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Design and Analysis of Constant Function Market Makers

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Abstract

The rise of blockchain technology and its widespread adoption as infrastructure for deploying financial and non-financial decentralized applications brings the need for efficient and censorship-resistant digital asset exchange mechanisms to the forefront. In response to this evolving landscape, this work embarks on a comprehensive exploration of Constant Function Market Makers (CFMMs), pivotal entities in the realm of decentralized finance (DeFi) and broader blockchain utility. This work begins with an extensive primer on convex analysis, establishing the necessary notation that will be later employed to frame CFMMs within the scope of convex analysis, and develops from there a characterization of each core component of a CFMM. This work shows that every qualitative aspect of a CFMM is mapped into a certain property of its core component. This characterization allows to introduce a set of propositions which can be used as a toolkit in designing path-independent CFMMs. Most of the propositions of the presented toolkit are based on a polar correspondence between the invariant function and the portfolio value function of a CFMM, a correspondence which is rooted on the fact the convex set generating the hypograph of one function corresponds to the reverse polar of the convex set generating the hypograph of the other function. By designing one core component is possible to induce all the remaining ones to obtain a fully functional CFMM. As a practical application of the toolkit, this work shows how it is possible to induce the core components of Uniswap V2-like CFMMs, one of the most popular types of CFMMs in the DeFi ecosystem.

Chapter 1

Introduction

1.1 Decentralization and digital goods

The advent and proliferation of blockchain and distributed ledger technology (DLT) in general has ignited a profound infrastructural revolution that is silently but dramatically altering the foundations of industries, institutions, and economies. At its core, DLT embodies the promise of decentralization—an unprecedented paradigm shift that holds the potential to reshape the very essence of trust, governance, and economic exchange. In the traditional setting, the dynamics of trust and authority have long been characterized by a burdensome reliance on central authorities. These authorities, whether they take the form of financial institutions, governments, or intermediaries, have been tasked with the solemn responsibility of ensuring the integrity and security of transactions and records. However, this fiduciary role carries with it an inherent vulnerability: the dependence on central entities to act honestly in order to safeguard the interests of participants. Most of the time there is an alignment of incentives between such central authorities and the participants because a damage to the integrity of the system would be detrimental to both parties. However, when this alignment weakens or breaks down, the system is left vulnerable to the whims of the central authorities. In essence, these central authorities are the linchpin that maintains the fragile balance of trust within the system. Besides its efficiency, centralization has several points of failure both on the micro and macro levels. On the micro level, trusting a central authority for getting access to a particular service implies relying on the integrity, resilience and security of the platform on which the service is provided and which, most of the time, is not directly knowable by the user. Any malicious action or negligence on the part of the central authority that exposes the security of the platform to external attacks could result in, at best, denial of service, but possibly also a total compromise of the safeguarding of its users. On the macro level, the centralization of power in the hands of a few entities can lead to the creation of monopolies that can exploit their dominant position to the detriment of the users and the market in general. Having the same central authorities providing services to most of users generate negative network effects resulting in entry barriers for new competitors and, consequently, in a reduction of the incentives in delivering the best possible service because of the lack of competition. At the same time, in case of denial of service or other malicious actions, the centrality of the platforms for most of people implies that the damage is spread to all the users of the platform, inducing a systemic risk that could be avoided by a more distributed architecture. The rise of distributed ledger technologies challenges this status quo in a distributed and censorship-resistant fashion. It introduces a radical departure from the traditional centralized model by minimizing the need for intermediaries and central authorities. Instead, DLT systems, such as blockchain, operate as decentralized, censorship-resistant ledgers where trust is rooted in mathematics and consensus algorithms rather than in the intentions of centralized entities. This shift fundamentally alters the dynamics of trust, enabling economic interactions that are trustless in nature—transactions that occur without the need to trust any single party.

Reshaping digital goods and digital currencies The blockchain revolution has ushered in a transformative era for the concept of “digital goods", fundamentally redefining these intangible entities and revolutionizing the notion of ownership. In the pre-blockchain era, digital goods often existed solely as data files registered on third-party owned platforms, susceptible to replication and easy dissemination, raising questions about their authenticity and the legitimacy of ownership. With blockchain technology, digital goods become simple records registered on a decentralized ledger typically via an account based model. The ownership of digital assets is now intricately woven into the fabric of the blockchain and their validity is based on consensus among participating nodes. This paradigm shift provides a deterministic and tamper-resistant proof of ownership for digital goods. As a result, blockchain has not only redefined the concept of digital goods but has also introduced a novel dimension of trust, where ownership is no longer a matter of trust in centralized authorities but a product of mathematical certainty within a decentralized network. Before blockchain, digital goods were typically registered on isolated and non-composable proprietary platforms, often closed-ended in nature. This meant that all rights and ownership associated with these digital assets were confined within the boundaries of these platforms and couldn’t transcend their proprietary constraints. However, blockchain-based digital goods usher in a new paradigm, offering a shared playground where ownership and rights are not confined to closed platforms built on top of a certain blockchain. Instead, they become interoperable, portable, and exist independently of any specific platform built on top of that particular blockchain. This shift empowers users with greater control and flexibility over their digital assets, enabling them to traverse a broader digital landscape and participate in a shared, decentralized economy. Digital currencies are the most prominent example of fungible, divisible, blockchain-based digital goods and they are typically associated with the original use of blockchain as a decentralized system of payments [Nak09]. Naturally, the existence of different digital currencies surges with the need to find a way to exchange them in a decentralized fashion. Of course, interacting with a platform managed by a centralized exchange (CEX) which has the custody of the digital accounts of its users is still a possible route for exchanging digital currencies. However, even if this has been the most popular way of exchanging digital currencies so far, this approach is not in line with the decentralized nature of blockchain and it is also subject to the same problems as the traditional centralized model, most of the time exacerbated by the lack of regulations caused by the difficulty of framing such entities from a regulatory perspective, resulting in financial collapses like the FTX case in 2022 [CCH22].

1.2 Blockchain-based automated market makers

Decentralized exchange A decentralized venue for exchanging digital goods is typically called "decentralized exchange" (DEX) and it comes with several issues. The most obvious is the lack of a central entity managing it or the lack of a dedicated authority for liquidity provisioning via market making activity. Moreover, there is also the problem of "importing" price information of such digital currencies from external markets (like those ran by CEXs) to avoid arbitrage opportunities. This problem is typically mentioned as the "oracle problem" [Cal20] and, generally speaking, refers to the problem of importing in a decentralized ecosystem any information that is not natively available on-chain: indeed, the figure of the oracle is typically associated with the one of a trusted third party that provides information to the blockchain and this creates a centralization point exposed to all the risks mentioned above. To avoid these issues, a desirable DEX should be a headless and oracleless system which incentives individual market participants to provide liquidity to the market in a decentralized fashion. These properties are typical of the so-called "automated market makers" (AMMs) which are inventory-based systems allowing market participants to deposit, withdraw and exchange goods according to a "scoring rule" which maps the inventory of the AMM to the marginal prices of the assets which are negotiated [Han03]. The literature of AMMs is older than blockchain itself but finds with this technology a practical application.

Constant Function Market Makers The term "Constant Function Market Maker" (CFMM) has been introduced for the first time in [AC20] to describe a blockchain-based AMM which could be used as a decentralized venue for exchanging digital currencies and so far it remains the most popular approach for designing DEXs. Some working examples are Uniswap [AZR20] [Ada21], Balancer [MM19], Curve [Ego21] which share the same core aspects of a CFMM but differ in the way they are designed. Together with lending protocols and other blockchain-based financial applications, CFMMs are part of the so-called "Decentralized Finance" (DeFi) ecosystem which is one of the most popular use cases of blockchain technology. In its principles, a CFMM corresponds to a "smart contract" (i.e. a program running on a blockchain) which collects a set of methods through which market agents are capable of depositing and withdrawing digital currencies to and from the CFMM and exchanging them according to a certain pricing rule. The platform charges a fee for each exchange and the collected fees are typically distributed pro-rata to the market agents who provided liquidity to the CFMM, also called "liquidity providers" (LPs): this corresponds to the main incentive for market agents to provide liquidity to the CFMM. Even if the main purpose remains to be a decentralized venue used for exchanging digital currencies, the CFMM itself becomes a price oracle and, at the same time, it becomes a device for passive replication of concave payoffs for liquidity providers. However, the surge of lending protocols allowing users to borrow and repay loans within a single blockchain transaction (flash-loans) made it unsafe for a blockchain application to rely naively on a single CFMM for collecting price information of a certain asset. Indeed the capability offered by such lending protocols in disposing temporarily of huge amounts of assets at low cost could be exploited by malicious agents to manipulate temporarily the price "oracled" (in the sense of "signaled") by the CFMM and hack the blockchain application relying on that CFMM as price oracle [CZC21]. At the same time, being a blockchain-based AMM, the general architecture of a CFMM expands its scope also to a wide range of financial applications besides being a decentralized venue for exchanging digital currencies, like prediction markets [FPW23].

Portfolio value function Providing liquidity to the CFMM implies that LPs expose their portfolio holdings to a certain adverse-selection effect. Indeed, as a result of the trading activity performed by external agents, the portfolio dynamics of the assets deposited in a CFMM increase the exposure on the worst performing assets by decreasing the exposure on the best performing assets, simply because market participants have the incentive of pulling out from the CFMM the assets which are being appreciated on an external market in exchange of the assets which are relatively depreciating. This implies that the value of the portfolio of assets dedicated to the pool experiences eventually sublinear growth as the prices of the assets increase and, because of that, liquidity providers suffer the opportunity cost of providing liquidity compared to simply holding the assets in their portfolio (in this case, the portfolio value would experience linear growth). This opportunity-cost is typically defined as "impermanent loss" [AEC20] and the fees collected by the CFMM could be seen as a compensation for it. On the other hand, because of trading activity and arbitrageur forces, it is possible to recover deterministically the amounts of each asset at any price level and, from there, the value of the resulting portfolio of assets dedicated to the pool. The mapping between the external prices and the value of the portfolio of assets dedicated to the pool is typically called "portfolio value function" $\hat{V} : \mathbb{R}_+^n \to \mathbb{R}_+$ and it is a fundamental component of a CFMM. Because of the way a CFMM is designed, this work shows that any closed, non-negative, non-decreasing, origin-vanishing, positive-homogenous concave function could be conceived as a portfolio value function of a CFMM.

Feasible trades, reachable reserves and invariant function Taking the perspective of a CFMM, the trading activity is nothing more than a sequence of rebalance operations of the portfolio of assets dedicated to the pool such that the resulting portfolio satisfies always a certain condition. In the common case, introducing $\hat{L} : \mathbb{R}^n_+ \to \mathbb{R}_+$ as a utility function mapping the inventory of the CFMM to some utility level, the condition is that the utility associated with the portfolio of assets $x_1 \in \mathbb{R}^n_+$ after a trade is performed is greater or equal than the utility associated with the portfolio of assets $x_0 \in \mathbb{R}^n_+$ before the trade is performed. In other words $y \in \mathbb{R}^n$ is a feasible trade and $x_1 = x_0 + y \in \mathbb{R}^n_+$ is a reachable reserve if and only if $\hat{L}(x_1) \geq \hat{L}(x_0)$: this simple condition defines at the same time the set of "feasible trades" $T(x_0) \subset \mathbb{R}^n_+$ (i.e. portfolio rebalances potentially accepted by the CFMM) and the set of "reachable reserves" $C \subset \mathbb{R}^n_+$ (i.e. the set of portfolio holdings of the CFMM which could be potentially reached by performing feasible trades). In this work, the utility function just mentioned will be denoted as "invariant function" and it will be characterized as a closed, non-negative, non-decreasing, origin-vanishing, positive-homogenous concave function. Moreover, the utility of the pool will be measured in terms of "liquidity units" and so the terms "utility" and "liquidity" will be considered interchangeable for CFMMs. The invariant function plays a fundamental role in the design of a CFMM, being the core of the feasibility condition of trades and embedding the scoring rule of the CFMM conceived as a blockchain-based AMM. In fact, consistently with what is said in [AC20], in this work it will be proved that the scoring rule embedded in a CFMM corresponds to a function $\Xi : \partial\hat{L}(x_0) \subset \mathbb{R}^n \to \mathbb{R}^n_+$, similar to the perspective function described in [BV04], which maps a generic supergradient of the invariant function $\hat{L}$ evaluating the current inventory x_0 to a vector of $n-1$ marginal prices (not n since one of the assets is used as numeraire, or "quote asset", in order to have a unit of measure for the prices of the other assets), which could be considered as the execution price for a infinitesimal trade, so that the *ex-post* reserves $x_1 \in \mathbb{R}^n_+$ remain in the neighborhood of the current inventory $x_0 \in \mathbb{R}^n_+$. Thus, since superdifferentiability is always granted because of the concavity of the invariant function $\hat{L}$, differentiability remains a desirable property for an invariant function because it ensures unambiguity in marginal prices at every vector of current reserves $x_0 \in \mathbb{R}^n_+$. Indeed, if the invariant function is differentiable, its superdifferential will be always the singleton of the gradient of the invariant function and the scoring rule will always map to a single vector of marginal prices. The characterization of the invariant function introduced in this work is stricter compared to its description in [AC20], however as mentioned in [ACD$^+$23] it is always possible to map a CFMM with a general invariant function to an equivalent CFMM with an invariant function satisfying the stricter characterization and this allows to define an equivalence class of invariant functions.

Literature review Being a fertile resource of many potential blockchain applications, the existing literature on CFMMs is quite vast: [GRGM23] defines a framework for mapping liquidity providers belief about future valuation of digital goods into an optimal choice of the CFMM invariant function to maximize their expected utility, [ACEB22] and [DRCA23] show how common financial problems involving CFMMs like optimal routing or arbitrage trading could be framed as convex optimization problems and solved efficiently via dual decomposition, [MMRZ23] introduces a synthetic risk metric for liquidity provisioning called "Loss-Versus-Rebalancing" (LVR) and suggests it could be the starting point of a dynamic fee architecture for rewarding fairly liquidity providers, [BN23] and [Cla23] show how it's possible to delta and gamma-hedge CFMMs liquidity position via negative replication of the portfolio value function embedded in CFMMs, [AEC21a] and [AEC21b] show that it's possible inducing invariant functions of CFMMs starting from generic non-negative, non-decreasing concave portfolio value functions.

1.3 Main goal of this work

Focusing on a CFMM as a decentralized venue for exchanging digital currencies and as a ETF-like investment vehicle for liquidity providers in replicating concave payoffs, it's possible to argue that the core components characterizing the mechanics of a CFMM are:

- The invariant function $\hat{L} : \mathbb{R}^n_+ \to \mathbb{R}_+$ for disciplining the trading activity and the liquidity provisioning/withdrawing
- The portfolio value function $\hat{V} : \mathbb{R}^n_+ \to \mathbb{R}_+$ for being the concave payoff passively replicated by liquidity providers
- The set of reachable reserves $C \subset \mathbb{R}^n_+$ for describing which portfolio holdings are reachable via the trading activity
- The set of feasible trades $T(x_0) \subset \mathbb{R}^n$ for describing the domain of the trading activity given inventory $x_0 \in \mathbb{R}^n_+$

Because of the properties encoded in the core components of a CFMM, the application of convex analysis provides a means to attain a holistic understanding and thorough characterization of each core component inherent to a CFMM. Indeed, the purpose of this work is to demonstrate how each core component is actually capable of inducing all the other core components by applying the theoretical framework of convex analysis, thus providing a powerful toolkit for designing and customizing CFMMs. This approach not only yields a comprehensive characterization but also offers significant freedom in the design of a CFMM, as starting from any one of these core components enables the construction of a fully functional and well-defined CFMM. This work is split into two main parts: the first part is dedicated to give some extensive primers on the theoretical framework of convex analysis fixing also the notation that will be used throughout the work, while the second part is dedicated to the characterization of the core components of a CFMM and to define the set of proposition dedicated to the induction of the core components of a CFMM starting from any of them.

1.4 First part overview

1.4.1 Fundamental sets and their representation

Most of the notation used in the second chapter is consistent with that one used in [Roc70]. The second chapter is organized as follows: the first subsection of the second chapter is dedicated to the discussion about affine sets, convex sets and cones as essential components of convex analysis. Each fundamental set is introduced after discussing about the associated fundamental operation of affine, convex and conic combinations. Regarding affine sets, hyperplanes are presented as prominent example of affine sets underlying how their set notation is easily via the orthogonal complement of their parallel subspace and, at the same time, are introduced several ways of representing a m-dimensional affine space as intersection of $n-m$ hyperplanes, as solution set of $n-m$ affine equations in n variables, as graph of an affine transformation $A : \mathbb{R}^m \to \mathbb{R}^{n-m}$ (also called Tucker representation) and as affine hull of $m+1$ of affinely independent points. Regarding convex sets, half-spaces and polyhedra in general are presented as prominent examples of convex sets and, after discussing about Minkowski metric function, the unit ball is introduced as another example of convex set extremely useful also for understanding topological properties of convex sets in general like the concepts of relative interior, closure and relative boundary of a convex set. Regarding cones, after introducing the main properties and the concept of Conic hull, it is discussed the concept of proper cones as a device for defining generalized inequalities, which are going to be extremely useful in the second subsection of the second chapter. Here is discussed a fundamental concept which is going to be heavily used throughout the work, which is the fact that any convex cone containing the origin $K \subset \mathbb{R}^{n+1}$ can be actually generated as the Conic hull of a convex set $C \subset \mathbb{R}^n$ represented in higher dimension [Roc70]. The discussion about cones ends with presenting the two subspaces associated with any convex cone and with the introduction of several examples of convex cone containing the origin which are going to be used in several proofs. Such cones are the normal cone to $C \subset \mathbb{R}^n$, the polar cone to K, the dual cone of K, the barrier cone of C and its recession cone. This first subsection of the second chapter ends with the discussion about hyperplane separation theorem and the concept supporting hyperplane used for discussing about smoothness and differentiability from the point of view of convex analysis.

1.4.2 Pareto Optimal Frontier

The second subsection of the second chapter is dedicated to the discussion about minimum points, minimal points and Pareto Optimal Frontier. This subsection is the only subsection of the second chapter which uses mainly the notation borrowed from [BV04]. Such concepts allow to give a very brief anticipation about the set of “efficient reserves” as the subset of reachable reserves where the inventory of a CFMM is expected to lie at every price level because of arbitrage forces. Indeed, such set corresponds to the set of minimal points (also defined as Pareto Optimal Frontier) using the non-negative orthant as proper cone of reference for the generalized inequality used in the definition of minimal point.

1.4.3 Convex and concave functions

The third subsection of the second chapter is dedicated to the discussion about convex and concave functions: after introducing the definitions of convex and concave functions the Jensen's inequality is discussed and applied for retrieving the famous inequality between arithmetic mean and geometric mean (which is going to be widely used in the last subsection of this work in recovering some core components of Uniswap V2-like CFMMs). Then, the concepts of lower semi-continuity and upper semi-continuity are introduced and the discussion moves to the properties of ordinary continuity and Lipschitzian continuity applied to convex and concave functions. Positive homogeneous function are introduced for understanding which properties are encoded in convex cones containing used as epigraphs of convex functions or as hypographs of concave function: this is fundamental for understanding most of the properties associated with invariant function and portfolio value function of a CFMM. Then, recalling some findings of [Ber09], the recession function is introduced as a device for understanding the asymptotic behavior of a convex function moving in certain directions: this turns out extremely important in inferring the asymptotic behavior of the invariant and portfolio value function as well as the properties of the respective convex sets generating their hypographs. This subsection ends with the discussion about support functions because of their importance in designing CFMMs since in later subsections it will be proved that both the invariant function and the portfolio value function of a CFMM can be characterized as the negative of the support function of certain convex sets.

1.4.4 Function builder

The fourth subsection of the second chapter is dedicated to the application of a powerful device introduced in [Roc70] for inducing a convex function $f : \mathbb{R}^n \to \mathbb{R}$ or a concave function $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ from a generic convex set $F \subset \mathbb{R}^{n+1}$

$$f = \inf \{\mu : (x, \mu) \in F\} \qquad \hat{f} = \sup \{\mu : (x, \mu) \in F\}$$

This device is going to be used to derive all the functional forms of the functions introduced in the previous subsections. This includes the functions which can be induced by manipulating the epigraph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ like the lower semicontinuous hull of f, the recession function of f or the positive homogenous convex function generated by f, but also the functions which can be obtained by performing some operations on the epigraphs of a collection of convex functions like the infimal convolution function. As discussed in the third chapter, the device introduced in this subsection is extremely important for inducing, similarly to [ACD$^+$23], the invariant function and the portfolio value function of a CFMM starting from convex cones containing the origin living on the non-negative orthant. This subsection ends with the discussion about some polar correspondences between a convex cone K generated in higher dimension by a convex set (i.e. $K = \text{cone}(\{(1,x) : x \in C\})$), its polar cones K° and its dual cone $K^\star$: this leads to a characterization of K°, $K^\star$, and their image under affine map $A : (\mu, x) \mapsto (-\mu, x)$ as epigraphs and hypographs of convex and concave support functions of the set $C \subset \mathbb{R}^n$ generating K. Finally, these polar correspondences reveal two important inequalities related to gauge-like functions (as defined in [Roc70]) induced by $A(K^\circ)$ and $A(K^\star)$ and the the gauge-like function induced by $K = \text{cone}(\{(1,x) : x \in C\})$: this is going to be extremely useful for characterizing the portfolio value function of a CFMM in terms of the invariant function of the same CFMM and vice versa as shown in [ACD+23]

1.4.5 Directional derivatives and subgradients

The fifth subsection of the second chapter introduces the concepts of one-sided directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ showing that it corresponds to the positive homogenous function generated by the "variation function"

$$P(y; x_0) = f(x_0 + y) - f(x_0)$$

The use of the variation function points out also the main differences between the one-sided directional derivative of f and the recession function of f showing how, in the first case, the behavior of f is studied along the direction of $y \in \mathbb{R}^n$ but remaining in the neighborhood of x_0 while in the second case, the behavior of f is studied targeting the "horizon points" defined by the direction of $y \in \mathbb{R}^n$. Subsequently, a characterization of the one-sided directional derivative of a convex function as support function of some convex set allows to introduce the subgradient inequality as set notation of the subdifferential of f: indeed, $\partial f(x_0)$ is introduced as the closed convex set supported by the one-sided directional derivative of f. This leads to the introduction of the concepts of subdifferentiability of convex functions, superdifferentiability of concave functions and differentiability in general showing how, in the latter case (when the subdifferential of the convex function or the superdifferential of the concave function resembles to the singleton of the gradient of the function), the one-sided directional derivative can be recovered as the inner product between the gradient of the function (evaluating the generic point $x_0 \in \mathbb{R}^n$) and the vector $y \in \mathbb{R}^n$ associated with the direction of the one-sided directional derivative. Finally, the subsection ends with a reformulation of the subgradient inequality in terms of the Fenchel conjugate $f^\star$ as an anticipation of the next subsection.

1.4.6 Fenchel conjugate

The last subsection of the second chapter deals with the discussion about the Fenchel conjugate of a convex function as described in [Roc70]. Firstly, the closure of a convex function f is reformulated as the convex function induced by the intersection of half-spaces fully containing $\text{epi}(f)$ taking advantage from the fact that any closed convex set has an "external" representation given by the intersection of half-spaces which are including the set. Then, after defining the collection of such half-spaces as the collection of the epigraphs of some affine function $g(x; x^\star, \mu^\star) = \langle x, x^\star \rangle - \mu^\star$ such that $f(x) \geq g(x; x^\star, \mu^\star)$ indexed by $(x^\star, \mu^\star) \in \mathbb{R}^{n+1}$, the Fenchel conjugate is defined as the convex function induced by the index set of such collection, which is

$$F^\star = \{(x^\star, \mu^\star) \in \mathbb{R}^n \times \mathbb{R} : f(x) \geq g(x; x^\star, \mu^\star),\ x \in \text{dom}(f)\}$$

The functional form of the Fenchel conjugate allows to introduce the famous Fenchel inequality.

1.5 Second part overview

1.5.1 Analysis of the core components of a CFMM

The second part of this work, starting from the first two subsections of the third chapter, deploys all the concepts introduced in the first part to provide an exhaustive framework about each core component of a CFMM and the associated properties. The basic set of reachable reserves $C \subset \mathbb{R}^n_+$ is characterized as the set of reserves which are reachable by performing feasible trades when the liquidity value of the CFMM is equal to one. This set is characterized as a closed unbounded convex set, at least two-dimensional, living on the non-negative orthant and not including the origin, such that its recession cone corresponds to the non-negative orthant. The reason behind each property is explained in detail in the dedicated subsection. This convex set does the service of generating the invariant cone $K_{\hat{L}} = \text{cone}(\{(1, x) : x \in C\})$ which corresponds to the hypograph of the invariant function $\hat{L} : \mathbb{R}^n_+ \to \mathbb{R}$, which in fact is a gauge-like concave function similarly to the portfolio value function $\hat{V} : \mathbb{R}^n_+ \to \mathbb{R}$. Starting from the arbitrage problem, the portfolio value function is naturally introduced as the "fair value" of the portfolio held by a CFMM so that, given the current inventory and the vector of external prices, the optimal arbitrage profit is annihilated. This is also the reason why the portfolio value function describes the concave payoff passively replicated by liquidity providers: because if the portfolio value held by the CFMM is different from the image of the portfolio value function given some external prices then there are arbitrage opportunities. This allows to characterize the portfolio value function as the negative of the support function of the symmetric reflection across the origin of the basic set of reachable reserves (i.e. $\hat{V}(p; 1) = -\delta^\star(p| - C)$). Throughout the dedicated section, it is also showed that invariant functions $\hat{L}$ and portfolio value functions $\hat{V}$ share the same properties being both gauge like concave functions induced by some convex cones containing the origin (the invariant cone $K_{\hat{L}}$ and the portfolio value cone $K_{\hat{V}}$). The polar correspondence between the portfolio value function and the invariant function according to which $\hat{L}(x)\hat{V}(p; 1) \leq \langle x, p \rangle\ \forall (x, p) \in \mathbb{R}^{2n}$ is caused by the fact that the portfolio value cone $K_{\hat{V}}$ (i.e. the hypograph of the portfolio value function) is actually the convex cone containing the origin generated in higher dimension by the reverse polar $C^\star$ [Zaf12] of the set of reachable reserves C (i.e. $C^\star = \{p : \langle x, p \rangle \geq 1, x \in C\}$), and C is also the convex set generating in higher dimension the invariant cone (i.e. the hypograph of the invariant function): this correspondence allows to characterize the portfolio value function in terms of invariant function and vice-versa and this is consistent with [ACD$^+$23]. In particular, the reverse polar $C^\star \subset \mathbb{R}^n_+$ generating the portfolio value cone $K_{\hat{V}} = \text{cone}(\{(1,p) : p \in C^\star\})$ shares the same properties of the basic set of reachable reserves $C \subset \mathbb{R}^n_+$. It is also shown that any convex set D sharing all the properties of a basic set of reachable reserves with the exception of unboundedness can do the service of generating an invariant cone $K_{\hat{L}}$ or a liquidity cone $K_{\hat{V}}$ and so to characterize the design of a CFMM: however, such set D will meet the first-upper level set of the invariant function (or of the portfolio-value function) only on the "lower boundary" (since the level set will be a sort of "unbounded version" of the set D). Finally, given $x_0 \in \mathbb{R}^n_+$ as current inventory, the set of feasible trades $T(x_0)$ is defined as the upper level set at level zero of the variation function $P : \mathbb{R}^n \to \mathbb{R}$ of the invariant (i.e. $T(x_0) = \{y; P(y; x_0) \geq 0\}$ where $P(y; x_0) = \hat{L}(x_0 + y) - \hat{L}(x_0))$ which is the concave function induced by the hypograph of the invariant function $\hat{L}$ under an affine map which "shifts" it so that the pair $(x_0, \hat{L}(x_0))$ is mapped into the origin.

1.5.2 A toolkit for designing CFMMs

Besides the characterization of the core components of a CFMM in terms of convex analysis, the core of this work is also providing a set of proposition which acts as a toolkit to design a CFMM starting from any of its core components, since the remaining ones will be immediately derivable. The toolkit is summarized as follows: the basic set of reachable reserves $C \subset \mathbb{R}^n_+$ can be seen as the upper-level set of the invariant function $\hat{L}$ at level one (i.e. $C = \left\{x : \hat{L}(x) \geq 1\right\}$), but also as the effective domain of the concave conjugate of the portfolio value function (i.e. $C = \text{dom}(\hat{V}^\star)$) and as the Minkowski sum between the set singleton of current inventory $x_0 \in \mathbb{R}^n_+$ and the set of feasible trades $T(x_0)$ (i.e. $C = T(x_0) + x_0$); the portfolio value function $\hat{V} : \mathbb{R}^n_+ \to \mathbb{R}_+$ can be seen as the negative of the support function of the symmetric reflection across the origin of the basic set of reachable reserves (i.e. $\hat{V}(p;1) = -\delta^\star(p| - C)$) or of the symmetric reflection across the origin of the Minkowski sum between the singleton of current inventory $x_0 \in \mathbb{R}^n_+$ and the set of feasible trades $T(x_0)$ (i.e. $\hat{V}(p;1) = -\delta^\star(-p|x_0 + T(x_0)))$, but also as the concave function induced by the cone dual to the hypograph of the invariant function under linear map $A : (\mu, x) \mapsto (-\mu, x)$, implying that $\hat{V}(p;1) = \inf_{x \succ 0} \frac{\langle x,p\rangle}{\hat{L}(x)}$; the invariant function $\hat{L} : \mathbb{R}^n_+ \to \mathbb{R}_+$ can be seen as the closed concave gauge-like function induced by $\text{cone}(\{(1,x) : x \in C\})$ (i.e. $\hat{L}(x) = \sup\{\lambda > 0 : x \in \lambda C\}$) or by $\text{cone}(\{(1,x) : x \in (x_0 + T(x_0))\})$ (i.e. $\hat{L}(x) = \sup\{\lambda > 0 : x \in \lambda(x_0 + T(x_0))\}$), but also as $\hat{L}(x) = \inf_{p \succ 0} \frac{\langle x,p\rangle}{\hat{V}(p;1)}$ consistently with what ahs been said for the portfolio value function. Finally, the set of feasible trades $T(x_0) \subset \mathbb{R}^n$, can be seen as the Minkowski sum between C and the singleton of the negative of current inventory $x_0 \in \mathbb{R}^n_+$ (i.e. $T(x_0) = C - x_0$), but also as the upper level set of the variation function $P : \mathbb{R}^n \to \mathbb{R}$ of the invariant function $\hat{L}$ at level zero (i.e. $T(x_0) = \{y; P(y; x_0) \geq 0\}$ where $P(y; x_0) = \hat{L}(x_0 + y) - \hat{L}(x_0))$ or, in terms of the portfolio value function, as the set $\left\{y : \hat{V}^{\star}(x_0 + y) \geq 0\right\}$. The toolkit introduced in this work is summarized in the following table:

C	$\hat{V}$
• $\hat{L}(x) = \sup\{\lambda > 0 : x \in \lambda C\}$ • $\hat{V}(p; 1) = -\delta^{\star}(-p\|C)$ • $T(x_0) = C - x_0$	• $C = \mathrm{dom}(\hat{V}^{\star})$ • $\hat{L}(x) = \inf_{p \succ 0} \frac{\langle p, x\rangle}{\hat{V}(p;1)}$ • $T(x_0) = \left\{y : \hat{V}^{\star}(x_0 + y) \geq 0\right\}$
$\hat{L}$	$T(x_0)$
• $C = \left\{x : \hat{L}(x) \geq 1\right\}$ • $\hat{V}(p; 1) = \inf_{x \succ 0} \frac{\langle p, x\rangle}{\hat{L}(x)}$ • $T(x_0) = \left\{y : \hat{L}(x_0 + y) - \hat{L}(x_0) \geq 0\right\}$	• $C = x_0 + T(x_0)$ • $\hat{L}(x) = \sup\{\lambda > 0 : x \in \lambda(x_0 + T(x_0))\}$ • $\hat{V}(p; 1) = -\delta^{\star}(-p\|x_0 + T(x_0))$

The last subsection shows an application of such toolkit in recovering all the core components of any Uniswap V2-like pool starting from any one of them.

Chapter 2

Convex Analysis

In this chapter are going to be introduced the basic concepts of convex analysis that will be used throughout the work. Most of the notation is referred to the book *Convex Analysis* by Rockafellar [Roc70], but also by [BV04] and [Ber09] mainly for the concepts of Pareto Optimal Frontier and recession function respectively.

2.1 Fundamental sets and their representations

Most of the properties of the sets that are object of study of convex analysis are based on their closedness under specific types of combinations of their elements. For this reason, it is worth to introduce the concept of affine, convex and conic combinations and the sets which are closed under such combinations.

2.1.1 Affine sets

Definition 1 (Affine combination)**.** Given the points $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$, the point y given by

$$y = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k, \qquad \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \in \mathbb{R},\ \ i = 1, \ldots, k$$

is said to be *affine combination* of $x_1, x_2, \ldots, x_k$.

This allows to introduce a very important geometric object in convex analysis: given $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$, the *line through* x_1 *and* x_2 is the set of points M given by:

$$\begin{aligned} M &= \{(1-\lambda)x_1 + \lambda x_2 : \lambda \in \mathbb{R}\} \\ &= \{x_1 + \lambda(x_2 - x_1) : \lambda \in \mathbb{R}\} \end{aligned}$$

Thus, every point onto this line can be expressed in terms of x_1 and x_2 , being an affine combination of them. This allows to introduce the category of *affine sets*

Definition 2 (Affine set)**.** A set $M \subseteq \mathbb{R}^n$ is said to be affine if for every $x_1, x_2 \in M$, the line through x_1 and x_2 is contained in M:

$$x_1 + \lambda(x_2 - x_1) \in M \qquad \forall x_1, x_2 \in M,\ \forall \lambda \in \mathbb{R}$$

In other words, a set $M \subseteq \mathbb{R}^n$ is said to be *affine* if it is closed under affine combinations of its elements.
Heuristically, this allows to visualize affine sets as "endless flat structures" in $\mathbb{R}^n$ which could be either a line, a plane or the whole space $\mathbb{R}^n$. Indeed, also $\emptyset$ and $\mathbb{R}^n$ are affine sets. Thus, generally speaking, if two points $x_1, x_2 \in \mathbb{R}^n$ are contained in an affine set M, then the whole line through x_1 and x_2 is contained in M as well.

Remark 1. *Each affine set $M \subseteq \mathbb{R}^n$ can be expressed as the translation of a vector subspace L in $\mathbb{R}^n$:*

$$M = L + x_0 = \{x + x_0 : x \in L\} \tag{2.1}$$

In fact, every affine set containing the origin is a vector subspace of $\mathbb{R}^n$ and it's possible to retrieve the parallel subspace L of an affine set M by taking the Minkowski sum of M with its symmetric reflection across the origin ($-M = (-1)M$)

$$L = M - M = \{x - y : x \in M, y \in M\}$$

This allows to define an equivalence class of affine sets where the equivalent relation is given by the subspace L parallel to the affine set M. Indeed, the dimension of an affine set M corresponds to the dimension of the subspace L parallel to M.
For example, every line in $\mathbb{R}^n$ can be seen as a translation of a one-dimensional subspace L in $\mathbb{R}^n$ (i.e. the "parallel" line passing through the origin), indeed lines are one-dimensional affine sets. Trivially, points $\{x_0\}$ and planes are zero-dimensional and two-dimensional affine sets respectively, being translates of the singleton of the origin (which is a vector subspace) and of a two-dimensional subspace in $\mathbb{R}^n$. Of course, it becomes difficult to visualize sets in $\mathbb{R}^n$ for $n > 3$, but concepts remain the same.

Definition 3 (Hyperplanes)**.** An affine set $H \subset \mathbb{R}^n$ of dimension $n-1$ is called a *hyperplane*

Hyperplanes a very important class of affine sets; indeed, every hyperplane is the translate of a $(n-1)$-dimensional vector subspace $L \subset \mathbb{R}^n$ and it's possible to retrieve a set notation for hyperplanes recalling that the *orthogonal complement* of $L \subset \mathbb{R}^n$ corresponds to:

$$\begin{aligned} L^\perp &= \{x \in \mathbb{R}^n : \langle x, y \rangle = 0, \quad y \in L\} \\ &= \{x \in \mathbb{R}^n : x \perp y, \quad y \in L\} \end{aligned}$$

Where $\langle \cdot\,, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the inner product in $\mathbb{R}^n$. In particular, given $L \subset \mathbb{R}^n$, one has that $\dim(L) + \dim(L^\perp) = n$.

This implies that the one-to-one correspondence between a hyperplane and the $(n-1)$-dimensional parallel subspace can be extended to a one-to-one correspondence between a hyperplane and the 1-dimensional orthogonal complement of the parallel subspace.

Proposition 1. *Given $b \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, any hyperplane $H \subset \mathbb{R}^n$ can be written as*

$$H = \{z : \langle z, b \rangle = \beta\} \tag{2.2}$$

Proof. Starting from the set notation of affine sets in (2.1) and specifying $L \subset \mathbb{R}^n$ as a $(n-1)$-dimensional subspace, it's possible to formulate a set notation for hyperplanes as follows:

$$\begin{aligned} H &= L + x_0 \\ &= \left\{x : x \perp y, \quad y \in L^\perp\right\} + x_0 \\ &= \left\{x : \langle x, y \rangle = 0, \quad y \in L^\perp\right\} + x_0 \\ &= \left\{x + x_0 : \langle x, y \rangle = 0, \quad y \in L^\perp\right\} \end{aligned}$$

Calling $z = x + x_0$ and defining b as the unique vector in the basis of $L^\perp$ (so that $\langle x, b \rangle = 0$ is equivalent to $\langle x, y \rangle = 0,\ y \in L^\perp$ since each $y \in L^\perp$ is simply a scalar multiple of b, being $L^\perp$ a 1-dimensional vector subspace), it's possible to write the set notation of a hyperplane as:

$$\begin{aligned} H &= \left\{x + x_0 : \langle x, y \rangle = 0, \quad y \in L^\perp\right\} \\ &= \{z : \langle z - x_0, b \rangle = 0\} \\ &= \{z : \langle z, b \rangle = \langle x_0, b \rangle\} \\ &= \{z : \langle z, b \rangle = \beta\} \end{aligned}$$

Where $\beta = \langle x_0, b \rangle$. □

Thus, every hyperplane would be characterized by a vector $b \in \mathbb{R}^n$ and a scalar $\beta \in \mathbb{R}$. The vector b is called *vector normal to the hyperplane* H and it's exactly the vector in the basis of the 1-dimensional orthogonal complement of the parallel subspace L, while $\beta \in \mathbb{R}$ is a sort of "offset" of the hyperplane H from the origin (indeed, notice that by setting $\beta = 0$, $H = \{z : \langle z, b \rangle = 0\}$ resemble to the parallel subspace L).
It follows that each hyperplane $H \subset \mathbb{R}^n$ is characterized by a double $(b, \beta) \in \mathbb{R}^n \times \mathbb{R}$ and it is recoverable via the set-valued map $(b, \beta) \mapsto \{x : \langle x, b \rangle = \beta\}$. Thus, throughout this work, the hyperplane "associated" with the pair $(b, \beta) \in \mathbb{R}^n \times \mathbb{R}$ will refer to $\{x \in \mathbb{R}^n : \langle x, b \rangle = \beta\}$
. At the same time, recalling the fact that $\beta = \langle x_0, b \rangle$, a hyperplane can be characterized also by the double $(b, x_0) \in \mathbb{R}^{2n}$ in the set-valued sense $(b, x_0) \mapsto \{x : \langle x, b \rangle = \langle x, x_0 \rangle\}$

The notation used in (2.2) for hyperplanes can be extended to any m-dimensional set supposing that the parallel subspace L is m-dimensional.

Proposition 2. *Given a collection of hyperplanes of the type $H_i = \{x : \langle x, b_i\rangle = \beta_i\} \subset \mathbb{R}^n$ indexed by $i \in [1, n-m]$, any affine set $M \subset \mathbb{R}^n$ such that dim$(M) = m$ can be written as*

$$M = \bigcap_{i=1}^{n-m} H_i = \{z : Bz = c\} \tag{2.3}$$

for some $B \in \mathbb{R}^{(n-m)\times n}$ and $c \in \mathbb{R}^{(n-m)}$

Proof. Since $M \subset \mathbb{R}^n$ with $\dim(M) = m$ it follows that $\dim(L) = m$ while $\dim(L^\perp) = n - m$. Assuming that $(b_1, \ldots, b_{n-m})$ is a basis for $L^\perp$, it's possible to write M as:

$$\begin{aligned} M &= L + x_0 \\ &= \left\{x : x \perp y, \quad y \in L^\perp\right\} + x_0 \\ &= \{x : \langle x, b_1\rangle = 0, \ldots, \langle x, b_{n-m}\rangle = 0\} + x_0 \end{aligned}$$

At this point, it's immediate to see that every m-dimensional set can be expressed as the finite intersection of $n - m$ hyperplanes, indeed:

$$\begin{aligned} M &= \{x : \langle x, b_1\rangle = 0, \ldots, \langle x, b_{n-m}\rangle = 0\} + x_0 \\ &= \{x + x_0 : \langle x, b_1\rangle = 0, \ldots, \langle x, b_{n-m}\rangle = 0\} \\ &= \{z : \langle z - x_0, b_1\rangle = 0, \ldots, \langle z - x_0, b_{n-m}\rangle = 0\} \\ &= \{z : \langle z, b_1\rangle = \langle x_0, b_1\rangle, \ldots, \langle z, b_{n-m}\rangle = \langle x_0, b_{n-m}\rangle\} \\ &= \{z : \langle z, b_1\rangle = \beta_1, \ldots, \langle z, b_{n-m}\rangle = \beta_{n-m}\} \\ &= \bigcap_{i=1}^{n-m} H_i \end{aligned}$$

Where $H_i = \{z : \langle z, b_i\rangle = \beta_i\}$.
Moving back from the notation of M as $M = \{x : \langle x, b_1\rangle = 0, \ldots, \langle x, b_{n-m}\rangle = 0\} + x_0$, it's convenient introducing the linear map $B : \mathbb{R}^n \to \mathbb{R}^{n-m}$ where

$$B : x \mapsto (\langle x, b_1\rangle, \ldots, \langle x, b_{n-m}\rangle)$$

and, being a linear map, there is a one-to-one correspondence with a matrix $B \in \mathbb{R}^{(n-m)\times n}$ such that $Bx = (\langle x, b_1\rangle, \ldots, \langle x, b_{n-m}\rangle)$ (because of the one-to-one correspondence, B refers both to the linear map and the associated matrix). Thus, it's possible to write the set notation of M as:

$$\begin{aligned} M &= \{x : \langle x, b_1\rangle = 0, \ldots, \langle x, b_{n-m}\rangle = 0\} + x_0 \\ &= \{x : Bx = 0\} + x_0 \\ &= \mathrm{Ker}(B) + x_0 \end{aligned}$$

This notation allows to appreciate the nature of the subspace L parallel to M as the null-space of the matrix B, where each row of B corresponds to a vector of the basis of $L^{\perp}$. Proceeding with the same reasoning, it's possible to define M as the set of solutions of a system of $n-m$ affine equations in n variables, indeed:

$$\begin{aligned} M &= \{x : Bx = 0\} + x_0 \\ &= \{x + x_0 : Bx = 0\} \\ &= \{z : B(z - x_0) = 0\} \\ &= \{z : Bz = Bx_0\} \\ &= \{z : Bz = c\} \end{aligned}$$

And, as before, by setting $c = 0 \implies M = \{z : Bz = 0\} = \text{Ker}(B) = L$ (i.e., the parallel subspace corresponds to the homogenous version of the system of equations). $\square$

Thus, according to the representation given by (2.3), every m-dimensional affine set can be expressed as the set of solutions of a system of $n-m$ affine equations in n variables. Notice that this notation is consistent with that one provided in (2.2) for hyperplanes because since $\dim(H) = n-1 \implies m = n-1 \implies B : \mathbb{R}^n \to \mathbb{R}$ and so $\{z : Bz = c\}$ resembles to $\{z : \langle z, b\rangle = \beta\} =: H$

From the notation just introduced, it is possible to derive another possible representation of affine sets, which is called "Tucker representation" which allows to represent any m-dimensional affine set in $\mathbb{R}^n$ as the graph of a certain affine transformation from $\mathbb{R}^m$ to $\mathbb{R}^{n-m}$.

Proposition 3 (Tucker representation). *Given an affine set $M \in \mathbb{R}^n$ such that dim$(M) = m$, it's possible to write M as*

$$M = \mathit{Graph}(A) = \{(x, \mu) : \mu = Ax\} \tag{2.4}$$

for some $A \in \mathbb{R}^{(n-m)\times m}$

Proof. Given $m < n$, it's possible to decompose $z = (\xi_1, \dots, \xi_m, \xi_{m+1}, \dots, \xi_n) \in \mathbb{R}^n$ so that

$$\begin{aligned} M &= \{z : Bz = c\} \\ &= \{(\xi_1, \dots, \xi_m, \xi_{m+1}, \dots, \xi_n) : B(\xi_1, \dots, \xi_m, \xi_{m+1}, \dots, \xi_n) = c\} \end{aligned}$$

The previous discussion showed that $L = \text{Ker}(B)$, and this implies that $\dim(M) = \dim(L) = \dim(\text{Ker}(B))$ where L denotes the subspace parallel to M. Thus, $\dim(M)$ corresponds to the *nullity* of B, that is the dimension of the null-space of the linear operator associated with B. The rank-nullity theorem states that the number of columns of B is equal to the sum of the rank of B and the nullity of B. Thus, since $B \in \mathbb{R}^{(n-m)\times n}$ and $\dim(\text{Ker}(B) = m$, it follows that $\text{rank}(B) = n - m$,

which indeed is the dimension of the image of an affine transformation $A : \mathbb{R}^m \to \mathbb{R}^{n-m}$.
Thus, since $(\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$ and $(\xi_{m+1}, \ldots, \xi_n) \in \mathbb{R}^{n-m}$ (of course, such tuples are not ordered and so the way of splitting z in two parts is not unique: that's the reason why there exist multiple (but finite) possible Tucker representation for an affine set), it's possible to write M as:

$$\begin{aligned} M &= \{(\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_n) : B(\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_n) = c\} \\ &= \{(\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_n) : \xi_{m+i} = a_{i,1}\xi_1 + \cdots + a_{i,m}\xi_m + \alpha_i, \quad i = 1, \ldots, n-m\} \end{aligned}$$

Which, by setting

$$A : (\xi_1, \ldots, \xi_m) \mapsto (a_{1,1}\xi_1 + \cdots + a_{1,m}\xi_m + \alpha_1, \quad \ldots, \quad a_{n-m,1}\xi_1 + \cdots + a_{n-m,m}\xi_m + \alpha_{n-m})$$

resembles to

$$\begin{aligned} &M = \{(\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_n) : \xi_{m+i} = a_{i,1}\xi_1 + \cdots + a_{i,m}\xi_m + \alpha_i, \quad i = 1, \ldots, n-m\} \\ &M = \{(x, \mu) : \mu = Ax\} =: \text{Graph}(A) \end{aligned}$$

□

Definition 4 (Affine hull)**.** Given a generic set $S \subset \mathbb{R}^n$ (affine or not), there always exist an affine set containing S with minimal dimension. Such set is called *affine hull* of S and it's denoted by $\text{aff}(S)$. The affine hull of a set S is the smallest affine set containing S and it can be expressed as the set of affine combinations of the elements of S:

$$\text{aff}(S) := \left\{ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n : \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \in \mathbb{R},\ x_i \in S,\ i = 1, \ldots, n \right\}$$

Of course, if S is affine, then $\text{aff}(S) = S$. Moreover, if S is n-dimensional, then $\text{aff}(S) = \mathbb{R}^n$.

Definition 5 (Affinely independent points)**.** A set of $(m + 1)$ points $S = \{x_0, \ldots, x_m\}$ is said to be affinely independent if $\text{aff}(S)$ is m-dimensional, meaning that the subspace L parallel to $\text{aff}(S)$ is actually $\text{Span}(x_1 - x_0, \ldots, x_m - x_0)$ (i.e., the m vectors $(x_1 - x_0, \ldots, x_m - x_0)$ are a basis of L).

Thus, the elements of $\{x_0, \ldots, x_m\}$ are affinely independent if and only if the vectors $\{x_1 - x_0, \ldots, x_m - x_0\}$ are linearly independent. This is important since every m-dimensional affine set $M \subset \mathbb{R}^n$ can be expressed as the affine hull of an affinely independent set of $m + 1$ points, implying that every point of M can be expressed via a unique linear combination of the type $(\lambda_1(x_1 - x_0) + \cdots + \lambda_m(x_m - x_0) + x_0) \in M$ and the unique vector of coefficients $(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ is called *barycentric coordinates* of the point in M.

To sum up, every m-dimensional affine set $M \subset \mathbb{R}^n$ can be represented as:

- Intersection of $n - m$ hyperplanes: $M = \bigcap_{i=1}^{n-m} \{z : \langle z, b_i \rangle = \beta_i\},\ b_i \in \mathbb{R}^n,\ \beta_i \in \mathbb{R}$

- Set of solutions of a system of $n-m$ affine equations in n variables $M = \{z : Bz = c\}$, $B \in \mathbb{R}^{n-m} \times \mathbb{R}^n$, $c \in \mathbb{R}^{n-m}$

- (Tucker representation) Graph of an affine transformation $M = \{(x, \mu) : \mu = Ax\}$, $A \in \mathbb{R}^{(n-m)\times m}$

- Affine hull of $m+1$ affinely independent points $M = \text{aff}(S)$ where
  $S = \{x_0, \ldots, x_m : (x_1 - x_0) \perp \cdots \perp (x_m - x_0)\}$

2.1.2 Convex sets

Definition 6 (Convex combination)**.** Given the points $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$, the point y given by

$$y = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k, \qquad \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \in \mathbb{R}_+,\ \ i = 1, \ldots, k$$

is said to be *convex combination* of $x_1, x_2, \ldots, x_k$.

It's evident how the definition of convex combination is a "constrained" version of the definition of affine combination: indeed, in this case the linear coefficients are constrained to be non-negative and to sum to one. This means that the linear coefficients involved in a convex combination are all non-negative and smaller than one, implying that they can be conceived as a vector of proportions or of probabilities. Every affine combination is also a convex combination but the converse it's not true. As done before, introducing the line through x_1 and x_2, the first important geometric object that is related to the convex combination of two points is the *(closed) line-segment between x_1 and x_2*, which is the set of points C given by:

$$\begin{aligned} C &= \{(1-\lambda)x_1 + \lambda x_2 : \lambda \in [0,1]\} \\ &= \{x_1 + \lambda(x_2 - x_1) : \lambda \in [0,1]\} \end{aligned}$$

Trivially, notice that $\{x_1 + \lambda(x_2 - x_1) : \lambda \in [0,1]\} \subset \{x_1 + \lambda(x_2 - x_1) : \lambda \in \mathbb{R}\}$ being just a "portion" of the line through x_1 and x_2. Every point onto the line-segment C can be expressed as a convex combination of x_1 and x_2. This allows to introduce the class of *convex sets*

Definition 7 (Convex set)**.** A set $C \subseteq \mathbb{R}^n$ is said to be convex if for every $x_1, x_2 \in C$, the line-segment between x_1 and x_2 is contained in C:

$$x_1 + \lambda(x_2 - x_1) \in C \qquad \forall x_1, x_2 \in C,\ \forall \lambda \in [0,1]$$

In other words, a set $C \subseteq \mathbb{R}^n$ is said to be *convex* if it is closed under convex combinations of its elements. Analogously to the definition of convex combination, every affine set is also convex, but the contrary is not true.

Definition 8 (Convex hull)**.** Given any set $S \subset \mathbb{R}^n$, there always exist a convex set containing S with minimal dimension. Such set is called *convex hull* of S and it's denoted by $\text{conv}(S)$. The convex hull of a set S is the smallest convex set containing S and it can be expressed as the set of convex combinations of the elements of S:

$$\text{conv}(S) := \left\{ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n : \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \in \mathbb{R}_+,\ x_i \in S,\ i = 1, \ldots, n \right\}$$

The dimension of a convex set is defined by the dimension of its affine hull (which refers to the dimension of its parallel subspace L). Thus, if C is convex, then $\dim(C) = \dim(\text{aff}(C))$ and of course $\dim(\text{conv}(S)) = \dim(\text{aff}(\text{conv}(S))) = \dim(\text{aff}(\text{S}))$.

