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Analysis and Extensions of Game Theoretical Network Generation

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1 Introduction

The study of network formation and stability has become increasingly significant in understanding complex social, economic, and technological systems. Networks can be seen everywhere, appearing in various forms such as social networks, economic trade networks, and the internet. The analysis of these networks often involves examining how individual agents (nodes) form connections (links) and how these connections influence overall network stability and efficiency.

In the last centuries, studies on game theory have made incredible evolutions, capable to reach better results regarding the generation of networks. A practical example is the Nash refinements, concepts based on one of the biggest milestones in the field of Game theory: the Nash equilibrium. This was firstly ideated by the Nobel winner and homonymous John Nash based on the decision of single agents looking for the highest possible gain by adapting to the strategies of all the other agents in the network. This starting point allowed the formulation of different refinements, new notions of equilibrium that rely on more clauses than the simple Nash equilibrium, capturing new behaviors and dynamics thanks to the consideration of more conditions for the network formation. These refinements can filter out a large number of Nash equilibriums that could have been considered weak regarding concepts different from the Nash one, maintaining a smaller set of possible configurations.

The focus of this thesis is the exploration of different stability concepts that can be considered in the formation of a network and how these can be combined in order to find solutions that are robust from different points of view.

Moreover, we will compare a set of different notions, both on similarity and differences, trying to understand which ones are stricter than the others.

The foundational work by Francis Bloch and Matthew O. Jackson (2005) on stability concepts and linking games provides the theoretical underpinning for this study. Their research delineates two main branches of network formation notions: stability concepts, which focus on the conditions for network stability, and linking games, which explore the strategic formation of links among agents. This thesis aims to deep into these branches, examining how different approaches to stability and link formation interact and lead to well-defined network structures.

The main subject for the stability concepts will be the notion of Pairwise stability ideated by Jackson and Wolinski in 1996 and the possible derivates of it.

The main concept that will be taken into account for the linking games will be the notions of **Strong stability**, a refinement of **Nash stability** that consider not only the single agents, but that capture the behavior of a coalition of agents taken as a single entity.

There will be also a hybrid subject, the **Pairwise Nash stability**, that refers to both the Nash equilibrium and the Pairwise stability concepts, with the aim to combine the different properties.

The feature that will be primarily explored is the reason for which an agent decides to form or severe a link, considering also how its decision can effectively impact the network configuration: if its choice is required or sufficient in order to make a change in the graph (we'll dive into this last concept later).

A deep reasoning will be done for the last refinements cited, the pairwise Nash stability, and how its premises can be expanded in order to go beyond the idea of pairs constructing a derived definition that can reach a higher number of agents into consideration. Moreover, there will be a formulation of the *Pairwise Nash* concept that will refer to a set of three agents, with the name of ***triwise Nash stability***.

Finally, the principal objectives of this work are the analysis of the possible interactions between the notions already formulated and the creation of new definitions building on existing ones.

The main goal of this research is to study the activity of most of these ideas on stability with an interest in both Nash stability and its modifications. This aims at giving an understanding of factors favoring stable networks which are also beneficial in the larger setting of network dynamics in different places thus increasing our knowledge about them.

In this research, the complexities of network formation will be discussed in detailed and practical examples provided to underscore its nuances. The implications of this study are valuable for both theoretical research in networks as well as designing concrete ones in a wide range of sectors.

2 Stability and equilibrium concepts

2.1 Basic concepts

Before starting our analysis, we're going to recall all the prerequired notions that will be necessary to understand the next analysis.
We'll start by the basic definitions regarding graphs.

GRAPH:

In social network analysis, a graph is a mathematical representation of social relationships, where the main subjects are the agents, the entities inside the network and the connections between them are links.

UTILITY FUNCTIONS:

In game theory, a utility function is a mathematical representation that assigns a numerical value (utility) to each possible outcome of a game, reflecting the preferences of a player. The numerical value is the payoff of the agent, that has as aim always to maximize it (we consider a total noncooperative game environment).

Our objective is to understand what graph can be expected given the preferences of the different agents. As we said before we can focus on two main approaches: stability and linking games.

2.2 Stability concepts

Stability concepts rely on a simple idea: given a graph, understand the source of instability.
The two possible ways a graph could be considered not stable are the deletion or the creation of some links.

PAIRWISE STABILITY:

A key notion is the one introduced by Jackson and Wolinski in 1996 called "*Pairwise stability*", that consider the possibility of creation or deletion of just one link, putting a limit to itself.

The idea is that if any agent would benefit from cutting an edge or two agents would benefit by adding one, the graph we have cannot be considered stable. Moreover, any agent can delete a link where it is, and any pair of agents can create a link if they both will gain from it.

"Jackson and Wolinsky (1996)'s pairwise stability is based on two considerations. A network is deemed to be stable if (i) no individual agent has an incentive to sever a link, and (ii) no pair of agents have an incentive to form a new link"[1].

In the next definition we will see the mathematical formalization of Pairwise stability, where we introduce the idea of deletion or creation of a link from a starting graph. This link is denoted by "ij" that is the link between the two agents (i and j). When we will find "g + ij", it will be referred to the addition of the link "ij" to the starting graph, whereas when we will find "g - ij", it will be referred to the deletion of the link "ij" from the starting graph (thus, the link was in the starting configuration).

