Department of Economics and Finance

Chair of Gambling: Probability and Decision

Poker: Where Strategic Decision Overcome Chance

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To all my friends who made me discover a new inner passion.

Contents

Introduction

Poker, often considered simply as a game of luck, is, in reality, something that goes far beyond the pure "gambling" aspect. This thesis seeks to dismantle the conventional misperceptions surrounding poker, demonstrating it as a rigorous blend of many fields from the mathematical one, to the psychological. During the discussion, we will delve into the subtleties of poker strategy, exploring both the theoretical foundations and their practical applications, positioning poker not only as a form of entertainment but as a serious psychological challenge.
Central to this discussion is the application of game theory, which provides a structured framework for understanding the intricate interactions among players. Game theory, a mathematical study of strategic interactions among rational decision-makers, helps players grasp the dynamics of poker, where each decision profoundly impacts the game's outcome. For example, when deciding which action to take in a precise moment during the game, a player must consider the potential reactions and strategies of their opponents, demonstrating the core principles of game theory. Each player should try to adjust based on the patterns they recognize by carefully analyzing the opponent's strategies.
This thesis thoroughly examines how principles such as probability, Nash equilibria, and game theory optimal (GTO) strategies form the backbone of expert poker play. Probability, the measure of the likelihood that an event will occur, is essential in poker for calculating the odds of winning a hand, drawing a specific

card, or making a successful play.
Nash equilibrium, a situation where no player can benefit by changing their strategy while others keep theirs unchanged, also plays a crucial role in poker. However, as we will find out, this concept illustrates the complexity behind this incredible game, in which the players should actually avoid such an equilibrium in order to succeed.
Finally, Game Theory Optimal (GTO) strategies further provide a strategic plan to follow that transcends the poker table. GTO strategies are mathematically balanced to be unexploitable by opponents, guiding players to make decisions that minimize losses and maximize profits regardless of their opponents' actions. Interestingly, this strategies are impossible to be perfectly applied by players in reality, which is exactly why sometimes it is better to deviate from them in order to actually achieve the best outcome in that scenario. GTO strategies assume the players are perfect from the Game Theory point, which is something far away from the reality. Poker players, as any other human being, continuously make mistake. The goal is to exploit the mistakes made by our opponents while minimizing our own.
By exploring these concepts, this thesis positions poker not only as a form of entertainment but as a serious mental challenge. These principles offer a strategic framework that enhances understanding and mastery of the game, providing insights that extend beyond the simple card aspect.
Moreover, we will explore the psychological intricacies of poker, from bluffing and risk management to the psychological warfare waged across the table. By analyzing real-game scenarios and strategic maneuvers, we aim to illustrate the practical implementation of complex theoretical concepts, providing insights that resonate in various decision-making scenarios beyond poker.
Each chapter is designed to challenge traditional views on poker, helping readers gain a clearer and more detailed understanding of its complex strategies.

By bridging theoretical rigor with practical insights, this thesis not only aims to refine the strategic approaches of experienced players but also to introduce new players to the vastness of challenges that arise while playing, showcasing the game as something that extends to various disciplines.

The objective of this thesis is to serve as a comprehensive guide through the many facets of poker. It is designed to function as a manual, allowing readers to easily access and delve deeply into specific areas of interest within the game. Through this work, I hope to transform the reader's perception of poker, fully explaining why it is such a fascinating and worthwhile topic of study and, most importantly, why it is a game "where strategic decision overcome chance".

Chapter 1

Understanding the Theory

Poker is a card game of 19th-century American origin with Persian and European ancestors which combines elements of both **stochasticity** (due to the randomness of card draws) and **imperfect information** (given that players do not have full knowledge of each other's cards).
This is the reason why over the last decade *Texas Hold'em* poker has become a challenge problem and common testbed for researchers studying artificial intelligence and computational game theory.
In the field of computational game theory, games are often compared in terms of their size. The size of a game is a simple heuristic that can be used to describe its complexity and compare it to other games, and a game's size can be measured in several ways. The most commonly used measurement is to count the number of **game states** in a game: the number of possible sequences of actions by the players or by chance, as viewed by a third party that observes all of the players' actions.
In the poker setting, this would include all of the ways that cards can be dealt and all of the possible betting sequences. This number allows us to compare a game against others such as chess or backgammon, which have 10^{47} and 10^{20} distinct

game states respectively [6].

In the realm of imperfect information games, a crucial concept to understand is the one of **information sets**, which serve as the backbone for strategic decision-making in such environments. Unlike perfect information games, where every past action and the complete state of the game are visible to all players, imperfect information games conceal certain elements, leaving players to navigate with partial knowledge. In this context, information sets, or decision points, emerge as fundamental constructs.

An information set encapsulates all the game states that a player cannot distinguish between, given their limited information. For instance, in a card game like poker, a player is unaware of the opponents' cards. Thus, **every possible hand the opponents might hold, compatible with the known information, forms an information set**. A player's strategy must be crafted around these information sets, focusing on the collective knowledge they represent rather than specific, unobservable game states.

In these games, players strategize by considering their information set at each juncture, instead of the precise state of the game. This approach entails devising a plan of action for every potential information set, dictating how to respond to each conceivable scenario that the limited information might represent. Such a strategy ensures that players are prepared for all variations of hidden information, enabling them to make informed decisions even in the face of uncertainty.

Another important aspect, we need to talk about concerns **infoset-actions**. Think of this as a way to measure how tricky a game is by counting up all the different moves you can make in every situation you might find yourself in, without knowing everything that's going on. Basically, it is like looking at the total number of moves you can pick from, across all the different scenarios you might not be able to tell apart.

This concept, which we may even find hard to understand theoretically, explains the complexity behind the analysis of imperfect information games like Poker.

1.1 Game Theory

Game theory is a branch of mathematics and economics that examines the strategic interactions among rational decision-makers. It seeks to understand and predict the outcomes of scenarios where the actions of multiple agents influence each other. Employed across various disciplines such as economics, political science, psychology, and biology, game theory models competitive situations where an individual's outcome depends on the collective decisions of all participants. **The theory explores how individuals can optimize their strategies to obtain the most favorable outcome for themselves**, considering the potential strategies of others. Games are structured with defined rules, and the models typically incorporate elements like players, strategies, payoffs, and the information available to each player.

In game theory, **a player's strategy is a complete plan of action for whatever situation might arise**; this fully determines the player's behaviour. A player's strategy will determine the action the player will take at any stage of the game, for every possible history of play up to that stage. [8]

Central to game theory are two fundamental assumptions about the players: they are **self-interested and rational**. Being "self-interested" in this context does not carry a negative connotation; it simply implies that players are focused on maximizing their own outcomes. Understanding Game Theory Optimal (GTO) strategies requires accepting that players not only prioritize their own benefits but also engage in logical and strategic planning, based on the potential strategies of others. Furthermore, we need to assume the following:

1. **Players have utility or payoff functions**, which quantify the value or satisfaction they derive from different outcomes.

