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The Efficiency of Pay-As-You-Go Social Security schemes under uncertainty

An OLG analysis based on Italian system

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Abstract

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This research investigates the efficiency implications of social welfare systems, focusing on the potential benefits of Pay-as-you-go redistribution schemes in uncertain settings, characterized by market failures. After deeply exploring the concepts of Pareto optimality and the related evaluation criteria in both deterministic and stochastic environments, I investigate the critical issues of the Italian Pay-As-You-Go pension system and propose an Overlapping Generations model to assess the efficiency potential improvements of such redistribution schemes in an economy characterized not only by dynamic inefficiency but also individual productivity risk. Through a calibration based on the Italian economy, the model makes an attempt in quantifying the welfare effects of the introduction of this pension system in terms of both inter-generational and infra-generational risk sharing, net of the canonical distortionary effects implied by such policies.

Contents

List of Tables

Introduction

The future of social welfare remains uncertain today. Despite the increased political focus on citizens' social welfare, questions about the effectiveness and proper construction of systems to meet these needs persist. The financial unsustainability for governments and citizens, who often feel the negative impacts of the distorting effect of taxes rather than the benefits, is a major concern. This work primarily refers to pension systems, where a clear picture of the most efficient and sustainable model has yet to emerge. The public pension scheme adopted by most nations is the Pay-as-you-go system, aimed at redistribution and risk-sharing among generations. However, with increasing life expectancy, declining birth rates, and production slowdowns, the sustainability of this system is in question. Yet, mandatory transfers from the active labor force to the retired population remain popular despite their presumed unsustainability.

There are several reasons for the existence of inter-generational social security transfers.The classic reason in favour for social security is based on paternalism and the idea that consumption needs in old age may be systematically underestimated by individuals who lack in perfect foresight and risks evaluation (Samuelson, 1975). Feldstein (1985) has considered this topic, discussing the optimality of social security in economies characterized by myopic individuals, who thus do not save for the future.

However, for the purpose of this work, the reasons related to individuals' bounded rationality are set aside. Indeed, if one adopts a neoclassical context, as in this case, where individuals are considered fully rational, the discussion about the potential for social security reforms takes a different turn. In particular, the benefits of social security are linked to the possibility of market failures and are typically analyzed in models of economies with overlapping generations, *a la Samuelson* (1958), where the impossibility of intergenerational trade, especially, but also the potential failure of the transversality condition given by the infinity of individuals, can result in a failure of the First Welfare Theorem. Indeed, as demonstrated in the updated version of Diamond's OLG model (1965), the decentralized equilibria might not be dynamically Pareto efficient, even when there are no adverse externalities and the markets are complete. Specifically, this occurs in economies characterized by very low interest rates, lower than the growth rate, a context in which the redistributive intervention of a social planner becomes more efficient than saving, and thus there is room for improvement with respect to the decentralized allocation.

However, this work goes further in analyzing the efficiency potentials in a stochastic environment. When considering uncertainty, a choice must be made on the evaluation criterion to adopt. How should the effects of introducing a reform like social security on individuals' welfare be evaluated? The potential of resource redistribution systems indeed changes depending on whether one considers a criterion of efficiency conditional on the state in which individuals are born (Muench, 1977) or if one considers an ex-ante planning, with individuals being unable to insure against risks before birth.

In the model proposed in this thesis, the evaluation of the ex-ante or constrained efficiency of introducing a social security system also takes into account the insurance component that such systems propose against individual risks, specifically productive shocks during the second period of life. Indeed, in the analyzed economy, characterized by uncertainty but yet sequentially complete markets, the idea is that markets remain limited in the possibility to provide insurance to individuals against these individual- specific shocks. Hence, under these specifications, the model proposed in this study makes an attempt in exploiting also this additional market inefficiency to assess the benefits (and costs) of introducing a Pay-As-You-Go social security redistribution mechanism.

The study develops as follows: The first chapter deals deeply with the theory behind Welfare theorems and the Pareto efficiency criteria, used in the literature and in the context of Overlapping Generations models. The second chapter offers an overview of the functioning of existing pension systems, with a focus on the characteristics and evolution of the Italian one. The last chapter is fully dedicated to the representation of the model, calibration and simulation of its main variables and discussion of the results.

Chapter 1

Theoretical Background

1.1 First Welfare Theorem and Pareto optimality

Social security has been and still is today a core topic in macroeconomic analysis. Up to this point, numerous authors have attempted to address the question of whether it is justifiable to maintain social security systems, striving to establish evaluation criteria and quantify their costs and benefits. The responses and opinions are varied, and to date, there is no consensus on the optimality of introducing and sustaining a Pay-as-you-go type of social security system. This is largely due to the fact that various evaluation criteria can be adopted, and different contexts may be considered that are more or less neoclassical. In discussing evaluation criteria for policies concerning social welfare, it is essential to question what these criteria are and, most importantly, their origins. Welfare Economics theory is concerned with developing those criteria through which "socially" evaluate alternative allocations that arise from situations of conflict between individuals. Bentham (1748-1832) and Pigou (1877-1959) define Social Welfare as a function of the well-being of individual members:

$$SW = f(U_1, U_2, \ldots, U_N)$$

In the literature, the political discussion about which distributive criterion to use to define the social welfare function is still open. The efficiency criteria employed in the model presented in this work are necessarily based on the theory of Vilfredo Pareto, which falls within the neoclassical or marginalist conception that developed from 1870 onwards. Pareto essentially defines efficiency as society's ability, given individual preferences and available resources, to get the most out of the latter and increase its well-being. (Pareto, 1919). A resource allocation is therefore defined as Pareto efficient if it improves the well-being of at least one individual or class of individuals without worsening that of the others. Thus, whenever there is a situation of Pareto efficiency, no resource transfer is possible that does not worsen the situation for at least one individual. The first welfare theorem, analytically formulated by Arrow and Debreu (1954), although already theorized by Pareto (1896), states that every general economic equilibrium (GEE) of perfect competition is a Pareto optimum. The underlying intuition is that in a decentralized, perfectly competitive market situation, it is possible to reach an equilibrium where economic resources are optimally allocated among all individuals. Hence, in the sense of Pareto efficiency, the resource allocation is considered optimal if, starting from it, no redistribution can increase the utility of some without decreasing that of at least another agent. According to Tirole (1988), the same result would be much harder to achieve for a central, albeit "benevolent" and "fully informed" social planner, and it would therefore be preferable to rely on decentralized solutions, though this opens a different debate. Leaving aside the discussion on the usefulness or otherwise of state intervention, the welfare economics theorems and Pareto criteria are used by economists as powerful means of resolution for economic models. While the Pareto criterion has garnered consensus among many economists, the debate has not ceased and has indeed shifted to the timing of evaluations. The discussion is currently primarily focused on choosing between an ex-post policy evaluation criterion and an ex-ante one. In 1973, Ross Starr highlighted a fundamental challenge in economic resource management (Starr, 1973). He emphasized that resource planning based on forecasts (ex-ante) might not align with an optimal allocation when analysing actual outcomes (ex-post). This discrepancy arises when individuals have varying expectations or perceptions about the future. This gap between forecast-based planning and the actual outcomes of a policy challenges what was traditionally customary in economic welfare assessment. Starr (1973) further commented on this, stating that "the achievement of an Arrow (ex-ante) optimum is a normative dead end. After all, we are not so much interested in expectations as in results."

Dutta (1990) defines the ex-ante optimality criterion as one that "imposes a veil of ignorance," while the ex-post would allow individuals to "know their place in history." Despite this divergent line of thought, many economists remain reluctant to abandon the ex-ante criterion. According to Harris (1978), although the operation and welfare consequences of a policy are necessarily judged ex-post, it is indeed impossible to plan an efficient resource allocation following a criterion other than ex-ante. In which contexts does a discussion on the choice of an efficiency criterion to evaluate a redistribution policy such as the introduction of a Pay-as-you-go system, makes sense? Certainly, in a deterministic and finite economy approached from a neoclassical perspective, which assumes complete rationality of individuals and the absence of market distortions, such a discussion would be irrelevant. In that context, economic optimality would be defined by the Pareto criterion and the First Welfare Theorem, and introducing a resource redistribution system at equilibrium would never result in a Pareto improvement. For this reason, in this work, as it is customary in the relevant literature, an overlapping generations model is adopted. For the intrinsic characteristics of this model, basically based on an intertemporal setting, the First Welfare Theorem might fail and this opens to the possibility of discussing the efficiency of redistributive schemes.

1.2 Pareto Optimality in the Overlapping Generations model

"We live in a world where new generations are always coming along" (Samuelson, 1958) This sentence encapsulates the idea that led to the creation of a model that revolutionized the history of economic thought, challenging the belief in one of the fundamental theorems of welfare economics. Indeed, by considering overlapping generations, even without accounting for typical distortions like market incompleteness, the author suggests that a competitive equilibrium is not necessarily Pareto efficient. In this section, I will summarize the model's characteristics and then delve deeper into its implications for welfare theory. Paul Samuelson introduced the OLG (Overlapping Generations) model in 1958 as a tool to analyse economic interactions between different generations in an economy. Unlike traditional economic models that considered agents with finite time horizons, the OLG model considers individuals living for two periods: young and old. The author presents an economic structure where individuals live for two periods. In the first, they are young, work, save, and consume. In the second, they are old, do not work, and consume what they saved during their youth. Samuelson's world consists of an infinite succession of generations. The perpetual renewal of cohorts (or, in the presence of uncertainty, the mere possibility of new cohorts appearing) is a crucial element of the overlapping generations model. Combined with the assumption of an infinite succession of generations, the hypothesis that generations consist of "new" agents implies that the total number of distinct economic agents, along with the number of dated goods, is infinite in the overlapping generations model. In contrast, in the Ramsey-Cass model (which serves as another cornerstone of dynamic macroeconomic theory), no new agent ever emerges: each individual is part of a pre-existing family. Death is certain. It could be assumed to occur randomly, as when Blanchard (1985) adopts Yaari's (1965b) simplifying assumption of age-independent death probabilities, or even with zero probability, as in the model of infinite overlapping families (Weil, 1989). However, none of this truly matters since the specificity of the overlapping generations model depends, qualitatively, on the arrival of new, unconnected agents rather than the exact length of lives. How and when consumers disappear is, for the economist wanting to understand why the overlapping generations model is different, of secondary interest. Samuelson (1958) divides lives into three periods but also briefly examines a two-period version called youth and old age. Most of the literature, following the lead of Cass and Yaari (1966), has adopted the two-period formalization because it technically eliminates intertemporal trade between two consecutive cohorts. When there are two life stages, I meet my ancestors only once: when I am young (and they are old). This single encounter excludes intergenerational exchange because of the execution of an intertemporal exchange. The absence of intergenerational trade implied by the two-period version is convenient because it makes equilibrium calculations easy. Fortunately, the specified number of periods is irrelevant for most purposes: more realistic overlapping generations models with an arbitrary number of periods, or where time flows continuously rather than discretely, offer similar insights even though they are much harder to handle. Lastly, it's convenient to assume that all agents born on a specific date are identical. This limits heterogeneity to that arising from the birth date. The economy produces a good that can be consumed or saved, and savings become capital for the next period. Naturally, output can be described by one of the common types of production functions, such as Cobb-Douglas. Weil (2008) further explains that to reveal the main features of this model, it is sufficient to consider two polar versions of preferences: economies with "infinitely patient consumers" and those with "infinitely impatient consumers." In this economy, the autarkic equilibrium is determined by the condition where the savings supply from the young generation equals the capital demand from companies and the consumption of the older generation. However, since there are two life periods, agents from different generations meet only once, so there is no possibility of trade in equilibrium.

To see this, let's consider a simple overlapping generations economy where all individuals live for 2 periods only, being young in the first period of life and old in the second one.

At all $t \geq 0$, a generation consisting of L_t identical two-period lived individuals is born. We denote with c_t^t the consumption at time t of an individual born at time t and with c_{t+1}^t the time t + 1 consumption of this same person and let $c^t = (c_t^t, c_{t+1}^t)$.
Individuals' utilities are

$$U_t(c_t) = u(c_t^t) + \beta u(c_{t+1}^t)$$

and individuals' lifetime endowments are $e^t = (e_t^t, e_{t+1}^t)$, such that $e_t > 0$, $e_{t+1}^t \geq 0$. The initial old generation has utility $U^{-1}(c^{-1}) = \beta u(c_0^{-1})$ and individual endowment $e^{-1} > 0$. For simplicity, we assume that population grows at a constant rate, n, i.e.,

$$L_{t+1} = (1 + n)L_t$$

Then, the resource feasibility for the economy is

$$L_{t-1}c_t^{t-1} + L_t c_t^t = e_t = L_{t-1}e_t^{t-1} + L_t e_t^t$$

From which we can derive the per-capita version

$$c_t^t \ + \ \frac{1}{1+n} \ c_t^{t-1} = e_t^t \ + \frac{1}{1+n} \ e_t^{t-1}$$

By referencing to the original model proposed by Samuelson (1958) and later analysed by Gale (1973), it is assumed that individuals are selfish, leave no bequests and therefore that assets or liabilities are birth are zero. Moreover, we assume for simplicity that no individuals invest in the second period of life, to avoid dying leaving debt outstanding. These two conditions are:

$$a_t^t \ = \ 0$$

$$a_{t+2}^t \ = \ 0$$

Then, we can define the 2 budget constraints, one for each period:

$$\frac{a_{t+1}^t}{1+r_{t+1}} \ + \ c_t^t \ = \ e_t^t$$

$$c_{t+1}^t \ = \ a_{t+1}^t \ + \ e_{t+1}^t$$

And that for the initial old:

$$c_0^{-1} \ = \ a_0^{-1} \ + \ e_0^{-1}$$

PV prices $p = \{p_t\}_{t=0}^{\infty}$ are defined recursively $p_{t+1} \ = \ \frac{p_t}{1+r_{t+1}}$. The first order conditions from individual optimization are

$$u\prime(c_t^t)/\beta u\prime(c_{t+1}^t) = \ 1 + r_{t+1} = \frac{p_t}{p_{t+1}}$$

and individual optimality for the initial old implies

$$p_0 \ (c_0^{-1} \ - \ \ e_0^{-1}) \ = \ p_0 \ a_0^{-1}$$

Competitive Equilibrium: A competitive equilibrium for this economy can be defined as a a consumption allocation $\{c_t^t, c_t^t + 1\}_{t=0}^{\infty}$, and a sequence of financial plans $\{a_t^t \ , a_{t+1}^t\}_{t=0}^{\infty}$, such that, for some price sequence, p, and some given $a_0^{-1} \geq 0$,

(INDIVIDUAL OPTIMALITY)

$$c_0^{-1} \ = \ a_0^{-1} \ + \ e_0^{-1}$$

and

$$(c^t, \ a_{t+1}^t)$$

are U^t optimal for all t

(RESOURCE FEASIBILITY)

$$c_t^t \ + \ \frac{1}{1+n} \ c_t^{t-1} = e_t^t \ + \frac{1}{1+n} \ e_t^{t-1}$$

holds for all t

(ASSETS MARKETS CLEARING)

$$a_t^t + \frac{a_t^{t-1}}{1+n} = 0$$

holds for all t

(INITIAL CLAIMS)

$$a_t^t \ = \ 0$$

holds for all t.

From the last two conditions of the definition, we can say that $a_t^{t-1} \ = \ 0$.

These results, together with the individual budget constraints, imply that the given equilibrium is indeed autarkic:

$$c_t^t = e_t^t$$

$$c_t^{t-1} = e1_t^{t-1}$$

$$1 + r_{t+1} = u\prime(e_t^t)/\beta u\prime(e_{t+1}^t)$$

The intuition behind this result is that young generation cannot borrow from the old one or, viceversa, the elderly people cannot lend to the young population as they would not be around next period for the repayment of the transfer, e.g. a loan. As previously mentioned, this a distinctive feature of the model: a competitive equilibrium in the OLG economy is autarkic in the sense that it is not characterized by exchange of assets between generations.

In this context, it is trivial to construct a sequence of intergenerational transfers from young to old that improves the situation of every generation: just perpetually confiscate a lump-sum amount x, with $0 \leq x \leq e_1$, from the young's endowment and transfer it lump-sum to the elderly. The elderly then consume, assuming a constant population and thus an equal number of young and old, $e_2 + x$ instead of e_2 in the competitive allocation. If the population grows at a constant rate n, such that the young are $1 + n$ times more numerous than the old, this sequence of perpetual transfers from young to old guarantees a consumption of $e_2 + (1 + n)x$ for every elderly person.
When there is a linear storage technology for the consumer good with a gross return of 1+r in the second period. If the young use this technology, consumption in old age becomes $e_2 + (1 + r)e_1$.

