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Course of Mathematical Finance

“Quantitative Methods for Option Pricing and Financial Modeling: Black-Scholes Model, Monte Carlo Simulation, and Lévy Processes”

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Ai miei fratelli, i miei genitori e mia nonna.

INDEX

Introduction *1*

Chapter 1: Introduction to Stochastic Calculus *2*

1.1.1 σ-algebra Definition 2

1.1.2 Probability Space Definition 2

1.1.3 Stochastic Process Definition 3

1.1.4 Martingale Definition 4

1.1.5 Markov Chain Definition 5

1.1.6 Random Walk Definition 5

1.2 Brownian Motion **6**

1.2.1 Brownian Motion Definition 6

1.2.2 Brownian Motion Properties 7

Chapter 2: Black Scholes Model *12*

2.1 Introduction to the model **12**

2.2 Assumptions **12**

2.3 Geometric Brownian Motion **14**

2.4 The Black-Scholes Differential Equation **16**

2.5 Risk Neutral Evaluation **18**

2.6 The Black-Scholes Formula for Option Pricing **19**

2.6.1 Black Scholes model implications 20

2.7 Volatility Estimation **21**

2.7.1 Historical Volatility Estimation 22

2.7.2 Implied Volatility Estimation 23

2.8 Drawbacks and Limitations of the Black-Scholes Model **24**

2.8.1 Lognormal Distribution and Stock Market 24

2.8.2 Constant Volatility Assumption 25

2.8.3 Volatility Smile 25

Chapter 3: Monte Carlo Simulation *27*

3.1 Monte Carlo Method **27**

3.2 Pseudorandom Sequences **28**

3.3 Geometric Brownian Motion for stock price simulation **28**

3.4 Monte Carlo simulation for option pricing on Excel **30**

3.5 Black-Scholes option pricing simulation on R-studio **34**

Chapter 4: Poisson Process and Lévy Process *38*

4.1 Poisson Process **38**

4.1.1 Poisson Distribution 38

4.1.2 Poisson Process Definition 39

4.1.3 Compensated Poisson Process 41

4.2 Compound Poisson Process **42**

4.2.1 Compound Poisson Process Definition and Properties 42

4.2.2 Convergence to the Normal Distribution 44

4.3 Lévy Process **45**

4.3.1 Introduction to Lévy Processes 45

4.3.2 Stable Distribution 46

4.3.3 Lévy Distribution 46

4.3.2 Lévy Process Definition 47

4.4 The Merton Jump Diffusion Model **48**

4.4.1 Introduction to the model 48

4.4.2 Hedging Strategies in Incomplete Markets 49

4.4.3 Formula derivation 50

Conclusion *56*

Bibliography *57*

LIST OF FIGURES

Figure 1: Symmetric Random Walk, Fair Coin Tossing 6

Figure 2: Comparison between Brownian Motion and Random Walk (Holton, 2013)............7

Figure 3: Lognormal Vs Normal Distribution (Ma, 2015) 13

Figure 4: Sample GBM values chart (Bhatia, 2016)..15

Figure 5: All time FTSE chart (Anon., 2023)...15

Figure 6: Volatility Smile (Mitchell, 2021) 25

Figure 7: Option Price Simulation on R 36

Figure 8: Black-Scholes Option Price computation on R...37

Figure 9: Poisson Distribution (Anon., 2020)..38

Figure 10: Poisson Process (Tankov & Cont, 2003)...41

Figure 11: While on the left two sample paths of a Wiener process with $\sigma = 1$ are illustrated, the graph on the right shows the sample path of a compensated Poisson process with intensity $\lambda = 5$, rescaled to have the same variance as the Wiener process (Tankov & Cont, 2003). 42

Figure 12: Compound Poisson Process (Tankov & Cont, 2003)..43

Figure 13: Comparison between leptokurtic and normal distribution (Lumholdt, 2018)........45

LIST OF TABLES

Table 1: Summary of the implications on option's price of an increase in the value of each variable..21

Table 2: Option Pricing simulation inputs on Excel...31

Table 3: Stock Price Simulation on Excel 32

Table 4: Option Payoff Simulation on Excel...32

Table 5: Option Price Simulation on Excel 33

Table 6: Black-Scholes Option Price Computation on Excel...34

Table 7: Option Price Simulation on R...42

Table 8: Black-Scholes Option Price computation on R 43

Introduction

The field of stochastic calculus plays a crucial role in the analysis and modeling of uncertain financial phenomena. This bachelor’s thesis delves into the realm of stochastic calculus and its practical applications in option pricing and risk management.

Chapter 1 serves as an introduction to stochastic calculus. The fundamental concepts are defined, starting with the σ-algebra and probability space, which lay the mathematical groundwork for probability theory. Stochastic processes, martingales, Markov chains, and random walks are then introduced as key components of stochastic calculus.

In Chapter 2, the focus shifts on the renowned Black-Scholes model. The model is introduced along with its assumptions, differential equation and the risk neutral evaluation framework. Additionally, Black-Scholes formula for option pricing and volatility estimation techniques are discussed.

Chapter 3 delves into Monte Carlo simulation as an alternative approach to option pricing. An overview of the Monte Carlo method is provided, elucidating the use of pseudorandom sequences. The implementation of Monte Carlo simulation for option pricing is then showcased through Excel and R-studio. It has been decided to use R-studio for its well-established role in statistical modeling and data analysis, while Excel was chosen for its widespread adoption in the financial industry.

Chapter 4 concentrates on Lévy processes. The Poisson process is explored, encompassing its distribution, definition, and properties. Then the compound Poisson process and its convergence to the normal distribution are examined. The Merton jump diffusion model, which combines Lévy processes with the Black-Scholes model, is investigated. Hedging strategies within the Merton framework in incomplete markets are explored, accompanied by a derivation of the model's formula.

The aim of this bachelor’s thesis is to to contribute to the field of quantitative finance by providing a comprehensive analysis of the beforementioned concepts and models, through which valuable insights into the complexities of financial markets can be gained.

Chapter 1: *Introduction to Stochastic Calculus*

This introductory section aims at clarifying some fundamental concepts of stochastic calculus that are needed for a comprehensive understanding of this thesis.

1.1.1 σ-algebra Definition

A *σ-algebra* $\mathcal{F}$ on Ω is a family of subsets of Ω such that

(1) $\emptyset, \Omega \in \mathcal{F}$

(2) if $A \in \mathcal{F}$, then so does the complement set $\Omega - A$

(3) if $A_1, A_2, \ldots, A_\infty$ is a sequence of sets in $\mathcal{F}$, then their union or intersection of countably many members of the algebra $\bigcup_{i=1}^{\infty} A_i$, $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

In other words, σ-algebra a is a collection of subsets of a set Ω that is closed under countable unions and contains the complements of its elements. Therefore, all the measurable sets in measure theory are factors of a given σ-algebra. The σ-algebra properties make sure that the collection of measurable sets forms a well-behaved structure for defining measures and conducting mathematical analyses in measure theory.

In a financial modelling context, Ω usually represents the set of scenarios that can occur in the market, where each scenario ω ∈ Ω is described in terms of the evolution of prices of different instruments.

1.1.2 Probability Space Definition

A *probability space* is the triple $(\Omega, \mathcal{F}, P)$, where Ω is a nonempty set, $\mathcal{F}$ is a σ-algebra of subsets of Ω and P is a function that, to every set $A \in \mathcal{F}$, assigns a number in [0,1], called the probability of A and written P(A).

It is required that:

- P(Ω) = 1
- (Countable additivity) whenever $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathcal{F}$, then P($\bigcup_{n=1}^{\infty} A_n$) = $\sum_{n=1}^{\infty} P(A_n)$.

When considering stochastic processes, the concept of a probability space is useful because it gives a precise framework for defining and quantifying probabilities associated with various events in the process. This framework enables the study and analysis of process behavior, as well as the capacity to generate probabilistic predictions and derive statistical features. The concept of a probability space provides a rigorous and mathematical foundation for the study of stochastic processes, allowing for the development of theories, models, and techniques to understand and predict random phenomena in a variety of fields including finance, physics, biology, and engineering.

1.1.3 Stochastic Process Definition

A *stochastic process* is a parameterized collection of random variables $\{X_t\}_{t\in[0,T]}$ indexed by time, defined on a probability space (Ω, F, P) and assuming values in $\mathbb{R}^n$.

X_t can be thought as the position of a particle *X* at time *t*, changing as t increases.

Stochastic processes are adopted when working with phenomena that possess an aleatory term. The resulting mathematical models aim to mimic relationships between variables by including a randomized term into a deterministic function.

Stochastic processes are mathematical models adopted when describing phenomena that involve randomness or uncertainty. They are often used to model real world systems, where the outcomes are not deterministic or completely predictable. These models typically involve a deterministic function that governs the evolution of the system over time, but they also include a probabilistic element that captures the effects of chance on the system. The randomness can arise from various sources, such as measurement errors, natural variability, or incomplete information. By incorporating stochasticity into the model, we can better understand the behavior of the system and make predictions about its future evolution.

Stochastic processes are divided into two main categories: discrete-time stochastic processes and continuous time stochastic processes. The evolution of a continuous time stochastic process occurs over continuous time intervals represented by a continuous variable, often denoted as "t." The process's state can change on a continuous basis, and thus is frequently characterized by a continuous function, such as a stochastic differential equation. Brownian motion and geometric Brownian motion are two examples of continuous time processes. Discrete time stochastic processes, on the other hand, evolve over discrete time intervals represented by discrete points or indices, which are frequently labeled as "n" or "k." At these discrete time periods, the state of the process is updated, which is commonly represented as a sequence of random variables or observations. Random walks and Markov chains are examples of discrete time processes.

The primary difference is in the underlying temporal representation. Continuous time processes represent systems in which the state changes continuously, allowing for real-time dynamics in applications such as stock market fluctuations or physical events. Discrete time processes, on the other hand, update the state at precise time points, making them appropriate for modeling systems with discrete observations, such as daily market returns or queueing system events.

Understanding the distinction between continuous and discrete time stochastic processes is critical for selecting acceptable models in a variety of domains. The decision is influenced by elements such as the process's nature, accessible data, computational concerns, and analytical aims. Both types have applications and advantages, and the choice should be based on the unique needs of the problem at hand.

1.1.4 Martingale Definition

An adapted stochastic process $\{X_0, X_1, X_2, \dots\}$ is a *Martingale* if:

$$X_t = E[X_{t+1} \mid \mathcal{F}_t]$$

For all $t \geq 0, \mathcal{F}_t = \{\{X_0, X_1, \dots, X_t\}$

A Martingale is the formalization of the price process for a fair game. It is a sequence of random variables $\{X_0, X_1, X_2, \dots, X_n\}$ in which the expectation of the next value, given all previous values, is equal to the current value. Therefore, the future behavior of the process is unpredictable and random, and there is no way to exploit any patterns or trends in the past data to predict future values.

1.1.5 Markov Chain Definition

Where Ω is the finite state space and P is the transition matrix $|\Omega| \times |\Omega|$

A sequence of random variables $\{X_0, X_1, X_2, ...\}$ is a Markov chain with finite state space Ω and transition matrix P if for all $n \geq 0$, and all sequences $(x_0, x_1, \dots, x_n, x_{n+1})$, we have that:

$$\mathbb{P}[X_{n+1} = x_{n+1} | X_0 = x_0, \dots, X_n = x_n] = \mathbb{P}[X_{n+1} = x_{n+1} | X_n = x_n] = P(X_{n+1}, x_{n+1})$$

A Markov Chain is characterized by its memoryless property, by which the probability of a future event depends only on the current state and is completely independent from all past events.

1.1.6 Random Walk Definition

Suppose that $X_1, X_2, ...$ is a sequence of $\mathbb{R}^d$ valued independent and identically distributed random variables. A random walk started at z ∈ $\mathbb{R}^d$ is the sequence $(S_n)_{n\geq 0}$ where $S_0 = \text{z}$ and

$$S_n = S_{n+1} + X_n, \qquad n \geq 1$$

The quantities (X_n) are referred to as steps of the random walk.

A special example of random walk, also called *symmetric random walk*, occurs in the case of the head or tails game, in which the player gains 1 with a probability p(gain)= ½ and loses 1 with a probability p2 = ½.

