale	TRN
UniCredit	UCG
UnipolSai	UNI

Finally, it is necessary to highlight the following significant cases of crises resulting in stock market collapses during the analyzed period, from 2021-01-01 to 2023-04-12, which have had a substantial impact on the stock value of each index component and, in some cases, on specific stocks.

First and foremost, the impact of the Russian armed attack on Ukraine needs to be emphasized. As a consequence, the strict sanctions imposed by the international community on Russia have further exacerbated the already present pandemic crisis, leading to a surge in energy costs and various commodities. This has particularly affected households and businesses.

According to data provided by an article from Banca Generali[2], during the first two weeks of the war, all major global stock markets experienced double-digit losses. The backlash was especially significant in Europe: from February 24 to March 8, the main Milan stock index (FTSE MIB) had lost 14.6%, Frankfurt lost 14%, Paris lost 12%, and London recorded relatively smaller losses but still in the range of 7 percentage points.

In practice, within a span of 14 days, all major international stock markets returned to levels seen two years earlier, effectively nullifying the entire attempt to recover from the pandemic crisis. However, the service specifies that the market retracement described above did not last long.

[2] Banca Generali SpA. "Russia-Ucraina: L'impatto sui mercati finanziari". Bancagenerali.com. https://www.bancagenerali.com/blog/mercati-finanziari-momenti-di-guerra (accessed May 02, 2022)

By the end of March, almost all stock markets had already returned to pre-war levels. A telling snapshot is the state of the indexes exactly one month after the outbreak of the conflict. On March 24, the Euro Stoxx 600 index (which comprises the most important stocks across European stock exchanges) had only lost 0.87%, with individual stock exchanges also approaching these values: Frankfurt was down 1.09%, Paris down 1.89%, and London down only 0.47%. The case of the American Dow Jones was even more emblematic. In a month, it had recorded an increase of almost 5 percentage points. It is worth noting that this recovery of the stock indexes occurred despite a steep inflationary ride not seen in years and significant pressure on central banks regarding interest rate policies. Furthermore, noteworthy stock market collapses were observed in the shares of BMPS (Banca Monte dei Paschi di Siena SpA) and SPM (Saipem Spa).
In the case of BMPS, the crisis was primarily caused by the capital increase carried out in October 2022. This process involved the issuance of 1,249,665,648 shares to be offered as an option to shareholders at a subscription price of 2 euros per new share, in a ratio of 374 for every 3 Mps shares held. Axa had acquired 7.976% of the capital of the Sienese bank, making it the second-largest shareholder after the Treasury[3]. Additionally, another significant downturn occurred in February 2023 when the French company announced the sale of 100 million shares, equivalent to approximately 7.94% of the capital, through an accelerated bookbuilding exclusively for institutional investors at a price of 2.33 euros per share, resulting in proceeds of 233 million euros[4].

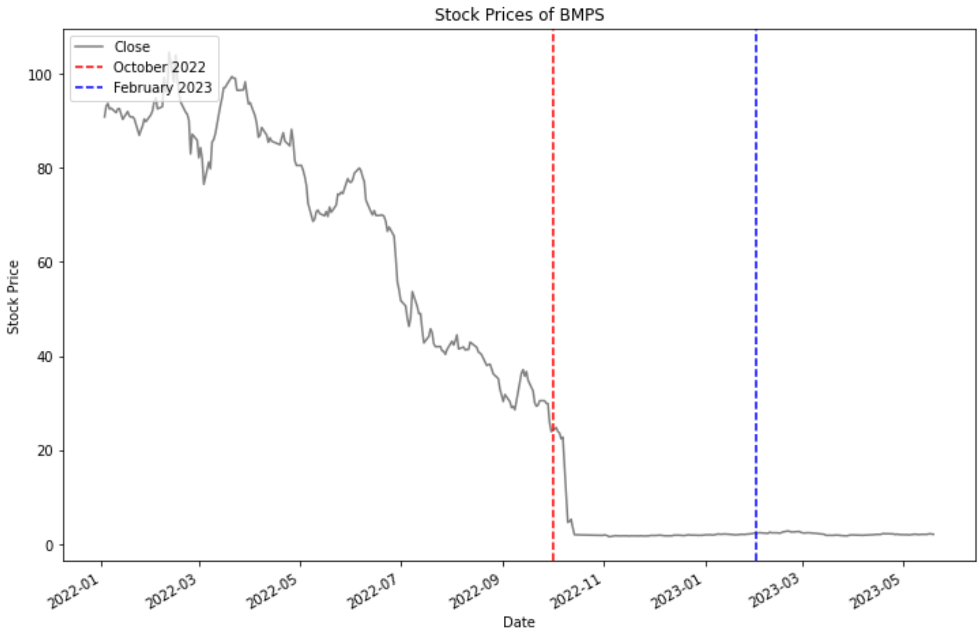

Figure 3: The Stock Prices of BMPS and its Downturns

As for Saipem, it experienced a notable decline in March 2020 due to the outbreak of the pandemic and the subsequent crisis in the oil industry. Eni lost 21%, Saipem 24%, and Tenaris 18.5%[5].

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[3] Rai - Radiotelevisione Italiana Spa. "Il Monte dei Paschi crolla in borsa: la ricapitalizzazione punisce i vecchi azionisti". Rainews.it. https://www.rainews.it/articoli/2022/10/monte-de-paschi-crolla-in-borsa-la-ricapitalizzazione-punisce-i-vecchi-azionisti-b209bc8d-5c33-4872-91f6-6152f2b089e8.html (accessed Oct. 13, 2022)
[4] Il Sole 24 Ore. "Mps precipita in Borsa dopo l'uscita di Axa dal capitale". Ilsole24ore.com. https://www.ilsole24ore.com/art/mps-precipita-borsa-l-uscita-axa-capitale-AEf1UBvC (accessed Febr. 28, 2023)
[5] Il Sole 24 Ore. "Borsa: Ftse Mib giu' dell'11%, crollo petroliferi con -24% Saipem e -21% Eni". Ilsole24ore.com. https://www.ilsole24ore.com/radiocor/nRC_09.03.2020_09.52_17831367 (accessed Mar. 09, 2020)

Another substantial drop occurred in July 2022 due to a new capital increase and the unexercised option rights (6,284,082), which were subsequently offered on the stock market starting from July 12th. The total value of the offering was approximately 0.6 billion euros at a price of 1.013 euros (compared to a share price of 2.52 euros) in a subscription ratio of 95 new shares for every unexercised right[6]. These events positioned these shares at a considerable distance from other index components due to lower expected values and a higher standard deviation.

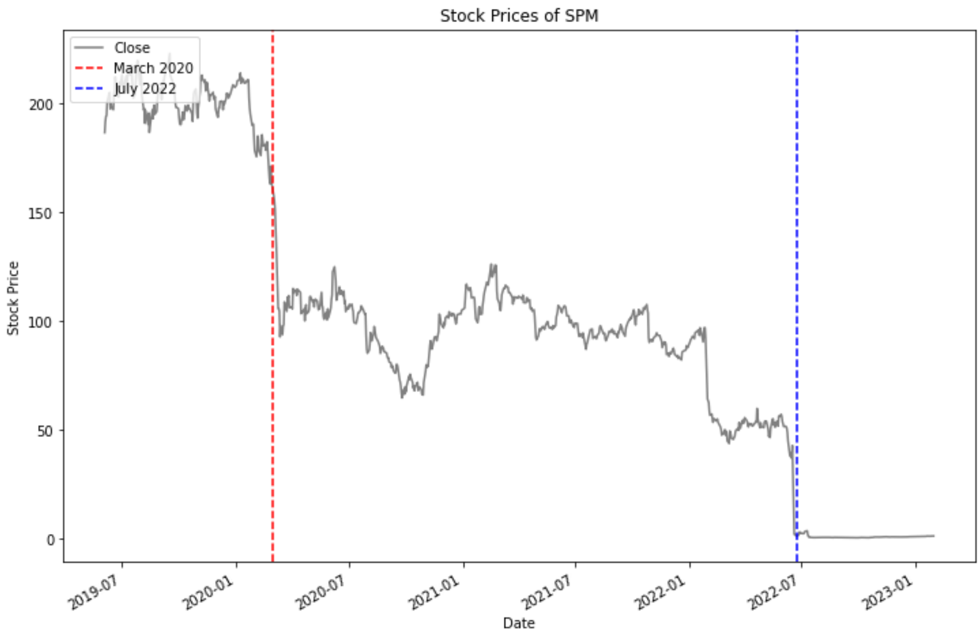

Figure 4: The Stock Prices of SPM and its Downturns

Chapter 3.6: Gathering the Data

The Data has been collected through the YahooFinance platform thanks to the Python's package yfinance. Such an API (application programming interfance) it's an open-source tool that uses Yahoo's publicly available APIs, and is intended for research and educational purposes[7]. The selected period, as mentioned, starts from 2021-01-01 to 2023-04-12, in this way we can collect sufficient observation for our analysis. Indeed, every year we observe 252 closing prices as for 252 trading days.

Chapter 3.7: Prices and Daily Returns

We start by selecting the closing daily prices of our stocks. Other types of available prices such as open, high, low or adjusted close ones are not relevant for our academic purposes.

[6] MilanoFinanza. "Saipem perde fino a quasi il 40%. Ecco perché deve ancora scendere". Milanofinanza.it https://www.milanofinanza.it/news/saipem-perde-fino-a-quasi-il-40-ecco-perche-deve-ancora-scendere-202207121001448384 (accessed July 12, 2022)
[7] Pypi. "yfinance 0.2.18". pypi.org. https://pypi.org/project/yfinance/ (accessed Apr. 16, 2023)

```
Stock: UNI.MI
             Open   High    Low  Close  Adj Close
Volume
Date
2021-01-04  3.934  3.970  3.872  3.890   3.266037
3082823
2021-01-05  3.886  3.924  3.854  3.880   3.257641
2099843
2021-01-06  3.918  3.992  3.858  3.986   3.346638
2514688
2021-01-07  4.012  4.044  3.988  4.026   3.380222
3512210
2021-01-08  4.040  4.106  4.004  4.024   3.378543
3119275
...           ...    ...    ...    ...        ...
...
2023-04-03  4.763  4.786  4.741  4.778   4.778000
934553
2023-04-04  4.794  4.848  4.772  4.791   4.791000
2757515
2023-04-05  4.810  4.810  4.758  4.805   4.805000
1027611
2023-04-06  4.828  4.914  4.824  4.914   4.914000
1731226
2023-04-11  4.930  5.004  4.923  5.004   5.004000
1990936

[582 rows x 6 columns]
```

Figure 5: The Available Stock Prices' Data (UNI.MI example)

By running the code available in the Appendix, the result is displayed in Figure 3. Indeed, by considering only the Adjusted Close prices (closing prices adjusted for eventual dividend distributions and all applicable splits), we create a vector containing 582 observations.