By considering a deviation from the assumed equilibrium stationary allocation, it is possible to derive a condition for Pareto optimality: By considering a deviation from the assumed equilibrium stationary allocation, it is possible to derive a condition for Pareto optimality:

$$c_t^{t-1} = e^o + \ \epsilon \ (1+n) \, , \ c_t^t = \ e^y - \ \epsilon,$$

for all t and $\epsilon \in (0, \ e^y)$.

This new allocation is a Pareto improvement if and only if

$$U\left(e^y - \ \epsilon\right) + \ \beta U\left(e^o + \ \epsilon \ \left(1+n\right)\right) - U(e^y) - \ \beta U\left(e^o\right) \equiv \ \Phi(\ \epsilon) > 0$$

$$\Rightarrow \ \Phi'\left(\ 0\right) = \ U(e^y)\left(\frac{1+n}{1+r^a} - 1\right) > 0 \Rightarrow \ r^a < n$$

Hence if $r^a < n$, the competitive equilibrium of the canonical OLG model is not Pareto efficient.

Essentially, the interest rate, which in equilibrium satisfies 1+r = u'(e1)/u'(e2), ensures that markets stabilize and can be very high or very low, depending on whether the economy is characterized by much or little impatience. When agents are patient and thus when the interest rate is very low, the competitive equilibrium is not Pareto optimal, and thus the first fundamental theorem of welfare fails. Gale (1973) defines economies with low interest rates as "Samuelsonian."

In this type of economy, individuals are solely concerned with consumption during old age. They would, therefore, like to trade their youth consumption for greater possessions to enjoy during old age, but unfortunately, this trade is not possible. For agents to be satisfied with the autarkic equilibrium allocation, the interest rate must be punitive enough to make the elderly content to consume their endowment. However, the equilibrium remains sub-optimal because the endowment of the young is wasted when considering non-storable consumer goods. It turns out that ff the population growth rate n exceeds the interest rate r, intergenerational redistribution provides a superior alternative that produces a higher implicit return rate n. As long as the interest rate r is lower than the population growth rate, the proposed sequence of transfers from young to old improves welfare in Pareto terms. For every generation, it is better to receive a transfer as elderly from the new generation of young rather than to store and hoard the consumer good. In a Samuelsonian economy, limiting the private storage capacity of the young results in a Pareto improvement in welfare.

Overlapping generations economies with high or "classical" interest rates are less interesting from a welfare perspective. If individuals are to be satisfied with the autarkic equilibrium allocation, the interest rate must be very high to eliminate consumers' inclination to borrow. The less agents care about consumption in old age, the more they want to borrow, and the higher the interest rate. This is sufficient to ensure that the equilibrium interest rate exceeds the population growth rate for a very low utility value of second-period consumption. The competitive allocation thus prescribes that the young consume their endowment in the first period, which they value, while the elderly consume their endowment, which they value very little. One might be tempted to argue that it is suboptimal in Pareto terms and that a centralized redistribution from old to young, symmetrically to the previous case, constitutes an improvement. However, in this situation, the initial generation of the elderly loses out, having to endure taxation without any benefit in return.

Hence, it should be emphasized that Pareto improvement can only occur in the presence of a transfer of resources from young to old and not the other way around. As Weil (2008) explains, "any transfer from old to young, be it implemented in a low or high interest rate economy, hurts the first generation of old that it affects" and this happens because in the assumed framework "there is an initial instant (the big bang, or today), but no last period" (Weil, 2008).

This comment brings the discussion on another peculiar element of the OLG model. All individuals' endowments have a finite value. However, in this model, the number of individuals is infinite, since a new person is born every period. Essentially, the economy is characterized by two distinct forms of infinity: an endless variety of goods, as seen in the Neoclassical model, and a continuous succession of generations or agents. This dual aspect of infinity, often referred to as the "double infinity" problem highlighted by K. Shell (1971), distinguishes OLG economies. The perpetual series of goods in the Neoclassical framework

does not by itself lead to inefficiencies. However, the additional layer of infinity from infinite generations introduces unique challenges. The OLG model's inclusion of infinitely many generations adds complexity by incorporating the temporal dimension of economic decisions across different lifespans. This aspect raises questions about how resources are allocated not just across different types of goods but also across different generations.
In other words, differently from other Neoclassical dynamic macroeconomic models, overlapping generations may lead to the failure of a fundamental boundary condition, that helps in ruling out economically implausible or non-optimal paths. In essence, it ensures that the intertemporal budget constraint is satisfied not just at each point in time but also in the aggregate over the infinite horizon, facilitating the selection of paths that are sustainable in the long run.

Social Transversality Condition (STC): The STC ensures that the present value of aggregate endowments doesn't explode:

$$\lim_{t \to \infty} \sum_{t=0}^{\infty} p_t \; y_t \; = 0$$

where p_t is the price of consumption when young in terms of consumption when old. Intuitively, real interest rates must be sufficiently large relative to the aggregate income growth for a sufficient number of periods. In the canonical OLG economy, social transversality may fail, with the PV of aggregate endowment over the infinite horizon likely not to be finite and real interest rates likely to be negative. Whenever the Transversality condition does not hold, there might be consequences in terms of welfare properties. Specifically, Pareto optimality of the equilibria may fail to hold.

In the OLG model all individuals' endowments have a finite value. However, the number of individuals is infinite, since a new person is born every period. This implies that the sum over individuals of the value of their lifetime endowments may not be finite, and we cannot compare allocations through present value prices. Furthermore, we cannot show that STC holds by relying on the finite valuation of individuals' endowments. Hence, equilibrium capital plans may be inefficient.

Suppose we are at a stationary autarky allocation of the type $c_t^t = e^y$ and $c_t^{t-1} = e^o$ for all t and equilibrium prices satisfy

$$1 + r^a = \frac{p_t}{p_{t+1}}$$

with

$$r^a = \; u\prime(e^y)/\beta u\prime(e^o)$$

Then

$$p_{t+1} = \frac{p_0}{(1+r^a)^t}$$

so that

$$\sum_{t=0}^{\infty} p_t \; < \infty \quad \Longleftrightarrow r^a > 1$$

In our case, defining time-t aggregate endowment as

$$y_t = \; L_{t-1}e^o + L_t e^y = \; L_t(\frac{e^o}{1+n} + e^y \;)$$

We derive that

$$\sum_{t=0}^{\infty} p_t \; y_t \; < \; \infty \quad \Longleftrightarrow \; r^a > n$$

The issue, therefore, seems to conveniently resolve itself in a discussion and comparison between the population growth rate and the market interest rate in the classic OLG framework.

1.3 Efficiency in OLG equilibria with & without uncertainty

So, if in a deterministic context the Pareto optimality of a stationary allocation is defined by the comparison between the equilibrium interest rate and the population growth rate, what analogous rule exists in a stochastic environment? Things become more complicated when we enter the realm of uncertainty, probability, and both endogenous and exogenous shocks that inevitably influence the factors of an economy and, above all, the fate and choices of individuals.

Differently from the canonical framework, when uncertainty is introduced, the competitive equilibrium of the OLG model is never optimal because those not yet born cannot participate in transactions involving risk-sharing with previous generations before coming into the world. For instance, if a negative shock to the availability of first-period consumption materializes just before the young generation comes into the world, they have no way to shield themselves from this shock. In an uncertain world, market participation failures, even in a sequentially complete Arrow-Debreu market that is free from any tax, will always render the OLG equilibrium allocation Pareto suboptimal. Adam Smith's invisible hand cannot restore market order and properly distribute risk across generations. In this context, Pareto sub-optimality cannot be uniquely reduced to dynamic efficiency issues and thus with over or under-saving. Transfers of resources from young to old that eliminate over-saving are beneficial, regardless of the presence of uncertainty. But, in the uncertain reality, even in an economy that has no dynamic efficiency problems and thus does not present these over-saving issues, intergenerational redistribution could still lead to a welfare improvement.

1.3.1 Conditional Pareto Efficiency

To understand the potential for social welfare of introducing a new resource transfer system, the assessment may change depending on which efficiency criterion is used when dealing with OLG models. Under uncertainty, a market failure may arise either because there exists room for Pareto improvement for some generation conditional on the state at which it is born (*conditional Pareto improvement*) or because there is room for Pareto improvements from an *ex-ante* perspective (when generations are only distinguished by the time at which they are born). The *ex-ante* optimality of social security is interpreted as the inability of generations to insure themselves against aggregate shocks before their birth[1]. To better mimic the uncertainties of reality, many authors would rather adopt an efficiency assessment criterion that in constrained on the realization of the state in which individuals are born. It is the Conditional Pareto optimality (CPO), proposed for the first time by Muench (1977), which indeed requires that individual welfare assessment is determined by considering individual utility conditional on the realized state at the time of assessment. With CPO, problems arise when considering reallocations. To understand whether a resource redistribution represents an improvement or not, not only the value of the exchange on a certain date must be considered, but also the consequent changes in terms of expenditure that may arise depending on the possible different events that can be realized on that same date.

Nevertheless, if the desired evaluation criterion is conditional, thus with the aim to consider individuals in a manner contingent upon the mode, timing, and state of the world in which people are born, then the classic criterion proposed by Aiyagari and Peled (1991) is certainly appropriate. Using the Perron-Frobenius theorem, the authors demonstrate that the necessary and sufficient condition for a stationary allocation to be conditionally Pareto efficient is that it be an equilibrium allocation and that the matrix of one-period state prices has a dominant root of the associated characteristic polynomial less than or equal to 1. Here, a simple application of this criterion is proposed.

Example: CPO criterion

Consider a standard OLG model with two period lived individuals whose preferences are defined by

$$U_t = u(c_t^y) + \mathbb{E}_t[v(c_{t+1}^o)].$$

[1] Quite clearly, conditional (or *ex-post*) Pareto inefficiency implies *ex-ante* inefficiency, but the converse is not true.

Aggregate output at each period t is a two-states iid stochastic variable y_t taking the value y^H in state H and y^L in state L, with $y^H \geq y^L$, and probabilities p and $1-p$, respectively. For some given pair of numbers, $(\alpha^H, \alpha^L) \in [0,1] \times [0,1]$, the young are assumed to own a share α^j of current output and the old the residual share $1 - \alpha^j$ in any state $j \in \{H, L\}$. Then, the budget constraints of an individual born at time t are

$$c_t^y + a_{t+1}/(1 + r_t) = \alpha_t y_t, \qquad c_{t+1}^o = a_{t+1} + (1 - \alpha_{t+1}) y_{t+1}, \tag{1.1}$$

where a denotes investment in a one period discount bond. Since there is one individual per generation, markets are sequentially complete and the competitive equilibrium is autarkic. In particular, using the above budget constraints, the resource feasibility constraint

$$c_t^y + c_t^o = y_t, \tag{1.2}$$

and utility maximization subject to (1.1), an equilibrium is such that

$$c_t^y = \alpha_t y_t, \qquad c_{t+1}^o = (1 - \alpha_{t+1}) y_{t+1}, \qquad \mathbb{E}_t[m_{t,t+1}^*] = \frac{1}{1 + r_t}, \tag{1.3}$$

where $m_{t,t+1}^*$ define the individual's stochastic discount factors at equilibrium, *i.e.*,

$$m_{t,t+1}^* = \frac{v'((1 - \alpha_{t+1}) y_{t+1})}{u'(\alpha_t y_t)}. \tag{1.4}$$

Note that, since markets are complete, the stochastic discount factors weighted by the probabilities of the next period event can be identified with the prices, $q_{t,t+1}^*$, of state contingent Arrow securities. More specifically, by the stated assumptions about the stochastic variables y_t, α_t, we have $q_{t,t+1} = q^{i,j}$ for $j \in \{H, L\}$, where

$$q^{HH} = p \frac{v'((1 - \alpha^H) y^H)}{u'(\alpha^H y^H)}, \quad q^{HL} = (1 - p) \frac{v'((1 - \alpha^L) y^L)}{u'(\alpha^H y^H)} \tag{1.5}$$

and

$$q^{LH} = p \frac{v'((1 - \alpha^H) y^H)}{u'(\alpha^L y^L)}, \quad q^{LL} = (1 - p) \frac{v'((1 - \alpha^L) y^L)}{u'(\alpha^L y^L)}. \tag{1.6}$$

The above imply that the time-t safe rates contingent on the time-t realizations of output are r^H and r^L such that

$$\frac{1}{1+r^H} = q^{HH} + q^{HL}, \qquad \frac{1}{1+r^L} = q^{LH} + q^{LL}. \tag{1.7}$$

For later reference, note that the "long-run" expectation of the GDP growth rate is

$$\mathbb{E}g = p^2(y^H/y^L) + (1-p)^2(y^L/y^H) + 2p(1-p) - 1. \tag{1.8}$$

According to the Ayiagari-Peled criterion, the competitive equilibrium is CPO if and only if the dominant eigenvalue of the matrix of state prices, $Q = (q^{ij})_{i,j=H,L}$ is smaller than one. Now assume that $u(c) = v(c) = \log c$ and let

$$\theta^{HH} = \frac{\alpha^H}{1-\alpha^H}, \quad \theta^{HL} = \frac{\alpha^H}{1-\alpha^L}, \quad \theta^{LH} = \frac{\alpha^L}{1-\alpha^H}, \quad \theta^{LL} = \frac{\alpha^L}{1-\alpha^L}, \quad \frac{y^H}{y^L} = \mu.$$

Then,

$$\frac{1}{1+r^H} = p\theta^{HH} + (1-p)\theta^{HL}, \qquad \frac{1}{1+r^L} = p\theta^{LH} + (1-p)\theta^{LL}, \tag{1.9}$$

and

$$Q = \begin{pmatrix} p\theta^{HH} & (1-p)\theta^{HL}\mu \\ p\theta^{LH}/\mu & (1-p)\theta^{LL} \end{pmatrix}.$$

Then, the characteristic polynomium is

$$P(\lambda) = \lambda^2 - (p\theta^{HH} + (1-p)\theta^{LL})\lambda + \left(\theta^{HH}\theta^{LL} - \theta^{HL}\theta^{LH}\right).$$

Since $\theta^{HH}\theta LL = \theta^{HL}\theta^{LH}$, the dominant eigenvalue is

$$\rho = p\theta^{HH} + (1-p)\theta^{LL}.$$

It follows that CPO holds for

$$p(1-\theta^{HH}) + (1-p)(1-\theta^{LL}) \geq 0. \tag{CPO}$$

Since θ^{jj} is a measure of how rich are the young relatively to the old in the "persistent" states jj, conditional Pareto optimality requires a distribution of endowments that makes the old sufficiently rich in these states, so that the young would not benefit by a transfer of resources from young to old age.

That $\rho < 1$ is a necessary condition for CPO follows from the Perron Frobenius Theorem. Suppose that we perturb the autarkic equilibrium by transferring some amount of consumption good, ϵ^j from the young to the old when the state is $j \in \{H, L\}$. Provided $\epsilon^j > 0$, this scheme makes the old better off and it does not make the young worse off if

$$\begin{aligned} U^H(\epsilon^H, \epsilon^L) \equiv u(\alpha^H y^H - \epsilon^H) + pu((1-\alpha^H)y^H + \epsilon^H) + (1-p)u((1-\alpha^L)y^L + \epsilon^L) &\geq 0, \\ U^L(\epsilon^H, \epsilon^L) \equiv u(\alpha^L y^L - \epsilon^L) + pu((1-\alpha^H)y^H + \epsilon^H) + (1-p)u((1-\alpha^L)y^L + \epsilon^L) &\geq 0. \end{aligned}$$

By differentiation at $\epsilon^H = \epsilon^L = 0$, and by a rearrangement of terms, the above are verified if

$$\begin{aligned} (q^{HH} - 1)\epsilon^H + q^{HL}\epsilon^L &\geq 0, \\ q^{LH}\epsilon^H + (q^{LL} - 1)\epsilon^L &\geq 0. \end{aligned}$$

Letting $\varepsilon' = (\epsilon^H, \epsilon^L)$, the above are equivalent to $Q\varepsilon \geq \varepsilon$. Since Q is a strictly positive matrix, the Perron-Frobenius theorem states that Q has a positive dominant eigenvalue $\rho > 0$ associated to a strictly positive eigenvector v. Then, if $\rho > 1$, we can take $\varepsilon = v$ and show $Q\varepsilon = \rho\varepsilon > \varepsilon$. Now assume that $\rho < 1$. By concavity,

$$\frac{U^i(\varepsilon') - U^i(0)}{u'(\alpha^i y^i)} \leq \sum_{i=H,L} q^{ij}\epsilon^j - \epsilon^i, \qquad i = H, L.$$

It follows that inefficiency can only occur if there exists ε such that

$$0 < \varepsilon \leq \varepsilon Q. \tag{*}$$

Now assume that such $\varepsilon > 0$ exists. Since the dominant root of Q is $\rho \in (0{,}1)$ and the associated eigenvector, $v > 0$, is determined up to a scalar factor, we can set $\epsilon \leq v$ and derive

$$0 < \varepsilon \leq Q\varepsilon \leq Qv = \rho v.$$

The above, together with (*) implies

$$0 < \varepsilon \leq Q\varepsilon \leq Q(\rho v) = \rho Qv = \rho^2 v.$$

Hence, by repeating this argument t times, we get that, for all $t > 0$, $0 < \varepsilon \leq \rho^t v$. Letting $t \to \infty$, we get a contradiction.