Definition: *Given the graph g, i and j the agents inside the graph and ij the link between the agents, g is said to be pairwise stable if:*

- $ij \in g \;\rightarrow u_i(g) > \; u_i(g - ij) \;\forall\, i, ij \;\in g$
- $ij \notin g \;\rightarrow if\; u_i(g + ij) > \; u_i(g)$ then $u_j(g + ij) < \; u_j(g).$[2]

2.3 Linking games

Linking games are strategic interactions where links are formed according the decisions of the agents. Every game has its own conditions for the formation of the link in the initial graph. Basically, each agent has its own strategy s_i, and so the network configuration will depend on the set of strategies of each agent: *g(s)*.

[1] This is the definition of pairwise stability in Bloch and Jackson (2005) taken from the older paper above mentioned of the last one and Wolinsky.

[2] Mathematical definition of pairwise stability in Jackson and Wolinski (1996).

MYERSON'S GAME:

In order to study the linking game approaches for a network, we first have to decide how this game should work in order to generate the graph. We now define the Myerson's game (ideated by the homonymous in 1991) [3].

We first define a vector strategy S_i for each player of N booleans, one per other agents. Values in this vector means the desire of agent i to form a link with the other agents: formally, this preference is expressed by a Boolean value, 0 and 1 (0 means that the agent i doesn't want to create a link with the other agent, whereas 1 means the opposite.

After this, we define a vector S that takes into account all the vector strategies for the single agents: from this last vector we can define which links will be formed into our network, more precisely all the pairs of agents that both declared 1 will form a link between them (Ex. If s_{ij} (declaration of agent i on agent j) = 1 and s_{ji} = 1, then the link ij will be formed).

NASH STABILITY:

One of the most important concepts regarding noncooperative games, Nash equilibrium take its name by John Nash, its author and also one of the most important mathematicians in the field of game theory.

The game on which we're going to explain the Nash stability concepts is the forementioned Myerson's game and so:

Definition: *A strategy profile s is a Nash equilibrium of Myerson's game if and only if:*

$$\forall\ i \in I, s'_i\ \in\ S_i, u_i\big(g(s)\big) \geq u_i(g(s'_i, s_{-i}))$$

Where I is the set of agents in the network and s'$_i$ is the deviation of agent i regarding the starting strategy.

From this, we can say that a graph g is said to be Nash stable if there exists at least one Nash equilibrium s such that g(s) = g.

The idea under the Nash equilibrium is that any agent by itself cannot achieve a higher payoff than the one it has by itself, and so it has to maintain its choice.

[3] The Myerson's game refers to: Myerson, R. (1991) *Game Theory: Analysis of Conflict*, Harvard University Press: Cambridge, MA.

The Nash equilibrium reflects the assumption of a noncooperative game, since each player has to move by only itself and need to adapt to other players strategies, the perfect base point for a deep analysis of the linking games approach.

NASH REFINEMENTS:

Since the Nash equilibrium leads to a large set of stable configurations, there are a lot of studies regarding equilibrium concepts that took as basis the classic Nash equilibrium criteria and then put in addition new conditions that can assure two main benefits:

- A smaller set of stable configurations thanks to the new parameters considered;
- The guarantee of new properties that evidence the quality of the network from some specific "*Game theory points of view"* (an example we'll see later is the robustness of a network with respect to the possible coalitions that some agents can make).

COALITION:

Before mentioning the concept of Strong stability, it's helpful to give a definition of coalitions first:

A coalition is a set of agents *C* inside a network (*C* must be of course a subset of *N*, the total set of agents in the network) that jointly decide to deviate from the network, excluding themselves from the formation of links with agents that are not in the coalition. Since we are in a noncooperative game, the word coalition refers not properly on a collective decision of a set of agents, but more on the union of the single choice by each agent. Moreover, the idea behind this coalition is that single agents want only reach the maximum payoff possible for their own. This concept is of course related to a sort of cooperative approach on the network, where agents decide something together but always with the main objective to become *richer*.

STRONG STABILITY:

Definition: *Defined a vector strategy S, a strategy s is said strongly stable if there doesn't exist a coalition C (subset of N, the agents in the network) that deviating in a joint way from the network will increase the payoff of the agents inside the coalition:*

$$\nexists\, C \subseteq N\,, s_c' \mid u_i\left(g\left(s_c', s_{N\setminus C}\right)\right) \geq u_i\left(g(s)\right) \forall\, i \in C$$

With strict inequality for at least one.

Strong stability is effectively the concept of Nash equilibrium adjusted to a set of agents treated as a single player in the network.

STRONG STABILITY OF ORDER 2:

There can be defined intermediate notions of different order of Strong stability by restricting the number of agents inside a coalition. The Strong stability of order 1 simply reflects the Nash equilibrium, the order 2 is the Nash equilibrium plus the detection of coalitions of 2 agents and so on.

Definition: *Defined a vector strategy S, a strategy s is said strongly stable of order 2 if it doesn't exist a coalition C of at most 2 agents that deviating in a joint way from the network will increase the payoff of all the agents inside the coalition.*

Weaker refinement of strong stability, it allows the cooperation of at most pair of agents:

$$\nexists\, C \subseteq N \; with \; C \leq 2, s'_c \mid u_i\left(g\left(s'_c, s_{N\setminus C}\right)\right) \geq \; u_i\big(g(s)\big) \; \forall \, i \; \in C$$

With strict inequality for at least one.

2.4 Mixed strategies between stability and linking games

PAIRWISE NASH STABILITY:

A bridge between stability and equilibrium notions, pairwise Nash stability takes the property of severance by the Nash equilibrium, but admits the cooperation on the creation of links by two agents, exactly as the pairwise stability.