Consequently, we calculate the log returns for each stock. This is common practice in empirical studies which can be justified on two grounds. On the one hand, the difference with the arithmetic returns is minimal at high frequencies, as these values are close to zero. On the other hand, using log returns these will typically possess a distribution which is more like that of a multi-normal random variable.

Chapter 3.8: The Correlation Matrix

Central to MPT is the concept of diversification, suggesting that a portfolio composed of diverse investments will, on average, yield higher returns and pose a lower risk than any individual investment found within the portfolio.

Indeed, the correlation matrix plays a pivotal role in diversification. It quantifies the degree to which the returns of two different investments move in relation to each other. Correlation coefficients, the elements of the correlation matrix, range from -1 to 1. A correlation of -1 implies that the returns of the two investments move in opposite directions (perfect negative correlation), while a correlation of 1 signifies that the returns move in the same direction (perfect positive correlation). A correlation of 0 indicates no linear relationship between the returns of the two investments.

Understanding these correlations provides critical insights into how different securities interact with each other within a portfolio, and it is a vital part of the process of creating a diversified portfolio that can optimize returns for a given level of risk. Given that not all securities move in the same direction at the same time, diversification reduces the overall risk of the portfolio [6].
However, the correlation matrix is not static; it changes over time as market conditions change. Therefore, periodic reviews and adjustments of the portfolio, considering the current correlation matrix, are required to maintain the level of risk and return in line with the investor's preferences [5].

By running the code available in the Appendix, the resulting Correlation Matrix is the following:

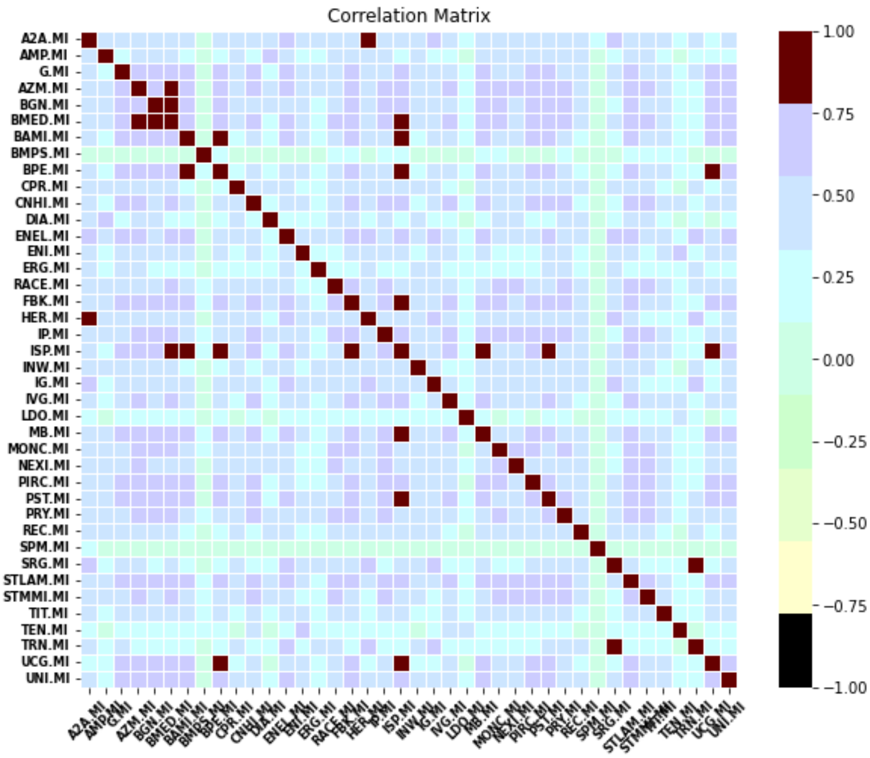

Figure 8: The Correlation Matrix

The main diagonal of the Correlation Matrix comprises only perfect positive correlations since the correlation of a certain stock with itself is equal to 1. Therefore, the main diagonal is all Bordeaux (such a color takes values from more than 0.75 up to 1 as described in the colorbar's palette).
The colors of the palette have been selected in such a way that particularly high or low correlation values stand out against the less aggressive and lighter tones assigned to intermediate values. Therefore, we can observe that Intesa Sanpaolo (ISP) stock is the most frequently and highly correlated with the rest of the index stocks. On the other hand, stocks such as Banca Monte dei Paschi di Siena (BMPS) and Saipem (SPM) exhibit notably lower levels of correlation, and the only significant correlation is observed along the main diagonal (represented by the only Bordeaux hue). This low level of correlation with the other stocks in the FTSE MIB index is likely linked to their considerable volatility and fluctuation in terms of price movement, as discussed earlier. It is possible to find the Covariance Matrix's analysis in the Appendix.

Chapter 3.9: Visualizing the MIB-40 Stocks in a Risk and Return graph

The next step consists in visualizing the MIB-40 Stocks in a Risk and Return graph in order to confirm the outcomes of the previous analysis, namely, the difference between the BMPS and SPM's metrics and the rest of the index.

Therefore, we can obtain the following picture:

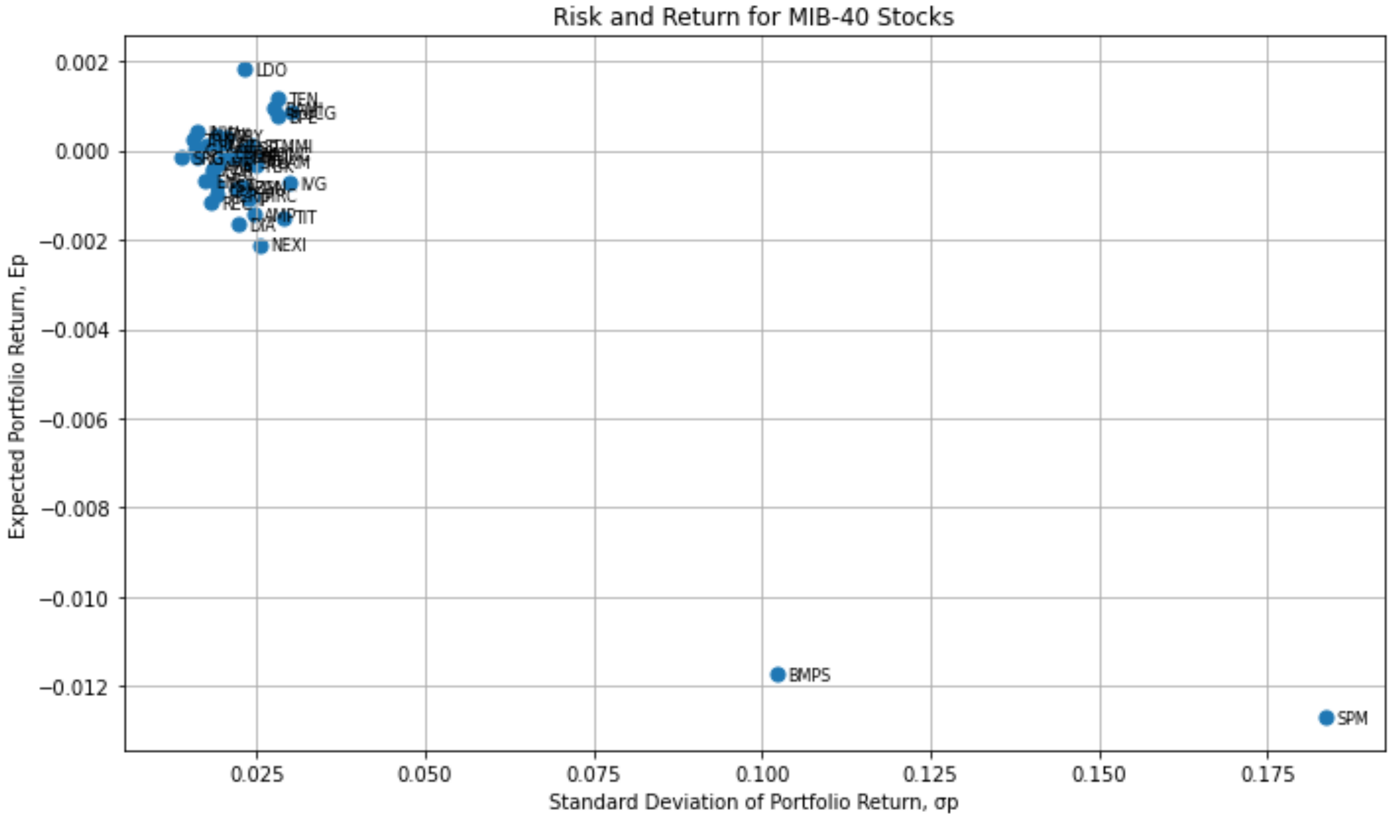

Figure 9: Risk and Return for MIB-40 Stocks

For the aforementioned issues, the BMPS and SPM stocks are, indeed, placed far away from the other 38 stocks in terms of standard deviation and expected return.

Indeed, in the future these stocks may better perform than the rest of the market.

In order to better visualize the placements of the rest of the index's stocks it is possible to focus on the top-left section of the Risk and Return graph.

Furthermore, by obtaining a better view on the other 38 stocks of the Milan Bourse, we are going to be able to locate the stocks with the highest values for the expected return and lowest values for the standard deviation.