The upshot of the above discussion is that, starting from a competitive equilibrium, a transfer scheme similar to social security (*i.e.*, transfers from young to old) is Pareto improving conditional on the state at which individuals are born if and only if $\rho > 1$.

The work of Aiyagari and Peled has been further developed and revised by various authors, adapting it to more complex scenarios beyond the original focus on stationary equilibria. Bloise and Reichlin (2023) expanded upon the dominant root criterion, deriving an explicit formula for the growth-adjusted spectral radius. Their innovation includes considering the stochastic component of growth, arguing that simply comparing the long-term interest rate with the average growth rate is not fully meaningful. Even low interest rates can be consistent with conditional Pareto efficiency when the stochastic component of growth, typically negatively correlated with the marginal utility of wealth, is taken into account.

1.3.2 Ex - Ante Pareto Efficiency

Although many authors opt for the conditional criterion for efficiency evaluation of resources allocations and redistributive schemes, others, such as Gottardi and Kubler (2011), believe that, by adopting an interim welfare judgement, the inefficiencies that could be exploited to sustain the benefits of redistributive schemes are limited to the possibility of financial market incompleteness.

Under uncertainty, the widely accepted assumption is that markets are necessarily incomplete and individuals cannot provide any form of insurance against potential positive or

negative shocks that may occur at the time of their birth. Indeed, many works in the field, such as those of Imrohoroglu et al. (1995) and Krueger and Kubler (2006), propose dinamically efficient economies with missing markets. Chattopadhyay and Gottardi (1999) believe that, in a stochastic setting, it appears impossible to find a sequential market structure, where agents can establish contracts only after their birth, that supports the same equilibrium allocation established in a situation where agents have unrestricted access to a complete set of markets from the start date. The authors elaborate on what they believe is the situation closest to that dictated by certainty and complete markets: individuals who can fully insure themselves during the two periods of their lives but only conditional on the state/event in which they were born. With this premise, the two authors find that competitive equilibria can only be Conditionally Pareto optimal and not Ex-ante Pareto optimal. Their work suggest that using an Ex-ante welfare assessment, it is more likely that competitive equilibria result to be inefficient and hence leave space for intervention in terms of risk sharing improving policies. Gottardi and Kubler (2011) sustain that when welfare is evaluated ex- ante, even when markets are complete, competitive equilibria in overlapping generations economies are suboptimal, although conditionally Pareto optimal. They propose indeed an economy with complete markets with capital accumulation and an additional long term asset, land. In their assumed setting, what they find is that a Pay-as-you go reform is welfare improving even if equilibria are interim Pareto efficient.
In the following lines, it will be provided an example in which the ex-ante inefficiency of competitive equilibria is tested. In other words, it is an attempt to explore the possibility that individuals may benefit from getting insurance against the state at which they are born, before knowing which state is it.

Example: Ex-Ante Efficiency

Now, let's show the possibility competitive equilibrium may be *ex-ante* inefficient. In particular, the idea is to explore the possibility that individuals may benefit from getting insurance against the state at which they are born, before knowing which state is this. Starting from the autarkic equilibrium, we introduce the following scheme: the t-individual pays $\epsilon^j > 0$ in young age if the state at which she is born is j and receives ϵ^j in old age if the state is j. We are not imposing any restriction on the sign of these transfers, so that ϵ^j can be positive or negative for all $j = H, L$. If, for example, $\epsilon^L < 0 < \epsilon^H$, this scheme provides insurance in young age at the cost of generating more risk in old age. How does the t-individual's utility change, in this case? To see this, define the *ex-ante* expected utilities

$$\begin{aligned}
\bar{V}(\varepsilon) &= pv((1-\alpha^H)y^H + \epsilon^H) + (1-p)v((1-\alpha^L)y^L + \epsilon^L), \\
\bar{U}(\varepsilon) &= pu(\alpha^H y^H - \epsilon^H) + (1-p)u(\alpha^L y^L - \epsilon^L) + \bar{V}(\varepsilon).
\end{aligned}$$

We say that the assumed transfer scheme is an *ex-ante* Pareto improvement if

$$\bar{V}(\epsilon) \geq \bar{V}(0), \tag{1.10}$$

$$\bar{U}(\epsilon) \geq \bar{U}(0), \tag{1.11}$$

with at least one inequality. Letting $u'_j = u'(\alpha^h y^j)$ and $\tilde{\epsilon}^j = u'_j \epsilon^j$, the above is verified for some $\varepsilon = (\epsilon^H, \epsilon^L)$ such that

$$q^{HH}\tilde{\epsilon}^H + q^{LL}\tilde{\epsilon}^L \geq p\tilde{\epsilon}^H + (1-p)\tilde{\epsilon}^L, \tag{1.12}$$

$$q^{HH}\tilde{\epsilon}^H + q^{LL}\tilde{\epsilon}^L \geq 0, \tag{1.13}$$

which reduce to

$$((1-p)q^{HH} - pq^{LL})\tilde{\epsilon}^H \geq 0, \qquad \tilde{\epsilon}^L \geq -\frac{q^{HH}}{q^{LL}}\tilde{\epsilon}^H.$$

Clearly, if we restrict transfers to be from young to old, as in a social security system, the above require $(1-p)q^{HH} \geq pq^{LL}$. In the case $u(c) = v(c) = \log c$, this condition is verified for:

$$\alpha^L y^L \geq \alpha^H y^H,$$

i.e., the young get a low endowment (relative to the old) in the good state. Note that this condition can be verified even if the autarkic equilibrium is CPO, provided that $\alpha^H < \alpha^L$.

Now, assume that we impose the restriction $\varepsilon > 0$. Note that $V(0)$ and $U(0)+V(0)$ are the *ex-ante* expected utilities of the old and the young at any generic node of the event tree at the autarkic equilibrium, and, by concavity, $V(\epsilon) \geq 0$ $(U(\epsilon) + V(\epsilon) \geq 0)$ for some $\epsilon > 0$

only if $V'(0) \geq 0$ ($U'(0) + V'(0) \geq 0$). In particular, the first order effects of ϵ at $\epsilon = 0$ are

$$\begin{aligned} V'(0) &= pv'((1-\alpha^H)y^H) - (1-p)v'((1-\alpha^L)y^L), \\ U'(0) &= -pu'(\alpha^H y^H) + (1-p)u'(\alpha^L y^L). \end{aligned}$$

We conclude that the assumed transfer scheme is *ex-ante* Pareto improving if:

$$\begin{aligned} \frac{p}{1-p} - \frac{v'((1-\alpha^L)y^L)}{v'((1-\alpha^H)y^H)} &\geq 0, \\ u'(\alpha^H y^H)\left(\frac{v'(\alpha^L y^L)}{v'(\alpha^H y^H)} - \frac{p}{1-p}\right) + v'((1-\alpha^H)y^H)\left(\frac{p}{1-p} - \frac{v'((1-\alpha^L)y^L)}{v'((1-\alpha^H)y^H)}\right) &\geq 0. \end{aligned}$$

Now assume, as above, that $u(c) = v(c) = \log c$. Then, by a simple manipulation of the above inequalities, we derive the following conditions for $\epsilon > 0$ to be an *ex-ante* Pareto improvement:

$$\frac{p}{1-p}\left(\frac{1-\alpha^L}{1-\alpha^H}\right) \geq \mu \geq 1, \tag{1.14}$$

$$\left(\frac{1-\theta^{LL}}{\theta^{LL}}\right)\mu \geq \frac{p}{1-p}\left(\frac{1-\alpha^L}{1-\alpha^H}\right)\left(\frac{1-\theta^{HH}}{\theta^{HH}}\right). \tag{1.15}$$

Note that, if $\alpha^H = \alpha^L$, the above can only be verified for $\theta \geq 1$, implying that the autarkic equilibrium cannot be a CPO.

Now let $\theta^{jj} < 1$ for $j = H, L$. Then, by (CPO), the autarkic equilibrium is conditionally efficient. In this case, conditions (1.14) and (1.15) can be verified for $\theta^{HH} > \theta^{LL}$, *i.e.*, for $\alpha^H > \alpha^L$.

In light of the above, the thematic distinction between dynamic efficiency and Pareto efficiency in the applications of Samuelson's model becomes evident. As this is a work that addresses the welfare implications of an economic model, the focus is inevitably on Vilfredo Pareto's criterion, being, among other things, more restrictive as a condition. Indeed, as already mentioned, when there is uncertainty, even dynamically efficient allocations are suboptimal in Pareto terms due to incorrect risk-sharing.

This is one of the motivations that led to the immense popularity of this model and its countless applications for research in the field of welfare economics and the evaluation of socio-economic policies. Ball and Mankiw (2007) offer an elegant characterization of the policies needed to achieve the optimal allocation, while in this work, as in many other works that I will further cite, an attempt is made to characterize an optimal social security system model.

Chapter 2

Social Security

2.1 The functioning of pension systems

Pension systems are redistributive mechanisms intended to transfer resources across generations, precisely from active population (population in their working age) to inactive population. If the system is public, then it is financed by workers' contributions that the entitled state entities collect.
There exist different types of pension system, but the first distinction that I will mention is that based on the financing typology:

- Pay-as-you-go system: in every period, the total contributions are dedicated to the financing of pensions distributed in the same period (intergenerational agreement): $C_t \Longrightarrow P_t$, where C_t is the contribution and P_t is the pension amount.

- Fully-funded system: the contributions that each worker pays during his active period are invested on capital markets. The related pension will correspond to the accumulated amount, collected in the form of an annuity (individual insurance): $P_{t+1} = C_t \ (1+i)$.

To better understand the functioning and in particular the yield characteristics of these systems, let's first define some variables:

- w_t is the wage at time t;

- n is the employment growth rate;
- g is the productivity growth rate;
- τ is the contribution rate;
- $w_{t+1} = w_t\,(1+n)\,(1+g)$ is the wage at time t+1.

In a Pay-as-you-go system, $C_t = \tau w_t = P_t$ while $C_{t+1} = \tau w_t\,(1+n)\,(1+g) = P_{t+1}$. The implicit yield for the generation working at time t and retiring at time t+1, is:

$$\frac{P_{t+1}}{C_t} - 1 = \frac{\tau w_t\,(1+n)\,(1+g)}{\tau w_t} - 1 \cong g + n$$

Hence, the implicit yield rate of a Pay-as-you-go pension system is almost equal to the sum of the employment rate and the productivity growth rate.

On the other hand, a funded system, the contributions $C_t = \tau w_t$ are directly invested in capital markets and therefore pension benefits have a yield rate i that is that of the markets and is not related to the economic conditions of the specific country. Therefore, the Pay-as-you-go system guarantees a higher yield if $g + n > i$ and vice versa. Another difference between these two alternatives in related to savings. In a funded system, individuals compulsory devolve part of their savings to the financing of their pensions. Since retirement saving is invested in capital markets, the overall level of savings in each period t is not affected by the existence of the social security system. In a pay-as-you-go system, instead, the level of savings of an economy is affected (reduced) by the introduction of such a system as the retirement contributions do not turn into investments. However, this system makes it possible to pay pension benefits immediately, at the time of its introduction, without the need of previous contribution collected from the first generation of beneficiaries. This phenomenon is also called as "First Generation Effect". A secondary classification is that related on the method of computation and determination of pension benefits. There exist two main methods:

- Defined Benefit (DB): it ensures intergenerational equity, that is it guarantees the same replacement rate (pension benefit/latest salary) to all individuals with the same working life duration. Annual pension is equal to a given ratio of pensionable salary multiplied by the number of working years. This ratio is called *Coefficiente di rendimento*. The pensionable salary can be the one of the final years of the last career, an average of the salaries of either the whole working life or just the "best" working years.

- Notional Defined Contribution (NDC): it ensures actuarial fairness, that is guaranteeing the same internal interest rate (the rate that equates the present value of the contributions to the present value of the cash flow of pension benefits) to all individuals. The contribution amount transforms into a pension annuity through the transformation coefficient. It resembles a private funded system but the savings remuneration is not given by the market interest rate but is defined by law. The transformation coefficient is defined in order to guarantee the sustainability of the system and hence the pension benefits diminish as life expectancy increases.

The funded system uses generally the Defined Contribution (DC) method, which differs from the NDC only in that contribution are invested in capital markets, interest rates are different, and the intergenerational equity is not always guaranteed. Social security systems, in general, inevitably deal with uncertainty and therefore risk. The two systems described by now present differences also in this context. The first source of risk that ought to be mentioned is demographic/employment risk, which is related to the variability of the ratio of the total number of pensioners to the total number of employed people. Changes in this ratio may be caused by a variety of reasons such as the lengthening of average life, decrease in birth rate as well as contribution evasion or employment rate reduction. In a Pay-as-you-go system, who bears this risk depends on the equilibrium contribution rate τ^* :

$$\tau^* = \frac{P}{R_l} \frac{N_{pt}}{N_{lt}}$$

where P is the per capita average pension, R_l is the per capita average salary, N_{lt} is the number of active workers at time t and N_{pt} is the number of pensioners at time t. if τ^* increases, then the demographic risk is transferred to the contributors, while if the ratio $\frac{P}{R_l}$ decreases, the risk is on the pensioners' shoulders. However, it is possible to spread the risk among the two categories.

In a Fully-funded system, the risk is suffered by the old people as the individual pension takes into account life expectancy of the population at the moment in which the amount is established. Inflation risk is not insurable in a Fully-funded system while it falls on the pensioners in the case of a Pay-as-you-go system with no inflation rate indexation of pensions. Regarding the salary risk, if the system is a Pay-as-you-go with indexation to the salaries growth rate, then the burden is suffered by the workers while in a Fully-funded system, this is suffered by the old people, since the salaries dynamics of active workers do not influence the pension amounts received. Finally, the risk of inadequacy of returns regards mainly Fully-funded systems and, in particular, pensioners, as their annuities are determined by the markets returns. On the other hand, the rate at which contributions are transformed into annuities in Pay-as-you-go systems are based on real quantities, less volatile.

2.2 The Evolution of the Italian pension system

In our country, the public pension system is structured according to the Pay-as-you-go principle: contributions made by workers and companies to pension institutions are used to pay the pensions of those who have retired from work. There is no provision for accumulating financial reserves to cover future pensions. Clearly, in such a system, the inflow of funds (from contributions) must balance the outflow (pensions paid). Over the past thirty years, the Italian pension system has undergone structural reforms aimed at:

- Controlling public pension expenditure to ensure it doesn't grow disproportionately compared to the Gross Domestic Product (GDP).
- Establishing a supplementary pension system alongside the public one.
- Introducing elements of flexibility in retirement, using supplementary pensions as a tool.

To understand the significance of these reforms, it's crucial to briefly trace the key developments in our country's pension system, for which COVIP (2022) provided an overview.

During the 1970s, like most Western countries, Italy experienced a significant economic slowdown, primarily due to the 1973-1976 oil crisis, which disrupted the country's economic landscape. The state had to support those unable to find employment and businesses in crisis, leading to a challenging situation for public finances, characterized by a sharp rise in public debt. In the 1980s, many industrialized countries recognized the need to rebalance public accounts by reducing current expenditure. In Italy, it was only at the end of the decade that a fiscal tightening measure was implemented to correct budget deficits. From the 1990s, structural reforms began, including in the pension sector. The gradual increase in the average life expectancy meant pensions had to be paid for longer. Additionally, negative demographic transition resulted in economic growth slowdown, reducing contribution revenues. To address this, several reforms were implemented, all aimed at ensuring the sustainability of public finances:

- Minimum requirements for pension eligibility were raised, both in terms of age and contribution years.
- Pension amounts were linked to total contributions made throughout one's working life, GDP growth and life expectancy at retirement.
- The pension revaluation system changed, now linked only to inflation trends.
- Foundations were laid for the creation of supplementary pension funds, allowing workers to receive a more comprehensive pension in old age and diversify risks.