Definition: *Defined a vector strategy S, a strategy s is said pairwise Nash stable if it is Nash stable (no agent can increase its payoff by just its decision) and there doesn't exist a link outside the graph that can increase the payoff of both the agents involved:*

$$u_i\big(g(s)\big) > u_i(g(S'_i, S_{-i})$$

and

$$\nexists\ ij \notin g \mid u_i(g(s)+ij) \geq u_i\big(g(s)\big)\ ;\ u_j(g(s)+ij) \geq u_j\big(g(s)\big).$$

With at least one strict.[4]

[4] Pairwise Nash Stability has its roots in two different works: Calvo-Armengol and Ilkilic (2004) firstly defined sufficient and necessary condition for PNS, while Gilles and Sarangi (2004) defined Pairwise Nash Stable networks as *Strongly Pairwise Stable*.

3 Literature review

3.1 “Definitions of Equilibrium in Network Formation Games”.

The study will rely mostly on the acute article written by Francis Bloch and Matthew O. Jackson in 2005.

The paper talks about the different notions regarding the generation of networks that are mentioned and defined in Chapter 2.

After an introduction of the different families of concepts and their members, the study is oriented to their different interactions.

First of all, an analysis is conducted to the strictness of all the definitions: when and how a definition is stricter than others, and so also the consideration of whether two different notions can be compared (for instance, when one is a subset of the other).

A huge section is dedicated to how the concept of stability and linking games would change if transfers (negotiation between agents in order to form a link) are allowed in the network, but for sake of simplicity all this part will not be analyzed from us since our aim isn’t related to it.

Nevertheless, the first part before mentioned will give us a lot of key concepts for our needing.

3.2 Explanation of inclusions between the sets

The first proposition analyzed in the paper regards all the stability and equilibrium notions that are mentioned above. Moreover, the different notions are compared regarding their inclusion and distinction.

From this proposition our aim is to prove that the different notions are comparable, their sets are included and they can also be distinct.

As said before, most of the notions will be compared:

The Strong stability (**SS**), the Strong stability of order 2 (**SS2**) and the Pairwise Nash stability (**PNS**), for which we are going to say also if it can be defined as the intersection of Pairwise stability (**PS**) and Nash stability (**NS**).

For any N and profile of utility functions u:

$$SS(u) \subseteq SS2(u) \subseteq PNS(u) = PS(u) \cap NS(u)$$

Starting from:

$$PNS(u) = PS(u) \cap NS(u)$$

We easily can say that PNS has to fulfill both Nash stability and Pairwise stability requirements, and so it can be seen as the intersection of the 2 sets.

Now, regarding the first statement:

$$SS(u) \subseteq SS2(u)$$

We need to recall the concept of Strong stability of order 2, that is the detection of coalitions of at most two agents, while the Strong stability in wider sense it's the detection of coalitions of each number of agents, so in the general Strong stability we include also the Strong stability of order 2.
For this we can say that Strong stability is a stricter requirement than Strong stability of order 2.

Finally, we have to prove the mid statement:

$$SS2(u) \subseteq \; PNS(u)$$

The main problem of this inclusion is that the two notions are not properly comparable, since SS2 is a linking game while pairwise Nash stability is based on a simple stability concept.
For this, we have to rewrite one of them in order to allow the comparison with the other.
Let's rewrite the pairwise stability in a linking game approach, starting from the consideration of the link inside the graph:

$$\forall\, i, ij \;\in g(s),\; u_i(g(s) - ij)) \;\leq\; u_i(g(s))$$

We have to consider the possible deviations of *i*: removing the link *ij* in a "linking game sense" means that we have to let the product of the strategies of *i* and *j* equal to zero.
By the *i* point of view, we can easily modify its strategy by saying 0 according to the link *ij:*

$$\text{Starting condition: } i_j \cdot j_i = 1 \ \rightarrow \ i_j = j_i = 1$$

Where i_j means the decision of agent i on whether to form a link with agent j.

The deviation will be:

$$Strategy\ s_i'(t) \begin{cases} s_i(t) \ \ if\ t \neq j \\ \ \ 0\ if\ t = j \end{cases}$$

Where "t" refers to the agents in the network.

Arriving to:

$$u_i(g(s_i', s_{-i})) \leq u_i(g(s))$$

That is equal to:

$$u_i(g(s) - ij)) \ \leq \ u_i(g(s))$$

And so:

$$u_i\big(g(s_i', s_{-i})\big) \leq u_i(g(s_i, s_{-i})$$

Now regarding the links outside the graphs, recall that for pairwise Nash stability:

$$\forall\ ij \notin g \ \rightarrow if\ u_i(g + ij) > \ u_i(g)\ then\ u_j(g + ij) < \ u_j(g)$$

In this case we need to use the reverse approach of the one did before: we want that the product of the strategies of *i* and *j* to be one, starting from zero:

$$Starting\ condition: i_j \cdot j_i = 0$$

There are three possible solutions to make this product equal to zero:

- Agent *i* doesn't want to form the link whereas agent *j* does;
- Agent *j* doesn't want to form the link whereas agent *i* does;
- Both agents don't want to form the link.

The mathematic formulation of these three possible causes is:

$$i_j \cdot j_i = 0 \begin{cases} i_j = 0, j_i = 1 \\ i_j = 1, j_i = 0 \\ i_j = j_i = 0 \end{cases}$$

While the first two possibilities are specular (i will make the deviation in the first as j will do in the second) the third one will need the deviation of both the agents.