The result is the following:

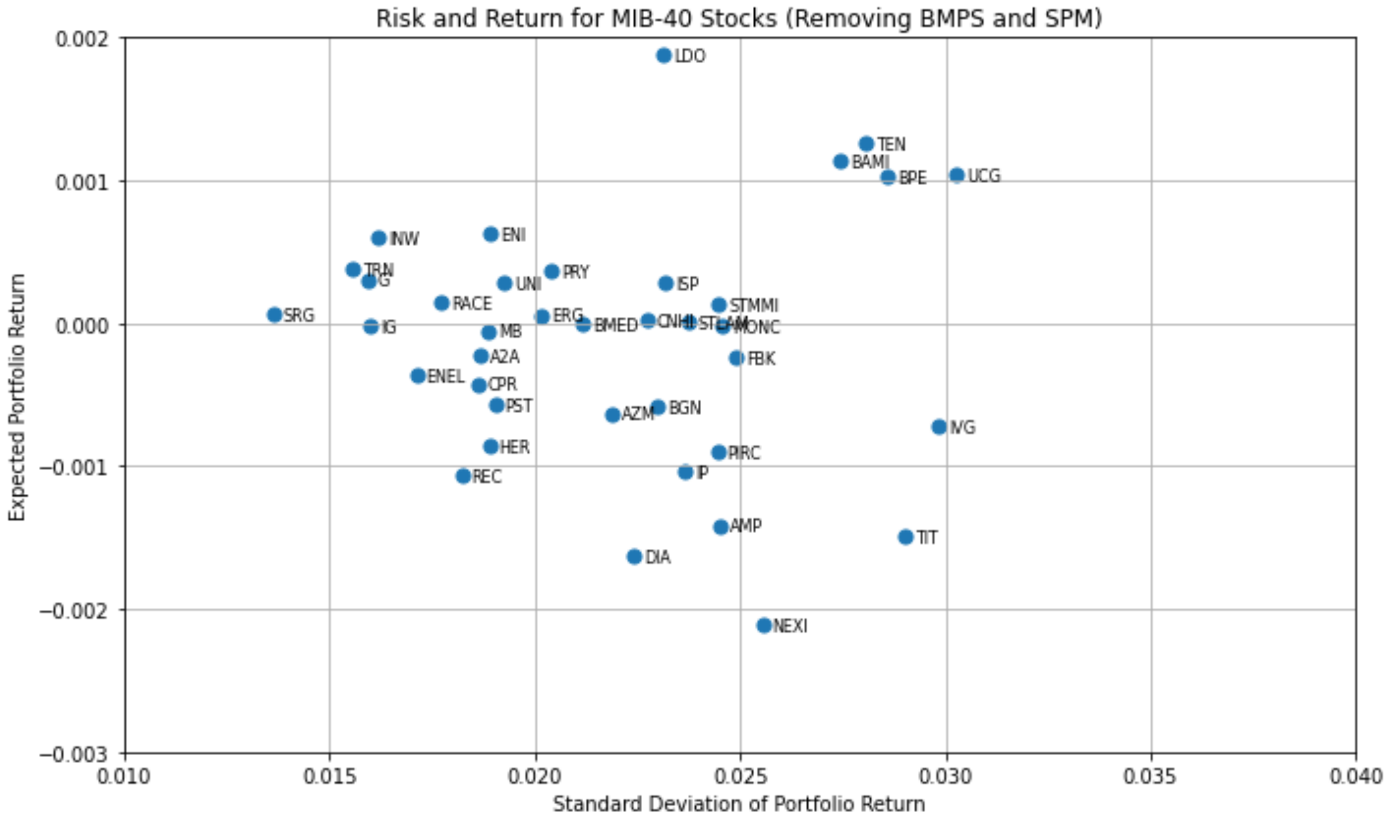

Figure 10: Risk and Return for MIB-40 Stocks (Removing BMPS and SPM)

The rest of the index is therefore included in a range circa between 0.013 and 0.032 for the standard deviation and -0.0022 and 0.002 for the expected return. Furthermore, the most performing stock in terms of daily expected return is LDO (Leonardo), reaching a value of circa 0,2% and the safest asset with the lowest daily standard deviation is SRG (Snam) with a value of circa 1,3%.

Chapter 3.10: Plotting the Portfolio Frontier

Finally, all the information needed to plot the portfolio frontier is now available. The upper branch of such frontier is defined as the Efficient Frontier, because for the same level of standard deviation it is possible to achieve a higher expected portfolio return rather than the one linked to the lower branch. Such branches are divided thanks to the presence of the Minimum Variance Portfolio, that will be consequently plotted in the graph as a red dot. Indeed, for such a minimum global level of St. Dev. no other expected return is available and so that portfolio is unique and of utmost importance in our analysis. Subsequently, we will also calculate and analyse its composition (the weights attached to the underlying stocks).

Analyzing the portfolio frontier allows investors to make informed decisions by comparing risk and return trade-offs among different portfolios. These comparisons guide investors in their selection process, thereby aligning their investment decisions with their risk tolerance and return objectives [28]. Therefore, the construction of the portfolio frontier serves as a crucial step in portfolio optimization.

However, it's essential to remember that the portfolio frontier is derived based on historical data and certain assumptions, such as investors being rational and markets being efficient. Changes in market conditions or the invalidation of these assumptions may cause deviations from the predicted outcomes [11].
Therefore, the Portfolio Frontier of the MIB-40 index is the following:

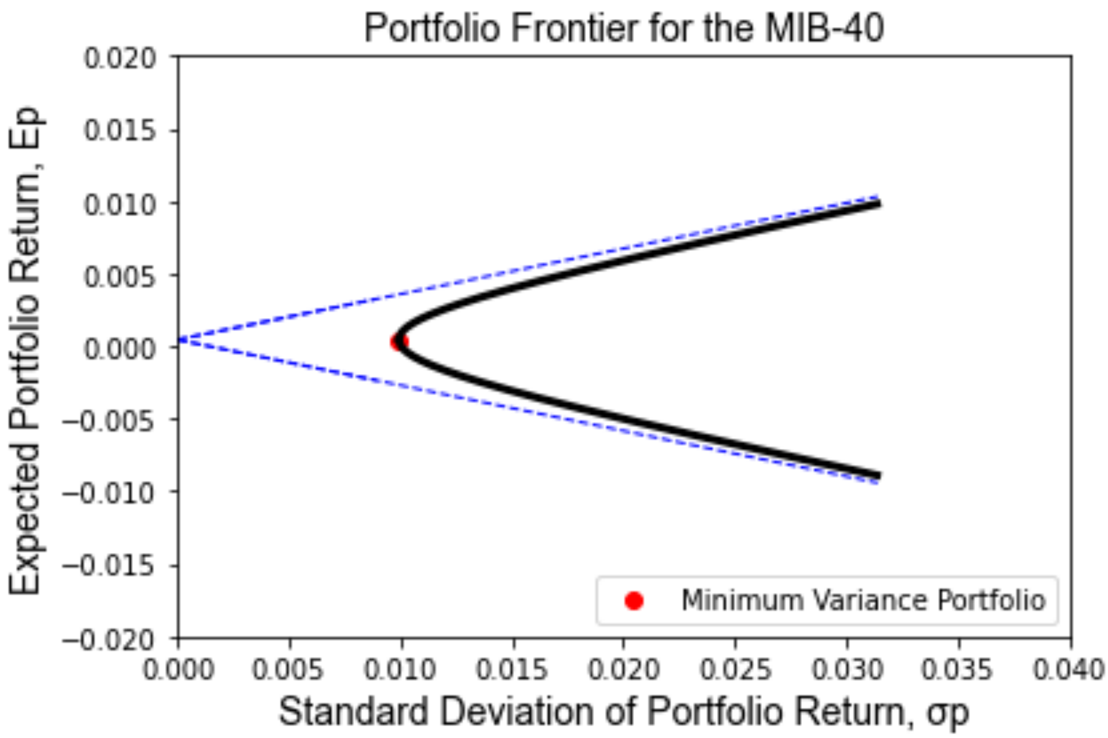

Figure 11: The Portfolio Frontier for the MIB-40 index

As described in Chapter 1, the portfolio frontier is an hyperbola, with center (0, A/C) and the dotted lines are its asymptotes. Finally, the Minimum Variance portfolio has the coordinates ($1/\sqrt{C}$, A/C).

Chapter 3.11: Plotting the Capital Market Line (CML)

The Capital Market Line (CML) is a fundamental concept in finance that represents the relationship between risk and return for a portfolio of risky assets in the context of the Capital Asset Pricing Model (CAPM) [28] [22]. It is a graphical representation of the risk-return tradeoff for an efficient portfolio that includes both risk-free assets and risky assets.

The CML is derived from the combination of the risk-free rate of return and the efficient frontier, which is the set of portfolios that offer the highest expected return for a given level of risk. It starts at the risk-free rate and extends upward, reflecting higher returns associated with increasing levels of risk.

Furthermore, by combining the risk-free asset with the portfolio that lies on the efficient frontier and has the highest Sharpe ratio, it is possible to construct the Tangent Portfolio. The Tangent Portfolio represents the optimal combination of the risk-free asset and the risky portfolio, considering both risk and return and the Sharpe ratio measures the excess return earned per unit of risk taken by an investment.

Furthermore, the Tangent Portfolio demonstrates the benefits of diversification.

By combining the risk-free asset with the Tangent Portfolio, investors can achieve an optimal risk-return tradeoff and maximize their expected return for a given level of risk. It highlights the importance of diversifying investments to reduce risk without sacrificing returns.
The risk-free rate can be obtained by the following: taking the sample mean on the one-month inter-bank offer rate in Italy between December 1994 and April 2005, that on a yearly basis such a return is equal to 5.028%. On a daily basis, this corresponds to 0.0195% or 0.000195.
In the following plot such a portfolio is depicted as a green dot, while the CML is the black line that starts from a blue dot, the risk-free rate. The stocks of the FTSE MIB index are also included, displayed as light blue dots.