Until December 1992, a worker registered with INPS received a pension linked to their salary in the last years of work. With an average revaluation of 2 % for each contribution year, a pension equivalent to about 80 % of the last salary was granted after 40 years of contributions. This pension was subsequently revalued based on two main factors: price increases and real wage growth. Supplementary pension schemes were mainly present in banks and some companies with specific pension funds created only for their employees.

Italy has undergone a series of reforms in both the public pension system and the supplementary pensions system. This brought many innovations during the years, starting from 1992, with the Amato reform (Legislative Decree 503/1992), that changed the whole scenario: retirement age increased, and the contribution period for pension calculation was gradually extended to cover one's entire working life. The salaries used to determine the pension amount were revalued at 1 %. The automatic revaluation of pensions in payment was limited to price dynamics (not also to real wages). The Amato reform harmonized rules across different pension schemes and effectively reduced pension coverage compared to the last salary received. This led to the need to introduce a comprehensive regulation of supplementary pensions with the establishment of collective bargaining and open pension funds (Legislative Decree 124/1993).

With the 1995 Dini reform (Law 335/1995), the system transitioned from a wage-based (DB) to a contribution-based (NDC) regime. The difference between the two is substantial. In the wage-based system, the pension corresponds to a percentage of the worker's salary, depending on contribution years and salaries, especially those received in the last period of working life, which are generally the highest. In the contribution-based system, the pension amount depends on the total contributions made by the worker throughout their working life. The transition from one calculation system to another was gradual, distinguishing workers based on contribution years. This created three different situations: workers who had at least 18 years of contributions by the end of 1995 retained the wage-based system; workers with less than 18 years of contributions by that date were given a mixed system, wage-based up to 1995 and contribution-based for subsequent years (pro-rata contribution method); finally, those hired after 1995 were given the contribution-based calculation system. Furthermore, pensions were revalued based on the inflation rate. These changes led to a significant reduction in the ratio between the first pension instalment and the last work income (replacement rate) compared to what was previously granted by the wage-based system. The Legislative Decree 47/2000 then enabled improvements in the tax treatment for those joining a pension fund, as well as new opportunities for those wishing to join individually by subscribing to an open pension fund or an Individual Pension Plan (known as PIP).

Another crucial reform occurred in 2004, with the Maroni delegation law 243/2004, through which incentives were established for those who decided to postpone early retirement: those who chose to delay - limited to workers who met the requirements by December 31, 2007 – could benefit from a super bonus consisting of the payment of pension contributions that would have been paid to the pension institution (an amount equivalent to about a third of the salary). Moreover, the retirement age for early and old-age pensions increased and the delegation criteria for a comprehensive pension reform were established. Key elements of the delegation were: better alignment between different supplementary pension forms, the transfer of TFR by employees to supplementary pensions also tacitly, the unity and homogeneity of supervision over the sector attributed to COVIP. With the 2007 Prodi reform (Law 247/2007), the so-called "quotas" for access to early retirement were introduced, determined by the sum of age and years worked: in 2009 the quota to be reached was 95 (with at least 59 years of age), from 2011 it went to quota 96 (with at least 60 years of age), while from 2013 it rose to 97 (with at least 61 years of age); the automatic and triennial revision of the mandatory pension calculation coefficients was also introduced based on the average life calculated on ISTAT data. Another crucial year for the evolution of the Italian pension system was 2011, when the "Save Italy" manoeuvre (Law 214/2011) was enacted by the Monti government. From 2012, a variety of modifications were made. The retirement age for both men and women will be adjusted every two years based on the increase in life expectancy, the minimum contribution period required for retirement was increased to 42 years and 10 months for men and 41 years and 10 months for women and the "quotas" system for early retirement was abolished. In addition, the pension amount was linked to the entire working life, and the revaluation of pensions was limited to inflation dynamics. Finally, with the 2019 manoeuvre (Law 145/2018), the "quota 100" was introduced, allowing retirement with 62 years of age and 38 years of contributions, but only for those born before 1959.

2.3 The inefficiencies in Pay-as-you-go social security systems

By virtue of what has been described so far in this chapter, it is clear that pension systems, in particular Pay-as-you-go system, inevitably present gaps in terms of efficiency. It is therefore interesting to study the phenomenon through the concepts presented in the first chapter of this work.

Pay-as-you-go (PAYG) social security programs typically accomplish two objectives at the same time: inter-generational and infra-generational risk sharing. The first, most commonly studied, consists in transferring resources from the young to the old, possibly contingent on the realization of aggregate uncertainty; the second is achieved by insuring old individuals against the risks associated to unintended early retirement, individual-specific health or productivity shocks.

Lack of insurance for individual-specific labor productivity shocks calls for government action but one could argue that governments may address *inter* and *infra*-generations risk sharing using "independent" tools. However, due to the limited range of instruments that governments can use to address these potential sources of inefficiencies, the extent of inter and infra-generations transfers are typically related. For example, pension benefits accruing to old age individuals with higher labor income (or staying in the labor force for more years) are reduced to compensate old age individuals who have lower labor income (or are unable to work). Virtually, all social security systems include minimum guaranteed or means-tested old age pensions, survivor benefits and cost of living adjustments that give higher percentage increases to lower income retirees[1].

When discussing the adoption of a Pay-As-You-GO pension system, the core arguments in favour are related to potential dynamic inefficiency of the economy and bounded rationality and time inconsistency in individuals' behaviour in the realistic environment, that features lack of perfect information. Regarding dynamic efficiency, there is a widespread consensus (e.g. (Abel et al., 1989)) that dynamic inefficiency is unlikely (as it requires exceptionally

[1]For instance, according to the European Central Bank (Rodríguez-Vives and Kezbere, 2019), expenditure on means-tested old age pensions in 2016 as a percentage of total government expenditure on old age pensions is 12% in Spain, 8% in the Netherlands and Ireland and about 5% in Portugal.

and long lasting low rates of returns on safe assets) and that the welfare losses due to the market distortions (reduced capital accumulation and labor supply) generated by the social security taxes and benefits are likely to outweigh any welfare benefit from intergenerational transfers. Feldstein (Feldstein, 1985) argued that the main reason to introduce social security is individuals' limited foresight and the main cost is the "welfare loss that results from reductions in private saving (p. 303)", *i.e.*, a lower capital accumulation. Kubler and Kruguer (Krueger & Kubler, 2006) provide a quantitative evaluation of introducing social security in an overlapping generations model with fully rational individuals and incomplete markets. By assuming dynamic efficiency, they conclude that the welfare cost from the crowding out of capital stock is likely to outweigh the benefit from the enhanced risk sharing provided by social security transfers. Interestingly, if one abstracts from the crowding out effect, social security becomes welfare improving even though the economy is dynamically efficient.

In general, economists hold different opinions about the welfare effects of introducing Pay-As-You-GO social security retirement benefits, on which there is still an open debate.

Chapter 3

Social security under uncertainty

To assess the social welfare benefits of introducing pay-as-you go social security, I consider a standard overlapping generations model with production and stochastic aggregate productivity shocks. Individuals live two periods only, they all supply a unit of labor inelastically in young age and, when old, are subject to an idiosyncratic individual specific productivity (or “health”) shock. If the shock is positive, they supply labor elastically, otherwise they retire early. Hence, the economy is subject to two sources of uncertainty: the idiosyncratic productivity (or health) shocks and an aggregate productivity shock affecting output and wages. We impose a specific Pay-As-You-Go social security system where the contribution rate is constant and retirement income is proportional to the current wage (or GDP). This feature implies that the young’s contributions are immune from the risk that the current wage is low relative to the promised retirement income of the current old, so as to make it more likely that the system is not welfare diminising for “unlucky” generations. This type of Pay-As-You-Go program can be accomplished in a defined benefit (DB) (which is most widely adopted across advanced economies) or through a notional defined contribution (NDC) system where retired income is based on past contributions augmented by some proportion of the effective wage growth (growth-adjustment) (as in Italy and Sweden). Since I assume that social security promises some income (as a share of the current wage) to the old individuals that have been hit by an adverse productivity shock, retirement income is marked down so as to verify a balanced budget condition. Hence, social security has a redistributive component and it may be helpful to improve the degree of risk sharing across and within generations from an ex ante and, possibly, from an ex post perspective. However, as social security contributions affect aggregate savings and investment, and since old age labor supply is elastic, the system generates distortions that may have adverse effects on individuals' welfare through a lower level of wages. The purpose of the model is to provide an estimate of the welfare benefits of social security in this type of environment based on some relevant parameters, such as the degree of risk aversion, the safe interest rate and the mean and variance of output growth and a social security policy characterized by a fixed contribution rate, τ , and a constant degree of redistribution, ζ, defined as the benefit paid to non working old individuals as a share of the contribution rate. To keep the analysis tractable and avoid as much as possible reliance on computational methods, I consider a specific parametrization of individuals' preferences, defined by a class of Epstein-Zin recursive utility with a unitary intertemporal elasticity of substitution. Assuming zero public debt, this parametrization generates a time-invariant marginal propensity to save and a sharp characterization of the conditions under which social security is welfare enhancing. It turns out that a first order approximation of the expected welfare benefit enjoyed by any generation of individuals, conditional on the state at birth, can be decomposed into three distinctive terms, all of which can be estimated.

3.1 The Model

3.1.1 Technology

Consider a canonical overlapping generations economy with a constant unit mass of *ex-ante* identical two period lived individuals. In young age, a fraction $n^y \in [0,1]$ of them offer a unit of labor inelastically, whereas a variable labor time is offered elastically in old age. In particular, as in Michel and Pestieau (Michel & Pestieau, 2013), the old age time span is divided into two sub-periods. The second is called "retirement age", the time span in which no individual is able or allowed to work. In the first sub-period, instead, old individuals come to the labor market endowed with a labor productivity ϵ, an i.i.d. random variable whose realizations, $\mathcal{E} = \{\epsilon^0, .., \epsilon^m\}$, are known (and publicly observable) at the beginning of the second period of an individual's life. $\mathcal{E}$ is normalized so that with $\epsilon^0 = 0 < \epsilon^1 \leq .. \leq \epsilon^m = 1$. From now on we refer to an old individual with productivity ϵ^i as a type-i worker. Letting p_i be the *ex ante* probability that an individual is of type i, and h_t^i her (intensive margin) labor supply, the law of large numbers implies that total labor supply at each date t is

$$L_t = n^y + n_t^o,$$

where $n_t^o = \sum_i p_i \epsilon^i h_t^i$. Output, y_t, is produced by using a Cobb-Douglas technology using capital, K and labor, L. Production function is represented as

$$y_t = a_t K_{t-1}^{\alpha_k} L_t^{\alpha_l}, \tag{3.1}$$

where $\alpha_k + \alpha_l \leq 1$ and a_t is a stochastic TFP variable. Note that, if the sum of the factor shares is less than one, the technology allows for the existence of pure rents. Now let w and R be the wage rate and the rental rates of capital and land, respectfully. By profit maximization and perfect competition, we get the marginal productivity conditions

$$w_t = \alpha_l y_t / L_t, \qquad R_t = \alpha_k y_t / K_{t-1}, \qquad \Pi_t = (1 - \alpha_k - \alpha_l) y_t, \tag{3.2}$$

at all time periods $t \geq 0$, where Π defines the non negative *pure rents*.

3.1.2 Preferences

We assume that preferences have a recursive structure of the Epstein-Zin variety. Let c_t^y, c_{t+1}^o be the age and state contingent consumptions of an individual born at time t and h_{t+1} the labor effort she supplies in old age. We consider the following class of "lifetime" utility functions

$$U_t = (1 - \beta) u(c_t^y) + \beta u \left(H^{-1} \left(\mathbb{E}_t H(v(c_{t+1}^o, h_{t+1})) \right) \right), \tag{3.3}$$

where $u(.)$ and $v(.)$ are increasing and strictly concave functions and, for some $\gamma > 0$, different from one,

$$H(v) = v^{1-\gamma}/1-\gamma. \tag{3.4}$$

With these preferences, the young household exhibits aversion towards volatility in old age utility. Note that $\gamma = -H''(v)v/H'(v)$ is the (constant) coefficient of relative risk aversion with respect to risks in old age utility and

$$\mu_t = H^{-1}\left(\mathbb{E}_t H(v)\right) = \left(\mathbb{E}_t v^{1-\gamma}\right)^{\frac{1}{1-\gamma}}$$

is the *certainty equivalent* of old age utility. For analytical convenience, the form of the per period utilities are specified as follows

$$u(c) = \begin{cases} c^{1-\sigma}/1-\sigma & \text{for } \sigma > 0,\ \sigma \neq 1, \\ \log c & \text{otherwise,} \end{cases} \qquad v(c^o, h) = \frac{(c^o)^{1-\eta}\left(1-\kappa(1-\eta)h^{1+\chi}\right)^{\eta}}{1-\eta}, \tag{3.5}$$

where $\eta \in (0,1)$, $\kappa \geq 1$, $\chi > 0$. Note that σ represents the intertemporal elasticity of substitution. The functional form for $v(.)$ is also considered by Trabandt and Uhlig, 2011, and it features a constant Frish elasticity of labor supply equal to $1/\chi$. Note that the stochastic discount factor for the class of utility functions that we are considering is

$$m_{t+1} = \left(\frac{\beta}{1-\beta}\right) u'\left(\left(\mathbb{E}_t v^{1-\gamma}\right)^{\frac{1}{1-\gamma}}\right)\left(\mathbb{E}_t v^{1-\gamma}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{v^{-\gamma} v_1(c^o_{t+1}, h_{t+1})}{u'(c^y_t)}\right). \tag{3.6}$$

3.1.3 Budget Constraints and the Social Security System

The only available insurance against individuals' old age productivity shocks is provided through fiscal transfers. In particular, the income transferred to old individuals at retirement by the social security system has two components, a pension benefit based on past contributions and a redistributive component inversely related to an individual's labor productivity. As in the NDC systems, pension benefits at time t are equal to the worker's past contributions, based on a fixed contribution rate $\tau \in [0,1]$, appreciated by a notional rate of return defined as a proportion of the actual wage growth from $t-1$ to t, as in the following formula

$$\text{pension benefits to type-}i\text{ indiv.} = \underbrace{\tau n^y w_{t-1}}_{\text{past contrib.}} \times \rho \times \underbrace{\left(\frac{w_t}{w_{t-1}}\right)}_{\text{wage growth}} + \tau \epsilon^i h^i,$$

where the term $\rho \in [0,1]$ is the degree to which past contributions are re-evaluated. This adjustment is necessary to balance the per period social security inflows and outflows, due to the redistributive component of the system. In fact, in addition to pension benefits, old individuals at time t receive a transfer unrelated to past contributions serving the purpose of insuring against idiosyncratic productivity shocks. The latter is proportional to the current wage and defined by

$$b \times (1 - \epsilon^i) \times w_t,$$

where $b \in [0,1]$. Under this scheme, the overall income at retirement from the PAYG social security system is a fixed proportion of the current wage rate. In particular, letting P_t^i be the total transfers from social security to a type-i individual, we have

$$P_t^i = [\tau(n^y \rho + \epsilon^i h_t^i) + b(1 - \epsilon^i)]w_t. \tag{3.7}$$

This model assumptions about the social security benefit scheme is quite special and designed to minimize distortions. In particular, the (downward) adjustment ρ on pension benefits is only applied to the contributions paid in young age and the subsidized component, b, is not related to h^i, so that old age labor supply is undistorted by current contributions and benefits. The proportionality of retirement income to current wages implies that the retirees share the risk of aggregate shocks at retirement with the (young) workers. This risk would entirely fall on young workers if pension benefits were independent of the current wages at retirement (as in some defined contribution or defined benefit schemes). Quite clearly, a social security retirement income proportional to the current wage may be replicated through alternative programs, such as a defined benefit scheme indexed to current wages at retirement. What really matters in our analysis is that the old are subject to the old age aggregate risks related to wage and output variability. In practice, income redistribution within social security systems is often implemented through progressive benefit formulas, minimum guaranteed benefits, cost of living adjustments, taxation of pension income and other means.