2. Given all other conditions being constant, **players aim to maximize their utility or minimize their costs**.

3. **Players are capable of logical and mathematical reasoning**, enabling them to analyze and strategize within the game's framework.

4. Each player is aware of the game's rules, the strategies and utility functions of all players, and the fact that all players are rational.

This leads to the concept of **common knowledge of rationality**, where it is understood that not only all players know each other's strategies and utility functions, but they also know that everyone else knows this information, and so on. Moreover it is important to remark that each player has to choose a strategy in order to maximize his payoff. The most important thing is to choose a strategy which we think our opponent cannot exploit, otherwise we will give him an important edge.

Definition 1.1. *A strategy a_i of player i is strictly dominated by a strategy b_i if no matter what the other players do, the payoff by player i obtained while playing a_i is strictly smaller than the payoff obtained by playing b_i.*

Formally,

$$\forall x_{-i} \in X_{-i}, \qquad u_i(a_i, x_{-i}) < u_i(b_i, x_{-i}).$$

Let us now clarify what does exactly this equation states by analyzing each of the term. Concerning the strategies that the player can take we have: a_i which represents the strategy chosen by player i and then we have b_i which is an alternative strategy available to the player. Then we have the set of strategies of all the possible different strategies by the other players which is X_{-i}. Finally we have the two possible payoffs for the player: $u_i(a_i, x_{-i})$ and $u_i(b_i, x_{-i})$. The equation as you now may have understood exactly capture the idea of a *dominated strategy*. Strategy a_i is strictly dominated by strategy b_i because b_i always yields a higher payoff regardless of what the other players do.

1.2 Nash Equilibrium and Optimal Strategies

In the literature of Game theory, a Nash Equilibrium solution is often referred to as an *optimal strategy*. However, the adjective *optimal* is dangerously misleading when applied to the poker world because it implies that an equilibrium strategy will perform better than any other possible solution.
A Nash equilibrium strategy is one in which no player has an incentive to deviate from it, because the alternatives could lead to a worse result. This simply maximizes the minimum outcome since it implicitly assumes the opponent is perfect is some sense (of course this is not the case of poker). However, it is important to immediately notice that **a Nash equilibrium player will not necessarily defeat a non-optimal opponent.** For this reason, we need to introduce the concepts of **pure and mixed strategies**:

Definition 1.2. *A pure strategy provides a complete definition of how a player will play a game, determining the move a player will make in any situation they could face.*

Definition 1.3. *A mixed strategy is an assignment of a probability to each pure strategy, allowing a player to randomly select a pure strategy based on these probabilities.*

This introduces an element of unpredictability into the player's actions, making it harder for opponents to anticipate their moves. Since probabilities can take any

value within a continuous range from 0 to 1, there are infinitely many mixed strategies available to a player.
Now, let us consider as an example the game of rock-paper-scissors; the equilibrium strategy is to choose to play 1/3 of the times each action, this means that no one can defeat you in the long-term, but it also means that you will not win, since you have an expectation of winning equal to zero [1] against any other strategy.
In contrast, **a maximal player, which is player that aimes at maximize his payoff, can make moves that are not-optimal (in the GTO sense) when he believes that such a move has a higher EV.** Consider the case of rock-paper-scissors, where an opponent has played“rock“ 100 times in a row. A Nash equilibrium player/program is completely indifferent to the other player's tendencies and does not attempt to punish predictable play, since its goal is to avoid any decision that could lower the EV. A maximal, on the other hand, will attempt to exploit perceived patterns or biases even if this implies some small risk, such as the possibility that our opponent has set us a trap with the intention of playing "paper" on the 101st hand. **The main aspect of a maximal player is that he is willing to accept a small risk if it brings some positive EV**.
From a game theory standpoint, the most important thing about games with hidden information, like poker, is that **their optimal strategies involve using mixed strategies**.

1.2.1 A Useful Example: Cops and Robbers

Before going on to directly analyze poker scenarios, it is interesting to discuss a game called Cops and Robbers.

[1] Note that in Game theory analysis often such expectation of winning is referred to as Expected Value (EV). We will use this notation many times throughout the discussion of this essay.

In this game, one player is the "Cop", and the other player is the "Robber". The cop has two strategy options: to patrol or not to patrol. The robber has two strategy options as well: to attempt to rob a store, or not to. If the robber decides to rob, he wins 1 unit if the cop is not patrolling, but loses 1 unit if the cop patrols because he gets arrested. If the robber stays home, the cop loses 1 unit (resources, etc.) by patrolling, but nothing if he does not patrol.[2]

In order to better understand the reasoning behind this game, we will utilize a **payoff matrix** which is a tool used in game theory that outlines the potential outcomes or payoffs for each player based on the different actions each one might take. We can think of it as a table where each row represents one player's possible moves, and each column represents another player's moves. At every intersection, the table shows the outcome for both players if those specific actions are chosen. This helps players understand the consequences of their choices and strategize accordingly.

The payoff matrix for the Cops and Robbers game is the following:

	Robber	
Cop	**Rob**	**Don't Rob**
Patrol	(1, -1)	(-1, 1)
Don't Patrol	(-1, 1)	(0, 0)

Table 1.1: Payoff Matrix

To find the optimal strategy we pick a player and compute his payoff for every possible case.

Taking in consideration the Cop we know that he will choose a percentage x of the time to patrol, which means that he will not patrol $1 - x$ times. So we have

[2]This example is taken by Bill Chen and Jerrod Ankenman[2]

the following situation for the robber:

(i) if the **robber effectively robs**, his payoff will be:

$$(-1)(x) + (1)(1-x)$$
$$= 1 - 2x$$

The reason is that if the robber robs he will have a payoff of (-1) every time that the cop patrols, while he will have a payoff of 1 when the cop does not patrol

(ii) if the **robber does not rob**, his payoff will be:

$$(1)(x) + 0(1-x)$$
$$= x$$

Given that the robber is indifferent between his choices when these two values are equal, solving the equation $1 - 2x = x$ we can easily find that $x = 1/3$ which tells us that **the cop's optimal strategy is to patrol** $1/3$ **of the time**.
Following the same reasoning, now we consider that the robber will choose a percentage x of the time to rob and $1 - x$ not to rob.

(i) if the **cop patrols**, his payoff will be:

$$(1)(x) + (-1)(1-x)$$
$$= 2x - 1$$

(ii) if the **cop does not patrol**, his payoff will be:

$$(-1)(x) + 0(1-x)$$
$$= -x$$

Solving the equation $2x - 1 = -x$ we immediately notice that $x = 1/3$ also in this case, which means that also **the robber optimal strategy is to rob** $1/3$ **of the time**.