Proposition 4. *Considering a collection of non-empty convex sets $\{C_i : i \in I\}$ (where I denotes a generic index set), the convex hull of the union of the collection corresponds to the union of the convex combinations of the elements of the collection:*

$$\textit{conv}\left(\bigcup_{i \in I} C_i \right) = \bigcup \left\{ \sum_{i \in I} \lambda_i C_i : \sum_{i \in I} \lambda_i = 1, \lambda_i \in \mathbb{R}_+,\ i \in I \right\}$$

Proof.

$$\text{conv}\left(\bigcup_{i \in I} C_i \right) = \left\{ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k : \sum_{j=1}^{k} \lambda_j = 1, \lambda_j \in \mathbb{R}_+, x_j \in \bigcup_{i \in I} C_i, j = 1, \ldots, k \right\}$$

Notice that $x_j \in \bigcup_{i \in I} C_i$ means that x_j is allowed to belong to any C_i of the collection. Thus, it's possible to reformulate the set notation considering the union of all the possible combinations (ranging over $\sum_{i \in I} \lambda_i = 1, \lambda_i \in \mathbb{R}_+$) of elements taken from different sets C_i

$$= \bigcup \left\{ \lambda_1 x_1 + \lambda_2 x_2 + \cdots : \sum_{i \in I} \lambda_i = 1, \lambda_i \in \mathbb{R}_+, x_i \in C_i, i \in I \right\}$$

By calling $z = \lambda_1 x_1 + \lambda_2 x_2 + \ldots$ so that $z \in \lambda_1 C_1 + \lambda_2 C_2 + \ldots$

$$= \bigcup \left\{ z : \sum_{i \in I} \lambda_i = 1, \lambda_i \in \mathbb{R}_+,\ i \in I, z \in \lambda_1 C_1 + \lambda_2 C_2 + \ldots \right\}$$

$$= \bigcup \left\{ \lambda_1 C_1 + \lambda_2 C_2 + \cdots : \sum_{i \in I} \lambda_i = 1, \lambda_i \in \mathbb{R}_+,\ i \in I \right\}$$

$$= \bigcup \left\{ \sum_{i \in I} \lambda_i C_i : \sum_{i \in I} \lambda_i = 1, \lambda_i \in \mathbb{R}_+,\ i \in I \right\}$$

□

A similar operation to the convex hull of a collection of convex sets is the *inverse sum* of a collection of convex sets, which corresponds to

$$C_1 \# C_2 \# \cdots = \bigcup \left\{ \bigcap_{i \in I} \lambda_i C_i : \sum_{i \in I} \lambda_i = 1, \lambda_i \in \mathbb{R}_+,\ i \in I \right\}$$

Every hyperplane $H \subset \mathbb{R}^n$ partitions $\mathbb{R}^n$ into two convex sets called *half-spaces*, arranged into two possible pairs of convex sets:

$$\begin{cases} \{x \in \mathbb{R}^n : \langle x, b\rangle \geq \beta\} \\ \{x \in \mathbb{R}^n : \langle x, b\rangle < \beta\} \end{cases} \text{or} \quad \begin{cases} \{x \in \mathbb{R}^n : \langle x, b\rangle \leq \beta\} \\ \{x \in \mathbb{R}^n : \langle x, b\rangle > \beta\} \end{cases}$$

The half-spaces having weak inequality in the set notation are called *closed half-spaces* while the other ones are called *open half-spaces*.

Definition 9 (Half-space)**.** Given $b \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, any convex set $\tilde{H} \subset \mathbb{R}^n$ which can be represented as

$$\{x : \langle x, b\rangle \leq \beta\} \quad (\{x : \langle x, b\rangle < \beta\})$$

or

$$\{x : \langle x, b\rangle \geq \beta\} \quad (\{x : \langle x, b\rangle > \beta\})$$

is called *closed (open) half-space*

For the sake of representation, half-spaces stands to convex sets as hyperplanes stands to affine sets. Indeed, the intersection operation is convexity-preserving and every *closed* convex set $C \subset \mathbb{R}^n$ can be expressed as the intersection of all the half-spaces containing it.
By intersecting a finite number of half-spaces and hyperplanes, it's possible to retrieve a particular kind of convex set called *polyhedron*.

Definition 10 (Polyhedron)**.** Given a finite collection of half-spaces of type $\tilde{H}_i = \{x : \langle x, b_i\rangle \leq \beta_i\}$ indexed by $i \in I$, and a finite collection of hyperplanes of type $H_j = \{x : \langle x, b_j\rangle = \beta_j\}$ indexed by $j \in J$, the convex set $P \subset \mathbb{R}^n$ defined by

$$P = \left(\bigcap_{i\in I} \tilde{H}_i\right) \cap \left(\bigcap_{j\in J} H_j\right)$$

is called *polyhedron*

Thus, a polyhedron can be represented as the set of solutions of a system of a finite number of affine weak inequalities and equalities:

$$\begin{aligned} P &= \{x \in \mathbb{R}^n : \langle x, b_i\rangle \leq \beta_i,\ \langle x, c_i\rangle = d_i, i \in I\} \\ &= \{x \in \mathbb{R}^n : Bx \preceq \beta,\ Cx = d\} \end{aligned}$$

Where the $\preceq$ symbol denotes the element-wise inequality (or, more precisely, the generalized inequality using $\mathbb{R}^n_+$ as proper cone of reference, as it will be discussed in the next section).
Heuristically, trying to retrieve a convex set C by intersecting just some of the half-spaces containing it, leads to a polyhedron which simply"approximates" C from the outside and this gives an initial

intuition about the possibility of retrieving "external" representations of a convex set C via the half-spaces containing it,

However, to appreciate this fact, it's necessary to introduce the topological concepts of *closure, interior* and *relative interior* of a set in $\mathbb{R}^n$. For this purpose, it's necessary introducing the concept of norm, metric and ball.

Definition 11 (Norm function)**.** A norm is a convex function $||\cdot|| : \mathbb{R}^n \to \mathbb{R}$ satisfying the following properties:

- $||x|| \geq 0 \quad \forall x \in \mathbb{R}^n$ (non-negativity)
- $||x|| = 0 \iff x = 0$ (vanishing at zero)
- $||\alpha x|| = |\alpha| \cdot ||x|| \quad \forall x \in \mathbb{R}^n, \ \forall \alpha \in \mathbb{R}$ (homogeneity)
- $||x + y|| \leq ||x|| + ||y|| \quad \forall x, y \in \mathbb{R}^n$ (triangular inequality)

Throughout this work, the Euclidean norm function (a particular kind of norm function) is going to be extremely important in understanding several concepts. Indeed, this function encodes several properties which have yet to be mentioned. To anticipate some of them, which will result more clear throughout this work, it is possible to say that the Euclidean norm is a closed positively homogenous convex function corresponding both to the support function and gauge function of the Euclidean unit ball. From these simple though still obscure observations will derive very important findings. For example, the "triangular inequality" property mentioned above is an extension of the *Jensen's inequality* property (which is typical for convex functions) to the case of positive homogeneous convex functions (i.e. when the epigraph of such functions is a convex cone).

A very popular collection of norm functions is the p-norm family (ℓ_p), defined as:

$$\left\{ ||x||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} : p \geq 1 \right\}$$

For example, considering $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$:

- $p = 1$: $||x||_1 = \sum_{i=1}^n |\xi_i|$
- $p = 2$: $||x||_2 = \sqrt{\sum_{i=1}^n \xi_i^2}$ (Euclidean norm)
- $p = \infty$: $||x||_\infty = \max_{i=1,\ldots,n} |\xi_i|$ (Tchebycheff norm)

The Euclidean norm is the most popular norm function and it's the one that will be used in this work if not otherwise specified.

Besides norms, it's possible to introduce the *metric* function

Definition 12 (Metric function)**.** A metric function on $\mathbb{R}^n$ is a convex function $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following properties:

- $\rho(x, y) \geq 0 \quad \forall x, y \in \mathbb{R}^n$ (non-negativity)
- $\rho(x, y) = 0 \iff x = y$
- $\rho(x, y) = \rho(y, x) \quad \forall x, y \in \mathbb{R}^n$ (commutativity in the arguments)
- $\rho(x, y) \leq \rho(x, z) + \rho(z, y) \quad \forall x, y, z \in \mathbb{R}^n$

If some other properties are added to such definition, it is possible to have a refinement of the metric function, that is the *Minkowski metric* function.

Definition 13 (Minkowski metric function)**.** If a metric function $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following properties:

- $\rho(x + z, y + z) = \rho(x, y) \quad \forall x, y, z \in \mathbb{R}^n$ (translation invariance)
- $\rho(x, (1 - \lambda)x + \lambda y) = \lambda\rho(x, y)$ (linear behavior along line segments)

it is called *Minkowski metric* function on $\mathbb{R}^n$

Metric functions play a fundamental role because they express a way of measuring the distance between two points in $\mathbb{R}^n$. Moreover, there is a one-to-one correspondence between norm functions and Minkowski metric functions, in the sense that every norm function *induces* a metric function in the form of

$$\rho(x, y) = ||x - y||$$

Trivially, the Euclidean distance $d(x, y)$ is the Minkowski metric function induced by the Euclidean norm (i.e. $d(x, y) = ||x - y||_2$), and it is also the most popular metric function.
Having a tool for measuring distance between points allows to introduce the concept of *ball* in the sense of a set of points where the rule of inclusion is given by the distance between the points and a certain point called *center* of the ball.

Definition 14 (Unit ball)**.** Given a metric function $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the convex set $B \subset \mathbb{R}^n$ defined by

$$B = \{x : \rho(0, x) \leq 1\}$$

is called *unit ball*

The most known unit ball is the "Euclidean unit ball" which is the ball induced by the Euclidean distance, centered in the origin and with radius equal to one:

$$\begin{aligned} B &= \{x \in \mathbb{R}^n : d(0, x) \leq 1\} \\ &= \{x \in \mathbb{R}^n : ||x||_2 \leq 1\} \end{aligned}$$

In the section dedicated to convex functions, it will be seen that the Euclidean unit ball can be seen as the 1-sub-level set of the Euclidean norm function which, being convex, establishes the convexity of the Euclidean unit ball.
By transforming the Euclidean unit ball is possible to obtain a Euclidean balls centered in some point $x_0 \in \mathbb{R}^n$ with some radius $r > 0$ (i.e. the points that are distant from x_0 no more than r):

Proposition 5. *Let $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Minkowski metric function. Let $B \subset \mathbb{R}^n$ be the unit ball associated with the metric function ρ. The set of points $x \in \mathbb{R}^n$ which are not distant more than $\epsilon > 0$ from a certain $x_0 \in \mathbb{R}^n$ corresponds to*

$$B(x_0; \epsilon) = x_0 + \epsilon B \tag{2.5}$$

Proof. Without loss of generality, one can pick the Euclidean distance as Minkowski metric function of reference since, as shown in [Roc70], any Minkowski metric function can be expressed as a "rescaled" version of the Euclidean distance for a positive scalar

$$\begin{aligned} B(x_0; r) &:= \{x \in \mathbb{R}^n : d(x, x_0) \leq r\} \\ &= \{x \in \mathbb{R}^n : ||x - x_0||_2 \leq r\} \end{aligned}$$

By calling $y = x - x_0$

$$\begin{aligned} B(x_0; r) &= \{y + x_0 : ||y||_2 \leq r\} \\ &= x_0 + \{y : ||y||_2 \leq r\} \end{aligned}$$

By calling $\frac{y}{r} = z$ and exploiting the homogeneity of the Euclidean norm:

$$\begin{aligned} B(x_0; r) &= x_0 + \{rz : ||z||_2 \leq 1\} \\ &= x_0 + r\{z : ||z||_2 \leq 1\} \\ &= x_0 + rB \end{aligned}$$

□

The expression $B(x_0; r)$ is intentionally used to define the Euclidean ball centered in x_0 with radius r as a set-valued function on $\mathbb{R}^n$ (parameterizing $r \in \mathbb{R}$) Analogously, it's possible to define the set of points that are distant from a convex set $C \subset \mathbb{R}^n$ no more than r:

Proposition 6. *Let $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Minkowski metric function. Let $B \subset \mathbb{R}^n$ be the unit ball associated with the metric function ρ. The set of points $x \in \mathbb{R}^n$ which are not distant more than $\epsilon > 0$ from at least one element of C corresponds to*

$$B(C; \epsilon) = C + \epsilon B$$

Proof. As before, also in this case one can pick the Euclidean distance as Minkowski metric function of reference

$$\begin{aligned} B(C;r) &:= \{x \in \mathbb{R}^n : \exists y \in C, d(x,y) \le r\} \\ &= \{x \in \mathbb{R}^n : \exists y \in C, ||x-y||_2 \le r\} \\ &= \{z + y : \exists y \in C, ||z||_2 \le r\} \\ &= \{rz + y : \exists y \in C, ||z||_2 \le 1\} \\ &= C + \{rz : ||z||_2 \le 1\} \\ &= C + rB \end{aligned}$$

Which, using the Minkowski sum of sets, could be interpreted as the "ball centered in C with radius r". □

As previously mentioned, such concepts are useful to appreciate several topological components of convex sets. Indeed, the Euclidean ball becomes a tool for a geometric understanding of the position of points and the relative distance from others.

Definition 15 (Interior of a convex set)**.** Let $C \subset \mathbb{R}^n$ be a convex set. The *interior* of C is defined as:

$$\begin{aligned} \text{int}(C) &:= \{x \in C : \exists r > 0, B(x;r) \subset C\} \\ &= \{x \in C : \exists r > 0, x + rB \subset C\} \end{aligned}$$

However, given $C \subset \mathbb{R}^n$ such that $\dim(C) < n$, one trivially has $\text{int}(C) = \emptyset$. Indeed, imagining a line-segment C in $\mathbb{R}^2$, there isn't any Euclidean ball with positive radius centered in a point of C that is fully contained in C. Thus, to relax the definition of interior, it's possible to introduce the concept of *relative interior* of a convex set, by considering just the intersection of such Euclidean balls with the affine hull of the set

Definition 16 (Relative interior of a convex set)**.** Let $C \subset \mathbb{R}^n$ be a convex set. The *relative interior* of C is defined as:

$$\begin{aligned} \text{ri}(C) &:= \{x \in C : \exists r > 0, B(x;r) \cap \text{aff}(C) \subset C\} \\ &= \{x \in C : \exists r > 0, x + rB \cap \text{aff}(C) \subset C\} \end{aligned}$$

Taking the previous example, it's immediate to see that $\text{aff}(C)$ corresponds to the line passing through the endpoints of the line-segment C and so that the intersection of such line with the ball $B(x;r)$ is fully included in C for some points of C. The first topological property derivable for convex sets is that $C \subseteq \mathbb{R}^n$ is said to be *relatively open* if $C = \text{ri}(C)$. On the other hand, the topological property of *closedness* (i.e. the fact that C "contains its boundary points") can be introduced by defining the *closure* of a convex set, again via the use of Euclidean balls:

Definition 17 (Closure of a convex set)**.** Let $C \subset \mathbb{R}^n$ be a convex set. The *closure* of C is defined as:

$$\begin{aligned} \text{cl}(C) &:= \{x \in \mathbb{R}^n : \exists y \in C, d(x,y) \leq r, r > 0\} \\ &= \bigcap_{r>0} C + rB \\ &= \bigcap_{r>0} B(C;r) \end{aligned}$$

Not surprisingly, C is *closed* if $C = \text{cl}(C)$. Being "endless flat surfaces" on $\mathbb{R}^n$, affine sets are both closed and relatively open by definition. Moreover, $\text{ri}(C)$ and $\text{cl}(C)$ have the same affine hull (hence the same dimension) of C. The fundamental relations linking C with $\text{ri}(C)$ and $\text{cl}(C)$ are:

$$\begin{aligned} &\text{ri}(\text{cl}(C)) = \text{ri}(C) \\ &\text{cl}(\text{ri}(C)) = \text{cl}(C) \\ &\text{ri}(C) \subseteq C \subseteq \text{cl}(C) \end{aligned}$$

Finally, in force of the inclusion described by the latter relation, it's useful characterizing the set of points which do belong to the closure of C without living on its relative interior. Such set of points is called *relative boundary* of C

Definition 18 (Relative boundary of C)**.** Let $C \subset \mathbb{R}^n$ be a convex set. The *relative boundary* of C is defined as:

$$\text{cl}(C) \setminus \text{ri}(C)$$

Proposition 7. *Let $C \subset \mathbb{R}^n$ be a convex set. Defining z as any convex combination between a generic $x \in ri(C)$ and $y \in cl(C)$ (excluding the case where $z = y$) one has that $z \in ri(C)$. In other words:*

$$(1-\lambda)x + \lambda y \in ri(C) \qquad \forall \lambda \in [0,1),\ \forall x \in ri(C),\ \forall y \in cl(C) \tag{2.6}$$

Proof. This statement claims that considering $x \in \text{ri}(C)$ and $y \in \text{cl}(C)$, any point lying on the line segment joining x and y excluding y itself is contained in $\text{ri}(C)$ (and so, in C since $\text{ri}(C) \subseteq C \subseteq \text{cl}(C)$). To prove this, supposing that C is n-dimensional (so that $\text{ri}(C) = \text{int}(C)$), is sufficient to prove that $\exists \epsilon > 0$ such that $z + \epsilon B \subset C$ where $z = (1-\lambda)x + \lambda y$. Indeed, by definition of relative interior point, this is equivalent to say that the point retrieved as convex combination is "sufficiently inside" C so that exists a positive radius, also infinitesimal, such that the Euclidean ball centered in that point is still contained in C. More formally, if $(1-\lambda)x + \lambda y \in \text{ri}(C)$, it means that

$$\exists \epsilon > 0 : ((1-\lambda)x + \lambda y) + \epsilon B \subseteq C \quad \forall \lambda \in [0,1),\ \forall x \in \text{ri}(C),\ \forall y \in \text{cl}(C)$$

This can be proved algebraically recalling that

$$y \in \text{cl}(C) \implies y \in (\cap_{\epsilon>0} C + \epsilon B) \implies y \in (C + \epsilon B)\ \forall \epsilon > 0$$

meaning that one can rewrite

$$\begin{aligned}((1-\lambda)x+\lambda y)+\epsilon B &= (1-\lambda)x+\lambda(C+\epsilon B)+\epsilon B\\ &= \lambda C+(1-\lambda)x+\epsilon(1+\lambda)B\\ &= \lambda C+(1-\lambda)\left[x+\epsilon\frac{(1+\lambda)}{(1-\lambda)}B\right]\\ &= \lambda C+(1-\lambda)B\left(x;\ \epsilon\frac{(1+\lambda)}{(1-\lambda)}\right)\end{aligned}$$

As said before, supposing that C is n-dimensional, one has that $\mathrm{ri}(C)=\mathrm{int}(C) \implies x\in\mathrm{int}(C)$ and so, by definition of interior point, $\exists\epsilon>0 : B\left(x;\ \epsilon\frac{(1+\lambda)}{(1-\lambda)}\right)\subset C$, meaning that $\lambda C+(1-\lambda)B\left(x;\ \epsilon\frac{(1+\lambda)}{(1-\lambda)}\right)\subseteq\lambda C+(1-\lambda)C=C$. Thus,

$$\begin{cases}\lambda C+(1-\lambda)B\left(x;\ \epsilon\frac{(1+\lambda)}{(1-\lambda)}\right)=((1-\lambda)x+\lambda y)+\epsilon B\\ \exists\epsilon>0 : \lambda C+(1-\lambda)B\left(x;\ \epsilon\frac{(1+\lambda)}{(1-\lambda)}\right)\subseteq C\end{cases}\implies \exists\epsilon>0 : ((1-\lambda)x+\lambda y)+\epsilon B\subseteq C$$

□

A further characterization of points $x\in\mathrm{ri}(C)$ is that, using any $y\in C$ as "source of light" there is always at least one point in the "shadow" of x (i.e. the half-line starting from x, but not including it, whose affine hull is the line passing through x and y) that is still contained in $\mathrm{ri}(C)$ (i.e. it is still "sufficiently inside" C)

Proposition 8. *Let $C\subset\mathbb{R}^n$ be a convex set and $x\in ri(C)$, then*

$$\exists\mu>1 : ((1-\mu)y+\mu x)\in ri(C)\ \forall y\in C \tag{2.7}$$

In other words:

$$\forall x\in ri(C), \exists\mu>1 : ((1-\mu)y+\mu x)\in ri(C)\ \forall y\in C$$

Proof. Supposing that exists a $z\in\mathrm{ri}(C)$ which can be expressed as $z=(1-\mu)y+\mu x$ for some $\mu>1$ with $x,y\in C$, then recalling 2.6 proving this statement is equivalent to prove that $x\in\mathrm{ri}(C)$. Noticeably, it's possible to express $x=\frac{z-(1-\mu)y}{\mu}=\frac{z}{\mu}-\frac{y}{\mu}+y=\frac{z}{\mu}+(1-\frac{1}{\mu})y$ and setting $\lambda=\frac{1}{\mu}$, it's evident how x is actually any point on the line-segment joining y and z excluding the two endpoints, i.e. $x=(1-\lambda)y+\lambda z,\ \lambda\in(0,1)$ (since $\mu>1$). Since the claim here is that $z\in\mathrm{ri}(C)$, then it must be that $x\in\mathrm{ri}(C)$ since, as previously discussed in (2.6), any point on the line-segment joining a relative interior point (z in this case) with any other point of the closure of C (y in this case, recalling that $C\subseteq\mathrm{cl}(C)$), excluding such point, is contained in $\mathrm{ri}(C)$, thus $x\in\mathrm{ri}(C)$.

□

Closedness is a very desirable property for convex sets, because it allows to have an "external" representation of the set as the intersection of the closed half-spaces containing it. By definition, such external representation is always possible for $\mathrm{cl}(C)$. Because of this external representation, when a closed convex set $C \subset \mathbb{R}^{n+1}$ acts as epigraph of some convex function $f : \mathbb{R}^n \to \mathbb{R}$, it encodes several property of the function f itself.

Closed convex sets can have an "external" representation, but for any convex set it is always possible to have its "internal" representation. Indeed, like any m-dimensional affine sets, which can be always expressed as affine hulls of $m+1$ affinely independent points, also every convex set could be expressed as the convex hull of a finite set of points.

2.1.3 Cones

Definition 19 (Cone)**.** A set $K \subseteq \mathbb{R}^n$ is said to be a *cone* if it is closed under positive scalar multiplication:

$$\lambda x \in K \qquad \forall x \in K,\ \forall \lambda > 0$$

Some common examples of cones are the positive orthant $\{(\xi_1, \dots, \xi_n) : \xi_i > 0,\ i = 1, \dots, n\}$ or the negative orthant $\{(\xi_1, \dots, \xi_n) : \xi_i < 0,\ i = 1, \dots, n\}$.
Notice that a cone doesn't have to contain the origin (indeed the requirement for scalar multiplication has strict inequality) and most importantly it doesn't have to be convex. For example, the union of the positive orthant and the negative orthant is still a cone but it is non-convex and it doesn't contain the origin.
Given a generic set $S \subset \mathbb{R}^n$it's possible to introduce the set of half-lines emanating from the origin and passing through at least one point of S as an example of non-convex cone including the origin:

$$\mathrm{ray}(S) = \{\lambda x : x \in S,\ \lambda \in \mathbb{R}_+\}$$

However, most of the theory in convex analysis is built around convex cones, which are cones that are also convex sets.
Some authors [BV04] put the inclusion of the origin as a requirement for convex cones, simplifying the definition of *convex cones* to any set closed under *conic combinations* of its elements.
Convex cones including the origin are extremely important also in the analysis of a CFMM since both the invariant function and the portfolio value function describing its mechanics are "induceable" by closed convex cones containing the origin.

Definition 20 (Conic combination)**.** Given the points $x_1, x_2, \dots, x_k \in \mathbb{R}^n$, the point y given by

$$y = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k, \qquad \lambda_i \in \mathbb{R}_+,\ i = 1, \dots, k$$

is said to be a *conic combination* of $x_1, x_2, \dots, x_k$.

As in convex combinations here the coefficients are constrained to be non-negative, but they don't have to sum to one. Keeping the generic definition of convex cones, it can't be said that convex cones are closed under conic combinations of their elements, unless they contain the origin.

Proposition 9. *Let $K \subset \mathbb{R}^n$ be a convex cone, then K is closed under addition and positive scalar multiplication. Indeed, given $x_1 \in K$ then $\lambda x_1 \in K\ \forall \lambda > 0$. In general, defining I as an finite index set, given $x_i \in K\ \forall i \in I$ and $\lambda_i > 0\ \forall i \in I$ one has that*

$$\sum_{i \in I} \lambda_i x_i \in K$$

Proof. The closure under positive scalar multiplication is trivial from the definition, while the closure under addition could be proved as follows: considering K as a convex cone and taking $x_1, x_2 \in K$ one of course has that $\lambda_1 x_1, \lambda_2 x_2 \in K,\ \forall \lambda_i > 0$. Thus, one could pick $z = \frac{\lambda_1}{\lambda_1+\lambda_2} x_1 + \frac{\lambda_2}{\lambda_1+\lambda_2} x_2$ being confident that $z \in K$ because of the convexity of K. This point could be rewritten as

$$z = \frac{1}{\lambda_1 + \lambda_2} \left(\lambda_1 x_1 + \lambda_2 x_2\right)$$

and this allows to rewrite

$$\begin{aligned}
&\lambda_1 x_1 + \lambda_2 x_2,\ \lambda_i > 0 \\
&= (\lambda_1 + \lambda_2) \frac{1}{\lambda_1 + \lambda_2} \left(\lambda_1 x_1 + \lambda_2 x_2\right) \\
&= (\lambda_1 + \lambda_2) z \\
&= \lambda_3 z,
\end{aligned}$$

Where of course $\lambda_3 > 0$ being the sum of positive numbers and $z \in K$ being a convex combination of x_1 and x_2, thus $(\lambda_1 x_1 + \lambda_2 x_2) \in K$. □

Definition 21 (Conic hull)**.** Given any set $S \subset \mathbb{R}^n$, there always exist a convex cone containing S and the origin with minimal dimension. Such set is called *cone generated by* S (sometimes called also *Conic hull*) and it's denoted by cone(S). The cone generated by a set S is the smallest convex cone containing S and it can be expressed as the set of conic combinations of the elements of S:

$$\text{cone}(S) = \{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k : \lambda_i \in \mathbb{R}_+,\ x_i \in S,\ i = 1, \ldots, n\}$$

This notation reminds that one of ray(S), however notice that ray(S) contains the origin but it might not be convex. Nevertheless, the similarity between these two sets is given by the following proposition:

Proposition 10. *Let $S \subset \mathbb{R}^n$, then*

$$cone(ray(S)) = cone(S)$$

Proof.

$$\operatorname{conv}(\operatorname{ray}(S)) = \left\{ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k : \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \in \mathbb{R}_+,\ x_i \in \operatorname{ray}(S),\ i = 1, \ldots, n \right\}$$
$$= \left\{ \lambda_1 \mu_1 z_1 + \lambda_2 \mu_2 z_2 + \cdots + \lambda_k \mu_k z_k : \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \in \mathbb{R}_+,\ z_i \in S, \mu_i \in \mathbb{R}_+\ i = 1, \ldots, n \right\}$$

setting $\alpha_i = \lambda_i \mu_i$

$$= \{\alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_k z_k : \alpha_i \in \mathbb{R}_+,\ z_i \in S,\ i = 1, \ldots, n\} =: \operatorname{cone}(S)$$

□

Introducing convex cones is quite useful for having an additional representation of a generic convex set $C \subset \mathbb{R}^n$. Indeed, given $C \subset \mathbb{R}^n$ one could define $\{(1, x) : x \in C\} \subset \mathbb{R}^{n+1}$ as a convex set in $\mathbb{R}^{n+1}$ having the same dimension of C (i.e., the same set C represented in a higher dimension). From here, one could generate a convex cone in $\mathbb{R}^{n+1}$ from $\{(1, x) : x \in C\}$ obtaining:

$$\begin{aligned} K &= \operatorname{cone}(\{(1, x) : x \in C\}) \\ &= \{\lambda_1(1, x_1) + \lambda_2(1, x_2) + \cdots + \lambda_k(1, x_k) : \lambda_i \in \mathbb{R}_+,\ x_i \in C,\ i = 1, \ldots, n\} \\ &= \{(\lambda_1, \lambda_1 x_1) + (\lambda_2, \lambda_2 x_2) + \cdots + (\lambda_k, \lambda_k x_k) : \lambda_i \in \mathbb{R}_+,\ x_i \in C,\ i = 1, \ldots, n\} \\ &= \{(\lambda_1 + \lambda_2 + \cdots + \lambda_k, \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k) : \lambda_i \in \mathbb{R}_+,\ x_i \in C,\ i = 1, \ldots, n\} \end{aligned}$$

setting $z = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k$

$$= \{(\lambda_1 + \lambda_2 + \cdots + \lambda_k, z) : \lambda_i \in \mathbb{R}_+,\ i = 1, \ldots, n,\ z \in (\lambda_1 C + \lambda_2 C + \cdots + \lambda_k C)\}$$

setting $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ and recalling that $\lambda_1 C + \lambda_2 C + \cdots + \lambda_k C = (\lambda_1 + \lambda_2 + \cdots + \lambda_k)C = \lambda C$

$$= \{(\lambda, z) : \lambda \in \mathbb{R}_+,\ z \in \lambda C\}$$

Of course $K \subset \mathbb{R}^{n+1}$ and it's still possible to recover C (even if represented in $\mathbb{R}^{n+1}$) as a "cross-section" of K at level $\lambda = 1$, i.e. as the intersection between K and the hyperplane $H = \{(1, z) : z \in \mathbb{R}^n\}$:

$$K \cap H = \{(\lambda, z) : \lambda \in \mathbb{R}_+,\ z \in \lambda C\} \cap \{(1, z) : 1, z \in \mathbb{R}^n\} = \{(1, z) : z \in C\}$$

This kind of representation will reveal extremely useful to build invariant functions for CFMMs starting from the design of the convex set of reachable reserves C.
Indeed it will be shown that the hypograph of the invariant function corresponds to the convex cone generated by the basic set of reachable reserves C, being C the first upper-level set of the invariant function conceived as a gauge-like concave function (in the sense of [Roc70]). Another source of usefulness of cones or, more precisely, of *proper cones* is the concept of *generalized inequality.*

Definition 22 (Proper cone)**.** A cone K is said to be *proper* if:

- K is convex and contains the origin (i.e. $\text{cone}(K) = K$)
- K is closed (i.e. $\text{cl}(K) = K$)
- K is solid (i.e. $\text{int}(K) \neq \emptyset$)
- K is pointed (i.e. $K \cap -K = \{0\}$, so it doesn't contain any line)

Proper cones can be used for defining a generalized inequality, which is a partial order relation on $\mathbb{R}^n$ defined by:

$$x \preceq_K y \iff y - x \in K$$
$$x \prec_K y \iff y - x \in \text{int}(K)$$

Notice that by setting $K = \mathbb{R}_+$ (which indeed is a proper cone) the generalized inequality resembles to the usual inequality (i.e. standard ordering) on $\mathbb{R}$, indeed:

$$x \preceq_{\mathbb{R}_+} y \iff y - x \in \mathbb{R}_+ \implies x \leq y$$
$$x \prec_{\mathbb{R}_+} y \iff y - x \in \text{int}(\mathbb{R}_+) \implies x < y$$

More generally, whenever $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and the inequality sign is used as "elementwise inequality" like in [DRCA23], it's possible to interpret it as a generalized inequality with respect to the non-negative orthant $K = \mathbb{R}^n_+$. Indeed

$$(x \preceq 0) = (x \preceq_{\mathbb{R}^n_+} 0) \iff 0 - x \in \mathbb{R}^n_+ \iff \xi_i \leq 0,\ i = 1, \ldots, n$$
$$(x \prec 0) = (x \prec_{\mathbb{R}^n_+} 0) \iff 0 - x \in \mathbb{R}^n_+ \setminus \{0\} \iff \xi_i < 0,\ i = 1, \ldots, n$$

However, differently from the standard ordering, a generalized inequality doesn't bring necessarily the concept of "less than" or "greater than" being just a partial ordering. Indeed, given $x, y \in \mathbb{R}^n$, it's possible that $x \preceq_K y$ and $y \preceq_K x$ but $x \neq y$.

The following proposition states that convex hull of the union of two convex cones containing the origin K_1 and K_2 leads to the ordinary sum of the two cones, while their inverse sum leads to their intersection.

Proposition 11. *Let $K_1 \subset \mathbb{R}^n$ and $K_2 \subset \mathbb{R}^n$ be two convex cones containing the origin, then*

$$\textit{conv}(K_1 \cup K_2) = K_1 + K_2$$
$$K_1 \# K_2 = K_1 \cap K_2$$

Proof.

$$\begin{aligned}
\operatorname{conv}(K_1 \cup K_2) &= \bigcup \left\{ \lambda_1 K_1 + \lambda_2 K_2 : \sum_{i=1}^{2} \lambda_i = 1, \lambda_i \in \mathbb{R}_+,\ i = 1, 2 \right\} \\
&= \bigcup_{\lambda \in [0,1]} \{(1-\lambda)K_1 + \lambda K_2\} \\
&= \{(1-\lambda)K_1 + \lambda K_2 : \lambda = 0\} \cup \{(1-\lambda)K_1 + \lambda K_2 : \lambda = 1\} \cup \bigcup_{\lambda \in (0,1)} \{(1-\lambda)K_1 + \lambda K_2\} \\
&= K_1 \cup K_2 \cup (K_1 + K_2) \\
&= K_1 + K_2
\end{aligned}$$

since both K_1 and K_2 contains the origin, $(K_1 \cup K_2) \subset (K_1 + K_2)$ implying that $(K_1 \cup K_2) \cup (K_1 + K_2) = K_1 + K_2$

On the other hand, regarding inverse sum:

$$\begin{aligned}
K_1 \# K_2 = &= \bigcup \left\{ \lambda_1 K_1 \cap \lambda_2 K_2 : \sum_{i=1}^{2} \lambda_i = 1, \lambda_i \in \mathbb{R}_+,\ i = 1, 2 \right\} \\
&= \bigcup_{\lambda \in [0,1]} \{(1-\lambda)K_1 \cap \lambda K_2\} \\
&= \{(1-\lambda)K_1 \cap \lambda K_2 : \lambda = 0\} \cup \{(1-\lambda)K_1 \cap \lambda K_2 : \lambda = 1\} \cup \bigcup_{\lambda \in (0,1)} \{(1-\lambda)K_1 \cap \lambda K_2\} \\
&= (K_1 \cap \{0\}) \cup (\{0\} \cap K_2) \cup (K_1 \cap K_2) \\
&= \{0\} \cup \{0\} \cup (K_1 \cap K_2) \\
&= K_1 \cap K_2
\end{aligned}$$

□

Every convex cone K containing the origin is associated with a pair of subspaces which can be retrieved from the ordinary sum and the inverse addition of K with its symmetric reflection across the origin $-K$:

- *The smallest subspace containing* K, that is $\operatorname{aff}(K)$ (this is the subspace that can be associated with any set, i.e. the subspace parallel to the affine hull of the set, but in this case the subspace coincides with the affine hull since K contains the origin, thus $\operatorname{aff}(K)$ contains the origin and any affine set containing the origin is actually a vector subspace)

$$L_1 = \operatorname{aff}(K) = K - K = \{x - y : x, y \in K\} \supset K$$

  Notice in fact that, being closed under positive scalar multiplication, K will contain the half-lines starting from the elements of K. By taking the Minkowski sum of K with its symmetric reflection across the origin $-K$, you obtain an affine set because now every line passing through each point of K is contained in $K - K$.

Notice also that such subspace can be characterized as the convex hull of the union between K and its symmetric reflection across the origin since $L_1 = K - K = K + (-K) = \text{conv}(K \cup (-K))$

- *The largest subspace contained in* K, also called *lineality space*

$$L_2 = K \cap -K = \{x \in K : -x \in K\} \subset K$$

Of course, if K is a proper cone (or more generally if K is pointed) then $L_2 = \{0\}$.

Analogously to before, notice also that $L_2 = K \cap -K = K \# (-K)$

There exist several convex cones containing the origin which can be referenced to convex sets in order to have a better understanding of their nature and properties. Thus, given $C \subset \mathbb{R}^n$, it's possible to anticipate some of them, which are going to be largely used in the upcoming sections:

- *Normal cone to* C *at* $x_0 \in C$: $N(x_0|C) := \{x^\star : \langle x - x_0, x^\star \rangle \leq 0,\ x \in C\}$.

  This set contains all the "normal points" $x^\star$ to C at x_0, which are all the points that are not making any acute angle with any point of C lying on any line segment having x_0 as one of its endpoints. Indeed, according to the Carnot theorem, $\langle x - x_0, x^\star \rangle = ||x - x_0|| \cdot ||x^\star|| \cdot \cos(\theta)$ where θ stands for the angle between vector $x - x_0$ and $x^\star$. Because of the non-negativity of the norm, if $\cos(\theta) \leq 0$ then $\langle x - x_0, x^\star \rangle \leq 0$ which occurs when $\theta \in [\frac{\pi}{2}, \frac{3}{2}\pi]$

- *Polar cone to* K: $K^\circ := \{x^\star : \langle x, x^\star \rangle \leq 0,\ x \in K\}$.

  It can be interpreted as the normal cone to K at the origin. Polar cones are typically introduced as a specific case of the more general class of polars of convex sets C°, that are the convex sets containing the origin (not necessarily cones) that are supported by the closure of gauge function of C. Of course, in the case of cones, the closure of the gauge function would correspond to the indicator function of the cone itself, and that's the reason why the polar cone K° reminds to the cone normal to K at the origin.

- *Dual cone of* K: $K^\star := \{x^\star : \langle x, x^\star \rangle \geq 0,\ x \in K\}$

  This convex cone containing the origin is given by the symmetric reflection across the origin of the polar cone K°. In other words, $K^\star = -K^\circ$ implying that $x^\star \in K^\star$ if and only if $-x^\star \in K^\circ$ (i.e., $-x^\star$ is normal to K at the origin).

- *Barrier cone of* C: $\{x^\star : \exists \beta \in \mathbb{R}, \langle x, x^\star \rangle \leq \beta, x \in C\}$.

  As it will be discussed in the dedicated section, the barrier cone of C is the effective domain of the support function of C (i.e. $\delta^\star(\cdot|C)$), because it defines the collection of half-spaces (indexed by $x^\star \in \{x^\star : \exists \beta \in \mathbb{R}, \langle x, x^\star \rangle \leq \beta, x \in C\}$) of the type $H_{x^\star} = \{x : \langle x, x^\star \rangle \leq \delta^\star(x^\star|C)\}$ such that $C \subset H_{x^\star}$. The barrier cone acts as index set of such collection and taking the intersection of half-spaces ranging over the barrier cone is possible to recover the external representation of $\text{cl}(C)$

- *Recession cone of* C: $0^+C := \{x^\star : (\lambda x^\star + C) \subseteq C, \lambda \geq 0\}$

  The elements of the recession cone of C are called *directions of recession* of C and correspond to the set of vectors $x^\star$ such that $(x + \lambda x^\star) \in C$ $\forall x \in C$, $\forall \lambda \in \mathbb{R}_+$ (i.e., the half-line starting from any point of C and pointing in the direction of $x^\star$ is fully contained in C). Directions of recession of C (or, more generally, the recession cone 0^+C) are particularly useful to understand the asymptotic behavior of C if it is an unbounded convex sets. On the other hand, if the set is bounded, the recession cone would correspond to the singleton of the origin, implying that C does not recede in any direction. The subspaces associated with the recession cone of C are the *lineality space of* C $L_2 = (-0^+C) \cap 0^+C$ and the affine hull $L_1 = 0^+C - 0^+C$ of C. Noticeably, when C is bounded, one has $0^+C = L_1 = L_2 = \{0\}$. The *lineality of* C corresponds to $\dim(L_2) = \dim((-0^+C) \cap 0^+C)$ while the *rank of* C corresponds to $\dim(L_1) - \dim(L_2) = \dim(\text{aff}(0^+C)) - \dim((-0^+C) \cap 0^+C)$ and it gives a measure of the non-linearity of C (for example, partial affine sets have null rank).

2.1.4 Hyperplane separation theorem

As mentioned in the previous sections, the simple existence of a hyperplane in $\mathbb{R}^n$ partitions the space in two half-spaces. This simple evidence becomes more interesting if referred to the relation between two convex sets, each living in a different subset of such partition.

Indeed, considering two convex sets $C_1 \subseteq \mathbb{R}^n$ and $C_2 \subseteq \mathbb{R}^n$, a hyperplane $H \subseteq \mathbb{R}^n$ is said to *separate* C_1 and C_2 if C_1 lives in one of the closed half-spaces associated with H and C_2 lives in the opposite one.

However, this generic situation would cover also the trivial case in which both sets are living on the separating hyperplane: imagine for example $H \subset \mathbb{R}^2$ as a line in $\mathbb{R}^2$ and C_1 and C_2 as the singleta of points, or segments or even half-lines with opposite direction lying on H. Thus, as further specification of such definition, a hyperplane $H = \{x \in \mathbb{R}^n : \langle x, b \rangle = \beta\}$ is said to:

- *properly* separate C_1 and C_2 if both are not contained in H
- *strictly* separate C_1 and C_2 if $\exists r > 0 : B(C_1, r) \subseteq \{\langle x, b \rangle < \beta\}$ and $B(C_2, r) \subseteq \{\langle x, b \rangle > \beta\}$, where $B(C_i, r)$ recalls the notation used in (2.5) for the set of points that are distant from C_i no more than r (i.e. $B(C_i, r) = \bigcap_{r>0} C_i + rB$).

In other words, in case of strict separation the two sets are not only living in different open half-spaces, but they are also living at a positive distance from the separating hyperplane, meaning that they are "sufficiently inside" the half-space in which they are living.

Trivially, the simple fact that $C_1 \cap C_2 = \emptyset$ is not a sufficient condition for strict separation. For example, taking $C_1 = \left\{(\xi_1, \xi_2) : \xi_2 \geq \xi_1^{-1}\right\}$ (which will correspond to the set of reachable reserves for Uniswap V2) and $C_2 = \{(\xi_1, \xi_2) : \xi_2 = 0, \xi_1 \geq 0\}$ (i.e. the horizontal half-line emanating from the origin in $\mathbb{R}^2$), the only candidate as separating hyperplane is $H = \{(\xi_1, \xi_2), \xi_2 = 0, \xi_1 \in \mathbb{R}\}$. In this case, C_1 and C_2 are disjoint but because of the asymptotic behavior of the boundary of C_1, $\nexists r > 0 : B(C_1, r) \subseteq \{(\xi_1, \xi_2), \xi_2 > 0, \xi_1 \in \mathbb{R}\}, B(C_1, r) \subseteq \{(\xi_1, \xi_2), \xi_2 < 0, \xi_1 \in \mathbb{R}\}$. However, C_1 and C_2 are properly separated by H (since proper separation allows C_2 to live in H, if C_1 is not living in it, and vice-versa).

Trivially, the existence of a proper separating hyperplane is referred to find a pair $(b, \beta) \in \mathbb{R}^n \times \mathbb{R}$ such that the associated hyperplane $H = \{x : \langle x, b\rangle = \beta\}$ is a proper separating hyperplane. One way to check this is to verify the existence of a normal vector $b \in \mathbb{R}^n$ having the properties stated by the following proposition:

Proposition 12 (Existence of a proper separating hyperplane)**.** *Let $C_1 \in \mathbb{R}^n$ and $C_2 \in \mathbb{R}^n$ be non-empty convex sets. Then, there exists a proper separating hyperplane for C_1 and C_2 if and only if there exists $b \in \mathbb{R}^n$ such that:*

$$\begin{cases} \inf_{x \in C_1} \langle x, b\rangle \geq \sup_{x \in C_2} \langle x, b\rangle \\ \sup_{x \in C_1} \langle x, b\rangle > \inf_{x \in C_2} \langle x, b\rangle \end{cases}$$

Moreover, from a topological perspective, this is equivalent to say that $ri(C_1) \cap ri(C_2) = \emptyset$.

Proof. The first requirement would allow to pick any $\beta \in \left[\sup_{x \in C_2} \langle x, b\rangle, \inf_{x \in C_1} \langle x, b\rangle\right]$ such that $H = \{x : \langle x, b\rangle = \beta\}$ is a separating hyperplane for C_1 and C_2, because $\langle x, b\rangle \geq \beta \; \forall x \in C_1$ (since $\beta < \inf_{x \in C_1} \langle x, b\rangle$, implying that $C_1 \subset \{x : \langle x, b\rangle \geq \beta\}$) and $\langle x, b\rangle \leq \beta \; \forall x \in C_2$ (since $\beta > \sup_{x \in C_2} \langle x, b\rangle$, implying that $C_2 \subset \{x : \langle x, b\rangle \leq \beta\}$), while the second requirement prevents the hyperplane from being a trivial separating hyperplane (i.e. when both C_1 and C_2 live on the same hyperplane). The existence of a proper separating hyperplane can be inferred also from the topological properties of the considered sets. Indeed, a necessary and sufficient condition for the existence of a proper separating hyperplane is that $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$. As a graphical intuition, one could think about C_1 as a cube in $\mathbb{R}^3$ and C_2 as a face of such cube. In this case, $C_2 \subset C_1$ but $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$ since C_2 lives on the relative boundary of C_1 (which, by definition, is $\mathrm{cl}(C_1) \setminus \mathrm{ri}(C_1)$): in this case, $\mathrm{aff}(C_2)$ is the hyperplane (being a $(n-1)$-dimensional affine set) that separates properly C_1 from C_2 fully containing C_2 but not C_1.

□

Proposition 13 (Existence of a strictly separating hyperplane)**.** *Let $C_1 \in \mathbb{R}^n$ and $C_2 \in \mathbb{R}^n$ be non-empty convex sets. Then, there exists a strictly separating hyperplane for C_1 and C_2 if and only if there exists $b \in \mathbb{R}^n$ such that:*

$$\inf_{x \in C_1} \langle x, b\rangle > \sup_{x \in C_2} \langle x, b\rangle$$

Moreover, from a topological perspective, this is equivalent to say that $\exists \epsilon > 0 : (C_1 + \epsilon B) \cap (C_2 + \epsilon B) = \emptyset$, implying that $0 \notin cl(C_1 - C_2)$.

Proof. The strict inequality $\inf_{x\in C_1}\langle x,b\rangle > \sup_{x\in C_2}\langle x,b\rangle$ implies that the interval

$$\left[\sup_{x\in C_2}\langle x,b\rangle,\ \inf_{x\in C_1}\langle x,b\rangle\right]$$

is surely not a singleton, implying that there exists a neighborhood with ray $\delta > 0$ centered in β (belonging to the interval) that it's still contained in this interval i.e.:

$$[\beta-\delta,\beta+\delta] \subseteq \left[\sup_{x\in C_2}\langle x,b\rangle,\ \inf_{x\in C_1}\langle x,b\rangle\right]$$

. This is useful to know since it would imply that:

- $\langle x,b\rangle \le \beta-\delta\ \forall x \in C_2$ (since $\beta-\delta \ge \sup_{x\in C_2}\langle x,b\rangle$) $\implies \langle x,b\rangle+\delta \le \beta\ \forall x\in C_2$. By picking $\epsilon > 0$ sufficiently small such that $\delta > \sup_{y\in\epsilon B}\langle y,b\rangle$ you have that $\langle x,b\rangle + \langle y,b\rangle < \beta\ \forall x \in C_2, \forall y \in \epsilon B \implies \langle x+y,b\rangle < \beta\ \forall x\in C_2, \forall y\in\epsilon B \implies \langle z,b\rangle < \beta\ \forall z \in (C_2+\epsilon B)$ implying that $C_2+\epsilon B \subset \{z : \langle z,b\rangle < \beta\}$

- $\langle x,b\rangle \ge \beta+\delta\ \forall x \in C_1$ (since $\beta+\delta \le \inf_{x\in C_1}\langle x,b\rangle$) $\implies \langle x,b\rangle-\delta \ge \beta\ \forall x\in C_1$. By picking $\epsilon > 0$ sufficiently small such that $-\delta > \inf_{y\in\epsilon B}\langle y,b\rangle$ you have that $\langle x,b\rangle + \langle y,b\rangle > \beta\ \forall x \in C_1, \forall y \in \epsilon B \implies \langle x+y,b\rangle > \beta\ \forall x\in C_1, \forall y\in\epsilon B \implies \langle z,b\rangle > \beta\ \forall z \in (C_1+\epsilon B)$ implying that $C_1+\epsilon B \subset \{z : \langle z,b\rangle > \beta\}$

In other words, the condition $\inf_{x\in C_1}\langle x,b\rangle > \sup_{x\in C_2}\langle x,b\rangle$ is sufficient to guarantee the existence of a hyperplane strictly separating C_1 from C_2 (so that C_1 lives in $\{z : \langle z,b\rangle > \beta\}$ while C_2 lives in $\{z : \langle z,b\rangle < \beta\}$).
On the other hand, if two convex sets are strictly separable, it means that $\exists\epsilon > 0$ arbitrarily small such that $(C_1+\epsilon B)\cap(C_2+\epsilon B) = \emptyset$. Of course, this means that there isn't any point that it's far more than ϵ from both to C_1 and C_2. Thus, since these two sets don't have any point in common, the Minkowski sum between $(C_1+\epsilon B)$ and the symmetric reflection across the origin of $(C_2+\epsilon B)$ should not contain the origin.
In other words, C_1 and C_2 are strictly separable if and only if $\exists\epsilon > 0 : 0 \notin (C_1+\epsilon B)-(C_2+\epsilon B)$, meaning that

$$0 \notin \bigcap_{\epsilon>0}(C_1+\epsilon B)-(C_2+\epsilon B)$$

Notice that $(C_1+\epsilon B)-(C_2+\epsilon B) = C_1-C_2+\epsilon B-\epsilon B = C_1-C_2+\epsilon(B+(-B))$. However, recalling that the Euclidean ball is symmetric around the origin ($-B = B$) this condition can be rewritten as

$$0 \notin \bigcap_{\epsilon>0} C_1-C_2+\epsilon 2B$$

calling $\delta = \epsilon 2$

$$\begin{aligned}&\bigcap_{\epsilon>0} C_1-C_2+\epsilon 2B\\ &= \bigcap_{\delta>0} C_1-C_2+\delta B\\ &= \mathrm{cl}(C_1-C_2)\end{aligned}$$

Thus, the topological condition for strict separation is that $0 \notin \text{cl}(C_1 - C_2)$.

□

Strict separation gives an important tool for understanding the "external representation" of closed convex sets as the intersection of the closed half-spaces containing it. Indeed, given $C \subset \mathbb{R}^n$ as a closed convex set and $\bar{C} = \{x : x \notin C\}$ as the complement of C, one has $0 \notin \text{cl}(C - \{a\})$ $\forall a \in \bar{C}$. This means that for any $a \in \bar{C}$, there exists a hyperplane separating strictly C from $\{a\}$, meaning that C only is fully contained in one of the closed half-spaces generated by such hyperplane. Thus, by collecting all such closed half-spaces ranging over all the elements of $\bar{C}$, and taking the intersection of such collection, one can recover the closed convex set C.

This fact guarantees also the non-emptiness of the *barrier cone* (i.e., non-emptiness of the support function $\text{dom}(\delta^\star(\cdot|C))$) of C, because if C is closed it is representable as the intersection of closed half-spaces containing it, meaning that, $\exists b \in \mathbb{R}^n : \langle x, b\rangle \leq \beta$ $\forall x \in C$ for some $\beta \in \mathbb{R}$ (recall that the barrier cone of C is given by $\{b : \exists \beta \in \mathbb{R}, \langle x, b\rangle \leq \beta, x \in C\}$).

Separating hyperplanes are extremely important also for describing the concept of tangency in convex analysis which is usually described in terms of *supporting half-spaces*.

Definition 23 (Supporting half-space)**.** Let $C \subset \mathbb{R}^n$ be a convex set. A closed half-space $\tilde{H}$ associated with $(b, \beta) \in \mathbb{R}^{n+1}$ is said to support C at some point $x_0 \in C$ if these two conditions hold:

$$\begin{cases} C \subseteq \tilde{H} \\ \langle x_0, b\rangle = \beta \end{cases}$$

the second condition is equivalent to say that $x_0 \in \text{cl}(\tilde{H}) \setminus \text{ri}(\tilde{H})$ where the relative boundary of $\tilde{H}$ is said to be a *supporting hyperplane* to C at x_0.

Thus, a *supporting half-space* to a convex set C is defined as a closed half-space fully containing C and having at least one point of C on its boundary, which is called *supporting hyperplane.* In other words, $(b, \beta) \in \mathbb{R}^n \times \mathbb{R}$ associated with $\tilde{H} = \{x : \langle x, b\rangle \leq \beta\}$ defines a supporting half-space to C if:

- $\langle x, b\rangle \leq \beta \; \forall x \in C$
- $\exists x \in C : \langle x, b\rangle = \beta$

These two specifications allow to understand that a supporting half-space is associated with the supremum of a linear function over C: indeed the first specification is equivalent to say that $\beta \geq \sup_{x\in C}\langle x, b\rangle$, while the second specification requires that such value β is attained for at least one point of C, implying that $\beta = \sup_{x\in C}\langle x, b\rangle$. Indeed, as it will be seen in the dedicated section, such "optimal offset" β for obtaining a supporting hyperplane to C specifying a certain vector b normal to the hyperplane is given by the support function of C, that is $\delta^\star(\cdot|C) : b \mapsto \sup_{x\in C}\langle x, b\rangle$, whose effective domain is indeed the barrier cone of C.