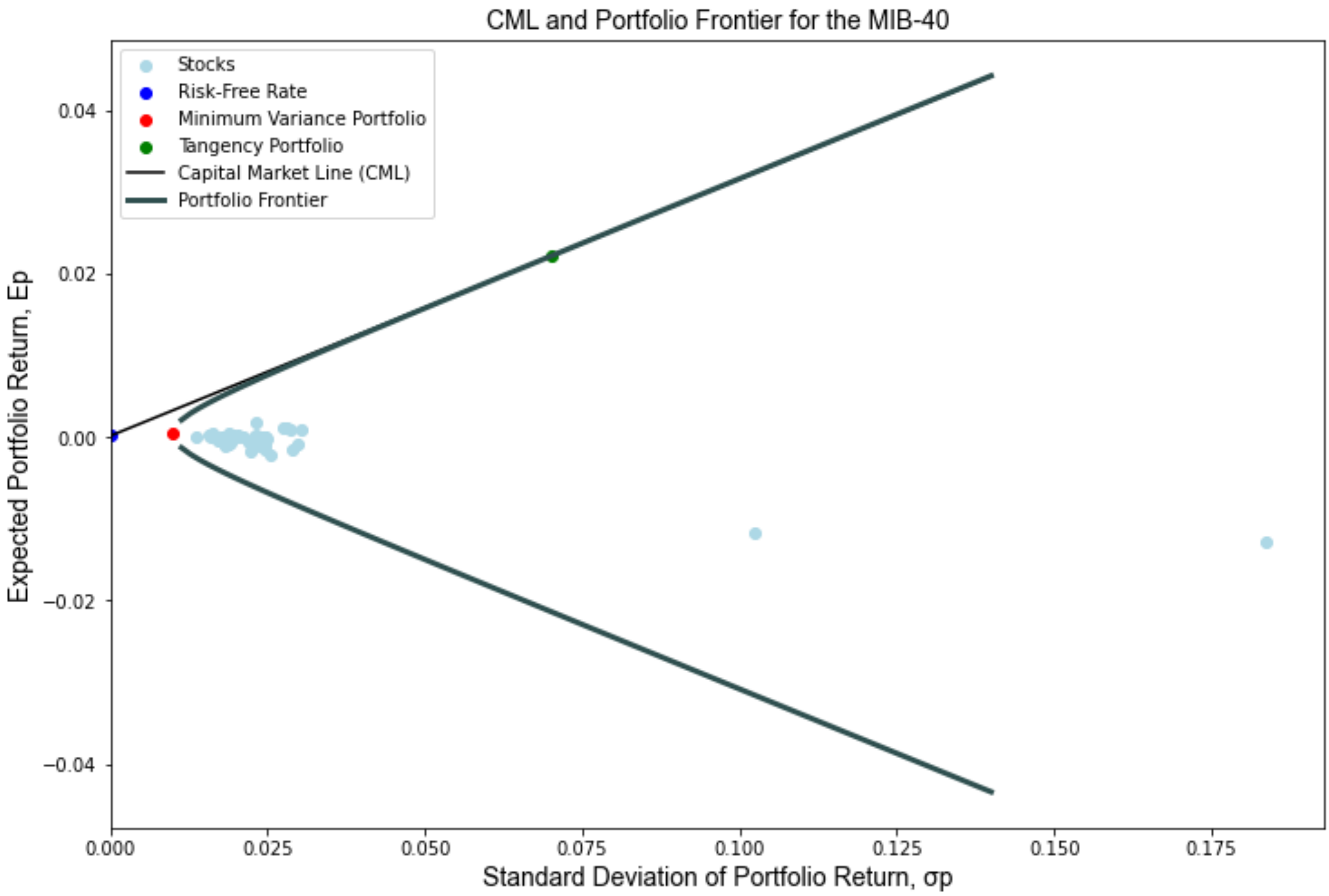

Figure 12: The CML, Portfolio Frontier and Tangency Portfolio for the MIB-40

Chapter 3.12: The composition of the Tangency and Minimum Variance Portfolios

By running the code available in the Appendix, it is possible to observe the weights of the TP and MVP. Furthermore, it is also possible to analyze the maximum value of the Sharpe Ratio, Exp. Ret., Var. and St. Dev. in those particular portfolios.
While analyzing the MVP and TP weights, we will observe negative values. In such cases, to construct such a portfolio we are short selling a certain asset. Namely, rather than investing (going long) a certain percentage of our wealth in a certain asset, we are increasing our overall wealth by selling (with the promise of buying it back in the future) that security, in order to invest more in other better performing ones.

Therefore, the following table represents the composition of the TP and MVP:

Table 2: MVP and TP metrics

	MVP Weights	Tangency Portfolio Weights
A2A.MI	0.15	0.99
AMP.MI	0.07	-0.44
G.MI	0.08	0.1
AZM.MI	-0.05	0.4
BGN.MI	-0.01	0.16
BMED.MI	-0.04	-0.52
BAMI.MI	0.0	-0.08
BMPS.MI	-0.01	-0.64
BPE.MI	-0.05	-0.48
CPR.MI	0.37	-0.53
CNHI.MI	0.05	-0.2
DIA.MI	0.1	-0.9
ENEL.MI	0.16	0.55
ENI.MI	-0.05	-1.14
ERG.MI	-0.05	0.33
RACE.MI	0.04	0.38
FBK.MI	-0.21	-0.36
HER.MI	0.15	1.8
IP.MI	-0.05	1.1
ISP.MI	0.01	-0.9
INW.MI	-0.02	-0.4
IG.MI	-0.05	0.46
IVG.MI	0.0	-0.27
LDO.MI	-0.03	1.06
MB.MI	-0.04	0.22
MONC.MI	0.03	0.68
NEXI.MI	0.06	-0.15
PIRC.MI	-0.07	-0.49
PST.MI	-0.02	-0.33
PRY.MI	0.21	1.08
REC.MI	0.01	-0.35
SPM.MI	-0.0	-1.46
SRG.MI	0.11	1.21
STLAM.MI	0.01	0.76
STMMI.MI	-0.03	-1.98
TIT.MI	-0.03	-1.72
TEN.MI	-0.06	1.05
TRN.MI	0.14	0.25
UCG.MI	-0.05	1.0
UNI.MI	0.14	0.77
Expected Return	0.0	0.02
Variance	0.0	0.0
Standard Deviation	0.01	0.07
Sharpe Ratio	nan	0.31

In order to construct the MVP, we short sell AZM, BGN, BMED, BMPS, BPE, ENI, ERG, FBK, IP, INW, IG, LDO, MB, PIRC, PST, SPM, STMMI, TIT, TEN and UCG, while we go long or simply do not invest in the remaining securities. Interestingly, most of the portfolio is invested in the following 5 assets (from the highest contribution to the lowest): CPR (+0,37), PRY (+0,21), ENEL (+0,16), A2A and HER (+0,15).

In order to construct the TP, we short sell AMP, BMED, BAMI, BMPS, BPE, CPR, CNHI, DIA, ENI, FBK, ISP, INW, IVG, NEXI, PIRC, PST, REC, SPM, STMMI and TIT, while we go long or simply do not invest in the remaining assets. Interestingly, most of the portfolio is invested in the following 5 securities (from the highest contribution to the lowest): HER (+1,8), SRG (+1,21), IP (+1,1), PRY (+1,08) and LDO (+1,06).

Chapter 3.13: Interpretation of the Results

The obtained results are consistent with the pre-analysis of the FTSE MIB index. Indeed, the BMPS and SPM stocks are the worst performing ones due to a combination of external factors and capital stock increases that further diluted the value of investors' existing shares.

Furthermore, the ER of the tangency portfolio is equal to 0.02.

Notice, as these are daily returns, this expected value is very, very large, as it indicates that, on average, over one year, the Tangency portfolio yields a return larger than 79%. On the other hand, the daily St. Dev. is equal to 0,07. Notice, such a value on a daily level is incredibly large, considering that the correspondent yearly value is equal to more than 111%, indicating that the tangent portfolio is particularly risky and indicating that the Milan Bourse is highly volatile.
Ultimately, the Tangency Portfolio (TP) attains the highest Sharpe Ratio value, reaching a level of 0.31.

CONCLUSIONS

In this thesis we both illustrated the theory behind the MPT, also in compliance with its mathematical foundations, and applied such concepts on a real dataset, that is, the benchmark index for the Milan Bourse. Thanks to the latter analysis, it is now possible to state that the Italian Stock Exchange market is highly volatile, considering, on the other hand, that, on average, the TP yields remarkably significant returns. Furthermore, it was possible to observe that the impact of the Covid-19 outbreaks and of the Russo-Ukrainian War negatively affected the overall performances, specifically of those company's shares which belonged to the energy sector.
These events have had far-reaching effects on businesses and financial markets, including the Italian stock market.

1. Covid-19: The pandemic significantly disrupted global supply chains, which had severe implications for Italian companies, especially those in the manufacturing sector that are heavily dependent on these chains. The imposed lockdowns also had a substantial impact on domestic demand as businesses were forced to close and consumer spending declined due to economic uncertainty [12]. Additionally, travel restrictions crippled Italy's tourism industry, which is a crucial sector of the country's economy [15]. The combined impact of these factors led to a contraction of the Italian economy, which in turn affected the performance of Italian stocks.
2. Russo-Ukrainian War: Italy has a close economic relationship with both Russia and Ukraine. The war led to a disruption in trade, which negatively affected Italian companies with significant exposure to these markets [4]. Additionally, the conflict created significant geopolitical uncertainty, which tends to increase market volatility and impact investor sentiment negatively. Consequently, this uncertainty put downward pressure on Italian stocks.

Moreover, the economic instability and uncertainty stemming from these events often lead to increased risk aversion among investors. Risk aversion can result in capital flight from equities to safer assets, leading to selling pressure on stocks and stock market declines [16].
It's important to note that these impacts can vary across sectors and individual companies, depending on their specific exposures to the risks associated with these events. The long-term effects also depend on a range of factors, including the duration and severity of the pandemic, the evolution of the geopolitical conflict, and policy responses at both the national and international levels.

Finally, the exploration of Modern Portfolio Theory (MPT) in this dissertation has provided a robust theoretical foundation for understanding portfolio diversification, risk management, and risk-return tradeoff. The application of MPT's principles to the FTSE MIB index has demonstrated its practical utility and enduring relevance in the financial markets, supporting the assertions of Bodie, Kane, & Marcus (2017) [3].