To insure balanced budget of the social security system, we use the law of large numbers and impose

$$\tau\left(n^y + n^o_t\right) = \sum_i p_i P^i / w_t = \tau\left(n^y \rho + n^o_t\right) + b\left(1 - \sum_i p_i \epsilon^i\right),$$

implying

$$\tau n^y (1-\rho) = (1-\bar{\epsilon}) b, \tag{3.8}$$

where $\bar{\epsilon} = \sum_i p_i \epsilon^i$ is the average old age productivity. It is useful, at this point, to define the “degree of (intergenerational) redistribution” of the social security system with the additional variable $\zeta = b/\tau$, so that the balanced budget condition implies

$$\rho = 1 - \zeta(1-\bar{\epsilon})/n^y. \tag{3.9}$$

Then, $\rho = 1$ implies $\zeta = 0$ and $\zeta > 0$ for all $\rho \in [0,1)$. In what follows we will be assuming that $b \leq \tau$ and $\rho \geq 0$, or, equivalently,

$$\rho \geq 1 - \frac{1-\bar{\epsilon}}{n^y}, \qquad \zeta \leq \frac{n^y}{1-\bar{\epsilon}}. \tag{3.10}$$

The value τ is a measure of the intensity of the social security program and the value ζ a measure of the degree of redistribution implied by program. We assume that $n^y + \zeta(\bar{\epsilon} - \epsilon^i) \geq 0$ for all i, which guarantees a positive net retirement income from social security for all types. This assumption boils down to the following restriction

$$\zeta \leq \frac{n^y}{1-\bar{\epsilon}}. \tag{3.11}$$

Given the above premises and letting c^y_t and $c^{o,i}_{t+1}$ be age and state contingent consumptions, the budget constraints of an individual born at t are

$$c_t^y = n^y w_t(1-\tau) - K_t, \tag{3.12}$$

$$c_{t+1}^{o,i} = R_{t+1}K_t + \Pi_{t+1} + \epsilon^i h_{t+1}^i w_{t+1}(1-\tau) + P_{t+1}^i. \tag{3.13}$$

Note that, by the assumption about technology, at equilibrium we have

$$R_{t+1}K_t + \Pi_{t+1} = (1-\alpha_l)y_{t+1}, \qquad y_{t+1} = L_{t+1}w_{t+1}/\alpha_l, \tag{3.14}$$

so that, letting $\pi = (1-\alpha_l)/\alpha_l$,

$$R_{t+1}K_t + \Pi_{t+1} = \pi L_{t+1}w_{t+1}. \tag{3.15}$$

The above derivation will be used extensively in the sequel. Using (3.7) and (3.9) in (3.13), we obtain

$$c_{t+1}^{o,i} = R_{t+1}K_t + \Pi_{t+1} + [\epsilon^i h_{t+1}^i + \tau(n^y + \zeta(\bar{\epsilon} - \epsilon^i))]w_{t+1}. \tag{3.16}$$

Hence, for $\zeta > 0$, the redistributive component of social security increases (decreases) the old age consumption of the individuals whose productivity is lower (higher) than the average productivity. Since the productivity (health) status of an old individual is verifiable and late retirement is mandatory, the system may not be actuarially fair and productive individuals may want to retire earlier (work less). In particular, note that, since $h^0 = 0$ (as $\epsilon^0 = 0$), a type-i individual attains a larger consumption than the lest productive individual under this mandatory retirement system only if

$$h_t^i \geq \tau\zeta. \tag{3.17}$$

In the sequel we will assume that τ and ζ are sufficiently small to guarantee that the above is always verified for all $i > 0$.

3.2 Equilibrium

3.2.1 Characterization

A young individual born at time t maximizes the lifetime utility defined in (3.3), (3.5) with respect to $(c_t^y, c_{t+1}^o, K_t, h_{t+1}^i)$ subject to the budget constraints (3.12), (3.13). The solution to this problem is characterized by the following first order conditions

$$\begin{aligned} \mathbb{E}_t\left[m_{t+1}R_{t+1}\right] - 1 &= 0, && (3.18)\\ \eta\kappa(1+\chi)c_{t+1}^{o,i}(h_{t+1}^i)^{\chi} - (1-\kappa(1-\eta)(h_{t+1}^i)^{1+\chi})\epsilon^i w_{t+1} &= 0, \qquad i = 0,..,m, && (3.19) \end{aligned}$$

where the stochastic discount factors, m_{t+1} are defined in (3.6). A *competitive equilibrium* with the (time invariant) social security policy (τ, ζ), is a sequence of output, capital, labor, rates of return on capital and wages,

$$\{y_t, K_t, h_t^i, R_t, w_t; i = 1,..,n\}_{t=0}^{\infty},$$

verifying the profit maximization conditions (3.2), the budget constraints (3.12), (3.13) and the first order conditions (3.18), (3.19).

By the assumed characterization of technology and individuals' preferences, and, in particular, by the specification in (3.5), I derive two important properties about individuals' behavior at equilibrium. First, the old age labor supplies, h^i, are time invariant. This follows crucially from the Cobb-Douglas specification, which implies that capital income is proportional to wage income and, then, the state contingent old age consumptions as a share of the current wage are time invariant and given by

$$z^i \equiv \frac{c_{t+1}^{o,i}}{w_{t+1}} = \pi L + \epsilon^i h^i + \tau(n^y + \zeta(\bar{\epsilon} - \epsilon^i)) \qquad \text{for } i = 0,..,m. \tag{3.20}$$

In particular, by the above and (3.19), the time invariant old age labor effort, h^i, is implicitly defined by the equation

$$\frac{\epsilon^i}{\kappa} = \left[(1+\eta\chi)\epsilon^i h^i + \eta(1+\chi)\left(\tau(n^y + \zeta(\bar{\epsilon} - \epsilon^i)) + \pi L\right)\right](h^i)^{\chi}. \tag{3.21}$$

Note that $h^i(\tau)$ are the type-i old age labor supplies once we take into account the indirect effect of each individual's labor supply on capital income, which is taken as given at the individual level. A second important result is that young individuals' savings depends on two time dependent variables. One is the net of contributions wage income in young age, $n^y w_t(1-\tau)$, the other is a term reflecting the moments of next period wage and next period per unit of wage consumptions. Namely, let, for all i,

$$\lambda^i = (1 - \kappa(1 - \eta)(h^i)^{1+\chi})^{(1-\gamma)\eta} \tag{3.22}$$

and define the variable

$$x_t = \left(\frac{\beta}{1-\beta}\right)(1-\eta)^\sigma \left(\mathbb{E}_t w_{t+1}^\theta\right)^{\frac{1-\sigma}{1-\gamma}} \frac{\mathbb{E}[\lambda z^{\theta-1}]}{(\mathbb{E}[\lambda z^\theta])^{\frac{\sigma-\gamma}{1-\gamma}}}, \tag{3.23}$$

where $\theta = (1-\gamma)(1-\eta)$ and $\mathbb{E}[\lambda z^\theta] = \sum_i p_i \lambda^i (z^i)^\theta$. Then, the solution, K_t, to the first order condition (3.18) is a *saving function* $S_t = S(n^y w_t(1-\tau), x_t) \in [0, n^y w_t(1-\tau)]$, implicitly defined by

$$S_t = (n^y w_t(1-\tau) - S_t)^\sigma x_t \left(\frac{\alpha_k L}{\alpha_l}\right). \tag{3.24}$$

Based on these premises, we can state the following.

Proposition 1. *At equilibrium, the old age labor supplies, $(h_t^i; i \in \mathcal{E})$, are time invariant functions, defined as $h_t^i = h^i(\tau)$. Then, letting*

$$n_t^o(\tau) = \sum_i p_i \epsilon^i h^i(\tau),$$

the equilibrium sequence of labor supply and capital and wage rates takes the following form

$$L_t = n^y + n_t^o(\tau) \equiv L(\tau), \tag{3.25}$$

$$K_t = S(n^y w_t(1-\tau), x_t). \tag{3.26}$$

Evidently, all the remaining variables, y_t, R_t, as well as individual consumptions, can be obtained from the set of conditions defined above. In particular, the sequence of equilibrium wage rates and capital returns are defined by

$$w_{t+1} = a_{t+1}\alpha_l \left(S(n^y w_t(1-\tau), x_t)\right)^{\alpha_k} L^{\alpha_l}, \tag{3.27}$$

$$R_{t+1} = (\alpha_k/\alpha_l) L w_{t+1}. \tag{3.28}$$

Observe that, for a unitary intertemporal elasticity of substitution, *i.e.*, for $\sigma = 1$, we have

$$x_t = \left(\frac{\beta}{1-\beta}\right)(1-\eta)^{\sigma} \frac{\mathbb{E}[\lambda z^{\theta-1}]}{\mathbb{E}[\lambda z^{\theta}]} \equiv x \tag{3.29}$$

and the saving function is

$$S_t = s \times n^y w_t(1-\tau), \tag{3.30}$$

where

$$s = \frac{\alpha_k L x}{\alpha_l + \alpha_k L x}$$

is the constant saving rate. Note, also, that old age labor supply is increasing in ϵ and decreasing in τ, as shown in the following proposition.

Proposition 2. *Assuming* (3.11)*, the labor supply functions, $h^i(\tau)$, are increasing in ϵ^i for all i and the aggregate labor supply, $L(\tau)$, is decreasing in τ and increasing in ζ.*

The proof of the proposition is in appendix A.I. Both effects follow from the normality of leisure. In particular, a larger τ (ζ) increases (decreases) old age non labor income thereby reducing the incentives to work. Note that the adverse effect of social security on labor supply is not a by product of tax distortions, because we have assumed that old individuals recognize the exact equivalence between contributions and pension benefits. The reason why $L'(\tau) < 0$ is that, once uncertainty is realized, productive individuals are richer than they would be without social security and, since leisure is a normal good, they tend to work less. The opposite effect holds when we increase ζ, since the latter makes productive individuals poorer. This type of mechanism has been analyzed by Marcet and Weil (Marcet et al., 2007).

Note that the idiosyncratic uncertainty and old age labor supply affects individuals' savings behavior in an ambiguous way. To see this, suppose $\epsilon_i = \bar{\epsilon}$ for all $i \in \mathcal{E}$. In this case, the per unit wage consumptions, z^i, are all equal to

$$\bar{z} = \pi L + \bar{\epsilon}\bar{h} + \tau n,$$

and

$$x_t = \left(\frac{\beta}{1-\beta}\right)(1-\eta)^{\sigma}\left(\mathbb{E}_t w_{t+1}^{\theta}\right)^{\frac{1-\sigma}{1-\gamma}} \bar{z}^{(1-\eta)(1-\sigma)-1} \equiv x_t^c.$$

Now assume that $(\sigma-\gamma)/(1-\gamma) > 0$ and let old age labor supply be inelastic ($\chi = -1$). Noting that $\theta < 1$, by Jensen's we have

$$\mathbb{E}[z^{\theta}] \leq \bar{z}^{\theta}, \qquad \mathbb{E}[z^{\theta-1}] \geq \bar{z}^{\theta-1}.$$

Then, for the case of inelastic labor supply and $(\sigma-\gamma)/(1-\gamma) > 0$ we have $x_t \geq x_t^c$. If this inequality holds, savings falls when we shut down idiosyncratic uncertainty. Hence, when inelastic labor supply and $(\sigma-\gamma)/(1-\gamma) > 0$, we have precautionary savings, but the elastic labor supply in old age and $(\sigma-\gamma)/(1-\gamma) < 0$ may overturn this result.

3.3 Welfare Benefits of Social Security

3.3.1 First Order Effects

In this section we provide a decomposition of the first order welfare effects of rising τ from zero on each generation. The effect on utility U_t of an individual born at t of increasing the social security contribution τ is given by

$$\frac{\partial U_t}{\partial \tau} = (1-\beta)u'(c_t^y)\left\{\frac{\partial c_t^y}{\partial \tau} + \mathbb{E}_t\left[m_{t+1}\frac{\partial c_{t+1}^{o,i}}{\partial \tau}\right]\right\}. \tag{3.31}$$

For this purpose, we first evaluate the "indirect" effect of τ on contingent consumptions, *i.e.*, the effect obtained by ignoring the impact on the decision variables, K_t and h_{t+1} ("direct effects"). The reason why we can ignore the latter effects is that these are going to vanish when we will evaluate the overall effects on the individuals' expected utility because of the envelope theorem. Recalling the budget constraints, the "indirect" effect of τ on c_t^y

and $c_{t+1}^{o,i}$ evaluated at $\tau = 0$ are

$$\left.\frac{\partial c_t^y}{\partial \tau}\right|_{\tau=0} = n^y \frac{\partial w_t}{\partial \tau} - n^y w_t, \tag{3.32}$$

$$\left.\frac{\partial c_{t+1}^{o,i}}{\partial \tau}\right|_{\tau=0} = \left(K_t \frac{\partial R_{t+1}}{\partial \tau} + \frac{\partial \Pi_{t+1}}{\partial \tau}\right) + \epsilon^i h^i \frac{\partial w_{t+1}}{\partial \tau} + [n^y - \zeta(\epsilon^i - \bar{\epsilon})] w_{t+1}. \tag{3.33}$$

We separate the effects on U_t for given wages and capital from the effects that arise through changes in these two variables. The latter are called *distortionary effects* of social security. Adopting this partition, using the above derivatives in (3.31), letting $1 + g_{t+1} = w_{t+1}/w_t$ and evaluating all effects at $\tau = 0$, we obtain

$$\left.\frac{\partial U_t}{\partial \tau}\right|_{\tau=0} = -(1-\beta) u'(c_t^y) w_t \left[n^y (1 - q_t^g) + \zeta q_t^\epsilon + \Delta_t\right], \tag{3.34}$$

where,

$$q_t^g = \mathbb{E}_t[m_{t+1}(1 + g_{t+1})], \tag{3.35}$$

$$q_t^\epsilon = \mathbb{E}_t\left[m_{t+1}(1 + g_{t+1})(\epsilon - \bar{\epsilon})\right], \tag{3.36}$$

$$\Delta_t = -n^y \frac{\log w_t}{\partial \tau} - \frac{1}{w_t} \mathbb{E}_t \left[m_{t+1} \left(K_t \frac{\partial R_{t+1}}{\partial \tau} + \frac{\partial \Pi_{t+1}}{\partial \tau} + \epsilon h \frac{w_{t+1}}{\partial \tau}\right)\right]. \tag{3.37}$$

The term q_t^g represents the benefit of intergenerational transfers and it can be interpreted as the price at t of a discount bond whose payoffs are indexed to wage (output) growth (a *GDP indexed bond*). The term q_t^ϵ, instead, represents the cost of insuring against old age (idiosyncratic) productivity shocks. We are now discussing the meaning, sign and size of these three terms in turn.

Inter-Generations Risk Sharing

Note that, by the Cobb-Douglas specification of technology,

$$w_{t+1} = \frac{\alpha_l K_t}{\alpha_k L} R_{t+1},$$

so that, recalling the first order condition (3.18),

$$q_t^g = \frac{1}{w_t}\mathbb{E}_t[m_{t+1}w_{t+1}] = \frac{\alpha_l K_t}{\alpha_k w_t L}\mathbb{E}_t[m_{t+1}R_{t+1}] = \frac{\alpha_l K_t}{\alpha_k w_t L}. \tag{3.38}$$

Since, for $\tau = 0$, $K_t \leq n^y w_t$, the above implies

$$q_t^g \leq \frac{\alpha_l}{\alpha_k} \times \frac{n^y}{n^y + n^o}.$$

Recalling from (3.30) that, if $\sigma = 1$, the saving rate is time invariant, from the above we derive that, in this case,

$$q_t^g = \frac{\alpha_l s n^y w_t (1-\tau)}{\alpha_k w_t L} = \frac{\alpha_l}{\alpha_k} \times \frac{n^y}{n^y + n^o} \times s \times (1-\tau). \tag{3.39}$$

Hence, $\sigma = 1$ implies that q^g is time invariant. Based on the dominant root criterion proposed in Aiyagari and Peled (1991) and the extension to stochastic growth and non stationary equilibria provided in Bloise and Reichlin (2023), conditional Pareto efficiency obtains when the dominant root of the growth adjusted state price matrix (in a stationary environment) or the spectral radius of the linear operator defined by the growth adjusted prices is smaller than one. More formally, the spectral radius is defined as

$$q^* \equiv \lim_{T\to\infty} \sqrt[T]{\mathbb{E}_t \prod_{j=1}^{T} m_{t+j}(1 + g_{t+j})} \tag{3.40}$$

and $q^* < 1$ is a necessary and sufficient condition for a competitive equilibrium without social security to be *conditionally Pareto efficient*. This notion, we recall, corresponds to the inability of any across generations contingent transfer policy to improve the utility of any generation conditional on the state at which it is born. Note that, by the law of iterated expectations, (3.40) can be rewritten as

$$q^* = \lim_{T\to\infty} \sqrt[T]{\mathbb{E}_t \prod_{j=0}^{T-1} q_{t+j}^g}. \tag{3.41}$$

Note that, for $\sigma = 1$, q^g is time invariant, so that the latter is equal to q^* and, then, in this case, $q^g < 1$ is a necessary a sufficient condition for competitive equilibria to be conditional Pareto efficient.