The first deviation will be:

$$Strategy\ s_i'(t) \begin{cases} s_i(t)\ if\ t \neq j \\ 1\ if\ t = j \end{cases}$$

Now let's rewrite the statement before in a linking game approach:

$$\forall\ ij \notin g\ \rightarrow if\ u_i(g(s_i', s_{-i})) > u_i(g)\ then\ u_j(g(s_i', s_{-i})) < u_j(g)$$

Comparing it to the strong stability of order 2:

$$\nexists\ C \subseteq N\ with\ C \leq 2 \mid u_i\left(g\left(s_C', s_{N\setminus C}\right)\right) \geq u_i\left(g(s)\right) \forall\ i\ \in C$$

From here we can easily see why the strong stability of order 2 is stricter than pairwise Nash equilibrium: while the PNS admit the possibility that one agent will gain from a link outside the graph (only if the other one will lose something) the strong stability of order 2 will not admit gain from a link outside the graph for any agent in the coalition (since we are weakening the strong stability to only the second order, the two agents involved in the link).

Let's do it for the other two cases:

$$Strategy\ s_j'(t) \begin{cases} s_j(t)\ if\ t \neq i \\ 1\ if\ t = i \end{cases}$$

As expected, the situation here is symmetric to the first deviation, just the swap of agent i with agent j:

$$\forall\, ij \notin g \;\rightarrow if\; u_i\big(g(s_i, s_j')\big) > \; u_i(g)\; then\; u_j\big(g(s_i, s_j')\big) < \; u_j(g)$$

Now let's write the third case. This time we have to consider the deviation of both:

$$Strategy\; s_i'(t) \begin{cases} s_i(t)\; if\; t \neq j \\ 1\; if\; t = j \end{cases} \cup\; Strategy\; s_j'(t) \begin{cases} s_j(t)\; if\; t \neq i \\ 1\; if\; t = i \end{cases}$$

The main difference from the other two is the combined use of the two deviations:

$$\forall\, ij \notin g \;\rightarrow if\; u_i\big(g(s_i', s_j', s_{-i,j}\;)\big) > \; u_i(g)\; then\; u_j\big(g(s_i', s_j', s_{-i,j}\,)\big) < \; u_j(g)$$

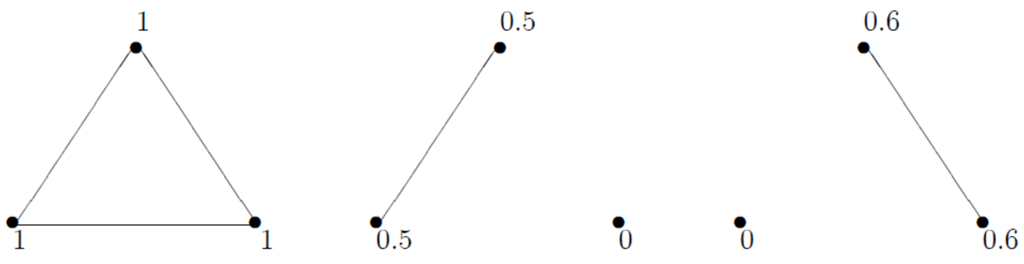
Finally, summing up all the analysis, we proved the proposition above.

3.3 Proof of distinction between the different sets

Now let's prove that all these sets:

$$SS \;\subset\; SS2 \;\subset\; PNS \;\subset\; NS$$

Can be distinct with an example:



Where all the other possible graphs will lead to a zero-payoff for all the agents.
From left to right, we have graph **A**, **B** and **C** and **D** will be the empty graph not pictured.
For any notion we'll see which of the graphs satisfy its conditions.

NASH STABILITY

All the graphs are Nash stable, since any change in configuration for any agents would lead to a zero-payoff. Moreover, while in graph **A**, **B** and **C** a change will make some players lose something from their initial configuration, in graph **D** the payoff of the agents would be the same as before, so there's still no incentive to change.

We can do a consideration regarding Nash stability, because it can be tricky in some cases. For example, looking at the graph **D**, imagine that the starting strategy for the top agent include the willingness to form a link with the bottom right agent, that instead doesn't want to form the link: a change in the second agent's strategy would permit the formation of the link and so an increase in payoff for the agent. For this we have to recall that a graph is considered Nash stable if at least one of the strategies that will make it as outcome is a Nash equilibrium. For instance, a Nash equilibrium that will lead to the empty network will be the strategy where all the agents declare the non-willingness to form the link with all the others: this is a Nash equilibrium because a single change will not make the difference.

PAIRWISE NASH STABILITY

Basing on the pairwise Nash stability conditions:

Graph **A** is stable, because no player has incentives to cut a link and no pair of agents would benefit from the creation of a link.

Graph **B** and graph **C** are also pairwise Nash stable, because as before no one has an incentive to cut and no pair wants to create. If we could use both the conditions of pairwise Nash stability together, we could say that in graph **B** the top agent would decide to cut the link with the bottom left agent and create the link with the bottom right one, making as final configuration the one in graph **C**, but the PNS condition will admit a change of status of just one link and so graph **B** is considered stable.

The graph **D** is not pairwise Nash stable because top agent can decide to cooperate with one of the bottom agents since both the agents included in the link will reach a higher payoff.

STRONG STABILITY OF ORDER 2

Going through the Strong stability of order 2:

Graph **A** is strongly stable of order 2, because no coalition of two agents would lead to a payoff increase for any agent.

The same is for graph **C**, since the other coalitions of two agents would not lead to higher payoffs.

Regarding graph **B**, top agent and bottom right agent would decide to cooperate, since both would reach a higher payoff by excluding themselves from the total network forming a link between each other.

As PNS is a subset of SS2, we can say a priori that the graph **D** will not be Strong stable of order 2.

STRONG STABILITY

Finally, we can see which graph will be considered as stable for the general Strong stability conditions.

As was before for graph **D**, we can exclude this time from our analysis graph **B** and graph **D**, because they don't satisfy the conditions of Strong stability of order 2 so they will not satisfy also the stricter conditions of Strong stability.