Through a rigorous application of MPT to the FTSE MIB index, we have illuminated the role of diversification in mitigating risk. We also elucidated the nuances of risk-return tradeoff in a dynamic and complex market environment. The empirical evidence from this study strengthens the claim that strategic portfolio diversification - consistent with the precepts of MPT - can lead to superior risk-adjusted returns, as argued by Fabozzi, Huang, & Zhou (2010) [7].

Critiques of MPT, its extensions, and the emergence of alternative theories such as Post-Modern Portfolio Theory (PMPT) were also thoroughly examined. This discussion reflected the vibrant academic debate in the field of portfolio management and risk analysis. Despite its limitations and criticisms, as highlighted by Michaud (1989) [23], MPT remains an integral part of this discourse.

However, this study underscores the necessity of continuously refining and expanding the MPT framework to better suit the evolving market conditions and the needs of modern investors. Lütje's (2009) study [18], which explores asset managers' attitudes towards herding, suggests the value of integrating behavioral finance insights into the MPT framework. Similarly, the work of Platanakis & Sutcliffe (2017) [25] emphasizes the potential for machine learning techniques to enhance portfolio optimization.

In particular, Platanakis and Sutcliffe suggest that machine learning, with its ability to learn from data and improve its predictions over time, could improve the accuracy and efficiency of risk-return forecasts that form the basis for portfolio optimization.

In the context of portfolio optimization, machine learning can be used in several ways:

1. Predictive Modelling: Machine learning can analyze vast amounts of historical data to identify patterns and trends that could forecast future price movements. These predictive models could be utilized to enhance the selection of assets for portfolio optimization [2].
2. Incorporating Non-Linear Relationships: Traditional models often assume linear relationships between variables, which may not always hold true. Machine learning techniques, such as neural networks and decision trees, can uncover complex, non-linear relationships within data, potentially providing more accurate predictions [1].
3. Mitigating Overfitting: Overfitting is a common problem in portfolio optimization where models fit too closely to historical data and fail to generalize to unseen data. Machine learning can employ techniques such as regularization and cross-validation to mitigate this issue [14].
4. Handling High Dimensionality: Machine learning is adept at managing high-dimensional data, which is often the case with financial data where we have many assets and numerous potential predictors. This can improve the robustness of portfolio optimization [25].

The potential of machine learning to enhance portfolio optimization is significant;

however, it's crucial to note that these techniques also come with challenges. Machine learning models can be complex and lack transparency, making them difficult to interpret – a problem often referred to as the 'black box' issue. Additionally, while machine learning can learn and adapt from data, they are still dependent on the quality of the input data.

In closing, this thesis reaffirms the enduring relevance of Harry Markowitz's pioneering work, while also acknowledging the need for its ongoing evolution.

Markowitz's principles provide both a starting point and a guiding light as we continue to explore the vast and complex terrain of portfolio management.

APPENDIX

Introduction – The FTSE MIB IT40 Index

BMPS Data:

```
import pandas as pd
import yfinance as yf
import matplotlib.pyplot as plt

# Download stock data for BMPS
btps = yf.download('BMPS.MI', start='2022-01-01', end='2023-12-31')

# Create a figure and axes
fig, ax = plt.subplots(figsize=(12, 8))

# Plot the stock prices
btps['Close'].plot(ax=ax, color='gray')

# Highlight October 2022 and February 2023 values
ax.axvline(pd.to_datetime('2022-10-01'), color='red', linestyle='--', label='October 2022')
ax.axvline(pd.to_datetime('2023-02-01'), color='blue', linestyle='--', label='February 2023')

# Add legend
ax.legend(loc='upper left')

# Set axis labels
ax.set_xlabel('Date')
ax.set_ylabel('Stock Price')

# Set plot title
plt.title('Stock Prices of BMPS')

# Show the plot
plt.show()
```

SPM Data:

```
import pandas as pd
import yfinance as yf
import matplotlib.pyplot as plt

# Download stock data for SPM
spm = yf.download('SPM.MI', start='2019-06-01', end='2023-02-01')

# Create a figure and axes
fig, ax = plt.subplots(figsize=(12, 8))

# Plot the stock prices
spm['Close'].plot(ax=ax, color='gray')

# Highlight March 2020 and July 2022 values
ax.axvline(pd.to_datetime('2020-03-01'), color='red', linestyle='--', label='March 2020')
ax.axvline(pd.to_datetime('2022-06-25'), color='blue', linestyle='--', label='July 2022')
```

```python
# Add legend
ax.legend(loc='upper left')

# Set axis labels
ax.set_xlabel('Date')
ax.set_ylabel('Stock Price')

# Set plot title
plt.title('Stock Prices of SPM')

# Show the plot
plt.show()
```

2.1 – Gathering the Data:

```python
import yfinance as yf # Package for downloading the Data from YahooFinance

# Define the ticker symbols for stocks on the Italian stock exchange (.MI)
ftse_mib_it40 = ["A2A.MI", "AMP.MI", "G.MI", "AZM.MI", "BGN.MI", "BMED.MI",
"BAMI.MI", "BMPS.MI", "BPE.MI", "CPR.MI", "CNHI.MI", "DIA.MI", "ENEL.MI",
"ENI.MI", "ERG.MI", "RACE.MI", "FBK.MI", "HER.MI", "IP.MI", "ISP.MI",
"INW.MI", "IG.MI", "IVG.MI", "LDO.MI", "MB.MI", "MONC.MI", "NEXI.MI",
"PIRC.MI", "PST.MI", "PRY.MI", "REC.MI", "SPM.MI", "SRG.MI", "STLAM.MI",
"STMMI.MI", "TIT.MI", "TEN.MI", "TRN.MI", "UCG.MI", "UNI.MI"]

# Download the historical stock data for each stock
stock_data = yf.download(ftse_mib_it40, start="2021-01-01", end="2023-04-12", group_by='ticker',
interval="1d")

# Display the downloaded data for each stock
for stock in ftse_mib_it40:
    print("Stock:", stock)
    print(stock_data[stock])
    print()
```

2.2 – Prices and Daily Returns:

```python
import numpy as np
import yfinance as yf

# Define the ticker symbols for stocks on the Italian stock exchange (.MI)
ftse_mib_it40 = ["A2A.MI", "AMP.MI", "G.MI", "AZM.MI", "BGN.MI", "BMED.MI",
"BAMI.MI", "BMPS.MI", "BPE.MI", "CPR.MI", "CNHI.MI", "DIA.MI", "ENEL.MI",
"ENI.MI", "ERG.MI", "RACE.MI", "FBK.MI", "HER.MI", "IP.MI", "ISP.MI",
"INW.MI", "IG.MI", "IVG.MI", "LDO.MI", "MB.MI", "MONC.MI", "NEXI.MI",
"PIRC.MI", "PST.MI", "PRY.MI", "REC.MI", "SPM.MI", "SRG.MI", "STLAM.MI",
"STMMI.MI", "TIT.MI", "TEN.MI", "TRN.MI", "UCG.MI", "UNI.MI"]

# Download the historical stock data for each stock
stock_data = yf.download(ftse_mib_it40, start="2021-01-01", end="2023-04-12", group_by='ticker',
interval="1d")

# Consider only the adj closing prices: 'Adj Close' column
close_prices = stock_data.iloc[:, stock_data.columns.get_level_values(1)=='Adj Close']
```

```
# Calculate log returns for each stock
log_returns = np.log(close_prices / close_prices.shift(1))

# Remove the first row containing NaN values
log_returns = log_returns.dropna()

# Display the downloaded data for each stock
for stock in ftse_mib_it40:
    print("Stock:", stock)
    print(log_returns[stock])
    print()
```

2.3 – The Covariance Matrix:

```
import numpy as np
import yfinance as yf
import seaborn as sns # Package for the colorbar and data visualisation
import matplotlib.pyplot as plt

# Define the ticker symbols for stocks on the Italian stock exchange (.MI)
ftse_mib_it40 = ["A2A.MI", "AMP.MI", "G.MI", "AZM.MI", "BGN.MI", "BMED.MI",
"BAMI.MI", "BMPS.MI", "BPE.MI", "CPR.MI", "CNHI.MI", "DIA.MI", "ENEL.MI",
"ENI.MI", "ERG.MI", "RACE.MI", "FBK.MI", "HER.MI", "IP.MI", "ISP.MI",
"INW.MI", "IG.MI", "IVG.MI", "LDO.MI", "MB.MI", "MONC.MI", "NEXI.MI",
"PIRC.MI", "PST.MI", "PRY.MI", "REC.MI", "SPM.MI", "SRG.MI", "STLAM.MI",
"STMMI.MI", "TIT.MI", "TEN.MI", "TRN.MI", "UCG.MI", "UNI.MI"]

# Download the historical stock data for each stock
stock_data = yf.download(ftse_mib_it40, start="2021-01-01", end="2023-04-12", group_by='ticker',
interval="1d")

# Consider only the adj closing prices: 'Adj Close' column
close_prices = stock_data.iloc[:, stock_data.columns.get_level_values(1)=='Adj Close']

# Calculate log returns for each stock
log_returns = np.log(close_prices / close_prices.shift(1))

# Remove the first row containing NaN values
log_returns = log_returns.dropna()

# Order the columns by the given list
log_returns = log_returns[ftse_mib_it40]

# Calculate the variance-covariance matrix
covariance_matrix = log_returns.cov()

# Plot the variance-covariance matrix
plt.figure(figsize=(10, 8))

# Set the Palette
cmap = sns.diverging_palette(220, 10, as_cmap=True, s=95, l=50, center='light')

# Hide the repeating symmetric triangle
mask = np.triu(np.ones_like(covariance_matrix, dtype=bool))
covariance_matrix_masked = covariance_matrix.mask(mask)

plt.imshow(covariance_matrix_masked, cmap=cmap)
```

```
cbar = plt.colorbar()
cbar.set_label('Covariance')

plt.xticks(range(len(ftse_mib_it40)), ftse_mib_it40, rotation=90, fontsize=8)
plt.yticks(range(len(ftse_mib_it40)), ftse_mib_it40, fontsize=8)
plt.xlabel('')
plt.ylabel('')

plt.title('Covariance Matrix', fontsize=12)
plt.show()
```