Building the random variable X_j as:

$$X_j = \begin{cases} 1 & if\ w_j = "Head" \\ -1 & if\ w_j = "Tail" \end{cases}$$

Then the stochastic process $\{M_k\}_{k=0}^{\infty}$, whose value at $M_0 = 0$ and $M_k = \sum_{j=1}^{k} X_j$ can be graphed as:

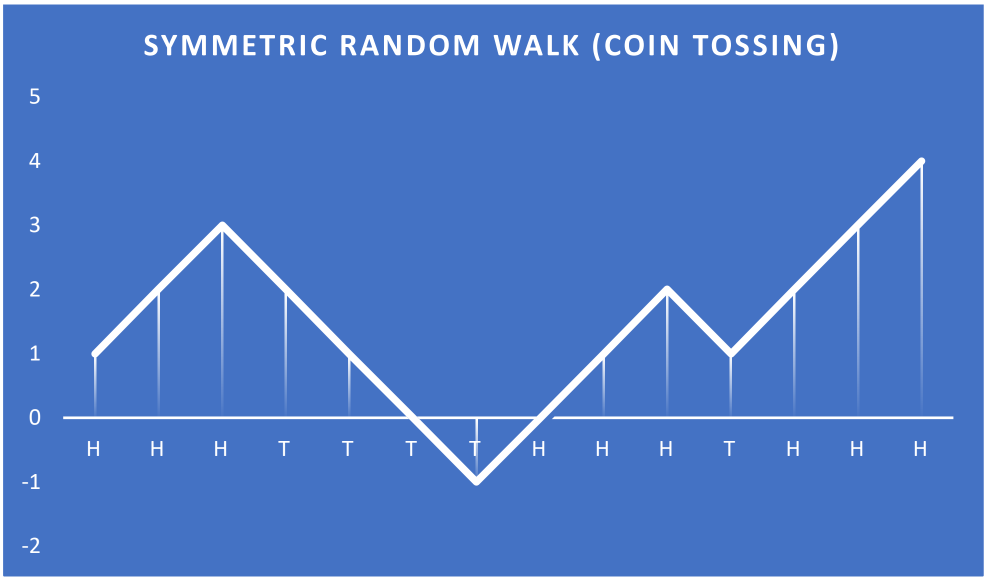


Figure 1: Symmetric Random Walk, Fair Coin Tossing

It is crucial to underline that for $0 = k_0 < k_1 < k_2 < \cdots < k_m$ each and every increment $(M_{k_1} - M_{k_0}), (M_{k_2} - M_{k_1}), \ldots, (M_{k_m} - M_{k_{m-1}})$, is an independent random variable.

1.2 Brownian Motion

Brownian motion is the most well-known stochastic method for predicting price variations. It includes a random process, indicated as W_t, with independent and stationary increments that follow a Gaussian Distribution.

Brownian motion holds a central position in stochastic analysis, captivating extensive research and serving as the foundation for modern advancements in the field.

1.2.1 Brownian Motion Definition

A *Brownian motion*, also denominated Wiener process, is a stochastic process W(t) with values in $\mathbb{R}$ defined for $t \in [0, \infty)$ such that the following conditions hold:

$W(0) = 0.$

If $0 < s < t$ then $W(t) - W(s)$ has a normal distribution $\sim N(0, t - s)$ with mean 0 and variance $(t - s)$.

If $0 \leq s \leq t \leq u \leq v$ (i.e., the two intervals $[s, t]$ and $[u, v]$ do not overlap) then $W(t) - W(s)$ and $W(v) - W(u)$ are independent random variables. In fact, the Wiener process is the only time-homogeneous stochastic process with independent and identically distributed increments that has continuous trajectories.

The probability density function of W(t) is:

$$f_{W(t)}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

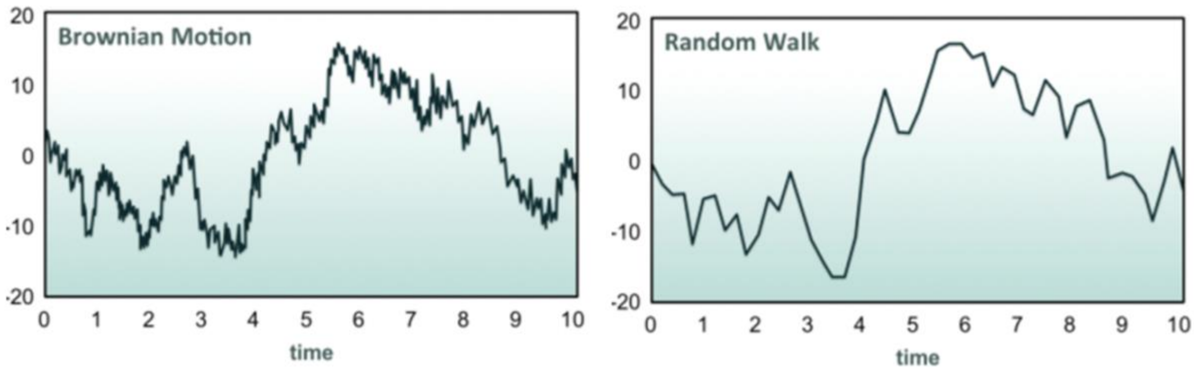


Figure 2: Comparison between Brownian Motion and Random Walk (Holton, 2013)

From figure 1.2 the deep relationship that links a Brownian Motion to a Random Walk can be visually inferred. This strong similarity lays its roots on the notion that a Brownian motion can be defined as a continuous stochastic process that exhibits the properties of a simple random walk.

1.2.2 Brownian Motion Properties

The Brownian motion is particularly popular among stochastic processes because of its special properties. Firstly, W_t is a Gaussian process, and therefore for all $0 \leq t_1 \leq t_2 \leq ... \leq t_k$ the random vector Z = (W_{t1} , ..., W_{tk}) ∈ ℝ has a multinormal distribution.

Moreover, W_t has stationary increments, which means that the distribution of the increments of the process is independent of time. More specifically, for any two time points t and $t + h$ (with $h > 0$), the distribution of the increment $W_{t+h} - W_t$ only depends on the time difference h and not on the specific values of t and t + h. This property implies that the statistical properties of the process remain the same over time, making it a fundamental characteristic of Brownian motion.

In addition, the Brownian motion is a stochastic process with two significant path characteristics. For starters, it has continuous paths, which implies that as time passes, its trajectory becomes a smooth, continuous curve. This means that there are no abrupt changes in its course, giving it a sense of continuity. Despite having continuous routes, the Brownian motion is not differentiable anywhere. This means that at any given point in time, it lacks a well-defined derivative. The cause for this is due to its irregular and random nature. Brownian motion is characterized by unpredictability and abrupt changes in direction, resulting in an irregular and jagged course.

Furthermore, the Brownian motion is a martingale, which indicates that its predicted value does not change over time on average. This trait is an important feature of Brownian motion and holds true regardless of the time instance under consideration. Brownian motion's martingale property has several applications in finance mathematics and other domains. It enables the formulation of pricing models for different derivative securities, like options and futures, by building portfolios that mimic the behavior of these instruments. The martingale property ensures that these derivatives' pricing is fair and free from arbitrage opportunities.

At last, the equation $Cov\ (W_s, W_t) = \ min\ (s, t)$ suggests that the covariance grows as the time points being compared get closer together. When t and s are near in time, their minimum value is smaller, and so their covariance is reduced as well. If s and t are further apart, their minimum value will be greater, resulting in a higher covariance. This characteristic of Brownian motion covariance has practical implications. It means that as the time interval between increments of a Brownian motion process increases, they become less linked. In other words, the Brownian motion values at remote time points tend to be more independent.

Furthermore, the resulting Brownian Motion distribution will have key properties as well. Firstly, the spatial homogeniety property, which states that the addition of a constant value x to the entire trajectory will result in a new Brownian Motion.

$W_t \ + \ x$ for any $x \ \epsilon$ is a Brownian motion.

This spatial homogeneity characteristic of the Brownian motion is beneficial because it allows us to shift the entire process along the spatial axis without changing its core statistical properties (randomness, continuity, stationary increments and gaussian distributed).

Secondly, the distribution is symmetric, which means that by mirroring the original Brownian motion process W_t across the horizontal axis, the new process that is obtained will possess the same statistical traits. This is a desired property since, if we are interested in studying the behavior of a financial derivative under a certain pricing model based on Brownian motion, analyzing the corresponding derivative with negative prices can provide useful insights.

Thirdly, its scaling property implies that the process appears qualitatively the same regardless of scale or time granularity. The statistical features and behavior of the process do not change as we zoom in or out on the time axis. Therefore:

$\sqrt{c}W_{t/c}\ \forall\ c > 0$ is a Brownian motion.

Furthermore, by reversing the direction of time, the new process that will be obtained still exhibits all the statistical traits of the original process (time inversion property).

$Z_t = \begin{cases} 0, & t = 0 \\ tB_{\frac{1}{t}}, & t > 0 \end{cases}$ is a Brownian motion.

In practice, time inversion contributes to determining the sensitivity of option prices to time changes. The impact of time decay (expresses by the Greek theta) on the option's value can be evaluated by analyzing the behavior of options under time reversal. This knowledge is essential for option traders and investors to manage their positions and estimate the possible risks and rewards connected with temporal changes.

At last, time reversibility asserts that the distribution of the process's trajectory up to time t is equal to the distribution of the time-reversed trajectory from time t back to 0. In other words, the distribution of these two paths is the same if we look at the road that the process takes from 0 to t and then reverse time and look at the path from t to 0.

Linear transform

A fundamental concept of stochastic calculus is the linear transformation of a Brownian motion. It involves applying a linear function to the original Brownian motion process, which results in a new stochastic process with distinct features. This transformation enables the investigation and modelling of a wide range of financial instruments and derivatives.

When a linear transformation is implemented to a Brownian motion, the original process is altered by the introduction of a linear function. This transformation allows for the investigation of the link between the original Brownian motion and the converted process, revealing fresh insights about financial variable dynamics.

The linear transform of the standard Brownian motion W_t, with drift b and volatility $\sigma > 0$ is:

$$B(t) = bt + \sigma W(t)$$

However, despite its many desirable properties, the linear transform of the Brownian motion is still not the right transform for stock price modeling because it can take negative values, which is not consistent with the behavior of real-world financial assets.

In order to address this issue, an alternative transform, also known as geometric Brownian motion, is adopted.

Exponential transform

The exponential transform resolves the negative prices issue by building the stochastic process as the Euler's number with the linear Brownian motion transform as its exponent.

$$Y(t) = e^{(B(t))} = e^{bt+\sigma W(t)}$$

Furthermore, $Y(t)$, for any fixed t, is the exponential of a Gaussian variable, so its marginal distributions are lognormals, which is ideal for stock prices modeling since their return's distribution is usually assumed to be normal (such as in the Black-Scholes model).

The volatility σ measures the random fluctuations in the asset's price, while the drift b estimates the expected rate of return of the underlying asset.

Moreover, it can be proven that the geometric Brownian motion differently from the simple Brownian motion, isn't always a Martingale. Therefore, after some computation, it can be demonstrated that the only scenario in which the geometric Brownian motion is a martingale is if its parameters satisfy the equation:

$$b = -\frac{\sigma^2}{2}$$

Therefore, the initial formula of a Geometric Brownian motion that is also a Martingale can be rewritten as:

$$Y(t) = e^{-\frac{\sigma^2 t}{2} + \sigma W(t)}$$

This last expression is crucial to determine the fair option price. In fact, after the construction of a portfolio that combines the underlying asset and the financial derivative, the implementation of the martingale property will allow the determination of the appropriate weights for the underlying asset and the option such that the portfolio value is independent of changes in the underlying asset (perfectly hedged portfolio).

Chapter 2: Black Scholes Model

2.1 Introduction to the model

In the early seventies, Fisher Black and Myron Scholes published, under the title "The Pricing of Options and Corporate Liabilities" on the *Journal of Political Economy,* the work from which they obtained the differential equation that revolutionized derivatives pricing. The Black-Scholes model was widely adopted by analysts to evaluate the price of European put and call options on non-income paying assets. Another common use for this formula was to derive the discount rate that corporate bonds should adopt because of the default possibility. Nowadays, it is still frequently used by financial markets operators because it only requires the estimation of one parameter: volatility.

2.2 Assumptions

The **assumptions** put forward to obtain the Black-Scholes differential equation are:

1. The underlying asset prices follow a geometric Brownian motion.

   As a result, the stock price of the underlying will be log-normally distributed with both mean and variance constant.

   When modelling stock returns, a normal distribution is usually chosen as stock returns can be either positive or negative. However, since stock prices cannot be negative, the lognormal distribution is more appropriate.

   A lognormal distribution has two important properties:

   - It has a lower bound of zero.
   - The distribution is right-skewed (i.e. it has a long right tail).

   Additionally, log-normality has been proven consistent with many observed phenomena in capital markets, such as the volatility clustering and the "fat tails" of returns.

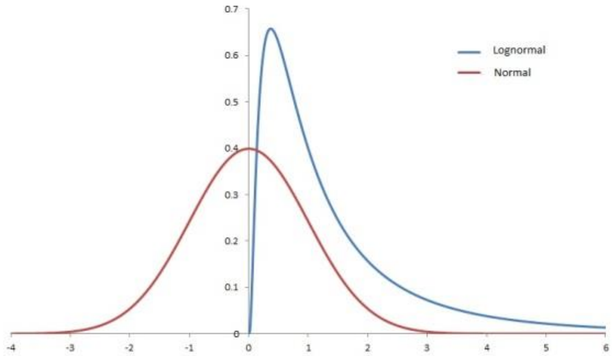


Figure 3: Lognormal Vs Normal Distribution (Ma, 2015)

2. The short-term interest rate is known and constant through time.
3. Constant volatility of the underlying stock
4. The option is "European"

   This model focuses on European-style options, which is a specific type of options that can only be exercised on the expiration date. Their counterpart are the American-style options which can be exercised at any time during the life of the option.
5. The underlying stock pays no dividends.