On the other hand, given the "offset" β of a supporting half-space, it's possible to decompose it as $\beta = \langle x_0, b\rangle$ for a given $x_0 \in C$: thus, the first requirement could be interpreted also as:

$$\langle x, b\rangle \leq \beta \; \forall x \in C$$

$$\langle x, b\rangle \leq \langle x_0, b\rangle \; \forall x \in C$$

$$\langle x - x_0, b\rangle \leq 0 \; \forall x \in C$$

Implying that:

- The vector b normal to the supporting hyperplane is also normal to the set C at some point $x_0 \in C$ (i.e. $b \in N(x_0|C)$)
- The normal cone to C at $x_0 \in C$ (i.e. $N(x_0|C) = \{b : \langle x - x_0, b\rangle \leq 0\}$) is the index set of the collection of supporting hyperplanes to C at $x_0 \in C$ in the sense that, given $x_0 \in C$, it's possible to retrieve all the supporting hyperplane to C at x_0 from the pairs $(b, \beta) = (b, \langle x_0, b\rangle) \in \mathbb{R}^n \times \mathbb{R} \quad \forall b \in N(x_0|C)$

These findings prepare the understanding of the following proposition, according to which there always exist at least one half-space supporting a convex set C at any of its boundary points. As it will be seen in the next sections, considering such sets as epigraphs (hypographs) of convex (concave) functions, this fact is extremely important for understanding the existence of subgradients (supergradients) of convex (concave) functions at any of their boundary points, i.e. the subdifferentiability (superdifferentiability) of convex (concave) functions at any point of the relative interior of their effective domain

Proposition 14 (Non-emptiness of normal cone at boundary points). *Let $C \subset \mathbb{R}^n$ be a convex set and let $N(x_0|C)$ be the cone normal to C at some $x_0 \in C$. Then,*

$$N(x_0|C) \setminus \{0\} \neq \emptyset \; \forall x_0 \in cl(C) \setminus ri(C)$$

This is equivalent to say that there always exists at least one supporting half-space to C at any of its boundary points, because there will be always at least one non-null vector normal to C for every $x_0 \in cl(C) \setminus ri(C)$

Proof. To prove this, one can think about the supporting hyperplane as a hyperplane properly separating C from $x_0 \in \mathrm{cl}(C) \setminus \mathrm{ri}(C)$. Given a subset of points $D \subset C$ disjoint from $\mathrm{ri}(C)$ (e.g. the singleton of a point a on the relative boundary of C), the supporting hyperplane to C fully containing D is the same hyperplane that separates properly C from D. Indeed, the fact that $D \cap \mathrm{ri}(C) = \emptyset$ implies $\mathrm{ri}(D) \cap \mathrm{ri}(C) = \emptyset$, that is the topological necessary and sufficient condition for the existence of a hyperplane properly separating D from C. This fact guarantees two important properties:

- There always exists a non-null vector normal to C at each boundary point of C (i.e. the normal cone to C is non-empty and it is not the zero vector alone at each boundary point of C, implying that you can always find a supporting hyperplane to C at each boundary point of C)

- $x \in \mathrm{cl}(C) \setminus \mathrm{ri}(C) \iff$ exists a linear function h non-constant on C achieving its maximum on C at x. That's because it's possible to find a supporting hyperplane to a convex set C at any of its boundary points and, since such hyperplane is characterized by $(b, \beta) \in \mathbb{R}^n \times \mathbb{R}$, the associated linear function is given by $h(x) = \langle x, b \rangle$ and the maximum achieved over C is $\beta = \sup_{x \in C} \langle x, b \rangle$

□

2.2 The Pareto Optimal Frontier

The discussion about minimum points, minimal points and Pareto Optimal Frontier of a generic set $S \subset \mathbb{R}^n$ requires the definition polar cone and dual cone of a generic cone $K \subset \mathbb{R}^n$, as anticipated at the end of the subsection dedicated to cones.

2.2.1 Polar cones and dual cones

Definition 24 (Polar cone)**.** Let $K \subset \mathbb{R}^n$ be a generic cone. the cone *polar* to K is defined as:

$$K^\circ = \{x^\circ : \langle x, x^\circ \rangle \leq 0,\ x \in K\}$$

Noticeably, independently from the properties of K as a cone, K° includes always the origin and it is always a closed convex set being the intersection of a collection of closed half-spaces having $x \in K$ as normal vector, indeed:

$$K^\circ = \bigcap_{x \in K} \tilde{H}_x^\circ$$

where $\tilde{H}_x^\circ = \{x^\circ : \langle x^\circ, x \rangle \leq 0\}$.
Alternatively, as an anticipation, K° can be seen as the zero-sub-level set of the closed convex function $\delta^\star(\cdot|K)$, which is the support function of K.

The definition of K° allows to characterize it as the cone normal to K at the origin, indeed

$$K^\circ = N(0|K)$$

Thus, recalling the findings discussed at the end of the previous section, the polar cone K° automatically indexes the collection of supporting hyperplanes to K at the origin, in the sense that, given $K \subset \mathbb{R}^n$ a generic cone, it's possible to retrieve all the supporting hyperplane to K at the origin from the pairs $(b, \beta) = (b, 0) \in \mathbb{R}^n \times \mathbb{R} \quad \forall b \in K^\circ$. This service is provided also by the dual cone $K^\star$ of K being simply the symmetric reflection across the origin of the polar cone K° as it can be induced by the definition of $K^\star$.

Definition 25 (Dual cone)**.** Let $K \subset \mathbb{R}^n$ be a generic cone. The cone *dual* to K is defined as:

$$K^\star = \{x^\star : \langle x, x^\star \rangle \geq 0,\ x \in K\}$$

Trivially, keeping the notation $\tilde{H}^\circ_x = \{x^\circ : \langle x^\circ, x \rangle \leq 0\}$, it follows that

$$K^\star = -K^\circ = -N(0|K) = \bigcap_{x \in K} (-\tilde{H}^\circ_x)$$

And this implies that $K^\star$ remains a closed, convex, origin-including cone as K°. Analogously to K°, $K^\star$ can be seen as the zero-upper-level of the closed concave function $-\delta^\star(\cdot | - K)$, which could be used for proving the properties just mentioned.

The dual cone $K^\star$ preserves the properness of the original cone K, in the sense that if K is a proper cone, then $K^\star$ is also a proper cone, which means that induces it also a generalized inequality defined as the "dual" of the generalized inequality $x \preceq_K y$.

Indeed, assuming that K is a proper cone, the dual correspondence between K and $K^\star$ can be extended to the dual correspondence between the respective induced generalized inequalities. The following proposition defined in [BV04] describes this kind of correspondence. In order to keep the same notation as in [BV04], $\lambda \succeq_{K^\star} 0$ will be considered equivalent to $\lambda \in K^\star$.

Proposition 15. *Let $K \subset \mathbb{R}^n$ be a proper cone. Then, the generalized inequality $x \preceq_K y$ is verified if and only if*

$$\langle \lambda, x - y \rangle \leq 0 \quad \forall \lambda \succeq_{K^\star} 0 \tag{2.8}$$

Proof. Because of the properness of K (which implies closedness, convexity and origin-inclusion) one has that $K = K^{\star\star}$. Moreover, $x \preceq_K y$ means that $y - x \in K$. Thus, by calling $z = y - x \in K$ one can characterize K as

$$K = K^{\star\star} = \{z : \langle \lambda, z \rangle \geq 0,\ \lambda \in K^\star\}$$

Implying that $z \in K \iff \langle \lambda, z \rangle \geq 0,\ \forall \lambda \in K^\star$ which, unwrapping z, leads to the following condition

$$\begin{aligned} y - x \in K \iff & \langle \lambda, y - x \rangle \geq 0,\ \forall \lambda \in K^\star \\ & \langle \lambda, x - y \rangle \leq 0,\ \forall \lambda \in K^\star \end{aligned}$$

Which, expressed in terms of generalized inequality, is equivalent to say that $x \preceq_K y \iff \langle \lambda, x - y \rangle \leq 0 \quad \forall \lambda \succeq_{K^\star} 0$ □

2.2.2 Minimum points and minimal points

As previously anticipated, generalized inequalities are partial ordering which gives more freedom than the standard ordering $\leq$ in the sense that it allows to define "custom" rules of comparison between elements according to the properties of the proper cone K specified. It follows that the definitions of "minimum" and "minimal" points are not true in absolute sense but always with respect to a specific generalized inequality, i.e. with the respect of a specific proper cone of reference.

Definition 26 (Minimum point of a set w.r.t. K)**.** Let $K \subset \mathbb{R}^n$ be a proper cone and let $S \subset \mathbb{R}^n$ be a generic set. The *minimum point* of S with the respect of the generalized inequality $\preceq_K$ is the unique point $x_0 \in S$ such that

$$x_0 \preceq_K y \quad \forall y \in S$$

Proposition 16. *Let $K \subset \mathbb{R}^n$ be a proper cone and let $S \subset \mathbb{R}^n$ be a generic set. A point x_0 is a minimum point of S with the respect of the generalized inequality $\preceq_K$ if and only if it is the unique point such that*

$$\langle \lambda, x_0 - y \rangle \leq 0 \quad \forall y \in S,\ \forall \lambda \in K^\star$$

Proof. Apply proposition (15) to the definition of minimum point of S with the respect of the generalized inequality $\preceq_K$. □

This means that x_0 is a minimum element of S w.r.t. K if and only if the entire collection of half-spaces of type $\tilde{H}_\lambda = \{y : \langle \lambda, y \rangle \geq \langle \lambda, x_0 \rangle\}$ indexed by $\lambda \in K^\star$ supports the set S at x_0. Thus, each half-space $\tilde{H}_\lambda$ must support S at x_0 $\forall \lambda \in K^\star$. If this is not true even for a single $\lambda \in K^\star = -N(0|K)$, then x_0 is not a minimum point. Of course this requirement is quite strict: a possible relaxation of such condition could be

$$\exists \lambda \in K^\star : \langle \lambda, x_0 - y \rangle \leq 0 \quad \forall y \in S$$

In this context, it is required that at least one (not all) half-space $\tilde{H}_\lambda$ supports S at x_0 for some $\lambda \in K^\star$. This relaxed condition is what is seen in the set notation of "minimal points" of a set S with the respect of the generalized inequality $\preceq_K$.

Definition 27 (Minimal point of a set w.r.t. K and Pareto Optimal Frontier)**.** Let $K \subset \mathbb{R}^n$ be a proper cone and let $S \subset \mathbb{R}^n$ be a generic set. The *minimal point* of S with the respect of the generalized inequality $\preceq_K$ is a point $x_0 \in S$ such that

$$\exists \lambda \in K^\star, \langle \lambda, x_0 - y \rangle \leq 0 \quad \forall y \in S$$

it follows that the set of minimal points, also called *Pareto Optimal Frontier* of S w.r.t. K, is

$$\{x \in S : \exists \lambda \in K^\star, \langle \lambda, x - y \rangle \leq 0, y \in S\}$$

Anticipating the notation, is possible to characterize the Pareto Optimal Frontier of S w.r.t. K in terms of the support function of S since

$$\begin{aligned}\{x \in S : \exists \lambda \in K^\star, \langle \lambda, x-y \rangle \leq 0, y \in S\} &= \{x \in S : \exists \lambda \in K^\star, \langle \lambda, x \rangle \leq \langle \lambda, y \rangle, y \in S\} \\ &= \left\{x \in S : \exists \lambda \in K^\star, \langle \lambda, x \rangle \leq \inf_{y \in S} \langle \lambda, y \rangle \right\} \\ &= \{x \in S : \exists \lambda \in K^\star, \langle \lambda, x \rangle \leq -\delta^\star(-\lambda|S)\} \\ &= \left(\bigcup_{\lambda \in K^\star} \{x \in S : \langle \lambda, x \rangle \leq -\delta^\star(-\lambda|S)\} \right) \subset S\end{aligned}$$

2.2.3 Basic set of efficient reserves

This subsection uses a lot of concepts which are not yet introduced and so the reader is not supposed to fully understand it at this stage. However, it is important to introduce it here because it is the main application of the concepts of Pareto Optimal Frontier to the core topic of the thesis. The concept of Pareto Optimal Frontier applied to CFMM is important because it corresponds to the subset of the set of reachable reserves C where the CFMM reserves are expected to be located at any external price level $p \in \mathbb{R}^n_+$ because of the effect of arbitrage forces.
Indeed, the set of possible prices corresponds to the non-negative orthant because of the nature of prices as non-negative quantities. Thus, one could take $K = \mathbb{R}^n_+$ as proper cone of reference with the peculiarity of being a "self-dual" cone since $K^\star = K = \mathbb{R}^n_+$.

At the same time, picking $S = C$ as the basic set of reachable reserves and anticipating the characterization of the portfolio value function as $\hat{V}(p; 1) = -\delta^\star(-p|C)$ one could define the basic set of "efficient reserves" (i.e. the Pareto Optimal Frontier of the basic set of reachable reserves w.r.t $K = \mathbb{R}^n_+$) as a subset of C equal to

$$\left(\bigcup_{p \in \mathbb{R}^n_+} \{x \in C : \langle p, x \rangle \leq V(p; 1)\} \right) \subset C$$

corresponding to the range of a set-valued map of the type

$$\Theta : p \in \mathbb{R}^n_+ \mapsto \{x \in C : \langle p, x \rangle \leq V(p; 1)\}$$

But since $V(p; 1) = \inf_{y \in C}$, if $x \in C \implies \langle p, x \rangle \geq V(p; 1)\ \forall p \in \mathbb{R}^n$, implying that the basic set of efficient reserves is actually equal to

$$\left(\bigcup_{p \in \mathbb{R}^n_+} \{x \in C : \langle p, x \rangle = V(p; 1)\} \right) \subset C$$

so that the set-valued map $\Theta(\cdot)$ mapping each price to the corresponding set of efficient reserves is given by

$$\Theta : p \in \mathbb{R}^n_+ \mapsto \{x \in C : \langle p, x \rangle = V(p; 1)\} \tag{2.9}$$

Which resembles to the solution set of a system of equalities (provided by $\langle p, x\rangle = V(p;1)$) and inequalities (provided by $x \in C$).

2.3 Convex Functions

2.3.1 Core definitions

Definition 28 (Epigraph and hypograph of a function)**.** Let f be a function $f : S \subset \mathbb{R}^n \to \mathbb{R}\cup\{\pm\infty\}$, the *epigraph* of f is defined as the set

$$\text{epi}(f) := \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq \mu\} \subset \mathbb{R}^{n+1}$$

while the *hypograph* of f is defined as the set

$$\text{hyp}(f) := \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} : f(x) \geq \mu\} \subset \mathbb{R}^{n+1}$$

Definition 29 (Convex and concave functions)**.** Let f be a function $f : S \subset \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$. The function f is said to be *convex* if epi(f) is a convex set, while it is said to be *concave* if hyp(f) is a convex set.

From now on, the hat "ˆ" symbol will be used for denoting concave functions (the symbol reminds the "hat-shaped" graph of the negative of the absolute value, that is a concave function).

Proposition 17. *Let $f : S \subset \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a convex function. Then, the function $\hat{f} = -f$ is concave.*

Proof. This follows from the fact that hyp$(\hat{f})$ corresponds to the epigraph of f under the transformation $A : (x, \mu) \mapsto (x, -\mu)$. Since linear maps are convexity preserving, hyp$(\hat{f})$ remains a convex set.

$$\begin{aligned} A(\text{epi}(f)) &= \{A(x,\mu) : (x,\mu) \in \text{epi}(f)\} \\ &= \{(x, -\mu) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq \mu\} \\ &= \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq -\mu\} \\ &= \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} : -f(x) \geq \mu\} \\ &= \left\{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} : \hat{f}(x) \geq \mu\right\} =: \text{hyp}(\hat{f}) \end{aligned}$$

□

Definition 30 (Level sets)**.** Let $f : S \subset \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a convex function and let $\alpha \in \mathbb{R}$. The *sub-level set* of f associated with α is defined as

$$\{x \in S : f(x) \leq \alpha\}$$

while the *upper-level set* (or *upper-level set*) of f associated with α is defined as

$$\{x \in S : f(x) \geq \alpha\}$$

Definition 31 (Effective domain)**.** Let $f : S \subset \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a convex (concave) function. The subset of S such that f takes finite values is called *effective domain* of f and is denoted as

$$\text{dom}(f) := \{x \in S : \exists \mu \in \mathbb{R}, (x, \mu) \in \text{epi}(f)\} \quad (\text{dom}(f) = \{x \in S : \exists \mu \in \mathbb{R}, (x, \mu) \in \text{hyp}(f)\})$$

Heuristically speaking, the effective domain of f could be defined as the "widest" sub-level (upper-level) set of f i.e.

$$\text{dom}(f) = \{x : f(x) < \infty\} \quad (\text{dom}(f) = \{x : f(x) > -\infty\})$$

The convexity of f implies the convexity of its effective domain since dom(f) could be interpreted as the image of epi(f) under a linear map $A : (x, \mu) \mapsto x$ and, again, linear maps preserves affinity and convexity of the sets they are mapping. The same can be said for concave functions, thus dom(f) is always a convex set whether f is convex or concave.

Introducing the notation of the effective domain of f allows to split the behavior of f in the so-called *extended-value extension* $\tilde{f}$ that is

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \text{dom}(f) \\ \infty & x \notin \text{dom}(f) \end{cases} \qquad \tilde{\hat{f}}(x) = \begin{cases} \hat{f}(x) & \text{if } x \in \text{dom}(\hat{f}) \\ -\infty & x \notin \text{dom}(\hat{f}) \end{cases}$$

2.3.2 Core and additional properties

Definition 32 (Properness of convex and concave functions)**.** Let $f : S \subset \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a convex function. The function f is said to be *proper* if epi(f) does not contain any vertical line, meaning that:

- $\exists x : f(x) < \infty$ (i.e. non-emptiness of the effective domain)
- $f(x) > -\infty \ \forall x \in \text{dom}(f)$

Analogously, let $\hat{f} : S \subset \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a concave function. The function $\hat{f}$ is said to be *proper* if hyp$(\hat{f})$ does not contain any vertical line, meaning that:

- $\exists x : \hat{f}(x) > -\infty$ (i.e. non-emptiness of the effective domain)
- $\hat{f}(x) < \infty \ \forall x \in \text{dom}(\hat{f})$

A fundamental theorem for convex functions (which is furtherly specified according to additional properties of epi(f)) is the so-called *Jensen's inequality*:

Theorem 1 (Jensen's inequality)**.** *Let $f : S \subset \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a convex function. Then, for any $x_1, \dots, x_k \in S$ and any $\lambda_1, \dots, \lambda_k \in \mathbb{R}_+$ such that $\sum_{i=1}^k \lambda_i = 1$ it holds that*

$$f(\lambda_1 x_1 + \cdots + \lambda_k x_k) \leq \lambda_1 f(x_1) + \cdots + \lambda_k f(x_k)$$

Analogously, let $\hat{f} : S \subset \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ *be a concave function, then it holds that*

$$\hat{f}(\lambda_1 x_1 + \cdots + \lambda_k x_k) \geq \lambda_1 \hat{f}(x_1) + \cdots + \lambda_k \hat{f}(x_k)$$

Proof. Since epi(f) is convex, then $\sum_{i=1}^k \lambda_i (x_i, \mu_i) \in \text{epi}(f)$ where $\lambda_i \geq 0 \; \forall i \in [1, k]$ and $\sum_{i=1}^k \lambda_i = 1$. this means that

$$\left(\sum_{i=1}^k \lambda_i x_i, \sum_{i=1}^k \lambda_i \mu_i \right) = (x, \mu) \in \text{epi}(f)$$

Thus the set condition becomes

$$\begin{aligned} \mu \geq f(x) \implies & \sum_{i=1}^k \lambda_i \mu_i \geq f\left(\sum_{i=1}^k \lambda_i x_i \right) \\ \implies & \sum_{i=1}^k \lambda_i f(x_i) \geq f\left(\sum_{i=1}^k \lambda_i x_i \right) \end{aligned}$$

For concave functions, the proof is analogous. □

The Jensen's inequality can be used also in proving the well-known inequality between the arithmetic and geometric mean of a set of positive numbers (AM-GM inequality) [Roc70]. Indeed, let $f(x) = -\log(x)$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_+$ such that $\sum_{i=1}^k \lambda_i = 1$, then f is a convex function and the Jensen's inequality implies that

$$\begin{aligned} -\log\left(\sum_{i=1}^k \lambda_i x_i \right) &\leq -\sum_{i=1}^k \lambda_i \log(x_i) \\ \log\left(\sum_{i=1}^k \lambda_i x_i \right) &\geq \sum_{i=1}^k \lambda_i \log(x_i) \\ \sum_{i=1}^k \lambda_i x_i &\geq e^{\sum_{i=1}^k \lambda_i \log(x_i)} \\ \sum_{i=1}^k \lambda_i x_i &\geq \prod_{i=1}^k x_i^{\lambda_i} \end{aligned}$$

By setting $\lambda_1 = \cdots = \lambda_k = \frac{1}{k}$ one has that

$$\frac{\sum_{i=1}^k x_i}{k} \geq \sqrt[k]{\prod_{i=1}^k x_i} \tag{2.10}$$

Noticeably, the Jensen's inequality for concave functions has the reversed weak inequality sign. This theorem, which is usually presented as the "algebraic definition" of convex functions, depends on the convexity of the epigraph, i.e. the fact that such set is closed under any convex combination of its elements $(x, \mu) \in \text{epi}(f)$. Indeed, the image of a convex combination of points will be always lower or equal than the convex combination of the images of each point. Thinking about a convex parabola in $\mathbb{R}^2$ as an example, it's evident that the chord (i.e. line segment) connecting $(x_1, f(x_1))$

and $(x_1, f(x_1))$ (on which lies the point $((1-\lambda)x_1 + \lambda x_2, (1-\lambda)f(x_1) + \lambda f(x_2)), \lambda \in [0,1])$ is always above the parabola itself, (on which lies the point $((1-\lambda)x_1+\lambda x_2, f((1-\lambda)x_1+\lambda x_2)), \lambda \in [0,1])$.

The Jensen's inequality allows to conclude also an important aspect of improper convex functions, that is their unboundedness from below for any point in ri(dom(f)).

Proposition 18. *Let $f : S \subset \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be an improper convex function. Then,*

$$f(x) = -\infty \; \forall x \in ri(dom(f))$$

Proof. By definition of improperness it means that $\exists x_0 \in \text{dom}(f) : f(x_0) = -\infty$. Thus, according to (2.7), picking any $x \in \text{ri}(\text{dom}(f))$ and setting $\mu > 1$ implies that it's surely possible finding $(1-\mu)x_0 + \mu x =: y \in \text{dom}(f)$ because x is a relative interior point. This implies that it's possible to rewrite any relative interior point x as a generic point on the line-segment between x_0 and y (excluding the endpoints) i.e.

$$x = \frac{1}{\mu}(y - (1-\mu)x_0)$$

setting $\lambda = \frac{1}{\mu} \implies \lambda \in (0,1)$

$$x = (1-\lambda)x_0 + \lambda y$$

Because of Jensen's inequality:

$$f(x) = f((1-\lambda)x_0 + \lambda y) < (1-\lambda)f(x_0) + \lambda f(y) \quad \lambda \in (0,1)$$

And since $f(x_0) = -\infty \implies f(x) < -\infty \; \forall x \in \text{ri}(\text{dom}(f))$, that is $f(x) = -\infty \; \forall x \in \text{ri}(\text{dom}(f))$ □

Definition 33 (Lower and upper semicontinuity)**.** Let $f : S \subset \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a function. The function f is said to be *lower-semicontinuous* at $x \in S$ if

$$\begin{aligned} f(x) &= \liminf_{y \to x} f(y) \\ &= \lim_{\epsilon \to 0^+} \inf\{f(y) : d(x,y) \le \epsilon\} \\ &= \lim_{\epsilon \to 0^+} \inf\{f(y) : ||x-y||_2 \le \epsilon\} \\ &= \lim_{\epsilon \to 0^+} \inf_{y \in B(x,\epsilon)} f(y) \end{aligned}$$

The function f is said to be *upper-semicontinuous* at $x \in S$ if

$$\begin{aligned} f(x) &= \limsup_{y \to x} f(y) \\ &= \lim_{\epsilon \to 0^+} \sup\{f(y) : d(x,y) \le \epsilon\} \\ &= \lim_{\epsilon \to 0^+} \sup\{f(y) : ||x-y||_2 \le \epsilon\} \\ &= \lim_{\epsilon \to 0^+} \sup_{y \in B(x,\epsilon)} f(y) \end{aligned}$$

The function f is said to be *lower-semicontinuous* if it is lower-semicontinuous at any $x \in S$, while it is said to be *upper-semicontinuous* if it is upper-semicontinuous at any $x \in S$.

As previously mentioned, several properties of the epigraph of a function f affect the behavior of f itself. For example, as stated in [Roc70], if $\text{epi}(f)$ is a closed set, this is equivalent to say that any α-sub-level set of f is closed and that f is a lower-semicontinuous function. On the contrary, the closedness of $\text{hyp}(f)$ implies the lower semicontinuity of f and the closedness of all its upper-level sets.

As a trivial example, suppose that

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x^2 + 1 & \text{if } x \geq 0 \end{cases}$$

in this case you have that

$$\begin{cases} f(0) = 1 \\ \liminf_{y \to 0} f(y) = \lim_{\epsilon \to 0^+} \inf_{y \in \epsilon B} f(y) = 0 \\ \limsup_{y \to 0} f(y) = \lim_{\epsilon \to 0^+} \sup_{y \in \epsilon B} f(y) = 1 \end{cases}$$

It's evident that the function is not ordinary continuous at $x = 0$ (in particular, it's only upper-semi continuous since $f(0) = \limsup_{y \to 0} f(y) \neq \liminf_{y \to 0} f(y)$), but it's less evident here that $\text{epi}(f)$ is actually not-closed. Indeed, recalling that $\text{epi}(f) := \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq \mu\}$, the vertical interval $\{(0, \mu) : \mu \in [0, 1]\}$, that is a portion of the boundary of $\text{epi}(f)$, is not contained in $\text{epi}(f)$ (because $\mu \geq f(0)$ is equivalent to $\mu \geq 1$), implying that $\text{epi}(f)$ is not closed.
On the other hand, the lower semicontinuous "version" of such function is simply

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ x^2 + 1 & \text{if } x > 0 \end{cases}$$

As it will be described in the next section, given any real-valued convex function f on $\mathbb{R}^n$ is possible to derive the *lower semicontinuous hull* of f as the function g such that $\text{epi}(g) = \text{cl}(\text{epi}(f))$. This function is the greatest lower-semicontinuous function majorized by f (i.e. $g \leq f$). In [Roc70] it's specified that if f is proper, the *closure* of f (i.e. $\text{cl}(f(x))$) corresponds to the *lower semicontinuous hull* of f, otherwise it's infinity. Under this setting, this means that for proper convex functions closedness implies (and is implied by) lower-semicontinuity.

$$\text{cl}(f(x)) = \begin{cases} \liminf_{y \to x} f(y) & \text{if } f \text{ proper} \\ -\infty & \text{if } f \text{ improper} \end{cases}$$

Notice that a proper convex function f and its closure $\text{cl}(f(x))$ agree everywhere on $\text{dom}(f)$ except (maybe) at some boundary points of $\text{dom}(f)$: indeed

$$\text{ri}(\text{epi}(\text{cl}f)) = \text{ri}(\text{cl}(\text{epi}(f))) = \text{ri}(\text{epi}(f)) := \{(x, \mu) : x \in \text{ri}(\text{dom}(f)), \mu \in (f(x), \infty)\}$$

And since $\text{ri}(\text{epi}(\text{cl}f)) = \{(x,\mu) : x \in \text{ri}(\text{dom}(\text{cl}f)), \mu \in (\text{cl}(f(x)), \infty)\}$ by definition, this implies that $\text{ri}(\text{dom})(\text{cl}f) = \text{ri}(\text{dom}(f))$ and that if $(x,\mu) \in \text{epi}(f) \implies (x,\mu) \in \text{epi}(\text{cl}f)\ \forall x \in \text{ri}(\text{dom}(f))$.

Notice that the function in the previous example is not convex because, even if the epigraph is closed, some convex combinations (portions of line-segments) lie outside the epigraph. Indeed, $f(x)$ is not a convex-function but heuristically speaking it reminds something of a convex function because it's "almost" convex. Indeed, this function belongs to a wider set of functions called *quasiconvex functions* which relax the requirement about the convexity of the epigraph demanding just that all the α-sub-level sets of f are convex sets.

Definition 34 (Quasiconvex and quasiconcave functions)**.** Let f be a function $f : S \subset \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$. The function f is said to be *quasiconvex* if all the α-sub-level sets of f are convex sets, i.e. $\{x : f(x) \leq \alpha\}$ is a convex set $\forall \alpha \in \mathbb{R}$.
Analogously, the function f is said to be *quasiconcave* if all the α-upper-level sets of f are convex sets, i.e. $\{x : f(x) \geq \alpha\}$ is a convex set $\forall \alpha \in \mathbb{R}$.

Keeping the same example as before, it's possible to see that $\{x : f(x) \leq \alpha\}$ correspond to line-segments on the x-axis, thus convex sets $\forall \alpha \in \mathbb{R}$. Analogously, a function g is said to be *quasiconcave* if all the α-upper-level sets of g are convex sets, i.e. $\{x : g(x) \geq \alpha\}$ is a convex set $\forall \alpha \in \mathbb{R}$.

Of course, the convexity of all the α-sub-level sets is always satisfied for convex functions (the extreme case is the convexity of $\text{dom}(f)$ as the ∞-sub-level set of f), since any α-sub-level set of f is actually the image of $\text{epi}(f) \cap \{(x,\mu) : x \in \mathbb{R}^n, \mu \leq \alpha\}$ under $A : (x,\mu) \mapsto x$ (recall that both intersection and linear transformations are convexity-preserving).
For quasiconvex functions, the *Jensen's inequality* becomes

$$f(\lambda_1 x_1 + \cdots + \lambda_k x_k) \leq \max\{f(x_1), \ldots, f(x_k)\} \quad \lambda \succeq 0, \langle \lambda, \mathbf{1} \rangle = 1$$

And using a trivial graphical example of $f : \mathbb{R} \to \mathbb{R}$ it means that the image of a convex combination of two points $f((1-\lambda)x_1 + \lambda x_2), \lambda \in [0,1]$ is always lower or equal than the greatest image associated with one of the two points of the chord connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

Another important example of how the properties of $\text{epi}(f)$ affect f occurs when $\text{epi}(f)$ is a convex cone $K \subset \mathbb{R}^{n+1}$. Indeed, when this is the case, the convex function f is said to be *positively homogeneous* of degree one,

Definition 35 (Positive homogenous functions)**.** Let f be a function $f : S \subset \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$. The function f is said to be convex *positively homogenous* of degree one if $\text{epi}(f)$ is a convex cone $K \subset \mathbb{R}^{n+1}$. Analogously the function f is said to be concave *positively homogenous* of degree one if $\text{hyp}(f)$ is a convex cone $K \subset \mathbb{R}^{n+1}$.

This category of functions is fundamental in this analysis since both the invariant function and the basic portfolio value function of a CFMM are typically positively homogenous concave functions of degree one.

The main property of positively homogenous functions of degree one is that $f(\lambda x) = \lambda f(x)\ \forall \lambda > 0$. Indeed, since now epi$(f)$ is considered as a convex cone, if $(x, \mu) \in \text{epi}(f) \implies (\lambda x, \lambda\mu) \in \text{epi}(f)\ \forall\lambda > 0$, and of course this means that since $(x, f(x)) \in \text{epi}(f)$ then $(\lambda x, \lambda f(x)) \in \text{epi}(f)\ \forall\lambda > 0$, implying that $f(\lambda x) = \lambda f(x)\ \forall\lambda > 0$. Notice in fact that a positive rescaling via $\lambda > 0$ can affect the epigraph of f affecting it as a whole but also only the range or only the effective domain of f:

- $A : (x, \mu) \mapsto (x, \lambda\mu) \implies A(\text{epi}(f)) = \{(x, \mu) : x \in \text{dom}(f), \mu \geq \lambda f(x)\}$ inducing the function $(\lambda f)(x) = \lambda f(x)$, also called *left-scalar multiplication* of f, i.e. the function obtained by rescaling the range of f.
- $A : (x, \mu) \mapsto (\lambda x, \mu) \implies A(\text{epi}(f)) = \left\{(x, \mu) : x \in \lambda\text{dom}(f), \mu \geq f\left(\frac{x}{\lambda}\right)\right\}$, inducing the function $f\left(\frac{x}{\lambda}\right)$, i.e. the function obtained by rescaling the effective domain of f.
- $A : (x, \mu) \mapsto (\lambda x, \lambda\mu) \implies A(\text{epi}(f)) = \lambda\text{epi}(f) = \left\{(x, \mu) : x \in \lambda\text{dom}(f), \mu \geq \lambda f\left(\frac{x}{\lambda}\right)\right\}$ inducing the function $(f\lambda)(x) = \lambda f\left(\frac{x}{\lambda}\right)$, also called *right-scalar multiplication* of f, i.e. the function obtained by rescaling the epigraph of f (so rescaling both the effective domain and the range of f by the same scalar).

As seen previously, when epi(f) is a convex cone one has that if $(x, \mu) \in \text{epi}(f) \implies \lambda(x, \mu) \in \text{epi}(f)\ \forall\lambda > 0$, meaning that $\text{epi}(f) \supseteq \lambda\text{epi}(f)\ \forall\lambda > 0$, which implies that positively homogeneous convex functions are invariant to right-scalar multiplication (i.e. $(f\lambda)(x) = f(x)\ \forall\lambda > 0$). Of course, this property could be derived algebraically from the first property discussed: indeed $(f\lambda)(x) = \lambda f(x\lambda^{-1})$ and since $\lambda^{-1} > 0 \implies \lambda f(x\lambda^{-1}) = \lambda\lambda^{-1} f(x) = f(x)$.

For positively homogenous convex functions the *Jensen's inequality* becomes

$$f(\lambda_1 x_1 + \cdots + \lambda_k x_k) \leq \lambda_1 f(x_1) + \cdots + \lambda_k f(x_k) \quad \forall\lambda \succ 0$$

A very popular class of positively homogenous convex function is (ℓ_p), i.e. the p-norms class functions (like the Euclidean or the Tchebycheff norm) or, more in general, any norm function as described in previous sections. Indeed, the "triangular-inequality" property, is nothing more than the Jensen's inequality for positively homogenous convex functions with $\lambda = \mathbf{1} \in \mathbb{R}^k$ (i.e. $||x_1 + \cdots + x_k|| \leq ||x_1|| + \cdots + ||x_k||$).

2.3.3 Continuity of convex functions

Convex functions (proper or improper) have the quality of being ordinary continuous (i.e. lower-semicontinuous and upper-semicontinuous at the same time) on any open subset of the effective domain (in particular, with the respect of ri(dom(f))).

Proposition 19. *Let $f : S \subset \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a convex (or concave) function. Then, f is continuous on ri(dom(f)).*

Proof. As previously discussed, if f is convex and improper it follows that $f(x) = -\infty \ \forall x \in$ ri(dom(f)) and continuity is trivially verified.
On the other hand, if f is convex and proper, it has been said that $\text{cl}(f(x)) = f(x) \ \forall x \in \text{ri}(\text{dom}(f))$, implying that f is lower-semicontinuous on ri(dom(f)). Thus, it's sufficient to prove that f it's also upper semicontinuous on ri(dom(f)), that is equivalent to say that any α-upper-level is closed, i.e. $\{x : f(x) \geq \alpha\} = \text{cl}\left(\{x : f(x) \geq \alpha\}\right) \ \forall \alpha \in \mathbb{R}$ (which would imply the upper-semicontinuity of f everywhere actually, not only with the respect of ri(dom(f))).
Noticeably, the closedness of $\{x : f(x) \geq \alpha\}$ implies (and is implied by) the openness of $\{x : f(x) < \alpha\}$ being its complement. By picking $A : (x, \mu) \mapsto x$ it's possible to see that

$$\{x : f(x) < \alpha\} = A\left(\{(x, \mu) : x \in \text{ri}(\text{dom}(f)), \mu \in (f(x), \infty)\} \cap \{(x, \mu) : x \in \mathbb{R}^n, \mu < \alpha\}\right)$$

calling D the closed half-space $\{(x, \mu) : x \in \mathbb{R}^n, \mu \leq \alpha\}$ so that $\{(x, \mu) : x \in \mathbb{R}^n, \mu < \alpha\} = \text{ri}(D)$, then

$$\{x : f(x) < \alpha\} = A\left(\text{ri}(\text{epi}(f)) \cap \text{ri}(D)\right)$$

since $\text{ri}\left(\bigcap_{i \in I} C_i\right) = \bigcap_{i \in I} \text{ri}(C_i)$, then

$$\{x : f(x) < \alpha\} = A\left(\text{ri}(\text{epi}(f) \cap D)\right)$$

since $A(\text{ri}(C)) = \text{ri}(AC)$, then

$$\{x : f(x) < \alpha\} = \text{ri}\left(A(\text{epi}(f) \cap D)\right)$$

Implying that $\{x : f(x) < \alpha\}$ is open because it's the relative interior of a convex set (which is $A(\text{epi}(f) \cap D)$). Thus, it follows that $\{x : f(x) \geq \alpha\}$ is closed $\forall \alpha \in \mathbb{R}$ because it's the complement of an open set which implies that f is upper-semicontinuous everywhere (and so, also with the respect of ri(dom(f))). □

It follows that, if f is finite throughout $\mathbb{R}^n$, is also continuous everywhere: this occurs because in this case $\text{dom}(f) = \mathbb{R}^n \implies \text{dom}(f) = \text{ri}(\text{dom}(f))$ (since any affine set is both closed and relatively open), meaning that f is continuous over all its effective domain (which coincides with $\mathbb{R}^n$). This finding is particularly important because the concerns about the continuity of f can be addressed to checking the finiteness of the function throughout $\mathbb{R}^n$. For example, taking $f(x, t)$ such that $f(\cdot, t)$ is convex $\forall t \in T$ and such that $\exists t \in T : f(x, t) < \infty \ \forall x \in \mathbb{R}^n$ (i.e. f as function of $t \in T$ is bounded from above for any $x \in C$), then the function

$$g(x) = \sup_{t \in T} f(x, t)$$

is convex (since, as it will be discussed in the next section, its epigraph is the intersection of epigraphs of a collection of functions convex in x and indexed by $t \in T$) and continuous since $\text{dom}(g) = \mathbb{R}^n = \text{ri}(\text{dom}(g))$ (i.e. $g(x) < \infty \; \forall x \in \mathbb{R}^n$).

On the contrary, supposing that $\text{dom}(f) \neq \mathbb{R}^n$, one has to pay attention to the behavior of f at the boundary points, because it's not granted that the function is continuous there. Suppose for example that

$$f(x) = \begin{cases} \frac{\xi_2^2}{2\xi_1} & \text{if } \xi_1 > 0 \\ 0 & \text{if } \xi_1 = 0, \xi_2 = 0 \\ \infty & \text{otherwise} \end{cases}$$

Noticeably, the relative interior of the effective domain is the open half-space

$$\text{ri}(\text{dom}(f)) = \{(\xi_1, \xi_2) : \xi_1 > 0, \xi_2 \in \mathbb{R}\}$$

while the effective domain of the functions is still the open half-space adjoined with the origin (that is just one point of the relative boundary of the half-space, indeed $\{0\} \subset \{(\xi_1, \xi_2) : \xi_1 = 0, \xi_2 \in \mathbb{R}\}$)

$$\text{dom}(f) = \{(\xi_1, \xi_2) : \xi_1 \in \mathbb{R}_+, \xi_2 \in \mathbb{R}\} = \text{ri}(\text{dom}(f)) \cup \{0\}$$

Thus, $0 \in \text{dom}(f)$ but $0 \notin \text{ri}(\text{dom}(f))$ and there is the risk that f is not continuous at 0. Indeed, different restrictions of the functions lead to different limits as $(\xi_1, \xi_2) \to 0$:

$$\lim_{\xi_2 \to 0} f(x)|_{\xi_1 = \frac{\xi_2^2}{2\alpha}} = \alpha$$
$$\lim_{\xi_2 \to 0} f(x)|_{\xi_1 = \xi_2} = 0$$

As a formal refinement of the ordinary continuity of convex functions on the relative interior of their effective domain, one could introduce of *Lipschitzian continuity* relative to a set $S \subseteq \mathbb{R}^n$.

Definition 36 (Lipschitzian continuity)**.** Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. The function f is said to be *Lipschitzian* relative to S if

$$\exists \alpha \geq 0 : ||f(x) - f(y)||_2 \leq \alpha ||x - y||_2 \quad \forall x, y \in S$$

The definition just introduced can be re-arranged in the following form taking advantage of the positive homogeneity of the Euclidean norm:

$$\exists \alpha \geq 0 : \left|\left| \frac{f(x) - f(y)}{x - y} \right|\right|_2 \leq \alpha \quad \forall x, y \in S$$

This notation makes more evident the fact that a function is Lipschitzian relative to $S \subseteq \mathbb{R}^n$ whenever the slope of every secant line passing through points $(x, f(x))$ and $(y, f(y))$ remains bounded in absolute value $\forall x, y \in S$.

Proposition 20. *Any function f that is Lipschitzian relative to $S \subseteq \mathbb{R}^n$ is also uniformly continuous relative to S:*

$$\forall \epsilon > 0 \; \exists \delta > 0 : ||x - y||_2 < \delta \implies ||f(x) - f(y)||_2 < \epsilon \quad \forall x, y \in S$$

Proof. Lipschitzian continuity is a particular case of uniform continuity: by setting $\delta = \frac{\epsilon}{\alpha} > 0$, the notation becomes

$$\forall \epsilon > 0 \; \exists \alpha > 0 : ||x - y||_2 < \frac{\epsilon}{\alpha} \implies ||f(x) - f(y)||_2 < \epsilon \quad \forall x, y \in S$$

Which, can be rearranged into

$$\forall \epsilon > 0 \; \exists \alpha > 0 : ||f(x) - f(y)||_2 < \alpha ||x - y||_2 < \epsilon \quad \forall x, y \in S$$

which resembles to

$$\exists \alpha > 0 : ||f(x) - f(y)||_2 \leq \alpha ||x - y||_2 \quad \forall x, y \in S$$

□

Proposition 21. *Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a proper convex function. Then, f is Lipschitzian relative to any closed bounded set $S \subseteq ri(dom(f))$.*

Proof. by picking $S \subset \text{ri}(\text{dom}(f))$ as a closed bounded set it's already known that f is ordinary continuous on S (since f is ordinary continuous $\forall x \in \text{ri}(\text{dom}(f))$).
Supposing that $\text{dom}(f)$ is n-dimensional, one has actually that $S \subset \text{int}(\text{dom}(f))$ (because the n-dimensionality of $\text{dom}(f)$ allows to write $\text{ri}(\text{dom}(f)) = \text{int}(\text{dom}(f))$) and by definition of interior of a convex set, there exists a $\epsilon > 0$ sufficiently small such that the closed bounded set $S + \epsilon B$ is still contained in $\text{int}(\text{dom}(f))$
Because of the definition of effective domain, one has that

$$\sup_{x \in (S + \epsilon B)} f(x) =: \alpha_2 < \infty$$

and since f is assumed to be proper:

$$\inf_{x \in (S + \epsilon B)} f(x) =: \alpha_1 > -\infty$$

meaning that α_1 and α_2 act as lower and upper bounds of f on $S + \epsilon B$ respectively.
By picking $z \in S + \epsilon B$ is possible to write $y \in S$ as a convex combination of z and $x \in S$, i.e. $y = (1 - \lambda)x + \lambda z$ where

$$\lambda = \frac{||y - x||_2}{\epsilon + ||y - x||_2}$$

(notice that $\lambda \in [0, 1)$ since the norm is non-negative and $\epsilon > 0$).
Thus, because of Jensen's inequality

$$f(y) = f((1 - \lambda)x + \lambda z) \leq (1 - \lambda)f(x) + \lambda f(z)$$

it follows that

$$f(y) - f(x) \leq \lambda(f(z) - f(x)) \leq \lambda(\alpha_2 - \alpha_1) \quad \forall x, y \in S$$

And since $\lambda = \frac{||y-x||_2}{\epsilon+||y-x||_2} \leq \frac{||y-x||_2}{\epsilon} \quad \forall x, y \in S$ the relation can be rewritten as

$$f(y) - f(x) \leq \lambda(\alpha_2 - \alpha_1) \leq \frac{||y-x||_2}{\epsilon}(\alpha_2 - \alpha_1) \quad \forall x, y \in S$$

Finally, by calling $\alpha = \frac{\alpha_2 - \alpha_1}{\epsilon}$ (notice that $\alpha \geq 0$), one has that

$$f(y) - f(x) \leq \alpha ||y-x||_2 \quad \forall x, y \in S$$

which resembles the inequality defining Lipschitzian continuity relative to S □

As a brief recap one could say that if f is a convex function (proper or improper) then it is ordinary continuous on ri(dom(f)); however, if f is also proper function then is also Lipschitzian continuous relative to any closed bounded set of ri(dom(f))

2.3.4 Recession function and recession cone of a convex function

Epigraphs of convex functions are typically unbounded convex sets, thus it is worth evaluating their recession cones for inferring additional properties of such functions. Indeed, the recession cone of epi(f) is not surprisingly denoted as 0^+epi(f) and this is the epigraph of the so called *recession function* $f0^+$. In other words

$$\text{epi}(f0^+) = 0^+\text{epi}(f)$$

Definition 37 (Recession function)**.** Let $f : S \subset \mathbb{R}^n \to \mathbb{R}$ be a convex function. The positive homogenous convex function $f0^+$ having 0^+epi(f) as epigraph is called *recession function* of f.

Since the epigraph of $f0^+$ is the recession cone of epi(f) (i.e. a convex cone containing the origin), the recession function is a positively homogenous convex function. As it will be more clear in the next section from the way it is derived, $(f0^+)(y)$ gives a lot of insights of how the function f variates along the direction y. Some books [Ber09] refer to the recession function as the "asymptotic slope" of f along the direction y to stress the fact $(f0^+)(y)$ tells what is going to be the definitive variation of f by moving indefinitely along the direction y independently from where the movement has started.

In other words, if $(f0^+)(y) \leq 0$, it means that keeping a generic point $x_0 \in$ dom(f) one has that $f(x_0 + \lambda y) \leq f(x_0)$ $\forall \lambda \geq 0$. Thus, moving towards the "horizon" of the half-line starting from the generic x_0 in the direction of y, the function f is non-increasing, implying that $f(x_0 + \lambda y)$ is a non-increasing function of λ. The set of directions y such that this condition holds is called *recession cone of f* and in fact it corresponds to the zero sub-level set of the recession function:

$$\text{recc}(f) = \{y : (y, 0) \in 0^+\text{epi}(f)\} = \{y : (f0^+)(y) \leq 0\} \subset \mathbb{R}^n$$

Noticeably, the recession cone of f it's different from the recession cone of epi(f) (which is indeed $0^+\text{epi}(f) \subseteq \mathbb{R}^{n+1}$). The elements of the recession cone are called *directions of recession of f* or *directions in which f receeds*, and it's possible to anticipate the derivation of the recession function from the qualitative understanding of the recession cone of f. Indeed, one has that

$$y \in \text{recc}(f) \iff f(x_0 + \lambda y) \leq f(x_0) \; \forall \lambda \geq 0 \; \forall x_0 \in \text{dom}(f)$$

Thus it's possible to rewrite the recession cone of f as

$$\begin{aligned}
\text{recc}(f) &= \{y : f(x_0 + \lambda y) \leq f(x_0), \; \lambda \geq 0, \; x_0 \in \text{dom}(f)\} \\
&= \{y : f(x_0 + \lambda y) - f(x_0) \leq 0, \; \lambda \geq 0, \; x_0 \in \text{dom}(f)\} \\
&= \left\{\frac{y}{\lambda} : f(x_0 + y) - f(x_0) \leq 0, \; \lambda \geq 0, \; x_0 \in \text{dom}(f)\right\} \\
&= \bigcap_{\lambda > 0} \frac{1}{\lambda} \{y : f(x_0 + y) - f(x_0) \leq 0, \; x_0 \in \text{dom}(f)\} \\
&= \bigcap_{\lambda > 0} \frac{1}{\lambda} \left\{y : \sup_{x_0 \in \text{dom} f} f(x_0 + y) - f(x_0) \leq 0\right\}
\end{aligned}$$

However, if y is a direction of recession of f, also $\frac{y}{\lambda}$ is a direction of recession of f $\forall \lambda > 0$. Thus, the intersection above is the intersection of the same set (indeed, recc(f) is a convex cone), thus

$$\text{recc}(f) = \left\{y : \sup_{x_0 \in \text{dom} f} f(x_0 + y) - f(x_0) \leq 0\right\}$$

and since $\text{recc}(f) = \{y : (f0^+)(y) \leq 0\}$ this introduces one of the possible notations of the recession function:

$$(f0^+)(y) = \sup_{x_0 \in \text{dom} f} f(x_0 + y) - f(x_0) \tag{2.11}$$

As any other convex cone containing the origin, recc(f) is paired with two subspaces:

$$\begin{aligned}
L_1 &= \text{aff}\left(\left\{y : (f0^+)(y) \leq 0\right\}\right) \\
L_2 &= \left\{y : (f0^+)(y) \leq 0\right\} \cap \left(-\left\{y : (f0^+)(y) \leq 0\right\}\right) \\
&= \left\{y : (f0^+)(y) \leq 0\right\} \cap \left\{y : (f0^+)(-y) \leq 0\right\} \\
&= \left\{y : (f0^+)(y) \leq 0, (f0^+)(-y) \leq 0\right\}
\end{aligned}$$

The subspace L_2, i.e. the largest subspace fully contained in the recession cone of f, is called *constancy space of f* and its elements are defined as "directions in which f is constant". Indeed, the function $f(x + \lambda y)$ is a constant function of λ $\forall x$ if and only if $(f0^+)(y) \leq 0, (f0^+)(-y) \leq 0$. For a graphical intuition, think about f as a real-valued convex function on $\mathbb{R}$ and these two possible scenarios:

- $y \in \text{recc}(f)$

  - $(f0^+)(y) \leq 0$ but $(f0^+)(-y) > 0 \implies f(x+\lambda y)$ is a non-increasing function of λ $\forall x$ (by looking at Graph(f), it means that the vertical intercept is in correspondence of $f(x)$ and from there $f(x+\lambda y)$ has a non-increasing behavior as λ increases, i.e. moving rightward)
  - $(f0^+)(y) \leq 0$ and $(f0^+)(-y) \leq 0 \implies f(x+\lambda y)$ is a constant function of λ $\forall x$ (it means that as λ increases, i.e. moving rightward, $f(x+\lambda y) = f(x)$ that is also the vertical intercept of the graph of the function)

- $y \notin \text{recc}(f)$. In this case you have that $f(x+\lambda y)$ is *eventually* a non decreasing function of λ, in the sense that $\lim_{\lambda\to\infty} f(x+\lambda y) = \infty$

Indeed, the directions of recession of f (i.e. $\{y : (f0^+)(y) \leq 0\}$) play a fundamental role in convex programming since they tell on which half-lines $\{x+\lambda y : \lambda \geq 0\}$ f is asymptotically non-increasing. In other words, recalling the explanation made in [Ber09]: starting from $x \in \text{dom}(f)$ and moving along a direction of recession y, the point $z = x+\lambda y$ reached by moving along that direction must live within each level set that contains x, implying that $f(z) \leq f(x)$: whenever the boundary of a particular α-sub-level set is crossed, it is never crossed-back again and proceeding along the direction of recession y means either remaining on such α-sub-level set or moving to a new one associated with a lower level (i.e. moving to α'-sub-level set with $\alpha' \leq \alpha$).

This behavior is captured by the fact that every α-sub-level sets of f (i.e. $\{x : f(x) \leq \alpha\}$ $\forall \alpha \in \mathbb{R}$) shares the same recession cone and lineality space, that is the recession cone of f and the constancy space of f respectively.

2.3.5 Support function

Before concluding this section, it is worth introducing a very important class of convex functions, that is the class of *support functions* of convex sets C, usually denoted as $\delta^\star(\cdot|C) : \mathbb{R}^n \to \mathbb{R}$. As it will be described in the next section, the support function of a convex set evaluates a vector b belonging to the barrier cone of C (i.e. $\{x^\star : \exists\beta \in \mathbb{R}, \langle x, x^\star\rangle \leq \beta, x \in C\}$), and returns the value of β such that $\langle x, b\rangle \leq \beta$ $\forall x \in C$.
In other words, $\text{dom}(\delta^\star(\cdot|C)) = \{x^\star : \exists\beta \in \mathbb{R}, \langle x, x^\star\rangle \leq \beta, x \in C\}$ and it is said to "support" C in the sense that the half-space $\tilde{H}_b = \{x : \langle x, b\rangle \leq \delta^\star(b|C)\}$ indexed by $b \in \text{dom}(\delta^\star(\cdot|C))$ acts as a supporting half-space of C at some point $z \in C$. A support function fully characterizes a convex set C, indeed it's equal to the Fenchel conjugate of the indicator function

$$\delta(x|C) = \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{if } x \notin C \end{cases}$$

as it will be discussed in the dedicated section (hence, the "star" notation for the support function). Indeed, as an anticipation, the intersection of the half-spaces introduced before ranging over the full barrier cone of C (i.e. the effective domain of its support function) resembles to the external representation of $\mathrm{cl}(C)$ indeed

$$\begin{aligned}\bigcap_{b\in\mathrm{dom}(\delta^\star(\cdot|C))} \tilde{H}_b &= \bigcap_{b\in\mathrm{dom}(\delta^\star(\cdot|C))} \{x : \langle x, b\rangle \le \delta^\star(b|C)\}\\ &= \{x : \langle x, b\rangle \le \delta^\star(b|C), b \in \mathrm{dom}(\delta^\star(\cdot|C))\}\\ &= \left\{x : \sup_{b\in\mathrm{dom}(\delta^\star(\cdot|C))} \langle x, b\rangle - \delta^\star(b|C) \le 0\right\}\\ &= \{x : \delta(x|C) \le 0\}\\ &= \mathrm{dom}(\delta(\cdot|C)) = \mathrm{cl}(C)\end{aligned}$$

As for recession functions, the derivation of the functional form of the support function will be covered in the next section. For now, it's possible to anticipate that every support function of a non-empty convex set C is a proper positively homogenous closed convex function, which implies that any proper positively homogenous closed convex function is actually the support function of some convex set.

Thus, the epigraph of a support function is always a convex cone in $\mathbb{R}^{n+1}$ not containing any vertical line and it can be recovered by thinking about the "external" representation of any non-empty closed convex set C. Indeed, it's known that every closed convex set $C \subset \mathbb{R}^n$ is given by the intersection of all the closed half-spaces of the type $\{x \in \mathbb{R}^n : \langle x, b\rangle \le \beta\}$ containing it. Such half-spaces could be supporting or non-supporting half-spaces of C, however it is worth considering the supporting ones as the "most efficient" in the sense that the inclusion of any translate to the supporting half-space doesn't add any additional benefit in defining C.