2.4 – The Correlation Matrix:

```
import numpy as np
import yfinance as yf
import seaborn as sns
import matplotlib.pyplot as plt

# Define the ticker symbols for stocks on the Italian stock exchange (.MI)
ftse_mib_it40 = ["A2A.MI", "AMP.MI", "G.MI", "AZM.MI", "BGN.MI", "BMED.MI",
"BAMI.MI", "BMPS.MI", "BPE.MI", "CPR.MI", "CNHI.MI", "DIA.MI", "ENEL.MI",
"ENI.MI", "ERG.MI", "RACE.MI", "FBK.MI", "HER.MI", "IP.MI", "ISP.MI",
"INW.MI", "IG.MI", "IVG.MI", "LDO.MI", "MB.MI", "MONC.MI", "NEXI.MI",
"PIRC.MI", "PST.MI", "PRY.MI", "REC.MI", "SPM.MI", "SRG.MI", "STLAM.MI",
"STMMI.MI", "TIT.MI", "TEN.MI", "TRN.MI", "UCG.MI", "UNI.MI"]
# Download the historical stock data for each stock
stock_data = yf.download(ftse_mib_it40, start="2021-01-01", end="2023-04-12", group_by='ticker',
interval="1d")
# Consider only the adj closing prices: 'Adj Close' column
close_prices = stock_data.iloc[:, stock_data.columns.get_level_values(1)=='Adj Close']
# Calculate log returns for each stock
log_returns = np.log(close_prices / close_prices.shift(1))
# Remove the first row containing NaN values
log_returns = log_returns.dropna()
# Order the columns by the given list
log_returns = log_returns[ftse_mib_it40]
# Calculate the correlation matrix
correlation_matrix = log_returns.corr()
# Plot the correlation matrix
f, ax = plt.subplots(figsize=(10, 8))
# Set the colours with their Hex code
colors = ['#000000', '#FFFFCC', '#E5FFCC', '#CCFFCC', '#CCFFE5', '#CCFFFF', '#CCE5FF',
'#CCCCFF', '#660000']
# Set the heatmap
sns.heatmap(correlation_matrix, cmap=colors, annot=False, linewidths=0.5, square=True,
cbar=True, ax=ax, vmin=-1, vmax=1)
# Show the stocks' tickers on the axes
ax.set_xticklabels(ftse_mib_it40, rotation=45, fontsize=8, fontweight='bold')
ax.set_yticklabels(ftse_mib_it40, fontsize=8, fontweight='bold')
# Set the title and remove the labels for the axes
plt.title('Correlation Matrix', fontsize=12)
ax.set_xlabel(None)
ax.set_ylabel(None)
# Show the plot
plt.show()
```

2.5 – Visualizing the MIB-40 Stocks in a Risk and Return graph (all the stocks):

```
import numpy as np
import yfinance as yf
import matplotlib.pyplot as plt

# Define the ticker symbols for stocks on the Italian stock exchange (.MI)
ftse_mib_it40 = ["A2A.MI", "AMP.MI", "G.MI", "AZM.MI", "BGN.MI", "BMED.MI",
"BAMI.MI", "BMPS.MI", "BPE.MI", "CPR.MI", "CNHI.MI", "DIA.MI", "ENEL.MI",
"ENI.MI", "ERG.MI", "RACE.MI", "FBK.MI", "HER.MI", "IP.MI", "ISP.MI",
"INW.MI", "IG.MI", "IVG.MI", "LDO.MI", "MB.MI", "MONC.MI", "NEXI.MI",
"PIRC.MI", "PST.MI", "PRY.MI", "REC.MI", "SPM.MI", "SRG.MI", "STLAM.MI",
"STMMI.MI", "TIT.MI", "TEN.MI", "TRN.MI", "UCG.MI", "UNI.MI"]

# Download the historical stock data for each stock
stock_data = yf.download(ftse_mib_it40, start="2021-01-01", end="2023-04-12", group_by='ticker',
interval="1d")

# Consider only the adj closing prices: 'Adj Close' column
close_prices = stock_data.iloc[:, stock_data.columns.get_level_values(1)=='Adj Close']

# Calculate log returns for each stock
log_returns = np.log(close_prices / close_prices.shift(1))

# Remove the first row containing NaN values
log_returns = log_returns.dropna()

# Calculate the expected returns of individual stocks
expected_returns = log_returns.mean()

# Calculate the standard deviation of portfolio returns
portfolio_std = np.sqrt(np.diag(log_returns.cov()))

filtered_std = portfolio_std
filtered_expected_returns = expected_returns
filtered_tickers = close_prices.columns.get_level_values(1)


# Plot the filtered MIB 40 stocks as dots
plt.figure(figsize=(10, 6))
plt.scatter(filtered_std, filtered_expected_returns, marker='o', s=50)
plt.xlabel('Standard Deviation of Portfolio Return, σp')
plt.ylabel('Expected Portfolio Return, Ep')
plt.title('Risk and Return for MIB-40 Stocks')

# Extract the stock symbols from the multi-indexed columns
stock_symbols = [col[0] for col in close_prices.columns]

# Create a list of stock names corresponding to the stock symbols
stock_names = [symbol.replace('.MI', '') for symbol in stock_symbols]

# Add labels for each stock
for i, (std, exp_ret, stock_name) in enumerate(zip(filtered_std, filtered_expected_returns,
stock_names)):
    plt.annotate(stock_name, (std, exp_ret), xytext=(5, -5), textcoords='offset points',
ha='left', va='bottom', fontsize=8)
```

```
plt.grid(True)
plt.tight_layout()
plt.show()
```

2.5 – Visualizing the MIB-40 Stocks in a Risk and Return graph (removing BMPS and SPM):

```
import numpy as np
import yfinance as yf
import matplotlib.pyplot as plt

# Define the ticker symbols for stocks on the Italian stock exchange (.MI)
ftse_mib_it40 = ["A2A.MI", "AMP.MI", "G.MI", "AZM.MI", "BGN.MI", "BMED.MI",
"BAMI.MI", "BMPS.MI", "BPE.MI", "CPR.MI", "CNHI.MI", "DIA.MI", "ENEL.MI",
"ENI.MI", "ERG.MI", "RACE.MI", "FBK.MI", "HER.MI", "IP.MI", "ISP.MI",
"INW.MI", "IG.MI", "IVG.MI", "LDO.MI", "MB.MI", "MONC.MI", "NEXI.MI",
"PIRC.MI", "PST.MI", "PRY.MI", "REC.MI", "SPM.MI", "SRG.MI", "STLAM.MI",
"STMMI.MI", "TIT.MI", "TEN.MI", "TRN.MI", "UCG.MI", "UNI.MI"]

# Download the historical stock data for each stock
stock_data = yf.download(ftse_mib_it40, start="2021-01-01", end="2023-04-12", group_by='ticker',
interval="1d")

# Consider only the adj closing prices: 'Adj Close' column
close_prices = stock_data.iloc[:, stock_data.columns.get_level_values(1)=='Adj Close']

# Calculate log returns for each stock
log_returns = np.log(close_prices / close_prices.shift(1))

# Remove the first row containing NaN values
log_returns = log_returns.dropna()

# Calculate the expected returns of individual stocks
expected_returns = log_returns.mean()

# Calculate the standard deviation of portfolio returns
portfolio_std = np.sqrt(np.diag(log_returns.cov()))

filtered_std = portfolio_std
filtered_expected_returns = expected_returns
filtered_tickers = close_prices.columns.get_level_values(1)  # Update this line

# Set the plot limits based on specified ranges
x_min, x_max = 0.01, 0.04
y_min, y_max = -0.003, 0.002

# Plot the filtered MIB 40 stocks as dots
plt.figure(figsize=(10, 6))
plt.scatter(filtered_std, filtered_expected_returns, marker='o', s=50)
plt.xlabel('Standard Deviation of Portfolio Return')
plt.ylabel('Expected Portfolio Return')
plt.title('Risk and Return for MIB-40 Stocks (Removing BMPS and SPM)')
plt.xlim(x_min, x_max)  # Set x-axis limits
plt.ylim(y_min, y_max)  # Set y-axis limits

# Extract the stock symbols from the multi-indexed columns
```

```
stock_symbols = [col[0] for col in close_prices.columns]

# Create a list of stock names corresponding to the stock symbols
stock_names = [symbol.replace('.MI', '') for symbol in stock_symbols]

# Add labels for each stock
for i, (std, exp_ret, stock_name) in enumerate(zip(filtered_std, filtered_expected_returns, 
stock_names)):
    plt.annotate(stock_name, (std, exp_ret), xytext=(5, -5), textcoords='offset points', 
ha='left', va='bottom', fontsize=8)

plt.grid(True)
plt.tight_layout()
plt.show()
```