   During the lifespan of the option, no dividends are paid out.
6. There are no transaction costs nor commission costs for buying or selling either stocks or options.
7. Short selling is allowed.
8. Asset trading is continuous.
9. There are no arbitrage opportunities.

   Markets are assumed efficient; the stock price behavior is assumed to follow a random walk. Therefore, it is impossible to predict future market movements and that future stock prices are independent of the past.

2.3 Geometric Brownian Motion

In 1900, the French mathematician Louis Bachelier completed his doctoral thesis titled "*Théorie de la spéculation"* (Theory of Speculation), modernly considered one of the founding research paper of quantitative finance. In this document he outlined that if stock prices exhibited any identifiable pattern (other than the long-term growth trend associated with macroeconomic expansion), speculators would find it and exploit it, thereby eliminating it. Therefore, after speculators have incorporated all the special insights into their trades, prices are expected to show unpredictable fluctuations, independent of their past history. This would thus behave like a Random Walk, or in certain contexts, a Martingale. The equations obtained by Bachelier from this model correspond to the Brownian Motion, which was later mathematically formalized by Norbert Wiener.

The Brownian motion is acclaimed by financial researchers because it is both the mathematical model that most securities markets are assumed to follow, and the tool on which all financial asset pricing and derivatives pricing models are based.

The Brownian motion is the leading mathematical model to work on financial securities because it introduces a random component into the financial model (since stock prices can be considered as random). Nevertheless, there are ways to adjust the random pattern followed by the Brownian motion. In fact, mean and variance can be calibrated to better fit the asset that is being modelled, since it is a random variable with a normal distribution. For instance, if we are modelling stock prices, and we know that they have a tendency to rise, the mean could be set as positive and different from 0, such that the Brownian motion will have a tendency to grow. On the other hand, when implementing the Brownian Motion for option pricing, we are not looking for the real stock prices but just for the fair ones, therefore the Brownian motion will lead to the best estimates if the mean is set to 0.

The Geometric Brownian (GBM) motion is widely adopted because it has several fundamental similarities with security prices.

1. The expected returns from GBM processes are independent of the values of the process just like it happens in financial markets.
2. GBM processes can only take positive values (same as equity prices).
3. GMB values exhibit volatility similar to equities.

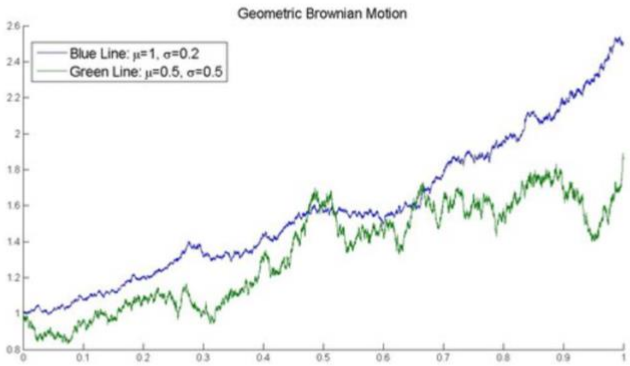


Figure 4: Sample GBM values chart (Bhatia, 2016)

The Blue sample path has lower standard deviation but more drift while the green one has higher volatility and lower drift.

Figure 5: All time FTSE chart (Anon., 2023)

There is a clear similarity between the chart in Figure 5 and the green line of the Figure 4 above.

However the GMB usage must be followed by solid due diligence since in three main scenarios it fails stock market common sense:

- In the first chart, the volatility for both lines are assumed constant, that's just an unrealistic assumption to make in the real markets.
- GBM processes are continuous, and assumes no jumps. On the other hand, stock prices open gap up or down 2 to 3% all the time, due to many reasons such as earnings release after market closure, holidays, or just unexpected macroeconomic events.
- GBM processes do not assume any extra 'costs' associated with changing values; stock returns are a function of transaction costs.

2.4 The Black-Scholes Differential Equation

The Geometric Brownian motion, adopted by the Black-Scholes model, allows for the representation of the evolution though time of the stock price and is described by the following stochastic differentiation equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{2.1}$$

Where μ and σ are constant values that represent respectively the drift and the volatility of the stock price. The volatility σ is always greater than zero and from a financial perspective it is the standard deviation of the annual log return. On the other hand, the drift μ can be described as the exponential growth of the average stock price. Furthermore, $S(t)$ is the function describing the price of the underlying asset at time t. At last, the term $W(t)$ is a Wiener Process, the stochastic component of the equation, that characterizes its usage to model problems of probabilistic random nature.

Supposing now that f is the value of a particular option (i.e. either c_t or p_t) based on the underlying S and time t and assuming that f is continuous with respect to S and t, the Ito's Lemma can be applied(which is often used to compute the differential of an equation in stochastic calculus) and the following equation obtained:

$$df(t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \mu S \frac{\partial f}{\partial S}\right)dt + \frac{\partial f}{\partial S}\sigma S dZ \tag{2.2}$$

At this point the stochastic term dZ can be eliminated by choosing a portfolio P with value defined as:

$$P_v = -f + \frac{\partial f}{\partial S} * S \tag{2.3}$$

Respectively for each derivative contract shorted the analyst must be long $\frac{\partial f}{\partial S}$ of the underlying asset. The variation in portfolio value during the time interval Δt is equal to:

$$\Delta P_v = -\Delta f + \frac{\partial f}{\partial S} * \Delta S$$

(2.4)

Substituting equations (2.1) and (2.2) in (2.4) it becomes:

$$df(t) = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt$$

(2.5)

It's noticeable from this last equation that the term dZ has been eliminated, which means that the resulting expression is no longer stochastic, but it is deterministic. The absence of stochastic terms in the last formula implicates that the portfolio is riskless, and therefore that its performance is equal to the risk-free rate r_f. Otherwise, there would be arbitrage opportunities to exploit, which would conflict with assumption 9 of the model. It is crucial here to underline that the portfolio P is riskless only for an infinitesimal moment, and that frequent adjustments of the proportion of the derivative to the underlying ($\frac{\partial f}{\partial S}$) is necessary to keep the portfolio risk-free. This practice is known as delta hedging.

Furthermore, equation (2.5) shows that the variations in portfolio value are independent of the drift μ, this is because the component representing the call option is perfectly offset by the effect that μ has on the underlying.

Therefore, the variation of portfolio value can be rewritten as the the risk-free rate times the current value of the portfolio and the narrow time interval Δt.

$$\Delta P_v = r_f * P_v * \Delta t$$

(2.6)

The combination of formulas (2.1), (2.2) and (2.6) leads to the Black-Scholes Merton differential equation:

$$r_f f = \frac{\partial f}{\partial t} + r_f \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}$$

(2.7)

Equation (2.7) has a unique solution for every derivative contract boundary condition.

For example, the boundary condition for a European Call option with strike price K and price S_t is:

$$c = max\,(S_t - K, 0)$$

(2.8)

In the case of a European put option the boundary condition would become:

$$p = max\,(K - S_t, 0)$$

(2.9)

2.5 Risk Neutral Evaluation

The notion of neutral risk evaluation, which is based on the binomial model, is critical when examining financial derivatives. This notion stems from a crucial aspect of the Black-Scholes-Merton differential equation (2.7), which excludes variables influenced by investors' risk preferences. The variables in the equation (current price of the underlying asset, time, volatility, and the risk-free interest rate) are all independent of investors' risk preferences.

However, if the expected return of the underlying asset (μ) were to be included in the equation, it would depend on investors' risk preferences, as more risk-averse investors would require a higher expected return. The fact that the differential equation is risk-free means that we can make any assumptions we like about investors' risk preferences, including the assumption that they are all risk neutral.

If this assumption is correct, the expected rate of return on all assets should be equal to the risk-free interest rate r_f, because investors do not demand a risk premium. This would simplify drastically the computation of future values and expected payoffs.

However, it is crucial to emphasize that this assumption is a simplification of reality, and the Black-Scholes equation can still be solved without it. When investors go from risk-neutral to risk-averse, two things happen: the expected return rate of the security changes, and the discount rate for calculating present values changes. These two impacts perfectly cancel each other out.

2.6 The Black-Scholes Formula for Option Pricing

The Black-Scholes expression that is drawn from equations (2.7) and (2.8) evaluates the theoretical price of a European, non-income paying call option at time $t \leq T$ is:

$$c_t = S_t \Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

(2.10)

The put theoretical price is derived from the same procedure but adopting as a boundary condition the equation describing the option payoff.

$$p_t = Ke^{-r(T-t)}\Phi(-d_2) - S_t \Phi(-d_1)$$

(2.11)

Where K is the strike price, S_t is the price of the underlying asset at time t, σ is the asset price volatility and, r is the risk-free interest rate (equal to the rate of return of a riskless asset with duration $T - t$, the maturity of the option). Φ is the cumulative distribution function of the standard normal distribution with null mean and unitary standard deviation. The values d_1 and d_2 are respectively equal to:

$$d_1 = \frac{ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$\Phi(d_2)$ is the probability that the option will be exercised at maturity, which is risk-neutral under the assumption put forward by Black and Scholes. $\Phi(d_1)$ describes the slope of the Black Scholes curve relatively to S_t, which represents the sensibility of the option's price relatively to movements in the underlying financial asset. Both $\Phi(d_1)$ and $\Phi(d_2)$ must have values that are included between 0 and 1.

2.6.1 Black Scholes model implications

According to the Black-Scholes option pricing equation, there are precisely 5 factors affecting the price of an option: underlying stock price, strike price, time to maturity, volatility, and the risk-free interest rate.

The underlying stock price is positively correlated with the price of a call option since if the stock price rises the probability that the option will be exercised will increase. Therefore, this movement is reflected with an increase in the price of the option. The same applies for put options.

The strike price defines the moment from which the option holder starts to earn from the contract. Moreover, if the holder is expected to profit from a rise in the stock price (call option) a higher strike price would erode those profits, which would make the option seller better off, who is therefore willing to sell the option at a lower price. A similar reasoning can be conducted for put options, where a higher strike price would be riskier for the option market maker, who is therefore asking for a higher option price in return for the contract.

The time to maturity for European options has an ambiguous effect on their price. In fact, differently from American options, whose prices often rise as the time to maturity increases due to a longer time interval during which the option can be exercised, European options do not possess a strict direct relationship with residual time to maturity. For example, in a scenario in which dividends are paid, if two call options with respective maturities of 3 months and 5 months are considered, and the dividend is paid out in 4 months, the option with the longer maturity will have a lower value than the one with a 3-month maturity.

Volatility is a key input in options pricing models, and changes in volatility can have a significant impact on option prices. When the volatility of the underlying asset rises, larger price moves occur, increasing the likelihood that the option will be in-the-money by the expiration date. This increase in the probability of the option becoming profitable improves its value because the option holder has the potential to profit more, causing the price of both call and put options to increase.

There are two main effects of a rise in risk-free interest rate. First, for option holders it decreases the present value of the future cash flows. On the other hand, an increase in the risk-free interest rate often leads to a greater expected rate of return for stocks. Both these effects tend to lower the value of a put, whose price will decrease. However, it can be proven that for call options the second effect positively outweigh the decrease caused by the first effect, therefore resulting in a price increment.

Variables	**European Call Option Price**	**European Put Option Price**
Stock Price	+	-
Strike Price	-	+
Time to Maturity	?	?
Volatility	+	+
Risk Free Interest Rate	+	-

Table 1: Summary of the implications on option's price of an increase in the value of each variable

Among all the parameters included in the Black-Scholes option pricing expression, the only one that needs estimation is the volatility. In the next section, this intriguing problem will be addressed, illustrating several options available to financial analysts to evaluate a feasible solution.

2.7 Volatility Estimation

Usually denoted by σ, the volatility of a security is a measure of the dispersion of its returns across their mean. It is defined as the standard deviation of logarithmic returns observed over fixed time intervals.

As previously stated, the only factor of the Black-Scholes model that cannot be observed directly on the market is the volatility of the underlying asset return. Therefore, its estimation is crucial for a correct implementation of the model and for realistic, unbiased final option prices. The two most widely adopted methods to do so are historical volatility estimation and implicit volatility estimation.

2.7.1 Historical Volatility Estimation

Historical Volatility is the annualized standard deviation of stock prices and measures the intensity by which past securities prices deviated from their average over a chosen period of time.

Its estimation begins with setting three key parameters: the basic period on which we compute the returns (1 day), the amount n of periods considered for the calculation (21 days – number of trading days in a month) and the quantity T of periods in a year (252 days – average number of trading days in a year), which will be used for annualizing the volatility in the end.

After that, follows the calculation of the continuously compounded logarithmic returns for each period (P_i is the closing price and P_{i-1} is the previous day closing price).

$$R_i = ln\left(\frac{P_i}{P_{i-1}}\right)$$

The third and last point involves first the evaluation of the average of the returns obtained in the previous step and then the computation of the annualized sample standard deviation thorough a small twist of the formula (the Excel function STDEV.S could be used as well to carry out this last step).

$$\bar{R} = \frac{\sum_{i=1}^{n} R_i}{n}$$

$$\sigma = \sqrt{\frac{T}{n-1}\sum\nolimits_{i=1}^{n}(R_i - \bar{R})^2}$$

2.7.2 Implied Volatility Estimation

Implied volatility describes the market sentiment on the likelihood of future changes in a security price. Differently from historical volatility, which is computed from past, realized data, implied volatility is forward looking, because it manages to reflect the beliefs of investors on how much volatility is expected in the future.