In this game, if the robber plays a pure strategy (see Definition 1.2) of always robbing, the cop can play a pure strategy of always patrolling. But if the cop does this, the robber can switch to a strategy of always staying home, and so on. These oscillating exploitive strategies tell us that the **optimal strategies will be mixed**.

In imperfect information games mixed strategies (see Definition 1.3) are necessary in order to have an edge on the opponents, and this is exactly what Poker players what to achieve: **a perfect mixed strategy**.

Chapter 2

Fundamental Aspects of the Game

2.1 Getting Familiar with Basic Concepts

For the readers new to this extraordinary world, the game starts with each player receiving two private cards, hidden from the other players. The idea is that each player tries to accumulate the most chips they can by exploiting their cards. As we will understand proceeding with the explanation of the basic concepts of the game, there are many ways to actually win the most money. We will start by introducing the general rules and notions to fully understand how to approach the game. First of all, to ensure there's always money on the table, two players, **The "small blind" and "big blind"**, are required to place forced bets before the cards are dealt, with the big blind typically contributing double the amount of the small blind (the reason behind the names of these two players is that they put the money blindly since they do it before receiving the cards). The first betting round begins with the player to the left of the big blind acting. This player has four possible actions based on his private cards. If no bets have been placed in a round, the player may either **check** (refrain from betting while staying in the game) **or place a bet. If the player bets, subsequent players can fold** (discard their cards and forfeit the pot), **call** (match the latest bet), **or raise** (increase the current bet). Check-raising, where a player checks early in a round and later raises after a bet is made, is a legitimate strategy. A betting round ends when all active players have checked or matched the latest bet or raise. This is done to ensure fairness with each active player having contributed the same amount of the pot. [1]

The tricky aspects about Texas Hold'em poker is that there are two main "stages" of the game: **pre-flop and post-flop** which are different in some aspects.

During the pre-flop phase the first player to act is the one to the left of the big blind [2], which means that actually the last to act is the big blind allowing him and also the small blind to have a little edge by being able to see the action before them. Once the big blind has act, three community cards are revealed (the flop), initiating another betting round. It is at this point that the post-flop action begins. **The critical aspect to keep in mind is that now, if they are still in play the small blind and the big blind have to act respectively first and second until the end of the hand, which is an important concern to take in consideration.**

Following the flop there is the reveal of the fourth (the turn) and fifth (the river) community cards, each followed by additional betting rounds. After the final betting round, if multiple players remain, a showdown occurs where the players reveal their hands to determine the winner. This is done by looking at who has the highest five-card combination using his two private cards and the community

[1]the term "pot" refers to the amount of chips currently on the table. Whoever wins the pot wins the chips.

[2]Generally referred to as "Under the Gun"(UTG) since it is the worst position to start because you have all the other players to act behind you.

cards.
In this chapter we will try to show how the theory applies to the practice through real-play relevant examples. [3]

2.2 Pot Odds: a Fundamental Aspect

In order to start our analysis we must start with the concept of **pot odds**. However, before proceeding with the explanation of such a concept, we need to define some terminology that we will use throughout the rest of this essay.

Definition 2.1. *We refer to a **made hand** as a poker hand that does not require any additional cards to improve its value. It is a strong hand that is likely to be the best on the current board, made up of a player's private cards and the community cards already dealt. For example, if a player has a flush after the flop or turn, they have a made hand because they do not need to draw more cards to have a complete, competitive hand.*

Definition 2.2. *We refer to a **draw** as a hand that can be made given certain community cards come out.*

Now, going back to the concept of pot odds, suppose that, with all cards out, the pot contains a units and a player is contemplating calling a bet of b units. Then we say that the pot is offering him odds of a to b, that is, the pot odds are a to b. Consider also that the player can estimate his probability of winning the hand as p, which is to say that the odds against winning are $1 - p$ to p, or $\frac{(1-p)}{p}$ to 1. Then the player's expected profit from making the call is $(1 - p)(-b) + pa$,

[3] Note that for seek of simplicity in this essay, we'll focus exclusively on Heads Up scenarios, a term used to describe situations where only two players are actively involved in the hand.

and this is positive if and only if:

$$p > \frac{b}{a+b} \qquad or \qquad \frac{a}{b} > \frac{1-p}{p}\,. \tag{2.1}$$

Let us consider for example a case in which following the "turn" in a game of Texas hold'em, a player holds a flush draw. He has reason to believe that he will have the best hand if and only if he makes his flush on the river. Since 46 cards remain unseen, of which 9 are of the flush suit, his odds against winning are 37 to 9. If the pot contains a units and the player is contemplating calling a bet of b units, he should call if and only if $\frac{a}{b} > \frac{37}{9} = 4.1$. Now we will see through an example how does this concept apply to real poker hands scenario.

Example 2.1. *Suppose there are two players in the hand: Alice and Bob.*
Alice has $A\diamondsuit A\spadesuit$ *(the best possible starting hand in Texas Hold'em Poker) and Bob has* $5\heartsuit 6\heartsuit$*. The community cards on the turn are* $K\diamondsuit 9\heartsuit 2\clubsuit Q\heartsuit$*. Alice has a made hand of a pair of aces and Bob has a flush draw (he currently holds two hearts in his hand, and there are an additional two hearts present on the board. This combination leaves him one heart short of completing a five-card flush).*
Now suppose there is already \$100 in the pot and Alice can either check or bet before the river card comes out. There are 9 hearts remaining in the deck, which would give Bob a flush, beating Alice. The remaining 35 cards would allow Alice's aces to hold. Pot odds derive form the computation of expected value (EV). The expected value is calculated as the probability of winning the pot times the new pot amount, deducted by the amount we need to call. Note that this calculation emphasizes that as soon as a player places a bet or calls, they should no longer consider that money to be theirs to lose, but rather part of the pot that they can win (sunk cost). Given this information we can easily compute the pot odds for Bob:

$$\mathbb{E}(B) = \frac{9}{44}(100 + 2x) - x$$
$$\approx 20 - 0.6x$$
$$x = 33.3$$

Bob should only call a bet of \$33 or $\frac{1}{3}$ of the pot.

This $\frac{1}{3}$ represents exactly the concept of pot odds*: since Bob expects to win the pot $\frac{1}{3}$ times he should only call an equivalent pot-sized bet.*

2.2.1 Implied Pot Odds

Going on with our practical analysis we must also consider the strategy of making a call when hitting our draw could significantly increase our winnings. Let us return to the first example, in which the pot contains a units and the player is contemplating calling a bet of b units following the turn. Now, suppose that, if the player makes his flush on the river and bets **c** units, he expects his opponent to call. This will add **c** units to the pot on the final betting round **with the result that the pot odds increase to a + c to b**. These odds are known as implied odds because they take future bets by the player's opponents into account. **It is often the case that a call that is not justified by the pot odds is justified by the implied odds**.