Infra-Generation Risk Sharing

The term q^ϵ is a measure of the individuals' net welfare cost from within generations risk sharing. Note that, by the independence of the idiosyncratic shocks,

$$q_t^\epsilon = q_t^g \left(\frac{\mathbb{E}\left[\lambda z^{\theta-1}(\epsilon - \bar{\epsilon})\right]}{\mathbb{E}\left[\lambda z^{\theta-1}\right]} \right) = q_t^g \left(\frac{\text{Cov}[\lambda z^{\theta-1}, \epsilon]}{\mathbb{E}\left[\lambda z^{\theta-1}\right]} \right). \tag{3.42}$$

Hence, $q^\epsilon > 0$ only if there is a positive covariance between the old age marginal utilities of consumption (normalized by the wage rates) and the idiosyncratic shocks. Importantly, these stochastic discount factors are affected by the terms λ^i, which depend on the old age labor supply, as shown in equation (3.22). These are i and h-independent only if $\gamma = 1$, and they are increasing (decreasing) in h^i for $\gamma > 1$ ($\gamma < 1$).

Hence, the sign of q^ϵ is ambiguous. At a first sight, a high productivity, ϵ, increases individuals' retirement income and, then, it lowers the (wage adjusted) marginal utility of consumption, $z^{\theta-1}$. This should make the covariance in (3.42) negative. In this case, we say that social security improves risk sharing. On the other hand, marginal utilities of consumption are weighted by the terms λ^i. As mentioned above, if $\gamma \neq 1$, these are affected by the old age labor supply, which, in turn, is increasing in the productivity shocks ϵ^i. Then, a higher value of ϵ lowers the marginal utility of consumption in that state, but also increases labor supply in the same state and, then, if $\gamma > 1$, it increases the value of λ. The intuition is that, through the utility aggregator, labor supply serves the scope of reducing fluctuations in old age utility. Hence, if individuals' risk aversion is large, the extra risk sharing in consumption across individuals of the same generation that is induced by a higher τ, generates less risk sharing in utility. To get a better sense of what is going on, suppose that $\epsilon \in \{0,1\}$ and let $\kappa = 1$, $\chi = 0$. Then, letting $h = h^1$, $\lambda^1 = \lambda$ and $p = p_1$,

$$q^\epsilon = q^g p(1-p) \left(\frac{\lambda(z^0 + h)^{\theta-1} - (z^0)^{\theta-1}}{p\lambda(z^0 + h)^{\theta-1} + (1-p)(z^0)^{\theta-1}} \right). \tag{3.43}$$

It follows that

$$q^{\epsilon} > 0 \qquad \Leftrightarrow \qquad (1-(1-\eta)h)^{\eta(1-\gamma)} > \left(\frac{\pi(n+ph)+h}{\pi(n+ph)}\right)^{\eta+\gamma(1-\eta)}, \tag{3.44}$$

an inequality that is never verified for $\gamma \leq 1$.

Distortionary effects

A characterization of the distortionary effect of social security, Δ_t, is more elaborate. In appendix B.I, we provide an expression for this variable that takes advantage of the Cobb-Douglas form of the production function and the stochastic independence of idiosyncratic and aggregate shocks. To get some intuition about the role of the various parameters, here we show a characterization of Δ_t for t large and for the case $\sigma = 1$ and $\alpha_k + \alpha_l = 1$. Remember that the last two conditions imply a time invariant saving rate, a time invariant value of the GDP indexed bond price, q^g, and the absence of pure rents. In particular, define the "normalized" covariance between the old age (per unit of wage) marginal utilities of consumption and the old age labor supply in efficiency units,

$$\omega = \frac{\mathrm{Cov}[\lambda z^{\theta-1}, \epsilon h]}{\mathbb{E}[\lambda z^{\theta-1}]}. \tag{3.45}$$

Then, as shown in appendix B.I, if $\sigma = 1$ and $\alpha_k + \alpha_l = 1$,

$$\lim_{t\to\infty} \Delta_t = -\left(\frac{\alpha_k}{1-\alpha_k}\right)[(1-q^g)n^y + q^g\omega]\,\frac{\partial \log q^g}{\partial \tau} + q^g L'(\tau). \tag{3.46}$$

where, by (3.39),

$$\frac{\partial \log q^g}{\partial \tau} = \frac{s'(\tau)}{s} - 1 - \frac{L'(\tau)}{L}. \tag{3.47}$$

Note that the right hand side of equation (3.46) is the sum of two components. The first is related to the crowding out effect of social security and, provided that $q^g \leq 1$ (conditional Pareto optimality) and ω small or positive, is positive as long as q^g falls with τ. We interpret the latter condition as a sort of "canonical effect of social security", since, as shown in (3.47), a fall in q^g comes about as consequence of a fall in saving, induced by a transfer of resources from young to old age. Note, however, that $L'(\tau) < 0$, so that a sufficiently elastic old age labor supply may overturn this result even when $s'/s - 1 < 0$. The second term is the negative direct effect on Δ of a fall in old age labor supply. We interpret this effect as a positive effect on welfare due to higher wages (and social security income) coming from a more scarse labor force, *i.e.*, a larger labor productivity.

Recall that, when $\sigma = 1$, $1 - q^g \geq 0$ is a necessary and sufficient condition for conditional Pareto efficiency, so that, in this case, the expression multiplying the derivative of $\log q^g$ with respect to τ is positive whenever ω is positive or small. This, in turn, implies that, for t large, and when $L'(\tau)$ is sufficiently small, the sign of Δ_t is positive whenever social security lowers the GDP indexed bond price, q^g. Furthermore, we have already mentioned that idiosyncratic uncertainty makes the sign of $s'(\tau)$ ambiguous. These are not, however, the only reasons why the sign of the distortionary effect of social security is ambiguous. An additional complication is due to the possibility that the covariance ω defined in (3.45) may be negative, so that the term multiplying the derivative of $\log q^g$ with respect to τ may be negative even when $q^g \leq 1$. To see what determines the sign of ω, note that

$$\omega = \sum_i p_i \left(\frac{\lambda^i (z^i)^{\theta-1}}{\sum_j p_j \lambda^j (z^j)^{\theta-1}} - 1 \right) \epsilon^i h^i. \tag{3.48}$$

In particular, consider the case $\epsilon \in \{0,1\}$. Then,

$$\omega = p(1-p) \left(\frac{\lambda (z^0 + h)^{\theta-1} - (z^0)^{\theta-1}}{p\lambda (z^0 + h)^{\theta-1} + (1-p)(z^0)^{\theta-1}} \right) = q^\epsilon h / q^g,$$

where q^ϵ for the two-states case is defined in (3.43). Then, we can state the following proposition.

Proposition 3. *Suppose that the no-social security equilibrium is conditionally Pareto efficient ($q^g < 1$) and the crowding out effect dominates over the factors substitution effect (< 0). Then, the distortionary effect of social security has a positive effect on welfare ($\Delta < 0$) if*

$$q^\epsilon h > (1 - q^g) n. \tag{3.49}$$

Following the above discussion and, in particular, equation (3.44), we know that $q^\epsilon > 0$ only if $\gamma > 1$.

3.3.2 Indirect Utility

We estimate the welfare gains/losses of social security by computing

$$V_t(\tau) = \max_{K_t} U_t$$

for all $t \geq 0$ and assuming that the stochastic variable $\log(a_{t+1}/a_t)$ is distributed as a normal $\mathcal{N}(\mu, \sigma^2)$. Using (3.5) and the saving function, we derive the indirect expected utility of the individual in generation t as

$$V_t(\tau) = (1-\beta)\log(n(1-s)(1-\tau)) + (1-\beta)\log w_t + \frac{\beta}{1-\gamma}\log \mathbb{E}_t\left[(u^o_{t+1})^{1-\gamma}\right].$$

Letting $Z(\tau) = \sum_i p_i \lambda_i (z^i)^\theta$, and using

$$\mathbb{E}_t[(u^o_{t+1})^{1-\gamma}] = \left(\frac{1}{1-\eta}\right)^{1-\gamma} \mathbb{E}_t\left[w^\theta_{t+1}\left(p\lambda(h_{t+1})z^{1-\gamma}_{p,t+1} + (1-p)z^\theta_{u,t+1}\right)\right]. \tag{3.50}$$

(3.50) in section A.I,

$$\log \mathbb{E}_t\left[(u^o_{t+1})^{1-\gamma}\right] = \log \mathbb{E}_t[w^\theta_{t+1}] + \log Z(\tau) - (1-\gamma)\log(1-\eta).$$

Now note that by (3.27) and, by log-normality,

$$\log w^\theta_{t+1} = \log(1+g_{t+1})^\theta + \theta \log w_t$$

$$\log(1+g_{t+1}) \sim \mathcal{N}(\mu + \alpha_k \log(1+g_t), \sigma^2). \tag{3.51}$$

From the above it follows that

$$\log \mathbb{E}_t\left[(1+g_{t+1})^\theta\right] = \theta(\mu + \alpha_k \log(1+g_t)) + \theta^2\sigma^2/2.$$

Hence, we derive

$$V_t(\tau) = A(\tau) + (1 - \beta\eta)\log w_t + \beta(1 - \eta)(\alpha_k \log(1 + g_t)), \quad (3.52)$$

where

$$\begin{aligned} A(\tau) &= (1 - \beta)\log(n(1 - s)(1 - \tau)) - \beta\log(1 - \eta) \\ &+ \frac{\beta}{1 - \gamma}\log Z(\tau) + \beta(1 - \eta)\left(\mu + \theta\frac{\sigma^2}{2}\right). \end{aligned}$$

Note that output and land growth is unaffected by the social security policy, whereas the evolution of w_t depends on τ and ζ. In particular, solving this equation backward and recalling that a hat on the variables means these are in log value, we derive

$$\log w_t = \omega(\tau)\left(\frac{1 - \alpha_k^t}{1 - \alpha_k}\right) + \sum_{j=0}^{t-1}\alpha_k^j \log a_{t-j} + \alpha^t \log w_0. \quad (3.53)$$

Then, the welfare effect of social security on the t-generation's utility is

$$\begin{aligned} V_t(\tau) - V_t(0) &= (1 - \beta)\left(\log\left(\frac{1 - s(\tau)}{1 - s(0)}\right) + \log(1 - \tau)\right) + \frac{\beta}{1 - \gamma}\log\frac{Z(\tau)}{Z(0)} \\ &+ (1 - \beta\eta)\left(\frac{1 - \alpha_k^t}{1 - \alpha_k}\right)\left(\alpha_k \log\frac{s(\tau)}{s(0)} + \alpha_k \log(1 - \tau) - (1 - \alpha_l)\log\frac{L(\tau)}{L(0)}\right). \end{aligned}$$

3.4 Calibration

3.4.1 Equations and Unknowns

To estimate the welfare effect of introducing social security, we restrict attention to the case of two possible realizations of the health shock. In particular, let $i = 0,1$, so that $\epsilon^0 = 0 < 1 = \epsilon^1$, $h^0 = 0$, $h^1 > 0$. To simplify the notation, we set $h^1 = h$, $p_1 = p \in (0,1)$, $p_0 = 1 - p$. The set of endogenous variables is $(z^0, z^1, \Psi, h, \lambda, s, q^g, q^\epsilon, q^h)$, to be obtained from the following equations:

$$
\begin{aligned}
\pi(n+ph)+(h-\tau\zeta)+\tau(n+p\zeta)-z^1 &= 0,\\
\pi(n+ph)+\tau(n+p\zeta)-z^0 &= 0,\\
\left(1-\kappa(1-\eta)h^{\chi+1}\right)^{\eta(1-\gamma)}-\lambda &= 0,\\
\left(\frac{\alpha_k}{\alpha_l}\right)(n+ph)\frac{\left(p\lambda(z^1)^{\theta-1}+(1-p)(z^0)^{\theta-1}\right)}{p\lambda(z^1)^{\theta}+(1-p)(z^0)^{\theta}}-\Psi &= 0,\\
\left[(1+\eta\chi+\eta(1+\chi)\pi p)h+\eta(1+\chi)(\pi n+\tau(n-(1-p)\zeta)\right]h^{\chi}-\frac{1}{\kappa} &= 0,\\
\frac{\beta(1-\eta)\Psi}{(1-\beta)+\beta(1-\eta)\Psi}-s &= 0,\\
\left(\frac{\alpha_l}{\alpha_k}\right)\frac{ns(1-\tau)}{n+ph}-q^g &= 0,\\
q^g p(1-p)\left(\frac{\lambda(z^1)^{\theta-1}-(z^0)^{\theta-1}}{p\lambda(z^1)^{\theta-1}+(1-p)(z^0)^{\theta-1}}\right)-q^{\epsilon} &= 0,\\
q^g\left(\frac{p\lambda(z^1)^{\theta-1}h}{\lambda p(z^1)^{\theta}+(1-p)(z^0)^{\theta-1}}\right)-q^h &= 0.
\end{aligned}
$$

where

$$\theta=(1-\gamma)(1-\eta).$$

The calibration is based on Italian data.

Employment

We assume that output, labor and capital are all normalized by the working age (15-64) population (WAP) and identify n and p with the employment to WAP ratio for the age 15-54 and 55-64 respectfully. Based on the 2023 third quarter data provided by ISTAT, we get $n = 0.47$ and $p = 0.14$. The intensive margin of old age labor supply, h, is derived based on these and other calibrated parameters with the only requirement that $h < 1$. It follows that $L \in [0.47,0.61]$.

Factor Shares

The labor share, α_l, is traditionally set at 65%, although the labor share estimated for the Italian economy is lower. Karabarbounis and Neiman (2018) documents the existence of a large share of value added that cannot be attributed to measurable inputs ("factorless income"). For the end of the last decade, they estimate the total capital share (arising from IT-capital, non IT-capital and residential capital) in the range 15%-20% and report an estimate of the profit share at about 10-15%. Similar findings are in Geerolf (Geerolf, 2018). However, for ease of computation, we assume profit shares to be incorporated in capital income share. The FRED (2023) Database of Economic Data of the Federal Bank of St. Louis that releases country-specific time-series estimates of these economic measures, reports historical measure of the Share of gross capital formation at current PPP for Italy around 26 % and estimates of the Share of labour compensation in GDP around 52%.

Policy parameters

According to the INPS report (2022), the 2020 "social pension benefit", *i.e.*, the minimum guaranteed pension income for elderly who did not contribute enough or at all is 5,897 euros yearly. The OECD (2023) estimate of the 2022 average wage is about 41,000 euros and the contribution rate, τ, is 0,33. then, as the parameter ζ is obtained from the ratio between the old age public transfer, bw, and the social security contributions, τw, we get

$$\zeta = \frac{bw}{\tau w} = \frac{5{,}897}{0.33 \times 41{,}000} = 0.43.$$

Preference parameters

The implicit time discount rate is $\beta/(1-\beta)$. As it is common in the literature, the latter is set by the condition

$$\frac{\beta}{1-\beta} = \frac{1}{1+r},$$

where r is a real interest rate matching the average real return on US Treasury of about 1.5% per year. Assuming that each of the two periods of a generation's life is 25 years, we set $r = 0.28$. This provides $\beta = 1/(2+r) = 0.44$. The value of η is taken by Trabandt and Uhlig (2011), $\chi \in [0{,}5{,}1]$ and κ is calibrated accordingly.