Unfortunately, it is not possible to find an explicit formula for σ from the Black-Scholes expression. One of the several approaches to solve this issue is to adopt an iterative procedure to find the implicit σ. It can be showed for example, that starting with $\sigma = 0.3$, having all the remaining data available, (S_t, K, r, $T - t$, and c_{price}), and computing the call price from the Black-Scholes equation, if the result obtained is greater than c_{price} then, since c_{price} is positively correlated with σ, we know that σ should have a lower value (e.g. $\sigma = 0.2$). Moreover, if when trying $\sigma = 0.2$ the value of σ is too small, we can logically draw the conclusion that $0.2 < \sigma < 0.3$, and therefore try with the midpoint $\sigma = 0.25$. This procedure continues until the value for σ that is found is deemed specific enough (e.g. $\sigma = 0.2569847$).

Often, financial markets operators extrapolate several implied volatilities from options with the same underlying, aiming to evaluate a general implied volatility. There are various approaches that apply this last strategy in order to obtain a final value that incorporates more information and is therefore more reliable.

Weighted Implied Standard Deviation (WISD)

Latané and Rendleman (1976) suggested a methodology that considers all the options with the same underlying, and computes the implied volatilities for each exercise price. After that, they build a weighted average that takes as weights the volumes of transactions occurred.

$$WISD_{i,t} = \frac{1}{V}\sum_{i=1}^{V} \sigma_i(t,T)n_i$$

V: total amount of transactions observed at t and with maturity T

$\sigma_i(t,T)$: implied volatility computed at t of the option with strike price K_i and maturity T.

n_i: volume of transactions on the option with specific strike price K_i

Moreover, the findings provided in Latané and Rendleman (1976) underlines both a strong correlation between WISD and actual standard deviations and that WISD is generally a a better predictor of future variability than standard deviation predictors based on historical data. On the other hand, the instability of WISD caused by factors that appear to affect all options in the same way, suggests that the model may not fully capture the process determining option pricing in the actual market.

Arithmetic Mean

Schmalensee and Trippi (1978) follows a similar approach, but excludes both deep in the money and deep out of the money options, underlying that those options barely reflect the general market sentiment on future volatility. Therefore, the authors only consider the most traded options and focus their attention on computing the arithmetic mean of the implied volatilities obtained from those options. For every stock s and week t, it is supposed to exist a value $\sigma_{s,t}$ that correctly captures the market's expectation of future volatily.

$$\sigma_{s,t} = \frac{1}{N_{s,t}} \sum_{i=1}^{N_{s,t}} \sigma_{i,s,t}$$

$N_{s,t}$: number of options available for some (s,t).

$\sigma_{i,s,t}$: value of σ computed on the ith option.

2.8 Drawbacks and Limitations of the Black-Scholes Model

2.8.1 Lognormal Distribution and Stock Market

While the Black-Scholes model assumes that stock prices are lognormally distributed, in practice, they are not perfectly lognormally distributed. There are various reasons for this. Firstly, stock prices exhibit "fat tails" (Westerfield, 1977), which means that extreme price movements occur more frequently than as predicted by a lognormal distribution, usually due to unexpected news and other macroeconomic factors. Secondly, stock prices possess skewness and kurtosis, suggesting that their distribution is not symmetrical and has a more profound peak than a lognormal distribution. This could be because favorable news, such as earnings announcements, can affect stock prices, resulting in a more significant positive skew. At last, security prices exhibit volatility clustering and non-stationarity (meaning that their statistical properties change over time) which are not accounted by the lognormal distribution of the Black-Scholes model.

2.8.2 Constant Volatility Assumption

Empirical studies have proven that the constant volatility assumption does not hold in practice. For example, in "The Variance Gamma Process and Option Pricing" (Madan, et al., 1998) is shown that the Black-Scholes model's assumption of constant volatility is inconsistent with observed option prices and proposes an alternative model that incorporates a stochastic volatility process. This inconsistency is agreed upon by a substantial body of academic research, including (Taylor, 1986).

2.8.3 Volatility Smile

The volatility smile effect is a phenomenon found in the implied volatility surface of options. It refers to the shape of the implied volatility curve that typically exhibits an upward sloping convex shape as a function of strike price, resulting in a "smile" or "smirk" shape. This indicates that the implied volatility of options with the same expiration date, but different strike prices is not constant, which is in contrast with the Black-Scholes model's assumption of constant volatility.

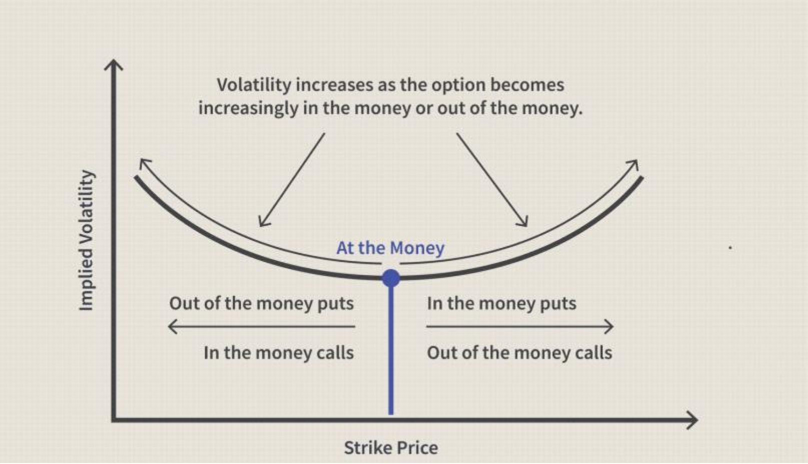


Figure 6: Volatility Smile (Mitchell, 2021)

The volatility smile effect occurs due to inefficiencies in the Black-Scholes model. Firstly, the Black-Scholes model suggests that the price of the underlying asset follows a geometric

Brownian motion with constant volatility. However, this assumption is not always correct since stock prices display volatility clustering and jumps, which can result in time-varying volatility. Furthermore, the volatility smile effect is correlated to market participants' risk perceptions. The implied volatility of an option represents the market's assessment of the future volatility of the underlying asset. When the implied volatility of options with different strike prices differs significantly, it indicates that market players perceive distinct levels of risk for different options. This could be due to a number of causes, including market sentiment, unexpected news, or market players' anticipation of future price changes. Lastly, the volatility smile may be linked to the erroneous assumption of normal distribution of underlying asset returns. As already mentioned, in practice the stock market returns are fat-tailed, meaning that by adopting the normal distribution the Black-Scholes model is underestimating the probability of extreme price movements. In conclusion, when the underlying asset's volatility is not fixed, the Black-Scholes model's supposition of constant volatility leads to option mispricing, resulting in the observed volatility smile effect.

Nevertheless, despite all the potential flaws in the model assumptions, analyses of market option prices do indicate that the Black-Scholes equation gives a very good approximation of market prices, especially for short-dated options.

Chapter 3: Monte Carlo Simulation

3.1 Monte Carlo Method

The Monte Carlo simulation is a computational technique for the analysis of complex systems that use random sampling. It is used to represent a system's behavior over time by simulating a large number of random events, which may then be used to assess the likelihood of certain outcomes occurring.

Monte Carlo simulation is frequently used to model and analyze complex systems that are difficult or impossible to evaluate analytically in finance, engineering, physics, and numerous other fields. It is extensively used in finance to mimic the behavior of stock prices and other financial assets, as well as to assess the value of financial derivatives like options.

The first stage of the Monte Carlo simulation involves the definition of the problem or system under consideration together with the identification of all the relevant variables that may influence its behavior.

Next, probability distributions are assigned to the system's unknown variables. These distributions describe the possible values for each variable as well as the chance of their occurrence.

Following the definition of the input distributions, random samples are generated from these distributions for each unknown parameter. The required degree of accuracy often determines the number of samples.

After that, simulations are run for every set of random samples. The defined mathematical model is used to calculate the desired output or outcomes. This entails passing the sampled data through the model to acquire the system responses.

The findings obtained from the simulations are then pooled and examined. This includes the computation of summary statistics such as the mean and the standard deviation of the outcomes.

To draw conclusions and make educated decisions, the aggregated results are examined and interpreted. Monte Carlo simulations provide information on the likelihood of various outcomes, the system's sensitivity to various variables, and the system's potential dangers.

The quantity of random samples created, and the quality of the probability distributions used to generate the samples determine the accuracy of a Monte Carlo simulation. The accuracy of the simulation generally improves as the number of samples grows, but at the expense of increasing computational time.

3.2 Pseudorandom Sequences

In essence, only a natural random source can produce truly random sequences of integers. It is nearly impossible to obtain entirely random sequences of numbers without any of errors and correlations while building software or programs using mathematical approaches.

As John Von Neumann (1951) affirmation further clarifies: "anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin".

This concept matters because it emphasizes the reality that producing fully random sequences of numbers using mathematical procedures without error is impossible. A true generator is a tool that can generate a non-deterministic (and consequently unpredictable) number sequence. When exploiting a totally deterministic computer, mathematical approaches must be used, resulting in sequences that are not entirely random. As a result, no computer can generate a purely random sequence of numbers, only pseudo-random sequences (all sequences generated by mathematical algorithms are called pseudorandom) using generators that can generate sequences of numbers that appear random and pass a series of statistical tests. These sequences are referred to as pseudorandom since the entire generated sequence can be determined if the algorithm and first element utilized (known as the seed) are known. The seed is the only fully random element in the entire number sequence.

3.3 Geometric Brownian Motion for stock price simulation

The financial instruments needed in the Black-Scholes model are only two, the bond B and the stock S. The Bond represents the riskless asset that is continuously paying a constant interest rate $r \geq 0$ by assumption, whose payoff is given by:

$$B(t) = e^{rt}$$

One the other hand, the stock S represents the risky asset, which is required to satisfy both the initial condition and the SDE.

$$\begin{cases} dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \\ S(0) = S_0 \end{cases}$$

According to SDEs general theory, this Cauchy problem has a unique solution, a GBM with appropriate parameters.

$$S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)} \tag{3.1}$$

Where:

- S_0 is the observed initial market price.
- $W(t)$ is a Weiner process $\sim N(0,1)$
- μ is the constant drift.
- σ is the constant volatility ($\sigma > 0$).

Moreover, the mean and standard deviation of the logarithm of the stock price are well-defined in a geometric Brownian motion (GBM) process and may be easily determined using the GBM equation.

Taking the natural logarithm on both sides of the equation (3.1) the following is obtained:

$$\ln\left(S(t)\right) = \ln\left(S_0\right) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)$$

Since the mean is the expected value of $\ln\left(S(t)\right)$ it can be computed as:

$$E\left[\ln\left(S(t)\right)\right] = \ln\left(S_0\right) + \left(\mu - \frac{\sigma^2}{2}\right)t$$

From this result it can be concluded that the logarithm of the stock price rises at an expected rate of $\left(\mu - \frac{\sigma^2}{2}\right)t$ per unit time.

The variance is given by:

$$Var\big(\ln\big(S(t)\big)\big) = \sigma^2 t$$

Therefore, the standard deviation, that is computed as the square root of the variance is:

$$SD\big(\ln\big(S(t)\big)\big) = \sigma\sqrt{t}$$

Now that standard deviation and mean of the Geometric Brownian motion are defined, the equation for the stock price of an option at maturity date T can be rewritten as:

$$S(T) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z)} \tag{3.2}$$

3.4 Monte Carlo simulation for option pricing on Excel

Excel is a spreadsheet tool that is well-known for its versatility and functionality. Its value in finance arises from its ability to manipulate and analyze data, conduct sophisticated computations, and generate visuals.

Excel is used in finance for a variety of objectives. It enables users to organize and manipulate data in an organized fashion, making it easier to evaluate and interpret financial data. It has an extensive number of formulas and procedures that allow users to execute financial modeling, valuation, and analysis computations. Excel's charting tools enable the production of visual representations of data, which aids in the identification of trends, patterns, and linkages.

For the purpose of this thesis, this software will be adopted to implement the Monte Carlo simulation on the Black-Scholes option pricing formula.

It all begins with fixing the parameters of the Black-Scholes equation: the risk-free rate, the time to maturity T, the volatility σ, the spot market price S_0 and the strike price K.

INPUTS	
RISK FREE RATE	2.5%
ST. DEV.	0,2
So	100
K	101
T	1

Table 2: Option Pricing simulation inputs on Excel

It is assumed that the stock is lognormally distributed, and therefore the stock price at time T=1 is described by the equation:

$$S_1 = S_0 e^{(r - \frac{\sigma^2}{2} + \sigma Z)} \tag{3.3}$$

Where Z is the random parameter, which is standard gaussian. The computation of this factor begins with the implementation of the function RAND(), which pseudo randomly generates a number between 0 and 1.