A good example of the concept of implied odds occurred on the final hand of the 1980 World Series of Poker main event (no-limit Texas hold'em). Doyle Brunson, who had \$232,500 in front of him, was heads up (i.e., only two players remained) against Stu Ungar, who had \$497,500. Brunson was dealt A♡-7♠ and Ungar was dealt 5♠-4♠. After the first betting round, the pot contained \$30,000. The flop came A-7-2, giving Brunson two pair and Ungar a straight draw. Ungar checked and Brunson bet \$17,000, a bet intended to keep Ungar around. Ungar's odds against making his straight on the turn were 43 to 4, or 10.75 to 1, while the pot was offering only 47 to 17, or about 2.765 to 1. But if he were to make his straight and Brunson had a big hand, he might be able to win Brunson's entire $232,500 plus his own $15,000 contribution to the pot. **This made his implied odds** 247.5 **to** 17**, or about** 14.559 **to** 1**, so he called**. Remarkably, the turn card was a 3, giving Ungar his straight, so he bet $40,000. Brunson raised all-in (betting all of his remaining $200,500) and Ungar called. Brunson failed to catch an ace or a 7 on the river, and Ungar was world champion [3].

2.3 Equity and Its Estimation

In this section, we will dive into the idea of **"equity" in poker, which is basically your chance of winning the pot based on the cards you have and the cards still to come.** It's like looking ahead to figure out how strong your hand could be when all the cards are out.

We will also discuss how this connects to the idea of pot odds, which is just about whether the money in the pot makes it worth sticking around in the game. If the pot's offering you good value for your bet, your equity should ideally match up with this. To make all this clearer, we'll walk through some examples that show how poker players use this concept to make smart decisions during a game.

Example 2.2. ***A flopped flush versus three of a kind*** *(see [4])*
Suppose that player A holds A♠K♠ while player B holds 10♣10♢ in a heads-up scenario. If the flop comes 2♠3♠10♠ both players hold fairly strong hands:
Player A has an ace-high flush, while Player B has three of a kind. Although Player A is currently ahead, Player B can still win the pot by making either a full house or four of a kind (i.e., "the board pairs") by the showdown.
Assuming that the hand is played to showdown, ***what is Player B's equity in***

this case? *Since there cannot be a tie, Player B's equity is the same as Player B's probability of winning the pot. In order to compute the probability of this event E we decompose it into three disjoint subevents:*

- *E_1: the event that the board pairs on the turn,*
- *E_2: the event that the board does not pair on the turn, but pairs on the river with a card that is neither an ace nor a king*
- *E_3: the event that the board does not pair on the turn, but pairs on the river with an ace or a king.*

For E_1 we have 7 of the remaining 45 cards that do the job, while for E_2 we have 32 cards that effectively do not pair the board and 10 cards which are neither an Ace or a king that do pair the board on the river. For E_3 instead, we have 6 aces/kings that have to come on the turn and then 9 total card that effectively pair the board on the river (the initial 7 cards + 2Aces or 2Kings). Thus, the computation is pretty straight-forward:

- $P(E) = P(E_1) + P(E_2) + P(E_3) = \frac{7}{45} + \left(\frac{32}{45}\right)\left(\frac{10}{44}\right) + \left(\frac{6}{45}\right)\left(\frac{9}{44}\right) \approx 0.344$*;*

We have found that player B has about 34.4% equity in this situation, which means that theoretically on the flop he/she should only call a bet of 34.4% of the pot.

As you may have noticed this proceeding is not so easy to apply during a real hand in which players have limited time to take decisions. Luckily enough there is a little trick that can be applied to assess immediately the equity of our hand.

2.3.1 The Rule of Four-Two

This interesting "rule" provides a simple way to estimate a player probability of winning at showdown given that *m outs* are available among the unseen cards.

The rule is applied as follows: given m *outs* after the flop, the probability of winning if the hand if played to showdown is approximately $4m$ percent, or $0.04m$, while given m *outs* after the turn, the probability of winning if the hand is played to showdown is approximately $2m$ percent, or $0.02m$.

The question we need to ask to ourselves is **why does this rule actually work?** In order to explain it we need to take in considerations the events that occur in a poker hand when we want to estimate our equity.

If m outs are available to a player after the flop, the event F of that player "hitting" an out on either the turn or the river can be decomposed into two disjoint sub-events:

- F_1 the event that the player hits an out on the turn
- F_2 the event that the player does not hit an out on the turn, but he hits it on the river

Considering that there are 47 hidden cards, the probability of event F which approximates the probability the player wins the hand in a showdown, is:

- $P(F) = P(F_1) + P(F_2) = \frac{m}{47} + \left(\frac{47-m}{47}\right)\left(\frac{m}{46}\right) = \frac{93m-m^2}{(47)(46)}$

Note that for typical values of m, m^2 is small relative to 93m . Thus, we use the approximation:

- $P(F) \approx \frac{93m}{(47)(46)} \approx 0.04m$

As you may have noticed, this rule plays a crucial role in real poker scenarios where players can rely on it in order to quickly estimate the equity of their hand, enabling them to assess whether to make a certain decision or another.[4]

[4]Example 2.2 highlights an interesting scenario in which the number of outs actually increases when the flop card is dealt (from 7 to 10 if no out is hit). Thus, it is reasonable to take the average number of outs, 8.5, in the calculation after the flop. **This yields a result of 4(8.5)% = 34%**, which compares quite favorably with the exact equity value.

2.4 Expected Value: a Measure of Favorable Play

In this section, we will try to analyze exactly the most crucial aspect that seems to govern the decision of a Game Theory Optimal (GTO) player, which is the concept of expected value. In statistics expected value (EV) is the average result of a given action if it was made hundreds (or even thousands) of times. But in poker it assumes a slightly different meaning.

In order to understand it let's have a look to a useful example: **The Steph Curry Bet.** Suppose you manage to get in touch with Steph Curry and he proposes you a $5 bet on his next shot. The natural question you should ask yourself it is whether it is convenient or not for you to take such a bet. Your first inclination is probably to decline, and you'd be correct to do so. **At even money this bet has a negative expected value (-EV)** since there are two possible outcomes in this scenario. [5]

(i) Curry makes the free throw and you lose $5, which will happen 90.1% of the time

(ii) He misses and you win $5, which will happen 9.9% of the time.

By looking at the two outcomes and their likelihoods, your can probably tell intuitively that this is a losing bet for you. But the question is: how much is it losing? This is exactly where it comes into play the concept of expected value. To calculate the EV of the bet, we use the simple statistical approach which consists in multiplying the probability of each outcome by its respective result, and add them together:

[5] It is very important to remark that we are considering this measure in terms of money, which is why we said it is negative in this case. Negative EV play means a play that makes us losing money.