Estimating q^g

A possible estimation of the value q^g can be based on the assumption that wage growth is log normally distributed. In particular, note that the time-varying short-term safe interest rate is given by

$$\left(\frac{1}{1+r_t}\right) = \mathbb{E}_t m_{t+1} = q^g \frac{\mathbb{E}_t\left(1+g_{t+1}\right)^{\theta-1}}{\mathbb{E}_t\left(1+g_{t+1}\right)^{\theta}}.$$

Assuming that the stochastic variable $\log a_{t+1}/a_t$ defined in (**??**) is $\mathcal{N}(\mu,\sigma^2)$, and recalling (**??**), we obtain

$$\log(1+r_t) = \mu + \alpha_k \log(1+g_t) + (2\theta-1)\sigma^2/2 - \log q^g. \tag{3.54}$$

Solving (**??**) backward, we get

$$\log(1+g_t) = \sum_{j=0}^{t-1} \alpha_k^j \log \frac{a_{t-j}}{a_{t-j-1}} + \alpha_k^t \log(1+g_0).$$

It follows that the unconditional distribution of $\log(1+g_t)$ is normal with mean $\mu_\xi/(1-\alpha_k)$ and variance $\sigma_\xi^2/(1-\alpha_k^2)$, so that

$$\log \mathbb{E}(1+g_t) = \frac{\mu}{1-\alpha_k} + \frac{\sigma^2}{2(1-\alpha_k^2)}. \tag{3.55}$$

Then, by (3.54), the unconditional distribution of $\log(1+r_t)$ is normal with mean

$$\frac{\mu}{1-\alpha_k} + (2\theta-1)\frac{\sigma^2}{2} - \log q^g$$

and variance $\alpha_k^2\sigma^2/(1-\alpha_k^2)$. Then,

$$\mathbb{E}(1+r_t) = \frac{\mathbb{E}(1+g_t)}{q^g} \exp\left(-(1-\theta)\sigma^2\right) \tag{3.56}$$

and, recalling that $\theta = (1-\eta)(1-\gamma)$,

$$q^g = \left(\frac{1+\mathbb{E}g}{1+\mathbb{E}r}\right) \exp\left(-(\eta+(1-\eta)\gamma)\sigma^2\right). \tag{3.57}$$

Using the above equality, we can derive an estimate of q^g based on the long-run mean of the safe rate and the wage (output) growth rate, an estimate of the long-run GDP variance together with the available calibrations of the parameters η and γ. Note that the above implies

$$\log q^g = \log(1 + \mathbb{E}g) - \log(1 + \mathbb{E}r) - (\eta + (1 - \eta)\gamma)\sigma^2.$$

Hence, since $\log(1 + \mathbb{E}g) \sim \mathbb{E}g$, $\log(1 + \mathbb{E}r) \sim \mathbb{E}r$,

$$q^g > 1 \qquad \Leftrightarrow \qquad \mathbb{E}r < \mathbb{E}g - (\eta + (1 - \eta)\gamma)\sigma_\xi^2.$$

Synthesis of the calibration

The structural parameters are:

π	(1-labor share)/labor share	0.92
α_k	capital share	0.26
α_l	labor share	0.52
n	employment rate age 20-54	0.47
p	employment rate 55-64	0.14
η	inverse of IES	0.3
κ	weight of old-age labour	3
χ	reciprocal of Frish labor elasticity of 55-64	[0.5,1]
γ	rel. degree of risk aversion	$[0 - 4]$
β	to be calibrated so that $\beta/(1 - \beta)$ = time disc. rate	0.44
τ	contribution rate	0.33
ζ	transfer over τw	0.43

3.4.2 Results

In this sections the results of the calibration will be presented. After having defined values for the fixed parameters of the model, the equations are simulated via Matlab Symbolic Math Toolbox, which provides functions for solving, plotting, and manipulating symbolic math equations.
First estimation is dedicated to the calibration of the unknown fundamental variables of the model for a range of values of the degree of risk aversion γ. Then, the focus shifts onto the effects of introducing the social security system under investigation. Hence, results on the quantification of the distortionary and overall welfare effects are presented.

Endogenous variables

The values presented in Tables 3.1 and 3.2 illustrate how the model's fundamental variables change with varying degrees of risk aversion. These variables are considered both at a social security contribution rate of 33 % (therefore, presumably with the system already in place) and at a contribution rate of zero. Specifically, analyzing the values of these variables aside from the contribution rate effect helps to better understand the impact of introducing such a system, particularly as q^g, q^ϵ, and q^h are variables that are especially explanatory for this purpose. A calibration for different values of the degree of risk aversion turns out to be crucial in the context of this model, as the time-independent values of the core endogenous variables of interest all depend on this parameter.
First, looking at the saving rate, the findings appear to align with economic theory, suggesting that increased risk aversion may lead individuals to favor financial security through heightened savings. Conversely, a higher contribution rate, by reducing disposable income and enhancing the perception of future insurance provided by the social security system, could result in decreased personal savings, which already gives an idea of the sign of the crowding out effect on capital accumulation.

Shifting the focus to q^ϵ, its value remains negative for any value of γ and τ, indicating as a preliminary observation the benefit in terms of within-generation risk sharing from the introduction of social security.For γ in [1,4], q^ϵ decreases further, becoming even more negative, while for τ=0.33 and γ in [0,1], this measure is closer to zero. It has been previously shown that $q^{\epsilon} > 0$ only if there is a positive covariance between the old age marginal utilities of consumption (normalized by the wage rates) and the idiosyncratic shocks. However, the sign of the covariance remain ambiguous as the marginal utilities of consumption in old age are affected by the λ^{i}, which, in turn, depend on old-age labour supply for all values of γ different from 1. For this reason it is relevant to propose a calibration for different values of this parameter, as this can result in different sign of the covariance that determines q^{ϵ}. However, the results presented in Table 3.1 and Table 3.2 seem to suggest a within-generation risk sharing improvement. In particular, from the discussion made within the model, high productivity traduces in higher retirement income, which in turn lowers wage-adjusted marginal utility of consumption. This effect seems to outweigh the positive relationship between productivity and labour supply, that should have increased marginal utility of consumption for certain values of γ. This result persists also for τ=0.

Another crucial variable to examine is q^{g}. As previously discussed, here q^{g} not only represents the price of a GDP indexed bond, but also constitute the indicator for the assessment of ex-post Pareto efficiency. Regarding the degree of inter-generational risk sharing, the value of q^{g} is less than 1 for any value of γ and τ, although for high values of γ and for τ=0, it reaches the highest values. However, from the results obtained and from the meaning that is given to this variable within the context of this model, the introduction of social security does not seem to generate that sequence of inter-generational transfers that increase the utility of all generations, conditional on the state and period in which they are born. In this model, for the evaluation of the first order effects of introducing social security, the redistributive effect is assumed to be based on the productivity shock realizations. In reality, producitivity is not easily observable. Therefore, it could be argued that, logically, the redistribution is likely to be witnessed through labour income. In this context, q^{h} might become the main variable of interest. q^{h} can simply be defined as

$$q^{h} = q^{g} n^{o} + q^{\epsilon} h \tag{3.58}$$

Clearly, this variable is affected by the values of q^g and q^ϵ, for which the comments above remain valid. Consequently, if q^ϵ is positive, the labour effect prevails on the risk sharing potentialities. The results, however, present an opposite scenario: q^ϵ is negative enough to make the labour effect almost irrelevant.

Table 3.1: Calibration on Degree of RRA (γ) for $\tau = 0.33$

γ	λ	θ	Φ	s	q^g	q^ϵ	q^h
0	0.522343	0.7	0.388491	0.176053	0.211333	-0.0144489	0.00606732
0.2	0.663283	0.56	0.386478	0.175301	0.21043	-0.0114444	0.00722816
0.5	0.839942	0.35	0.384469	0.174548	0.209527	-0.00843923	0.00839173
0.7	0.926884	0.21	0.383779	0.17429	0.209216	-0.00740726	0.00879186
0.9	0.98388	0.07	0.383589	0.174218	0.209131	-0.00712225	0.00890242
1	1	0	0.383675	0.174251	0.209169	-0.00725061	0.00885262
1.2	1.00575	-0.14	0.384194	0.174445	0.209403	-0.00802853	0.00855094
1.6	0.906293	-0.42	0.386538	0.175323	0.210457	-0.011534	0.00719352
2	0.661426	-0.7	0.390408	0.176768	0.212191	-0.0173053	0.00496589
2.5	0.186735	-1.05	0.3968	0.179144	0.215043	-0.0267934	0.00132238
3	-0.391298	-1.4	0.403798	0.181729	0.218147	-0.0371183	-0.00261717
3.5	-0.949822	-1.75	0.409759	0.183918	0.220774	-0.0458613	-0.00593343
4	-1.35517	-2.1	0.413032	0.185116	0.222212	-0.0506429	-0.00773981

[a] Own elaboration.

Table 3.2: Calibration on Degre of RRA (γ) for $\tau = 0$

γ	λ	θ	Φ	s	q^g	q^ϵ	q^h
0	0.528445	0.7	0.51514	0.220775	0.394894	-0.0288052	0.0107998
0.2	0.669474	0.56	0.511083	0.219418	0.392467	-0.0246223	0.0123801
0.5	0.844834	0.35	0.50796	0.21837	0.390592	-0.0213917	0.013603
0.9	0.985023	0.07	0.508389	0.218514	0.39085	-0.0218363	0.0134346
1	1	0	0.509189	0.218783	0.39133	-0.0226635	0.0131213
1.2	1.00342	-0.14	0.511493	0.219556	0.392712	-0.0250455	0.0122201
1.6	0.9	-0.42	0.51843	0.221872	0.396856	-0.032187	0.00952473
2	0.653789	-0.7	0.52746	0.224868	0.402215	-0.041421	0.00605398
2.5	0.18351	-1.05	0.539639	0.228872	0.409376	-0.0537621	0.00143954
3	-0.382314	-1.4	0.55025	0.232327	0.415556	-0.0644116	-0.0025214
3.5	-0.92264	-1.75	0.55709	0.234537	0.41951	-0.0712254	-0.00504593
4	-1.30877	-2.1	0.559266	0.235238	0.420763	-0.0733851	-0.0058446

[a] Own elaboration.

First Order Effects

The estimation of the main endogenous variables of the model, presented in Table 3.1 and 3.2 is also instrumental for the quantification of the welfare implications of the introduction of such a system in the economy. With the estimated variables, it is indeed possible to understand the sign of the first order effects, which account also for the potential distortions, and the evolution of the welfare effects for present and future generations. First of all, to estimate the first order effects, all variables of interest are considered for τ=0. Table 3.3 presents results for the main components of the equation characterizing first order effects for give values of γ, which, again, is a key parameter in this context. The effect on utility U_t of an individual born at t of increasing the social security contribution τ is given by equation 3.31, which simplifies into 3.34.

Given that $(1-\beta)u'(c_t^y)w_t$ will necessarily have positive sign, the direction of these first order effects will depend on the sign of the sum of the three components analyzed in Table 3.3. The terms ζq_t^ϵ and $n^y(1-q_t^g)$ are mainly dependent on q_t^ϵ and q_t^g, discussed above. Regarding the distortionary effects, these are captured by Δ_t. As discussed in Section 3.3.1, a characterization of Δ_t for t large and for the case $\sigma = 1$ and $\alpha_k + \alpha_l = 1$ could facilitate understanding the intuition about the role of the various parameters.

The two components summing in equation (3.46) represent, respectively, a measure for the crowding out effect of social security and the negative direct effect on Δ_t of a fall in old age labor supply. As shown in (3.47), a fall in q^g comes about as consequence of a fall in saving, induced by a transfer of resources from young to old age. However, as already discussed, since $L'(\tau) < 0$ as well, a sufficiently elastic old age labor supply may overturn this result even when $s'/s - 1 < 0$ and hence make this term negative. The idea is that the decrease in labour supply caused by the introduction of the social security contributions provokes an increase in wages, which could potentially limit the negative impact of the expected crowding out of capital induced by the Pay-as-you-go redistribution mechanism.

However, in estimating Δ_t, the negative impact on savings of raising the contribution rate appears to outweigh the impact on total labour supply. The related results are shown in Table 4 and 5 of the Appendix.

From the results obtained, it is clear that distortionary effects have an impact in decreasing the first order effect of introducing the system on individuals' utility. To clarify, the term $\frac{\partial U_t}{\partial \tau}|_{\tau=0}$ is not the overall first order effect, but only the sum of the terms Δ_t, ζq_t^{ϵ} and $n^y(1-q_t^g)$. Being the sign of this sum positive, the overall effect turns out to be negative, once the sum is multiplied by $-(1-\beta)u'(c_t^y)w_t$. This suggests that, conditionally on the state of the economy in which individuals find themselves at the time of an hypothetical introduction of a Pay-As-You-Go pension system, this policy cannot be considered Pareto improving., as its costs outweight the potential benefits in terms of inter-generational and infra-generational risk sharing.

Table 3.3: Distortionary and First Order effects

γ	Δ_t	ζq_t^{ϵ}	$n^y(1-q_t^g)$	$\frac{\partial U_t}{\partial \tau}\|_{\tau=0}$
0.5	0.103249	-0.0110491	0.254796	0.346996
1.2	0.102788	-0.0129324	0.253642	0.343498
1.6	0.100944	-0.01661	0.25139	0.335724
2	0.0983542	-0.0213587	0.248482	0.325477
4	0.087783	-0.0377404	0.238449	0.288492

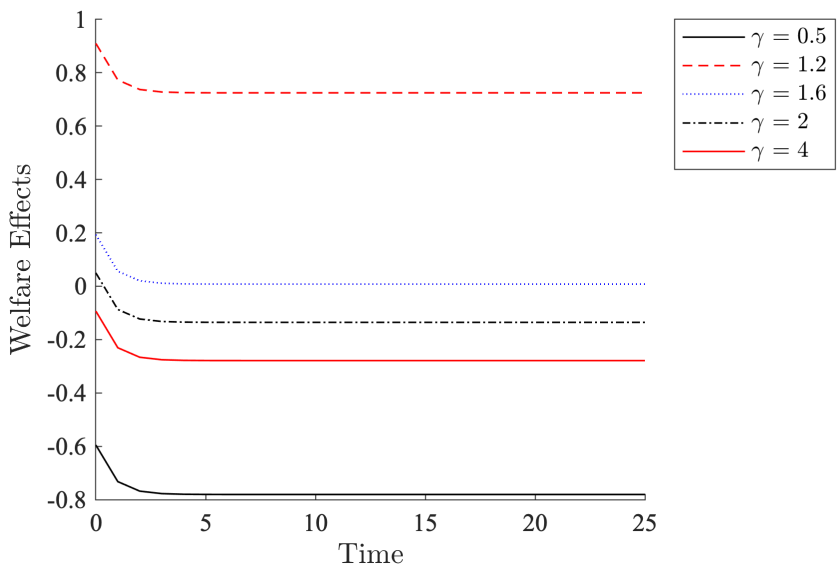
[a] Own elaboration.

Indirect Utility

Finally, welfare gains or costs are evaluated through indirect utility, as specified in 3.3.2. In Figure 3.1, the welfare effect of social security on the t-generation's utility is represented, for different values of the degree of risk aversion. For values of $\gamma \leq 1$, hence assuming risk-seeking agents, introducing social security never results in an improvement of individuals' welfare, regardless of the generation.

Interestingly, when risk aversion is assumed to be in the range (1, 1.6], benefits seem to overshadow the costs of this policy reform for all generations. For higher values of risk aversion [1.7 - 2], the introduction of the social security system seems to have a positive effect only at time 0, likely due to the first-generation effect: In fact, the older generation at time 0 receives the pension amount without having contributed during their working age, and the positive effect for this generation apparently outweighs the negative effect for the generation that is in working age at time 0. Overall, given underlying assumptions of the proposed economy and the customised parametrization, these results provide a positive insight on the long-run welfare effects when evaluated from an ex-ante perspective.

Figure 3.1: Welfare effects over time



Conclusions

In this work, the ever-relevant and divisive issue of the effectiveness of social security has been discussed, with a focus on the characteristics of the Italian Pay-As-You-Go system. Starting with a theoretical overview on the definition of a welfare improvement, the differences arising from the type of efficiency criterion adopted have been clarified, especially when considering a stochastic overlapping generations environment. This distinction is particularly significant in the context of this work, namely the discussion on the efficiency of a Pay-As-You-Go social security system, which imply a redistribution of resources between generations. Contrary to the general consensus on the unsustainability of such systems, the proposed model has made an attempt to further justify the presence of these redistributive schemes, considering an economy affected by both aggregate and idiosyncratic productivity uncertainty. The model's assumptions, particularly the specific parameterization of individuals' preferences, allow this evaluation to be reduced to solving equations that define welfare effect measures. Primarily, the hypothesis tested is that given the productive uncertainty in the second period of life, the negative effect on labor supply, which is reduced and thus likely leads to an increase in incomes and potentially an increase in savings, can outweigh the traditional "crowding out" effect on capital accumulation induced by the reduction of savings due to the payment of contributions by the young. Analyzing the welfare measures of the model, from an ex-post perspective, this does not occur, and thus we cannot claim potential improvements in terms of conditional Pareto efficiency. However, by evaluating the benefits from an ex-ante perspective for all future generations, for certain values of risk aversion, this work confirms that Pay-As-You-Go Social Security system can lead to welfare improvements for individuals of all generations.