The next step involves the implementation of the INV.NORM() function on that cell, specifying the mean and standard deviation of the normal distribution ($mean = 0$ and $standard\ deviation = 1$). This function returns the value of the inverse normal distribution function for the pseudo random value obtained in the previous step.

Since Z has been computed, it is now possible to calculate the stock price S_1 by implementing equation (3.3).

INPUTS	
RISK FREE RATE	0,025
ST. DEV.	0,2
So	100
K	101
T	1
t	0

Random (0,1)	Z	S1	CALL PAYOFF	PUT PAYOFF
0,618023125	0,300292898	=$D$5*EXP($D$3-($D$4^2)/2+$D$4*C11)		

Table 3: Stock Price Simulation on Excel

The following step is concerned with the simulation of the put and call option payoffs that are recalled from the previous chapter to be:

$$c = max\,(S_1 - K, 0)$$

$$p = max\,(K - S_1, 0)$$

In order to simulate their payoffs, the IF function is implemented.

The IF function is a common Excel function that allows logical comparisons between a number and its expected value. As a result, an IF statement can have two outcomes. So an IF statement can have two results. The first result is if your comparison is True, the second if your comparison is False.

In this scenario it first compares S_1 with the strike price K, and then computes the payoff only if complies with the condition specified. In the case of the call option payoff the IF function will compute the payoff $(S_1 - K)$ only if the stock price S_1 is greater than the strike price K, otherwise it will automatically set the payoff equal to 0.

INPUTS	
RISK FREE RATE	0,025
ST. DEV.	0,2
So	100
K	101
T	1
t	0

Random (0,1)	Z	S1	CALL PAYOFF	PUT PAYOFF
0,449439884	-0,127076605	97,9791661	=IF(D11>$D$6;D11-$D$6; 0)	

INPUTS	
RISK FREE RATE	0,025
ST. DEV.	0,2
So	100
K	101
T	1
t	0

Z	S1	CALL PAYOFF	PUT PAYOFF
1,630385768	139,246722	38,2467222	=IF(D11<$D$6;$D$6-D11; 0)

Table 4: Option Payoff Simulation on Excel

At this point the beforementioned steps are repeated for a minimum of 1000 times; the more the better, but Excel might not have enough computing power for vast numbers.

Below there is an illustration of how the Excel spreadsheet should look like at this point.

Random (0,1)	**Z**	**S1**	**CALL PAYOFF**	**PUT PAYOFF**
0,618023125	0,300292898	106,722154	5,722153995	0
0,298672239	-0,52822313	90,4254324	0	10,57456758
0,376670462	-0,314237158	94,379393	0	6,620607041
0,719676424	0,581880535	112,904948	11,90494759	0
0,58014044	0,202252771	104,649926	3,649925838	0

Table 5: Option pricing simulation table

The last step is concerned with the computation of the option price. It is evaluated as the present value of the average of all the option's payoffs. Where the average is obtained through the AVERAGE() function and the discount factor is simply e^{-rT}.

OUTPUTS	
CALL OPTION PRICE	=EXP(-D3*D7)*AVERAGE(E11:E2178)
PUT OPTION PRICE	6,89563248

OUTPUTS	
CALL OPTION PRICE	8,82044641
PUT OPTION PRICE	=EXP(-D3*D7)*AVERAGE(F11:F2178)

Table 6: Option Price Simulation on Excel

The computation of the predicted option prices through the Black-Scholes expressions (2.10) and (2.11) is essential to determine whether the results obtained are realistic or not.

This procedure begins with the computation of d1 and d2 through their corresponding formulas. Then it follows with the execution of the function NORM.DIST() to standardize d1 and d2 (null mean and unitary standard deviation) and find $\Phi(d_1), \Phi(-d_1), \Phi(d_2)$ and $\Phi(-d_2)$. At last, all the values found are inputted in the Black-Scholes formula to obtain the predicted option prices.

OUTPUTS	
CALL OPTION PRICE	8,64189465
PUT OPTION PRICE	6,97943434

D1	D2	N(D1)	N(D2)	BS CALL PRICE
0,175248346	-0,0247517	0,56955775	0,49012653	8,675223886

N(-D1)	N(-D2)	BS PUT PRICE
0,43044225	0,50987347	7,181525001

Table 7: Black-Scholes Option Price Computation on Excel

The approximation obtained is quite accurate since the simulated call price is 8,642 and the one predicted by the Black-Scholes equation is 8,675. However, by running again the same Excel sheet, different final values will be obtained, always in the same range but sometimes they might be slightly far from the predicted ones.

In the next section, where the same procedure will be carried out on R, the results obtained will acquire stability and accuracy because of the higher number of simulations run. This is thanks to the heavy computing power that programming languages such as R possess.

3.5 Black-Scholes option pricing simulation on R-studio

R is a programming language designed for statistical computing and data analysis. It provides a large range of tools and packages designed expressly for statistical analysis and data manipulation. R's main advantages are its vast functionality, great data processing skills, and sophisticated data visualization options. It interfaces well with other programming languages and software systems, allowing for quick data import/export and maximizing the strengths of multiple tools.

The algorithm begins the setting of a seed for reproducibility of the results. This makes sure that the same set of random numbers will be generated every time the code is run with the same seed value.

```
> set.seed(123)
```

After that, the input parameters for the option pricing model are fixed.

```
> S0 <- 100
```

```
> K <- 101
> r <- 0.025
> sigma <- 0.2
> T <- 1
```

Where S0 is the initial stock price, K is the strike price, r is the risk-free interest rate, sigma is the volatility of the stock price, and T is the time to maturity of the option.

In order to simulate the stock prices, the amount of Monte Carlo simulations to run (n) needs to be chosen. Usually, any number greater than one thousand leads to feasible results, but since it has been empirically proven that the greater n the more accurate the results and R grants enough computing power to do so, n is set to be 1mln.

```
> n <- 1000000
```

Furthermore, for the purpose of this simulation, a generator of n random numbers from a normal distribution with a mean of 0 and a standard deviation of 1 is required. This is accomplished through the rnorm function.

```
> Z <- rnorm(n)
```

Since all the parameters are set, it is now possible to execute the formula for the geometric Brownian motion (3.3) to simulate stock prices.

```
> S1 <- S0*exp(r - 0.5*sigma^2 + sigma*Z)
```

After that, since both the simulated S_1 and the strike price K are known, the call option payoff is computed with the pmax function, which takes the maximum value among those specified in the arguments. In the case of call option pricing, it evaluates $S_1 - K$ if $S_1 > K$ (scenario in which the option is ITM and therefore exercised at maturity), or it just gives 0 if $S_1 \leq K$ (ATM or OTM option that is not exercised). This process is repeated for each of the n stock prices S_1.

```
> payoff <- pmax(S1 - K, 0)
```

At this point, the discount factor is defined, and the option price is computed as the mean of the payoffs obtained previously times the discount factor.

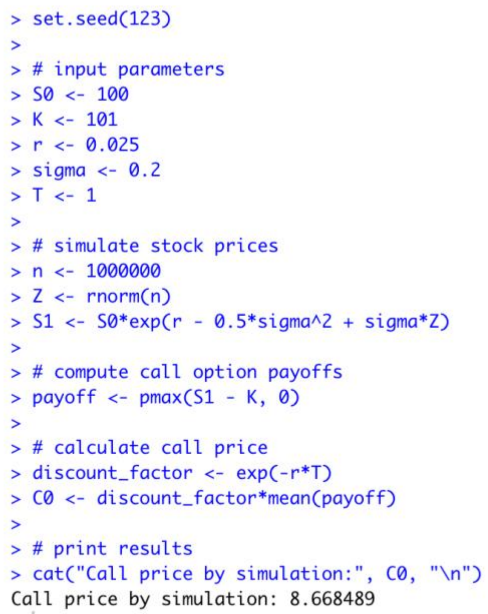
> discount_factor <- exp(-r*T)

> C0 <- discount_factor*mean(payoff)

At last, the cat function is adopted to print the results in a user-friendly manner.

> cat("Call price by simulation:", C0, "\n")

In summary, when the option price simulation algorithm is run on the R console the result obtained is shown below.



```
> set.seed(123)
>
> # input parameters
> S0 <- 100
> K <- 101
> r <- 0.025
> sigma <- 0.2
> T <- 1
>
> # simulate stock prices
> n <- 1000000
> Z <- rnorm(n)
> S1 <- S0*exp(r - 0.5*sigma^2 + sigma*Z)
>
> # compute call option payoffs
> payoff <- pmax(S1 - K, 0)
>
> # calculate call price
> discount_factor <- exp(-r*T)
> C0 <- discount_factor*mean(payoff)
>
> # print results
> cat("Call price by simulation:", C0, "\n")
Call price by simulation: 8.668489
```

Figure 7: Option Price Simulation on R

Now that the simulated option price is known, for comparison purposes the Black-Scholes predicted call price is calculated.

The procedure for this computation starts with the evaluation of d1 and d2 through their corresponding formulas.

> d1 <- (log(S0/K) + (r + 0.5*sigma^2)*T) / (sigma*sqrt(T))

> d2 <- d1 - sigma*sqrt(T)

After that the function N is defined, such that from an input x it will output the standardized values $\Phi(d_1)$ and $\Phi(d_2)$. Then it is implemented on the values d1 and d2.

> N <- function(x) {pnorm(x, mean = 0, sd = 1)}

> Nd1 <- N(d1)

> Nd2 <- N(d2)

Finally, the predicted call option price is computed through the Black-Scholes formula and the results are printed.

> C0_BS <- S0*Nd1 - K*exp(-r*T)*Nd2

> cat("Black-Scholes call price:", C0_BS, "\n")

The implementation of the R code is illustrated below.

```
> d1 <- (log(S0/K) + (r + 0.5*sigma^2)*T) / (sigma*sqrt(T))
> d2 <- d1 - sigma*sqrt(T)
> N <- function(x) {pnorm(x, mean = 0, sd = 1)}
> Nd1 <- N(d1)
> Nd2 <- N(d2)
>
> # calculate predicted Black-Scholes call price
> C0_BS <- S0*Nd1 - K*exp(-r*T)*Nd2
>
> # print results
> cat("Black-Scholes call price:", C0_BS, "\n")
Black-Scholes call price: 8.675224
```

Figure 8: Black-Scholes Option Price computation on R

In conclusion, through the R simulation algorithm the call price is 8.668 which is extremely close to the predicted price of 8.675. Moreover, this result, differently from the one in Excel, is extremely stable due to the vast number of simulations that are run (1mln on R versus 2 thousand on Excel).

Chapter 4: Poisson Process and Lévy Process

4.1 Poisson Process

4.1.1 Poisson Distribution

In probability theory and statistics, the Poisson distribution is a fundamental concept. It is named after Siméon Denis Poisson, a French mathematician who developed it in the early nineteenth century.

The distribution is defined by a single parameter, frequently represented as λ, which reflects the average rate or intensity of the event occurring. This parameter determines both the distribution's mean and variance. When dealing with events that are independent of each other, and exhibit a constant average rate, the Poisson distribution comes in handy.

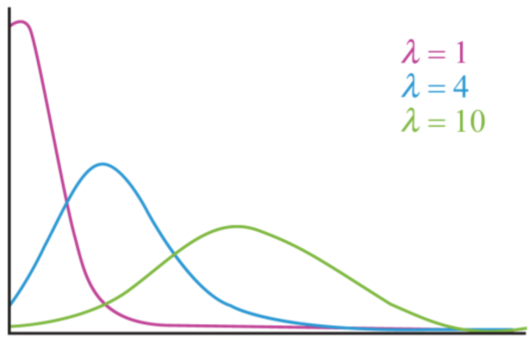


Figure 9: Poisson Distribution (Anon., 2020)

An integer valued random variable N is said to follow a Poisson distribution with parameter λ if:

$$\forall\, n \in \mathbb{N}, \mathbb{P}(N = n) = e^{-\lambda}\frac{\lambda^n}{n!}$$

An interesting feature of the Poisson distribution is its property of stability under convolution. It implies that when two independent Poisson variables, Y_1 and Y_2, with their corresponding parameters $\lambda_1$and λ_2, are added together, the resulting sum, $Y_1 + Y_2$, follows a Poisson distribution with the parameter $\lambda_1 + \lambda_2$.

As a consequence of the stability under convolution attribute, it can be proven that for any integer n, a Poisson random variable Y with parameter λ can be represented as the sum of n independent Poisson random variables, denoted as Y_i , each with a parameter of λ/n. This last property, known as infinite divisibility, suggests that a Poisson random variable can be divided into any desired number of independent and identically distributed (i.i.d.) random variables. In practice, this means that a Poisson distribution allows for the possibility to be decomposed into a collection of smaller and independent Poisson distributions. Infinite divisibility is a crucial property through which researchers and practitioners can effectively analyze complex systems involving event occurrences.

4.1.2 Poisson Process Definition

Let$(\tau_i)_{i\geq 1}$be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^{n} \tau_i$.

The process $(N_t\ , t \ \geq\ 0)$ defined by:

$$N_t = \sum_{n\geq 1} 1_{\ t \geq T_n}$$

is called a Poisson process with intensity λ.