For graph **C** we have to see if in the network there is a possible coalition of any number that would make to at least an agent the incentive to deviate: Since Strong stability of order 2 requirements were satisfied by the graph and the total number of agents is three, we precisely need to find a coalition of three agents that would lead to a benefit for at least an agent. Such coalition exist and it is equal to the complete network, that is also the graph **A**, for which all the agents will increase their payoff. For this reason, graph **C** is not Strongly stable.

Conversely, since the coalition of three mentioned is equal to the graph **A**, there's no other coalitions that would lead to higher payoff for any of the agents, and so the graph is the only one that satisfies the Strong Nash requirements.

4 Extension of pairwise Nash stability

4.1 A new concept: the *triwise Nash stability (3NS)*

Looking at the analysis done in Chapter 3, a plausible continuation could be a deeper understanding on the interactions between Strong stability refinements and mixed refinements like the pairwise Nash stability, in order to see if the two different strategies will enlarge their differences or build up new similarities.

Starting from this idea, a good comparation could be the one regarding the SS3 and a new notion that, based on pairwise Nash stability, will take into account not only the needs of two agents but consider a group of three, in order to be more useful to compare with the third order of strong stability.

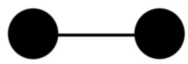
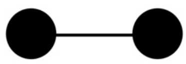
Thanks to this new *"triwise"* approach there will be interesting to analyze also how the concept of stability change internally, mostly regarding the possible configurations that a graph could reach and also the way links are generated or deleted, defining the strategies not only on the two nodes that form a link, but also to the other considered by the notion.

Since starting the reasoning directly from a *3-agents approach* could be tricky, a good basis point is understanding deeply how the simple PNS allows manipulation in a network. The graphical example below shows all the configurations that a pair of agents can have:

Here it is easy to understand that there are two possible configurations: the graph where the two nodes are linked and the one where they are not. Since there are just two configurations, there are also just two deviations in the graph, from one to the other disposition.

The first is the one from the connected to the unconnected graph: this deviation depends on the first condition of pairwise Nash stability because each player decides whether a link is useful or not for him, and it also can delete it by its own choice (this is due to the

fact that to form a link both the agents must say yes in order to create it, and so even a single declaration of not wanting the link would lead to its absence).
The reverse deviation is instead related to the second condition of the PNS, the one mostly related to the pairwise stability concept.
For this in accordance to create the link the choice of a single agent may not be sufficient, because both of them are needed to agree in order to create the new link (if in the starting situation one of the agents already said yes to the link then is sufficient that only the other one changes its opinion).
These two statements lead us to a big understanding that would have been possible to see also in the previous analysis:
"The creation of a link needs the agreement of both the agents while the deletion needs only the decision of one of them".
In the following table is expressed how the two conditions of PNS works in order to reach all the configuration from the other (from the configurations in the first column to the configurations in the first row):

From/To		
	*X*	*2nd condition*
	*1st condition*	*X*

Now we'll see the transposition of this properties in a three-agent world, starting by looking at all the possible situations that we can have:

The first important thing to consider is that we can have four different possible situations: 0,1,2 and 3 link created (considering also all the possible combinations of links).

A priori we could move from any configuration to any other in the graph, but we're going to define some restrictions to the possible actions that can be done in the network.
The restriction that we are going to apply from the simple PNS is that is possible to change the status of only one link (that is equal to the unique link in the pair approach).
While in the network with two nodes the two conditions allow to reach all the other possible networks (actually the other one), in the 3-agents world all the changes of disposition that require more than one link creation/deletion are not analyzed since just a deviation is considered.
For this reason, we can allow more transformations of the graph, one of that is the possibility of making two creations or two deletions, but not one of each type (we're going to ignore the changes of disposition of the links without modifying the number of them). In order to consider these new transformations, we need to put some new conditions that admit the creation or the deletion of two links (precisely, one for creations and one for deletions).

4.2 Mathematical conditions of the 3NS

We first need to show the subjects that will interact with these new conditions:

- The set of agents: agent ***i***; agent ***j*** and agent ***k***;
- The whole set of links that we can create with the agents: ***ij,ik,jk;***

The new conditions will capture the possible changes of exactly two links, since the change of just one status are totally explained by the conditions of the pairwise Nash stability.
Regarding the deletion condition, we have:

$$\nexists\ i,j,k\ \in N;\ ij,ik \in g \mid u_i(g - ij - ik) > u_i(g)$$

We have to recall that it's necessary just one of the two agents that is not happy in order to remove the link, and so that we need to cover not only the interest of one but of all the agents. The condition above must be adjusted including all the nodes:

$$\nexists\, i,j,k \in N;\ ij, ik \in g \text{ such that:}$$
$$u_i(g - ij - ik) > u_i(g) \ \ OR:$$
$$\begin{cases} u_j(g - ij - ik) \geq u_j(g) \\ u_k(g - ij - ik) \geq u_k(g) \end{cases}$$

With at least one of the last two strict.

Leaving for a moment the math behind, we could dive into the reasons on why the graph could deviate and so doesn't satisfy the stability requirements.
In the situation above the idea is that the two links will be deleted if one agent (precisely agent i) doesn't like to have a link with the other two agents and so decides to stay by himself, or that the other two agents (agents *j* and *k*) both wants to cut the link with the same agent, resulting in the same output. Other possibilities could be a mixed decision of the agents, such as for example the willingness of agent *i* to cut the link with the agent *j* and agent *k* that decides to cute the link with agent *i.*
All these examples are clear evidences of instability of a graph that we want to remove.
For this, there will be a in-depth study regarding the motives behind the deletion of the links.