Appendix

A Proofs

A.I Proof of Proposition 1

Let $V_{t+1} \equiv R_{t+1}K_t + \Pi_{t+1}$. Solving (3.19), we derive that, if $\epsilon^i > 0$, the old age labor supply is a value h^i verifying

$$\frac{\epsilon^i}{\kappa} = \left[(1+\chi\eta)\epsilon^i h^i + \eta(1+\chi)\left(\frac{V_{t+1}}{w_{t+1}} + \tau(n + \zeta(\bar{\epsilon} - \epsilon^i))\right)\right](h^i)^\chi, \tag{59}$$

whereas $h^i = 0$ if $\epsilon^i = 0$. Now note that, by (3.15), we have

$$V_{t+1}/w_{t+1} = \pi L_{t+1} = \pi\left(n + \sum_i p_i \epsilon^i h^i_{t+1}\right).$$

Using this in (59), we derive that, at equilibrium, $h^i = h^i(\tau)$, where, for all $i = 1, .., m$, $h^i(\tau)$ is the solution to (3.21). To solve (3.18), define

$$\lambda^i = \left(1 - \kappa(1-\eta)(h^i_{t+1})^{1+\chi}\right)^{\eta(1-\gamma)}, \qquad \theta = (1-\eta)(1-\gamma),$$

and note that

$$v^{1-\gamma} = (1-\eta)^{\gamma-1}(c^o)^\theta \lambda, \qquad v^{-\gamma}v_1 = (1-\eta)^\gamma (c^o)^{\theta-1}\lambda.$$

Hence, recalling (3.6), we derive the following characterization of the stochastic discount factors, contingent on the old age productivity shock:

$$m_{t+1} = \left(\frac{\beta}{1-\beta}\right) u' \left(\frac{(\mathbb{E}_t[(c^o_{t+1})^\theta \lambda])^{\frac{1}{1-\gamma}}}{1-\eta}\right) \frac{(\mathbb{E}_t[(c^o_{t+1})^\theta \lambda])^{\frac{\gamma}{1-\gamma}} (c^o_{t+1})^{\theta-1}\lambda}{u'(c^y_t)}. \tag{60}$$

Note that, since old age labor supply is wage independent and time invariant, old age consumption as a share of the current wage is also time invariant. In particular, recalling the definition of the "per unit of wage" contingent consumptions in (3.20), by the independence between the aggregate and the idiosyncratic shocks, and since $u'(c) = c^{-\sigma}$, we obtain

$$m_{t+1} = \left(\frac{\beta}{1-\beta}\right) (1-\eta)^\sigma (c^y_t)^\sigma \frac{(w_{t+1})^{\theta-1}\lambda z^{\theta-1}}{(\mathbb{E}_t[w^\theta_{t+1}]\mathbb{E}[\lambda z^\theta])^{\frac{\sigma-\gamma}{1-\gamma}}}. \tag{61}$$

Now note that, by (3.2), we have

$$R_{t+1} = \frac{\alpha_k}{\alpha_l}\left(\frac{w_{t+1}L}{K_t}\right).$$

Then, using the above in the expression for the stochastic discount factor in (61), we get

$$\mathbb{E}_t[m_{t+1}R_{t+1}] = \left(\frac{\beta}{1-\beta}\right) (1-\eta)^\sigma (c^y_t)^\sigma \left(\mathbb{E}_t[w^\theta_{t+1}]\right)^{\frac{1-\sigma}{1-\gamma}} \frac{\mathbb{E}[\lambda z^{\theta-1}]}{(\mathbb{E}[\lambda z^\theta])^{\frac{\sigma-\gamma}{1-\gamma}}} \left(\frac{\alpha_k L}{\alpha_l K_t}\right).$$

Finally, by the first order condition (3.18), we derive that the optimal demand for capital, K^*_t, is implicitly defined by

$$K^*_t = (n^y w_t(1-\tau) - K^*_t)^\sigma x_t \left(\frac{\alpha_k L}{\alpha_l}\right), \tag{62}$$

where

$$x_t = \left(\frac{\beta}{1-\beta}\right)(1-\eta)^\sigma \left(\mathbb{E}_t w^\theta_{t+1}\right)^{\frac{1-\sigma}{1-\gamma}} \frac{\mathbb{E}[\lambda z^{\theta-1}]}{(\mathbb{E}[\lambda z^\theta])^{\frac{\sigma-\gamma}{1-\gamma}}},$$

Letting $K^*_t = S(n^y w_t(1-\tau), x_t)$, it immediately verified that $S(.) \in [0, n^y w_t(1-\tau)]$, and the partial derivatives of S, denoted by S_w and S_x, are positive.

A.II Proof of proposition 2

Rewrite (3.21) as

$$\frac{1}{\kappa} = H(h^i, \epsilon^i, \bar{\epsilon}, L), \tag{63}$$

where

$$H(h^i, \epsilon^i, \bar{\epsilon}, L) = \left[(1+\eta\chi)h^i + \frac{\eta(1+\chi)}{\epsilon^i}\Big(\tau(n+\zeta(\bar{\epsilon}-\epsilon^i)) + \pi L\Big)\right](h^i)^{\chi}$$

and $L = n + \sum_j p_j \epsilon^j h^j$. Note that, by (3.11), $H(.)$ is increasing in h^i and L, and decreasing in ϵ^i. Since $\kappa \geq 1$,

$$\frac{1}{\kappa} = H(h^i, \epsilon^i, \bar{\epsilon}, L) \leq 1 \leq H(1, \epsilon^i, \bar{\epsilon}, L),$$

which guarantees that $h^i \in (0{,}1)$ for all i and $\tau \geq 0$, provided that $\epsilon^i > 0$. Now we prove that $\partial h^i/\partial \epsilon^i > 0$. By total differentiation of the function $H(h^i, \epsilon^i, \bar{\epsilon}, L)$ defined in (3.21), we obtain

$$\frac{\partial h^i}{\partial \epsilon^i}(H_1^i + H_4^i p_i \epsilon^i) = -H_2^i - H_3^i p_i - H_4^i \left(p_i h^i + \sum_{j \neq i} p_j \epsilon^j \frac{\partial h^j}{\partial \epsilon^i}\right), \tag{64}$$

and, for $j \neq i$,

$$\frac{\partial h^i}{\partial \epsilon^j}(H_1^i + H_4^i p_i \epsilon^i) = -H_3^i p_j - H_4^i \left(p_j h^j + \sum_{s \neq i} p_s \epsilon^s \frac{\partial h^s}{\partial \epsilon^j}\right), \tag{65}$$

where

$$\begin{aligned}
H_1^i &= \frac{\chi}{\kappa h^i} + (h^i)^{\chi}(1+\eta\chi) > 0, \\
H_2^i &= -\eta(1+\chi)(h^i)^{\chi}(\tau(n+\zeta\bar{\epsilon}) + \pi L)/(\epsilon^i)^2 < 0, \\
H_3^i &= \eta(1+\chi)(h^i)^{\chi}\tau\zeta/\epsilon^i > 0, \\
H_4^i &= \eta(1+\chi)(h^i)^{\chi}\pi/\epsilon^i > 0
\end{aligned}$$

are the partial derivatives of $H(h^i, \epsilon^i, \bar{\epsilon}, L)$ with respect to h^i, ϵ^i and L, respectively. Using the above in (64) and (65), we find that the "own partial derivatives" are

$$\frac{\partial h^i}{\partial \epsilon^i} = \frac{\eta(1+\chi)(h^i)^\chi}{(H_1^i + H_4^i p_i \epsilon_i)\epsilon_i^2} \left(n(\pi+\tau) + \pi \sum_{j \neq i} p_j \epsilon^j h^j \left(1 - \frac{\partial h^j / h^j}{\partial \epsilon^i / \epsilon^i}\right) + \tau\zeta \sum_{j \neq i} p_j \epsilon^j \right). \quad (66)$$

and, for $j \neq i$, the "cross partial derivatives" are

$$\frac{\partial h^j}{\partial \epsilon^i} = -\frac{\eta(1+\chi)(h^j)^\chi}{(H_1^j + H_4^j p_i \epsilon_j)\epsilon_j} \left(\pi \left(p_i h^i + \sum_{s \neq j} p_s \epsilon^s \frac{\partial h^s}{\partial \epsilon^i} \right) + \tau\zeta p_i \right). \quad (67)$$

Now assume that $\partial h^i / \partial \epsilon^i < 0$. Then, by (66),

$$0 < n(\pi+\tau) + \sum_{j \neq i} p_j \epsilon^j (\pi h^j + \tau\zeta) < \pi \epsilon^i \sum_{j \neq i} p_j \epsilon^j \frac{\partial h^j}{\partial \epsilon^i}.$$

By (67),

$$\frac{\partial h^j}{\partial \epsilon^i} > 0 \qquad \Leftrightarrow \qquad \pi \sum_{s \neq j} p_s \epsilon^s \frac{\partial h^s}{\partial \epsilon^i} < -p_i(\pi h^i + \tau\zeta) < 0.$$

This implies a contradiction. It follows that $\partial h^i / \partial \epsilon^i \geq 0$ for all i. Now we show that $L'(\tau) < 0$. By total differentiation of the function $H(.)$ defined in (3.21), we obtain

$$\frac{\partial h^i}{\partial \tau} \Big((1+\eta\chi)\epsilon^i h^i + \eta\chi(\tau(n + \zeta(\bar{\epsilon} - \epsilon^i)) + \pi L(\tau)) \Big) = -\eta h^i \Big(n + \zeta(\bar{\epsilon} - \epsilon^i) + \pi L'(\tau) \Big)$$

where $L'(\tau) = \sum_j p_j \epsilon^j \partial h^j / \partial \tau$. Now assume that $L'(\tau) \geq 0$. Then, since $n + \zeta(\bar{\epsilon} - \epsilon^i) > 0$, the above implies that $\partial h^i / \partial \tau < 0$ for all i. A contradiction.

B Additional computations

B.I Evaluation of the Distortionary Effects

Using (3.2), we have

$$\begin{aligned} R_{t+1} &= a_{t+1} \alpha_k K_t^{\alpha_k - 1} L^{\alpha_l}, \\ \Pi_{t+1} &= a_{t+1}(1 - \alpha_k - \alpha_l) K_t^{\alpha_k} L^{\alpha_l}, \end{aligned}$$

so that

$$K_t \frac{\partial R_{t+1}}{\partial \tau} + \frac{\partial \Pi_{t+1}}{\partial \tau} = -w_{t+1} L \left(\alpha_l \frac{L'(\tau)}{L} + \alpha_k \frac{\partial \log K_t}{\partial \tau} \right).$$

Moreover, it is readily verified that

$$\frac{\partial \log w_{t+1}}{\partial \tau} = \alpha_k \frac{\partial \log K_t}{\partial \tau} - (1-\alpha_l) \frac{L'(\tau)}{L}.$$

Then, from (3.37), we derive

$$\Delta_t = (\alpha_l L q_t^g + (1-\alpha_l) q_t^h) \frac{L'(\tau)}{L} - n^y \frac{\partial \log w_t}{\partial \tau} + \alpha_k \left(L q_t^g - q_t^h \right) \frac{\partial \log K_t}{\partial \tau}, \tag{68}$$

where

$$q_t^h = \mathbb{E}_t \left[m_{t+1} (1+g_{t+1}) \epsilon h \right]. \tag{69}$$

Note that, by the stochastic independence of the aggregate and idiosyncratic shocks, we can derive

$$q_t^h = q_t^g \left(\frac{\mathbb{E}[\lambda z^{\theta-1} \epsilon h]}{\mathbb{E}[\lambda z^{\theta-1}]} \right) = q_t^g \left(\mathbb{E}[\epsilon h] + \frac{\mathrm{Cov}[\lambda z^{\theta-1}, \epsilon h]}{\mathbb{E}[\lambda z^{\theta-1}]} \right).$$

Now let

$$\omega = \frac{\mathrm{Cov}[\lambda z^{\theta-1}, \epsilon h]}{\mathbb{E}[\lambda z^{\theta-1}]} \tag{70}$$

and recall that $\mathbb{E}[\epsilon h] = n^o$. Then,

$$q_t^h = (n^o + \omega) q_t^g. \tag{71}$$

Now note that, by (3.38),

$$\frac{\partial \log w_t}{\partial \tau} = \frac{\partial \log K_t}{\partial \tau} - \frac{L'}{L} - \frac{\partial \log q_t^g}{\partial \tau}.$$

Then, from (68), we derive

$$\frac{\Delta_t}{n^y} = \frac{\partial \log q_t^g}{\partial \tau} + A_t \frac{L'(\tau)}{L} - B_t \frac{\partial \log K_t}{\partial \tau}, \tag{72}$$

where

$$A_t = 1 + \left(\alpha_l + \frac{n^o}{n^y} + (1-\alpha_l)\frac{\omega}{n^y}\right) q_t^g, \qquad B_t = 1 - \alpha_k \left(1 - \frac{\omega}{n^y}\right) q_t^g. \tag{73}$$

Now use (3.38) and the Cobb-Douglas specification of the technology to derive

$$\log K_t = (\log \alpha_k + \alpha_l \log L) + \log q_t^g + \log a_t + \alpha_k \log K_{t-1},$$

from which we obtain

$$\frac{\partial \log K_t}{\partial \tau} = \alpha_l \left(\frac{1-\alpha_k^t}{1-\alpha_k}\right) \frac{L'(\tau)}{L} + \sum_{j=0}^{t-1} \alpha_k^j \frac{\partial \log q_{t-j}^g}{\partial \tau}.$$

Plugging the above in (72), we derive

$$\frac{\Delta_t}{n^y} = \left(A_t - B_t \alpha_l \left(\frac{1-\alpha_k^t}{1-\alpha_k}\right)\right) \frac{L'(\tau)}{L} + \frac{\partial \log q_t^g}{\partial \tau} - B_t \sum_{j=0}^{t-1} \alpha_k^j \frac{\partial \log q_{t-j}^g}{\partial \tau}, \tag{74}$$

where ω, A_t and B_t have been defined in (70) and (73).

We now characterize the above expression for the case $\sigma = 1$. Remember that $\sigma = 1$ implies that q^g is time invariant. Then, (74) reduces to

$$\frac{\Delta_t}{n^y} = \left(A - B\alpha_l \left(\frac{1-\alpha_k^t}{1-\alpha_k}\right)\right) \frac{L'(\tau)}{L} + \left(1 - B\left(\frac{1-\alpha_k^t}{1-\alpha_k}\right)\right) \frac{\partial \log q^g}{\partial \tau}.$$

By taking the limit and recalling (73):

$$\begin{aligned} \lim_{t\to\infty} \frac{\Delta_t}{n^y} \quad &= \quad \left[\left(\frac{1-\alpha_k-\alpha_l}{1-\alpha_k}\right) + q^g \left(\frac{\alpha_l}{1-\alpha_k} + \frac{n^o}{n^y} + \frac{\omega}{n^y}\left(\frac{1-\alpha_k-\alpha_l}{1-\alpha_k}\right)\right)\right] \frac{L'(\tau)}{L} \\ &\quad - \frac{\alpha_k}{1-\alpha_k}\left(1 - q^g + \frac{\omega}{n^y} q^g\right) \frac{\partial \log q^g}{\partial \tau}. \end{aligned} \tag{75}$$

By setting $\alpha_k + \alpha_l = 1$, we get (3.46).

B.II First Order effects on Labour supply and Savings

Derivatives of Old age labour supply (h), Total labour supply (L) and saving rate (s) with respect to τ are computed for increasing values of the contribution rate and for an intermediate value of γ, equal to 1.2.

Table 4: FO effects on h and L

τ	$\frac{h'(\tau)}{h}$	$\frac{L'(\tau)}{L}$
0	-0.01939	-0.00271568
0.05	-0.0193424	-0.00270794
0.1	-0.0192872	-0.0027002
0.15	-0.019232	-0.00269248
0.2	-0.0191769	-0.00268477
0.25	-0.0191219	-0.00267707
0.3	-0.019067	-0.00266938
0.35	-0.0190122	-0.0026617

Table 5: FO effects on s

τ	$\frac{s'(\tau)}{s}$
0	-0.19921
0.05	-0.183754
0.1	-0.170034
0.15	-0.1578
0.2	-0.146846
0.25	-0.137
0.3	-0.128118
0.35	-0.120078

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