Moreover this allows to take advantage of the topological property typical of convex sets according to which there always exist a non-null vector normal to C at any of its boundary points and this grants the existence of supporting half-spaces. At the same time, as previously discussed, this is equivalent to say that the function $\langle x, \cdot\rangle$ is bounded from above $\forall x \in C$, since

$$C \subset \{x : \langle x, b\rangle \le \beta\} \implies \langle x, b\rangle \le \beta \;\forall x \in C$$

Thus, ranging over $x \in C$, its possible to define a collection of affine functions of the type $\langle x, \cdot\rangle$ where each function, indexed by $x \in C$, is bounded from above and whose epigraph takes form $\{(b, \beta) \in \mathbb{R}^n \times \mathbb{R} : \langle x, b\rangle \le \beta\}$ where x is specific to that particular indexed function. By taking the intersection of all these epigraphs, it's possible to recover the epigraph of the support function of C

Definition 38 (Support function)**.** Let $C \subset \mathbb{R}^n$ be a convex set. Consider the collection of affine functions $\{h(b; x) = \langle x, b\rangle : x \in C\}$. The support function of C, denoted as $\delta^\star(\cdot|C) : S \subset \mathbb{R}^n \to \mathbb{R}$ is a closed proper positive homogenous convex function induced by the intersection of the collection of the epigraphs of the functions mentioned above ranging over C as index set. In other words:

$$\operatorname{epi}(\delta^\star(\cdot|C)) = \bigcap_{x\in C} \operatorname{epi}(h(\cdot;x))$$

Noticeably, setting any $\lambda > 0$, one has that $\lambda(b,\beta) \in \bigcap_{x\in C} \operatorname{epi}(\langle x,\cdot\rangle) \iff \langle x, \lambda b\rangle \leq \lambda\beta\ \forall x \in C \iff \lambda\langle x,b\rangle \leq \lambda\beta\ \forall x \in C \iff \langle x,b\rangle \leq \beta\ \forall x \in C \iff (b,\beta) \in \bigcap_{x\in C} \operatorname{epi}(\langle x,\cdot\rangle)$. Thus, the intersection of such collection is actually a closed convex cone containing the origin in $\mathbb{R}^{n+1}$ and this explains the closedness, the properness and the positive homogeneity of the support function.

$$\begin{cases} 0 \in \bigcap_{x\in C} \{(b,\beta) : \langle x,b\rangle \leq \beta\} \\ (b,\beta) \in \bigcap_{x\in C} \{(b,\beta) : \langle x,b\rangle \leq \beta\} \implies \lambda(b,\beta) \in \bigcap_{x\in C} \{(b,\beta) : \langle x,b\rangle \leq \beta\}\ \forall\lambda > 0 \end{cases}$$

Thus, as previously hinted with the example about continuity, the function whose epigraph is equal to the intersection of epigraphs of functions belonging to a collection indexed by a set I is the pointwise supremum of the functions belonging to that collection. Indeed, the support function is exactly the pointwise supremum of the functions $\langle x,\cdot\rangle$ and the convex set C acts as index set.

$$\delta^\star(\cdot|C) = \sup_{x\in C}\langle x,\cdot\rangle$$

Noticeably, support functions are a useful tool also for handling minimization of a linear function over a convex set C keeping the dependance of the optimal value to the external parameters wrapped in $x^\star \in \mathbb{R}^n$, indeed one has

$$\inf_{x\in C}\langle x, x^\star\rangle = -\sup_{x\in C}\langle x, -x^\star\rangle = -\delta^\star(-x^\star|C)$$

Of course, such function is concave being the negative of a convex function and to put the stress on this from now it will be used this notation

$$\hat{\delta}^\star(\hat{x}^\star|C) = -\delta^\star(-\hat{x}^\star|C) = \inf_{x\in C}\langle x, \hat{x}^\star\rangle$$

Thus, also $\hat{\delta}^\star(\hat{x}^\star|C)$ can be used for inducing a supporting half-space to C in the form of

$$\hat{H}_{\hat{x}^\star} = \left\{x \in \mathbb{R}^n : \langle x, \hat{x}^\star\rangle \geq \hat{\delta}^\star(\hat{x}^\star|C)\right\}$$

Because one has that $\langle x, \hat{x}^\star\rangle \geq \hat{\delta}^\star(\hat{x}^\star|C)\ \forall x \in C$ implying that $\hat{H} \supset C$. This device will be extremely useful in deriving the basic portfolio value function embedded in a CFMM from the basic set of reachable reserves.

Coming back to the "external" representation of any convex set C, it has been said that it's possible to recover $\operatorname{cl}(C)$ as

$$\operatorname{cl}(C) = \{x : \langle x,b\rangle \leq \delta^\star(b|C),\ b \in \operatorname{dom}(\delta^\star(\cdot|C))\} = \bigcap_{b\in\operatorname{dom}(\delta^\star(\cdot|C))} \tilde{H}_b$$

i.e. as the solution set of a system of weak linear inequalities ranging over the barrier cone of C (or, equivalently, the effective domain of the support function), and this notation is particularly useful as reference for recovering the closed convex set "supported" by a generic proper closed positively homogenous convex function. For example, the Euclidean norm satisfies the requirements for being considered as a support function of some closed convex set taking form

$$C = \{x : \langle x, b\rangle \leq ||b||_2,\ b \in \mathbb{R}^n\}$$

Meaning that the Euclidean norm must be consistent with the notation of the support function of the unknown convex set C:

$$||b||_2 = \sup_{x \in C} \langle x, b\rangle$$

According to the Cauchy-Schwarz inequality one has that

$$|\langle x, b\rangle| \leq ||x||_2 \cdot ||b||_2$$

Indeed one can write $\langle x, b\rangle = ||x||_2 \cdot ||b||_2 \cdot \cos(\theta)$ with $\cos(\theta) \in [-1, 1]$.
Thus, it's valuable analyzing the behavior of $\langle x, b\rangle$ for x lying inside and outside the unit circle. For example, considering any $||x|| \leq 1$ one has that $||x|| \cdot \cos(\theta) \leq 1\ \forall x \in B$ (where B denotes the euclidean unit ball). Thus, one has actually that

$$\langle x, b\rangle \leq ||b||_2\ \forall x \in B,\ \forall b \in \mathbb{R}^n$$

In other words:

$$x \in B \implies \langle x, b\rangle \leq ||b||_2\ \forall b \in \mathbb{R}^n$$

because on the left hand-side one would have the non-negative quantity $||b||_2$ rescaled by a factor not greater than 1. This is equivalent to say that

$$\sup_{x \in B} \langle x, b\rangle = ||b||_2\ \forall b \in \mathbb{R}^n \implies ||b||_2 = \delta^\star(b|B)$$

Because, given any $b \in \mathbb{R}^n$, the supremum of $\langle x, b\rangle$ over B is achieved by picking x at the boundary of B (so that $||x|| = 1$) pointing in the same direction of b (so that $\theta = 0$ and $\cos(\theta) = 1$). In other words, the supremum of $\langle x, b\rangle$ over B is achieved by picking $x = \frac{b}{||b||_2}$ (i.e. a "rescaled" version of b) and this implies that $\sup_{x \in B} \langle x, b\rangle = \langle \frac{b}{||b||_2}, b\rangle = \frac{||b||_2^2}{||b||_2} = ||b||_2$.

On the other hand, constraining $x \in B^c$ (where $B^c = \{x : ||x|| > 1\}$) and picking a generic $b \in \mathbb{R}^n$ one can always find a $x \in B^c$ such that $\langle x, b\rangle > ||b||_2$. In particular, one has that $\sup_{x \in B^c} \langle x, b\rangle = \infty$ because, given any $b \in \mathbb{R}^n$ one could pick any "explosive" x (in the sense that $||x|| \to \infty$) making an acute angle with b (so that $\cos(\theta) > 0$) and this would make $\langle x, b\rangle \to \infty$.

Coming back to the analysis of $x \in B$, the inequality $\langle x, b\rangle \leq ||b||_2 \ \forall x \in B, \ \forall b \in \mathbb{R}^n$ can be seen also as

$$\sup_{b\in\mathbb{R}^n} \langle x, b\rangle - ||b||_2 \leq 0 \ \forall x \in B \implies B = \left\{x : \sup_{b\in\mathbb{R}^n} \langle x, b\rangle - ||b||_2 \leq 0\right\}$$

And this is a "dual" proof of the fact that the Euclidean norm is the support function of the Euclidean unit ball. Indeed, one could write

$$||x||_2^\star = \sup_{b\in\mathbb{R}^n} \langle x, b\rangle - ||b||_2$$

So that $B = \{x : ||x||_2^\star \leq 0\}$. In particular, since from previous argument one has that $\langle x, b\rangle \geq ||b||_2 \ \forall x \in B^c, \ \forall b \in \mathbb{R}^n$, the conjugate of the euclidean norm is actually the indicator function of the Euclidean unit ball.

$$||x||_2^\star = \sup_{b\in\mathbb{R}^n} \langle x, b\rangle - ||b||_2 = \begin{cases} 0 & \text{if } x \in B \\ \infty & \text{if } x \notin B \end{cases}$$

And this is consistent with the fact that $\delta^\star(x^\star|C) = (\delta(x|C))^\star$. Indeed, since every closed proper positive homogeneous convex function can be framed as a support function, its Fenchel conjugate will be always an indicator function: more specifically, the Fenchel conjugate will be equal to the indicator function of the convex set it is supporting as support function. Generally speaking [Roc70], given f as a proper closed convex function, one has that

$$\delta^\star\left(x^\star | \{x : f(x) \leq 0\}\right) = (\mathrm{cl}(h))(x^\star) \text{ where } h(x^\star) = \inf_{\lambda>0} (f^\star\lambda)(x^\star)$$

Meaning that the support function of the 0-sub-level set of f corresponds to the closure of the positive homogenous convex function generated by the Fenchel conjugate of f, as it will result more clear in the next sections. Dually, one has that

$$\delta^\star\left(x | \{x^\star : f^\star(x^\star) \leq 0\}\right) = (\mathrm{cl}(g))(x) \text{ where } g(x) = \inf_{\lambda>0} (f^{\star\star}\lambda)(x) = \inf_{\lambda>0} (\mathrm{cl}(f)\lambda)(x) = \inf_{\lambda>0} (f\lambda)(x)$$

This follows from a more general theorem regarding the support function of $\mathrm{dom}(f)$ and $\mathrm{dom}(f^\star)$, which could be heuristically thought as the ∞-sub-level sets of f and $f^\star$. Indeed, given f closed proper convex (not necessarily positive homogenous), one has that

$$\delta^\star(y^\star|\mathrm{dom}(f)) = (f^\star 0^+)(y^\star)$$
$$\delta^\star(y|\mathrm{dom}(f^\star)) = (f 0^+)(y)$$

However, if f is positively homogenous and vanishing at the origin, it means that its epigraph will be a convex cone containing the origin, implying that $0^+\mathrm{epi}(f) = \mathrm{epi}(f)$ and f will be equal to its recession function. This is the reason why a positively homogenous convex function supports the effective domain of its Fenchel conjugate. To prove the equality $0^+\mathrm{epi}(f) = \mathrm{epi}(f)$ one could argue that the recession cone of a convex set C containing the origin can be retrieved as [Roc70]

$$0^+C = \bigcap_{\epsilon>0} \epsilon C$$

However, if C is a convex cone, then $\epsilon C \subseteq C \ \forall \epsilon > 0$ meaning that $\bigcap_{\epsilon>0} \epsilon C = C$. Thus, if f is positively homogenous, then $0^+\text{epi}(f) = \text{epi}(f)$ and f is equal to its recession function.

2.4 Designing Convex Functions

2.4.1 Convex and concave function builder

The previous subsection demonstrated how important the analysis of the epigraph of a function is in inferring the properties of the function itself. Thus, if the goal is designing functions with specific properties, it is worth understanding how to manipulate the epigraph of a function in order to "inherit" such desirable properties from those of its epigraph.

In other words, designing a convex set $F \subseteq \mathbb{R}^{n+1}$ with some specific properties is equivalent to design a convex function, having F as its epigraph, with some properties "encoded" in F.

To achieve this goal, one of the most powerful tools provided by convex analysis is designing a convex set $F \subset \mathbb{R}^{n+1}$ and then "inducing" a convex function having F as its epigraph. This tool could be heuristically defined as a "function builder" which defines the usual mapping $f : \mathbb{R}^n \to \mathbb{R}$ as

$$f(x) = \inf \{\mu : (x, \mu) \in F\}$$

However, the properness of the induced convex function is not always granted, but it can be always assessed in advance by checking the existence of vertical lines in the convex set F. On the other hand, the same convex set could be thought as the hypograph of a concave function, which can be induced as

$$\hat{f}(x) = \sup \{\mu : (x, \mu) \in F\}$$

This tool allows to derive the functional forms of all the functions discussed so far. Moreover, given any convex function f with some missing desirable properties, it's possible to obtain a "similar" function having those desirable properties by manipulating the epigraph of f and then deriving the associated function.

The derived function is "similar" in the sense that it is the greatest convex function majorized by f with the additional properties encoded in its epigraph.

In other words, calling $T : \mathbb{R}^n \to \mathbb{R}^n$ any convexity-preserving transformation which modifies the epigraph of f, one has that

$$g(x) = \inf\{\mu : (x, \mu) \in T(\text{epi}(f))\} \implies g \leq f$$

This was the case for example of the lower-semicontinuous hull of f, which was introduced as the greatest lower-semicontinuous function majorized by f and it was obtained by taking the closure of the epigraph of f.

2.4.2 Recovering the lower-semicontinuous hull of f

Thus, the lower-semicontinuous hull of f is built from cl(epi(f)) but of course, it's possible to build convex functions "similar" to f from λepi(f), 0^+epi(f) or also, given a generic f not necessarily convex, from conv(epi(f)) or cone(epi(f)). In the case of the lower-semicontinuous hull of f, one has the following proposition

Proposition 22. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then, the lower-semicontinuous hull of f can be obtained as*

$$clf(x) = \liminf_{\tilde{x}\to x} f(\tilde{x})$$

Proof. Supposing that f is proper, the convex function induced by cl(epi(f)) corresponds to cl(f), which could be expanded as follows:

$$\begin{aligned}
\text{cl}(f(x)) &= \inf \{\mu : (x,\mu) \in \text{cl}(\text{epi}(f))\} \\
&= \inf \left\{\mu : (x,\mu) \in \bigcap_{\epsilon>0} \text{epi}(f) + \epsilon B\right\}
\end{aligned}$$

Recalling that $\bigcap_{\epsilon>0} \text{epi}(f) + \epsilon B = \{(x,\mu) : \forall\epsilon > 0, \exists(\tilde{x},\tilde{\mu}) \in \text{epi}(f) :\ d((x,\mu),(\tilde{x},\tilde{\mu})) \leq \epsilon\}$ one has that:

$$\begin{aligned}
\text{cl}(f(x)) &= \inf \{\mu : (x,\mu) \in \{(x,\mu) : \forall\epsilon > 0, \exists(\tilde{x},\tilde{\mu}) \in \text{epi}(f) :\ d((x,\mu),(\tilde{x},\tilde{\mu})) \leq \epsilon\}\} \\
&= \inf \{\mu : \forall\epsilon > 0, \tilde{x} \in \text{dom}(f), \tilde{\mu} \geq f(\tilde{x}),\ ||(x-\tilde{x},\mu-\tilde{\mu})||_2 \leq \epsilon\} \\
&= \lim_{\epsilon\to 0^+} \inf \{\mu : \tilde{x} \in \text{dom}(f), \tilde{\mu} \geq f(\tilde{x}),\ ||(x-\tilde{x},\mu-\tilde{\mu})||_2 \leq \epsilon\}
\end{aligned}$$

Calling $\bar{\mu} = \mu - \tilde{\mu}$

$$\begin{aligned}
&= \lim_{\epsilon\to 0^+} \inf \{\mu : \tilde{x} \in \text{dom}(f), \mu - \bar{\mu} \geq f(\tilde{x}),\ ||(x-\tilde{x},\bar{\mu})||_2 \leq \epsilon\} \\
&= \liminf_{(\tilde{x},\bar{\mu})\to(x,0)} f(\tilde{x}) + \bar{\mu} \\
&= \liminf_{\tilde{x}\to x} f(\tilde{x})
\end{aligned}$$

Which not surprisingly resembles the limit inferior of f at x, coherently with the definition of the lower-semicontinuous hull of f. □

2.4.3 Recovering the recession function of f

Another function that was previously introduced was the recession function of f, which was defined as the function whose epigraph is equal to the recession cone of epi(f). Indeed:

$$(f0^+)(x) = \inf \left\{\mu : (x,\mu) \in 0^+\text{epi}f\right\}$$

Proposition 23. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then, the recession function of f can be obtained as*

$$(f0^+)(x) = \sup_{\lambda>0, \tilde{x}\in dom(f)} \frac{f(\lambda x + \tilde{x}) - f(\tilde{x})}{\lambda}$$

Or, alternatively

$$(f0^+)(x) = \sup_{\tilde{x}\in dom(f)} f(x + \tilde{x}) - f(\tilde{x})$$

Proof. Recalling that

$$\begin{aligned}
0^+\text{epi}f &= \{(x, \mu) : \lambda(x, \mu) + \text{epi}(f) \subseteq \text{epi}(f), \lambda \geq 0\} \\
&= \{(x, \mu) : \lambda(x, \mu) + (\tilde{x}, \tilde{\mu}) \in \text{epi}(f), \lambda \geq 0, (\tilde{x}, \tilde{\mu}) \in \text{epi}(f)\} \\
&= \{(x, \mu) : (\lambda x + \tilde{x}, \lambda\mu + \tilde{\mu}) \in \text{epi}(f), \lambda \geq 0, (\tilde{x}, \tilde{\mu}) \in \text{epi}(f)\} \\
&= \{(x, \mu) : \lambda x + \tilde{x} \in \text{dom}(f), \lambda\mu + \tilde{\mu} \geq f(\lambda x + \tilde{x}), \lambda \geq 0, (\tilde{x}, \tilde{\mu}) \in \text{epi}(f)\} \\
&= \left\{(x, \mu) : \lambda x + \tilde{x} \in \text{dom}(f), \mu \geq \frac{f(\lambda x + \tilde{x}) - \tilde{\mu}}{\lambda}, \lambda > 0, \tilde{x} \in \text{dom}(f), \tilde{\mu} \geq f(\tilde{x})\right\}
\end{aligned}$$

It's possible to remove some terms in the set notation by noticing that, since $\tilde{\mu} \geq f(\tilde{x})\ \forall \tilde{x} \in \text{dom}(f) \implies \frac{f(\lambda x+\tilde{x})-\tilde{\mu}}{\lambda} \leq \frac{f(\lambda x+\tilde{x})-f(\tilde{x})}{\lambda} \leq \mu\ \forall \tilde{x} \in \text{dom}(f)$, leading to:

$$0^+\text{epi}f = \left\{(x, \mu) : \lambda x + \tilde{x} \in \text{dom}(f), \mu \geq \frac{f(\lambda x + \tilde{x}) - f(\tilde{x})}{\lambda}, \lambda > 0, \tilde{x} \in \text{dom}(f)\right\}$$

The function builder notation can be rewritten as

$$(f0^+)(x) = \inf\left\{\mu : \lambda x + \tilde{x} \in \text{dom}(f), \mu \geq \frac{f(\lambda x + \tilde{x}) - f(\tilde{x})}{\lambda}, \lambda > 0, \tilde{x} \in \text{dom}(f)\right\}$$

Notice that this corresponds to the infimum of the majorants of $\frac{f(\lambda x+\tilde{x})-f(\tilde{x})}{\lambda}$, hence to the supremum of such function. Thus

$$(f0^+)(x) = \sup_{\lambda>0, \tilde{x}\in\text{dom}(f)} \frac{f(\lambda x + \tilde{x}) - f(\tilde{x})}{\lambda}$$

Sometimes the notation sets $\lambda = 1$ for the same reason why it's set like that in the notation of the recession cone of a generic set C. Indeed, $\lambda x + \tilde{x} \in \text{dom}(f)$ for $\lambda \geq 0$ and $\tilde{x} \in \text{dom}(f)$ is simply saying that $x \in 0^+\text{dom}(f)$ since

$$0^+\text{dom}(f) = \{x : \lambda x + \text{dom}(f) \subseteq \text{dom}(f), \lambda \geq 0\} = \{x : x + \text{dom}(f) \subseteq \text{dom}(f)\}$$

Indeed, since the recession cone of any convex set is a convex cone containing the origin, if $x \in 0^+\text{dom}(f) \implies \lambda x \in 0^+\text{dom}(f)\ \forall\lambda \geq 0$ (the weak inequality is allowed because of the inclusion of the origin)

Thus, an alternative notation for the recession function of f (which was previously anticipated in equation (2.11) from the qualitative understanding of the directions of recession of f) is

$$(f0^+)(x) = \sup_{\tilde{x}\in\text{dom}(f)} f(x + \tilde{x}) - f(\tilde{x})$$

□

Proposition 24. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then, the recession function of f can be obtained as*

$$(f0^+)(x) = \lim_{\lambda \to \infty} \frac{f(\lambda x + \tilde{x}) - f(\tilde{x})}{\lambda}$$

Proof. When x is a direction of recession of f, it's known that $f(\lambda x + \tilde{x})$ is actually a non-increasing function of λ, implying that $f(\lambda x + \tilde{x}) \leq f(\tilde{x})\ \forall \lambda \geq 0$ (think about $f(\tilde{x})$ as $f(\lambda x + \tilde{x})$ with $\lambda = 0$). Thus, $f(\lambda x + \tilde{x}) - f(\tilde{x})$ is a non-increasing function of λ, but the negative ratio $\frac{f(\lambda x + \tilde{x}) - f(\tilde{x})}{\lambda}$ is actually an increasing function of λ. Indeed, because of the convexity of f, the speed at which $f(\lambda x + \tilde{x}) - f(\tilde{x})$ decreases is sub-linear, implying that the numerator decreases at a lower rate compared to the increasing denominator. On the contrary, when x is not a direction of recession of f, one has $f(\lambda x + \tilde{x}) > f(\tilde{x}) \forall\ \lambda > 0$ and, because of the convexity of f, the speed at which $f(\lambda x + \tilde{x}) - f(\tilde{x})$ increases is super-linear, implying that also in this case the ratio $\frac{f(\lambda x + \tilde{x}) - f(\tilde{x})}{\lambda}$ is an increasing function of λ. An algebraic proof for this is provided in [Ber09]: consider $\mu > \lambda$ so that $\tilde{x} + \lambda x$ is actually a point lying on the line-segment connecting $\tilde{x}$ and $\tilde{x} + \mu x$; This means that such point could be expressed as convex combination of the two endpoints of the segment as follows:

$$\tilde{x} + \lambda x = \frac{\mu - \lambda}{\mu} \tilde{x} + \frac{\lambda}{\mu} (\tilde{x} + \mu x)$$

Thus, recalling Jensen's inequality, one has that:

$$\begin{aligned}
f(\tilde{x} + \lambda x) &\leq \frac{\mu - \lambda}{\mu} f(\tilde{x}) + \frac{\lambda}{\mu} f(\tilde{x} + \mu x) \quad \forall \lambda \in [0, \mu] \\
f(\tilde{x} + \lambda x) &\leq f(\tilde{x}) + \frac{\lambda}{\mu} \left(f(\tilde{x} + \mu x) - f(\tilde{x}) \right) \quad \forall \lambda \in [0, \mu] \\
\frac{f(\tilde{x} + \lambda x) - f(\tilde{x})}{\lambda} &\leq \frac{f(\tilde{x} + \mu x) - f(\tilde{x})}{\mu} \quad \forall \lambda \in [0, \mu]
\end{aligned}$$

This simple rearrangement of the Jensen's inequality shows that, independently from the nature of y as direction of the convex function f, the ratio $\frac{f(\lambda x + \tilde{x}) - f(\tilde{x})}{\lambda}$ is always an increasing function of λ. Of course, since the weak inequality sign in the Jensen's inequality is reversed for concave functions, one has that $\frac{\hat{f}(\tilde{x} + \lambda x) - \hat{f}(\tilde{x})}{\lambda}$ is actually a non-increasing function of λ given any direction of the concave function $\hat{f}$.

Thus, accounting for the non-decreasing nature of recalling the first notation of the recession function, it's possible to rewrite the recession function as

$$(f0^+)(x) = \sup_{\lambda > 0, \tilde{x} \in \mathrm{dom}(f)} \frac{f(\lambda x + \tilde{x}) - f(\tilde{x})}{\lambda} = \lim_{\lambda \to \infty} \frac{f(\lambda x + \tilde{x}) - f(\tilde{x})}{\lambda}$$

□

At this point, it's possible to introduce the "variation" function $P(x; \tilde{x})$

$$P(x; \tilde{x}) = f(\tilde{x} + x) - f(\tilde{x})$$

Where $x \in \mathbb{R}^n$ is a generic direction, not necessarily of recession, and $\tilde{x} \in \mathrm{dom}(f)$ is a parameter. Notice that this function is convex in x and its epigraph is obtained by translating the epigraph of f so that the point $(\tilde{x}, f(\tilde{x}))$ is mapped to the origin (indeed, $P(0; \tilde{x}) = f(\tilde{x}) - f(\tilde{x}) = 0$). This function is going to be extremely useful in understanding directional derivatives in the next section (in particular in comparing how the directional derivative focuses on the "local" behavior of f while the recession function focuses on the "asymptotic" behavior of f) but now it can be deployed to write a fourth notation for the recession function of f. Indeed, one could rewrite the ratio used in the previous notations as the left scalar multiplication of $P(x; \tilde{x})$ by λ^{-1}.

$$\frac{f(\lambda x + \tilde{x}) - f(\tilde{x})}{\lambda} = \lambda^{-1} P(\lambda x; \tilde{x}) = (P\lambda^{-1})(x; \tilde{x})$$

Implying that

$$(f0^+)(x) = \sup_{\lambda > 0, \tilde{x} \in \mathrm{dom}(f)} (P\lambda^{-1})(x; \tilde{x}) = \lim_{\lambda \to \infty} (P\lambda^{-1})(x; \tilde{x})$$

Proposition 25. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then, the recession function of f can be obtained as*

$$(f0^+)(x) = \lim_{\lambda \to 0^+} (f\lambda)(x)$$

Proof. The recession cone of of $\mathrm{epi}(f)$ could be interpreted as the set obtained by rescaling the original epigraph of f for a positive scalar that tends to zero (hence the notation $0^+\mathrm{epi}f$), and this gives a first intuition of the use of the right scalar multiplication in such notation.

Notice in fact that $0^+\mathrm{epi}(f) = \lambda\mathrm{epi}(f),\ \lambda \to 0^+$ and this can be seen from the following example. Suppose that $K = \mathrm{cone}\{(1, z) : z \in \mathrm{epi}(f)\}$ where $z = (x, \mu) \in \mathbb{R}^{n+1}$. As seen in previous sections, this convex cone containing the origin is actually

$$\begin{aligned} K &= \mathrm{cone}(\{(1, z) : z \in \mathrm{epi}(f)\}) \\ &= \bigcup_{\lambda \geq 0} \{(\lambda, z) : z \in \lambda\mathrm{epi}(f)\} \\ &= \{(\lambda, z) : \lambda > 0, z \in \lambda\mathrm{epi}(f)\} \subseteq \mathbb{R}^{n+2} \end{aligned}$$

And notice that $K \cap \left\{(\lambda, x) : \lambda > 0, x \in \mathbb{R}^{n+1}\right\} = K \setminus \{0\}$ meaning that, except for the origin, such convex cone lives in the open half-space $\left\{(\lambda, x) : \lambda > 0, x \in \mathbb{R}^{n+1}\right\}$. On the other hand, indeed, $K \cap \left\{(0, x) : x \in \mathbb{R}^{n+1}\right\} = \{(0, z) : z \in 0\mathrm{epi}(f)\} = \{0\}$. Thus, one could think about a cone $\tilde{K} = K \cup K_0$ where $K_0 \subset \left\{(0, x) : x \in \mathbb{R}^{n+1}\right\}$ is a convex cone containing the origin. Thus, it's possible to think about K_0 as a convex cone in $\mathbb{R}^{n+2}$ generated by some "unknown" convex set $C_0 \subseteq \mathbb{R}^{n+1}$ at level $\lambda = 0$, i.e. $K_0 = \mathrm{cone}(\{(0, z) : z \in C_0\})$.

For $\tilde{K}$, one would have

$$\begin{aligned} \tilde{K} &= K \cup K_0 \\ &= \{(\lambda, z) : \lambda \geq 0, z \in \lambda\mathrm{epi}(f)\} \cup \{(0, z) : \lambda \geq 0, z \in \lambda C_0\} \end{aligned}$$

But since K_0 includes the origin, we can omit the origin from the first set in the union

$$= \{(\lambda, z) : \lambda > 0, z \in \lambda \text{epi}(f)\} \cup \{(0, z) : \lambda \geq 0, z \in \lambda C_0\}$$
$$= K \setminus \{0\} \cup K_0$$

where

$$\tilde{K} \cap \left\{(\lambda, x) : \lambda > 0, x \in \mathbb{R}^{n+1}\right\} = K \setminus \{0\}$$
$$\tilde{K} \cap \left\{(0, x) : x \in \mathbb{R}^{n+1}\right\} = K_0$$

Thus, $\tilde{K}$ is a "larger" cone than K in the half-space $\{(\lambda, x) : \lambda \geq 0, x \in \mathbb{R}^n\}$ in the sense that both are living in such closed half-space and that $K \subset \tilde{K}$ (since $\tilde{K} = K \cup K_0$ and K_0 contains the origin). Noticeably, in order to be a convex cone, $\tilde{K}$ must be closed under addition and positive scalar multiplication, meaning that

$$\forall (1, z_1) \in K \; \exists (0, z_2) \in K_0 : (1, z_1) + (0, z_2) \in K$$

Which resembles to

$$\forall z_1 \in \text{epi}(f) \; \exists z_2 \in C_0 : z_1 + z_2 \in \text{epi}(f)$$

Which implies that z_2 is a direction of recession of $\text{epi}(f)$ and so that the unknown convex set C_0 generating K_0 is actually the recession cone $0^+\text{epi}(f)$. Thus $\tilde{K}$ can be rewritten as

$$\tilde{K} = \{(\lambda, z) : \lambda \geq 0, z \in \lambda \text{epi}(f)\} \cup \left\{(0, z) : z \in 0^+\text{epi}(f)\right\}$$

Or alternatively, in order to properly capture the nature of $0^+\text{epi}(f) = \lambda \text{epi}(f)$ for $\lambda \to 0^+$,

$$\tilde{K} = \{(\lambda, z) : \lambda > 0, z \in \lambda \text{epi}(f)\} \cup \left\{(0, z) : z \in 0^+\text{epi}(f)\right\}$$

since the recession cone of a convex set always contains the origin.

Now that it's clear that $0^+\text{epi}(f) = \lambda \text{epi}(f),\ \lambda \to 0^+$, it's possible to appreciate the fifth notation of the recession function of f by deriving the right scalar multiplication of f. As previously introduced, one has:

$$(f\lambda)(x) = \lambda f(x\lambda^{-1}) \quad \lambda > 0$$

And the claim here is that $\text{epi}(f\lambda) = \lambda \text{epi}(f)$. Thus, using the function builder to derive the function whose epigraph is a "rescaled" version of $\text{epi}(f)$ for some scalar $\lambda > 0$ should lead to the right scalar multiplication. Indeed:

$$\begin{aligned}(f\lambda)(x) &= \inf\{\mu : (x,\mu) \in \lambda\text{epi}(f)\}\\ &= \inf\left\{\mu : \lambda^{-1}(x,\mu) \in \text{epi}(f)\right\}\\ &= \inf\left\{\mu : (\lambda^{-1}x, \lambda^{-1}\mu) \in \text{epi}(f)\right\}\\ &= \inf\left\{\mu : \frac{\mu}{\lambda} \geq f\left(\frac{x}{\lambda}\right)\right\}\\ &= \inf\left\{\mu : \mu \geq \lambda f\left(\frac{x}{\lambda}\right)\right\}\end{aligned}$$

As before, the infimum of the majorants of a function is actually the supremum of the function. However, since λ is *a priori* defined according to the rescaling of epi(f) and x is the variable of the function, there aren't additional parameters to optimize over. Thus, this resembles to the function itself, i.e.

$$(f\lambda)(x) = \lambda f\left(\frac{x}{\lambda}\right)$$

And this is the reason why

$$(f0^+)(x) = \inf\left\{\mu : (x,\mu) \in 0^+\text{epi}(f)\right\} = \lim_{\lambda\to 0^+}(f\lambda)(x)$$

Right scalar multiplication is one of the cases in which the epigraph of the derived function corresponds to the image of epi(f) (or any other convex set) under a linear transformation $A : \mathbb{R}^k \to \mathbb{R}^m$. Indeed, as seen in the previous section, in the case of right scalar multiplication one has $A : (x,\mu) \mapsto \lambda(x,\mu)$ for some $\lambda > 0$ or, alternatively, $A : \text{epi}(f) \mapsto \lambda\text{epi}(f)$.

□

2.4.4 Infimal convolution of convex functions

Another important example is given by the *infimal convolution* of a collection of convex functions $\{f_i : i \in I\}$, which is the associated function when $A : ((x_1,\mu_1),(x_2,\mu_2),\dots) \mapsto (x_1,\mu_1) + (x_2,\mu_2),\dots$. Thus, the linear transformation in this context maps the direct sum of the epigraphs of the collection into the ordinary sum of the epigraphs of the collection, i.e.

$$A : \bigoplus_{i\in I}\text{epi}(f_i) \mapsto \sum_{i\in I}\text{epi}(f_i)$$

Definition 39. Let $\{f_i : i \in I\}$ be a collection of convex functions, where I is an index set. Then, the *infimal convolution* of such collection is the function $f_1\square\dots\square f_p$ induced by the sum of the epigraphs of the functions in the collection, i.e.

$$(f_1\square\dots\square f_p)(x) = \inf\left\{\mu : (x,\mu) \in \sum_{i=1}^{p}\text{epi}(f_i)\right\}$$

Proposition 26. *Let $\{f_i : i \in I\}$ be a collection of convex functions, where $I = [1, p]$ is an index set. Then, the infimal convolution of such collection can be obtained as*

$$(f_1 \square \ldots \square f_p)(x) = \inf \{f(x_1) + \cdots + f(x_p) : x = x_1 + \cdots + x_p\}$$

Proof.

$$\begin{aligned}(f_1 \square \ldots \square f_p)(x) &= \inf \{\mu : (x, \mu) \in \operatorname{epi}(f_1) + \cdots + \operatorname{epi}(f_p)\} \\ &= \inf \{\mu : x = x_1 + \cdots + x_p, \mu = \mu_1 + \cdots + \mu_p, \mu_i \geq f(x_i), i = 1, \ldots, p\} \\ &= \inf \{\mu_1 + \cdots + \mu_p : x = x_1 + \cdots + x_p, \mu_i \geq f(x_i), i = 1, \ldots, p\} \\ &= \inf \{f(x_1) + \cdots + f(x_p) : x = x_1 + \cdots + x_p\}\end{aligned}$$

□

Notice that if the collection of functions corresponds to the same function and $I = [1, \lambda]$ (i.e. the collection of functions is $\{f : i \in [1, \lambda]\}$) the infimal convolution of such collection resembles the right scalar multiplication. Indeed, this would mean that $(x_1, \mu_1) = \cdots = (x_\lambda, \mu_\lambda) = (\tilde{x}, \tilde{\mu})$ so that

$$\begin{aligned}(f \square \ldots \square f)(x) &= \inf \left\{ \sum_{i=1}^{\lambda} f(\tilde{x}) : x = \sum_{i=1}^{\lambda} \tilde{x} \right\} \\ &= \inf \{\lambda f(\tilde{x}) : x = \lambda \tilde{x}\} \\ &= \lambda f\left(\frac{x}{\lambda}\right) =: (f\lambda)(x)\end{aligned}$$

2.4.5 Image of f under affine map

Infimal convolution and right scalar multiplication are examples in which the affine transformation is applied to the whole epigraph. However, one could generate convex functions also by applying A just to the effective domain of the convex function $f : \mathbb{R}^n \to \mathbb{R}$. This means that the set used in the function builder notation would be, for some affine transformation $A : \mathbb{R}^n \to \mathbb{R}^m$

$$F = \{(Ax, \mu) : (x, \mu) \in \operatorname{epi}(f)\}$$

Definition 40 (Image of f under affine map)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $A : \mathbb{R}^n \to \mathbb{R}^m$ be an affine transformation. Then, the *image of f under A* is the function $(Af) : \mathbb{R}^m \to \mathbb{R}$ induced by the set $F = \{(Ax, \mu) : (x, \mu) \in \operatorname{epi}(f)\}$

Proposition 27. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $A : \mathbb{R}^n \to \mathbb{R}^m$ be an affine transformation. Then, the image of f under A can be obtained as*

$$(Af)(x) = \inf_{A\tilde{x}=x} f(\tilde{x})$$

Proof.

$$\begin{aligned}(Af)(x) &= \inf\{\mu : (x,\mu) \in \{(A\tilde{x},\tilde{\mu}) : (\tilde{x},\tilde{\mu}) \in \text{epi}(f)\}\} \\ &= \inf\{\mu : (x,\mu) \in \{(A\tilde{x},\tilde{\mu}) : \tilde{x} \in \text{dom}(f), \tilde{\mu} \geq f(\tilde{x})\}\} \\ &= \inf\{\mu : x = A\tilde{x}, \mu = \tilde{\mu} \geq f(\tilde{x})\} \\ &= \inf_{A\tilde{x}=x} f(\tilde{x})\end{aligned}$$

□

If A is invertible this resembles to the *inverse image* of f under A^{-1}:

$$\begin{aligned}(Af)(x) &= \inf_{\tilde{x}=A^{-1}x} f(\tilde{x}) \\ &= f(A^{-1}x) =: (fA^{-1})(x)\end{aligned}$$

An example of this function was introduced in the previous section as the function induced by rescaling the effective domain of f for some scalar $\lambda > 0$. Indeed, in that case the linear map is $T : x \mapsto \lambda x$ which is invertible since $T^{-1} : x \mapsto \frac{x}{\lambda}$

$$(Tf)(x) = \inf_{T\tilde{x}=x} f(\tilde{x}) = \inf_{\lambda\tilde{x}=x} f(\tilde{x}) = \inf_{\tilde{x}=\frac{x}{\lambda}} f(\tilde{x}) = \inf_{\tilde{x}=T^{-1}x} f(\tilde{x}) = f(T^{-1}x) = (fT^{-1})(x) = f\left(\frac{x}{\lambda}\right)$$

This function is extremely useful to demonstrate the convexity of element-wise minimization over a convex set.

Proposition 28. *Let $f : S \subset \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a function convex in both arguments. Then, the partial minimization of f over a convex set $C_y \subset \mathbb{R}^n$*

$$\inf_{y\in C_y} f(x,y)$$

is a convex function

Proof. Given $f(x,y)$ convex both on x and y, one actually has that $\text{dom}(f) = C_x \oplus C_y$ is actually a convex set. One could then apply an affine transformation of the type $A : (x,y) \mapsto x$ to the effective domain, meaning that $A(\text{dom(f)}) = A(C_x \oplus C_y) \mapsto C_x$ This would resemble to

$$(Af)(x) = \inf\{f(x,y) : A(x,y) = x\}$$

But since the map $(x,y) \mapsto x$ is valid $\forall y \in C_y$ this is equivalent to taking the infimum of $f(x,y)$ evaluating with the second argument ranging over C_y, i.e.

$$\begin{aligned}(Af)(x) &= \inf\{f(x,y) : A(x,y) = x,\} \\ &= \inf\{f(x,y) : y \in C_y\} \\ &= \inf_{y\in C_y} f(x,y)\end{aligned}$$

And since any function which can be written as the image of a convex function under an affine transformation is actually a convex function, this implies that the element-wise minimization of a convex function over a convex set is a convex function. □

Another very important function which can be derived via the function builder notation is the function induced by the intersection of epigraphs of a collection of convex functions. One could frame the support function or any other Fenchel conjugate as a function induced in such way, proving at the same time their convexity (being functions induced by a convex set).
Indeed, suppose that $\{f_i : i \in I\}$ is a collection of convex functions such that $f_i : \mathbb{R}^n \to \mathbb{R}$ and I is a generic index set. The function induced by the intersection of the epigraphs of such collection is

$$\begin{aligned} g(x) &= \inf \left\{ \mu : (x, \mu) \in \bigcap_{i \in I} \text{epi}(f_i) \right\} \\ &= \inf \left\{ \mu : (x, \mu) \in \text{epi}(f_i), i \in I \right\} \\ &= \inf \left\{ \mu : x \in \text{dom}(f_i), \mu \geq f_i(x), i \in I \right\} \end{aligned}$$

Notice that this is the infimum of the majorants of all $f_i(x)$, hence the supremum of such functions. Thus, as anticipated in the previous sections, the function induced by the intersection of the epigraphs of convex sets is:

$$g(x) = \sup_{i \in I} f_i(x)$$

Notice that one could be tempted to see a "hidden" relationship between the convexity of $\sup_{y \in S} f(x, y)$ and the convexity of $\inf_{y \in C} f(x, y)$ but the interpretation is actually different. Indeed, for what regards $\sup_{y \in S} f(x, y)$, f is required to be convex just on x since y acts as a index ranging over a index set S which of course is not required to be convex: indeed the final function is induced by the intersection of epigraphs of convex functions in a collection, where each function of such collection is indexed by y. On the other hand, regarding $\inf_{y \in C} f(x, y)$, f is required to be convex both on x and y and the set C is required to be convex, simply because the final function is induced by a particular kind of affine transformation (of the type $A : (x, y) \mapsto x$) applied to the effective domain of the original function, which must be convex in order to have a convex epigraph (and effective domain).

Moreover, one can retrieve an additional information from the following equality recalling that the negative of a concave function is convex and vice-versa:

$$g(x) = \sup_{i \in I} f_i(x) = -\inf_{i \in I}(-f_i(x)) = -\inf_{i \in I} \hat{f}_i(x)$$

Since $\{f_i : i \in I\}$ is a collection of convex functions, $\{-f_i : i \in I\} = \left\{\hat{f}_i : i \in I\right\}$ must be a collection of concave functions. On the other hand, since $-\inf_{i \in I}(-f_i(x))$ is a convex function (being equal to the supremum of a family of a convex functions), then $\inf_{i \in I}(-f_i(x))$ must be a concave function. Indeed, the element-wise infimum of a family of concave functions (ranging over a generic index set) is always a concave function.

2.4.6 Positive homogenous convex functions generated by f

Finally, it's possible to define the *positive homogenous convex function* generated by a function f as the function induced by the convex cone generated by the epigraph of f

Definition 41 (Positive homogenous convex functions generated by f)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then, the *positive homogenous convex function generated by* f is the function induced by the Conic hull of epi(f), i.e. cone(epi(f))

Proposition 29. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then, the positive homogenous convex function generated by f can be denoted as*

$$h(x) = \inf_{\lambda>0}(f\lambda)(x)$$

Proof.

$$\begin{aligned}
h(x) &= \inf\{\mu : (x,\mu) \in \text{cone}(\text{epi}(f))\} \\
&= \inf\left\{\mu : (x,\mu) \in \bigcup_{\lambda\geq 0} \lambda\text{epi}(f)\right\} \\
&= \inf\{\mu : (x,\mu) \in \{\lambda(\tilde{x},\tilde{\mu}) : \lambda > 0, (x,\mu) \in \text{epi}(f)\}\} \\
&= \inf\{\mu : x = \lambda\tilde{x}, \mu = \lambda\tilde{\mu}, \tilde{x} \in \text{dom}(f), \tilde{\mu} \geq f(\tilde{x}), \lambda > 0\} \\
&= \inf\{\lambda\tilde{\mu} : x = \lambda\tilde{x}, \tilde{x} \in \text{dom}(f), \tilde{\mu} \geq f(\tilde{x}), \lambda > 0\} \\
&= \inf\left\{\lambda f\left(\frac{x}{\lambda}\right) : \frac{x}{\lambda} \in \text{dom}(f), \lambda > 0\right\} \\
&= \inf_{\lambda>0}(f\lambda)(x)
\end{aligned}$$

□

Noticeably cone(epi(f)) $\supset \lambda$epi(f) $\forall\lambda \geq 0$ and this is graphically explained from the fact that the epigraph of the function induced by the non-negative rescaling of epi(f) is always contained in the convex cone generated by epi(f).

This device allows to introduce the gauge function $\gamma(\cdot|C)$ as the positive homogeneous convex function generated by $f(\cdot) = \delta(\cdot|C)+1$. Indeed, generally speaking, a *gauge* γ is defined as a non-negative positive homogenous convex function vanishing at the origin (i.e. $\gamma(0|C) = 0$) [Roc70] meaning that it's epigraph is a convex cone containing the origin. Indeed, considering

$$f(x) = \delta(x|C) + 1$$

one has that epi(f) $= \{(x,\mu) : \mu \geq 1, x \in C\}$ implying that

$$\text{cone}(\text{epi}(f)) = \bigcup_{\lambda\geq 0} \text{epi}(f) = \{(x,\mu) : \exists\lambda \geq 0, \mu \geq \lambda, x \in \lambda C\}$$

. Thus, the gauge function is simply the convex function induced by cone(epi(f)), i.e.

$$\begin{aligned}\gamma(x|C) &= \inf\{\mu : \exists\lambda \geq 0, \mu \geq \lambda, x \in \lambda C\} \\ &= \inf\{\lambda \geq 0 : x \in \lambda C\}\end{aligned}$$

2.4.7 Polars of convex sets

Definition 42 (Polar of a convex set)**.** Let $C \subset \mathbb{R}^n$ be a convex set. The polar of C is the set

$$C^\circ = \{x^\star : \langle x, x^\star\rangle \leq 1, x \in C\}$$

Proposition 30. *Let $C \subset \mathbb{R}^n$ be a convex set. The polar C° is always a closed origin-including convex set. Moreover, the polar relationship binding C and C° is the following*

$$cl(\gamma(\cdot|C)) = \delta^\star(\cdot|C^\circ)$$

Proof. Starting from

$$f(x) = \delta(x|C) + 1$$

one has that the positive homogenous convex function generated by such f is actually the gauge function of C as previously discussed, i.e.

$$\gamma(x|C) = \inf_{\lambda \geq 0}(f\lambda)(x) = \inf\{\lambda : x \in \lambda C, \lambda \geq 0\}$$

As already mentioned, the closure of the positive homogenous convex function generated by f is the support function of the 0-sub-level set of the conjugate $f^\star$. Thus, $\gamma(x|C)$ must be the support function of $\{x^\star : f^\star(x^\star) \leq 0\}$ where

$$\begin{aligned}f^\star(x^\star) &= \sup_{x \in \mathbb{R}^n} \langle x, x^\star\rangle - (\delta(x|C) + 1) \\ &= -1 + \sup_{x \in C}\langle x, x^\star\rangle \\ &= \delta^\star(x^\star|C) - 1\end{aligned}$$

Thus, one can reformulate the 0-sub-level set of the Fenchel conjugate (i.e. the set supported by cl($\gamma(\cdot|C)$)) as follows which resembles to C°

$$\begin{aligned}\{x^\star : f^\star(x^\star) \leq 0\} &= \{x^\star : \delta^\star(x^\star|C) \leq 1\} \\ &= \{x^\star : \langle x, x^\star\rangle \leq 1,\ x \in C\} \\ &=: C^\circ\end{aligned}$$

Since the support function completely characterizes the supported set, one can rewrite the polar of C as

$$\begin{aligned}C^\circ &= \{x^\star : \langle x, x^\star\rangle \leq \delta^\star(x|C^\circ), x \in \mathrm{dom}(\delta^\star(\cdot|C^\circ))\} \\ &= \{x^\star : \langle x, x^\star\rangle \leq \mathrm{cl}(\gamma(x|C)), x \in \mathrm{dom}(\mathrm{cl}(\gamma(\cdot|C)))\}\end{aligned}$$

It is evident from the notation that $0 \in C^\circ$, thus one has that $C = C^{\circ\circ}$ if and only if $0 \in C$ and $\mathrm{cl}(C) = C$. The closedness of C° follows from the fact that it is the intersection of a collection of closed half-spaces indexed by C indeed

$$C^\circ = \bigcap_{x \in C} \{x^\star : \langle x, x^\star \rangle \leq 1\}$$

□

Thus, the important caveat here is the following: given any closed gauge function of a convex set $C \subset \mathbb{R}^n$ (i.e., a non-negative positive homogeneous convex function), its closure corresponds to the support function of another convex set $C^\circ \subset \mathbb{R}^n$ (its polar). On the contrary, given the support function of a convex set $C^{\circ\circ} \subset \mathbb{R}^n$ (which could be any closed convex set containing the origin), this corresponds also to the closure of the gauge function of another convex set $C^\circ \subset \mathbb{R}^n$.

Notice that the notation just introduced is consistent with that one provided for polar convex cones K°. Indeed, if C is a convex cone containing the origin (i.e. $C = K$), one has that the gauge function obtained as generated positive homogenous convex function resembles to the indicator function of the convex cone (because if $x \in K \implies \lambda x \in K \ \forall \lambda \geq 0$). Indeed, starting from $f(x) = \delta(x|K) + 1$ one has

$$\begin{aligned}
\gamma(x|K) &= \inf_{\lambda \geq 0} (f\lambda)(x) \\
&= \inf_{\lambda \geq 0} \lambda + \lambda\delta(x|\lambda K) \\
&= \inf\{\lambda : \lambda \geq 0, x \in K\} \\
&= \delta(x|K)
\end{aligned}$$

Thus, for convex cones, the dual relationship becomes

$$\delta(x|K) = \mathrm{cl}(\gamma(\cdot|K)) = \delta^\star(\cdot|K^\circ)$$

and one has

$$\begin{aligned}
K^\circ &= \{x^\star : \langle x, x^\star \rangle \leq \delta^\star(x|K^\circ)\} \\
&= \{x^\star : \langle x, x^\star \rangle \leq \mathrm{cl}(\gamma(x|K))\} \\
&= \{x^\star : \langle x, x^\star \rangle \leq \delta(x|K)\} \\
&= \{x^\star : \langle x, x^\star \rangle \leq 0, \ x \in K\}
\end{aligned}$$

Which resembles the notation originally introduced for polar convex cones.
Recalling that any convex cone containing the origin $K \subset \mathbb{R}^{n+1}$ can be generated from a convex set $C \subset \mathbb{R}^n$ it is interesting how it's possible to generate K° and $K^\star$ directly from C. Noticeably, one

has that

$$
\begin{aligned}
K^\star &= \{x^\star : \langle x, x^\star\rangle \geq 0,\ x \in K\} \\
&= \{x^\star : \langle x, x^\star\rangle \geq 0,\ x \in \operatorname{cone}(\{(1,x) : x \in C\})\} \\
&= \{(\xi_1^\star, \xi_2^\star) : \langle(\xi_1, \xi_2), (\xi_1^\star, \xi_2^\star)\rangle \geq 0,\ (\xi_1, \xi_2) \in \operatorname{cone}(\{(1,\zeta) : \zeta \in C\})\} \\
&= \{(\xi_1^\star, \xi_2^\star) : \langle(1, \zeta), (\xi_1^\star, \xi_2^\star)\rangle \geq 0,\ \zeta \in C\} \\
&= \{(\xi_1^\star, \xi_2^\star) : \xi_1^\star + \langle\zeta, \xi_2^\star\rangle \geq 0,\ \zeta \in C\} \\
&= \{(\xi_1^\star, \xi_2^\star) : \xi_1^\star \geq \langle\zeta, -\xi_2^\star\rangle,\ \zeta \in C\} \\
&= \left\{(\xi_1^\star, \xi_2^\star) : \xi_1^\star \geq \sup_{\zeta \in C}\langle\zeta, -\xi_2^\star\rangle\right\} \\
&= \{(\xi_1^\star, \xi_2^\star) : \xi_1^\star \geq \delta^\star(-\xi_2^\star|C)\} \\
&= \{(\xi_1^\star, \xi_2^\star) : \xi_1^\star - \delta^\star(\xi_2^\star| - C) \geq 0\} \\
&= \operatorname{epi}(\delta^\star(\cdot| - C))
\end{aligned}
$$

This finding is extremely important because it shows that the dual cone $K^\star$ is simply the epigraph of the support function of $-C$ (where C is the convex set generating K in higher dimension). On the contrary, for what regards the polar cone K°, one has that

$$
\begin{aligned}
K^\circ &= -K^\star \\
&= \{(-\xi_1^\star, -\xi_2^\star) : \xi_1^\star - \delta^\star(\xi_2^\star| - C) \geq 0\} \\
&= \{(\xi_1^\star, \xi_2^\star) : -\xi_1^\star - \delta^\star(-\xi_2^\star| - C) \geq 0\} \\
&= \{(\xi_1^\star, \xi_2^\star) : -\delta^\star(\xi_2^\star|C) \geq \xi_1^\star\}
\end{aligned}
$$

Which could be interpreted as the hypograph of the negative of the support function of C.
On the other hand, the support function of $-C$ defines a link between the dual cone $K^\star$ and the polar convex set $-C^\circ$. Indeed, since

$$-C^\circ = \{x^\star : \delta^\star(x^\star| - C) \leq 1\}$$

one can see that $-C^\circ$ is actually the sub-level set of $\delta(\cdot| - C)$ at level one, and since $K^\star = \operatorname{epi}(\delta^\star(\cdot| - C))$ one can express $-C^\circ$ as the lower dimensional projection of the intersection between $K^\star = \operatorname{epi}(\delta^\star(\cdot| - C))$ and the hyperplane $\{(1, x) : x \in \mathbb{R}^n\}$.