The conditions that must be satisfied by a counting process to be a Poisson process are:

1. $N(0) = 0$
2. $\forall\ t_1 \leq t_2 \leq s_1 \leq s_2$ the random variables $N(t_2) - N(t_1)$ and $N(s_2) - N(s_1)$ are independent.

   The number of occurrences in one time interval is unrelated to the number of occurrences in other time intervals (independent increments).
3. There ∃ a $\lambda > 0$ such that given any $0 \leq t_1 \leq t_2$, $E[N(t_2) - N(t_1)] = \lambda(t_2 - t_1)$

   The expected number of occurrences between intervals of the same size is constant, known as stationary increments.
4. If $P(s) = \mathbf{P}\{[\mathrm{N}(\mathrm{t}+\mathrm{s}) - \mathrm{N}(\mathrm{t})] > 2\}$, then $\lim_{s\to 0} \frac{P(s)}{s} = 0$

   Formalization of the notion that events happen one at a time, where the limit affirms that the probability of two or more events occurring in an interval of length s is much smaller than s.

Therefore, the Poisson process is random process that counts the number of random times (T_n) which occur between 0 and t, where $(T_n\ -\ T_{n-1})_{n\geq 1}$ is an i.i.d. sequence of exponential variables.

The online sales made by Amazon is a simple example of this type of process, assuming that the number of purchases occurring in each consecutive time interval is independent and that the price of each sale is the same.

The Poisson process is commonly adopted by researchers and practitioners to model patterns whose arrivals occurs in a completely random way.

The properties that make this stochastic process so popular are several.

Firstly, the process is memoryless (Markov property), implying that future behavior is determined solely by the current state and is unaffected by previous events. The next event's timing is independent of all previous events, which simplifies analysis and modeling.

Secondly, a Poisson process's number of events within a given time interval follows a Poisson distribution. This distribution describes the likelihood of a given number of events occurring within a given time interval, providing a probabilistic framework for comprehending event occurrences.

Moreover, by its homogeneous intensity property, the process maintains a constant intensity over time, with the probability of an event occurring in a short interval proportional to the length of the interval. This property enables the modeling of systems with a consistent pattern of event arrival.

Furthermore, the Poisson process increments are independent, meaning that the number of occurrences in disjoint time intervals is unrelated.

In addition to being independent, the process's increments are also stationary, implying that statistical properties are unaltered for intervals of the same length. The distribution of events and the average event rate remain constant over time, allowing for consistent analysis across time periods. Here its crucial to emphasize that it is the only counting process with stationary independent increments.

Moreover, by knowing that the sum of independent Poisson random variables is Poisson distributed, it can be empirically proven that the addition or merger of independent Poisson processes is a Poisson process.

Another interesting feature of Poisson processes is the so-called thinning property. If a Poisson process $(N_t)_{t\geq1}$ is considered, and a new process X_t is defined by "thinning" N_t, which means that having taken all the jump events, they are kept with probability p or deleted with probability $1 - p$, independently from each other. At this point, the points that haven't been eliminated are ordered $T_1', \dots, T_n', \dots$ and X_t is defined as:

$$X_t = \sum_{n \geq 1} 1_{T_n' \geq t}$$

Then the process X is a Poisson process with intensity $p\lambda$

Stated in a simpler manner, when a Poisson process is thinned, it means that events are removed or filtered selectively based on certain chosen criteria. It is therefore a powerful property that allows for the selective inclusion or exclusion of events based on specific criteria, which can have far-reaching implications in a variety of fields.

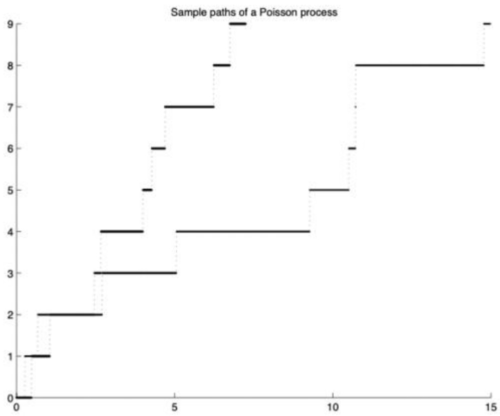


Figure 10: Poisson Process (Tankov & Cont, 2003)

In figure 9 above, two sample paths of a Poisson process can be seen. On the horizontal axis there is time, while on the vertical axis the number of arrivals. It is crucial to notice that each jump is of equal intensity λ.

4.1.3 Compensated Poisson Process

It is crucial to underline that the simple Poisson process is not a Martingale. However, if suitable adjustments are applied to the process, it can become a Martingale.

Such is the case for a compensated Poisson process, where through a compensator λt the Poisson process $(N_t)_{t \geq 1}$ is modified to possess the martingale property.

By definition, the compensated Poisson process is a more centered version of the Poisson process $(N_t)_{t \geq 1}$ and is described by the equation:

$$\widetilde{N_t} = N_t - \lambda t$$

Furthermore, it is relevant to emphasize that this new process $\widetilde{N_t}$ is no longer a counting process.

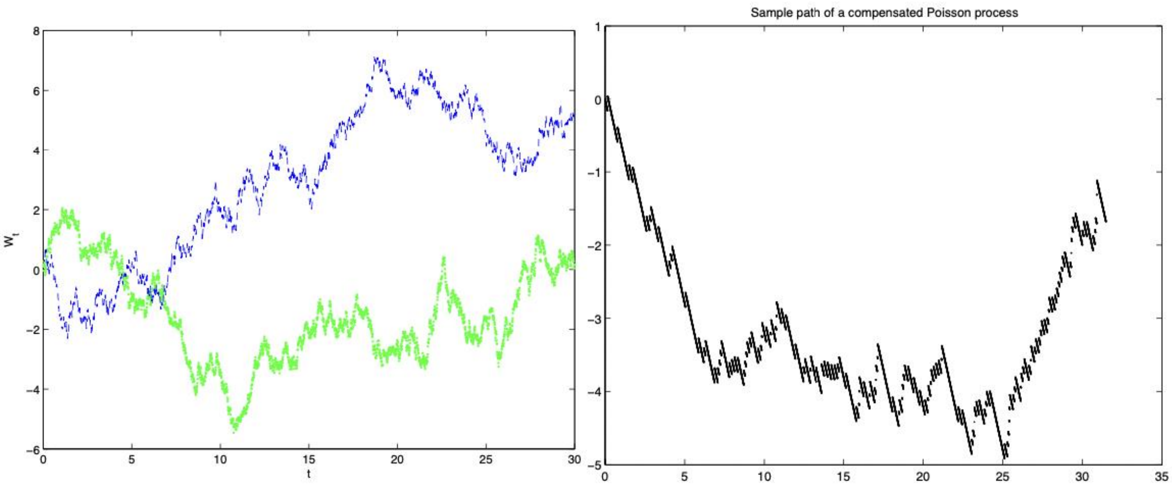


Figure 11: While on the left two sample paths of a Wiener process with σ = 1 are illustrated, the graph on the right shows the sample path of a compensated Poisson process with intensity λ = 5, rescaled to have the same variance as the Wiener process (Tankov & Cont, 2003).

In Figure 11 the rescaled version $\widetilde{N_t}/\lambda$ is compared to standard Wiener process. A solid bond between the two graphs can be inferred. This is because, as the intensity of its jumps increases, the (interpolated) compensated Poisson process's distribution converges to that of a Wiener process.

Next the compound Poisson process will be outlined, where together with the time of arrival, even the intensity of the jump will vary.

4.2 Compound Poisson Process

4.2.1 Compound Poisson Process Definition and Properties

A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f is a stochastic process X_t defined as:

$$X_t = \sum_{i=1}^{N_t} Y_i$$

Y_i is referred to as the jump size, which is i.i.d. with distribution f. Furthermore, N_t is a Poisson process with intensity λ, independent from $(Y_i)_{i\geq 1}$.

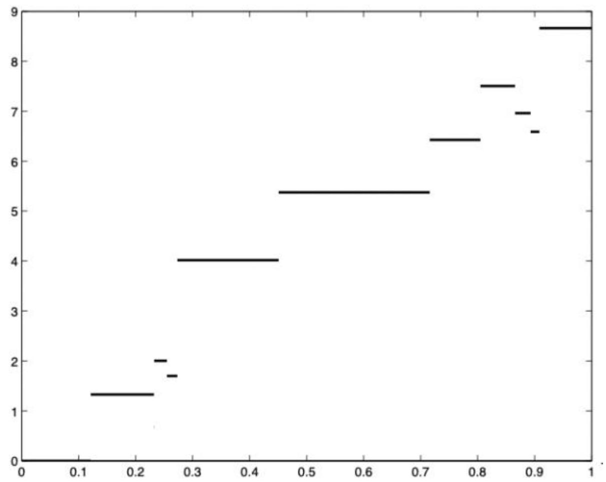


Figure 12: Compound Poisson Process (Tankov & Cont, 2003)

From Figure 12, the main trait that makes the compound Poisson process different from the simple one can be deduced: the possibility of having non-constant jump sizes. The Poisson process can be seen as a compound Poisson process whose Y_i is constant and equal to 1.

One crucial property of the compound Poisson process states that its sample paths are cadlag piecewise constant functions. Cadlag is an abbreviation for right continuous with left limits, meaning that the process has a right limit at each point and a left limit, which may differ from the right limit. The term piecewise constant refers to the fact that the process remains constant between consecutive event times and changes only when the Poisson process arrives. Considering how X_t accumulates the intensities of the events at the arrival times, the process remains constant between consecutive events, reflecting the absence of any additional events during that time interval, but when an event occurs, the process jumps to a new value based on the magnitude of the event, and then remains constant until the next event occurs. This is what causes the left limit to differ from the right one.

Moreover, the jump times $(T_i)_{i\geq 1}$ follow the same law as the jump times of the Poisson process N_t. Therefore, they can be expressed as partial sums of independent exponential random variables with parameter λ.

At last, the jump sizes $(Y_i)_{i\geq 1}$ are independent and identically distributed random variables, whose intensity is drawn from the same probability distribution *f*. Therefore, the value of one jump size is completely uncorrelated to the value of another jump size, while their statistical properties are identical.

A simple and intuitive example of compound Poisson process is described by the cumulative sales made on Amazon. In fact, the time at which sales take place is completely random and the purchase price differs from sale to sale, therefore resulting in non-constant jumps in the total revenue earned.

4.2.2 Convergence to the Normal Distribution

A compound Poisson process arises when events follow a Poisson process, and each event adds a random amount Y to the cumulative sum. Considering an event of type *j*, that is realized each time that the amount α_j is added, with $j \geq 1$. Meaning that the ith event of the Poisson process is of type j if it has $Y_i = \alpha_j$. Furthermore, denoting with $N_j(t)$ the number of type *j* events by time *t* it can be demonstrated that $N_j(t), j \geq 1$ are independent Poisson random variables with mean described as:

$$E\left[N_j(t)\right] = \lambda p_j t$$

At this point, since the amount α_j is added to the cumulative sum a total of $N_j(t)$ times by time *t*, the cumulative sum at time t can be expressed as:

$$X_t = \sum_j \alpha_j N_j(t)$$

(4.1)

Computing now the mean of X_t as:

$$E[X_t] = E\left[\sum_j \alpha_j N_j(t)\right] = \sum_j \alpha_j E\left[N_j(t)\right] = \sum_j \alpha_j \lambda p_j t = \lambda t E[Y_1]$$

It is now crucial to explain that in the case of a Poisson random variable, its distribution becomes more symmetrical and bell-shaped as its mean increases. This results from the notion that the mean of a Poisson distribution determines its shape, with larger means producing more concentrated distribution around the mean. Moreover, as the mean of a Poisson distribution increases, so do the number of individual events that contribute to the sum.

Therefore, equation (4.1) allows for the conclusion that as t increases, the distribution of X_t converges to the normal distribution. This is because, as t increases, each of the random variables $N_j(t)$ converges to a normal random variable. In addition, since they are independent, and the sum of independent normal random variables is also normal, X_t approaches a normal distribution as t increases.

4.3 Lévy Process

4.3.1 Introduction to Lévy Processes

As anticipated in chapter 2, there is strong empirical evidence indicating that most financial asset prices exhibit a distribution with heavy tails and a high peak. This kind of distribution is denominated leptokurtic, and it is used to describe distributions characterized by a large kurtosis, usually greater than three. To make a comparison with the normal distribution, that has a kurtosis equal to three, the leptokurtic distribution will differ for the higher peak and the heavier tails as shown in Figure 13 below.

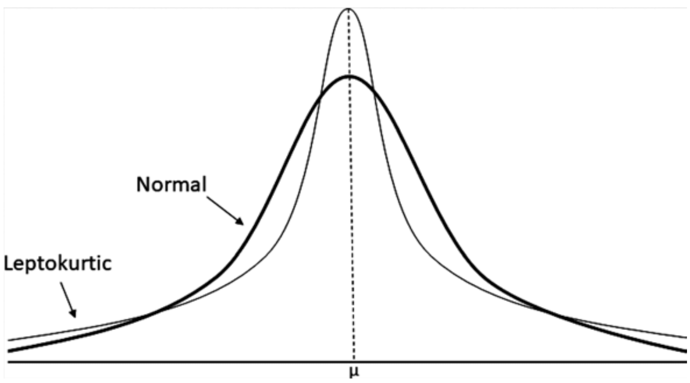


Figure 13: Comparison between leptokurtic and normal distribution (Lumholdt, 2018)

The traditional geometric Brownian motion model, which prices the stock as $S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$, fails to take into account this feature, since as assumed by the model the returns follow a normal distribution. Lévy processes on the other hand, have been proposed to incorporate the leptokurtic attribute.