2.6 – Plotting the Portfolio Frontier:

```
import numpy as np
import yfinance as yf
import matplotlib.pyplot as plt

# Define the ticker symbols for stocks on the Italian stock exchange (.MI)
ftse_mib_it40 = ["A2A.MI", "AMP.MI", "G.MI", "AZM.MI", "BGN.MI", "BMED.MI", 
"BAMI.MI", "BMPS.MI", "BPE.MI", "CPR.MI", "CNHI.MI", "DIA.MI", "ENEL.MI", 
"ENI.MI", "ERG.MI", "RACE.MI", "FBK.MI", "HER.MI", "IP.MI", "ISP.MI", 
"INW.MI", "IG.MI", "IVG.MI", "LDO.MI", "MB.MI", "MONC.MI", "NEXI.MI", 
"PIRC.MI", "PST.MI", "PRY.MI", "REC.MI", "SPM.MI", "SRG.MI", "STLAM.MI", 
"STMMI.MI", "TIT.MI", "TEN.MI", "TRN.MI", "UCG.MI", "UNI.MI"]

# Download the historical stock data for each stock
stock_data = yf.download(ftse_mib_it40, start="2021-01-01", end="2023-04-12", group_by='ticker', 
interval="1d")

# Consider only the adj closing prices: 'Adj Close' column
close_prices = stock_data.iloc[:, stock_data.columns.get_level_values(1)=='Adj Close']

# Calculate log returns for each stock
log_returns = np.log(close_prices / close_prices.shift(1))

# Remove the first row containing NaN values
log_returns = log_returns.dropna()

# We'll start with an equally weighted portfolio and then proceed to find the optimal portfolio 
weights.

# Step 1: Calculate the expected returns of individual stocks

# Formula:

# ee = E[R]
# where ee represents the expected returns and E[R] is the mean return.

# Code:

# python
expected_returns = log_returns.mean()
```

```python
# Step 3: Calculate the covariance matrix of returns

# Formula:

# VV = cov(R)
# where VV represents the covariance matrix and cov(R) is the covariance matrix of returns.

# Code:

# python
covariance_matrix = log_returns.cov()

# Calculate the expected returns of individual stocks
expected_returns = log_returns.mean()

# Calculate the covariance matrix of returns
covariance_matrix = log_returns.cov()

# Convert expected returns to numpy array
ee = np.array(expected_returns)

# Calculate inverse covariance matrix
invVV = np.linalg.inv(covariance_matrix)

# Create a vector of ones
one1 = np.ones(len(ee))

# Calculate variables A, B, C, D
A = np.dot(np.dot(one1.T, invVV), ee)
B = np.dot(np.dot(ee.T, invVV), ee)
C = np.dot(np.dot(one1.T, invVV), one1)
D = B * C - A**2

# Calculate gg and hh
gg = (1 / D) * (B * np.dot(invVV, one1) - A * np.dot(invVV, ee))
hh = (1 / D) * (C * np.dot(invVV, ee) - A * np.dot(invVV, one1))

# Calculate the minimum variance portfolio weights
wmvp = gg + hh * (A / C)

# Calculate the minimum variance portfolio mean return
emvp = np.dot(wmvp.T, ee)

# Calculate the variance of return on minimum variance portfolio
vmvp = np.dot(np.dot(wmvp.T, covariance_matrix), wmvp)

# Calculate the standard deviation of minimum variance portfolio
sdmvp = np.sqrt(vmvp)

# Generate data for plotting
ssigma = np.arange(1 / C, abs(10 * (1 / C)), abs((1 / C) / 50))
effeplus = A / C + np.sqrt(D * (C * ssigma - 1)) / C  # Upper branch of portfolio frontier
effepmin = A / C - np.sqrt(D * (C * ssigma - 1)) / C  # Lower branch of portfolio frontier
```

```
x2 = np.arange(0, np.sqrt(1 / C), np.sqrt(1 / C) / 10)  # Values for standard deviation of
portfolio returns
limsup = A / C + np.sqrt(D / C) * x2  # Upper limit
liminf = A / C - np.sqrt(D / C) * x2  # Lower limit
x1 = np.sqrt(ssigma)
y1 = effeplus
y2 = effepmin
l1 = limsup
l2 = liminf

# Plotting
plt.plot(x1, y1, '-k', x1, y2, '-k', linewidth=3)
plt.plot(x2, l1, '--b', x2, l2, '--b', linewidth=1)
plt.gca().set_prop_cycle('color', ['magenta'])
plt.gcf().set_facecolor('white')
plt.axis([0, 0.04, -0.02, 0.02])
plt.title('Portfolio Frontier for the MIB-40', fontname='Arial', fontsize=14)
plt.xlabel('Standard Deviation of Portfolio Return, σp', fontname='Arial', fontsize=14)
plt.ylabel('Expected Portfolio Return, Ep', fontname='Arial', fontsize=14)
plt.scatter(sdmvp, emvp, color='r', marker='o', label='Minimum Variance Portfolio')
plt.legend(loc='lower right')

# Calculate the endpoint values for the asymptotes
x_end = np.sqrt(10 / C)
y_end_1 = A / C + np.sqrt(D / C) * x_end
y_end_2 = A / C - np.sqrt(D / C) * x_end

# Add asymptotes
plt.plot([0, x_end], [A / C, y_end_1], '--b', linewidth=1, clip_on=False)
plt.plot([0, x_end], [A / C, y_end_2], '--b', linewidth=1, clip_on=False)

plt.show()
```

2.7 – Plotting the Capital Market Line (CML):

```
import numpy as np
import yfinance as yf
import matplotlib.pyplot as plt

# Define the ticker symbols for stocks on the Italian stock exchange (.MI)
ftse_mib_it40 = ["A2A.MI", "AMP.MI", "G.MI", "AZM.MI", "BGN.MI", "BMED.MI", "BAMI.MI",
"BMPS.MI", "BPE.MI", "CPR.MI", "CNHI.MI", "DIA.MI", "ENEL.MI", "ENI.MI", "ERG.MI", "RACE.MI",
"FBK.MI", "HER.MI", "IP.MI", "ISP.MI", "INW.MI", "IG.MI", "IVG.MI", "LDO.MI", "MB.MI",
"MONC.MI", "NEXI.MI", "PIRC.MI", "PST.MI", "PRY.MI", "REC.MI", "SPM.MI", "SRG.MI", "STLAM.MI",
"STMMI.MI", "TIT.MI", "TEN.MI", "TRN.MI", "UCG.MI", "UNI.MI"]

# Download the historical stock data for each stock
stock_data = yf.download(ftse_mib_it40, start="2021-01-01", end="2023-04-12", group_by='ticker',
interval="1d")

# Consider only the adj closing prices: 'Adj Close' column
close_prices = stock_data.iloc[:, stock_data.columns.get_level_values(1)=='Adj Close']

# Calculate log returns for each stock
log_returns = np.log(close_prices / close_prices.shift(1))
```

```
# Remove the first row containing NaN values
log_returns = log_returns.dropna()

# Calculate the expected returns and covariance matrix of individual stocks
expected_returns = log_returns.mean()
covariance_matrix = log_returns.cov()

# Convert expected returns to numpy array
ee = np.array(expected_returns)

# Calculate inverse covariance matrix
invVV = np.linalg.inv(covariance_matrix)

# Create a vector of ones
one1 = np.ones(len(ee))

# Calculate variables A, B, C, D
A = np.dot(np.dot(one1.T, invVV), ee)
B = np.dot(np.dot(ee.T, invVV), ee)
C = np.dot(np.dot(one1.T, invVV), one1)
D = B * C - A**2

# Calculate gg and hh
gg = (1 / D) * (B * np.dot(invVV, one1) - A * np.dot(invVV, ee))
hh = (1 / D) * (C * np.dot(invVV, ee) - A * np.dot(invVV, one1))

# Calculate the minimum variance portfolio weights
wmvp = gg + hh * (A / C)

# Calculate the minimum variance portfolio mean return
emvp = np.dot(wmvp.T, ee)

# Calculate the variance of return on minimum variance portfolio
vmvp = np.dot(np.dot(wmvp.T, covariance_matrix), wmvp)

# Calculate the standard deviation of minimum variance portfolio
sdmvp = np.sqrt(vmvp)

# Define the risk-free rate
annual_rate = 0.05028  # The annual risk-free rate
daily_rate = (1 + annual_rate)**(1/252) - 1
rf = daily_rate

# Calculate the tangent portfolio weights
wtan = gg + hh * ((B / A) - rf)

# Calculate the tangent portfolio mean return
etan = np.dot(wtan.T, ee)

# Calculate the variance of return on the tangent portfolio
vtan = np.dot(np.dot(wtan.T, covariance_matrix), wtan)

# Calculate the standard deviation of the tangent portfolio
sdtan = np.sqrt(vtan)
```

```python
# Generate the data for the Capital Market Line (CML)
x = np.linspace(0, sdtan*2, 100)
y_cml = rf + (x * (etan - rf) / sdtan)

# Generate the data for the efficient frontier
std = np.linspace(0, max(x), len(x))
ef_returns = np.zeros(len(std))
for i in range(len(std)):
    under_sqrt = A**2 * std[i]**4 - (std[i]**2 - B) * (std[i]**2 - C * A) * D
    if under_sqrt >= 0:
        ef_returns[i] = rf + (std[i] * (etan - rf) / sdtan) if std[i] > sdtan else ((A *
std[i]**2 + np.sqrt(under_sqrt)) / (std[i]**2 - C * A))
    else:
        ef_returns[i] = None

# Create a new figure
plt.figure(figsize=(12,8))

# Plot the individual stocks
plt.scatter(np.sqrt(np.diag(covariance_matrix)), expected_returns, c='lightblue', marker='o',
label='Stocks')

# Plot the risk-free rate
plt.scatter(0, rf, c='blue', marker='o', label='Risk-Free Rate')

# Plot the minimum variance portfolio
plt.scatter(sdmvp, emvp, c='red', marker='o', label='Minimum Variance Portfolio')

# Plot the tangent portfolio
plt.scatter(sdtan, etan, c='green', marker='o', label='Tangency Portfolio')

# Plot the CML
plt.plot(x, y_cml, c='black', linestyle='-', label='Capital Market Line (CML)')

# Plot the Portfolio Frontier
x1 = np.linspace(0, max(x), 100)
y1 = A / C + np.sqrt(D * (C * x1**2 - 1)) / C  # Upper branch of portfolio frontier
y2 = A / C - np.sqrt(D * (C * x1**2 - 1)) / C  # Lower branch of portfolio frontier
plt.plot(x1, y1, color='darkslategray', linewidth=3, label='Portfolio Frontier')
plt.plot(x1, y2, color='darkslategray', linewidth=3)

# Set the title and labels
plt.title('CML and Portfolio Frontier for the MIB-40', fontname='Arial', fontsize=14)
plt.xlabel('Standard Deviation of Portfolio Return, σp', fontname='Arial', fontsize=14)
plt.ylabel('Expected Portfolio Return, Ep', fontname='Arial', fontsize=14)

# Set the limits for the x-axis
plt.xlim(left=0)

# Add a legend
plt.legend(loc='upper left')

# Show the plot
plt.show()
```