EV(MISS): 0.099 x \$5 = \$0.495

EV(MAKE): 0.901 x -\$5 = -\$4.505

EV(TOT)= EV(MAKE) + EV(MISS): -\$4.505 + \$0.495 = -\$4.01

EV(TOT) = -\$4.01

Our intuition was right. If we made the bet 100 times over, we should expect to lose a lot of money: more than \$4 per shot. But now suppose that he offered you 20:1 odds: his \$100 to your \$5. What should you do? If we re-apply the above reasoning we get that:

EV(MISS): 0.099 x \$100 = \$9.90

EV(MAKE): 0.901 x -\$5 = -\$4.505

EV(TOT)= EV(MAKE) + EV(MISS): -\$4.505 + \$9.90 = +\$5.395

EV(TOT)= \$5.39

It is useless to say that we should accept this bet since it has +EV, but this of course does not mean that we will always win. As a matter of fact we expect to lose 90% of the time.

So, now you are wondering why would we take a bet that we will loose $\frac{9}{10}$ times ? The answer is that if we make this bet **often enough** we will realize our expected value for sure, generating a profit. **The point is that we are focusing on the long run, we don't care about some small losses in the short-run.**

2.4.1 Computing Expected Value in Poker

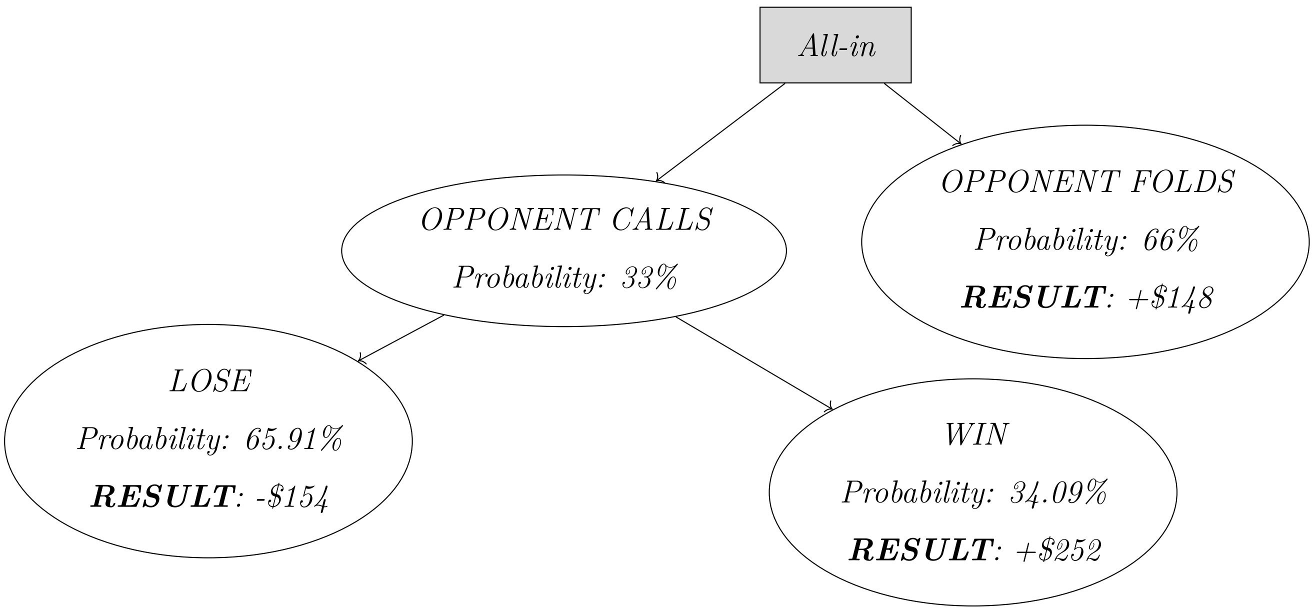
As we have seen through the concepts of pot odds and implied pot odds (Section 2.2 and Subsection 2.2.1), there is a way to compute the profitability of a certain play, however it is not as easy as it seems because we need to take in consideration more complex aspects. In the following example we will see step by the step the reasoning behind this computation:

Example 2.3. *Suppose we start a hand with a \$200 stack. An opponent raises to*

$16 from early position and we elect to call with J♢-9♢. Both blinds fold leaving us heads up. The pot is $38.

*The flop comes 5♢10♢2♣ and villain [6] bets $30. We decide to call, given the fact that with our flush draw we are getting the right odds (note that **here we are considering implied pot odds** since we are not getting the right "true" pot odds due to the fact that the opponent has bet around the size of the pot).*

The pot now is $98, and we have $154 remaining in our stack. The turn brings the 7♠ and the villain bets $50 which means the pot size increases to $148. Now it's our turn and we need to decide whether to just call or raising all-in. In this example we will demonstrate why in such scenario we should choose the second option. Let's assume we're familiar with Villain's game, and know that he's very capable of putting on the pressure also with medium hands. Therefore we are more than justified to think that if we go all-in he might fold 66% of the time. On the other hand, if Villain calls, we will need to hit our combo draw to win the pot.



[6] Villain it a very broad poker terminology used to refer to an opponent.

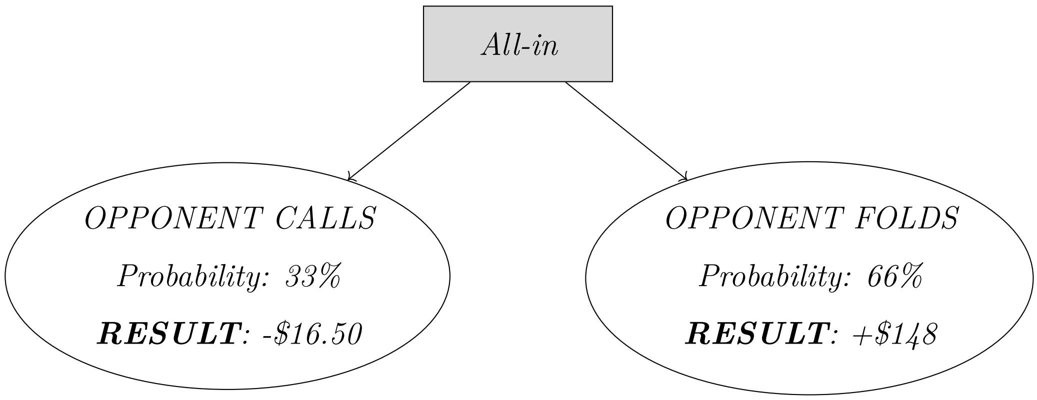
As we can see from the tree there are three possible outcomes:

(i) *Villain folds and we win $148 which guarantees us an EV=0.66 × $144=$97.68*

(ii) *Villain calls and we miss our draw, which results in a loss of $154 and a consequent EV of 0.6591 × -$154 = -$101.5014*

(iii) *Villain calls and we hit our draw which results in a win of $252 and an EV=0.3409 × $252 = $85.9068*

If we plug this values in our decision tree we will get:



Which means that:

EV(VILLAIN CALLS): 0.33 × -$16.50 = -$5.45

EV(VILLAIN FOLDS): 0.66 × $148 = $97.68

EV(ALL-IN)= -$5.45 + $97.68 = ***$92.23***

We have proved that in this scenario it is indeed profitable to raise all-in.