Now that we have the condition for the severance of two links, we're going to ideate one for the creation of them, remembering the necessity of both the agents to increase their payoffs in order to form the link:

$$\nexists\, i,j,k \in N;\ ij, ik \notin g \text{ such that:}$$
$$\begin{cases} u_i(g + ij + ik \ \geq u_i(g) \\ u_j(g + ij + ik \ \geq u_j(g) \\ u_k(g + ij + ik \ \geq u_k(g) \end{cases}$$

With at least one strict.

Once again, the reasons about the deviation can be investigated.
In this case if all the agents could benefit from the creation of the links then they will be formed, letting us understand also this time that the initial graph cannot be considered stable. Moreover, in the case of creation there is the unanimous will to change the configuration.

The double creation/deletion allows to analyze more deviations than before (now we can analyze the relation between graph with two links of difference) but there are still some of them that are not considered. For example, under these assumptions we do not consider the possibility to go from the empty graph to the complete graph or the converse, because we cannot do a change in status for three links.

For this, we're going to think about a new possible condition that will take into account these two deviations but before, we can analyze the situations already comparable in order to get some useful insights.

According to what we were saying before, an interesting reasoning can be done on the motives for which a link is created or deleted.

While creation happens just in a single case (both the agents agree on the link), the severance can come up from different configurations as we saw in **3.2**.

The link *ij* is not formed if:

- Agent *i* doesn't want to form the link whereas agent *j* does;
- Agent *j* doesn't want to form the link whereas agent *i* does;
- Both agents don't want to form the link.

Transposing this analysis on a 3-agents world can be useful to understand the behavior of the agents, since there can be more than one link possible and so also more changes of status. We can see this by an example:

We want to study the change from the first figure, a complete graph, to the second figure, a graph with just one link.

In the second configuration we can see two distinct components: the bottom left agent alone as first one and the other two as the second.

Ignoring the deviations where both the agents agree on the severance of the link since they just came out from the unanimous decision, we can try to understand what can be the causes of the isolation of the bottom-left agent.
We can define three different situations:

- Bottom-left agent decides to exclude himself by the network and so decides to cut both the links;
- Top agent and bottom-right agent decide to exclude the bottom-left node and so they both express the willingness to cut the links with it;
- Bottom-left agent decides to cut the link with just one agent of the other two, while maintaining its willingness to form a link with the other. This last one will instead express its preference to cut the link with it and so both the links will be cut.

We can distinct the first two possibilities from the last one.
In the first two cases, the isolation of the agent derives from a sort of "robust" decision, because a set of agents (in the first case only the bottom-left one, in the second the other two agents) decides all by itself to remove the links. These type of deviations are similar to the idea of Strong stability, because it's effectively a coalition in the network that decides to exclude itself.
On the other hand, the last deviation, that can be specular with respect to the two agents of the second component (it's indifferent which of the two decides to cut and vice versa), can be seen as a "soft" decision, because the final configuration will depend on different decisions in the networks: the bottom-left agent is not isolated or decide to isolate itself, it just wants to cut a link. The reason is equal for the second component, because the agents inside it don't want to totally exclude the first one, but just one of the two expressed this willingness.

After this analysis of severance, we can go back to the unexplained relations of our graph. As we were saying before, given the fact that we are allowed to do just 2 changes of status until now, we cannot reach a 3-step change.
For this purpose, we have to add two new conditions, following the pattern of the previous ones that this time will admit three changes of status.

The first one, according to deviate from a complete to an empty graph, is that if at least two of the agents would have a higher payoff by removing themselves from the network (remember that all the agents have the objective to maximize their profit, ignoring the total "wealth" of the coalition) then they will decide to cut both links and so the final result will be the empty graph:

$$\nexists\ i,j,k\ \in N;\ ij,ik,jk \in g\ such\ that:$$
$$\begin{cases} u_i(g-ij-ik-jk) \geq u_i(g) \\ u_j(g-ij-ik-jk) \geq u_j(g) \end{cases}$$

With at least one strict.

We also have to consider that this reasoning has to be applied to all the possible pairs of agents, and so we will have three different systems of equations.

On the other hand, in order to form a complete network from an empty one we need to have all the three agents that will reach a bigger payoff by joining the whole-linked network:

$$\nexists\ i,j,k\ \in N;\ ij,ik,jk\ \notin g\ such\ that:$$
$$\begin{cases} u_i(g+ij+ik\ \geq u_i(g) \\ u_j(g+ij+ik\ \geq u_j(g) \\ u_k(g+ij+ik\ \geq u_k(g) \end{cases}$$

With at least one strict.

It's easy to see how these two conditions are heavily related to the concept of Strong stability:
In the first condition, at least two of the three agents decide to form a coalition of one (i.e. stay on their own), while in the second one all the agents agree in forming a coalition of three.