Among the simplest examples of Lévy process are the Brownian motion, the standard Poisson process, and the compound Poisson process. However, one of the most well-known Lévy process is probably the jump diffusion process that combines the Brownian motion with the compound Poisson process.

Similarly to the Brownian motion, Lévy processes are adopted in modeling continuous movements, while, like the Poisson process, they capture sudden changes through jumps. This ability to incorporate abrupt moves by jumps is a crucial characteristic with significant implications for many practical scenarios.

4.3.2 Stable Distribution

A random variable is said to be stable if a linear combination of two independent copies of it has the same distribution as the random variable. X is said to be stable if for any positive numbers a and b, there exist a positive number c and a real number d such that:

$$aX_1 + bX_2 \sim cX + d$$

Where X_1 and X_2 are independent copies of X and the symbol $\sim$ means that they have the same identical distribution.

4.3.3 Lévy Distribution

Among the most popular stable distributions, there is the Lévy distribution, that has $\alpha = 1/2$ and $\beta = 1$. Therefore, the probability density function (PDF) of X would be given by:

$$f_x\left(x; \frac{1}{2}, 1, \sigma, \mu\right) = \left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(x-\mu)^{3/2}} e^{-\frac{\sigma}{2(x-\mu)}} \quad \text{with} \quad \mu < x < \infty$$

This distribution is leptokurtic which gives it a huge advantage over the Gaussian PDF when modelling stock prices, since their distribution has been proven to be highly leptokurtic. For instance, if we were to model the SPX from Jan 2, 1980 to Dec 31, 2005 and compute the kurtosis it would be about 42 (Kou, s.d.), which is extremely higher than the kurtosis of 3 of the normal distribution.

Furthermore, while a GBM can adequately describe stock price evolution most of the time, it was discovered that a large jump may occur from time to time (for instance when the stock market closes for holidays and new information is made known to market participants), which cannot be adequately captured by a GBM.

4.3.2 Lévy Process Definition

A cadlag stochastic process $(X_t)_{t\geq 1}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$ such that $X_0 = 0$ is called a Lévy process if it possesses the following properties:

1. Independent increments: for every increasing sequence of times $t_{0,} \dots, t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.
2. Stationary increments: the law of $X_{t+h} - X_t$ does not depend on t.
3. Stochastic continuity: $\forall \varepsilon > 0 \ \lim_{h\to 0} \mathbb{P}(| X_{t+h} - X_t| \geq \varepsilon) = 0$

Lévy processes constitute a wide class of stochastic processes whose sample paths can be continuous, continuous with occasional discontinuities, and purely discontinuous.

A noteworthy property of the Lévy process is the infinite divisibility, which states that its distribution can be expressed as the convolution of infinitely many i.i.d. random variables. This property adds to the flexibility of the Lévy processes in capturing complex dynamics.

Moreover, furtherly adding to the flexibility, Lévy processes are time homogeneous, meaning that their statistical features are constant over time. This property allows the usage of time-independent statistical measures which are crucial to correctly analyze the data and capture its characteristics.

Furthermore, as a consequence of having independent and stationary increments, the Lévy process has the Markov property, which states that conditionally on X_t, the evolution of the process in the future is completely independent of its past.

As previously mentioned, Lévy processes are stable processes. Therefore, the sum of independent Lévy processes remains a Lévy process when appropriately scaled and shifted. In other words, if two or more independent Lévy processes were taken, their parameters were adjusted, and then they were summed together, the resulting process will still have the characteristics of a Lévy process.

A key element to gain understanding of the tail behavior of the Lévy process and its moments it the Lévy measure, which is defined as:

Let $(X_t)_{t\geq 1}$ be a Lévy process on $\mathbb{R}^d$. The measure υ on $\mathbb{R}^d$ defined by:

$$\upsilon(A) = E[\# \{t \in [0,1]: \Delta X_t \neq 0, \Delta X_t \in A\}], \qquad A \in \mathfrak{B}(\mathbb{R}^d)$$

Is called the Lévy measure of X: $\upsilon(A)$ is the expected number, per unit time, of jumps whose size belongs to A.

Through the Lévy Khintchine representation, which is a specific form of the Lévy characteristic function, it can be understood that a d dimensional Lévy process X distribution is determined by the characteristic triplet (A, υ, γ). The Lévy Khintchine representation is as follows:

$$E\left[e^{iz.X_t}\right] = e^{t\psi(z)}, z \in \mathbb{R}^d$$

$$with\ \psi(z) = \frac{1}{2} z.Az + i\gamma.z + \int_{\mathbb{R}^d} (e^{iz.x} - 1 - iz.x1_{|x|\leq 1})\upsilon(dx)$$

In this expression, γ is the drift term, the positive semidefinite matrix A describes the Brownian motion component of X, and υ is a measure on $\mathbb{R}^d$ such that $\upsilon(A)$ is the rate at which jumps $\Delta X \in A$ of X take place. Therefore, the Lévy Khintchine representation encodes key information about the process's drift, diffusion, and jump characteristics.

4.4 The Merton Jump Diffusion Model

4.4.1 Introduction to the model

The Black-Scholes equation has worked as a solid foundation of many option pricing models. In fact, it has been demonstrated (Merton, 1973), that with the right adjustments the model could be adopted even in scenarios such as where the stock pays dividends, when the option is American and when the interest rate is stochastic. Moreover, it has been furtherly proven (Thorp, 1973), that the model is still a valid base in the case of stock dividends and of restrictions against the use of proceeds of short sales. However, the model fails to be consistent if the stock price dynamics are not representable by a stochastic process with continuous sample path. In essence, the Black-Scholes formula's validity is determined by whether or not stock price changes satisfy a 'local' Markov property (i.e. the stock price can only move by a small amount in a short period of time).

A jump stochastic process defined in continuous time would be the antipathetic process to this continuous stock price motion, because it would mean that, an unexpected jump large in magnitude could occur with a probability greater than zero.

The jump diffusion model, developed by Robert Merton in 1976, is a stock price behaviour model that incorporates small day-to-day "diffusive" movements (geometric Brownian motion) as well as larger, randomly occurring "jumps" (compound Poisson process). Because the inclusion of jumps allows for more realistic "crash" scenarios, the standard dynamic replication hedging approach of the standard Black-Scholes model no longer works. This causes option prices to rise in comparison to the Black-Scholes model and to be dependent on investors' risk aversion.

4.4.2 Hedging Strategies in Incomplete Markets

To introduce the jump, the Merton model considers one notable feature that distinguishes it from the standard Black-Scholes model: it makes the market incomplete, and there is no perfect hedging of options in this case.

Furthermore, when the jump size is continuous, the set of probability measure under which the discounted stock price is a martingale is infinite. Therefore, an infinite number of derivatives on the underlying would be needed for a perfect hedge to exist. Since the number of available derivatives on the underlying asset are finite and the transaction costs that would arise from this trading would be extreme, it is impossible to perfectly hedge the jump risk under a jump diffusion framework.

A common approach to hedging is through the usage of option Greeks. It involves adjusting options or their underlying assets to lessen the impact of changes in certain risk factors known as the Greeks. Delta, gamma, vega, theta, and rho are Greeks that measure an option's sensitivity to underlying price, volatility, time decay, and interest rate changes. The most popular Greeks hedging technique is the delta hedging outlined in the Black-Scholes model. However, adopting a hedging approach bases on Greeks has a major drawdown in jump diffusion frameworks. This is because the Greek letters are only effective for hedging the diffusive part of the process, and since the hedger is unable to rebalance the portfolio through a jump, the positions in the hedging instrument will be incorrect, resulting in significant hedging errors.

Adopting the Gauss-Hermite Quadrature, a different hedging technique that uses options with shorter maturity than the target option can be developed (Carr & Wu, 2004). Unlike dynamic delta hedging, which may struggle when confronted with significant random jumps in price dynamics, the static hedge suggested in this paper performs well under both continuous and discontinuous price movements. The results of this paper show that when the price dynamics are purely continuous, discretized static hedges with as few as three to five options outperform dynamic delta hedging with underlying futures and daily updating. When the underlying price process includes random jumps, static hedges significantly outperform the daily delta hedge. To back up their findings, the authors conducted a historical analysis on S&P-500 index options spanning over 13 years. This analysis confirms the static hedging strategy's superior performance in real-world scenarios.

Another strategy proposed to address this hedging issue involves the minimization of the squared change in portfolio value at some specific future point in time (He, 2006). The authors emphasize that in real markets, the availability of options with a specific maturity is limited to a few strikes. Since no other options can be chosen in the application, the strategy uses these strikes to minimize hedging error for a range of possible future stock prices, each weighted by some probability density function (PDF). The hedger can express the importance of minimizing the hedging error arising for specific values of Y using the PDF.

4.4.3 Formula derivation

The Black-Scholes model assumed that the stock price followed a diffusion process (geometric Brownian motion) defined as:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + JUMP\ COMPONENT \tag{4.1}$$

The Merton jump diffusion model adds a jump component to this equation, which has to be defined in size and arrival time. The size is denoted as Y_t and is lognormally distributed (so

$Y_t > 0$). Therefore, the stock price will be S_t before the jump and Y_tS_t after the jump. This means that if $Y_t > 1$ the jump will be positive, while if $Y_t < 1$ the jump will result in a lower final stock price. Knowing this, the change in price caused by the jump can be written as:

$$dS_t = Y_tS_t - S_t$$

So, the percentage change in price due to the jump will be:

$$\frac{dS_t}{S_t} = Y_t - 1$$

Now that the size of the jump has been defined, there is the need to model when the jump will take place. This can be easily done through a simple Poisson process N_t. Recalling that the probability of a jump is equal to $\lambda\Delta t$, the change in N_t can be modelled as:

$$dN_t = \begin{cases} 1 \; with \; probability \; \lambda dt \\ 0 \; with \; probability \; 1 - \lambda dt \end{cases}$$

At this point equation 4.1 can be rewritten as:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + (Y_t - 1)dN_t$$

Where $(Y_t - 1)dN_t$ represents the jump component that will be added to the relative change in stock price if the jump occurs ($dN_t = 1$), otherwise it will be negligible ($dN_t = 0$). Moreover, it can be inferred that $(Y_t - 1)dN_t$ is exactly a compound Poisson process, and therefore that the stock price change is determined by a combination of the geometric Brownian motion and the compound Poisson process.

However, it is important to notice now that the jump component that has been added introduces a drift as well. This can be seen by taking the expected value of the jump component as following:

$$E[(Y_t - 1)dN_t] = E[Y_t - 1] * E[dN_t] = k * \lambda dt$$

Where k is the expected value of the relative jump size and is computed as:

$$k = e^{\mu_y + \frac{1}{2}\sigma_y^2} - 1$$

The drift component is subtracted so that the jump component contribution can now be purely random jumps.

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + (Y_t - 1)dN_t - \ k * \lambda dt$$

At this point, in order to make the model flexible for larger time intervals, where more than one jump can occur, $(Y_t - 1)$ is adjusted as following:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + \prod_{j=1}^{dN_t}(Y_j - 1) - \ k * \lambda dt$$

Factorizing by dt and multiplying by S_t it becomes:

$$dS_t = (\mu - k\lambda)S_t dt + \sigma S_t dW_t + S_t\left(\prod_{j=1}^{dN_t}(Y_j - 1)\right)$$

Implementing the Ito's Lemma and making some computations it converts to:

$$d \ \ ln(S_t) = (\mu - k\lambda - \frac{1}{2}\sigma^2)dt + \sigma dW_t + \sum_{j=1}^{dN_t} ln \ (Y_j)$$

Now integrating from 0 to t:

$$ln(S_t) - \ lnS_0 = (\mu - k\lambda - \frac{1}{2}\sigma^2)t + \sigma(W_t - W_0) + \sum_{j=1}^{N_t - N_0} \ln(Y_j)$$

Since both the Brownian motion and the compound Poisson process begin at zero, both W_0 and N_0 are equal to 0. Rewriting the above expression, to reflect this information:

$$ln(S_t) - lnS_0 = (\mu - k\lambda - \frac{1}{2}\sigma^2)t + \sigma W_t + \sum_{j=1}^{N_t} \ln(Y_j)$$

(4.2)

Recalling now that Y_j was assumed to follow the log-normal distribution, it can be stated that $\ln(Y_j)$ follows a normal distribution $\sim N(\mu_y, \sigma_y^2)$. Furthermore, since $\sum_{j=1}^{N_t} \ln(Y_j)$ is a sum of normally distributed random variables, the result will be still normally distributed with mean $n\mu_y$ and variance $n\sigma_y^2$.