This example offers a clear understanding of how the expected value (EV) concept applies in actual play scenarios. Yet, we must consider that our scenario relies on certain assumptions based on our understanding of the villain's play style. What if the villain is a very tight player, one who typically has strong hands? Such an opponent would likely call our bets more often, significantly reducing our EV and making the decision to go all-in less effective than previously calculated. This illustrates why our decisions must be flexible and informed by our knowledge of the opponent to ensure we maintain an edge. Now, the natural question we should ask ourselves is: **if the villain adapts to our strategy, how do we adjust?** Keeping the villain uncertain is key, and this strategic dilemma is exactly what we will discuss in the following chapter.

Chapter 3

GTO Strategies

3.1 Exploiting the Opponent

In the previous sections, we talked about how to make the best choices for just one poker hand at a time, aiming to win as much as possible from that one hand (optimizing EV). However, poker isn't just about winning once; it's about prevailing over many scenarios. A good move for one hand might not always make you money in the long run. For example, imagine you're playing against someone who only bets with really good hands but still calls your bets with not-so-good hands, which is something beginners who don't like to take risks often do. You can easily beat this kind of player by folding (not playing) whenever they bet or raise, and only betting when you have good hands yourself. But, if they start to notice you always fold, they might start bluffing (pretending to have a good hand) to make you fold even when they have bad cards. On the other hand, if someone bluffs too much, you can play more hands than usual, because you'll have a good chance to win big when you actually have a strong hand. This leads to the idea of mixing up the kinds of hands you play, so your opponent can't easily figure out your strategy and take advantage of it. **This idea can be incorporate into**

the concept of balancing our range, or deciding the hands we play in a given situation such that an opponent cannot exploit our strategy. [1]

3.2 Balance

In order to play non-exploitable GTO poker, we should always "balance" our range meaning that we should play exactly in the same way our "value" hands and our bluff hands. In this way we have a variety of possible hands in the eyes of the opponent in any situation making very difficult for him to acknowledge when he has us beat. [2]
However, before proceeding with a very useful example of this concept applied in game we need to define the notions of **defensive value and indifference** which governs the actions of GTO players.

Definition 3.1. *We define defensive value as the expected value of a strategy against the opponent's most exploitative strategy.*

Basically we can think about it as reducing the difference between the defensive value (as explained in Definition 3.1) and the expected value (EV). In other words, the expected payoff of the strategy in a given hand should not change over time as your strategy is gradually exposed to your opponent: your opponent plays the same way regardless whether your strategy is known to them.

Definition 3.2. *Indifference refers to a game state where a player gets the same expected payoff regardless what strategy is chosen.*

[1] In poker the term "range" refers to the set of hands we or our opponent can have in a precise spot, it is used in order to narrow the field of possible hands that play such hand. We will see more about it going on with our discussion.

[2] In this section we will follow the discussion proposed by [7]

In game theory, it's a well-known fact that every multiplayer game with a limited set of payouts contains at least one Nash equilibrium (as we discussed in section 1.2). Poker, which is identified as a zero-sum game, also follows a known principle, which is that all two-player zero-sum games have an optimal strategy if mixed strategies are included 1.2.1.

Definition 3.3. *A zero-sum game is a type of game in game theory where one player's gain or loss is exactly balanced by the losses or gains of other players.*

This idea leads us to the principle of indifference. By setting the expected value equations from different strategies equal to one another, we can solve for values of certain parameters. These values make a player equally inclined to select any of the available strategies, thus reaching a point of indifference. The parameter's value that is derived from solving these equations is what we call the **indifference threshold** which is exactly what we want to find out in order to exploit our opponent. If we know such value in a certain way we leave our opponent with nothing else than a guess since he he will not be able to know the best decision to take. Basically when we find such parameter our opponent will take his decision based on the flip of a coin since either decision which could be calling or folding are perfectly equal for him. In the following example we will demonstrate exactly what we mean by this.

Example 3.1. *Assume there are two players (Alice and Bob) playing a hand where the board is K♡A♢6♡2♡K♢. Bob has K♣10♣ which means he has three of a kind and he is only scared of Alice having a flush. Furthermore, given the previous action we can estimate Alice's range to contain 20% of flush combinations (mostly composed by any Ax of hearts* [3] *since it is more logical that she is still in the*

[3]Ax of hearts means any combination of two hearts containing at the same time an Ace, so for example A♡Q♡ or A♡J♡ and all the others.

pot by the river). Now, suppose there is $300 in the pot and Alice can choose to bet a fixed amount of $100. ***The question is how often can Alice bluff in this spot?*** *If Alice bets $100, Bob can pay $100 to potentially win $400 which is obviously very favorable, he is getting 4 to 1 pot odds. Suppose Alice only bets when she has the flush. Bob can exploit this strategy by folding every time Alice bets, preventing her from getting any additional value from hitting her flush and taking the pot 80% of the time.* ***Alice has a defensive value of*** $0.2 \times \$300 = \$60$ ***with this strategy, since she only profits when she has a flush.*** *On the other hand, if she bets all of her hands here Bob EV will be* $0.8 \times \$400 - \$100 = \$220$ *when he calls and 0 if he folds. So* ***it is clear that Bob will exploit Alice's strategy by always calling****. Note also that in this case Alice's defensive value is* $0.2 \times \$400 - \$100 = -\$20$. *The two strategies mentioned so far (always checking a dead hand and always betting a dead hand) are what are known as* ***pure strategies*** *and neither is optimal for Alice in this situation. We know this, because both are exploitable since Bob can change his strategy and increase his payoff. This indicates we are not at a equilibrium point, meaning the best option is to use* ***mixed strategies****.*

Now if we want to compute Bob's EV for calling when Alice bets we can think of $P(A, \mathit{bluff})$ *as the fraction of all hands Alice has on the river and she bluffs with, making very easy to construct Bob's EV equation:*

$$E_B(B, \mathit{call}) = \frac{P(A, \mathit{bluff})}{0.2 + P(A, \mathit{bluff})} \cdot (\$400 - \$100)$$