Finally, we have all the conditions required to define our new stability concept, that will be named as *"triwise Nash stability":*

Definition: *A network is said to be triwise Nash stable if it satisfies the following conditions:*

I. $u_i\big(g(s)\big) > u_i(g(S'_{i,}S_{-i});$

II. $\nexists\, ij \notin g \mid u_i(g(s)+ij) \geq u_i\big(g(s)\big)\,;\, u_j(g(s)+ij) \geq u_j\big(g(s)\big)$ *with one strict;*

III. $\nexists\, i,j,k \in N;\ ij, ik \in g\ such\ that$:

$u_i(g-ij-ik) > u_i(g)\ \ OR$:

$\begin{cases} u_j(g-ij-ik) \geq u_j(g) \\ u_k(g-ij-ik) \geq u_k(g) \end{cases};$

IV. $\nexists\, i,j,k \in N;\ ij, ik, \notin g\ such\ that$:

$\begin{cases} u_i(g+ij+ik \geq u_i(g) \\ u_j(g+ij \geq u_j(g) \\ u_k(g+ik \geq u_k(g) \end{cases}$ *with one strict*;

V. $\nexists\, i,j,k \in N;\ ij, ik, jk \in g\ such\ that$:

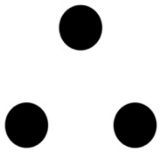
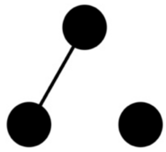
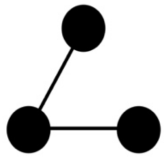
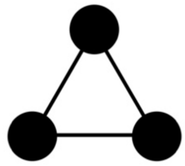
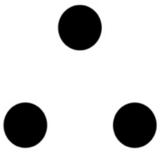
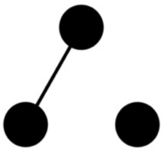
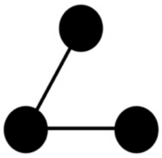
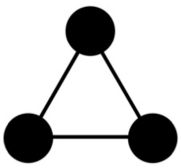
$\begin{cases} u_i(g-ij-ik-jk) \geq u_i(g) \\ u_j(g-ij-ik-jk) \geq u_j(g) \end{cases};$

VI. $\nexists\, i,j,k \in N;\ ij, ik, jk \notin g\ such\ that$:

$\begin{cases} u_i(g+ij+ik \geq u_i(g) \\ u_j(g+ij+jk \geq u_j(g) \\ u_k(g+ik+jk \geq u_k(g) \end{cases}$ *with one strict*;

Now that we have all the conditions together, we can see how the different configurations interact.

As before for the PNS, below there is a table that maps all the possible changes in configurations related to the precise conditions that allow them.
This time we will use all the six conditions we formulated for 3NS, in order to see if they will be sufficient to explain the possible changes in configuration that can happen in with three agents:

From/to				
	*X*	*2nd condition*	*4th condition*	*6th condition*
	*1st condition*	*X*	*2nd condition*	*4th condition*
	*3rd condition*	*1st condition*	*X*	*2nd condition*
	*5th condition*	*3rd condition*	*1st condition*	*X*

Except the diagonal, all the other relations are finally explained.

4.3 Inclusions for the 3NS

Now that the new stability concept is properly written, an interesting thing to see is how it interacts with all the others before mentioned. In order to do it, a good starting point is the proposition we have proven in Chapter 3:

$$SS(u) \subseteq SS2(u) \subseteq PNS(u) = PS(u) \cap NS(u)$$

We can re-arrange the subjects of the proposition in order to adapt it to the 3NS.

As for pairwise Nash equilibrium interaction with the Strong stability of order 2, for triwise Nash stability will be useful to analyze how is its behavior regarding Strong stability of order 3: the expectation is that the SS3 is included in 3NS because its constraints are stricter.

STRONG STABILITY OF ORDER 3

Recalling the SS3 conditions:

$$\nexists\ C \subseteq N \ with\ C \leq 3 \mid u_i\left(g\left(s_c', s_{N\setminus C}\right)\right) \geq\ u_i\left(g(s)\right) \forall\ i\ \in C$$

With one stricter.

Once again, the idea behind Strong stability of order 3 is that the graph is robust to any possible deviations based on coalitions of at most three agents, because no one of the agents in the coalition can have an increase by a different configuration.

This last statement is the empirical prove that our expectation was correct:

While 3NS creation conditions will admit the possibility of a higher payoff for some of the agents considered, as in pairwise Nash stability was, strong stability has stricter requirements because no player has the possibility to gain from a deviation of one up to three agents. Finally, we can say that the Strong stable of order 3 networks will be a subset of the triwise Nash stable:

$$SS3 \subseteq 3NS$$

STRONG STABILITY OF ORDER 2

Going on with the proposition, we now want to compare our new notion with the Strong stability of order 2.

Considerations for the Strong stability of order 2 will be equal to the ones done for the third order of the same refinement, but this time the attention is on coalitions with at most two agents.

Here we can see a problem regarding their comparison:

If we look at severance conditions of 3NS is clear that are taken into account all the three agents inside the analysis and so some of network that are detected as stable by SS2 will fail the conditions of 3NS.

On the other hand, we can look for a moment at the creation conditions of 3NS.

As we said before, 3NS can admit increase in payoffs for some of the agents considered, with the condition that at least one of them would have a loss.

This condition is weaker than the SS2, because in this last refinement there's no agents that can gain from a deviation from the network, so the comparison regarding the inclusion of 3NS and SS2 is not possible, because both the notions will be stricter than the other from some point of view, with the existence of networks that are only Strongly stable of order 2 and networks that are only triwise Nash stable.

PAIRWISE NASH STABILITY

We can proceed in the comparison with the pairwise Nash stability.

We have to recall that the triwise Nash stability is a derivative of the Pairwise Nash stability and it also takes the all the requirements of PNS and adding up more conditions, so it is easy to understand that networks that are triwise Nash stable must also be Pairwise Nash stable, leading to the fact that the set of 3NS is a subset of PNS.