Furthermore, since it's known that $W_t \sim N(0, t)$, the linear transform σW_t will keep the normal distribution with mean zero but variance $\sigma^2 t$.

Therefore, since the sum of independent normals is still normally distributed, the summation:

$\sigma W_t + \sum_{j=1}^{N_t} \ln(Y_j)$ is $\sim Normal(n\mu_y, \sigma^2 t + n\sigma_y^2)$.

By factorizing the variance for t, it becomes:

$$\sim Normal\left(n\mu_y, \left(\sigma^2 + \frac{n\sigma_y^2}{t}\right)t\right)$$

Which now can be redrafted in terms of the standard normal variable Z as:

$$\sim n\mu_y + \sqrt{\sigma^2 + \frac{n\sigma_y^2}{t}}\sqrt{t} * Z$$

In addition, since $\sqrt{t}Z$ has the same distribution of W_t it can furtherly be rephrased as:

$$\sim n\mu_y + \sqrt{\sigma^2 + \frac{n\sigma_y^2}{t}}W_t$$

At this point, the two random components in equation 4.2 can be replaced by the linear transform of a single Brownian above and the expression would be:

$$ln(S_t) - \ lnS_0 = (\mu - k\lambda - \frac{1}{2}\sigma^2)t + \ n\mu_y + \sqrt{\sigma^2 + \frac{n\sigma_y^2}{t}} W_t$$

(4.3)

At this stage, in order to obtain a final equation easily comparable to the Black-Scholes expression, σ_n is defined as:

$$\sigma_n = \sqrt{\sigma^2 + \frac{n\sigma_y^2}{t}}$$

σ_n represents the total volatility per unit time due to the diffusion and the jump.

This would further simplify equation (4.3) that after some simple computations would become:

$$ln(S_t) - \ lnS_0 = \left(-k\lambda + \ n\mu_y + \frac{n\sigma_y^2}{2}\right) + (\mu - \frac{1}{2}\sigma_n^2)t + +\sigma_n W_t$$

Exponentiating both sides:

$$S_t = S_0 e^{\left(-k\lambda + \ n\mu_y + \frac{n\sigma_y^2}{2}\right) + (\mu - \frac{1}{2}\sigma_n^2)t + +\sigma_n W_t}$$

(4.4)

Absorbing the extra terms in the initial price defined as:

$$S_0^n = S_0 e^{\left(-k\lambda + \ n\mu_y + \frac{n\sigma_y^2}{2}\right)}$$

S_0^n is defined as the scaled version of the initial stock price to account for some of the effects of the jumps. Substituting it in equation (4.4) the following is obtained:

$$S_t = S_0^n e^{(\mu - \frac{1}{2}\sigma_n^2)t + +\sigma_n W_t}$$

Which is now extremely akin to the Black-Scholes equation (3.1):

$$S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$$

Comparing the two it can be inferred that they match perfectly except for the definition of initial stock price and the volatility. These differences are caused by the fact that the Merton model embeds the jumps into the process.

According to the Merton jump diffusion model the call option price of a European option is:

$$C(S_0^n, T \mid N_T = n)$$

Notice that this is conditional to having n jumps, but this can be adjusted by taking the iterated expectation such that the following is obtained:

$$C(S_0^n, T) = \sum_{n=0}^{\infty} C(S_0^n, T \mid N_T = n)\ P[N_T = n]$$

Where the number of possible jumps is conditioned, the option price is calculated, then weighted by the probability of jumps taking that value and at last summed across all possible values.

Since in the Merton model the number of jumps follows a Poisson process, and the Poisson density is given by:

$$P[N_T = n] = \frac{\lambda T^n}{n!} e^{-\lambda T}$$

The Poisson density can be substituted in the formula such that the final Merton formula for the price of a European call option is obtained.

$$C(S_0^n, T) = \sum_{n=0}^{\infty} C(S_0^n, T \mid N_T = n)\ \frac{(\lambda T)^n}{n!} e^{-\lambda T}$$

Conclusion

In this thesis a quantitative approach is undertaken to model financial instruments and evaluate options fair price. It all begins in chapter one with a foundation of stochastic calculus that will be useful when analyzing more complex stochastic models. In particular, the concept of Wiener process is introduced, which through the right adjustments mimics the stock price fluctuations and lays the base for one of the key assumptions of the Black-Scholes framework. In the following chapter, the Black-Scholes model is examined in detail through its assumptions, implications, and limitations. Some of the most popular methodologies to estimate volatility (the only parameter that cannot be directly observed in the markets) are illustrated, together with relevant research papers in which they are implemented, and the results analyzed. Moreover, benefits and drawbacks of adopting the geometric Brownian motion to model securities prices are discussed. In chapter three the Monte Carlo simulation approach is adopted to price derivatives (call and put options) and the notion of pseudo-random sequences is introduced. This method is implemented both through an algorithm on R and in an Excel spreadsheet, to illustrate the accuracy that a higher computing power can obtain. In the last chapter of this thesis Lévy processes (Poisson and compound Poisson processes) are examined and their properties outlined. Moreover, the Merton jump diffusion model is presented, together with the new hedging strategy for incomplete markets and the formula derivation. The Merton framework is introduced as an advanced Black-Scholes model since it holds for stock price jumps thanks to the implementation of a Lévy process that combines the geometric Brownian motion with the compound Poisson processes to model the security's price. The objective of this bachelor's thesis is to contribute to the field of quantitative finance by conducting an exhaustive analysis of the beforementioned concepts and models. By doing so, this thesis offers valuable insights into the intricate nature of mathematical finance.

Bibliography

Tankov, P. & Cont, R., 2003. *Financial Modelling and Jump Processes.* s.l.:Chapman & Hall.

UCLA, n.d. *Math UCLA edu.* [Online].
Available at: https://www.math.ucla.edu/~biskup/PDFs/PCMI/PCMI-notes-1

Fidelity, n.d. *Historical Volatility.* [Online]
Available at: https://www.fidelity.com/learning-center/trading-investing/technical-analysis/technical-indicator-guide/historical-volatility#:~:text=Description,over%20a%20period%20of%20time

Kenton, W., 2021. *Investopedia.* [Online]
Available at: https://www.investopedia.com/terms/l/log-normal-distribution.asp

Turney, S., 2022. *Scribbr.* [Online]
Available at: https://www.scribbr.com/statistics/poisson-distribution/

Daniel, J. W., 2019. *Poisson processes (and mixture distributions).* [Online].

Ermogenous, A., 2016. *Brownian Motion and Its Applications In The Stock Market*, s.l.: Undergraduate Mathematics Day, Electronic Proceedings.

Ambrož, 2002. *Oceňování opcí,* s.l.: Praha: C.H. Beck.

Madan, D. B., Chang, E. C. & Carr, P. P., 1998. The Variance Gamma Process and Option Pricing. *European Finance Review,* pp. 79-105.

Jones, C. S., 2003. The dynamics of stochastic volatility: evidence from underlying and options markets. *Journal of Econometrics,* pp. 181-224.

Westerfield, R., 1977. The Distribution of Common Stock Price Changes: An Application of Transactions Time and Subordinated Stochastic Models. *The Journal of Financial and Quantitative Analysis,,* pp. 743-765.

Taylor, S., 1986. *Modelling Financial Time Series.* s.l.:World Scientific Publishing Company.

Jandačka, M. & Ševčovič, D., 2004. On the risk-adjusted pricing-methodology-based valuation of vanilla options and explanation of the volatility smile. *Journal of Applied Mathematics.*

Scholes, M. & Black, F., 1973. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy,* pp. 637-654.

Krome, C., 2017. *Black-Scholes Model: an Analysis of the Influence of Volatility,* s.l.: University of Wisconsin-Milwaukee.

Von Neumann, J., 1951. The General and Logical Theory of Automata. *Journal of Modern Physics,,* pp. 1-41.

Lee, C., 2015. *Stochastic Processes I.* [Online]
Available at: https://www.youtube.com/watch?v=TuTmC8aOQJE

Roy, S., 2016. *Martingales.* [Online]
Available at: https://www.youtube.com/watch?v=qTg0mqxuGeA

Roy, S., 2016. *Brownian Motion-I.* [Online]
Available at: https://www.youtube.com/watch?v=4-Tas6Uw_OQ

Roy, S., 2016. *Brownian Motion II.* [Online]
Available at: https://www.youtube.com/watch?v=8aesZvl_P_k

Roy, S., 2016. *Brownian Motion III.* [Online]
Available at: https://www.youtube.com/watch?v=-MMnSlhGGTg&t=537s

Roy, S., 2016. *Stochastic Processes II.* [Online]
Available at: https://www.youtube.com/watch?v=PPl-7_RL0Ko

Tsitsiklis, J., 2012. *Poisson Processes I.* [Online]
Available at: https://www.youtube.com/watch?v=jsqSScywvMc

Tsitsiklis, J., 2012. *Poisson Processes II.* [Online]
Available at: https://www.youtube.com/watch?v=XsYXACeIklU

Davis, M. & Etheridge, A., 2006. *Louis Bachelier's Theory of Speculation: The Origins of Modern Finance.* s.l.:Princeton University Press.

Black, F. & Scholes, M., 1973. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy,* pp. 637-654.

Schmalensee, R. & Trippi, R. R., 1978. Common Stock Volatility Expectations Implied by Option Premia. *The Journal of Finance,* pp. 129-147.

Latané, H. A. & Rendleman Jr., R. J., 1976. Standard Deviations of Stock Price Ratios Implied in Option Prices. *The Journal of Finance,* pp. 369-381.

Anon., n.d. *Lévy Process.* [Online]
Available at: https://www.sciencedirect.com/topics/economics-econometrics-and-finance/levy-process

Winkel, M., 2004. *Introduction to Lévy processes.* [Online]
Available at: https://www.stats.ox.ac.uk/~winkel/lp1.pdf

Kou, S. G., n.d. *Lévy processes in Asset Pricing.* [Online]
Available at: http://www.columbia.edu/~sk75/levy.pdf

Ibe, O. C., 2013. Special Random Processes. In: *Markov Processes for Stochastic Modeling* . s.l.:Elsevier Science.

Wu, D. H., 2015. *Introduction to Stochastic Processes.* [Online]
Available at: https://ocw.mit.edu/courses/18-445-introduction-to-stochastic-processes-spring-2015/download/

Mcquighan, P., 2010. *Simulating the Poisson Process.* [Online]
Available at:
https://www.math.uchicago.edu/~may/VIGRE/VIGRE2010/REUPapers/Mcquighan.pdf

Lowther, G., 2010. *Poisson Processes.* [Online]
Available at: https://almostsuremath.com/2010/06/24/poisson-processes/

Anon., 2020. *Poisson Distribution in Stat.* [Online]
Available at: https://calcworkshop.com/discrete-probability-distribution/poisson-distribution/

Mitchell, C., 2021. *What Is a Volatility Smile and What Does It Tell Options Traders?.* [Online]
Available at: https://www.investopedia.com/terms/v/volatilitysmile.asp

Anon., 2023. *FTSE 100.* [Online]
Available at:
https://finance.yahoo.com/quote/%5EFTSE/chart?p=%5EFTSE#eyJpbnRlcnZhbCI6Im1vbnRoIiwicGVyaW9kaWNpdHkiOjEsImNhbmRsZVdpZHRoIjoyLjM0NzQ1NzYyNzExODY0NCwiZmxpcHBlZCI6ZmFsc2UsInZvbHVtZVVuZGVybGF5Ijp0cnVlLCJhZGoiOnRydWUsImNyb3NzaGFpciI6dHJ1ZSwiY2hhcnRUeXBlIjoibGl

Bhatia, A., 2016. *GBM usage in the market.* [Online]
Available at: https://www.quora.com/How-do-you-test-whether-stock-prices-follow-a-geometric-Brownian-Motion

Ma, D., 2015. *Introducing the Lognormal Distribution.* [Online]
Available at: https://probabilityandstats.wordpress.com/tag/lognormal-distribution/

Holton, G., 2013. *Brownian Motion.* [Online]
Available at: https://www.glynholton.com/notes/brownian_motion/

Ross, S. M., 2019. *Introduction to probability models.* s.l.:Academic Press.

Lumholdt, H., 2018. *Strategic and Tactical Asset Allocation: An Integrated Approach.* s.l.:Palgrave Macmillan.

Anon., 2013. *Stable Distribution.* [Online]
Available at: https://www.sciencedirect.com/topics/mathematics/stable-distribution

Kou, S. G., n.d. *Lévy processes in Asset Pricing.* [Online]
Available at: http://www.columbia.edu/~sk75/levy.pdf

Merton, R. C., 1973. An Intertemporal Capital Asset Pricing Model. *Econometrica,* p. 867–887.

Thorp, E., 1973. *Extensions of the Black-Scholes option model.* Vienna, s.n.

Wilmott, P., 2006. *Paul Wilmott on Quantitative Finance.* s.l.:John Wiley & Sons Inc.

He, C. e. a., 2006. Calibration and Hedging under jump diffusion. *Review of Derivarives Research,* pp. 1-35.

Carr, P. & Wu, l., 2004. Static Hedging of Standard Options. *Journal of Financial Econometrics,* pp. 3-46.