Alternatively, Bob can fold when Alice bets.

$$E_B(B, \mathit{fold}) = \$0$$

Exploiting the fact that we know that the GTO strategies is the one for which Bob EV for calling and folding are equal, we can solve $E_B(B, \text{call})$=$E_B(B, \text{fold})$ in order to find out the optimal bluff frequency for Alice which turns out to be 6.7%. However, now we want to generalize this concept in order to have a more wider "Game theory" view of the scenario and we will do it through another example:

Example 3.2. *Assume that Alice and Bob are in a scenario in which Alice can make a bet of size 1 and Bob can call or fold if Alice bets. To visualize the payouts for each actions that the two player can do for this specific case it is very useful to construct the payout matrix which will be the following:*

	Bob	
Alice	Check-call	Check-fold
Winning hand	Bet $P+1$	P
	Check P	P
Dead hand	Bet -1	P
	Check 0	0

Table 3.1: Payout Matrix

As we can see it is always convenient for Alice to bet when she has a winning hand. On the other hand, it is more tricky to understand what she should do when she has a dead hand. The optimal decision depends on Bob's calling versus folding frequency. According to what we demonstrate in example 3.1 Alice wants to choose a bluffing frequency such that Bob's EV for calling is equal to his EV for calling. Let $b=\frac{bluffs}{bluffs+value}$ *bets we have:*

$$E_B(\mathit{call}) = b(P+1) - 1$$

$$E_B(\mathit{fold}) = 0$$

$E_B(\mathit{call}) = E_B(\mathit{fold})$ *when:*

$$b = \frac{1}{P+1} = \alpha$$

Similarly, Bob should choose a calling frequency such that Alice is indifferent to checking versus bluffing her dead hands. Let c be the frequency with which Bob calls.

$$E_A(check) = 0$$
$$E_A(bluff) = (1 - c)(P + 1) - 1$$

By setting these two EVs equal we find the value of c.

$$c = \frac{P}{P + 1} = 1 - \alpha$$

This can be generalized to any bet size which makes it incredibly useful when analyzing real game scenario. Suppose Alice can bet any fraction of the pot xP, we have that:

$$\begin{aligned} b &= \frac{xP}{P + xP} \\ &= \frac{x}{1 + x}, \end{aligned}$$

where x represents the actual bet made by a player allowing us to directly plug in the bet size into the equation.

As you may have noticed, so far we have just considered examples with one round of action which are much easier to analyze since they did not take in consideration any previous action, allowing us to directly assess the information we need. However, our analysis needs to consider one more interesting aspect which is actually the one of “Multi-street”. “Multi-street” refers to the fact that we are going to consider also everything (or at least something) that happened in different rounds of action.

3.2.1 Multi-Street Hands Scenario

In real play scenarios, every action that we take is guided by past information we gathered about our opponent. It is very interesting the fact that most of our adjustment should be made during the hands, and even if we don't really know the villain we can estimate his holding based on what he did during that specific session.

In the following example we will proof that the analysis of "single street hands" actually misses some very important considerations that make the result change.

Example 3.3. *Imagine that our two players, Alice and Bob are playing another hand. This time the board shows: K♢9♡2♣Q♡ and Alice has a pair of aces while Bob has a hand from a distribution which contains $\frac{1}{10}$ hands with two hearts and $\frac{9}{10}$ dead hands.*

The pot contains $4 and the players can either check or bet $1. Alice is first to act and Bob is confident that he will win the hand only if he hits the flush which will happen 20% of the times. Taking in considerations what we discussed before, Alice should bet and Bob should call only if he has the odds. [4]

Now, assume they make it to the river, and there is $6 in the pot (Alice bets on the turn and Bob calls). Alice has no reason to bet here, because we have assumed that Bob knows whether he has the winning hand at this point. Thus Alice checks and Bob can choose to either bet or check. According to example 3.2 Bob should bluff with frequency $\alpha = \frac{1}{7}$ as many hands as he value bets with and Alice should call with a frequency of $1 - \alpha = \frac{6}{7}$.

Nevertheless, this conclusion is not entirely accurate, as it is based on calculations specific to a single street game. Let's reevaluate this different scenario: if Bob contemplates a bluff on the river, this would mean he has previously called Alice's turn bet with a dead-hand. Additionally, Bob's bluff on the river is only possible if a heart is dealt. Hence, in this game that spans multiple streets, Alice must consider the equilibrium of Bob's decision to either fold or proceed with a dead-hand through

[4] Note that implied odds should be considered here rather than just pot odds, because Bob can get more value on the river by hitting his flush.

the successive stage of play. Consider c as Alice's optimal calling frequency on the river, we can do same computations in order to obtain the new calling frequency considering also past action.

$$
\begin{aligned}
E_B(\textit{dead hand, fold}) &= 0 \\
E_B(\textit{dead hand, play}) &= P(\textit{flush})P(\textit{Alice calls})(-2) \\
&\quad + P(\textit{flush})P(\textit{Alice folds})(5) \\
&\quad + P(\textit{no flush})(-1) \\
&= (0.2)(-2)c + (0.2)(5)(1-c) + (0.8)(-1) \\
c &= \frac{1}{7}
\end{aligned}
$$

Now, the actual calling frequency is $\frac{1}{7}$ rather than $\frac{6}{7}$. It is evident that the result show some discrepancy with the previous example where we consider the action just on the river. This is way we need to understand the difference between theoretical and more practical examples. By analyzing a single street game, we are able to reason about strategies, but the determined frequencies cannot be blindly applied to multi-street games where there are added layers of complexity.

3.3 GTO: a Safe Harbour to Avoid Exploitability

The examples that we have seen through the discussion of this chapter are very useful to understand the mathematical reasoning behind the principles of balance and indifference, key to reduce exploitability. However we need to make few considerations: when encountering an opponent whose style is largely unknown, adopting a strategy close to GTO is prudent; it serves as a defensive action that prepares for the worst by minimizing the potential for your approach to be exploited. GTO is predicated on the notion of an opponent who either plays optimally or is adept at leveraging strategy weaknesses. However, **the ideal of a perfectly rational**

opponent, always adhering to GTO, is more theoretical than practical. Real-world players often deviates from this ideal, presenting opportunities for the observant player to capitalize on. Rarely does one encounter adversaries who maintain a perfectly balanced game, hence they exhibit vulnerabilities. When armed with sufficient information about an opponent's tendencies, history, and strategy, and considering the dynamics of the game's earlier stages, playing exploitatively can be advantageous. Yet, caution is advised as deviating significantly from GTO could leave one's own strategy open to exploitation.