Noticeably, if $K = \operatorname{cone}(\{(1, x) : x \in C\}) \subset \mathbb{R}^{n+1}$, the convex function induced by K is the gauge function $\gamma(x|C)$ indeed

$$
\begin{aligned}
\inf\{\lambda : (\lambda, x) \in K\} &= \inf\{\lambda : x \in \lambda C, \lambda \geq 0\} \\
&=: \gamma(x|C)
\end{aligned}
$$

Thus, it could be interesting analyzing the *polar* of such gauge, that is the gauge function of the polar convex set C°

Proposition 31. *Let $C \subset \mathbb{R}^n$ be a convex set. Let $K = cone(\{(1,x) : x \in C\}) \subset \mathbb{R}^{n+1}$ be a convex cone containing the origin. Let $A : (\lambda, x) \mapsto (-\lambda, x)$ be a linear transformation. Then, the epigraph of the gauge function of C° is equal to*

$$epi(\gamma(\cdot|C^\circ)) = cone\,\{(1,x^\circ) : x^\circ \in C^\circ\} = A(K^\circ) \cap \{(\lambda, x) : \lambda \geq 0, x \in \mathbb{R}^n\}$$

Moreover, one has the following "polar inequality"

$$\gamma(x^\circ|C^\circ)\gamma(x|C) \geq \langle x^\circ, x\rangle \quad \forall (x^\circ, x) \in \mathbb{R}^{2n}$$

Proof. Applying the definition of gauge function, one has

$$\begin{aligned}
\gamma(x^\circ|C^\circ) &= \inf\{\lambda : (\lambda, x^\circ) \in \text{cone}(\{(1,x^\circ) : x^\circ \in C^\circ\})\} \\
&= \inf\{\lambda : x^\circ \in \lambda C^\circ, \lambda \geq 0\} \\
&= \inf\{\lambda : x^\circ \in \lambda\{x^\circ : \langle x^\circ, x\rangle \leq 1, x \in C\}, \lambda \geq 0\} \\
&= \inf\left\{\lambda : \langle x, \frac{x^\circ}{\lambda}\rangle \leq 1, x \in C, \lambda \geq 0\right\} \\
&= \inf\left\{\lambda : \langle \mu x, \frac{x^\circ}{\lambda}\rangle \leq \mu, x \in C, \lambda \geq 0, \mu \geq 0\right\} \\
&= \inf\left\{\lambda : \langle x, \frac{x^\circ}{\lambda}\rangle \leq \mu, x \in \mu C, \lambda \geq 0, \mu \geq 0\right\} \\
&= \inf\{\lambda : \langle x, x^\circ\rangle \leq \lambda\mu, x \in \mu C, \lambda \geq 0, \mu \geq 0\} \\
&= \inf\{\lambda : \langle(-\lambda, x^\circ), (\mu, x)\rangle \leq 0, (\mu, x) \in K, \lambda \geq 0\}
\end{aligned}$$

And recalling that $K^\circ = \{(\lambda, x^\circ) : \langle(x^\circ, \lambda), (x, \mu)\rangle \leq 0, (x, \mu) \in K\}$, one has that

$$\begin{aligned}
\{(\lambda, x^\circ) : \langle(x^\circ, -\lambda), (x, \mu)\rangle \leq 0, (\mu, x) \in K\} &= \{(-\lambda, x^\circ) : \langle(x^\circ, \lambda), (x, \mu)\rangle \leq 0, (\mu, x) \in K\} \\
&= A(K^\circ)
\end{aligned}$$

where A is a linear transformation of the type $A : (\lambda, x) \mapsto (-\lambda, x)$. In other words, given the gauge function $k(x) = \gamma(x|C)$ (i.e. the convex function induced by $K = \text{cone}(\{(1,x) : x \in C\}) \subset \mathbb{R}^{n+1}$) the polar of such gauge $k^\circ(x^\circ) = \gamma(x^\circ|C^\circ)$ is the convex function induced by $A(K^\circ) \cap \{(\lambda, x) : \lambda \geq 0, x \in \mathbb{R}^n\}$ where $A : (\lambda, x) \mapsto (-\lambda, x)$ indeed

$$\gamma(x^\circ|C^\circ) = \inf\{\lambda : (\lambda, x^\circ) \in A(K^\circ) \cap \{(\lambda, x) : \lambda \geq 0, x \in \mathbb{R}^n\}\}$$

Recalling that $K = \text{epi}(\gamma(\cdot|C))$, one can compute $\gamma(\cdot|C^\circ)$ directly from $\gamma(\cdot|C)$ as shown in [Roc70], indeed

$$\begin{aligned}
\gamma(x^\circ|C^\circ) &= \inf\{\lambda : \langle x, x^\circ\rangle \le \lambda\mu, x \in \mu C, \lambda \ge 0, \mu \ge 0\} \\
&= \inf\{\lambda : \langle x, x^\circ\rangle \le \lambda\mu, \lambda \ge 0, (x,\mu) \in \mathrm{epi}(\gamma(\cdot|C))\} \\
&= \inf\{\lambda : \langle x, x^\circ\rangle \le \lambda\gamma(x|C), \lambda \ge 0, x \in \mathrm{dom}(\gamma(\cdot|C))\} \\
&= \inf\left\{\lambda : \frac{\langle x, x^\circ\rangle}{\gamma(x|C)} \le \lambda, \lambda \ge 0, x \in \mathrm{dom}(\gamma(\cdot|C))\right\}
\end{aligned}$$

That is, as usual, the infimum of the majorants of the function $\frac{\langle x,x^\circ\rangle}{\gamma(x|C)}$, i.e. its supremum

$$\gamma(x^\circ|C^\circ) := \sup_{x\neq 0} \frac{\langle x, x^\circ\rangle}{\gamma(x|C)}$$

Noticeably, this notation allows to infer a very nice inequality, that is

$$\gamma(x^\circ|C^\circ)\gamma(x|C) \ge \langle x^\circ, x\rangle \quad \forall (x^\circ, x) \in \mathbb{R}^{2n}$$

□

And the inequality mentioned in proposition (31) is "hidden" inside very important relations such as the Cauchy-Schwarz inequality as mentioned in [Roc70]: indeed, as already discussed, the Euclidean norm acts as support function of the Euclidean unit ball but, at the same time, it corresponds also its gauge function. Indeed, calling B the Euclidean unit ball and $\tilde{K} = \mathrm{cone}(\{(1,x) : x \in B\})$, the function induced by $\tilde{K}$ is exactly the Euclidean norm indeed:

$$\begin{aligned}
\gamma(x|B) &= \inf\left\{\lambda : (\lambda, x) \in \tilde{K}\right\} \\
&= \inf\{\lambda : x \in \lambda B, \lambda \ge 0\} \\
&= \inf\{\lambda : x \in \lambda\{x : ||x||_2 \le 1\}, \lambda \ge 0\} \\
&= \inf\{\lambda : ||x||_2 \le \lambda, \lambda \ge 0\} \\
&= ||x||_2
\end{aligned}$$

Moreover, recalling that $||x||_2 = \delta^\star(x|B)$, one realizes that the polar of the Euclidean unit ball is the Euclidean unit ball itself, indeed:

$$\begin{aligned}
B^\circ &= \{x : \delta^\star(x|B) \le 1\} \\
&= \{x : ||x||_2 \le 1\} \\
&= B
\end{aligned}$$

And this implies that the polar of the Euclidean norm (conceived as a gauge function) is the Euclidean norm itself, indeed $\gamma(x^\circ|B^\circ) = ||x^\circ||_2$ and this resembles to the Cauchy-Schwarz inequality since

$$\gamma(x^\circ|B^\circ)\gamma(x|B) \ge \langle x^\circ, x\rangle \quad \forall (x^\circ, x) \in \mathbb{R}^{2n} \implies ||x^\circ||_2||x||_2 \ge \langle x^\circ, x\rangle \quad \forall (x^\circ, x) \in \mathbb{R}^{2n}$$

Recalling again that $K = \text{epi}(\gamma(\cdot|C))$, i.e. the epigraph of a convex function, and that the polar of such gauge function is the function induced by $A(K^\circ) \cap \{(\lambda, x) : \lambda \geq 0, x \in \mathbb{R}^n\} = \text{epi}(\gamma(\cdot|C^\circ))$, i.e. the polar of the epigraph of the convex function under a certain linear map intersected with a half-space, a natural generalization of the things discussed so far is that given f as a non-negative origin vanishing convex function, the polar of f denoted with f° is the function induced by $A((\text{epi}(f))^\circ)$ where $A : (\mu, x) \mapsto (-\mu, x)$. Noticeably, because of the non-negativity of f, one has that $A((\text{epi}(f))^\circ) = A((\text{epi}(f))^\circ) \cap \{(\lambda, x) : \lambda > 0, x \in \mathbb{R}^n\}$

Definition 43 (Polar of a non-negative origin-vanishing convex function)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non-negative origin-vanishing convex function. The function $f^\circ : \mathbb{R}^n \to \mathbb{R}$ is called the *polar* of f and it is a non-negative origin-vanishing closed convex function induced by $A((\text{epi}(f))^\circ)$ where $A : (\mu, x) \mapsto (-\mu, x)$.

Proposition 32. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non-negative origin-vanishing convex function and let $f^\circ : \mathbb{R}^n \to \mathbb{R}$ be the polar of f. Then, the following inequality holds*

$$1 + f(x)f^\circ(x^\circ) \geq \langle x, x^\circ \rangle, \quad \forall (x, x^\circ) \in \mathbb{R}^{2n}$$

Proof. Since

$$(\text{epi}(f))^\circ = \{(\mu^\circ, x^\circ) : \langle (\mu, x), (\mu^\circ, x^\circ) \rangle \leq 1, (\mu, x) \in \text{epi}(f)\}$$

One has that

$$\begin{aligned} A((\text{epi}(f))^\circ) &= \{(-\mu^\circ, x^\circ) : \langle (\mu, x), (\mu^\circ, x^\circ) \rangle \leq 1, (\mu, x) \in \text{epi}(f)\} \\ &= \{(\mu^\circ, x^\circ) : \langle (\mu, x), (-\mu^\circ, x^\circ) \rangle \leq 1, (\mu, x) \in \text{epi}(f)\} \\ &= \{(\mu^\circ, x^\circ) : \langle x, x^\circ \rangle \leq 1 + \mu^\circ \mu, (\mu, x) \in \text{epi}(f)\} \end{aligned}$$

Leading to

$$\begin{aligned} f^\circ(x^\circ) &= \inf \{\mu^\circ : (x^\circ, \mu^\circ) \in A((\text{epi}(f))^\circ)\} \\ &= \inf \{\mu^\circ : \langle x, x^\circ \rangle \leq 1 + \mu^\circ \mu, (\mu, x) \in \text{epi}(f)\} \\ &= \inf \{\mu^\circ : \langle x, x^\circ \rangle \leq 1 + \mu^\circ f(x), x \in \text{dom}(f)\} \end{aligned}$$

That is the same notation used in[Roc70]. Of course, now the induced inequality is

$$1 + f(x)f^\circ(x^\circ) \geq \langle x, x^\circ \rangle, \quad \forall (x, x^\circ) \in \mathbb{R}^{2n}$$

□

Proposition 33. *Let $C \subset \mathbb{R}^n$ be a convex set. Let $K = cone\{(1, x).x \in C\}$ and let $A : (\lambda, x) \mapsto (-\lambda, x)$. Then, the following inequalities hold:*

$$\begin{aligned} \lambda\lambda^\circ &\geq \langle x^\circ, x \rangle \quad \forall (\lambda, x) \in K, \forall (\lambda^\circ, x^\circ) \in A(K^\circ) \\ \lambda^\star \lambda &\leq \langle x^\star, x \rangle \quad \forall (\lambda, x) \in K, \forall (\lambda^\star, x^\star) \in A(K^\star) \end{aligned}$$

Proof. It's worth showing how to recover K°, $K^\star$, $A(K^\circ)$ and $A(K^\star)$ directly from

$$K = \text{cone}(\{(1,x) : x \in C\}) = \{(\lambda, x) : \lambda \geq 0, x \in \lambda C\}$$

. In this way it's possible to see quite immediately what are the functions (convex or concave) induced by such sets. Recall that $A : (\lambda, x) \mapsto (-\lambda, x)$. In the following passages, it must be recalled that the first coordinate of $(\lambda, x) \in K$ is a non-negative scalar since $K = \text{cone}(\{(1,x) : x \in C\})$

$$K^\circ = \{(\lambda^\circ, x^\circ) : \langle(\lambda^\circ, x^\circ), (\lambda, x)\rangle \leq 0, (\lambda, x) \in K\} = \left\{(\lambda^\circ, x^\circ) : \lambda^\circ \leq \frac{\langle -x^\circ, x\rangle}{\lambda}, (\lambda, x) \in K\right\}$$
$$K^\star = -K^\circ = \left\{(\lambda^\star, x^\star) : -\lambda^\star \leq \frac{\langle x^\star, x\rangle}{\lambda}, (\lambda, x) \in K\right\} = \left\{(\lambda^\star, x^\star) : \lambda^\star \geq \frac{\langle -x^\star, x\rangle}{\lambda}, (\lambda, x) \in K\right\}$$
$$A(K^\circ) = \left\{(\lambda^\circ, x^\circ) : -\lambda^\circ \leq \frac{\langle -x^\circ, x\rangle}{\lambda}, (\lambda, x) \in K\right\} = \left\{(\lambda^\circ, x^\circ) : \lambda^\circ \geq \frac{\langle x^\circ, x\rangle}{\lambda}, (\lambda, x) \in K\right\}$$
$$A(K^\star) = \left\{(\lambda^\star, x^\star) : -\lambda^\star \geq \frac{\langle -x^\star, x\rangle}{\lambda}, (\lambda, x) \in K\right\} = \left\{(\lambda^\star, x^\star) : \lambda^\star \leq \frac{\langle x^\star, x\rangle}{\lambda}, (\lambda, x) \in K\right\}$$

In a more compact way, one has actually

$$K^\circ = \left\{(\lambda^\circ, x^\circ) : \lambda^\circ \leq \inf_{(\lambda,x)\in K} \frac{\langle -x^\circ, x\rangle}{\lambda}\right\}$$
$$K^\star = \left\{(\lambda^\star, x^\star) : \lambda^\star \geq \sup_{(\lambda,x)\in K} \frac{\langle -x^\star, x\rangle}{\lambda}\right\}$$
$$A(K^\circ) = \left\{(\lambda^\circ, x^\circ) : \lambda^\circ \geq \sup_{(\lambda,x)\in K} \frac{\langle x^\circ, x\rangle}{\lambda}\right\}$$
$$A(K^\star) = \left\{(\lambda^\star, x^\star) : \lambda^\star \leq \inf_{(\lambda,x)\in K} \frac{\langle x^\star, x\rangle}{\lambda}\right\}$$

In particular, this allows to expand what has been said about the inequality of polar gauges. Indeed, from the last two expressions it's possible to see that

$$\lambda\lambda^\circ \geq \langle x^\circ, x\rangle \quad \forall(\lambda, x) \in K, \forall(\lambda^\circ, x^\circ) \in A(K^\circ)$$
$$\lambda^\star\lambda \leq \langle x^\star, x\rangle \quad \forall(\lambda, x) \in K, \forall(\lambda^\star, x^\star) \in A(K^\star)$$

□

Proposition (33) is extremely useful because it allows to define "agnostically" two important inequalities (the first one was previously introduced as the inequality of polar gauges). Indeed, it doesn't matter whether K, $A(K^\star)$, $A(K^\circ)$ are epigraphs of convex functions or hypographs of convex functions: as long as some functions (convex or concave) are induceable from such sets, such inequalities will hold. For example, $\gamma(\cdot|C), \gamma(\cdot|C^\circ)$ are the convex functions induced by K and $A(K^\circ) \cap \{(\lambda, x) : \lambda > 0, x \in \mathbb{R}^n\}$ respectively and indeed they satisfy the first inequality

$$\gamma(x|C)\gamma(x^\circ|C^\circ) \geq \langle x, x^\circ\rangle \quad \forall(x, x^\circ) \in \mathbb{R}^{2n}$$

On the contrary, for the sake of this work, it will be showed that the invariant function $\hat{L}(x)$ and the portfolio value function $\hat{V}(p; 1)$ are the concave functions induced by K and $A(K^\star) \cap$

$\{(\lambda, x) : \lambda > 0, x \in \mathbb{R}^n\}$ respectively and indeed they satisfy the second inequality

$$\hat{L}(x)\hat{V}(p;1) \leq \langle x, p\rangle \quad \forall (x,p) \in \mathbb{R}^{2n}$$

And this inequality is going to be extremely useful in recovering the basic portfolio value function of a CFMM from its invariant function and vice-versa as discussed in [ACD+23].

Proposition 34. *Let $C \subset \mathbb{R}^n$ be a convex set. Let $K = cone\{(1,x).x \in C\}$ and let $A : (\lambda, x) \mapsto (-\lambda, x)$. Considering $\delta^\star(\cdot|D) : \mathbb{R}^n \to \mathbb{R}$ as the support function of the set D, one has that*

$$\begin{aligned} K^\circ &= hyp(-\delta^\star(\cdot|C)) \\ K^\star &= epi(\delta^\star(\cdot|-C)) \\ A(K^\circ) &= epi(\delta^\star(\cdot|C)) \\ A(K^\star) &= hyp(-\delta^\star(\cdot|-C)) \end{aligned}$$

Proof.

$$\begin{aligned} K^\circ &= \{(\lambda^\circ, x^\circ) : \langle(\lambda^\circ, x^\circ), (\lambda, x)\rangle \leq 0, (\lambda, x) \in \mathrm{cone}(\{(1,x) : x \in C\})\} \\ &= \{(\lambda^\circ, x^\circ) : \langle(\lambda^\circ, x^\circ), (1, x)\rangle \leq 0, x \in C\} \\ &= \{(\lambda^\circ, x^\circ) : \lambda^\circ \leq \langle -x^\circ, x\rangle, x \in C\} \\ &= \left\{(\lambda^\circ, x^\circ) : \lambda^\circ \leq \inf_{x\in C}\langle -x^\circ, x\rangle\right\} \\ &= \left\{(\lambda^\circ, x^\circ) : \lambda^\circ \leq -\sup_{x\in C}\langle x^\circ, x\rangle\right\} \\ &= \{(\lambda^\circ, x^\circ) : \lambda^\circ \leq -\delta^\star(x^\star|C)\} \\ &= \mathrm{hyp}(-\delta^\star(\cdot|C)) \end{aligned}$$

$$\begin{aligned} K^\star &= \{(\lambda^\star, x^\star) : \langle(\lambda^\star, x^\star), (\lambda, x)\rangle \geq 0, (\lambda, x) \in \mathrm{cone}(\{(1,x) : x \in C\})\} \\ &= \{(\lambda^\star, x^\star) : \langle(\lambda^\star, x^\star), (1, x)\rangle \geq 0, x \in C\} \\ &= \{(\lambda^\star, x^\star) : \lambda^\star \geq \langle -x^\star, x\rangle, x \in C\} \\ &= \left\{(\lambda^\star, x^\star) : \lambda^\star \geq \sup_{x\in C}\langle -x^\star, x\rangle\right\} \\ &= \{(\lambda^\star, x^\star) : \lambda^\star \geq \delta^\star(-x^\star|C)\} \\ &= \mathrm{epi}(\delta^\star(\cdot|-C)) \end{aligned}$$

$$\begin{aligned}
A(K^\circ) &= \{(\lambda^\circ, x^\circ) : \langle(-\lambda^\circ, x^\circ), (\lambda, x)\rangle \leq 0, (\lambda, x) \in \text{cone}(\{(1, x) : x \in C\})\} \\
&= \{(\lambda^\circ, x^\circ) : \langle(-\lambda^\circ, x^\circ), (1, x)\rangle \leq 0, x \in C\} \\
&= \{(\lambda^\circ, x^\circ) : -\lambda^\circ \leq \langle -x^\circ, x\rangle, x \in C\} \\
&= \left\{(\lambda^\circ, x^\circ) : \lambda^\circ \geq \sup_{x \in C} \langle x^\circ, x\rangle\right\} \\
&= \{(\lambda^\circ, x^\circ) : \lambda^\circ \geq \delta^\star(x^\star | C)\} \\
&= \text{epi}(\delta^\star(\cdot | C))
\end{aligned}$$

$$\begin{aligned}
A(K^\star) &= \{(\lambda^\star, x^\star) : \langle(-\lambda^\star, x^\star), (\lambda, x)\rangle \geq 0, (\lambda, x) \in \text{cone}(\{(1, x) : x \in C\})\} \\
&= \{(\lambda^\star, x^\star) : \langle(-\lambda^\star, x^\star), (1, x)\rangle \geq 0, x \in C\} \\
&= \{(\lambda^\star, x^\star) : -\lambda^\star \geq \langle -x^\star, x\rangle, x \in C\} \\
&= \left\{(\lambda^\star, x^\star) : \lambda^\star \leq \inf_{x \in C} \langle x^\star, x\rangle\right\} \\
&= \{(\lambda^\star, x^\star) : \lambda^\star \leq -\delta^\star(-x^\star | C)\} \\
&= \text{hyp}(-\delta^\star(\cdot | -C))
\end{aligned}$$

□

And also the notation introduced with this proposition is going to be extremely useful: once that it will be clear that the portfolio value function, representing the value of a CFMM pool after the arbitrage activity, corresponds to the negative of the support function of the symmetric reflection of the set of reachable reserves, this notation will allow to recover easily the epigraph of the portfolio value function and, from here, it will be possible to recover the associated invariant function thanks to the inequality between K and $A(K^\star)$.

2.5 Directional Derivatives and Subgradients

Sometimes it's useful understanding what is the behavior of a convex function f moving from a point x in a certain direction y. In other words, it can be insightful understanding what is the variation of the convex function by moving from a reference point x to some other points lying on the half line starting from x in the direction of y. The recession function was introduced as a tool for understanding such behavior "asymptotically", i.e. keeping the focus only on the "horizon points" of such direction, and it has been underlined how such behavior was not dependent on the original reference point x but only on the direction y, dealing with the unboundedness of the epigraph of the function.

However, one could find useful understanding what is the "local" behavior of f considering the variation of the function in a given point of x. This time, instead of moving from a generic point

x tending towards very distant points in the direction of y, the perspective is reversed: the local behavior is studied by moving from "particular points" in the neighborhood of x tending towards x itself. These particular points are those lying in the neighborhood of x which actually lie at the same time on the half line starting from x in the direction of y (thus, they are all proportional to each other). In such sense, the local behavior of f is described by the *one-sided directional derivative* of f at x in the direction of y, which corresponds to the limit of the following incremental ratio

$$f'(x; y) = \lim_{\lambda \to 0^+} \frac{f(x + \lambda y) - f(x)}{\lambda}$$

Intuitively, such directional derivative is "one-sided" because the movement towards x is studied considering just the points of the neighborhood of x which are on the "side" specified by the sign of the direction of y. Indeed, the behavior of f reaching x from the the points belonging to the "other side" of the neighborhood of x (still in the same direction of y) is captured by $f'(x; -y)$.

Of course, if the directional derivative is *two-sided*, it means that the variation of f approaching x with infinitesimal distance from one side must be the negative of the variation of f approaching x with infinitesimal variation from the other side, implying that the directional derivative, expressed as function of the direction y is homogenous of order one:

$$f' \text{ is two-sided} \iff f'(x; -y) = -f'(x; y) \quad \forall y \in \mathbb{R}^n$$

And this implies that in case of two-sided directional derivative one has $f'(x; y) = -f'(x; -y)$. In the one-dimensional case, $y \in \mathbb{R}$ and the only two meaningful directions are $y = 1$ and $y = -1$ [Roc70]: indeed, the neighborhood of any x would be line-segment and the only two possible directions are "left" ($y = -1$) and "right" ($y = 1$) (in the two dimensional case instead, the neighborhood of x would be a disk and there would be way more possible directions to evaluate). In other words, in the one-dimensional case, the directional derivative evaluating any positive direction would be equal to the "right-derivative" $f_+(x) = f'(x; 1)$, while the directional derivative evaluating any negative direction would be equal to the "left-derivative" $f_-(x) = -f'(x; -1)$ and the directional derivative of f would be actually two-sided in that particular point x if $f_+(x) = f_-(x)$.

By focusing on the incremental ratio and recalling the previously introduced "variation" function P, it's immediate to see the difference between the recession function and the one-sided directional derivative of f at x in the direction of y

$$\begin{aligned} f'(x; y) &= \lim_{\lambda \to 0^+} \frac{f(x + \lambda y) - f(x)}{\lambda} = \lim_{\lambda \to 0^+} (P\lambda^{-1})(y; x) \\ (f0^+)(y) &= \lim_{\lambda \to \infty} \frac{f(x + \lambda y) - f(x)}{\lambda} = \lim_{\lambda \to \infty} (P\lambda^{-1})(y; x) \end{aligned}$$

Moreover, recalling that the ratio $\frac{f(x+\lambda y)-f(x)}{\lambda}$ is a monotonically non-decreasing function of λ, it means that it will approach it's infimum as λ becomes infinitesimal while it will approach its supremum as λ becomes infinite. Thus, one has

$$f'(x;y) = \lim_{\lambda \to 0^+} \frac{f(x+\lambda y) - f(x)}{\lambda} = \lim_{\lambda \to 0^+} (P\lambda^{-1})(y;x) = \inf_{\lambda > 0} (P\lambda^{-1})(y;x)$$
$$(f0^+)(y) = \lim_{\lambda \to \infty} \frac{f(x+\lambda y) - f(x)}{\lambda} = \lim_{\lambda \to \infty} (P\lambda^{-1})(y;x) = \sup_{\lambda > 0} (P\lambda^{-1})(y;x)$$

Notice that, since for concave functions $\frac{\hat{f}(x+\lambda y) - \hat{f}(x)}{\lambda} =: (\hat{P}\lambda^{-1})(y;x)$ is a monotonically non-increasing function of λ, the relationship is actually reversed and one has that

$$\hat{f}'(x;y) = \lim_{\lambda \to 0^+} \frac{\hat{f}(x+\lambda y) - \hat{f}(x)}{\lambda} = \lim_{\lambda \to 0^+} (\hat{P}\lambda^{-1})(y;x) = \sup_{\lambda > 0} (\hat{P}\lambda^{-1})(y;x)$$
$$(\hat{f}0^+)(y) = \lim_{\lambda \to \infty} \frac{\hat{f}(x+\lambda y) - \hat{f}(x)}{\lambda} = \lim_{\lambda \to \infty} (\hat{P}\lambda^{-1})(y;x) = \inf_{\lambda > 0} (\hat{P}\lambda^{-1})(y;x)$$

The use of the variation function allows to give a very insightful information about the nature of the directional derivative. Indeed, in the case of convex functions, f' corresponds to the *positive homogenous convex function generated by* $P(\cdot;x)$, implying that its epigraph corresponds to the convex cone containing the origin generated by the epigraph of $P(\cdot;x)$, i.e. from the epigraph of f translated in such a way that $(x, f(x)) \mapsto (0,0)$ (here stands the dependance of $P(\cdot;x)$ on x as parameter). Indeed, one has to notice that $\inf_{\lambda>0}(P\lambda^{-1})(y;x)$ is equivalent to $\inf_{\lambda>0}(P\lambda)(y;x)$ because the final output is always the convex function induced by the convex cone generated by $\text{epi}(P(\cdot;x))$ by ranging over positive rescalings.
Analogously, in the case of concave functions, $\hat{f}'$ corresponds to the *positive homogenous concave function generated by* $\hat{P}(\cdot;x)$, implying that its hypograph corresponds to the convex cone containing the origin generated by $\text{hyp}(\hat{P}(\cdot;x))$.
Thus, the convex cone generated by $\text{epi}(f)$ (i.e., epigraph of $\inf_{\lambda>0}(f\lambda)(x)$) and the convex cone generated by $\text{epi}(P(\cdot;x))$ (i.e., the epigraph of $f'(y;x)$) are "similar" in the sense that they are both generated by the epigraph of f, but the second one is generated actually by a translation of the epigraph of f.
This observation will be extremely important once that it will be introduced the set of feasible trades as the 0-upper level set of the variation function of the CFMM invariant.

Noticeably, since $f'(x_0;\cdot)$ is a positively homogenous convex function of y, its closure $(\text{cl}f')(x_0;\cdot)$ must support some convex set $C \subset \mathbb{R}^n$. Thus, it could be interesting deriving such set without knowing anything in advance about it:

$$
\begin{aligned}
C &= \{y^\star : \langle y, y^\star \rangle \leq (\mathrm{cl} f')(x_0; y), y \in \mathbb{R}^n\} \\
&= \left\{ y^\star : \langle y, y^\star \rangle \leq \inf_{\lambda > 0} (P\lambda^{-1})(y; x_0), y \in \mathbb{R}^n \right\} \\
&= \left\{ y^\star : \langle y, y^\star \rangle \leq (P\lambda^{-1})(y; x_0), y \in \mathbb{R}^n, \lambda > 0 \right\} \\
&= \left\{ y^\star : \langle y, y^\star \rangle \leq \frac{f(x_0 + \lambda y) - f(x_0)}{\lambda}, y \in \mathbb{R}^n, \lambda > 0 \right\} \\
&= \{y^\star : \langle \lambda y, y^\star \rangle + f(x_0) \leq f(x_0 + \lambda y), y \in \mathbb{R}^n, \lambda > 0\}
\end{aligned}
$$

Now, calling $z = x_0 + \lambda y$, one has that

$$
C = \{y^\star : \langle y^\star, z - x_0 \rangle + f(x_0) \leq f(z), z \in \mathbb{R}^n\}
$$

And it's evident how the set supported by the directional derivative of f expressed as function of y is actually dependant on the point $x_0 \in \mathrm{dom}(f)$ at which the directional derivative is evaluated, implying that one could think about such set as a map $x_0 \mapsto \{y^\star : \langle y^\star, z - x_0 \rangle + f(x_0) \leq f(z), z \in \mathbb{R}^n\}$.

Noticably, the rule of belongingness in the set notation of C is actually used for introducing the concept of subgradients. Indeed, a point $x^\star \in \mathbb{R}^n$ is said to be *subgradient* of a convex real-valued function on $\mathbb{R}^n$ f at $x_0 \in \mathbb{R}^n$ if it satisfies the so called *subgradient inequality*, that is:

$$
f(z) \geq f(x_0) + \langle x^\star, z - x_0 \rangle \quad \forall z \in \mathrm{dom}(f)
$$

The set of subgradients of f at a point x_0 is defined by a set-valued mapping called *subdifferential of f at x_0* and denoted by $\partial f(x_0)$:

$$
\partial f : x_0 \mapsto \{x^\star : f(z) \geq f(x_0) + \langle x^\star, z - x_0 \rangle,\ z \in \mathrm{dom} f\}
$$

Now, by comparing the set notations of C and $\partial f(x_0)$ it's evident to see that the set supported by $f'(x_0; \cdot)$ is actually $\partial f(x_0)$. Thus, one can actually introduce an alternative notation for the closure of $f'(x_0; \cdot)$ in terms of support function:

$$
(\mathrm{cl} f')(x_0; y) = \sup_{y^\star \in \partial f(x_0)} \langle y, y^\star \rangle
$$

And of course this allows to rewrite the subdifferential as

$$
\partial f(x_0) = \{y^\star : \langle y, y^\star \rangle \leq (\mathrm{cl} f')(x_0; y),\ y \in \mathbb{R}^n\}
$$

Implying that

$$
x^\star \in \partial f(x_0) \iff \langle y, x^\star \rangle \leq f'(x_0; y),\ y \in \mathbb{R}^n
$$

The function f is said to be *subdifferentiable* at $x_0 \in \mathrm{dom}(f)$ if $\partial f(x_0) \neq \emptyset$.

Proposition 35. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper convex function. Then,*

- $\partial f(x_0)$ *is non-empty* $\forall x_0 \in ri(dom(f))$
- $f'(x_0; y)$*, expressed as function of* y*, is a closed and proper* $\forall x_0 \in ri(dom(f))$

Proof. The first argument implies that f is subdifferentiable at any point belonging to its relative interior, while the second argument implies that the directional derivative corresponds to its closure and so that, expressed as function of the direction y, it supports $\partial f(x_0)$. To prove this, is sufficient proving that $f'(x_0; \cdot)$ is closed and proper, because in this case the directional derivative will correspond to the support function of $\partial f(x_0)$ and the non-emptiness of $\partial f(x_0)$ will follow from the fact that $\nexists y \in \mathbb{R}^n : f'(x_0; y) = -\infty$ (recall that, by definition, a support function is $-\infty$ if and only if it is supporting the $\emptyset$). The closedness of $f'(x_0; \cdot)$ follows from the fact that $\text{dom}(f'(x_0; \cdot))$ is a translate of $\text{aff}(\text{dom}(f))$ thus it's still an affine set: because of that, it doesn't have any boundary point and since a function agrees with its closure everywhere on $\text{dom}(f)$ apart maybe from the boundary points, here $f'(x_0; \cdot)$ must agree everywhere with its closure, implying that the function is closed. On the other hand, since $f'(x_0; \cdot)$ is the function induced by the convex cone containing the origin generated by $\text{epi}(P(\cdot; x_0))$, $f'(x_0; 0) = 0$ (origin-vanishing) and this is sufficient to conclude that $f'(x_0; \cdot)$ is proper: indeed, as discussed in the section introducing convex functions, if f is convex and improper one has $f(x) = -\infty \forall x \in \text{ri}(\text{dom}(f))$ and since $f'(x_0; 0) = 0$ with $0 \in \text{ri}(\text{dom}(f))$ (in particular, being an affine set, one has $\text{dom}(f) = \text{ri}(\text{dom}(f))$) this evidences that $f'(x_0; \cdot)$ is proper. □

Noticeably, by re-arranging the subgradient inequality, one is capable of deriving alternative forms for understanding the subdifferential of f at x_0.

Proposition 36. *Let f be a proper convex function. Let $x_0 \in ri(dom(f))$. Then the cone normal to $epi(f)$ at $(x_0, f(x_0))$ is the convex cone generated by $\partial f(x_0)$ in the following sense:*

$$N((x_0, f(x_0)|epi(f))) = cone(\{(x^\star, -1) : x^\star \in \partial f(x_0)\})$$

Proof. Firstly, one has that

$$\begin{aligned}
\partial f(x_0) &= \{x^\star : f(z) \geq f(x_0) + \langle x^\star, z - x_0 \rangle,\ z \in \text{dom} f\} \\
&= \{x^\star : f(z) - f(x_0) - \langle x^\star, z - x_0 \rangle \geq 0\ z \in \text{dom} f\} \\
&= \{x^\star : f(z) - f(x_0) - \langle x^\star, z - x_0 \rangle \geq 0\ z \in \text{dom} f\} \\
&= \{x^\star : -f(z) + f(x_0) + \langle x^\star, z - x_0 \rangle \leq 0\ z \in \text{dom} f\} \\
&= \{x^\star : \langle (x^\star, -1), (z - x_0, f(z) - f(x_0)) \rangle \leq 0\ z \in \text{dom} f\}
\end{aligned}$$

And recalling that the normal cone of $\text{epi}(f)$ at the point $(x_0, f(x_0))$ is equal to

$$N((x_0, f(x_0)|\text{epi}(f))) = \{(x^\star, \mu^\star) : \langle (x^\star, \mu^\star), (z - x_0, f(z) - f(x_0)) \rangle \leq 0\ z \in \text{dom} f\}$$

one has that the subdifferential of f at x_0 could be interpreted as the lower-dimensional projection of the intersection between the normal cone of epi(f) at the point $(x_0, f(x_0))$ and the hyperplane $\{(x^\star, -1) : x^\star \in \mathbb{R}^n\}$. In other words, defining $A : (x^\star, \mu^\star) \mapsto x^\star$, one would have

$$\partial f(x_0) = A(N((x_0, f(x_0)|\text{epi}(f))) \cap \{(x^\star, -1) : x^\star \in \mathbb{R}^n\})$$

This finding follows from the fact that $N((x_0, f(x_0))|\text{epi}(f))$ could be seen as the convex cone generated by $\partial f(x_0)$ at level -1, indeed:

$$\begin{aligned}\text{cone}\left(\{(x^\star, -1) : x^\star \in \partial f(x_0)\}\right) &= \{\lambda(x^\star, -1) : \lambda > 0, f(z) \geq f(x_0) + \langle x^\star, z - x_0\rangle, z \in \text{dom}(f)\} \\ &= \{(\lambda x^\star, -\lambda) : \lambda > 0, \langle (x^\star, -1), (z - x_0, f(z) - f(x_0))\rangle \leq 0,\ z \in \text{dom} f\}\end{aligned}$$

Calling $\tilde{x} = \lambda x^\star$ and $\tilde{\lambda} = -\lambda \implies (x^\star, -1) = \left(\frac{\tilde{x}}{\lambda}, \frac{\tilde{\lambda}}{\lambda}\right) = \frac{1}{\lambda}(\tilde{x}, \tilde{\lambda})$

$$\begin{aligned}\text{cone}\left(\{(x^\star, -1) : x^\star \in \partial f(x_0)\}\right) &= \left\{(\tilde{x}, \tilde{\lambda}) : \tilde{\lambda} < 0, \lambda > 0, \frac{1}{\lambda}\langle (\tilde{x}, \tilde{\lambda}), (z - x_0, f(z) - f(x_0))\rangle \leq 0,\ z \in \text{dom} f\right\} \\ &= \left\{(\tilde{x}, \tilde{\lambda}) : \tilde{\lambda} < 0, \langle (\tilde{x}, \tilde{\lambda}), (z - x_0, f(z) - f(x_0))\rangle \leq 0,\ z \in \text{dom} f\right\} \\ &= N((x_0, f(x_0)|\text{epi}(f)))\end{aligned}$$

□

Proposition 37. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper convex function. Let $h(x; x^\star, x_0) = f(x_0) + \langle x^\star, x - x_0\rangle$. Then:*

$$(x_0, x^\star) \in ri(dom(f)) \times \partial f(x_0) \implies f(x) \geq h(x; x^\star, x_0) = f(x_0) + \langle x^\star, x - x_0\rangle \quad \forall x \in dom(f) \tag{2.12}$$

In particular, epi$(h(\cdot; x^\star, x_0))$ is one of the (possibly many) supporting half-spaces of epi(f) at $(x_0, f(x_0))$

Proof. As previously mentioned, the collection half-spaces supporting a convex set C at a generic point $z_0 \in C$ is given by

$$\{\{z \in \mathbb{R}^n : \langle b, z - z_0\rangle \leq 0\} :\ b \in N(z_0|C)\}$$

Thus, since the point $(x_0, f(x_0))$ lives at the relative boundary of epi(f), one could derive the set of supporting hyperplanes of epi(f) at $(x_0, f(x_0))$ as

$$\{\{(x, \mu) \in \mathbb{R}^n : \langle (x^\star, -1), (x - x_0, \mu - f(x_0))\rangle \leq 0\} :\ (x^\star, -1) \in N((x_0, f(x_0)|\text{epi}(f)))\}$$

Or alternatively,

$$\{\{(x, \mu) \in \mathbb{R}^n : \langle (x^\star, -1), (x - x_0, \mu - f(x_0))\rangle \leq 0\} :\ x^\star \in \partial f(x_0)\}$$

Being convex sets, one could pick a generic supporting half-space in such collection (i.e. choosing arbitrarily a subgradient of the function at a particular point x_0) and induce a convex function, whose graph would be a hyperplane supporting epi(f) at $(x_0, f(x_0))$. Thus, defining $H(x^\star, x_0) = \{(x, \mu) \in \mathbb{R}^n : \langle (x^\star, -1), (z - x_0, f(z) - f(x_0)) \rangle \leq 0\}$ as the chosen half-space in the collection, one has

$$\begin{aligned} h(x; x^\star, x_0) &= \inf \{\mu : (x, \mu) \in H(x^\star, x_0)\} \\ &= \inf \{\mu : \langle (x^\star, -1), (x - x_0, \mu - f(x_0)) \rangle \leq 0\} \\ &= \inf \{\mu : \langle x^\star, x - x_0 \rangle - \mu + f(x_0) \leq 0\} \\ &= \inf \{\mu : \mu \geq f(x_0) + \langle x^\star, x - x_0 \rangle\} \\ &= f(x_0) + \langle x^\star, x - x_0 \rangle \end{aligned}$$

Thus, the subgradient inequality can be rewritten as $f(x) \geq h(x; x^\star, x_0)\ \forall x \in \text{dom}(f)$. Notice that h is an affine function majorized by f and, since epi($h(\cdot; x^\star, x_0)$) acts as supporting half-space for epi(f), then Graph($h(\cdot; x^\star, x_0)$) acts as supporting hyperplane for epi(f) [Roc70]. □

Noticeably, $h(x; x^\star, x_0) = f(x_0) + \langle x^\star, x - x_0 \rangle$ reminds the first-order Taylor expansion of f around x_0 even if $x^\star$ is not necessarily the gradient of f evaluating x_0. However, since the gradient of f corresponds to the unique subgradient of f at x_0 when the function is differentiable at x_0 (implying that $\exists! x^\star \in \mathbb{R}^n : f(z) \geq f(x_0) + \langle x^\star, z - x_0 \rangle\ \forall z \in \text{dom}(f)$ and such $x^\star = \nabla f(x_0)$), one can conclude that the first order Taylor expansion of f at any point is actually a global underestimator of f. Moreover, it's important to notice that if the function f is differentiable on its effective domain, this implies that $\partial f(\cdot)$ is actually a single-valued map $\partial f : x_0 \mapsto \{\nabla f(x_0)\}$. Thus, imagining a real-valued convex function f on $\mathbb{R}$, one could guess the "smoothness" of f just by looking at Graph(∂f) $\subset \mathbb{R}^n$: indeed, if Graph(∂f) experiences a "vertical jump" in correspondence of a particular x_0, this implies that f is not differentiable at x_0 since this point is mapped into more than one subgradient of f. Moreover, the higher is the verticality of such "jump" experienced by the subdifferential, the higher is the lack of smoothness of f at x_0 [BV04].

Proposition 38. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function differentiable at $x_0 \in ri(dom(f))$. Then, the one-sided directional derivative is actually two-sided and equal to*

$$f'(x; y) = \langle \nabla f(x_0), y \rangle$$

Proof. As previously said, when a function is differentiable at some point x_0 it means that the subdifferential is actually the singleton of the gradient of the function evaluating x_0. In particular [Roc70], if f is differentiable at x_0, it means that $\nabla f(x_0)$ exists and one has that

$$f(z) = f(x_0) + \langle \nabla f(x_0), z - x_0 \rangle + o(||z - x_0||_2)$$

Which is equivalent to say that

$$\lim_{z \to x_0} \frac{f(z) - f(x_0) - \langle \nabla f(x_0), z - x_0 \rangle}{||z - x_0||_2} = 0$$

By applying the usual substitution $z = x_0 + \lambda y$ with $\lambda > 0$

$$\lim_{\lambda \to 0^+} \frac{f(x_0 + \lambda y) - f(x_0) - \langle \nabla f(x_0), \lambda y \rangle}{||\lambda y||_2} = 0$$
$$\lim_{\lambda \to 0^+} \frac{f(x_0 + \lambda y) - f(x_0)}{\lambda ||y||_2} - \frac{\lambda \langle \nabla f(x_0), y \rangle}{\lambda ||y||_2} = 0$$
$$\frac{f'(x_0; y)}{||y||_2} - \frac{\langle \nabla f(x_0), y \rangle}{||y||_2} = 0$$

And this relation is satisfied if and only if $f'(x_0; y) = \langle \nabla f(x_0), y \rangle$ for all $y \in \mathbb{R}^n$.
This relation is particularly insightful because it tells that if f is differentiable at x_0 the directional derivative is actually two-sided since it is homogenous in y, indeed:

$$f'(x_0; -y) = \langle \nabla f(x_0), -y \rangle = -\langle \nabla f(x_0), y \rangle = -f'(x_0; y)$$

Moreover, recalling the support-notation for the closure of the directional derivative and the fact that, in case of differentiability at x_0, $\nabla f(x_0)$ is the unique subgradient of f at x_0 (i.e. $\partial f(x_0) = \{\nabla f(x_0)\}$), one has that the directional derivative is also closed if f is differentiable at x_0:

$$(\mathrm{cl} f')(x_0; y) = \sup_{y^\star \in \partial f(x_0)} \langle y, y^\star \rangle = \langle y, \nabla f(x_0) \rangle = f'(x_0; y)$$

But, as seen in proposition (35), the closedness (and properness) of the directional derivative at any $x_0 \in \mathrm{ri}(\mathrm{dom}(f))$ is always granted if f is proper and convex. □

A second possible rewriting of $\partial f(x_0)$ implies the use of the Fenchel conjugate of f, thus it will result more clear after reading the next section. Thus, this notation is introduced here just to be recalled later:

Proposition 39. *Let f be a proper convex function. Then, the subdifferential of f at x_0 can be rewritten as*

$$\partial f(x_0) = \{x^\star : f^\star(x^\star) + f(x_0) \leq \langle x^\star, x_0 \rangle\}$$

Proof.

$$\begin{aligned}
\partial f(x_0) &= \{x^\star : f(z) \geq f(x_0) + \langle x^\star, z - x_0 \rangle,\ z \in \mathrm{dom} f\} \\
&= \{x^\star : f(z) - \langle x^\star, z \rangle \geq f(x_0) - \langle x^\star, x_0 \rangle,\ z \in \mathrm{dom} f\} \\
&= \{x^\star : \langle x^\star, z \rangle - f(z) \leq \langle x^\star, x_0 \rangle - f(x_0),\ z \in \mathrm{dom} f\} \\
&= \left\{ x^\star : \sup_{z \in \mathrm{dom} f} \langle x^\star, z \rangle - f(z) \leq \langle x^\star, x_0 \rangle - f(x_0) \right\} \\
&= \{x^\star : f^\star(x^\star) + f(x_0) \leq \langle x^\star, x_0 \rangle\}
\end{aligned}$$

□

And this notation, paired with the Fenchel inequality, will provide nice dual correspondences between the subdifferential of f and its conjugate $f^\star$.

For concave functions $\hat{f}$ the relation described by the subgradient inequality is reversed and so the subdifferential of the function at x_0 is defined as *superdifferential* of $\hat{f}$ at x_0 and the elements of the superdifferential are called *supergradients* of $\hat{f}$ at x_0. For the sake of consistency, the hat-notation will be used for denoting the superdifferential of a concave function. In other words, given $\hat{f}$ as a real-valued concave function on $\mathbb{R}^n$, one has

$$\hat{\partial}\hat{f} : x_0 \mapsto \left\{ x^\star : \hat{f}(z) \leq \hat{f}(x_0) + \langle x^\star, z - x_0 \rangle,\ z \in \mathrm{dom}(\hat{f}) \right\}$$

2.6 The Fenchel Conjugate

As seen in the previous sections, any closed convex set C has a "dual" representation: an "internal" representation as the convex hull of $m+1$ affinely independent points (if the convex set is has dimension m) and an "external" representation as the intersection of all the closed half-spaces containing the closed convex set. Pairing the concept of "external" representation with the fact that at any point of the relative boundary of a convex set C there exists a non-null vector normal to the set (thus, a half-space properly supporting the convex set), one has that the "efficient external" representation would correspond to the intersection of just those half-spaces which are properly supporting the convex set at any point of its relative boundary. For this reason, inspecting the cones normal to C at each point z_0 of its relative boundary is particularly insightful for recovering $\mathrm{cl}(C)$, since each cone will map to a set of closed hyperplanes supporting C at that point z_0 and taking the intersection of such hyperplanes ranging over the relative boundary of C will result in the "efficient external" representation of $\mathrm{cl}(C)$.

This kind of external representation can be extended also to closed functions: indeed, in the case of convex functions, since $\mathrm{cl}(\mathrm{epi}(f))$ ($= \mathrm{epi}(\mathrm{cl}f)$ if f is proper) can be represented as the intersection of the collection of closed half-spaces supporting it, the function itself will be induced by such intersection and the graph of the function will correspond to the envelope of the supporting hyperplanes.

Thus, to recover "efficiently" $\mathrm{cl}(\mathrm{epi}(f))$, one has to range over the relative boundary of $\mathrm{epi}(f)$, collect all the supporting hyperplanes at each point on the relative boundary and take the intersection of such collection.