2.8 – The Composition of the Tangency and Minimum Variance Portfolios:

```
import numpy as np
import pandas as pd
import yfinance as yf
import matplotlib.pyplot as plt

# Define the ticker symbols for stocks on the Italian stock exchange (.MI)
ftse_mib_it40 = ["A2A.MI", "AMP.MI", "G.MI", "AZM.MI", "BGN.MI", "BMED.MI", "BAMI.MI",
"BMPS.MI", "BPE.MI", "CPR.MI", "CNHI.MI", "DIA.MI", "ENEL.MI", "ENI.MI", "ERG.MI", "RACE.MI",
"FBK.MI", "HER.MI", "IP.MI", "ISP.MI", "INW.MI", "IG.MI", "IVG.MI", "LDO.MI", "MB.MI",
"MONC.MI", "NEXI.MI", "PIRC.MI", "PST.MI", "PRY.MI", "REC.MI", "SPM.MI", "SRG.MI", "STLAM.MI",
"STMMI.MI", "TIT.MI", "TEN.MI", "TRN.MI", "UCG.MI", "UNI.MI"]

# Download the historical stock data for each stock
stock_data = yf.download(ftse_mib_it40, start="2021-01-01", end="2023-04-12", group_by='ticker',
interval="1d")

# Consider only the adj closing prices: 'Adj Close' column
close_prices = stock_data.iloc[:, stock_data.columns.get_level_values(1)=='Adj Close']

# Calculate log returns for each stock
log_returns = np.log(close_prices / close_prices.shift(1))

# Remove the first row containing NaN values
log_returns = log_returns.dropna()

# Calculate the expected returns and covariance matrix of individual stocks
expected_returns = log_returns.mean()
covariance_matrix = log_returns.cov()

# Convert expected returns to numpy array
ee = np.array(expected_returns)

# Calculate inverse covariance matrix
invVV = np.linalg.inv(covariance_matrix)

# Create a vector of ones
one1 = np.ones(len(ee))

# Calculate variables A, B, C, D
A = np.dot(np.dot(one1.T, invVV), ee)
B = np.dot(np.dot(ee.T, invVV), ee)
C = np.dot(np.dot(one1.T, invVV), one1)
D = B * C - A**2

# Calculate gg and hh
gg = (1 / D) * (B * np.dot(invVV, one1) - A * np.dot(invVV, ee))
hh = (1 / D) * (C * np.dot(invVV, ee) - A * np.dot(invVV, one1))

# Calculate the minimum variance portfolio weights
wmvp = gg + hh * (A / C)

# Calculate the minimum variance portfolio mean return
emvp = np.dot(wmvp.T, ee)
```

```python
# Calculate the variance of return on minimum variance portfolio
vmvp = np.dot(np.dot(wmvp.T, covariance_matrix), wmvp)

# Calculate the standard deviation of minimum variance portfolio
sdmvp = np.sqrt(vmvp)

# Define the risk-free rate
annual_rate = 0.05028  # The annual risk-free rate
daily_rate = (1 + annual_rate)**(1/252) - 1
rf = daily_rate

# Calculate the tangent portfolio weights
wtan = gg + hh * ((B / A) - rf)

# Calculate the tangent portfolio mean return
etan = np.dot(wtan.T, ee)

# Calculate the variance of return on the tangent portfolio
vtan = np.dot(np.dot(wtan.T, covariance_matrix), wtan)

# Calculate the standard deviation of the tangent portfolio
sdtan = np.sqrt(vtan)

# Compute the Sharpe ratio for the Tangency Portfolio
sharpe_ratio_tan = (etan - rf) / sdtan

# Create a DataFrame for the MVP
df_mvp = pd.DataFrame(wmvp, index=ftse_mib_it40, columns=['MVP Weights'])
df_mvp.loc['Expected Return'] = emvp
df_mvp.loc['Variance'] = vmvp
df_mvp.loc['Standard Deviation'] = sdmvp

# Create a DataFrame for the Tangency Portfolio
df_tan = pd.DataFrame(wtan, index=ftse_mib_it40, columns=['Tangency Portfolio Weights'])
df_tan.loc['Expected Return'] = etan
df_tan.loc['Variance'] = vtan
df_tan.loc['Standard Deviation'] = sdtan
df_tan.loc['Sharpe Ratio'] = sharpe_ratio_tan

# Concatenate the two DataFrames
df = pd.concat([df_mvp, df_tan], axis=1)

# Rounding up to have only 2 decimals
df = df.round(2)

# Print the DataFrame
print(df)

fig, ax = plt.subplots(figsize=(12, 8)) # set size frame
ax.axis('tight')
ax.axis('off')
ax.table(cellText=df.values, colLabels=df.columns, rowLabels = df.index, cellLoc = 'center',
loc='center')

plt.savefig('df.png')
```

The Covariance Matrix

We obtain the maximum covariance with a value of 0.03378 and the minimum covariance with a value of -0.00013.

Furthermore, to properly visualize the covariance and variance's values in a single plot with a colorbar, such values must be normalized since the latter are higher than the former. The resulting plot does not show the real values of the dataset; however, such a matrix clearly depicts the relationship between the stock's covariances and variances.

Finally, we only show the lower triangle since the variance-covariance matrix is symmetric, that is, the values below the main diagonal (representing the variances of the stocks as the covariance of that stock with itself corresponds, namely, to its variance) are the same as those above.

Therefore, the Normalized Covariance Matrix is the following:

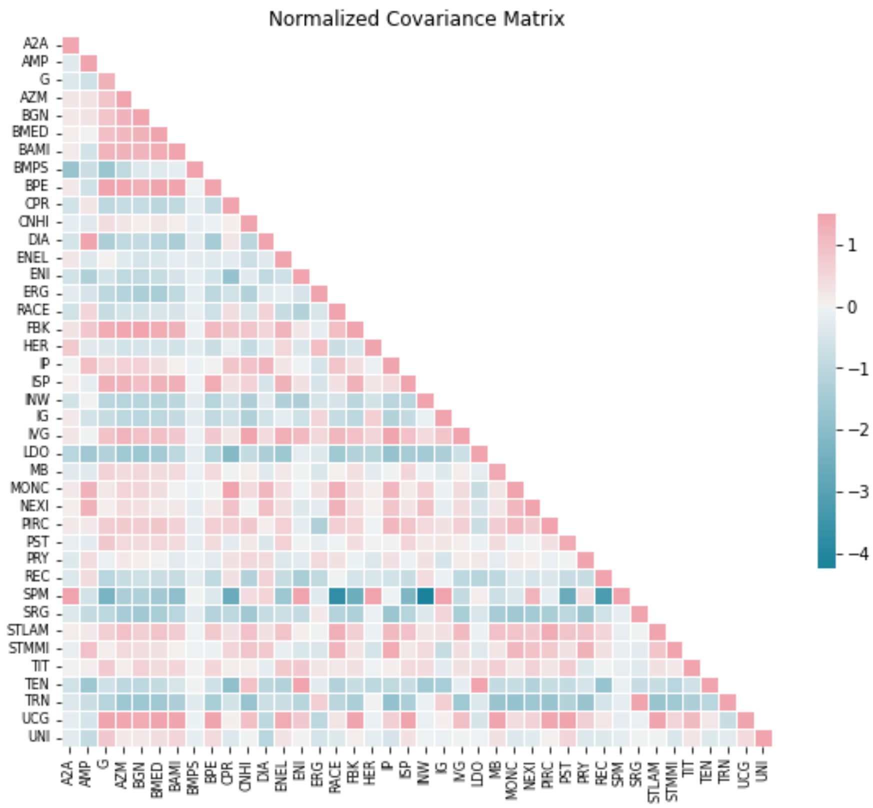

Figure 6: The Normalized Covariance Matrix

Indeed, we can also plot the Covariance Matrix by removing the main diagonal so eliminating the variances' values that will impede the interpretation of our matrix due to the consequent variation in the scale of the colorbar values on the side.

The following plot will represent the true covariances' metrics of our stocks.

Therefore, the Covariance Matrix is the following:

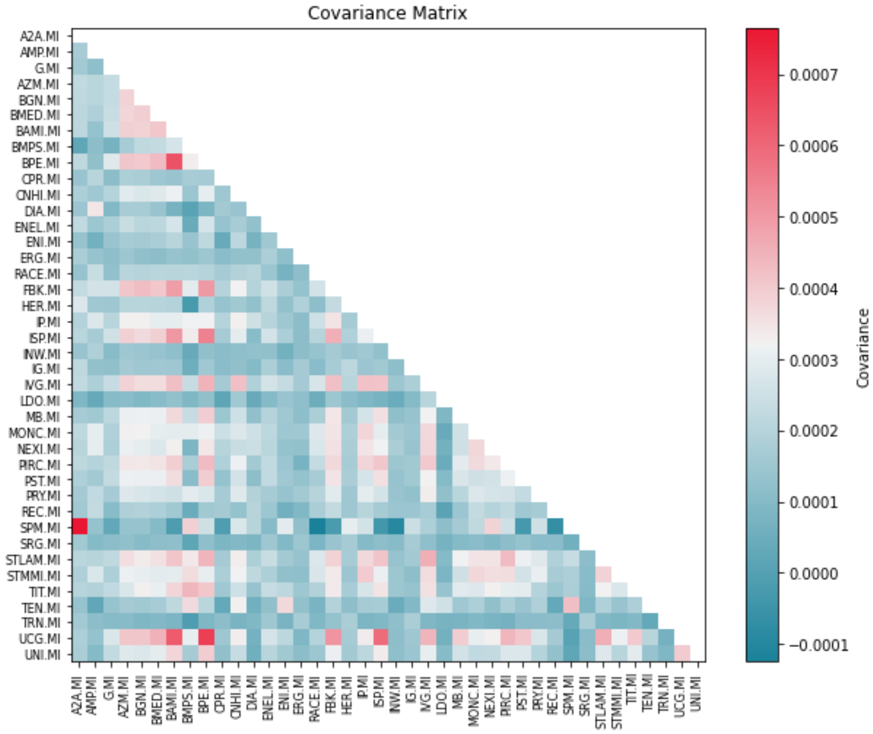

Figure 7: The Covariance Matrix (Removing the main diagonal – Variance)

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