In conclusion, we found that inside the first proposition of inclusion 3NS will be between SS3 and PNS exactly like SS2 but since these notions cannot be comparable, we have to put them in two different branches that then will re-converge in just one:

$$SS(u) \subseteq SS3(u) \begin{array}{c} \subsetneq SS2 \subsetneq \\ \subsetneq 3NS \subsetneq \end{array} PNS(u) = PS(u) \cap NS(u)$$

An interesting consideration we can do is the one regarding the comparison between SS3 and 3NS. While in the 2-agents environment the two notions of SS2 and PNS would be different by just the opportunity of one agent to increase a payoff, allowed in PNS but not in SS2, in the 3-agents world the distance between these two notions will be higher, since the number of agents that can reach a higher payoff by a deviation for 3NS will increase (precisely, N-1 agents can have a better payoff), while Strong stability will always admit a zero gain for all the N agents. With the increase of N, the distance will always be higher, making the requirements of SSN (Strong stability of order N) being really far from the ones of NNS (N-wise Nash stability).

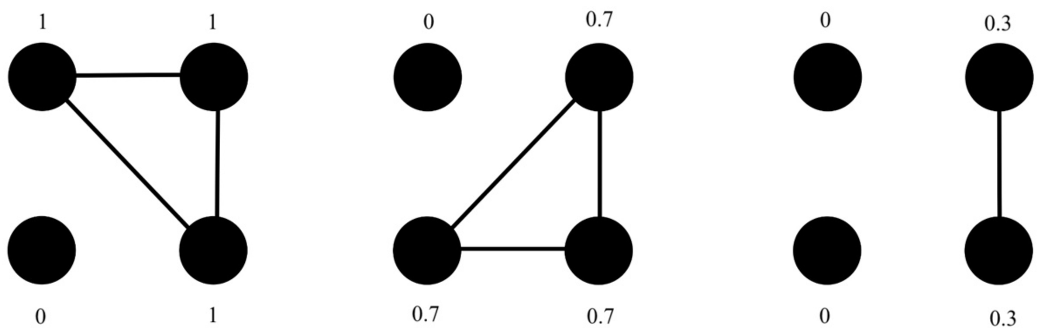
4.4 Proof of distinction for 3NS

Another trivial thing to check is as we did before (see **3.3**) the possibility that the different sets are distinct also including the 3NS in the environment.

This time the notions taken will be adjusted according to understand the behavior of 3NS: SS2 will not be considered while SS3 will be used instead of the more generic Strong Stability, PNS will be considered while the NS analysis would not lead to new information:

$$SS3 \subset 3NS \subset PNS$$

A practical graphical example is:



All other configurations will lead to a zero-payoff for all the agents in the network.

Once again, from left to right we have graph **A**, **B** and **C**, while the empty network is defined as graph **D**.

As before we're going to analyze all the notions with respect to which graphs fulfill their requirements.

PAIRWISE NASH STABILITY

All the pictured network are pairwise Nash stable because there's no link that an agent could sever in order to reach a higher profit and no link that a pair of agents could create in order to have an increase in payoff by both sides.

TRIWISE NASH STABILITY

We can finally see the new notion in a practical example.

Graph **B** is triwise Nash stable because there's no way that a pure strategy (only creation or deletion of links) lead to a re-arrangement such that the agents have incentive to deviate.
Graph **A** is still triwise Nash stable, because no matter how you change the configuration, the agents that are connected would not reach a higher payoff and so they will not deviate.
The situation is different for graph **C**: a pure strategy, precisely the creation of two links, will lead to two different configurations that are better with respect to the starting one, so the agents have incentive to deviate. Moreover, the final deviation will be the reaching the configuration of graph **A**, since the two agents that already form a link (the right side of the graph) will reach a payoff bigger than the one that the graph **B** would guarantee them.

STRONG STABILITY OF ORDER 3

The final analysis will be done on Strong stability of order 3:
Looking at graph **B**, we can see that there is a coalition of three that excluding itself from the network will guarantee an increase in payoff for the player inside the coalition: this coalition is indeed the one that we see in graph **A**, because the three agents excluding the bottom-left one would decide to exclude themselves by the network in order to form a coalition that will increase the payoff of all of them.
For this reason, the only graph that satisfies the conditions of SS3 is the graph **A** because, as we were saying before regarding the 3NS, there's no configuration that will lead to a higher payoff for the three agents that are linked.
By this example it's proven that all the sets can be distinct and so that all the different notions can be seen as single entities well defined in the social network analysis.

5 Conclusions

The exploration of network formation and stability through game theory brought out interesting insights regarding behaviors and interactions of agents.
This thesis explores Bloch and O. Jackson (2005)'s work, in order to analyze and explain the various different stability and equilibrium concepts with the aim to extend the study toward new notions between these two branches of the networks generation.
Moreover, a deep analysis is conducted on the Pairwise Nash Stability (PNS), a refinement that takes the fundamental conditions of both Pairwise Stability and Nash equilibrium, resulting in the perfect intersection between the two notions.
The main finding here is the formulation of the *triwise Nash stability* (3NS), a notion that relies on PNS fundamentals, shifting them on groups of three agents. The 3NS allows to find networks that fulfill both stability and equilibrium properties also regarding triplets of agents.
Just like the Strong stability of order 3 (SS3) does with the Strong stability of order 2 (SS2), the 3NS considers in its set of conditions also the ones that are already in PNS, making us understand that the two notions work in a parallel way, therefore there could be a possible generalization of the Pairwise Nash stability according to the number of agents analyzed, exactly like the different orders of Strong Stability (SS).
Despite their similar evolutions, the two concepts of Strong stability of order N (SSN) and *N-wise Nash Stability* (NNS) will increasingly differ as the number of agents considered grows, since SSn will always maintain a zero gain for any agent, while the NNS will admit some gains between the agents.
From this starting point it could be useful to find some new notions between these two,that rely on the stability conditions of PNS but that can also limit the possible gains in the networks, in order to define stable a smallest set of networks.

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