Proposition 40. *Let f be a proper convex function. Let $h(x; x^\star, x_0) = f(x_0) + \langle x^\star, x - x_0 \rangle$, then*

$$cl(epi(f)) = \bigcap_{x_0 \in ri(dom(f))} \left(\bigcap_{x^\star \in \partial f(x_0)} epi(h(\cdot; x^\star, x_0)) \right)$$

Proof. Because of the subdifferentiability of proper convex function as stated in proposition (35) one could start from proposition (37) to map each point $(x_0, f(x_0))$ to a set of supporting half-spaces tangent to epi(f) at $(x_0, f(x_0))$. Indeed, defining $h(x; x^\star, x_0) = f(x_0) + \langle x^\star, x - x_0 \rangle$, given $x_0 \in \text{ri}(\text{dom}(f))$ and $x^\star \in \partial f(x_0)$, one has that epi$(h(\cdot; x^\star, x_0))$ is one of the half-spaces supporting epi(f) at $(x_0, f(x_0))$ because of the subgradient inequality:

$$(x_0, x^\star) \in \text{ri}(\text{dom}(f)) \times \partial f(x_0) \implies f(x) \geq f(x_0) + \langle x^\star, x - x_0 \rangle \; \forall x \in \mathbb{R}^n$$

On the other hand, the collection of half-spaces supporting epi(f) at $(x_0, f(x_0))$ is actually a collection of half-spaces where $\partial f(x_0)$ acts as index set. Indeed, picking $\partial f(x_0)$ as index set is totally equivalent to pick $N(x_0, f(x_0)|\text{epi}(f))$ as index set because the latter is the convex cone generated in higher dimension by the first (proposition (36))

$$\begin{aligned}
&\{\text{epi}(h(\cdot; x^\star, x_0)) : x^\star \in \partial f(x_0)\} \\
&= \{\{(x, \mu) : f(x_0) + \langle x^\star, x - x_0 \rangle \leq \mu\} : x^\star \in \partial f(x_0)\} \\
&= \{\{(x, \mu) : \langle (x^\star, -1), (x - x_0, f(x) - f(x_0)) \rangle \leq 0\} : x^\star \in \partial f(x_0)\} \\
&= \{\{(x, \mu) : \langle (x^\star, \mu^\star), (x - x_0, f(x) - f(x_0)) \rangle \leq 0\} : (x^\star, \mu^\star) \in N(x_0, f(x_0)|\text{epi}(f))\}
\end{aligned}$$

which implies that

$$\bigcap_{x^\star \in \partial f(x_0)} \text{epi}(h(\cdot; x^\star, x_0)) = \bigcap_{(x^\star, \mu^\star) \in N(x_0, f(x_0)|\text{epi}(f))} \{(x, \mu) : \langle (x^\star, \mu^\star), (x - x_0, f(x) - f(x_0)) \rangle \leq 0\}$$

Implying that, given a certain $x_0 \in \text{ri}(\text{dom}(f))$ each subgradient $x^\star \in \partial f(x_0)$ indexes a certain half-space of type $\{(x, \mu) : f(x_0) + \langle x^\star, x - x_0 \rangle \leq \mu\}$ which supports epi(f) at $(x_0, f(x_0))$.
Thus, one could also range over the relative interior of the effective domain and take the "intersection of all the intersections" of the half-spaces supporting epi(f) at each $(x_0, f(x_0))$: this would lead to the "efficient" external representation of cl(epi(f)). In other words

$$\begin{aligned}
\text{cl}(\text{epi}(f)) &= \bigcap_{x_0 \in \text{ri}(\text{dom}(f))} \left(\bigcap_{x^\star \in \partial f(x_0)} \text{epi}(h(\cdot; x^\star, x_0)) \right) \\
&= \bigcap_{x_0 \in \text{ri}(\text{dom}(f))} \left(\bigcap_{x^\star \in \partial f(x_0)} \{(x, \mu) : f(x_0) + \langle x^\star, x - x_0 \rangle \leq \mu\} \right)
\end{aligned}$$

Indeed in this case one is capable of recovering Graph(f) as the envelope of tangents (i.e. supporting hyperplanes) of epi(f) at each point $(x_0, f(x_0))$ ranging over the relative interior of the effective domain of f. □

Proposition 41. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper convex function. Let $g(x; x^\star; \mu^\star) = \langle x, x^\star \rangle - \mu^\star$. Then*

$$\exists (x^\star, \mu^\star) \in \mathbb{R}^n \times \mathbb{R} : f(x) \geq g(x; x^\star, \mu^\star) \; \forall x \in \mathbb{R}^n$$

Implying that

$$\exists (x^\star, \mu^\star) \in \mathbb{R}^n \times \mathbb{R} : epi(f) \subset epi(g(\cdot; x^\star, \mu^\star))$$

Proof. One could rephrase the function used in proposition (40) as follows

$$\begin{aligned} h(x; x^\star, x_0) &= f(x_0) + \langle x^\star, x - x_0 \rangle \\ &= \langle x^\star, x \rangle - (\langle x^\star, x_0 \rangle - f(x_0)) \\ &= \langle x^\star, x \rangle - \langle (x^\star, -1), (x_0, f(x_0)) \rangle \end{aligned}$$

At this point, one could remove the dependence of $h(x; x^\star, x_0)$ from the tangent point by "wrapping" it into an parameter $\mu^\star = \langle (x^\star, -1), (x_0, f(x_0)) \rangle$ leading to

$$\begin{aligned} g(x; x^\star, \mu^\star) &= \langle x^\star, x \rangle - \mu^\star \\ &= \langle (x^\star, \mu^\star), (x, -1) \rangle \end{aligned}$$

This kind of transformation leads to a loss of information about the tangency point $(x_0, f(x_0))$. Because of that, there is also a loss of information about $x^\star$ which before was considered as a subgradient of the proper convex function. However, one is sure of the fact that $\exists (x^\star, \mu^\star) \in \mathbb{R}^n \times \mathbb{R} : f(x) \geq g(x; x^\star, \mu^\star)\ \forall x \in \mathbb{R}^n$ because when f doesn't take finite values the inequality is trivially verified, while when f takes finite values one can always pick $\mu^\star = \langle (x^\star, -1), (x_0, f(x_0)) \rangle$ and $x^\star \in \partial f(x_0)$. In this context, one could pick also any $\mu^\star > \langle (x^\star, -1), (x_0, f(x_0)) \rangle$ which will still result in a closed half-space $\tilde{H}_{(x^\star, \mu^\star)} = \{(x, \mu) : \langle x, x^\star \rangle - \mu^\star \leq \mu\} = \text{epi}(g(\cdot; x^\star, \mu^\star))$ which will still contain completely epi(f) but it may not support epi(f) being not tangent to it. □

Proposition 42. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper convex function. Let $g(x; x^\star; \mu^\star) = \langle x, x^\star \rangle - \mu^\star$. The set*

$$F^\star = \{(x^\star, \mu^\star) \in \mathbb{R}^n \times \mathbb{R} : f(x) \geq g(x; x^\star, \mu^\star),\ x \in dom(f)\}$$

is a non-empty, closed convex set and one has

$$cl(epi(f)) = \bigcap_{(x^\star, \mu^\star) \in F^\star} epi(g(\cdot; x^\star, \mu^\star))$$

Proof. The condition stated in proposition (41) doesn't hold $\forall (x^\star, \mu^\star) \in \mathbb{R}^{n+1}$, and, because of this, it can be useful defining the set $F^\star$ containing the vectors $(x^\star, \mu^\star)$ for which such inequality holds. Indeed, the set $F^\star$ is actually defining the collection of affine function $g(\cdot; x^\star, \mu^\star)$ which are majorized by f and $(x^\star, \mu^\star) \in F^\star$ acts as index for each affine function in such collection [Roc70]

$$\begin{aligned} F^\star &= \{(x^\star, \mu^\star) \in \mathbb{R}^n \times \mathbb{R} : \text{epi}(f) \subset \text{epi}(g(\cdot; x^\star, \mu^\star))\} \\ &= \{(x^\star, \mu^\star) \in \mathbb{R}^n \times \mathbb{R} : f(x) \geq g(x; x^\star, \mu^\star),\ x \in \text{dom}(f)\} \end{aligned}$$

The non-emptiness of $F^\star$ follows from 41, while closedness and convexity follow from the fact that $F^\star$ is retrievable as the intersection of closed half-spaces indexed by $x \in \text{dom}(f)$ indeed

$$F^\star = \bigcap_{x \in \text{dom}(f)} \{(x^\star, \mu^\star) \in \mathbb{R}^n \times \mathbb{R} : \langle (x, -1), (x^\star, \mu^\star) \rangle \leq f(x)\}$$

Noticeably, each half-space in such collection has $(x, -1)$ as normal vector and $f(x)$ as offset. Because of the equivalence between propositions (40) and (41), one can actually retrieve the "inefficient" external representation of cl(epi(f)) ("inefficient" in the sense that it evaluates in the intersection also those half-spaces which, even if they are not supporting epi(f), they are still including it) as

$$\mathrm{cl}(\mathrm{epi}(f)) = \bigcap_{(x^\star,\mu^\star)\in F^\star} \mathrm{epi}(g(\cdot; x^\star, \mu^\star))$$

□

Noticeably, $F^\star \subset \mathbb{R}^{n+1}$ is a closed convex set, thus it's possible to induce a convex lower-semicontinuous function which is going to be called *Fenchel conjugate* of f

Definition 44 (Fenchel conjugate)**.** Let f be a proper convex function. Let $g(x; x^\star; \mu^\star) = \langle x, x^\star\rangle - \mu^\star$ and $F^\star = \{(x^\star, \mu^\star) \in \mathbb{R}^n \times \mathbb{R} : f(x) \geq g(x; x^\star, \mu^\star),\ x \in \mathrm{dom}(f)\}$. The closed convex function induced by $F^\star$ is called *Fenchel conjugate* of f and it is denoted by $f^\star$

Proposition 43. *Let f be a proper convex function. The Fenchel conjugate of f can be obtained as*

$$f^\star(x^\star) = \sup_{x\in dom(f)} \langle x, x^\star\rangle - f(x)$$

And this defines the so-called Fenchel inequality:

$$x \in dom(f) \implies f^\star(x^\star) + f(x) \geq \langle x^\star, x\rangle \quad \forall x^\star \in \mathbb{R}^n$$

Proof.

$$\begin{aligned}
f^\star(x^\star) &= \inf\{\mu^\star : (x^\star,\mu^\star) \in F^\star\}\\
&= \inf\{\mu^\star : (x^\star,\mu^\star) \in \{(x^\star,\mu^\star) : f(x) \geq \langle (x,-1),(x^\star,\mu^\star)\rangle,\ x\in\mathrm{dom}(f)\}\}\\
&= \inf\{\mu^\star : (x^\star,\mu^\star) \in \{(x^\star,\mu^\star) : f(x) \geq \langle x,x^\star\rangle - \mu^\star,\ x\in\mathrm{dom}(f)\}\}\\
&= \inf\left\{\mu^\star : (x^\star,\mu^\star) \in \bigcap_{x\in\mathrm{dom}(f)} \{(x^\star,\mu^\star)\in\mathbb{R}^n\times\mathbb{R} : \mu^\star \geq \langle x,x^\star\rangle - f(x)\}\right\}\\
&= \sup_{x\in\mathrm{dom}(f)} \langle x,x^\star\rangle - f(x)
\end{aligned}$$

Thus, given a generic $x \in \mathrm{dom}(f)$, one has

$$f^\star(x^\star) + f(x) \geq \langle x, x^\star\rangle\ \forall x^\star \in \mathbb{R}^n$$

Indeed, when $x^\star \in \mathrm{dom}(f^\star)$ the inequality holds with finite values, while when $x^\star \notin \mathrm{dom}(f^\star)$ then the inequality holds trivially since $f^\star(x^\star) = \infty$ □

Proposition 44. *Let f be a proper convex function. Then $cl(f) = f^{\star\star}$ indeed*

$$cl(f(x)) = \sup_{x^\star\in dom(f^\star)} \langle x, x^\star\rangle - f^\star(x^\star)$$

Proof. Recalling proposition (42) and applying the definition of Fenchel conjugate (i.e. $F^\star = \text{epi}(f^\star)$) one has that

$$\begin{aligned}
\text{cl}(f(x)) &= \inf\left\{\mu : (x,\mu) \in \bigcap_{(x^\star,\mu^\star)\in F^\star} \text{epi}(g(\cdot; x^\star, \mu^\star))\right\} \\
&= \inf\left\{\mu : (x,\mu) \in \bigcap_{(x^\star,\mu^\star)\in F^\star} \{(x,\mu) : \langle x, x^\star\rangle - \mu^\star \leq \mu\}\right\} \\
&= \inf\left\{\mu : (x,\mu) \in \bigcap_{x^\star\in \text{dom}(f^\star)} \{(x,\mu) : \langle x, x^\star\rangle - f^\star(x^\star) \leq \mu\}\right\} \\
&= \inf\left\{\mu : (x,\mu) \in \bigcap_{x^\star\in \text{dom}(f^\star)} \{(x,\mu) : \langle (x,-1), x^\star\rangle - f^\star(x^\star) \leq \mu\}\right\} \\
&= \sup_{x^\star\in\text{dom}(f^\star)} \langle x, x^\star\rangle - f^\star(x^\star)
\end{aligned}$$

□

A further inspection of the closed convex sets inducing $f^\star$ and $\text{cl}(f)$ allows to create a nice dual correspondence between such functions. Indeed, one has that

$$\begin{cases} \text{cl}(\text{epi}(f)) = \bigcap_{x^\star\in\text{dom}(f^\star)} \{(x,\mu) : \langle (x^\star,-1),(x,\mu)\rangle \leq f^\star(x^\star)\} \\ \text{epi}(f^\star) = \bigcap_{x\in\text{dom}(f)} \{(x^\star,\mu^\star) : \langle (x,-1),(x^\star,\mu^\star)\rangle \leq f(x)\} \end{cases}$$

Thus, the external representation of $\text{cl}(\text{epi}(f))$ is based on the intersection of a collection of closed half-spaces having $\text{dom}(f^\star)$ as index set: indeed, taking $x^\star \in \text{dom}(f^\star)$ for each half-space the normal vector is $(x^\star, -1)$ while the offset is $f^\star(x^\star)$. On the contrary, the external representation of $\text{epi}(f^\star)$ comes from the intersection of a collection of half-spaces indexed by $\text{dom}(f)$ such that, taking $x \in \text{dom}(f)$, $(x, -1)$ acts as normal vector while $f(x)$ is the offset

Another fundamental dual correspondence is provided by relating the subgradient inequality with the Fenchel inequality.

Proposition 45. *let f be a proper convex function, Let $x_0 \in ri(dom(f))$ and let $x_0^\star \in \partial f(x_0)$, then*

$$f^\star(x_0^\star) + f(x_0) = \langle x_0^\star, x_0\rangle$$

Proof. Recalling the notation used in proposition (39) for the subdifferential of f at x_0, one has that

$$\begin{cases} x^\star \in \partial f(x_0) \implies f^\star(x^\star) + f(x_0) \leq \langle x^\star, x_0\rangle \\ x \in \text{dom}(f) \implies f^\star(x^\star) + f(x) \geq \langle x^\star, x\rangle \ \forall x^\star \in \mathbb{R}^n \end{cases}$$

Which implies that:

$$(x_0, x_0^\star) \in \text{ri}(\text{dom}(f)) \times \partial f(x_0) \implies f^\star(x_0^\star) + f(x_0) = \langle x_0^\star, x_0\rangle$$

But this follows from proposition (40). Indeed, as evidenced in the proof of proposition (41), if $x_0^\star$ is a subgradient of the function computed in x_0, the most "efficient" choice for $\mu^\star$ is $\mu_0^\star = \langle (x_0^\star, -1), (x_0, f(x_0)) \rangle$ because in this way $\text{epi}(g(\cdot; x_0^\star, \mu_0^\star))$ results in a hyperplane tangent to $\text{cl}(\text{epi}(f))$, implying that $f^\star(x_0^\star)$ is exactly equal to the negative of the vertical intercept of the first order Taylor expansion of f around x_0 using $x_0^\star$ as subdifferential. □

Several convex functions can be expressed "dually" as the Fenchel conjugate of another convex function. An illustrative example is the support function $\delta^\star(\cdot|C)$ which, consistently with the notation, is the Fenchel conjugate of the indicator function $\delta(\cdot|C)$. Indeed, $\delta(x_0|C)$ is finite (equal to zero actually) if and only if $x_0 \in C$ implying that $\text{dom}(\delta(\cdot|C)) = C$. Thus, by calling $f = \delta(\cdot|C)$ one has that

$$\delta^\star(b|C) = \sup_{x \in C} \langle x, b \rangle = \sup_{x \in C} \langle x, b \rangle - \delta(x|C) = \sup_{x \in \text{dom}(f)} \langle x, b \rangle - f(x) = f^\star(b)$$

Moreover, recall that $\delta^\star(b|C)$ is the "optimal offset" such that the half-space $\{x : \langle x, b \rangle \leq \delta^\star(b|C)\}$ supports C at some point "encoded" in $\delta^\star(b|C)$.

As a trivial example, think about $C = [0, 1] \subset \mathbb{R}$. One has that:

$$\delta(x|C) = \begin{cases} 0 & \text{if } x \in [0, 1] \\ +\infty & \text{otherwise} \end{cases}$$

It follows that

$$\delta^\star(b|C) = \sup_{x \in [0,1]} xb = \begin{cases} b & \text{if } b > 0 \text{ (supremum attained at } x = 1) \\ 0 & \text{if } b \leq 0 \text{ (supremum attained at } x = 0) \end{cases} = \max(0, b)$$

Thus, picking any $b \in \mathbb{R}^n$ one has that

$$\tilde{H}(b) = \{(x, \mu) : \langle (b, -1), (x, \mu) \rangle \leq \max(0, b)\}$$

is a supporting half-space for $\text{epi}(\delta(\cdot|C))$. The position of the tangency point between $\text{epi}(\delta(\cdot|C))$ and $\tilde{H}(b)$ can be found by inspecting $\partial\delta(\cdot|C)$. Because of the triviality of the example, is possible to easily analyze the behavior of the subdifferential at each point of $\text{ri}(C)$ and of $\text{cl}(C) \setminus \text{ri}(C)$ by inspecting the subgradient inequality.
For what regards the relative interior, one has that

$$x_0 \in (0, 1) \implies \delta(x|C) \geq \delta(x_0|C) + \langle b, x - x_0 \rangle \ \forall x \in [0, 1]$$
$$0 \geq b(x - x_0) \forall x \in [0, 1] \iff b = 0$$

Indeed $b = 0$ is the solution of a homogenous system of infinitely many weak linear inequalities (one inequality for each $x \in (0, 1) \subset \mathbb{R}$): this follows from the intersection of the solution sets of the following subsystems:

$$\begin{cases} 0 \geq b(x-x_0) \forall x \in (0,x_0) \iff b \leq 0 \text{ (since } x < x_0) \\ 0 \geq b(x-x_0) \forall x \in (x_0,1) \iff b \geq 0 \text{ (since } x > x_0) \\ 0 \geq b(x-x_0) \forall x \in \{x_0\} \iff b \in \mathbb{R} \end{cases} \implies (-\infty,0] \cap [0,+\infty) \cap (-\infty,+\infty) = \{0\}$$

Instead, for what regards the relative boundary, one has that

$$\begin{aligned} x_0 = 0 &\implies \delta(x|C) \geq \delta(x_0|C) + \langle b, x-x_0 \rangle \ \forall x \in [0,1] \\ &0 \geq bx \ \forall x \in [0,1] \iff b \leq 0 \\ x_0 = 1 &\implies \delta(x|C) \geq \delta(x_0|C) + \langle b, x-x_0 \rangle \ \forall x \in [0,1] \\ &0 \geq b(x-1) \ \forall x \in [0,1] \iff b \geq 0 \end{aligned}$$

Thus, it's possible to recover the full behavior of $\partial\delta(\cdot|[0,1])$ as

$$\partial\delta(x_0|C) = \begin{cases} (-\infty,0) & \text{if } x_0 = 0 \\ \{0\} & \text{if } x_0 \in (0,1) (= x_0 \in \text{ri}(C)) \\ (0,+\infty) & \text{if } x_0 = 1 \end{cases}$$

This is insightful because one can start from a generic subgradient b_0 and recover the subdifferential of belongingness. Indeed

- when $b_0 < 0 \implies b_0 \in \partial\delta(0|C)$, meaning that $\tilde{H}(b_0) = \{(x,\mu) : \langle (b_0,-1),(x,\mu)\rangle \leq \max(0,b_0)\} = \{(x,\mu) : \langle (b_0,-1),(x,\mu)\rangle \leq 0\}$ is supporting $\text{epi}(\delta(\cdot|C))$ at $(x_1,0) = (0,0)$ (indeed, with this example, whenever $b_0 < 0$ one has that the supporting hyperplane associated with $\tilde{H}(b_0)$ is a vector subspace, being a 1-dimensional affine set passing through the origin). Thus, giving $b_0 < 0$, the half-space $\tilde{H}(b_0)$ associated with $((b_0,-1),0)$ is one of the infinitely many half-spaces passing through the origin ranging from the "horizontal" one and tending to the "vertical" one (i.e. those having $(0,-1)$ and $(-\infty,0)$ as normal vectors respectively).

- When $b_0 = 0 \implies b_0 \in \partial\delta(x_0|C)$, $x_0 \in C$, meaning that the "horizontal" half-space $\tilde{H}(0) = \{(x,\mu) : \langle (0,-1),(x,\mu)\rangle \leq 0\} = \{(x,\mu) : \langle (0,-1),(x-x_0,\mu-0)\rangle \leq 0\}$ is supporting $\text{epi}(\delta(\cdot|C))$ at $(x_0,0)$ $x_0 \in C$ (and one has actually that $\tilde{H}(0) \supset \text{Graph}(\delta(\cdot|C))$).

- When $b_0 > 0 \implies b_0 \in \partial\delta(1|C)$, meaning that $\tilde{H}(b_0) = \{(x,\mu) : \langle (b_0,-1),(x,\mu)\rangle \leq \max(0,b_0)\} = \{(x,\mu) : \langle (b_0,-1),(x,\mu)\rangle \leq b\}$ $= \{(x,\mu) : \langle (b_0,-1),(x-1,\mu-0)\rangle \leq 0\}$ is supporting $\text{epi}(\delta(\cdot|C))$ at $(x_1,0) = (1,0)$. Thus, giving $b_0 > 0$, the half-space $\tilde{H}(b_0)$ associated with $((b_0,-1),0)$ is one of the infinitely many half-spaces passing trough the origin ranging from the "horizontal" one and tending to the "vertical" one (i.e. those having $(0,-1)$ and $(+\infty,0)$ as normal vectors respectively).

This example, despite its triviality, showed how the Fenchel conjugate of a proper convex function can map generic vectors b into half-spaces properly supporting $\mathrm{cl}(\mathrm{epi}(f))$ at some generic point of its boundary $(\tilde{x}, \mathrm{cl}(f(\tilde{x})))$ which remains unknown ("encoded" in $f^\star(b)$) unless one knows fully ∂f and, noticing that $b \in \partial f(x_0)$ one could know in advance that the supporting half-space is actually tangent to $\mathrm{cl}(\mathrm{epi}(f))$ at $(x_0, \mathrm{cl}(f(x_0)))$.
Thus, knowing $f^\star$ and ∂f allows to have full control on the supporting half-spaces of $\mathrm{cl}(\mathrm{epi}(f))$

- if you want a hyperplane supporting the epigraph at $(x_0, \mathrm{cl}(f(x_0)))$ you can pick any $b \in \partial f(x_0)$ so that $(b, -1)$ is the vector normal to the supporting hyperplane while $f^\star(b)$ is its offset
- if you have a hyperplane in the form of $\{(x, \mu) : \langle (b, -1), (x, \mu) \rangle \leq f^\star(b)\}$ and you want to know at which point (or points) it supports $\mathrm{cl}(\mathrm{epi}(f))$ you can check the range of the subdifferential to see for which x_0 you have actually that $b \in \partial f(x_0)$

On the other hand, for concave functions $\hat{f}$, the subdifferential inequality has reversed inequality sign, implying that the condition defined in proposition (41) has reversed inequality sign, indeed:

$$\exists (b, \mu^\star) \in \mathbb{R}^n \times \mathbb{R} : \hat{f}(x) \leq \tilde{h}(x; b, \mu^\star) \ \forall x \in \mathrm{dom}(\hat{f})$$

This allows to understand that $\mathrm{cl}(\mathrm{hyp}(\hat{f}))$ can be drawn as the intersection of the hypographs of affine functions $\tilde{h}(\cdot; b, \mu^\star)$ (indexed by some $(b, \mu^\star)$) such that $\tilde{h}(x; b, \mu^\star) \geq \hat{f}(x) \ \forall x \in \mathrm{dom}(\hat{f})$. As before, one can define the index set $\hat{F}^\star$ of the collection of affine functions majorizing $\hat{f}$ as

$$\begin{aligned} \hat{F}^\star &= \left\{ (b, \mu^\star) : \hat{f}(x) \leq \tilde{h}(x; b, \mu^\star), \ x \in \mathrm{dom}(\hat{f}) \right\} \\ &= \bigcap_{x \in \mathrm{dom}(\hat{f})} \left\{ (b, \mu^\star) : \langle (x, -1), (b, \mu^\star) \rangle \geq \hat{f}(x) \right\} \\ &= \bigcap_{x \in \mathrm{dom}(\hat{f})} \hat{W}(x) \end{aligned}$$

where $w(b; x) = \langle b, x \rangle - \hat{f}(x)$ and $\hat{W}(x) = \mathrm{hyp}(w(\cdot; x))$. Thus, $\mathrm{cl}(\hat{f}(x))$ is the concave function induced by the set

$$\begin{aligned} F &= \bigcap_{(b, \mu^\star) \in F^\star} \mathrm{hyp}(\tilde{h}(\cdot; b, \mu^\star)) \\ &= \bigcap_{(b, \mu^\star) \in F^\star} \{(x, \mu) : \mu \leq \langle x, b \rangle - \mu^\star\} \\ &= \bigcap_{(b, \mu^\star) \in F^\star} \{(x, \mu) : \langle (b, -1), (x, \mu) \rangle \geq \mu^\star\} \end{aligned}$$

Analogously to the previous discussion, the index set $\hat{F}^\star$ of the affine functions $\tilde{h}(\cdot; b, \mu^\star)$ majorizing $\hat{f}$ results in the intersection of a collection of half-spaces indexed by $x \in \mathrm{dom}(f)$ (that is the index set of the affine functions $w(\cdot; x)$ majorizing $\hat{f}^\star$). The dual correspondence between $\mathrm{cl}(\hat{f})$ and $\hat{f}^\star$ is depicted by the fact that $\mathrm{cl}(\hat{f})$ is the concave function induced by F, the intersection of the hypographs of the affine functions $\tilde{h}(\cdot; b, \mu^\star)$ indexed by $(b, \mu^\star) \in \hat{F}^\star$, while the conjugate $\hat{f}^\star$ is the concave function (and this is the reason why it is usually defined as the *concave conjugate of* $\hat{f}$) induced directly by the index set $\hat{F}^\star$ (which, in turn, is the intersection of the hypographs of the affine functions $w(b; x) = \langle b, x\rangle - \hat{f}(x)$ indexed by $x \in \mathrm{dom}(\hat{f})$). Thus, considering $\hat{W}(x) = \mathrm{hyp}(w(\cdot; x))$, one has that

$$
\begin{aligned}
\hat{f}^\star(b) &= \sup\{\mu^\star : (b, \mu) \in \hat{F}^\star\} \\
&= \sup\{\mu^\star : (b, \mu) \in \bigcap_{x \in \mathrm{dom}(f)} \hat{W}(x)\} \\
&= \sup\{\mu^\star : (b, \mu) \in \bigcap_{x \in \mathrm{dom}(f)} \Big\{(b, \mu^\star) : \langle x, b\rangle - \hat{f}(x) \geq \mu^\star\Big\}\} \\
&= \inf_{x \in \mathrm{dom}(f)} \langle x, b\rangle - \hat{f}(x)
\end{aligned}
$$

Thus, since it's evident now that $\mathrm{hyp}(f^\star) = \hat{F}^\star$, it's possible to see that the closure of the hypograph of $\mathrm{cl}(f(x))$ is the intersection of the hypograph of a collection of affine functions where the index set is actually the hypograph of the concave conjugate, indeed:

$$
\begin{aligned}
\mathrm{cl}(\hat{f}(x)) &= \sup\{\mu : (x, \mu) \in F\} \\
&= \sup\{\mu : (x, \mu) \in \bigcap_{(b, \mu^\star) \in F^\star} \{(x, \mu) : \langle (b, -1), (x, \mu)\rangle \geq \mu^\star\}\} \\
&= \sup\{\mu : (x, \mu) \in \bigcap_{(b, \mu^\star) \in \mathrm{hyp}(f^\star)} \{(x, \mu) : \langle (b, -1), (x, \mu)\rangle \geq \mu^\star\}\} \\
&= \sup\left\{\mu : (x, \mu) \in \bigcap_{b \in \mathrm{dom}(\hat{f}^\star)} \Big\{(x, \mu) : \langle (b, -1), (x, \mu)\rangle \geq \hat{f}^\star(b)\Big\}\right\} \\
&= \sup\left\{\mu : (x, \mu) \in \bigcap_{b \in \mathrm{dom}(\hat{f}^\star)} \Big\{(x, \mu) : \langle b, x\rangle - \hat{f}^\star(b) \geq \mu\Big\}\right\} \\
&= \inf_{b \in \mathrm{dom}(\hat{f}^\star)} \langle b, x\rangle - \hat{f}^\star(b)
\end{aligned}
$$

The dual correspondences between $\mathrm{cl}(\hat{f})$ and $\hat{f}^\star$ are analogous to the ones between $\mathrm{cl}(f)$ and $f^\star$ (indeed, the only difference is the inequality sign in the subdifferential inequality and the focus on the hypograph instead of the epigraph).

Chapter 3

Constant Function Market Makers

The previous sections allowed to introduce the theoretical framework needed for deploying consciously the concepts of convex analysis in designing Constant Function Market Makers. This chapter is the core of this work and it deals with the application of such concepts. Several findings presented here are analogous to those in [ACD$^+$23] published in these days.
The toolkit presented here gives the freedom of starting from one of the four components of a CFMM and build the remaining ones. For example, one could start from designing the convex set of reachable reserves and the properties encoded in such set will affect the nature of the remaining components. On the contrary, someone might want to replicate a concave payoff in a oracle-less way [AEC21a] [AEC21b] and start from designing the portfolio value function of the CFMM, inducing the other core components via the toolkit presented here.

3.1 Definition and introduction to core components

As described in [AC20], CFMMs are a class of automated market makers (AMMs) pioneered in the last decade as a way for exchanging digital assets without the need of a trusted third-party. Several decentralized exchanges (DEXs) like Uniswap [AZR20], Balancer [MM19] or Curve [Ego21] became very popular and today they represent the main alternative to centralized exchanges for exchanging digital assets. The theory around AMMs is way older than the blockchain technology itself and, in a nutshell, it is based on the idea of allowing passive market participants to provide liquidity to a market by depositing a pair of assets in a "pool", that is an independent and deterministic driver used for holding the assets and pricing them according to a "scoring rule" mapping the total deposited amounts to marginal prices, like the "Logarithmic Market Scoring Rule" (LMSR) presented by Hanson [Han03]. In the context of blockchain, the pricing mechanism of AMMs revealed to be an effective way for oracling external prices of digital assets practiced without the need of a trusted third-party actively importing such information in a closed and decentralized ecosystem such as the blockchain. The "pools" involved in blockchain-based AMMs are simply smart contracts which, on top of deterministic and (sometimes) immutable code-written rules, regulate the provision and the withdraw of liquidity as well as the exchange mechanism of the deposited assets. Market agents adding liquidity to (i.e. depositing assets) and removing liquidity from (i.e. withdrawing assets previously deposited) the CFMM are called "liquidity providers" (LP) and have the incentive to participate in the market making activity to earn a share of the fees collected by the CFMM, proportionally to the relative amounts they deposited.

In the context of CFMMs, a "trade" correspond to the action of tendering a basket of assets to the pool (which are added to the reserves of the pool) in exchange for another basket of assets (which are withdrawn from the pool). A trade is said to be "feasible" if it satisfies the feasibility condition of the CFMM, based on the evaluation of the concave "invariant function" of the CFMM $\hat{L} : \mathbb{R}^n \to \mathbb{R}$ which maps the amounts of the reserves of the pool to a real number. Indeed, the trade is feasible if the difference between the image of the invariant function evaluating the post-trade reserves and the pre-trade reserves is non-negative. Conceiving the invariant function as a sort of "utility function" for the CFMM, it means that a trade is feasible if the utility of the CFMM (measured in "liquidity units") is not worsened after the trade is performed. Typically, in order to maximize her own utility, a trader quotes a trade such that the difference just mentioned is as close as possible to zero, because otherwise she would receive less assets than those she could receive thanks to the amounts of the assets she is tendering. For this reason, assuming that traders are utility-maximizer agents, the image of the invariant function remains the same after the trade is performed and this is the reason why some authors [AAE$^+$21] specify this as feasibility condition. However, this is the typical reason why $\hat{L}(x)$ is defined as "invariant function" and this kind of blockchain-based AMMs are defined as "Constant Function Market Makers". The invariant function is the first core component of a CFMM and it is the main driver of the pricing mechanism of the CFMM. Indeed, the "scoring rule" of the CFMM, conceived as AMM, is embedded in the invariant function since it is possible to recover the marginal price of the i-th asset (base asset) in the pool in terms of the j-th asset (quote asset) by taking the ratio between the i-th component and the j-th component of the supergradient of the invariant function evaluating the current amounts of reserves, as it will be shown later in this section. As it will discussed, defining $x_0 \in \mathbb{R}^n$ as the vector of reserves currently deposited in the CFMM, the basic set of reachable reserves C is derivable as the upper-level set of the invariant function at level one, while the set of feasible trades $T(x_0)$ corresponds to the upper-level set of the variation function of the invariant function at level zero.

The four core components of a CFMM, which are going to be described in details in the dedicated subsections, are the following

- *Basic set of reachable reserves* $C \subset \mathbb{R}^n_+$: it is a non-empty (i.e. $C \neq \emptyset$), closed (i.e. $C = \text{cl}(C)$), convex (i.e. $x_0 \in C, x_1 \in C \implies (1-\lambda)x_0 + \lambda x_1 \in C \ \forall \lambda \in [0,1]$), unbounded set (i.e.

$\nexists \epsilon > 0 : C \subset \epsilon B$) such that $0^+C = \mathbb{R}^n_+$, not containing the origin $0 \notin C$ and at least two-dimensional (i.e. $\dim(C) = \dim(\mathrm{aff}(C)) \geq 2$) describing the reserves of CFMM which can be reached by performing feasible trades when the liquidity level is equal to one.

- *Basic portfolio value function* $\hat{V}(\cdot;1) : \mathbb{R}^n_+ \to \mathbb{R}_+$: it is a concave (i.e. $p_0 \in \mathrm{hyp}(\hat{V}(\cdot;1)), p_1 \in \mathrm{hyp}(\hat{V}(\cdot;1)) \implies (1-\lambda)p_0 + \lambda p_1 \in \mathrm{hyp}(\hat{V}(\cdot;1))\ \forall \lambda \in [0,1]$), closed (thus proper, i.e. $\exists p \in \mathbb{R}^n_+ : \hat{V}(p;1) > -\infty$ and $\hat{V}(p;1) < \infty\ \forall p \in \mathbb{R}^n_+$ and upper semicontinuous i.e. $\limsup_{y \to p} \hat{V}(y;1) = \hat{V}(p;1)\ \forall p \in \mathbb{R}^n_+$), positive homogenous (i.e. $(\hat{V}\lambda)(p;1) = \hat{V}(p;1)\ \forall \lambda > 0$), non-negative (i.e. $\hat{V}(p;1) \geq 0\ \forall p \in \mathbb{R}^n_+$), non-decreasing (i.e. $p_1 \preceq p_2 \implies \hat{V}(p_1;1) \leq \hat{V}(p_2;1)$), origin-vanishing (i.e. $\hat{V}(0;1) = 0$) function describing the portfolio dynamics of the assets deposited in the CFMM. In other words, it maps the vector of prices $p \in \mathbb{R}^n_+$ of the assets deposited in the CFMM to a non-negative scalar, corresponding to the portfolio value. For liquidity providers, the basic portfolio value function describes the payoff passively replicated by providing liquidity to the pool.

- *Invariant function* $\hat{L} : \mathbb{R}^n_+ \to \mathbb{R}_+$: it shares the same properties of the portfolio value function. Indeed it is a concave (i.e. $x_0 \in \mathrm{hyp}(\hat{L}), x_1 \in \mathrm{hyp}(\hat{L}) \implies (1-\lambda)x_0 + \lambda x_1 \in \mathrm{hyp}(\hat{L})\ \forall \lambda \in [0,1]$), closed (thus proper, i.e. $\exists x \in \mathbb{R}^n_+ : \hat{L} > -\infty$ and $\hat{L} < \infty\ \forall x \in \mathbb{R}^n_+$ and upper semicontinuous i.e. $\limsup_{y \to x} \hat{L}(y) = \hat{L}(x)$), positive homogenous (i.e. $(\hat{L}\lambda)(x) = \hat{L}(x)$), non-negative (i.e. $\hat{L}(x) \geq 0\ \forall x \in \mathbb{R}^n_+$), non-decreasing (i.e. $x_0 \preceq x_1 \implies \hat{L}(x_0) \leq \hat{L}(x_1)$), origin-vanishing (i.e. $\hat{L}(0) = 0$) function. These similarities with the portfolio value function follow from the fact that both are concave functions induced by convex cones containing the origin living in the non-negative orthant $\mathbb{R}^{n+1}_+$. Moreover, as noticed in [ACD$^+$23], these two cones are in a sort of "dual" correspondence which will be furtherly discussed. Differentiability (i.e. $\exists! x^\star \in \mathbb{R}^n : \hat{L}(z) \leq \hat{L}(x_0) + \langle x^\star, z - x_0 \rangle\ \forall z \in \mathrm{dom}(\hat{L})$, or simply that $\partial \hat{L}(x) = \left\{ \nabla \hat{L}(x) \right\}\ \forall x \in \mathrm{dom}(\hat{L})$) is a desirable property for invariant functions because, on a scoring rule perspective, it ensures the uniqueness of the marginal price vector oracled by the CFMM, but it's not a strictly demanding property recalling that superdifferentiability of the invariant function is always verified on $\mathrm{ri}(\mathrm{dom}(\hat{L}))$ thanks to concavity (and so there will always be at least one oracled marginal price vector for any vector of current reserves).

- *Set of feasible trades* $T(x_0) \subset \mathbb{R}^n$: it shares several (but not all) properties of the basic set of reachable reserves. Indeed $T(x_0)$ is a non-empty (i.e. $T(x_0) \neq \emptyset$), closed (i.e. $T(x_0) = \mathrm{cl}(T(x_0))$), convex (i.e. $x_0 \in T(x_0), x_1 \in T(x_0) \implies (1-\lambda)x_0 + \lambda x_1 \in T(x_0)\ \forall \lambda \in [0,1]$) unbounded set (i.e. $\nexists \epsilon > 0 : T(x_0) \subset \epsilon B$) such that $0^+T(x_0) = \mathbb{R}^n_+$. Taking the perspective of the pool (so that positive entries are amounts tendered to the pool while negative entries are amounts withdrawn from the pool), the set of feasible trades corresponds to the set of trades which are feasible for the CFMM given the current vector of reserves x_0 (or, equivalently, given the current liquidity level $\hat{L}(x_0)$). Differently from C, $T(x_0)$ is not constrained to live in the non-negative orthant (if so, traders won't be allowed to pull out assets from the pool) and, at the same time, $0 \in T(x)\ \forall x \in \mathbb{R}^n_+$, because the null-trade (i.e. the trade which doesn't move the current reserves) must be always possible. Finally, one has also that $T(x_0) \cap \mathbb{R}^n_{--} = \emptyset$, otherwise the CFMM would allow traders to pull out assets from the pool without exchanging them with other assets (i.e. the CFMM would allow traders to withdraw assets without paying anything).

Each component completely characterizes a CFMM in the sense that the other core components are derivable from any of them, and knowing all of them allows to fully understand the math of a CFMM.

3.2 Analysis of core components

Now that the general picture of CFMMs has been introduced, it's possible to analyze each core component of a CFMM in order to recover the toolkit needed for designing a CFMM protocol. The following analysis will be dedicated to "path independent" CFMMs as defined in [AC20] being a good starting point for understanding the mechanisms behind the design of a CFMM. In a nutshell, "path independence" refers to the fact that a trader is totally indifferent between performing an aggregated feasible trade or decomposing it into sequentially feasible sub-trades since the amounts spent and the amounts collected by interacting with the pool will be the same in both cases. The existence of a fee-structure (which is, at the same time, the main remuneration schema for a liquidity provider) compromises path-independence since a small portion of the tendered assets is not added to the reserves of the CFMM but it's put apart in order to be distributed pro-rata among all the liquidity providers. Thus, the amount pulled out from the CFMM is based on the amount materially added to the pool rather than the overall amount tendered to the pool. In this subsection, the CFMM core components are introduced and a qualitative description of their main properties is provided.

3.2.1 Basic set of reachable reserves

As anticipated, the main properties of a basic set of reachable reserves C are:

- $C \subset \mathbb{R}^n_+$ (non-negativity)
- $C \neq \emptyset$ (non-emptiness)
- $C = \text{cl}(C)$ (closedness)
- $x_0 \in C, x_1 \in C \implies (1-\lambda)x_0 + \lambda x_1 \in C\ \forall \lambda \in [0,1]$ (convexity)
- $0^+C = \mathbb{R}^n_+$ ("upward" unboundedness)
- $0 \notin C$ (origin not included)

- $\dim(C) \geq 2$ (at least two-dimensionality)

The first property $C \subset \mathbb{R}^n_+$ captures the trivial fact that portfolio holdings of a CFMM are necessarily non-negative. Indeed, traders can't withdraw more than the amounts of the reserves of the assets deposited in the CFMM and, canonically, a CFMM can't borrow assets from other markets (so that a negative amount of an asset would represent a debt position for the CFMM). The non-emptiness is assumed to avoid the trivial case of a CFMM where it's not possible to perform any trade while the closedness (i.e. $C = \text{cl}(C)$) implies that $\text{cl}(C)\setminus\text{ri}(C) \subset C$, implying that the relative boundary, which corresponds to the Pareto Optimal Frontier as it will be shown, is included in the set of reachable reserves.

The condition $0^+C = \mathbb{R}^n_+$ implies that, given $x_0 \in C$, then $\{y : x_0 \preceq y\} \subset C$ and this property is called "upward closedness" in [ACD+23]. Practically, it means that any vector of reserves having at least one component higher than the index-matching component of the vector of current reserves must be considered reachable. Roughly speaking, from the point of view of the CFMM such vector is "unambiguously" better than the current vector of reserves and this is captured by the use of the generalized inequality in the set notation. Indeed, notice that this generalized inequality considers the non-negative orthant as proper cone of reference as usually occurs when the proper cone is not specified. Because of the non-negativity of prices (meaning that the set of possible prices is $\mathbb{R}^n_+$) one could interpret prices also as directions of recession of C since $0^+C = \mathbb{R}^n_+$. C is called in "set of reachable reserves" because, given $x_0 \in L(x_0)C$ such that $L(x_0) = 1$, C corresponds to the set of reserves which are "reachable" from x_0 by performing a feasible trade. In other words, assuming that the amount of liquidity of the pool remains unchanged (i.e. no LP deposits or withdraws liquidity), this set describes all the possible future reserves of the CFMM, even if those lying on $\text{ri}(C)$ are "less efficient" than those lying on the relative boundary of C.

Indeed, as it will furtherly discussed in the subsection dedicated to the basic portfolio value function, considering $x_0 \in \text{ri}(C)$ and $p_0 \in \mathbb{R}^n_+$ as vector of prices of the n assets tradable via the CFMM, an arbitrageur would be incentivized to perform a trade in which she tenders zero assets rebalancing the reservers to $\{x_1\} = \{x : x_0 + \lambda p_0, \lambda \in \mathbb{R}\} \cap \text{cl}(C) \setminus \text{ri}(C)$, because in this way she would maximize her risk-free profit.

Notice that $0 \notin C$ otherwise arbitrageurs would be allowed to drain completely the reserves: indeed, if $0 \in C$ it would mean that rebalancing the reserves from the vector of current reserves $x_0 \in C$ to 0 would be considered a feasible trade. Moreover, $0 \notin C$ guarantees that this simple condition is verified at any positive liquidity level since $0 \notin C \implies 0 \notin \lambda C \; \forall \lambda > 0$, i.e. whenever the reserves of the CFMM are not equal to zero. On the contrary, $0 \in \lambda C \iff \lambda = 0$ which occurs when liquidity providers withdraw completely the reserves from the CFMM (i.e. when the liquidity level is null since $x_0 = 0 \implies \hat{L}(x_0) = \lambda = 0$).

C is defined as "basic" set of reachable reserves because it corresponds to the set of reachable reserves at liquidity level equal to 1, (i.e. $\hat{L}(x_0) = 1$): generally speaking, the set of reachable reserves associated with liquidity level $\lambda = \hat{L}(x_0)$ will be denoted as λC being the basic set of reachable reserves "rescaled" by a non-negative scalar equal to the image of the invariant function evaluating the current vector of reserves x_0. Indeed, as it will be discussed in details later, the closed convex cone containing the origin generated in higher dimension by C (i.e. $\text{cone}(\{(1, x) : x \in C\})$) corresponds to $\text{hyp}(\hat{L})$ and this ensures the positive homogeneity and non-negativity of $\hat{L}$. For this reason, C can be recovered as the upper-level set of $\hat{L}$ associated with level 1 and, more generally, given $\lambda = \hat{L}(x_0)$ one has always

$$\lambda C = \left\{x : \hat{L}(x) \geq \lambda\right\} = \left\{x : \hat{L}(x) \geq \hat{L}(x_0)\right\}$$

Being the hypograph of the invariant function $\hat{L}$, the cone generated by the basic set of reachable reserves C in higher dimension will be called *invariant cone* and will be denoted by

$$K_{\hat{L}} = \text{cone}(\{(1, x) : x \in C\}) \subseteq \mathbb{R}_+^{n+1}$$

(notice that this set is defined as "liquidity cone" in [ACD$^+$23]). The main properties of such cone, which encodes the main properties of the invariant function, will be described in the section dedicated to the invariant function, but it's sufficient anticipating that $K_{\hat{L}} \subseteq \mathbb{R}_+^{n+1}$ is a closed convex cone containing the origin.

Noticeably, thanks to the closedness of $\text{hyp}(\hat{L})$, for $\lambda \to 0^+$ (i.e. as the current reserves x_0 become infinitesimal since $\lambda = \hat{L}(x_0)$), one has that $\lambda C \mapsto 0^+ C = \mathbb{R}_+^n$, meaning that for very low values of liquidity (i.e. when the reserves of the CFMM are almost drained), the set of reachable reserves tends to its recession cone $\mathbb{R}_+^n$. This fact is consistent with the behavior of convex sets after infinitesimal rescaling as described in the sections at the beginning of this work but in this context it captures the very simple fact that any trade which allows the reserves to be simply non-negative would be allowed by the CFMM.

Finally, the requirement for $\dim(C) \geq 2$ is trivial because the CFMM must allow the negotiation of at least a trading pair but it is not limited to just two assets. Indeed, as in the case of Curve pools [Ego21], a single CFMM can allow for negotiations of a trading group rather just a trading pair. This also means that C admits representation in $\mathbb{R}_+^n$ with $n > 1$. This simple observation leads to a very important fact, that is that the set of reachable reserves itself is capable of inducing a convex function $\Gamma : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ such that $\text{epi}(\Gamma) = C$:

$$\Gamma = \inf \{\mu : (x, \mu) \in C\}$$

Of course, $\Gamma(\cdot)$ inherits the properties encoded in C, like properness (because $C \subset \mathbb{R}^n_+$, meaning that its epigraph doesn't contain vertical line) and lower semi-continuity (since $C = \text{cl}(C)$), implying that $\Gamma(\cdot)$ is always a closed convex function (i.e. $\Gamma(\cdot) = \text{cl}(\Gamma(\cdot))$).

3.2.2 Basic Portfolio Value Function

As anticipated, the main properties of the basic portfolio value function $V(\cdot;1)$ are

- $\hat{V}(\cdot;1) : \mathbb{R}^n_+ \mapsto \mathbb{R}_+ \; n > 1$ (non-negativity)
- $p_0 \in \text{hyp}(\hat{V}(\cdot;1)), p_1 \in \text{hyp}(\hat{V}(\cdot;1)) \implies (1-\lambda)p_0 + \lambda p_1 \in \text{hyp}(\hat{V}(\cdot;1)) \; \forall \lambda \in [0,1]$ (concavity)
- $\hat{V}(\cdot;1) = \text{cl}(\hat{V}(\cdot;1))$ (closedness)
  - $\exists p \in \mathbb{R}^n_+ : \hat{V}(p;1) > -\infty$ and $\hat{V}(p;1) < \infty \; \forall p \in \mathbb{R}^n_+$ (properness)
  - $\limsup_{y \to p} \hat{V}(y;1) = \hat{V}(p;1) \; \forall p \in \mathbb{R}^n_+$ (upper semi-continuity)
- $(\hat{V}\lambda)(p;1) = \hat{V}(p;1) \; \forall \lambda > 0$ (positive homogeneity)
- $p_1 \preceq p_2 \implies \hat{V}(p_1;1) \leq \hat{V}(p_2;1)$ (non-decreasingness)
- $\hat{V}(0;1) = 0$ (origin vanishingness)

The derivation of the portfolio value function can be performed qualitatively analyzing the activity performed by arbitrageurs as rational agents. It has been said that, given $x_0 \in C$, the whole set C defines the reserves which are reachable from x_0 by performing a feasible trade. Thus the trade itself, from the point of view of the pool, is simply a portfolio rebalancing which maps the current reserves to other reachable reserves. Keeping the idea of a feasible trade as a portfolio rebalancing of the assets deposited on the CFMM, an arbitrageur could ask herself if it's possible to perform a trade which allows her to incur in a risk-free profit by simply rebalancing the reserve amounts of the pool. Indeed, supposing that $p \in \mathbb{R}^n_+$ is a vector of prices practiced by external markets, $x_0 \in C$ is the current vector of reserves while $y \in T(x_0)$ is a feasible trade proposed to the CFMM (so that the reserves after the trade is performed would be equal to $z = x_0 + y \in C$), the profit function of an arbitrageur can be expressed as

$$\Pi(z;p) = \langle p, x_0 - z \rangle \quad z \in C$$

Indeed, if $z \in C$, $x_0 - z$ represents the net amounts that the arbitrageur was capable of pulling out from the CFMM. For example, if $x_0 - z \succeq 0$ it means that the arbitrageur managed in pulling out some assets from the CFMM without tendering any asset: this situation typically occurs when $x_0 \in \text{ri}(C)$ and the arbitrage-trade moved the reserves to some point $z \preceq x_0$; as it will be seen via the arbitrage problem, the best choice for z (in the sense that it maximizes the arbitrage profit) is $\{z\} = \{x : x_0 + \lambda p, \lambda \in \mathbb{R}\} \cap \text{cl}(C) \setminus \text{ri}(C)$, which is the vector of reserves lying on the relative boundary of C in the direction of p moving from x_0.

Noticeably, if $\langle p, x_0 - z\rangle \geq 0$, it means that the total amount spent for purchasing from an external market the assets deposited into the CFMM is lower than the total amount collected by selling to the external market the assets pulled out from the CFMM. The parametrization of the vector of prices p follows from the fact that the arbitrageur takes the prices as an exogenous information since she has not the capability of affecting external prices.

Moreover, constraining $z \in C$, it's interesting noticing that the set of prices such that the profit function is non-negative corresponds to the normal cone of C at x_0, indeed

$$\begin{aligned}\{p : \Pi(z; p) \geq 0, z \in C\} &= \{p : \langle p, x_0 - z\rangle \geq 0, z \in C\}\\ &= \{p : \langle p, z - x_0\rangle \leq 0, z \in C\}\\ &= N(x_0|C)\end{aligned}$$

This gives the additional insight according to which, when $x_0 \in \mathrm{cl}(C) \setminus \mathrm{ri}(C)$ (and this can occur since $C = \mathrm{cl}(C)$), the vectors of prices such that the profit function is non-negative (constraining $z \in C$) define at the same time the collection of hyperplanes supporting C at x_0.

Alternatively, it's possible to express the profit function as function of feasible trades. Indeed, $y \in T(x_0) \implies z = x_0 + y \in C$ by definition of feasible trade and so $-y = x_0 - z$. This means that the profit function can be rewritten as

$$\Pi(y; p, x_0) = \langle p, -y\rangle \quad y \in T(x_0)$$

And this function makes sense recalling that, differently from [ACD+23], the set of feasible trades is defined using the point of view of the pool (thus, positive entries are amounts spent by the trader and collected by the pool). At the same time, it is possible to characterize again the set of prices leading to non-negative profits as

$$\begin{aligned}\{p : \Pi(y; p, x_0) \geq 0, z \in C\} &= \{p : \langle p, -y\rangle \geq 0, y \in T(x_0)\}\\ &= \{p : \langle p, y\rangle \leq 0, z \in T(x_0)\}\\ &= N(0|T(x_0))\end{aligned}$$

This implies also that $N(x_0|C) = N(0|T(x_0))$ as it will be more clear after introducing the characterization $C = T(x_0) + x_0$.
The "arbitrage problem" as defined in [AC20] is based on the maximization of the profit function and can be framed as a linear program being the maximization of a linear function over a convex

set. Indeed, the optimal profit for the arbitrageurs corresponds to

$$\begin{aligned}\Pi^{\circledast} &= \sup_{z\in C} \Pi(z;p) \\ &= \sup_{z\in C} \langle p, x_0 - z\rangle \\ &= \langle p, x_0\rangle + \sup_{z\in C}\langle p, -z\rangle \\ &= \langle p, x_0\rangle - \inf_{z\in C}\langle p, z\rangle\end{aligned}$$

Thus, the image of the solution of the optimal problem can be written in the following form:

$$\Pi^{\circledast} = \langle p, x_0\rangle - \inf_{z\in C}\langle p, z\rangle = \sup_{y\in T(x_0)} \Pi(y;p,x_0) = \sup_{y\in T(x_0)} \langle p, -y\rangle$$

On the other hand, it is insightful expressing the optimal value of the profit function as function of the parameter $p \in \mathbb{R}^n_+$, obtaining a function of prices which could be called "optimal profit function" (defined in [ACD+23] as "arbitrage function") equal to

$$\Pi^{\circledast}(p;x_0) = \langle p, x_0\rangle - (-\delta^\star(-p|C)) = \langle p, x_0\rangle + \delta^\star(-p|C) = \delta^\star(-p|T(x_0))$$

Noticeably, the optimal profit function is a convex function corresponding to the support function of the symmetric reflection across the origin of the current set of feasible trades. In the other notation, it is expressed as sum of a linear function and a convex function, thus a sum of two convex functions which is still convex. Both notations are insightful: indeed, the second notation implies the Fenchel conjugate of the optimal profit function allows to recover the indicator function of $-T(x_0)$, indeed

$$(\Pi^{\circledast})^\star(x;x_0) = \sup_{p\in\mathbb{R}^n_+} \langle p, x\rangle - \Pi^{\circledast}(p;x_0) = \sup_{p\in\mathbb{R}^n_+} \langle p, x\rangle - \delta^\star(p|-T(x_0)) = \delta(p|-T(x_0))$$

while the other notation gives another insight: supposing market efficiency, the law of one price imposes that a certain external price $p \in \mathbb{R}^n_+$ is the same price practiced by all the external markets (otherwise there would be arbitrage opportunities) and, at the same time, the optimal profit function evaluating this unique external price should be equal to zero $\Pi^{\circledast}(p;x_0) = 0$ (otherwise, again, it would imply the existence of an arbitrage opportunity). Thus, using the other notation for the optimal profit function, according to the arbitrage-free assumption one has that

$$\Pi^{\circledast}(p;x_0) = 0 \iff \langle p, x_0\rangle = -\delta^\star(-p|C)$$

Thus, in a sense, the "observed" portfolio value of the assets deposited in the CFMM is expected to be equal to the negative of the support function of the symmetric reflection across the origin. If this is not true, then it must be that $\langle p, x_0\rangle > -\delta^\star(-p|C)$ and so it is possible to perform an arbitrage trade with positive gain since $\Pi^{\circledast}(p;x_0) > 0$ (notice that $\langle p, x_0\rangle < -\delta^\star(-p|C)$ does never occur for $x_0 \in C$ since $-\delta^\star(-p|C) = \inf_{z\in C}\langle z, p\rangle$).

Thus, whenever the actual portfolio value $\langle p, x_0\rangle$ is different from the "expected portfolio value" under arbitrage free assumption, then it's possible to perform an arbitrage trade leading to a risk-free profit equal to the difference of the two portfolio values. In other words, the decomposition of the optimal profit function shows that it's important considering $-\delta^\star(-p|C)$ as the expected CFMM portfolio value under the arbitrage-free assumption, because arbitrage forces will systematically rebalance the CFMM (incurring in risk-free profits) so that the value of the deposited assets replicates pointwisely (i.e. for any price level) the function $-\delta^\star(-p|C)$. In other words, $-\delta^\star(-p|C)$ defines the portfolio value function which is passively replicated by liquidity providers by the time they deposit their assets to the CFMM. Because of its nature, such function takes the name of "portfolio value function" and it's the second core component of a CFMM:

$$\hat{V}(p;1) = -\delta^\star(-p|C) = \inf_{z\in C}\langle p, z\rangle$$

Thus, the design of the basic set of reachable reserves C is extremely important because it deeply affects the portfolio passively replicated by liquidity providers. It's possible to characterize the portfolio value function also in terms of the set of feasible trades indeed

$$\begin{aligned} &\Pi^{\circledast}(p;x_0) = \langle p, x_0\rangle - \hat{V}(p;1) = \delta^\star(p| - T(x_0)) \\ \implies &\hat{V}(p;1) = \langle p, x_0\rangle - \delta^\star(p| - T(x_0)) \end{aligned}$$

This notation is consistent with what will be provided in the section dedicated to the set of feasible trades since

$$\begin{aligned} \hat{V}(p;1) &= \langle p, x_0\rangle - \delta^\star(p| - T(x_0)) \\ &= \langle p, x_0\rangle + \inf_{y\in T(x_0)}\langle p, y\rangle \\ &= \inf_{y\in T(x_0)}\langle p, y + x_0\rangle \\ &= \inf_{z-x_0\in T(x_0)}\langle p, z\rangle \\ &= \inf_{z\in x_0+T(x_0)}\langle p, z\rangle \\ &= -\delta^\star(-p|x_0 + T(x_0)) \end{aligned}$$

which, ones again, anticipates the characterization $C = x_0 + T(x_0)$. However, to capture the "qualitative" fact about the non-negativity of prices, from now on the portfolio value function will be conceived in the following extended real-valued sense in order to restrict its effective domain to a subset of the non-negative orthant:

$$\hat{V}(p;1) = \begin{cases} -\delta^\star(-p|C) & \text{if } p \in \mathbb{R}^n_+ \\ -\infty & \text{if } p \notin \mathbb{R}^n_+ \end{cases}$$

Being the negative of a convex function, it implies that $V(\cdot;1)$ is a concave function as noticed in [AEC21a], underlying the fact that the portfolio value function is always the pointwise infimum of a family of affine functions ranging over a convex set. On the other hand, the notation used here allows to immediately derive further information about $V(\cdot;1)$. Indeed, being the negative of the support function, the portfolio value function is also always closed and positive homogenous. Closedness follows from the fact that $V(\cdot;1)$ can be expressed as the negative of the Fenchel conjugate of $\delta(x|-C)$ while positive homogeneity depends on the fact that the portfolio value function can be seen as the concave function induced by a convex cone in $\mathbb{R}^{n+1}$ as said in proposition (34). The properties of such cone can be anticipated by looking at the behavior of $V(\cdot;1)$ as pointwise infimum of a family of linear functions indexed by C. Noticeably, prices are non-negative by their nature (i.e. $p \in \mathbb{R}^n_+$) while the basic set of reachable reserves was designed to live in the non-negative orthant (recall in fact that $C \subset \mathbb{R}^n_+$), thus:

$$\langle p, z\rangle \geq 0 \quad \forall (p,z) \in \text{dom}(\hat{V}(\cdot;1)) \times C \implies \inf_{z\in C} \langle p, z\rangle = \hat{V}(p;1) \geq 0 \quad \forall p \in \text{dom}(\hat{V}(\cdot;1))$$

Moreover, since $0 \notin C \implies \inf_{z\in C}\langle p,z\rangle = \hat{V}(p;1) = 0 \iff p = 0$. From this findings it follows immediately that this support function is also always non-negative and origin-vanishing and this implies that the convex cone containing the origin which induces the portfolio value function (i.e. $\text{hyp}(V(\cdot;1))$) is actually living inside the non-negative orthant.

The fact that $\text{hyp}(\hat{V}\cdot;1)$ is a convex cone containing the origin living in the non-negative orthant implies that $0^+\text{hyp}(\hat{V}\cdot;1) = \text{hyp}(\hat{V}\cdot;1)$. This means that the concave function induced by the recession cone of the hypograph of the basic portfolio value function is the basic portfolio value function itself. Indeed:

$$(\hat{V}0^+)(p;1) = \sup\left\{\lambda : (\lambda, x) \in 0^+\text{hyp}(\hat{V}\cdot;1)\right\} = \sup\left\{\lambda : (\lambda, x) \in \text{hyp}(\hat{V}\cdot;1)\right\} = \hat{V}(p;1)$$

The zero upper-level set of this function (which in the convex case takes the name of recession function [Roc70] as seen in the previous sections) corresponds to the set of directions in which $\hat{V}(\cdot;1)$ is non-decreasing (differently from the convex case, where the zero sub-level set of the recession function corresponds to the directions of recession of the associated convex function, i.e. the directions in which the function is non-increasing). Indeed given $\hat{g}$ concave, using a derivation analogous to that one used in the previous sections for obtaining the recession cone of the epigraph of a convex function, one has that the recession cone of the hypograph of a concave function $\hat{g}$ is given by

$$\begin{aligned}
0^+\text{hyp}(\hat{g}) &= \{(\mu,x) : (\hat{\mu}+\mu, \hat{x}+x) \in \text{hyp}(\hat{g}),\ (\hat{\mu},\hat{x}) \in \text{hyp}(\hat{g})\} \\
&= \{(\mu,x) : (\hat{\mu}+\mu, \hat{x}+x) \in \{(\hat{\mu},\hat{x}) : \hat{\mu} \leq \hat{g}(\hat{x})\},\ (\hat{\mu},\hat{x}) \in \text{hyp}(\hat{g})\} \\
&= \{(\mu,x) : \hat{\mu}+\mu \leq \hat{g}(\hat{x}+x),\ (\hat{\mu},\hat{x}) \in \text{hyp}(\hat{g})\} \\
&= \{(\mu,x) : \mu \leq \hat{g}(\hat{x}+x) - \hat{\mu},\ (\hat{\mu},\hat{x}) \in \text{hyp}(\hat{g})\} \\
&= \{(\mu,x) : \mu \leq \hat{g}(\hat{x}+x) - \hat{g}(\hat{x}),\ \hat{x} \in \text{dom}(\hat{g})\}
\end{aligned}$$

And it follows that the concave function induced by $0^+\text{hyp}(\hat{g})$ is

$$\begin{aligned}(\hat{g}0^+)(x) &= \sup\left\{\mu : (\mu, x) \in 0^+\text{hyp}(\hat{g})\right\} \\ &= \inf_{\hat{x}\in\text{dom}(\hat{g})} \hat{g}(\hat{x}+x) - \hat{g}(\hat{x})\end{aligned}$$

This implies that, given $\hat{x} \in \text{dom}(\hat{g})$, one has that $(\hat{g}0^+)(x) \leq \hat{g}(\hat{x}+x) - \hat{g}(\hat{x})\ \forall x$. Thus, if $(\hat{g}0^+)(x) \geq 0 \implies \hat{g}(\hat{x}+x) - \hat{g}(\hat{x}) \geq 0\ \forall \hat{x} \in \text{dom}(\hat{g})$ implying that the zero upper-level set of $\hat{g}0^+$ defines the directions in which $\hat{g}0^+$is "asymptotically" non-decreasing

$$\left\{x : (\hat{g}0^+)(x) \geq 0\right\} = \{x : \hat{g}(\hat{x}+x) \geq \hat{g}(\hat{x}), \hat{x} \in \text{dom}(\hat{g})\}$$

In the case of the portfolio value function, $(\hat{V}0^+)(p;1) = \hat{V}(p;1)$ implying that the zero-upper level set of $\hat{V}(p;1)$ corresponds to the directions in which $\hat{V}(p;1)$ is non-decreasing. However, being non-negative and origin-vanishing, one has actually that

$$\left\{p : (\hat{V}0^+)(p;1) \geq 0\right\} = \left\{p : \hat{V}(p;1) \geq 0\right\} = \text{dom}(V(\cdot;1))$$

And this implies that the basic portfolio value function $\hat{V}(\cdot;1)$ is non-decreasing on its maximal domain. Indeed, picking $p_1 \in \text{dom}(V(\cdot;1))$ and $p_0 \in \text{dom}(V(\cdot;1))$ so that $p_2 = p_0 + p_1$, one has that $p_2 \succeq p_1$ and that $\hat{V}(p_2;1) = \hat{V}(p_0+p_1;1) \geq \hat{V}(p_1;1)$ since $p_0 \in \left\{p : (\hat{V}0^+)(p;1) \geq 0\right\}$.

As previously discussed, the hypograph of the portfolio value function is a convex cone living in the non-negative orthant: thus, the portfolio value function itself is expected to be *gauge-like* in the sense of [Roc70] indeed $\hat{V}(0;1) = \inf \hat{V}(\cdot;1)$ and all the upper-level sets of $\hat{V}(\cdot;1)$ are proportional. Thus, analogously to the reasoning behind the set of reachable reserves, one could start from the upper level set of the portfolio value function at (liquidity) level one and then induce the upper-level set associated to all the other non-negative liquidity levels (because of the non-negativity of the portfolio value function). One could call such set $C^\star$ and it would define the set of prices such that the portfolio value function, associated to liquidity level 1, is higher than one.

$$\begin{aligned}C^\star &= \left\{p : \hat{V}(p;1) \geq 1\right\} \\ &= \left\{p : \inf_{z\in C}\langle p, z\rangle \geq 1\right\} \\ &= \{p : \langle p, z\rangle \geq 1,\ z \in C\} \\ &= \bigcap_{z\in C} \{p : \langle p, z\rangle \geq 1\}\end{aligned}$$

In some works [Zaf12] the set $\{p : \langle p, z\rangle \geq 1,\ z \in C\}$ is defined as the "reverse polar" of C and it is introduced as a different notion of polarity. Indeed, the convex set $C^\star$ reminds somehow the polar $C^\circ = \{p : \langle p, z\rangle \leq 1,\ z \in C\}$ but it is actually different since the inequality sign is reversed. On the other hand, both $C^\star$ and C° are closed convex sets being the intersection of closed half-spaces but C° is always origin-including while $C^\star$ is not. Since $C^\star \subset \text{dom}(\hat{V}) \subset \mathbb{R}^n_+$, $C^\star$ has all the properties seen previously with C. Since the portfolio is gauge-like, one could recover the hypograph of the portfolio value function by generating a cone in higher dimension from $C^\star$: such cone will be called *portfolio value cone* and will be denoted by

$$K_{\hat{V}} = \text{cone}(\{(1,x) : x \in C^\star\}) \subset \mathbb{R}^{n+1}$$

Since $V(\cdot;1) = -\delta^\star(\cdot| - C)$ one would expect that $\text{hyp}(V(\cdot;1)) = A(K^\star)$ where $A : (\lambda, x) \mapsto (-\lambda, x)$ as described in proposition (34). However, accounting also for the non-negativity of $V(\cdot;1)$, the hypograph of such function should include only the positive part of $A(K^\star)$, i.e. $A(K^\star) \cap \{(\lambda, x) : \lambda > 0, x \in \mathbb{R}^n\}$. Thus, one should expect that

$$\text{hyp}(V(\cdot;1)) = K_{\hat{V}} = \text{cone}(\{(1,x) : x \in C^\star\}) = A(K^\star) \cap \{(\lambda, x) : \lambda > 0, x \in \mathbb{R}^n\}$$

Indeed

$$\begin{aligned}
\text{cone}(\{(1,p) : p \in C^\star\}) &= \{(\lambda, p) : p \in \lambda C^\star, \lambda > 0\} \\
&= \{(\lambda, p) : \langle p, x\rangle \geq \lambda, x \in C, \lambda > 0\} \\
&= \{(\lambda, p) : \langle p, \mu x\rangle \geq \lambda\mu, x \in C, \lambda > 0, \mu > 0\} \\
&= \{(\lambda, p) : \langle p, x\rangle \geq \lambda\mu, x \in \mu C, \lambda > 0, \mu > 0\} \\
&= \{(\lambda, p) : \langle(-\lambda, p), (\mu, x)\rangle \geq 0, x \in \mu C, \lambda > 0, \mu > 0\}
\end{aligned}$$

And, recalling that $K_{\hat{L}} = \text{cone}(\{(1,x) : x \in C\})$ one has that

$$\text{cone}(\{(1,p) : p \in C^\star\}) = \left\{(\lambda, p) : \langle(-\lambda, p), (\mu, x)\rangle \geq 0, \lambda > 0, (\mu, x) \in K_{\hat{L}}\right\}$$

And since $K^\star_{\hat{L}} = \left\{(\lambda, p) : \langle(\lambda, p), (\mu, x)\rangle \geq 0, (\mu, x) \in K_{\hat{L}}\right\}$, recalling that $A : (\lambda, x) \mapsto (-\lambda, x)$, one has that

$$K_{\hat{V}} = \text{cone}(\{(1,p) : p \in C^\star\}) = A(K^\star_{\hat{L}}) \cap \{(\lambda, x) : \lambda > 0, x \in \mathbb{R}^n\}$$

Thus, it's possible to recover the hypograph of the portfolio value function by applying a linear map to the dual invariant cone (i.e. the dual of the hypograph of the invariant function, which is a convex cone).

As mentioned at the beginning of this subsection, $\hat{V}(\cdot;1)$ denotes the basic portfolio value function, i.e. the portfolio value function associated to liquidity level 1. However, it's possible to define the portfolio value function associated to any liquidity level $\lambda > 0$ as

$$\hat{V}(p;\lambda) = \inf_{z \in \lambda C} \langle p, z\rangle = -\delta^\star(-p|\lambda C)$$

But because of the positive-homogeneity of the portfolio value function this resembles to the left scalar multiplication (by $\lambda > 0$) of the basic portfolio value function, indeed

$$\begin{aligned}\hat{V}(p;\lambda) &= \inf_{z\in\lambda C}\langle p, z\rangle \\ &= \inf_{z\in C}\langle p, \lambda z\rangle \\ &= \lambda \inf_{z\in C}\langle p, \lambda z\rangle \\ &= (\lambda\hat{V})(p;1)\end{aligned}$$

Implying that the portfolio value function associated with any liquidity level $\lambda > 0$ is actually the function induced by rescaling the range of the portfolio value function by factor $\lambda > 0$

3.2.3 Invariant function

As anticipated, the main properties of the invariant function $\hat{L}$ are:

- $\hat{L} : \mathbb{R}^n_+ \mapsto \mathbb{R}_+ \; n > 1$ (non-negativity)
- $a_0 \in \text{hyp}(\hat{L}), a_1 \in \text{hyp}(\hat{L}) \implies (1-\lambda)a_0 + \lambda a_1 \in \text{hyp}(\hat{L}) \; \forall\lambda \in [0,1]$ (concavity)
- $\hat{L} = \text{cl}(\hat{L})$ (closedness)
  - $\exists x \in \mathbb{R}^n_+ : \hat{L}(x) > -\infty$ and $\hat{L}(x) < \infty \; \forall x \in \mathbb{R}^n_+$ (properness)
  - $\limsup_{y\to x} \hat{L}(y) = \hat{L}(x) \; \forall x \in \mathbb{R}^n_+$ (upper semi-continuity)
- $(\hat{L}\lambda)(x) = \hat{L}(x) \; \forall\lambda > 0$ (positive homogeneity)
- $x_1 \preceq x_2 \implies \hat{L}(x_1) \leq \hat{L}(x_2)$ (non-decreasingness)
- $\hat{L}(0) = 0$ (origin vanishingness)

The central role of the invariant function is due to the fact that it embeds the entire mechanics of the CFMM as a decentralized exchange of digital goods. Indeed, conceivable as a sort of utility function for the CFMM, it disciplines the way of providing and withdrawing liquidity to and from the CFMM as well as the trades performed by market agents against the pool of assets. The invariant function is the concave function induced by the invariant cone. Indeed, as mentioned in the section dedicated to the basic set of reachable reserves, calling $K_{\hat{L}} = \text{cone}(\{(1,x) : x \in C\})$, one has that

$$\hat{L}(x) = \sup\left\{\mu : (x,\mu) \in K_{\hat{L}}\right\}$$

Noticeably, the invariant function shares the same properties of the portfolio value function, but this is because of the similarity between the invariant cone $K_{\hat{L}}$ and the portfolio value cone $K_{\hat{V}}$. Indeed, both are closed convex cones containing the origin living in the non-negative orthant. Actually, the similarities between such cones (which encode the same properties of the invariant function and portfolio value function) follow from the fact that the convex sets generating such cones, which are respectively C and $C^\star$, have the same properties. Indeed, any non-empty unbounded closed convex set not containing the origin living in the non-negative orthant is capable of generating both the invariant cone and the portfolio value cone. However, if the final goal is to induce an invariant function, the convex set of reference D is allowed to be bounded: however, such set will not correspond to the basic set of reachable reserves since the upper-level set of the invariant function at level one will be necessarily unbounded. More precisely, D will meet C for the "lower boundary part", i.e. for the basic set of efficient reserves, which corresponds to the set of minimal points of C w.r.t of the non-negative orthant as proper cone of reference. An example of a bounded convex set D capable of generating the invariant function is the Euclidean unit ball in $\mathbb{R}^2$ centered in the point $(1,1)$

$$D = \left\{(\xi_1,\xi_2) : \sqrt{(\xi_1-1)^2+(\xi_2-1)^2} \leq 1\right\}$$

Indeed, apart from unboundedness, D shares all the properties said for the basic set of reachable reserves. Thus, it's possible to create a cone which is not going to be exactly the hypograph of the invariant function, but it's still possible to induce a invariant function indeed:

$$\begin{aligned}
\hat{L}(x) &= \inf\{\lambda : (\lambda, x) \in \text{cone}(\{(1,x) : x \in D\})\} \\
&= \inf\{\lambda : \lambda > 0, x \in \lambda D\} \\
&= \left\{\lambda : \lambda > 0, \sqrt{(\xi_1-\lambda)^2+(\xi_2-\lambda)^2} \leq \lambda\right\} \\
&= \inf\{\lambda : \lambda > 0, \lambda^2 - 2\lambda(\xi_1+\xi_2) + \xi_1^2 + \xi_2^2 \leq 0\} \\
&= \inf\left\{\lambda : \lambda > 0, \lambda \in \left[\xi_1+\xi_2-\sqrt{2\xi_1\xi_2},\ \xi_1+\xi_2+\sqrt{2\xi_1\xi_2}\right]\right\} \\
&= \xi_1+\xi_2+\sqrt{2\xi_1\xi_2}
\end{aligned}$$

Which indeed is a non-negative, non-decreasing, closed, origin vanishing, positive homogenous concave function. Thus, one has that the basic set of reachable reserves is equal to

$$C = \left\{(\xi_1,\xi_2) : \xi_1+\xi_2+\sqrt{2\xi_1\xi_2} \geq 1\right\}$$

Which of course is different from the original set D, even if they share the same lower boundary (i.e. set of efficient reserves).

The invariant function embeds the scoring rule used by the CFMM conceived as AMM: indeed, considering $x_0 \in \mathbb{R}^n_+$ as the vector of the current reserves of the CFMM, under the *arbitrage free assumption* (so that x_0 lies actually on the boundary of λC being the point of tangency between λC and the supporting hyperplane having the vector of external prices p as normal vector) the superdifferential $\partial\hat{L}(x_0)$ contains the "implicit" marginal prices oracled by the CFMM according to the current value of those reserves. Using the i-th asset as quote asset (where $i \in [1,n]$), the *marginal price function* $\Xi(\cdot;x_0,i) : \partial\hat{L}(x_0) \to \mathbb{R}^{n-1}_+$ it evaluates a generic supergradient of the invariant function (evaluating x_0) and returns the vector of marginal prices of the other $n-1$ assets expressed in terms of the i-th asset. Indeed, picking $(\xi_1, \dots, \xi_i, \dots, \xi_n) \in \partial \hat{L}(x_0)$ the marginal price function corresponds to the following map

$$\Xi(\cdot; x_0, i) : (\xi_1, \dots, \xi_i, \dots, \xi_n) \in \partial \hat{L}(x_0) \mapsto \left(\frac{\xi_1}{\xi_i}, \dots, \frac{\xi_n}{\xi_i} \right)$$

The concavity of the invariant function ensures its superdifferentiability at any point $x_0 \in \mathrm{ri}(\mathrm{dom}(\hat{L}))$ (proposition (35)) and so, given any vector of current reserves x_0, it's always possible to apply the marginal price function to obtain at least one vector of oracled prices. Moreover, it's interesting noticing that $\partial \hat{L}(0^+) = \mathbb{R}^n_+$ implying that in presence of a CFMM that is about to be completely drained in its reserves, the prices oracled by the CFMM are all the possible prices in nature (indeed, in this context the set of reachable reserves corresponds to $0^+C = \mathbb{R}^n_+$ and all the hyperplanes ranging from the "vertical" to the "horizontal" one are supporting $0^+C = \mathbb{R}^n_+$ at the origin). Thus, without further specifications, a CFMM implementing an invariant function with the properties mentioned above is always capable of oracling at least one marginal price vector of the n deposited assets but, at the same time, oracling more than one marginal price vector could create confusion if the purpose is also that one of importing price information from external markets. Indeed, an invariant function might not be differentiable and this would cause disambiguation in correspondence of vector of current reserves where the invariant function is not differentiable and the degree of disambiguation (i.e. proliferation of multiple oracled prices) is directly proportional to the degree of non-differentiability of the function in that point. Thus, to avoid disambiguation, differentiability is a desirable property for invariant functions (even if it's not strictly demanded). Recall that, being differentiable, it means that

$$\exists! x^\star \in \mathbb{R}^n : \hat{L}(z) \leq \hat{L}(x_0) + \langle x^\star, z - x_0 \rangle \ \forall z \in \mathrm{dom}(\hat{L})$$

or simply that $\partial \hat{L}(x) = \left\{ \nabla \hat{L}(x) \right\} \ \forall x \in \mathrm{dom}(\hat{L})$. In this context, the oracled marginal price will be always unique and, under the arbitrage free assumption, it oracles the prices practiced by external markets unambiguously.

3.2.4 Set of feasible trades

As anticipated, the main properties of the set of feasible trades $T(x_0) \subset \mathbb{R}^n$ are:

- $T(x_0) \neq \emptyset$ (non-emptiness)
- $T(x_0) = \mathrm{cl}(T(x_0))$ (closedness)
- $y_0 \in T(x_0), y_1 \in T(x_0) \implies (1 - \lambda) y_0 + \lambda y_1 \ \forall \lambda \in [0, 1]$ (convexity)
- $0^+T(x_0) = \mathbb{R}^n_+$ ("upward" unboundedness)
- $0 \in T(x_0)$ (origin included)

- $\dim(T(x_0)) \geq 2$ (at least two-dimensionality)
- $T(x_0) \cap \mathbb{R}^n_{--} = \emptyset$ (disjointedness from the negative orthant)

Most of the properties of the set of feasible trades are the same seen with the basic set of reachable reserves. However, differently from C, $T(x_0)$ is not constrained to live in $\mathbb{R}^n_+$ and, at the same time $0 \in T(x_0)$. The reason for the first difference is due to the fact that negative entries of $y \in T(x_0)$ corresponds to quantities received by the trader from the pool: thus, if $T(x_0)$ was enforced to live in the non-negative orthant, there wouldn't be any feasible trade in which the trader pulls out some assets from the pool and so the CFMM would lose its function of decentralized exchange of digital goods. On the other hand, the inclusion of the origin in such set is due to the fact the null trade is always considered feasible, since there isn't the risk of decreasing the utility of the CFMM expressed in terms of liquidity units. Finally, disjointedness from the negative orthant follows from the fact that otherwise traders will be allowed (and incentivized) to pull out assets from the pool without the need of exchanging them with other assets.
However, on a geometrical perspective, the set of feasible trades corresponds to a simple "shift" of the basic set of reachable reserves and, more precisely, it corresponds to the upper-level set at level zero of the variation function of the invariant function parameterizing the current vector of reserves $x_0 \in \mathbb{R}^n_+$. Of course, the set of feasible trades is strictly dependent on the local information about the current vector of reserves deposited in the CFMM, because this vector encodes a certain liquidity value and, at the same time, defines of the boundaries of the amounts which can be withdrawn from the pool. This is the reason why, denoting $C \subset \mathbb{R}^n_+$ as the basic set of reachable reserves, $x_0 \in \mathbb{R}^n_+$ it's possible to define the set of feasible trades via the following set-valued notation:

$$T(x_0) = C - x_0$$

Indeed, this definition is simply mapping all the points of the set of reachable reserves in the contrarian direction of the current vector of reserves x_0, because traders are not allowed to pull out from the pool more than what it's currently available.
The introduction of a notation for the set of feasible trades allows to define more precisely the meaning of *sequentially feasible trades* as stated in the definition of a path-independent CFMM. Indeed, a vector $(\Delta_1, \dots, \Delta_m) \in \mathbb{R}^{n \times m}$ is said to be a vector of *sequentially feasible trades* if

$$\Delta_i \in T\left(x_0 + \sum_{j=1}^{i-1} \Delta_j\right) \ \forall i = 1, \dots, m$$

implying that the i-th trade remains feasible even after performing all the previous $i - 1$ trades.

Another possible notation of the set of feasible trades is in terms of invariant function, which could be considered as "the set of traders which doesn't worsen" the utility of the CFMM expressed in liquidity terms. Thus, one has also

$$T(x_0) = \left\{ y : \hat{L}(x_0 + y) \geq \hat{L}(x_0) \right\}$$

Alternatively, one could use the concave variation function, i.e. the function induced by shifting the hypograph of $\hat{L}$ so that the point $(x_0, \hat{L}(x_0))$ is mapped into the origin:

$$P(y; x_0) = \hat{L}(x_0 + y) - \hat{L}(x_0)$$

This allows inducing the one-sided directional derivative of the invariant function, as well as the directions in which the invariant function is non-decreasing (what was seen as recession cone with convex function, with reversed meaning) and finally the set of feasible trades.

$$\hat{L}'(x_0; y) = \lim_{\lambda \to 0^+} \frac{\hat{L}(x_0 + \lambda y) - \hat{L}(x_0)}{\lambda} = \lim_{\lambda \to 0^+} (P\lambda)(y; x_0)$$
$$(L\hat{0}^+)(y) = \lim_{\lambda \to \infty} \frac{\hat{L}(x_0 + \lambda y) - \hat{L}(x_0)}{\lambda} = \lim_{\lambda \to \infty} (P\lambda)(y; x_0)$$

Now it's possible to characterize the upper-level sets at level zero of such functions recalling [Roc70] that the one-sided directional derivative of a concave function is the concave support function of the superdifferential of the function (evaluating $x_0 \in \mathbb{R}^n$), while the concave recession function is equal to the concave support function of the effective domain of the concave conjugate.

$$\begin{aligned}
\left\{ y : \lim_{\lambda \to 0^+} (P\lambda)(y; x_0) \geq 0 \right\} &= \left\{ y : \hat{L}'(x_0; y) \geq 0 \right\} \\
&= \left\{ y : -\delta^\star(-y | \partial \hat{L}(x_0)) \geq 0 \right\} \\
&= \left\{ y : \langle y, x \rangle \geq 0, x \in \partial \hat{L}(x_0) \right\} \\
\{ y : P(y; x_0) \geq 0 \} &= T(x_0) \\
\left\{ y : \lim_{\lambda \to \infty} (P\lambda)(y; x_0) \geq 0 \right\} &= \left\{ y : (L\hat{0}^+)(y) \geq 0 \right\} \\
&= \left\{ y : -\delta^\star(-y | \mathrm{dom}(\hat{L}^\star)) \geq 0 \right\} \\
&= \left\{ y : \langle y, x \rangle \geq 0, x \in \mathrm{dom}(\hat{L}^\star) \right\} \\
&= \{ y : \langle y, x \rangle \geq 0, x \in C^\star \}
\end{aligned}$$

Noticeably, $\left\{ y : \langle y, x \rangle \geq 0, x \in \partial \hat{L}(x_0) \right\}$ is a polyhedron being the intersection of a collection of half-spaces passing through the origin and indexed by the supergradients of the invariant function evaluated at x_0: if $\hat{L}$ is differentiable, this resembles to a half-space supporting $T(x_0)$ at the origin (since both include the origin). Moreover, under the arbitrage free assumption, the supergradient oracles the external prices and if y is infinitesimal (i.e. lives in the neighborhood of 0), one has actually that $y \in T(x_0)$ and $y \in \left\{ y : \langle y, x \rangle \geq 0, x \in \partial \hat{L}(x_0) \right\}$. On the contrary, because of the properties of $C^\star$, one has always that $\{ y : \langle y, x \rangle \geq 0, x \in C^\star \} = \mathbb{R}^n_+ = 0^+C$ that is the recession cone of the basic set of reachable reserves.

As mentioned in the subsection dedicated to the portfolio value function, it's possible to characterize the "optimal profit function" $\Pi^{\circledast}$ in terms of the set of feasible trades $T(x_0)$, indeed $\Pi^{\circledast}(p; x_0) = \delta^{\star}(p| - T(x_0))$. This function is called "arbitrage function" (denoted with $\text{arb}(c)$) in [ACD+23] and it's implicitly defined as the support function of $T(x_0)$: also in this case, the difference here is that in this work it is taken the perspective of the pool and not of the trader, thus the positive entries of $y \in T(x_0)$ are amounts spent by the arbitrageur (and not collected, as in [ACD+23]). One could introduce the *trading cone* as

$$K_T(x_0) = \text{cone}(\{(1, x) : x \in T(x_0)\})$$

So that, recalling proposition (34), one has actually

$$\Pi^{\circledast}(p; x_0) = \delta^{\star}(p| - T(x_0)) = \inf \{\lambda : (\lambda, p) \in K_T^{\star}(x_0)\}$$

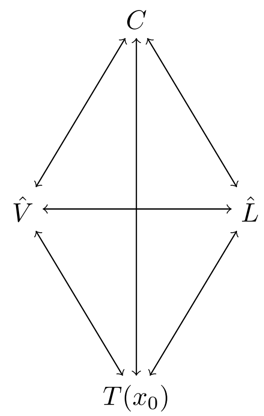
Meaning that an arbitrageur could have a screener of CFMM pools so that, provided external price information $p_0 \in \mathbb{R}^n_+$ and the current vector of reserves $x_0 \in \mathbb{R}^n_+$, she could spot arbitrage opportunities by evaluating persistently the optimal profit function and trigger an arbitrage trade if $\Pi^{\circledast}(p_0; x_0) > 0$.

Besides the induction of the optimal profit function, the set of feasible trades $T(x_0)$ is useful for visualizing the positive externalities generated by more LPs depositing assets to the pool. Indeed, as x_0 increases (so that the variation of $\hat{L}(x_0)$ is non-negative because of the non-decreasing behavior of L), on a graphical perspective the set $T(x_0)$ gets "wider" in the sense that, fixing a certain amount deposited to the pool it's possible to pull out a higher amount of assets from the pool.
To appreciate this fact on a quantitative perspective, taking advantage of the fact that $\dim(T(x_0)) \geq 2$ it's possible to consider $T(x_0)$ as the epigraph of a certain closed convex function called *get-amount-out function* $\Omega : \mathbb{R}^{n-1} \to \mathbb{R}$ which is very useful for quoting trades (as always, positive entries are paid by traders while negative quantities are collected by the trader)

$$\Omega(x; x_0) = \inf \{\mu : (x, \mu) \in T(x_0)\}$$

3.3 Designing a CFMM

The previous subsection defined the core components of a CFMM and their properties with some extensions as hints for understanding how they are all related together. This subsection instead introduces some propositions which, as a whole, provide a toolkit for designing CFMMs so that, starting from one component all the other core components are immediately induced.



C	$\hat{V}$
• $\hat{L}(x) = \sup\{\lambda > 0 : x \in \lambda C\}$ • $\hat{V}(p;1) = -\delta^\star(-p\|C)$ • $T(x_0) = C - x_0$	• $C = \mathrm{dom}(\hat{V}^\star)$ • $\hat{L}(x) = \inf_{p \succ 0} \frac{\langle p,x\rangle}{\hat{V}(p;1)}$ • $T(x_0) = \left\{y : \hat{V}^\star(x_0+y) \geq 0\right\}$
$\hat{L}$	$T(x_0)$
• $C = \left\{x : \hat{L}(x) \geq 1\right\}$ • $\hat{V}(p;1) = \inf_{x \succ 0} \frac{\langle p,x\rangle}{\hat{L}(x)}$ • $T(x_0) = \left\{y : \hat{L}(x_0+y) - \hat{L}(x_0) \geq 0\right\}$	• $C = x_0 + T(x_0)$ • $\hat{L}(x) = \sup\{\lambda > 0 : x \in \lambda(x_0 + T(x_0))\}$ • $\hat{V}(p;1) = -\delta^\star(-p\|x_0 + T(x_0))$

3.3.1 Inducing core components from C

Proposition 46. *Let $C \subset \mathbb{R}^n$ be a basic set of reachable reserves of a CFMM. Then, calling $K_{\hat{L}} = cone(\{(1,x) : x \in C\})$, the invariant function $\hat{L} : \mathbb{R}^n_+ \to \mathbb{R}_+$ of that CFMM is equal to*

$$\hat{L}(x) = \sup \left\{ \lambda : (\lambda, x) \in K_{\hat{L}} \right\}$$

In other words, $\hat{L}$ is the positive homogenous concave function generated by $\hat{f}(x) = -\delta(x|C) + 1$

Proof. Being a closed convex cone containing the origin fully included in $\mathbb{R}^n_+$, $K_{\hat{L}}$ is complaint to be the hypograph of a closed, concave, non-negative, non-decreasing, origin vanishing, positive homogenous concave function, i.e. an invariant function. Noticeably, the concave function induced by cone$(\{(1,x) : x \in C\})$ is equal to the concave function induced by the Conic hull of the hypograph of another concave function $\hat{f}$, implying that the induced function is "generated" by $\hat{f}$ according to proposition (29). By picking

$$\hat{f}(x) = -\delta(x|C) + 1 = \begin{cases} 1 & x \in C \\ -\infty & x \notin C \end{cases}$$

one has that

$$\begin{aligned} \text{cone}(\text{hyp}(\hat{f})) &= \bigcup_{\lambda \geq 0} \left\{ (\mu, x) : (\mu, x) \in \lambda \text{hyp}(\hat{f}) \right\} \\ &= \left\{ (\mu, x) : \exists \lambda \geq 0, (\mu, x) \in \lambda \text{hyp}(\hat{f}) \right\} \\ &= \{ (\mu, x) : \exists \lambda \geq 0, x \in \lambda C, \lambda \geq \mu \} \end{aligned}$$

Implying that

$$\begin{aligned} \sup \left\{ \mu : (\mu, x) \in \text{cone}(\text{hyp}(\hat{f})) \right\} &= \sup \{ \mu : \exists \lambda \geq 0, x \in \lambda C, \lambda \geq \mu \} \\ &= \sup \{ \lambda : \exists \lambda \geq 0, x \in \lambda C \} \\ &= \sup \left\{ \lambda : (\lambda, x) \in \bigcup_{\lambda \geq 0} \{ (\lambda, x) : x \in \lambda C \} \right\} \\ &= \sup \{ \lambda : (\lambda, x) \in \text{cone}(\{(1,x) : x \in C\}) \} \\ &= \sup \left\{ \lambda : (\lambda, x) \in K_{\hat{L}} \right\} \end{aligned}$$

And this means that

$$\hat{L} = \sup_{\lambda > 0} (\hat{f}\lambda)(x)$$

where $\hat{f}(x) = -\delta(x|C) + 1$ □

Proposition 47. *Let $C \subset \mathbb{R}^n$ be a basic set of reachable reserves of a CFMM. Let $K_{\hat{L}} = cone(\{(1,x) : x \in C\})$ and let $C^\star = \{p : \langle p, x \rangle \geq 1, x \in C\}$.*

Then, calling $K_{\hat{V}} = cone(\{(1,x) : x \in C^\star\})$, the basic portfolio value function $\hat{V} : \mathbb{R}^n_+ \to \mathbb{R}_+$ of that CFMM is equal to

$$\hat{V}(p;1) = \sup\left\{\lambda : (\lambda, p) \in K_{\hat{V}}\right\}$$

In other words, $\hat{V}$ is the positive homogenous concave function generated by $\hat{g}(p) = -\delta(p|C^\star) + 1$. Moreover, considering the linear map $A : (\lambda, x) \mapsto (-\lambda, x)$, one has that

$$K_{\hat{V}} = A(K^\star_{\hat{L}}) \cap \{(\lambda, x) : \lambda > 0, x \in \mathbb{R}^n\}$$

Proof. The first part of the proposition is analogous to proposition (46) since invariant function and portfolio value function share the same properties encoded in their respective hypographs. However, for the second part, as seen in the subsection dedicated to the basic portfolio value function, one has that

$$\begin{aligned}
\text{cone}(\{(1,p) : p \in C^\star\}) &= \{(\lambda, p) : p \in \lambda C^\star, \lambda > 0\} \\
&= \{(\lambda, p) : \langle p, x\rangle \geq \lambda, x \in C, \lambda > 0\} \\
&= \{(\lambda, p) : \langle p, \mu x\rangle \geq \lambda\mu, x \in C, \lambda > 0, \mu > 0\} \\
&= \{(\lambda, p) : \langle p, x\rangle \geq \lambda\mu, x \in \mu C, \lambda > 0, \mu > 0\} \\
&= \{(\lambda, p) : \langle(-\lambda, p), (\mu, x)\rangle \geq 0, x \in \mu C, \lambda > 0, \mu > 0\}
\end{aligned}$$

And, recalling that $K_{\hat{L}} = \text{cone}(\{(1,x) : x \in C\})$ one has that

$$\text{cone}(\{(1,p) : p \in C^\star\}) = \left\{(\lambda, p) : \langle(-\lambda, p), (\mu, x)\rangle \geq 0, \lambda > 0, (\mu, x) \in K_{\hat{L}}\right\}$$

And since $K^\star_{\hat{L}} = \left\{(\lambda, p) : \langle(\lambda, p), (\mu, x)\rangle \geq 0, (\mu, x) \in K_{\hat{L}}\right\}$, recalling that $A : (\lambda, x) \mapsto (-\lambda, x)$, one has that

$$K_{\hat{V}} = \text{cone}(\{(1,p) : p \in C^\star\}) = A(K^\star_{\hat{L}}) \cap \{(\lambda, x) : \lambda > 0, x \in \mathbb{R}^n\}$$

□

Proposition 48. *Let $C \subset \mathbb{R}^n$ be a basic set of reachable reserves of a CFMM and let $C^\star = \{p : \langle p, x\rangle \geq 1, x \in C\}$. Then, the basic portfolio value function $\hat{V} : \mathbb{R}^n_+ \to \mathbb{R}_+$ and the invariant function $\hat{L} : \mathbb{R}^n_+ \to \mathbb{R}_+$ of that CFMM are equal to*

$$\begin{aligned}
\hat{V}(p;1) &= \sup\{\lambda > 0 : p \in \lambda C^\star\} = -\delta^\star(-p|C) \\
\hat{L}(x) &= \sup\{\lambda > 0 : x \in \lambda C\} = -\delta^\star(-x|C^\star)
\end{aligned}$$

Moreover, one has that

$$\hat{V}(p;1)\hat{L}(x) \leq \langle p, x\rangle \quad \forall (p,x) \in \mathbb{R}^{2n}$$

Proof. The proof here is analogous to the proof of proposition (30). Indeed, the goal is to show that the gauge-like function $\sup\{\lambda > 0 : x \in \lambda C\}$ is actually the negative of the support function of $C^\star$. This underlines a relationship between C and $C^\star$ analogous to the "polar" relationship between C and C°. Proposition (46) showed that $\hat{L}$ is actually the positive homogenous concave function generated by $\hat{f}(x) = -\delta(x|C)+1$ and, following the proof of proposition (30), its negative is expected to be the support function of the symmetric reflection across the origin of the upper-level set at level zero $\left\{x^\star : \hat{f}^\star(x^\star) \geq 0\right\}$ where $\hat{f}^\star$ denotes the concave conjugate of $\hat{f}$:

$$\begin{aligned}\hat{f}^\star(x^\star) = \inf_{x\in\mathbb{R}^n} \langle x, x^\star\rangle - \hat{f}(x) = \inf_{x\in\mathbb{R}^n} \langle x, x^\star\rangle + \delta(x|C) - 1 \\ = -1 - \sup_{x\in\mathbb{R}^n} \langle x, -x^\star\rangle - \delta(x|C) \\ = -1 - \delta^\star(-x^\star|C)\end{aligned}$$

This implies that the upper-level set of $\hat{f}^\star$ at level zero is actually $C^\star$, indeed

$$\begin{aligned}\left\{x^\star : \hat{f}^\star(x^\star) \geq 0\right\} &= \{x^\star : -1 - \delta^\star(-x^\star|C) \geq 0\} \\ &= \left\{x^\star : \inf_{x\in C} \langle x, x^\star\rangle \geq 1\right\} \\ &= \{x^\star : \langle x, x^\star\rangle \geq 1, x \in C\} \\ &=: C^\star\end{aligned}$$

And, as anticipated, since $\hat{L}(x) = \sup_{\lambda>0}(\hat{f}\lambda)(x)$ according to proposition (46) one has that

$$\hat{L}(x) = -\delta^\star\left(x^\star| - \left\{x^\star : \hat{f}^\star(x^\star) \geq 0\right\}\right) = -\delta^\star(x^\star| - C^\star) = -\delta^\star(-x^\star|C^\star)$$

The proof for the basic portfolio value function moves from proposition (47) and it is analogous to the proof just shown for the invariant function: the only thing to be proved is that $C^{\star\star} = C$, which in the case of polar set C° it is true only if C contains the origin (recall in fact that C is closed by definition of basic set of reachable reserves).

$$\begin{aligned}C^{\star\star} &= \{x : \langle x, x^\star\rangle \geq 1, x^\star \in C^\star\} \\ &= \{x : -\delta^\star(-x^\star|C^\star) \geq 1\} \\ &= \{x : \sup\{\lambda \geq 0 : x \in \lambda C\} \geq 1\} \\ &= \{x : \exists\lambda \geq 1, x \in \lambda C\} \\ &= C\end{aligned}$$

Indeed, since $0^+C = \mathbb{R}^n_+$ and $C \subset \mathbb{R}^n_+$, one has that $C \supset \lambda C\ \forall\lambda > 1$ (because λC means summing C to itself λ times and $C \subset 0^+C$). Indeed, one has that

$$\exists\lambda \geq 1 : x \in \lambda C \implies x \in C$$

Now that it's proved that $C^{\star\star} = C$, one has that

$$\sup\{\lambda > 0 : p \in \lambda C^\star\} = -\delta^\star(-p|C^{\star\star}) = -\delta^\star(-p|C)$$

Since $\hat{L}$ is the concave function induced by $K_{\hat{L}} = \text{cone}(\{(1,x) : x \in C\})$ while, calling $A : (\lambda, x) \mapsto (-\lambda, x)$, $\hat{V}$ is the concave function induced by $K_{\hat{V}} = \text{cone}(\{(1,p) : p \in C^\star\}) = A(K_{\hat{L}}^\star) \cap \{(\lambda, x) : \lambda > 0, x \in \mathbb{R}^n\}$ (proposition (47)), one can apply proposition (33) to conclude that

$$\hat{V}(p;1)\hat{L}(x) \leq \langle p, x\rangle \quad \forall (p,x) \in \mathbb{R}^{2n}$$

□

Proposition 49. *Let $C \subset \mathbb{R}^n$ be a basic set of reachable reserves of a CFMM. Let $x_0 \in cl(C) \setminus ri(C)$ be the vector of current reserves in the CFMM. Then, the set of feasible trades $T(x_0)$ is equal to*

$$T(x_0) = C - x_0$$

Proof. The fact that $x_0 \in \text{cl}(C) \setminus \text{ri}(C)$ implies implicitly the arbitrage-free assumption (indeed, an arbitrage profit it's possible if $x_0 \in \text{ri}(C)$). Thus, because of the arbitrage-free assumption, it can be safely argued that the current liquidity level is equal to one, i.e. $\hat{L}(x_0) = 1$. Analogously, if $x_0 \in \text{cl}(\lambda C) \setminus \text{ri}(\lambda C)$ for some $\lambda \geq 0$, according to the arbitrage-free assumption one has that $\hat{L}(x_0) = \lambda$.

The feasibility condition implies that the trading activity maps the current vector of reserves x_0 to a new vector of reserves $x_0 + y$ belonging to any set λC with $\lambda \geq \hat{L}(x_0) = 1$ (so that the utility of the pool, measured in liquidity units λ, does not decrease). However, as remIndeed in the proof of proposition (48), since $0^+C = \mathbb{R}^n_+$ and $C \subset \mathbb{R}^n_+$, one has that $\hat{L}(x_0)C \supset \lambda C \ \forall \lambda > \hat{L}(x_0) = 1$, meaning that the set of feasible trades (i.e. the set of y such that the feasibility condition holds) can be written as

$$T(x_0) = \{y : x_0 + y \in C\}$$

By calling $z = x_0 + y$, one has that

$$\begin{aligned} T(x_0) &= \{z - x_0 : z \in C\} \\ &= C - x_0 \end{aligned}$$

Moreover, since $x_0 \in C$, this notation shows immediately that $0 \in T(x_0)$ as stated in the definition of set of feasible trades. □

3.3.2 Inducing core components from $\hat{V}$

Proposition 50. *Let $\hat{V} : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a portfolio value function of a CFMM. Then, calling $\hat{V}^\star$ the concave conjugate of $\hat{V}$, the basic set of reachable reserves $C \subseteq$ is equal to*

$$C = dom(\hat{V}^\star)$$

Analogously, considering $\hat{L} : \mathbb{R}^n_+ \to \mathbb{R}_+$ *as the invariant function of the CFMM and calling* $\hat{L}^\star$ *the concave conjugate of* $\hat{L}$*, one has that*

$$C^\star = dom(\hat{L}^\star)$$

Proof. In general, since $\delta^\star(x^\star|C) = \sup_{x\in C}\langle x, x^\star\rangle$, one has that

$$x \in C \implies \delta^\star(x^\star|C) \geq \langle x, x^\star\rangle \quad \forall x^\star \in \mathbb{R}^n$$

And this can be interpreted as the set condition of C (recalling that $C = \text{cl}(C)$ by definition) since

$$C = \{x : \delta^\star(x^\star|C) \geq \langle x, x^\star\rangle, x^\star \in \mathbb{R}^n\}$$

And from here it's possible to recover C as the effective domain of the indicator function of C (which is indeed the Fenchel conjugate of the support function of C)

$$\begin{aligned} C &= \{x : -\langle x, x^\star\rangle + \delta^\star(x^\star|C) \geq 0, x^\star \in \mathbb{R}^n\} \\ &= \left\{x : \sup_{x^\star\in\mathbb{R}^n} \langle x, x^\star\rangle - \delta^\star(x^\star|C) \leq 0\right\} \\ &= \{x : \delta(x|C) \leq 0\} \\ &= \text{dom}(\delta(\cdot|C)) \end{aligned}$$

On the contrary, since $-\delta^\star(-x^\star|C) = \inf_{x\in C}\langle x, x^\star\rangle$, one has that

$$x \in C \implies -\delta^\star(-x^\star|C) \leq \langle x, x^\star\rangle \quad \forall x^\star \in \mathbb{R}^n$$

Implying that the set condition becomes

$$C = \{x : -\delta^\star(-x^\star|C) \leq \langle x, x^\star\rangle, x^\star \in \mathbb{R}^n\}$$

According to proposition (48) one has that $V(p;1) = -\delta^\star(-p|C)$, implying that the set C can be rewritten as

$$C = \left\{x : \hat{V}(p;1) \leq \langle x, p\rangle, p \in \mathbb{R}^n_+\right\}$$

And this set can be characterized in terms of the concave conjugate of $\hat{V}$ since

$$\begin{aligned} C &= \left\{x : \hat{V}(p;1) - \langle x, p\rangle \leq 0, p \in \mathbb{R}^n_+\right\} \\ &= \left\{x : \inf_{p\in\mathbb{R}^n_+} \langle x, p\rangle - \hat{V}(p;1) \geq 0,\right\} \\ &= \left\{x : \hat{V}^\star(x) \geq 0,\right\} \\ &= \text{dom}(\hat{V}^\star) \end{aligned}$$

Indeed, since $\hat{V}$ is the negative of a support function, its concave conjugate $\hat{V}^\star$ turns out to be the negative of a indicator function (for Uniswap V2 this can be proved via the arithmetic mean-geometric mean inequality).

The proof for the invariant function is analogous since $\hat{L}(x) = -\delta^\star(-x|C^\star) = \inf_{p\in C^\star}\langle x, x^\star\rangle$ implying that

$$C^\star = \left\{p : \hat{L}(x) \leq \langle x, p\rangle, x \in \mathbb{R}^n_+\right\}$$

Which leads to

$$C^\star = \text{dom}(\hat{L}^\star)$$

An alternative proof could be based on interpreting $\hat{L}(x)$ as the negative of the support function of $-C^\star$ (i.e. $\hat{L}(x) = -\delta^\star(x| - C^\star)$) so that it's possible to recover $-C^\star$ as

$$-C^\star = \left\{p : \langle x, p\rangle \leq -\hat{L}(x), x \in \mathbb{R}^n\right\}$$

Meaning that

$$\begin{aligned}
-(-C^\star) = C^\star &= \left\{-p : \langle x, p\rangle \leq -\hat{L}(x), x \in \mathbb{R}^n\right\} \\
&= \left\{p : \langle x, -p\rangle \leq -\hat{L}(x), x \in \mathbb{R}^n\right\} \\
&= \left\{p : \langle x, p\rangle - \hat{L}(x) \geq 0, x \in \mathbb{R}^n\right\} \\
&= \left\{p : \inf_{x\in\mathbb{R}^n} \langle x, p\rangle - \hat{L}(x) \geq 0\right\} \\
&= \left\{p : \hat{L}^\star(p) \geq 0\right\} \\
&= \text{dom}(\hat{L}^\star)
\end{aligned}$$

□

Proposition 51. *Let $\hat{V} : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a portfolio value function of a CFMM. Then, the invariant function $\hat{L} : \mathbb{R}^n_+ \to \mathbb{R}_+$ of that CFMM is equal to*

$$\hat{L}(x) = \inf_{p\succ 0} \frac{\langle p, x\rangle}{\hat{V}(p; 1)}$$

Proof. The "reverse-polar" relationship between C and $C^\star$, which are the convex sets generating in higher dimension the hypographs of $\hat{L}$ and $\hat{V}$ respectively, lead to the inequality presented in proposition (48):

$$\hat{V}(p; 1)\hat{L}(x) \leq \langle p, x\rangle \quad \forall(p, x) \in \mathbb{R}^{2n}$$

Thus, because of the non-negativity of $\hat{V}(p; 1)$ for $p \in \text{dom}(\hat{V}) = \mathbb{R}^n_+$ (in fact, since $\hat{V}$ is defined in the extended value sense, $p \notin \mathbb{R}^n_+ \implies \hat{V}(p; 1) = -\infty$), one can write

$$\hat{L}(x) \leq \frac{\langle p, x\rangle}{\hat{V}(p; 1)} \quad \forall(p, x) \in \mathbb{R}^n_+ \times \mathbb{R}^n$$

Thus, recalling that $\text{hyp}(\hat{L})$ is

$$\text{hyp}(\hat{L}) = \left\{(\lambda, x) : \lambda \leq \hat{L}(x)\right\}$$

One can actually rewrite it as

$$\text{hyp}(\hat{L}) = \left\{(\lambda, x) : \lambda \leq \frac{\langle p, x\rangle}{\hat{V}(p;1)}, p \in \mathbb{R}^n_{++}\right\}$$

Setting $p \in \mathbb{R}^n_{++}$ (which is equivalent to the use of the generalized inequality $p \succ 0$ setting $\mathbb{R}^n_{++}$ as proper cone of reference) makes sure that $\hat{V}(p;1) > 0$, so that the induced function $L(x)$ takes finite values, indeed

$$\begin{aligned}\hat{L}(x) &= \sup\left\{\lambda : (\lambda, x) \in \text{hyp}(\hat{L})\right\} \\ &= \sup\left\{\lambda : \lambda \leq \frac{\langle p, x\rangle}{\hat{V}(p;1)}, p \succ 0\right\} \\ &= \inf_{p \succ 0} \frac{\langle p, x\rangle}{\hat{V}(p;1)}\end{aligned}$$

□

Proposition 52. *Let $\hat{V} : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a portfolio value function of a CFMM. Let $x_0 \in \mathbb{R}^n_+$ be the vector of current reserves in the CFMM living on the relative boundary of the unknown basic set of reachable reserves. Then, the set of feasible trades $T(x_0)$ is equal to*

$$T(x_0) = dom(\hat{V}^\star) - x_0 = \left\{y : \hat{V}^\star(x_0 + y) \geq 0\right\}$$

Proof. According to the feasibility condition, the trading activity is not allowed to lower the liquidity level of the pool. Recalling proposition (49), when $x_0 \in C$ the set of feasible trades is $T(x_0) = C - x_0$. Analogously, picking a generic liquidity level $\lambda > 0$ so that $x_0 \in \lambda C$, one has that

$$T(x_0) = \lambda C - x_0$$

As discussed in the proof of proposition (49), assuming that x_0 is living on the relative boundary of the unknown set of reachable reserves implies that the arbitrage-free assumption is holding, that the current liquidity level is $\lambda = L(x_0)$ and it is equal to one since x_0 belongs to the basic set of reachable reserves

$$T(x_0) = C - x_0$$

Applying proposition (50) one has that

$$C = \text{dom}(\hat{V}^\star) = \left\{x : \hat{V}^\star(x) \geq 0\right\}$$

And plugging this notation in the previous one one has that

$$T(x_0) = \text{dom}(\hat{V}^\star) - x_0$$

Which can be expanded into

$$\begin{aligned} T(x_0) &= \left\{x : \hat{V}^\star(x) \geq 0\right\} - x_0 \\ &= \left\{x : \hat{V}^\star(x) \geq 0\right\} - x_0 \\ &= \left\{x - x_0 : \hat{V}^\star(x) \geq 0\right\} \\ &= \left\{y : \hat{V}^\star(x_0 + y) \geq 0\right\} \end{aligned}$$

□

3.3.3 Inducing core components from $\hat{L}$

Proposition 53. *Let $L : \mathbb{R}^n_+ \to \mathbb{R}_+$ be an invariant function of a CFMM. Then, the basic set of reachable reserves $C \subseteq \mathbb{R}^n$ is equal to*

$$C = \left\{x : \hat{L}(x) \geq 1\right\}$$

Moreover, considering $\hat{V} : \mathbb{R}^n_+ \to \mathbb{R}_+$ as the portfolio value function of the CFMM, one has that

$$C^\star = \left\{p : \hat{V}(p; 1) \geq 1\right\}$$

Proof. C is the upper-level set of the invariant function at liquidity level one since the hypograph of $\hat{L}$ is the invariant cone $K_{\hat{L}}$, i.e. the Conic hull of $\{(1, x) : x \in C\}$ which makes $\hat{L}$ a gauge-like function. Indeed, the set of reachable reserves associated with any liquidity level $\lambda = \hat{L}(x_0) \geq 0$ can be recovered as

$$\lambda C = \left\{x : \hat{L}(x) \geq \lambda\right\}$$

One could characterize the invariant cone both as the Conic hull of $\{(1, x) : x \in C\}$

$$K_{\hat{L}} = \{(\lambda, x) : \lambda \geq 0, x \in \lambda C\}$$

But also as the hypograph of $\hat{L}$:

$$K_{\hat{L}} = \left\{(\lambda, x) : \lambda \leq \hat{L}(x)\right\}$$

Thus

$$\{(\lambda, x) : \lambda \geq 0, x \in \lambda C\} = \left\{(\lambda, x) : \lambda \leq \hat{L}(x)\right\}$$

The first notation shows explicitly that one can recover $\{(1, x) : x \in C\}$ by taking a cross-section of the invariant cone, intersecting it with the hyperplane $\{(1, x) : x \in \mathbb{R}^n\}$ indeed

$$\begin{aligned} K_{\hat{L}} \cap \{(1, x) : x \in \mathbb{R}^n\} &= \{(\lambda, x) : \lambda \geq 0, x \in \lambda C\} \cap \{(1, x) : x \in \mathbb{R}^n\} \\ &= \{(1, x) : x \in C\} \end{aligned}$$

Which implies that

$$\begin{aligned}\{(1,x) : x \in C\} &= \left\{(\lambda, x) : \lambda \leq \hat{L}(x)\right\} \cap \{(1,x) : x \in \mathbb{R}^n\} \\ &= \left\{(1,x) : 1 \leq \hat{L}(x)\right\}\end{aligned}$$

Meaning that

$$C = \left\{x : \hat{L}(x) \geq 1\right\}$$

The proof for $C^\star$ is analogous recalling that $K_{\hat{V}}$ is both the Conic hull of $\{(1,p) : p \in C^\star\}$ and the hypograph of $\hat{V}$. □

Proposition 54. *Let $L : \mathbb{R}^n_+ \to \mathbb{R}_+$ be an invariant function of a CFMM. Then, the portfolio value function $\hat{V} : \mathbb{R}^n_+ \to \mathbb{R}_+$ of that CFMM is equal to*

$$\hat{V}(p;1) = \inf_{x \succ 0} \frac{\langle p, x\rangle}{\hat{L}(x)}$$

Proof. Analogous to proposition (51) □

Proposition 55. *Let $L : \mathbb{R}^n_+ \to \mathbb{R}_+$ be an invariant function of a CFMM. Let $x_0 \in \mathbb{R}^n_+$ be the vector of current reserves in the CFMM living on the relative boundary of the unknown set of reachable reserves at some liquidity level $\lambda > 0$. Then, under the arbitrage-free assumption, the set of feasible trades $T(x_0)$ is equal to*

$$T(x_0) = \left\{y : \hat{L}(x_0 + y) \geq \hat{L}(x_0)\right\}$$

Or alternatively, using the variation function $P(y;x_0) = \hat{L}(x_0 + y) - \hat{L}(x_0)$

$$T(x_0) = \{y : P(y;x_0) \geq 0\}$$

Proof. As discussed in the proof of proposition (52), picking $\lambda > 0$ so that $x_0 \in \lambda C$, one has that

$$T(x_0) = \lambda C - x_0$$

The arbitrage-free assumption makes sure that $x_0 \in \mathrm{cl}(\lambda C) \setminus \mathrm{ri}(\lambda C)$ so that $\lambda = \hat{L}(x_0)$ (indeed, if $x_0 \in \mathrm{ri}(\lambda C)$, an arbitrage is possible and $\lambda < \hat{L}(x_0)$). This implies that under the arbitrage-free assumption the set of feasible trades can be rewritten as

$$T(x_0) = \hat{L}(x_0)C - x_0$$

Differently from proposition (52), here $\hat{L}(x_0)$ is not necessarily equal to one because x_0 is said to be on the relative boundary of the unknown set of reachable reserves at some liquidity level $\lambda > 0$ which is not necessarily equal to one (in this case, the set of reachable reserves would be equal to the basic set of reachable reserves). Now, applying proposition (53) one has that

$$\hat{L}(x_0)C = \left\{y : \hat{L}(y) \geq \hat{L}(x_0)\right\}$$

Plugging this notation inside the previous one, one has that

$$
\begin{aligned}
T(x_0) &= \left\{x : \hat{L}(x) \geq \hat{L}(x_0)\right\} - x_0 \\
&= \left\{x - x_0 : \hat{L}(x) \geq \hat{L}(x_0)\right\} \\
&= \left\{y : \hat{L}(y + x_0) \geq \hat{L}(x_0)\right\}
\end{aligned}
$$

Trivially, defining $P(y; x_0) = \hat{L}(y + x_0) - \hat{L}(x_0)$, one has that

$$
\begin{aligned}
T(x_0) &= \left\{y : \hat{L}(y + x_0) \geq \hat{L}(x_0)\right\} \\
&= \left\{y : \hat{L}(y + x_0) - \hat{L}(x_0) \geq 0\right\} \\
&= \{y : P(y; x_0) \geq 0\}
\end{aligned}
$$

□

3.3.4 Inducing core components from $T(x_0)$

Proposition 56. *Let $x_0 \in \mathbb{R}^n_+$ be the vector of current reserves in a CFMM living on the relative boundary of the unknown basic set of reachable reserves. Let $T(x_0)$ be the set of feasible trades. Then, the basic set of reachable reserves $C \subseteq \mathbb{R}^n$ is equal to*

$$C = x_0 + T(x_0)$$

Proof. Trivial from proposition (49) □

Proposition 57. *Let $x_0 \in \mathbb{R}^n_+$ be the vector of current reserves in a CFMM living on the relative boundary of the unknown basic set of reachable reserves. Let $T(x_0)$ be the set of feasible trades. Then, the invariant function $\hat{L} : \mathbb{R}^n_+ \to \mathbb{R}_+$ of that CFMM is equal to*

$$\hat{L}(x) = \sup \{\lambda > 0 : x \in \lambda(x_0 + T(x_0))\}$$

Proof. According to proposition (56), it's possible to define the invariant cone as

$$K_{\hat{L}} = \{(\lambda, x) : \lambda \geq 0, x \in \lambda(x_0 + T(x_0))\}$$

And, since this cone is the hypograph of $\hat{L}$, according to proposition (46) one has that

$$\hat{L}(x) = \sup \{\lambda > 0 : x \in \lambda(x_0 + T(x_0))\}$$

□

Proposition 58. *Let $x_0 \in \mathbb{R}^n_+$ be the vector of current reserves in a CFMM living on the relative boundary of the unknown basic set of reachable reserves. Let $T(x_0)$ be the set of feasible trades. Then, the basic portfolio value function $\hat{V} : \mathbb{R}^n_+ \to \mathbb{R}_+$ of that CFMM is equal to*

$$\hat{V}(p; 1) = -\delta^\star(-p | x_0 + T(x_0))$$

Proof. Trivial by applying proposition (56) and recalling proposition (48) □

3.3.5 Equivalent CFMMs

In [AC20] the invariant function is characterized as non-negative, non-decreasing, origin-vanishing quasiconcave function $\hat{\phi}$ because this ensures that its upper-level sets behave like sets of reachable reserves. At the same time, starting from $\hat{\phi}$, it is said that every quasiconcave function obtained from $\hat{g} = h \circ \hat{\phi}$, where h is a non-decreasing transformation, is an equivalent CFMM in the sense that the upper-level sets of $\hat{g}$ are proportional to those of $\hat{\phi}$. This means that the set of reachable reserves induceable from $\hat{g}$ are equal to those induceable from $\hat{\phi}$ at a different liquidity level with constant proportionality. This creates an "equivalence" between $\hat{g}$ and $\hat{\phi}$ as invariant functions because there won't be any difference in the mechanics of the CFMM by picking one or the other.

Adding positive-homogeneity to the characterization of invariant functions reduces ambiguity in defining the class of equivalent CFMMs. Indeed, one could map each non-negative, non-decreasing, origin-vanishing quasiconcave function to a precise invariant function as defined in this work. This means that one could consider the set of invariant functions as index set of a collection of sets where each indexed set corresponds to a class of non-negative, non-decreasing, origin-vanishing quasiconcave functions which are equivalent if deployed as invariant functions for a CFMM.

Proposition 59 (Uniqueness of the invariant function). *Let $\hat{\phi}$ be any non-negative, non-decreasing, origin-vanishing quasiconcave function. Then, given a certain $k > 0$, there exist a unique invariant function $\hat{L} : \mathbb{R}^n_+ \to \mathbb{R}_+$ such that, calling $D = \left\{x : \hat{\phi}(x) \geq 1\right\}$ and $C = \left\{x : \hat{L}(x) \geq 1\right\}$, one has that*

$$D = kC$$

Implying that

$$K_{\hat{L}} = cone(\{(1, x) : x \in C\}) = cone(\{(k, x) : x \in D\})$$

Or, alternatively, that

$$\hat{L}(x) = k \sup \{\lambda > 0 : x \in \lambda D\}$$

Proof. Because of quasiconcavity, the upper-level set $D = \left\{x : \hat{\phi}(x) \geq 1\right\}$ is a convex set and the fact that $\hat{\phi}$ is non-negative and non-decreasing makes D a possible set of reachable reserves. On a geometrical perspective, it's always possible to recover D as a cross-section at level one of the cone $K_{\hat{\varphi}} = \{(\lambda, x) : \lambda > 0, x \in \lambda D\} \subset \mathbb{R}^{n+1}$. The positive-homogenous concave function induced by $K_{\hat{\varphi}}$ is equal to

$$\hat{\varphi}(x) = \sup \{\lambda > 0 : x \in \lambda D\}$$

Fixing a certain $k > 0$, one can define

$$C = \frac{1}{k} D$$

So that $\hat{\varphi}(x)$ can be rewritten as

$$\begin{aligned}\hat{\varphi}(x) &= \sup\{\lambda > 0 : x \in \lambda k C\} \\ &= \sup\left\{\frac{\lambda}{k} > 0 : x \in \lambda C\right\} \\ &= \frac{1}{k}\sup\{\lambda > 0 : x \in \lambda C\}\end{aligned}$$

By calling $\hat{L}(x) = \sup\{\lambda > 0 : x \in \lambda C\}$, one has that $C = \left\{x : \hat{L}(x) \geq 1\right\}$ and one can rewrite the previous notation as

$$\hat{\varphi}(x) = \frac{\hat{L}(x)}{k}$$

Or alternatively

$$\hat{L}(x) = k\hat{\varphi}(x) = k \sup\{\lambda > 0 : x \in \lambda D\}$$

□

The same reasoning can be applied to portfolio value functions as well sharing the same properties of invariant functions. Indeed, applying the toolkit presented in the previous subsection one could design a CFMM starting from the portfolio value function passively replicated by liquidity providers rather than starting from the invariant function. However, every "equivalent" portfolio value function will correspond to the same positive-homogeneous invariant function at a different liquidity level. In other words, one could design a non-negative, non-decreasing, quasiconcave function of price describing the total value of the assets deposited on the pool at every price level which could be achieved once that the CFMM reaches a sufficiently large liquidity level according to the corresponding positive-homogeneous portfolio value function

Proposition 60 (Uniqueness of the invariant function). *Let $\hat{\phi}$ be any non-negative, non-decreasing, origin-vanishing quasiconcave function. Then, given a certain $k > 0$, there exist a unique basic portfolio value function $\hat{V} : \mathbb{R}^n_+ \to \mathbb{R}_+$ such that, calling $D = \left\{p : \hat{\phi}(p) \geq 1\right\}$ and $C^\star = \left\{p : \hat{V}(p;1) \geq 1\right\}$, one has that*

$$D = kC^\star$$

Implying that

$$K_{\hat{V}} = cone(\{(1,p) : p \in C^\star\}) = cone(\{(k,p) : p \in D\})$$

Or, alternatively, that

$$\hat{V}(p;1) = k \sup\{\lambda > 0 : p \in \lambda D\}$$

Proof. Analogous to proposition (59) □

Defining $\hat{\phi} : \mathbb{R}^n_+ \to \mathbb{R}^n$ as a generic non-negative, non-decreasing, origin-vanishing quasiconcave function, proposition (59) allows to characterize the collection of equivalent invariant functions (indexed by a generic positive-homogenous invariant function $\hat{L}_i : \mathbb{R}^n_+ \to \mathbb{R}_+$) as

$$E_{\hat{L}_i} = \left\{ \phi : \exists k > 0 \left\{ x : \hat{\phi}(x) \geq 1 \right\} = k \left\{ x : \hat{L}(x) \geq 1 \right\} \right\}$$

Thus, given a certain positive-homogenous invariant function $\hat{L}_0$, picking $\phi_1 \in E_{\hat{L}_0}$ or $\phi_2 \in E_{\hat{L}_0}$ is totally equivalent in the sense that the upper-level sets of ϕ_1 and ϕ_2 are proportional to each other being different cross-sections of the same invariant cone $K_{\hat{L}_0}$ (i.e. hyp($\hat{L}_0$)).

In general, given $\phi \in E_{\hat{L}_0}$, proposition (59) suggests that the the most straightforward way for recovering the associated positive-homogenous invariant function $\hat{L}_0$ is to induce a concave function from the Conic hull of $\{(1, x) : x \in D\}$, where D is the upper-level set of ϕ at level one, and to remove from the induced function the positive rescaling factor k.

At the same time, defining $\mathcal{K}$ the set of non-negative, non-decreasing, origin-vanishing, closed, positive-homogenous concave functions (which are not simply proportional), one can define the collection of invariant functions as

$$\left\{ E_{\hat{L}_i} : \hat{L}_i \in \mathcal{K} \right\}$$

Since invariant functions and portfolio value functions share the same properties, one could overwrite the notation just used for portfolio value functions as well, so that the set of equivalent portfolio value functions can be defined as

$$E_{\hat{V}_i} = \left\{ \phi : \exists k > 0 \left\{ p : \hat{\phi}(p; 1) \geq 1 \right\} = k \left\{ p : \hat{V}(p; 1) \geq 1 \right\} \right\}$$

And the collection of portfolio value functions as

$$\left\{ E_{\hat{V}_i} : \hat{V}_i \in \mathcal{K} \right\}$$

3.3.6 Impermanent loss

For liquidity providers, depositing assets on a path-independent CFMM implies inevitably the replication of a concave portfolio value function. The concave nature of the replicated payoff isn't desirable for liquidity providers because it implies that the portfolio value function is sub-linear. This means that, in the long run, the portfolio value function will grow slower and decrease faster than the linear growth and decay of the constant portfolio induced by naive holding of the assets. This is explained by the concavity of the portfolio value function but it can be intuitively understood from the fact that arbitrageurs and traders in general have the incentive to pull out from the pool the most expensive asset and to deposit the cheapest one according to the price practiced by external markets. For liquidity providers this means that the exposure of their portfolio is marginally increasing on the worst performing assets and marginally decreasing on the best performing ones. This phenomenon is called impermanent loss and it is a typical argument in favor of the "inefficiency" related to liquidity provisioning. Practitioners usually refer to the impermanent loss formula as the relative performance of the portfolio value function of the CFMM benchmarked against the constant portfolio induced by naive holding of the assets. This formula can be induced directly from proposition (48) recalling that

$$\hat{V}(p;\lambda) = -\delta^{\star}(-p|\lambda C) = \inf_{x\in\lambda C}\langle x, p\rangle$$

Implying that

$$x_0 \in \lambda C \implies \hat{V}(p;\lambda) \leq \langle x_0, p\rangle \ \forall p \in \mathbb{R}^n$$

Which, recalling that $\langle x_0, p\rangle$ is non-negative since both reserves and prices are non-negative, one can actually rewrite the expression as

$$x_0 \in \lambda C \implies \frac{\hat{V}(p;\lambda)}{\langle x_0, p\rangle} - 1 \leq 0 \ \forall p \in \mathbb{R}^n_+$$

One could call $I(p;x_0) = \frac{\hat{V}(p;\lambda)}{\langle x_0,p\rangle} - 1$ the *impermanent loss function* which, defining intuitively $\hat{V}_N(p;x_0) = \langle p, x_0\rangle$ as the portfolio value function associated with the naive holding of the asset, can be rewritten as the relative performance of the portfolio value function of the CFMM benchmarked against the constant portfolio induced by naive holding of the assets:

$$I(p;x_0) = \frac{\hat{V}(p;\lambda)}{\langle x_0, p\rangle} - 1 = \frac{\hat{V}(p;\lambda) - \hat{V}_N(p;x_0)}{\hat{V}_N(p;x_0)}$$

The major property of the impermanent loss function is that

$$x_0 \in \lambda C \implies I(p;x_0) \in [-1, 0] \quad \forall p \in \mathbb{R}^n_+$$

Indeed, proposition (48) describes the non-positivity of $I(p;x_0)$ when $x_0 \in \lambda C$ while the infimum is reached when $p \to 0^+$ or when $p \to \infty$: because of concavity, as $p \to 0^+$ $\hat{V}(p;1)$ approaches zero faster than $\langle x_0, p\rangle$ while the latter diverges to infinity faster than the former as $p \to \infty$. Of course, as $I(p;x_0)$ remains in the neighborhood of zero, the opportunity-cost suffered by liquidity providers is neglectable. However, as $I(p;x_0)$ tends to -1, the opportunity-cost suffered by liquidity providers becomes more and more relevant.

3.4 Designing Uniswap V2

A practical example of the toolkit presented in this section is the design of Uniswap V2. This is one of the most famous examples of Constant Function Market Makers which implements as invariant function the constant product of the reserves of a pair of assets deposited on the pool [Ada21] [AZR20]. Thus, Uniswap V2-like pools allows to exchange a pair of assets per-pool using as invariant function

$$\hat{\phi}(\xi_1, \xi_2) = \xi_1\xi_2$$

Thus, one could define $D = \{(\xi_1, \xi_2) : \xi_1\xi_2 \geq 1, \xi_1 > 0\} \subset \mathbb{R}^2_+$ and, for a generic $\lambda > 0$, one has

$$\begin{aligned}\lambda D &= \{(\lambda\xi_1, \lambda\xi_2) : \xi_1\xi_2 \geq 1, \xi_1 > 0\} \\ &= \left\{(\xi_1, \xi_2) : \frac{\xi_1}{\lambda}\frac{\xi_2}{\lambda} \geq 1, \frac{\xi_1}{\lambda} > 0\right\} \\ &= \left\{(\xi_1, \xi_2) : \xi_1\xi_2 \geq \lambda^2, \xi_1 > 0\right\}\end{aligned}$$

this is useful for generating a convex cone $K_{\hat{\phi}} \subset \mathbb{R}^n_+$ in higher dimension:

$$\begin{aligned}K_{\hat{\phi}} &= \text{cone}(\{(1, \xi_1, \xi_2) : (\xi_1, \xi_2) \in D\}) \\ &= \{(\lambda, \xi_1, \xi_2) : \lambda > 0, (\xi_1, \xi_2) \in \lambda D\} \\ &= \left\{(\lambda, \xi_1, \xi_2) : \lambda > 0, \xi_1\xi_2 \geq \lambda^2, \xi_1 > 0\right\} \quad \text{(recall that } \lambda > 0) \\ &= \left\{(\lambda, \xi_1, \xi_2) : \lambda > 0, \sqrt{\xi_1\xi_2} \geq \lambda, \xi_1 > 0\right\}\end{aligned}$$

Thus, applying proposition (59) and setting $k = 1$ it's possible to induce the associated positive-homogenous invariant function $\hat{L} : \mathbb{R}^n_+ \to \mathbb{R}_+$ which, in the case of Uniswap V2, it corresponds to the geometric mean of the reserves, indeed:

$$\begin{aligned}\hat{L}(\xi_1, \xi_2) &= k \sup \{\lambda > 0 : (\xi_1, \xi_2) \in \lambda D\} \\ &= \sup \left\{\lambda > 0 : \sqrt{\xi_1\xi_2} \geq \lambda, \xi_1 > 0\right\} \\ &= \sqrt{\xi_1\xi_2}\end{aligned}$$

From here it's possible to recover immediately the basic set of reachable reserves $C \subset \mathbb{R}^2_+$ by applying proposition (53) indeed

$$C = \left\{(\xi_1, \xi_2) : \sqrt{\xi_1\xi_2} \geq 1\right\}$$

This information allows to apply proposition (50) to recover the concave conjugate of the basic portfolio value function (still unknown). Indeed, since $C = \text{dom}(\hat{V}^\star)$ one has that

$$\hat{V}^\star(\xi_1, \xi_2) = \begin{cases} 0 & \text{if } \sqrt{\xi_1\xi_2} \geq 1 \\ -\infty & \text{if } \sqrt{\xi_1\xi_2} < 1 \end{cases}$$

Moreover, picking $(\xi_{1,0}, \xi_{2,0}) \in C$, it's possible to induce the set of feasible trades $T(x_0)$ at level $\lambda = \hat{L}(\xi_{1,0}, \xi_{2,0})$ from proposition (55) indeed

$$T(x_0) = \left\{(\delta_1, \delta_2) : \sqrt{(\xi_{1,0} + \delta_1)(\xi_{2,0} + \delta_2)} \geq \sqrt{\xi_{1,0}\xi_{2,0}}\right\}$$

Recovering $C^\star$ is possible by computing the concave conjugate $\hat{L}^\star$ (which is an indicator function because of the positive-homogeneity of $\hat{L}$) since $C^\star = \text{dom}(\hat{L}^\star)$ according to proposition (50) where

$$\hat{L}^\star(\pi_1, \pi_2) = \inf_{(\xi_1, \xi_2) \in \mathbb{R}^2_+} \pi_1\xi_1 + \pi_2\xi_2 - \sqrt{\xi_1\xi_2}$$

Noticeably, if $(\pi_1, \pi_2) \prec 0 \implies \hat{L}^\star(\pi_1, \pi_2) = -\infty$ because the function becomes non-positive. Thus, it must be defined the set of prices (π_1, π_2) such that the origin-vanishing objective $\pi_1\xi_1 + \pi_2\xi_2 - \sqrt{\xi_1\xi_2}$ is non-negative, so that $\inf_{(\xi_1,\xi_2)\in\mathbb{R}^2_+} \pi_1\xi_1 + \pi_2\xi_2 - \sqrt{\xi_1\xi_2} = 0$. In other words,

$$C^\star = \left\{(\pi_1, \pi_2) : \pi_1\xi_1 + \pi_2\xi_2 \geq \sqrt{\xi_1\xi_2}, (\xi_1, \xi_2) \in \mathbb{R}^2_+\right\}$$

To do so, it's possible recalling the AM-GM inequality described in equation (2.10) which implies that

$$\frac{\pi_1\xi_1 + \pi_2\xi_2}{2} \geq \sqrt{\pi_1\xi_1\pi_2\xi_2}$$

Which could be trivially rewritten as

$$\pi_1\xi_1 + \pi_2\xi_2 \geq 2\sqrt{\pi_1\pi_2}\sqrt{\xi_1\xi_2}$$

Now, if $2\sqrt{\pi_1\pi_2} \geq 1$, this implies that

$$\pi_1\xi_1 + \pi_2\xi_2 \geq 2\sqrt{\pi_1\pi_2}\sqrt{\xi_1\xi_2} \geq \sqrt{\xi_1\xi_2}$$

And so that

$$(\pi_1\pi_2) \in \{(\pi_1\pi_2) : 2\sqrt{\pi_1\pi_2} \geq 1\} \implies \pi_1\xi_1 + \pi_2\xi_2 \geq \sqrt{\xi_1\xi_2} \quad \forall(\xi_1, \xi_2) \in \mathbb{R}^2_+$$

Which resembles the set condition of $C^\star$, meaning that

$$C^\star = \left\{(\pi_1, \pi_2) : \pi_1\xi_1 + \pi_2\xi_2 \geq \sqrt{\xi_1\xi_2}, (\xi_1, \xi_2) \in \mathbb{R}^2_+\right\} = \{(\pi_1, \pi_2) : 2\sqrt{\pi_1\pi_2} \geq 1\}$$

And that

$$\hat{L}^\star(\pi_1, \pi_2) = \begin{cases} 0 & \text{if } 2\sqrt{\pi_1\pi_2} \geq 1 \\ -\infty & \text{if } 2\sqrt{\pi_1\pi_2} < 1 \end{cases}$$

Finally, the basic portfolio value function, which is the last core component of Uniswap V2 as a CFMM, can be induced from $C^\star$ as described in proposition (48):

$$\begin{aligned} V(\pi_1, \pi_2; 1) &= \sup\{\lambda > 0 : (\pi_1, \pi_2) \in \lambda C^\star\} \\ &= \sup\{\lambda > 0 : 2\sqrt{\pi_1\pi_2} \geq \lambda\} \\ &= 2\sqrt{\pi_1\pi_2} \end{aligned}$$

Thus, the Uniswap V2 core components can be summarized as follows

$$C = \mathrm{dom}(\hat{V}^\star) = \left\{(\xi_1, \xi_2) : \sqrt{\xi_1\xi_2} \geq 1\right\} \tag{3.1}$$

$$\hat{V}(\pi_1, \pi_2; 1) = 2\sqrt{\pi_1\pi_2} \tag{3.2}$$

$$\hat{L}(\xi_1, \xi_2) = \sqrt{\xi_1\xi_2} \tag{3.3}$$

$$T(x_0) = \left\{(\delta_1, \delta_2) : \sqrt{(\xi_{1,0} + \delta_1)(\xi_{2,0} + \delta_2)} \geq \sqrt{\xi_{1,0}\xi_{2,0}}\right\} \tag{3.4}$$

Moreover, even if it wasn't specified as a core component, one has

$$C^\star = \mathrm{dom}(\hat{L}^\star) = \{(\pi_1, \pi_2) : 2\sqrt{\pi_1\pi_2} \geq 1\}$$

3.4.1 Starting from C

In this subsection it's shown how to recover the core components of Uniswap V2 starting from the basic set of reachable reserves C defined in equation (3.1):

$$C = \operatorname{dom}(\hat{V}^\star) = \left\{(\xi_1, \xi_2) : \sqrt{\xi_1\xi_2} \geq 1\right\}$$

Applying proposition (46) the previous subsection showed that $\hat{L}$ is immediately recoverable as

$$\begin{aligned}
\hat{L}(\xi_1, \xi_2) &= \sup\{\lambda > 0 : (\xi_1, \xi_2) \in \lambda C\} \\
&= \sup\left\{\lambda > 0 : \sqrt{\xi_1\xi_2} \geq \lambda\right\} \\
&= \sqrt{\xi_1\xi_2}
\end{aligned}$$

On the other hand, applying proposition (48), one has that

$$V(\pi_1, \pi_2; 1) = -\delta^\star(-\pi_1, -\pi_2|C) = \inf_{(\xi_1,\xi_2)\in C} \xi_1\pi_1 + \xi_2\pi_2$$

This is equivalent to the solution (keeping the dependance of the optimal value to the parameters $(\pi_1, \pi_2) \in \mathbb{R}^{2n}_+$) of the following problem

$$\begin{aligned}
\text{minimize} \quad & \xi_1\pi_1 + \xi_2\pi_2 \\
\text{subject to} \quad & 1 - \sqrt{\xi_1\xi_2} \leq 0
\end{aligned}$$

Which can solved via the Lagrangian function

$$\mathcal{L}(\xi_1, \xi_2, \lambda) = \xi_1\pi_1 + \xi_2\pi_2 + \lambda\left(1 - \sqrt{\xi_1\xi_2}\right)$$

Noticeably, this function is convex being the sum of functions convex in (ξ_1, ξ_2) (for $\lambda \geq 0$), thus applying the first order condition allows to recover the optimal value of the problem. The gradient of the Lagrangian function is

$$\nabla\mathcal{L}(\xi_1, \xi_2, \lambda) = \begin{pmatrix} \pi_1 - \frac{\lambda\sqrt{\xi_2}}{2\sqrt{\xi_1}} \\ \pi_2 - \frac{\lambda\sqrt{\xi_1}}{2\sqrt{\xi_2}} \\ 1 - \sqrt{\xi_1\xi_2} \end{pmatrix}$$

Applying the first order condition, one has that the Lagrangian function is minimized when

$$\nabla\mathcal{L}(\xi_1, \xi_2, \lambda) = 0 \iff \begin{cases} \pi_1 - \frac{\lambda\sqrt{\xi_2}}{2\sqrt{\xi_1}} = 0 \\ \pi_2 - \frac{\lambda\sqrt{\xi_1}}{2\sqrt{\xi_2}} = 0 \\ 1 - \sqrt{\xi_1\xi_2} = 0 \end{cases}$$

From the third equation one has that $\frac{1}{\sqrt{\xi_1}} = \sqrt{\xi_2}$ implying that

$$\begin{cases} \lambda = \frac{2\pi_1}{\xi_2} & \implies \lambda = 2\sqrt{\pi_1\pi_2} \\ \pi_2 = \frac{1}{2}\frac{2\pi_1}{\xi_2}\xi_1 \implies \xi_2 = \xi_1\frac{\pi_1}{\pi_2} & \implies \xi_2 = \sqrt{\frac{\pi_1}{\pi_2}} \\ \frac{1}{\sqrt{\xi_1}} = \sqrt{\xi_2} & \implies \xi_1 = \sqrt{\frac{\pi_2}{\pi_1}} \end{cases}$$

Implying that

$$
\begin{aligned}
V(\pi_1, \pi_2; 1) = \inf_{(\xi_1,\xi_2)\in C} \xi_1\pi_1 + \xi_2\pi_2 &= \mathcal{L}(\xi_1, \xi_2, \lambda)|_{\left(\xi_1=\sqrt{\frac{\pi_2}{\pi_1}}, \xi_2=\sqrt{\frac{\pi_1}{\pi_2}}, \lambda=2\sqrt{\pi_1\pi_2}\right)} \\
&= \left(\sqrt{\frac{\pi_2}{\pi_1}}\right)\pi_1 + \left(\sqrt{\frac{\pi_1}{\pi_2}}\right)\pi_2 + 2\sqrt{\pi_1\pi_2}\left(1 - \sqrt{\sqrt{\frac{\pi_2}{\pi_1}}\sqrt{\frac{\pi_1}{\pi_2}}}\right) \\
&= \sqrt{\pi_2\pi_1} + \sqrt{\pi_1\pi_2} + 2\sqrt{\pi_1\pi_2}\,(1-1) \\
&= 2\sqrt{\pi_1\pi_2}
\end{aligned}
$$

And this result is consistent with equation (3.2).

Finally, applying proposition (49), one has that given $(\xi_{1,0}, \xi_{2,0}) \in \mathrm{cl}(C) \setminus \mathrm{ri}(C)$ as vector of current reserves in the Uniswap V2-like pool, the set of feasible trades is defined as

$$
\begin{aligned}
T(\xi_{1,0}, \xi_{2,0}) &= C - (\xi_{1,0}, \xi_{2,0}) \\
&= \left\{(\xi_1 - \xi_{1,0}, \xi_2 - \xi_{2,0}) : \sqrt{\xi_1\xi_2} \geq 1\right\} \\
&= \left\{(\delta_1, \delta_2) : \sqrt{(\xi_{1,0} + \delta_1)(\xi_{2,0} + \delta_2)} \geq 1\right\}
\end{aligned}
$$

According to proposition (53), one has that $C = \left\{(\xi_1, \xi_2) : \hat{L}(\xi_1, \xi_2) \geq 1\right\}$ and this implies that $\mathrm{cl}(C) \setminus \mathrm{ri}(C) = \left\{(\xi_1, \xi_2) : \hat{L}(\xi_1, \xi_2) = 1\right\}$: thus, since $(\xi_{1,0}, \xi_{2,0}) \in \mathrm{cl}(C) \setminus \mathrm{ri}(C) \implies \hat{L}(\xi_{1,0}, \xi_{2,0}) = 1$ This means that the previous result can be rewritten as

$$
\begin{aligned}
T(\xi_{1,0}, \xi_{2,0}) &= \left\{(\delta_1, \delta_2) : \sqrt{(\xi_{1,0} + \delta_1)(\xi_{2,0} + \delta_2)} \geq \hat{L}(\xi_{1,0}, \xi_{2,0})\right\} \\
&= \left\{(\delta_1, \delta_2) : \sqrt{(\xi_{1,0} + \delta_1)(\xi_{2,0} + \delta_2)} \geq \sqrt{\xi_{1,0}\xi_{2,0}}\right\}
\end{aligned}
$$

And this result is consistent with equation (3.4).

3.4.2 Starting from $\hat{V}$

In this subsection it's shown how to recover the core components of Uniswap V2 starting from the basic portfolio value function $\hat{V}$ defined in equation (3.2):

$$
\hat{V}(\pi_1, \pi_2; 1) = 2\sqrt{\pi_1\pi_2}
$$

This is the typical situation in which one prefers designing the CFMM starting from the concave function describing the payoff which is going to be passively replicated by liquidity providers while committing their assets to the CFMM. Applying proposition (50) it's possible to recover the basic set of reachable reserves C as the effective domain of the concave conjugate $\hat{V}^\star$

$$
\hat{V}^\star(\xi_1, \xi_2) = \inf_{(\pi_1,\pi_2)\in\mathbb{R}^2_+} \xi_1\pi_1 + \xi_2\pi_2 - 2\sqrt{\pi_1\pi_2}
$$

For a similar argument as the one presented in the previous subsection, the infimum is 0 for the values of $(\xi_1, \xi_2) \in \mathbb{R}^2_+$ such that $\xi_1\pi_1 + \xi_2\pi_2 - 2\sqrt{\pi_1\pi_2} \geq 0 \ \forall(\pi_1, \pi_2) \in \mathbb{R}^2_+$, otherwise the infimum if $-\infty$

$$C = \text{dom}(\hat{V}^\star) = \left\{(\xi_1, \xi_2) : \xi_1\pi_1 + \xi_2\pi_2 - 2\sqrt{\pi_1\pi_2} \geq 0, (\pi_1, \pi_2) \in \mathbb{R}^2_+\right\}$$

Recalling again the AM-GM inequality from equation (2.10), one has that

$$\frac{\xi_1\pi_1 + \xi_2\pi_2}{2} \geq \sqrt{\xi_1\xi_2}\sqrt{\pi_1\pi_2}$$
$$\xi_1\pi_1 + \xi_2\pi_2 \geq \sqrt{\xi_1\xi_2}2\sqrt{\pi_1\pi_2}$$

By setting $\sqrt{\xi_1\xi_2} \geq 1$, one has that

$$\xi_1\pi_1 + \xi_2\pi_2 \geq \sqrt{\xi_1\xi_2}2\sqrt{\pi_1\pi_2} \geq 2\sqrt{\pi_1\pi_2}$$
$$\xi_1\pi_1 + \xi_2\pi_2 \geq 2\sqrt{\pi_1\pi_2}$$

Implying that, consistently with what mentioned in the previous subsection

$$\hat{V}^\star(\xi_1\xi_2) = \begin{cases} 0 & \text{if } \sqrt{\xi_1\xi_2} \geq 1 \\ -\infty & \text{if } \sqrt{\xi_1\xi_2} < 1 \end{cases}$$

Implying that

$$C = \text{dom}(\hat{V}^\star) = \left\{(\xi_1, \xi_2) : \sqrt{\xi_1\xi_2} \geq 1\right\}$$

Which is consistent with equation (3.1).

On the other hand, applying proposition (51), one has that

$$\hat{L}(\xi_1, \xi_2) = \inf_{(\pi_1,\pi_2)\succ 0} \frac{\xi_1\pi_1 + \xi_2\pi_2}{2\sqrt{\pi_1\pi_2}}$$

calling $f(\pi_1, \pi_2) = \frac{\xi_1\pi_1 + \xi_2\pi_2}{2\sqrt{\pi_1\pi_2}}$ one has that the gradient of the function f is

$$\nabla f = \begin{pmatrix} \frac{\xi_1}{4\sqrt{\pi_1\pi_2}} - \frac{\xi_2\sqrt{\pi_2}}{4\pi_1\sqrt{\pi_1}} \\ \frac{\xi_2}{4\sqrt{\pi_1\pi_2}} - \frac{\xi_1\sqrt{\pi_2}}{4\pi_2\sqrt{\pi_2}} \end{pmatrix}$$

Applying the first order condition, one has that the function f is minimized when

$$\nabla f = 0 \iff \begin{cases} \frac{\xi_1}{4\sqrt{\pi_1\pi_2}} - \frac{\xi_2\sqrt{\pi_2}}{4\pi_1\sqrt{\pi_1}} = 0 \\ \frac{\xi_2}{4\sqrt{\pi_1\pi_2}} - \frac{\xi_1\sqrt{\pi_2}}{4\pi_2\sqrt{\pi_2}} = 0 \end{cases}$$

Which is true whenever $\pi_1 = \pi_2\frac{\xi_2}{\xi_1}$. Thus, one has that

$$\begin{aligned} \inf_{(\pi_1,\pi_2)\succ 0} \frac{\xi_1\pi_1 + \xi_2\pi_2}{2\sqrt{\pi_1\pi_2}} &= \inf_{(\pi_1,\pi_2)\succ 0} \frac{\xi_1}{2}\sqrt{\frac{\pi_1}{\pi_2}} + \frac{\xi_2}{2}\sqrt{\frac{\pi_2}{\pi_1}} \\ &= \frac{\xi_1}{2}\sqrt{\frac{1}{\pi_2}}\left(\sqrt{\pi_2\frac{\xi_2}{\xi_1}}\right) + \frac{\xi_2}{2}\sqrt{\pi_2}\left(\sqrt{\frac{\xi_1}{\pi_2\xi_2}}\right) \\ &= \frac{\sqrt{\xi_1\xi_2}}{2} + \frac{\sqrt{\xi_1\xi_2}}{2} \\ &= \sqrt{\xi_1\xi_2} =: \hat{L}(\xi_1, \xi_2) \end{aligned}$$

And this result is consistent with equation (3.3).

Finally, applying proposition (52), one has that given $(\xi_{1,0}, \xi_{2,0}) \in \mathrm{cl}(C) \setminus \mathrm{ri}(C)$ as vector of current reserves in the Uniswap V2-like pool, the set of feasible trades is defined as

$$T(x_0) = \left\{(\delta_1, \delta_2) : \hat{V}^\star(\xi_{1,0} + \delta_1, \xi_{2,0} + \delta_1) \geq 0\right\}$$

Where,as seen before, $\hat{V}^\star$ is the following concave indicator function

$$\hat{V}^\star(\xi_{1,0} + \delta_1, \xi_{2,0} + \delta_1) = \begin{cases} 0 & \text{if } \sqrt{(\xi_{1,0} + \delta_1)(\xi_{2,0} + \delta_2)} \geq 1 \\ -\infty & \text{if } \sqrt{(\xi_{1,0} + \delta_1)(\xi_{2,0} + \delta_2)} < 1 \end{cases}$$

Thus

$$T(x_0) = \left\{(\delta_1, \delta_2) : \sqrt{(\xi_{1,0} + \delta_1)(\xi_{2,0} + \delta_2)} \geq 1\right\}$$

however, as seen in the previous subsection, since $(\xi_{1,0}, \xi_{2,0}) \in \mathrm{cl}(C) \setminus \mathrm{ri}(C) \implies \hat{L}(\xi_{1,0}, \xi_{2,0}) = 1$, it means that $T(x_0)$ is

$$\begin{aligned} T(x_0) &= \left\{(\delta_1, \delta_2) : \sqrt{(\xi_{1,0} + \delta_1)(\xi_{2,0} + \delta_2)} \geq \hat{L}(\xi_{1,0}, \xi_{2,0})\right\} \\ &= \left\{(\delta_1, \delta_2) : \sqrt{(\xi_{1,0} + \delta_1)(\xi_{2,0} + \delta_2)} \geq \sqrt{\xi_{1,0}\xi_{2,0}}\right\} \end{aligned}$$

And this is consistent with equation (3.4).

3.4.3 Starting from $\hat{L}$

In this subsection it's shown how to recover the core components of Uniswap V2 starting from the positive-homogenous invariant function $\hat{L}$ defined in equation (3.3):

$$\hat{L}(\xi_1, \xi_2) = \sqrt{\xi_1 \xi_2}$$

Applying proposition (53) it's possible to recover the basic set of reachable reserves C as the upper-level set at level one of the invariant function, indeed

$$C = \left\{(\xi_1, \xi_2) : \sqrt{\xi_1 \xi_2} \geq 1\right\}$$

And this notation is consistent with equation (3.1). On the other hand, applying proposition(54), one has that

$$V(\pi_1, \pi_2; 1) = \inf_{(\xi_1, \xi_2) \succ 0} \frac{\xi_1 \pi_1 + \xi_2 \pi_2}{\sqrt{\xi_1 \xi_2}}$$

Analogously as before, the optimal value is achieved at $\xi_1 = \xi_2 \frac{\pi_2}{\pi_1}$, meaning that

$$\begin{aligned} \inf_{(\xi_1, \xi_2) \succ 0} \frac{\xi_1 \pi_1 + \xi_2 \pi_2}{\sqrt{\xi_1 \xi_2}} &= \inf_{(\xi_1, \xi_2) \succ 0} \pi_1 \sqrt{\frac{\xi_1}{\xi_2}} + \pi_2 \sqrt{\frac{\xi_2}{\xi_1}} \\ &= \pi_1 \sqrt{\frac{1}{\xi_2}} \left(\sqrt{\xi_2 \frac{\pi_2}{\pi_1}}\right) + \pi_2 \sqrt{\xi_2} \left(\sqrt{\frac{\pi_1}{\xi_2 \pi_2}}\right) \\ &= \sqrt{\pi_1 \pi_2} + \sqrt{\pi_1 \pi_2} \\ &= 2\sqrt{\pi_1 \pi_2} =: \hat{V}(\pi_1, \pi_2; 1) \end{aligned}$$

And this notation is consistent with equation (3.2). Finally, applying proposition (55), one has that given $(\xi_{1,0}, \xi_{2,0}) \in \text{cl}(\hat{L}(\xi_{1,0}, \xi_{2,0})C) \setminus \text{ri}(\hat{L}(\xi_{1,0}, \xi_{2,0})C)$, one has that

$$\begin{aligned} T(\xi_{1,0}, \xi_{2,0}) &= \left\{ (\delta_1, \delta_2) : \hat{L}(\xi_{1,0} + \delta_1, \xi_{2,0} + \delta_2) \geq \hat{L}(\xi_{1,0}, \xi_{2,0}) \right\} \\ &= \left\{ (\delta_1, \delta_2) : \sqrt{(\xi_{1,0} + \delta_1)(\xi_{2,0} + \delta_2)} \geq \sqrt{\xi_{1,0}\xi_{2,0}} \right\} \end{aligned}$$

And this notation is consistent with equation (3.4)

3.4.4 Starting from $T(x_0)$

In this subsection it's shown how to recover the core components of Uniswap V2 starting from the set of feasible trades $T(\xi_{1,0}, \xi_{2,0})$ defined in equation (3.4) and setting $\lambda = \hat{L}(\xi_{1,0}, \xi_{2,0})$:

$$\begin{aligned} (\xi_{1,0}, \xi_{2,0}) \in \text{cl}(C) \setminus \text{ri}(C) &\implies T(\xi_{1,0}, \xi_{2,0}) = \left\{ (\delta_1, \delta_2) : \sqrt{(\xi_{1,0} + \delta_1)(\xi_{2,0} + \delta_2)} \geq 1 \right\} \\ (\xi_{1,0}, \xi_{2,0}) \in \text{cl}(\lambda C) \setminus \text{ri}(\lambda C) &\implies T(\xi_{1,0}, \xi_{2,0}) = \left\{ (\delta_1, \delta_2) : \sqrt{(\xi_{1,0} + \delta_1)(\xi_{2,0} + \delta_2)} \geq \sqrt{\xi_{1,0}\xi_{2,0}} \right\} \end{aligned}$$

Applying proposition (56) one has that

$$\begin{aligned} C &= T(\xi_{1,0}, \xi_{2,0}) + (\xi_{1,0}, \xi_{2,0}) \\ &= \left\{ (\delta_1 + \xi_{1,0}, \delta_2 + \xi_{2,0}) : \sqrt{(\xi_{1,0} + \delta_1)(\xi_{2,0} + \delta_2)} \geq 1 \right\} \\ &= \left\{ (\xi_1, \xi_2) : \sqrt{(\xi_{1,0} + \xi_1 - \xi_{1,0})(\xi_{2,0} + \xi_2 - \xi_{2,0})} \geq 1 \right\} \\ &= \left\{ (\xi_1, \xi_2) : \sqrt{\xi_1 \xi_2} \geq 1 \right\} \end{aligned}$$

This notation is consistent with equation (3.1) and from here it's trivial retrieving the other core components as defined in equations (3.2) and (3.3) indeed one can follow the same stapes for inducing the portfolio value function and the invariant function from the basic set of reachable reserves

$$\begin{aligned} &\hat{V}(\pi_1, \pi_2; 1) = -\delta^\star(-\pi_1, -\pi_2 | x_0 + T(x_0)) = -\delta^\star(-\pi_1, -\pi_2 | C) = 2\sqrt{\pi_1 \pi_2} \\ &\hat{L}(\xi_1, \xi_2) = \sup\{\lambda > 0 : (\xi_1, \xi_2) \in \lambda(x_0 + T(x_0))\} = \sup\{\lambda > 0 : (\xi_1, \xi_2) \in \lambda C\} = \sqrt{\xi_1 \xi_2} \end{aligned}$$

3.4.5 Additional components

After having discussed about how to retrieve all the core components of Uniswap V2 starting from a generic component, it's possible to discuss about the "additional components" of Uniswap V2 as the ancillary functions introduced throughout the third chapter. The first additional component is the marginal price function Ξ, which allows to assess which are the prices oracled by the pool. Because of the differentiability of the geometric mean, one can compute the gradient of the invariant function at any value of the vector of current reserves (the superdifferential evaluating such point is actually the singleton of the gradient evaluating the point).

$$\hat{L}(\xi_{1,0}, \xi_{2,0}) = \sqrt{\xi_{1,0}\xi_{2,0}} \implies \nabla\hat{L}(\xi_{1,0}, \xi_{2,0}) = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{\xi_{2,0}}{\xi_{1,0}}} \\ \frac{1}{2}\sqrt{\frac{\xi_{1,0}}{\xi_{2,0}}} \end{pmatrix}$$

Since $\nabla\hat{L}(\xi_{1,0}, \xi_{2,0}) \in \mathbb{R}^2$, one has that the marginal price function is actually a map of the type $\Xi(\cdot; \xi_{1,0}, \xi_{2,0}, i) : \mathbb{R}^2 \to \mathbb{R}$. For example, if the asset indexed as second in the CFMM is considered as quote asset, the marginal price function corresponds to

$$\Xi(\nabla\hat{L}(\xi_{1,0}, \xi_{2,0}); \xi_{1,0}, \xi_{2,0}, 2) = \frac{1}{2}\sqrt{\frac{\xi_{2,0}}{\xi_{1,0}}} 2\sqrt{\frac{\xi_{2,0}}{\xi_{1,0}}} = \frac{\xi_{2,0}}{\xi_{1,0}}$$

Thus, in the case of Uniswap V2, the marginal price of the first token expressed in unit terms of the second token corresponds To

$$\pi_1 = \frac{\xi_{2,0}}{\xi_{1,0}}$$

which is the simple ratio between the reserves of the quote asset and the reserves of the base asset. On the contrary, for what regards the basic set of efficient reserves, one has that

$$\Theta : p \in \mathbb{R}^n_+ \mapsto \{x \in C : \langle p, x\rangle = V(p; 1)\}$$

Which in this case becomes

$$\begin{aligned} \Theta(\pi_1, \pi_2) &= \{(\xi_1, \xi_2) \in C : \xi_1\pi_1 + \xi_2\pi_2 = 2\sqrt{\pi_1\pi_2}\} \\ &= \left\{(\xi_1, \xi_2) : \xi_1\pi_1 + \xi_2\pi_2 = 2\sqrt{\pi_1\pi_2}, \sqrt{\xi_1\xi_2} \geq 1\right\} \end{aligned}$$

The set notation can be rephrased considering each equality term as part of a system of an equality and an inequality. The inequality implies that $\xi_1 \geq \xi_2^{-1}$ and this condition, plugged into the first equation, leads to

$$\begin{aligned} &\frac{\pi_1}{\xi_2} + \xi_2\pi_2 = 2\sqrt{\pi_1\pi_2} \\ &\pi_1 + \xi_2^2\pi_2 - 2\xi_2\sqrt{\pi_1\pi_2} = 0 \\ &(\sqrt{\pi_1} - \xi_2\sqrt{\pi_2})^2 = 0 \\ &\xi_2 = \sqrt{\frac{\pi_1}{\pi_2}} \end{aligned}$$

Analogously, one has that $\xi_1 = \sqrt{\frac{\pi_2}{\pi_1}}$. Thus, the set of efficient reserves can be rewritten as the following singleton:

$$\Theta(\pi_1, \pi_2) = \left\{\left(\sqrt{\frac{\pi_2}{\pi_1}}, \sqrt{\frac{\pi_1}{\pi_2}}\right)\right\}$$

Since the set of efficient reserves is a single-valued map, one can actually recover the portfolio value function as the inner product between p and $\Theta(p)$ indeed

$$\begin{aligned}\hat{V}(p;1) &= \langle p, \Theta(p)\rangle \\ &= \left\langle (\pi_1,\pi_2)\,, \left(\sqrt{\frac{\pi_2}{\pi_1}}, \sqrt{\frac{\pi_1}{\pi_2}}\right)\right\rangle \\ &= 2\sqrt{\pi_1\pi_2}\end{aligned}$$

At the same time, the function Γ whose epigraph corresponds to the basic set of reachable reserves, corresponds to

$$\Gamma(\xi_1) = \inf\left\{\xi_2 : \sqrt{\xi_1\xi_2} \geq 1\right\} = \frac{1}{\xi_1}$$

On the contrary, the get-amount-function (i.e. the convex function whose epigraph is equal to the set of feasible trades) corresponds to

$$\begin{aligned}\Omega(\delta_1;\xi_{1,0},\xi_{2,0}) &= \inf\left\{\delta_2 : \sqrt{(\xi_{1,0}+\delta_1)\,(\xi_{2,0}+\delta_2)} \geq \sqrt{\xi_{1,0}\xi_{2,0}}\right\} \\ &= \inf\left\{\delta_2 : \delta_2 \geq -\frac{\xi_{2,0}\delta_1}{(\xi_{1,0}+\delta_1)}\right\} \\ &= -\frac{\xi_{2,0}\delta_1}{(\xi_{1,0}+\delta_1)}\end{aligned}$$

Finally, for what regards the impermanent loss function, one has that

$$I(\pi_1,\pi_2;\xi_{1,0},\xi_{2,0}) = \frac{2\hat{L}(\xi_{1,0},\xi_{2,0})\sqrt{\pi_1\pi_2}}{\xi_{1,0}\pi_1+\xi_{2,0}\pi_2} - 1$$

Conventionally, the impermanent loss for Uniswap V2 is reported setting $(\xi_{1,0},\xi_{2,0}) = (1,1) \in C$ and reporting asset with index one as quote asset, so that the impermanent loss formula becomes

$$I(1,\pi_2;1,1) = \frac{2\sqrt{\pi_2}}{1+\pi_2} - 1$$

Noticeably, the main properties of impermanent loss function are satisfied, indeed:

$$\begin{aligned}&\frac{2\sqrt{\pi_2}}{1+\pi_2} - 1 \leq 0 \quad \forall \pi_2 \in \mathbb{R}_+ \\ &\lim_{\pi_2\to 0^+} \frac{2\sqrt{\pi_2}}{1+\pi_2} - 1 = \lim_{\pi_2\to\infty} \frac{2\sqrt{\pi_2}}{1+\pi_2} - 1 = -1\end{aligned}$$

Chapter 4

Conclusion

This work showed how the theoretical framework of convex analysis could be deployed for having an extensive and exhaustive characterization of the core components of a Constant Function Market Maker, conceived as a blockchain-based automated market maker. Moreover, the analysis of the core components of a CFMM under the lens of convex analysis allowed to introduce a set of propositions which can be used as a toolkit in designing path-independent CFMMs. Indeed, the set of reachable reserves $C \subset \mathbb{R}^n_+$, $n > 1$ could be seen as the non-empty closed, unbounded, convex set not containing the origin which generates in higher dimension the hypograph of the invariant function $\hat{L} : \mathbb{R}^n_+ \to \mathbb{R}_+$ also called "invariant cone" $K_{\hat{L}} \subset \mathbb{R}^{n+1}_+$. At the same time, this work shows that $C^\star = \{p : \langle x, p\rangle \geq 1, x \in C\}$, which is the reverse polar of C, is capable of generating the hypograph of the portfolio value function $\hat{V} : \mathbb{R}^n_+ \to \mathbb{R}_+$, also called "portfolio value cone" $K_{\hat{V}} \subset \mathbb{R}^{n+1}_+$. This leads to two important findings: the first one is that both invariant and portfolio value functions can be characterized always as closed concave gauge-like functions, implying that λC and $\lambda C^\star$ are recovereable as the upper-level sets at level $\lambda > 0$ of the invariant function and portfolio value function respectively, while the second one is that it is always possible to characterize the invariant function in terms of the portfolio value function and vice-versa thanks to the polar correspondence of the sets $C \subset \mathbb{R}^n_+$ and $C^\star \subset \mathbb{R}^n_+$, which is extended to $K_{\hat{L}} \subset \mathbb{R}^{n+1}_+$ and $K_{\hat{V}} \subset \mathbb{R}^{n+1}_+$ and so to the invariant function $\hat{L}$ and the portfolio value function $\hat{V}$. This dual correspondence is explicitly shown by expressing the invariant function $\hat{L}$ and the portfolio value function $\hat{V}$ as the negative of the support function of the symmetric reflection across the origin of $C^\star$ and C respectively. Another implicit way for pointing out this dual correspondence is to express C as the effective domain of $\hat{V}^\star$, which is the concave conjugate of the portfolio value function. Analogously, one can express $C^\star$ as the effective domain of $\hat{L}^\star$, which is the concave conjugate of the invariant function. Indeed, using the characterization of $\hat{L}$ and $\hat{V}$ as negative of support functions, it's immediate to see that $\hat{L}^\star$ and $\hat{V}^\star$ are the negative of indicator functions of $C^\star$ and C. Finally, given inventory $x_0 \in C$, this work shows that inducing the core-components of a CFMM from its set of feasible trades $T(x_0) \subset \mathbb{R}^n$ is straightforward by applying the feasibility condition of a trade: indeed, a trade $y \in \mathbb{R}^n$ is feasible if and only if $x_0 + y \in C$ implying that one can recover the set of feasible trades via the Minkowski sum $T(x_0) = C - x_0$. This characterization shows that $T(x_0)$ always includes the origin (since traders are allowed to perform a null trade) and all the propositions for inducing the other core components from $T(x_0)$ are simple generalizations of those referred to $C = T(x_0) + x_0$.

Once that the core components are derived, also the other additional components of a CFMM are immediately deduceable. For example, one can define the set of "efficient reserves" (i.e. the set of inventories which are expected to be held by the CFMM given external vector of prices under arbitrage-free assumption) as the set of reserves such that the inner product between the inventory and the external prices is equal to the portfolio value function. Other additional components are the convex functions induceable from C and $T(x_0)$. For example, the convex function induced by $T(x_0)$ is typically defined as the "get-amount-out" function and it's useful for quoting the amounts of a certain asset which can be pulled out from the pool by tendering a certain amount of other assets. Finally, this work shows that is possible to retrieve the "impermanent-loss function" (which is the concave function that quantifies the opportunity-cost of liquidity providers, as function of the prices of the assets deposited on the CFMM) as the ratio of the portfolio value function over the inner product between the vector of current prices and the vector of initial holdings (committed to the CFMM by providing liquidity) minus one. This work ends with the application of the toolkit presented in this work to Uniswap-V2 showing how every core component of such CFMM can be induced by any other core component. Finally the additional components of Uniswap-V2 are derived as well.

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