If we imagine two rational players beginning with different, exploitable strategies, a theoretical journey would see them adapting to and countering each other's moves over time. This iterative process would lead their strategies to evolve, asymptotically approaching GTO. To wrap up, while strict adherence to GTO might not guarantee maximal profit against all types of players, it serves as a robust baseline. It is a safe harbor against the unpredictability of various opponents, ensuring that one's strategy remains as unassailable as possible. We are humans, we make errors by nature. **No real human being can be as perfect as it is needed to follow the theory behind GTO strategies**, this is why nowadays we have computers that do that for us.

Chapter 4

Inside the Game

4.1 How to Play with a Criterion

In this last chapter we will try to give to the reader a sort of "guide" to follow in order to understand what should they actually do while playing.
Before we delve into it, it is important to remark that what we will go through is not the best way to approach the game since as we understood during the whole discussion of this essay, there is no real best way given the fact that being able to perfectly apply the theory to the practice can be considered impossible. Our goal is to provide the basic knowledge in order to play poker and not be played by it.

4.2 Pre-Flop Action

Pre-flop it is probably the most important stage of the game because it is the stage in which we choose which hand to play and how to play it.
It is fundamental to understand that the way in which we act during this phase will give our opponent the most information about our hand, since we solely rely on our private cards to bet. This is crucial in order to estimates the opponents

ranges allowing us to adapt our later stage game on such an estimate. Anyway we should always be careful, since good players will balance their strategy as well, trying to fool us by playing a portion of their range we do not expect them to do.

The general idea to approach such a stage of the game follows three cardinal strategies:

(i) Which hands to play.

(ii) Whether to raise or call with those hands.

(iii) What amount to raise with hands that will raise.

Imagine we approach the game with a straightforward tactic: we raise with any hand that's good enough to play and fold the rest. The key decision then is to figure out which hands are strong enough to merit this raise, considering they're part of the range of hands that we always play this way. Alternatively, we might choose to always call (or 'limp'[1]) with the hands we want to play, folding the ones that don't make the cut. In this case, our decision comes down to whether each hand is worth playing, based on how well it performs as part of a set of hands that always calls pre-flop.
There's also a third strategy we might adopt, which is mixing it up between calling and raising. If we go down this path, we must carefully manage the composition of our hands when we raise and when we call. This is crucial to keep our strategy from becoming predictable and thus exploitable.
The most important aspect to understand when first approaching the game is the one of avoiding calling from early position such as UTG or UTG+1. In such positions you should fold the vast majority of your range since the likelihood of our opponents to have stronger hands than us is very high. If you want to play

[1]the term "limp" is used to describe the action of calling the blind, most of the time in order to get a cheap flop.

hands in these positions you have to raise or at least call being prepared for an incoming raise from late positions players. This is the most common mistake that new players commit, especially because they don't realize that most of the time if you play hands from early position you could experiment one of the toughest situation that arise in the game which is the one of being dominated.

Definition 4.1. *In poker being dominated refers to the situation in which a player has a hand which contains one card equal to the one of the opponent and the other one weacker, which results in a very bad situation because in order to win the hand the "dominated" player has to hit the weacker card.*

Furthermore, in the event that actually both players hit the strong card the dominated player will likely loose a big chunk of money especially if they both make a good hand. Let's consider the following example in order to understand this tricky concept:

Example 4.1. *Assume Alice has A♠3♣, a reasonable hand to play, and she decides to call from UTG. Bob is on the button and has A♣Q♣. It is evident that Bob here has a clear advantage. Suppose that both players arrive to the river with the board that shows: A♢10♢5♣7♣2♠, Alice considering such a board could be confident that she has the best hand given her top pair, but she is oblivious that in reality she is beaten. If bob elects to go all-in on the river and we assume Alice is a novice player most of the time she will call the bet (given her top-pair) resulting in a huge loss.*

The reason why we don't want to play marginal hands from the beginning is exactly this. We want to avoid the possibility to end up in a tough spot where we have a mediocre hand that we feel like it is the best even if it is very far from so. The example explains the so-called "kicker" problem.[2]

[2]In poker terminology, a "kicker" is the unpaired card that accompanies your pair, playing a

4.2.1 The Power of Position

The next major concept to understand is that of position. Play from the front is perhaps the most relevant example of the power of position. Nine other (presumed near-optimal) players will act after us, and each of them has the possibility of picking up a strong hand. Entering the pot with weak or marginal hands will likely result in being raised or re-raised when one of those players has a strong hand. Therefore, we must choose wisely the hands that we select to play. As we move around toward the button, there are fewer and fewer random hands behind us, and so we can raise with weaker and weaker hands. On the button, then, we can play some wide variety of hands that rate to have positive expectation against the blinds, including the value of position on later streets. Still, though, we must take care to play only loosely enough to add hands that enhance the value of our distribution. It is crucial to understand such a concept also because playing in position gives you a big advantage especially post flop since you can apply pressure on the opponents in front of you and even better you don't have to worry about the possibility of facing big bets after you (of course if you bet first the other opponents could raise you but in general it is more common to face a raise from whoever has the position).

4.2.2 Understanding Raising Amounts

In poker, deciding how much to raise before the flop, especially when you are the first to raise, involves a strategic balance. This decision mainly revolves around the interaction with the player in the big blind position, who is often more likely

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crucial role in determining the strength of your hand, especially in scenarios where two or more players have the same pair. The kicker serves as a tiebreaker, with a higher kicker giving your hand a competitive edge.

to match the raise due to the initial bet they are already committed to.
There are two key situations to consider:

(i) When other players have strong hands, they might raise again or at least match our bet. In this case, it's wise to keep our initial raise smaller to reduce potential losses on hands we might end up folding if faced with a re-raise.

(ii) When it's likely just us against the big blind, we aim for a raise amount that applies pressure but still offers good value. Raising too much might scare off the big blind from calling with hands that actually stand a chance, while raising too little makes it an easy decision for them to call.

Our strategy involves adjusting our raise amount based on our position at the table and not on the strength of our hand to avoid giving away too much information. As we get closer to the button (a later position), we tend to raise more, leveraging the reduced risk of facing strong hands from the remaining players. Typically good players will have a standard raising amount based on their position, for example from early position many players elect to raise to $2.1BB$ [3] where instead from late position they choose a slightly larger sizing, for example $3.2BB$. However the size of the raise depends on the effective stack size we have. The deeper we are the more we want to raise (especially to apply pressure to shorter stacks).
The most common mistake to avoid is to choose the raise amount based on the hand we have, which is exactly what many players tend to do at the beginning, resulting in an extremely unbalanced strategy. Those players tend to raise big just with the top of their range which is composed by very few hands such as AA,KK,QQ and AKs,AQs, making them easily exploitable by a good player who observes carefully the action at the table. In reality AA and KK should merit a specific discussion.

[3] BB is the notation used to indicate big blinds when treating pot size.

Most of the time novice players do not want to scare others from entering the pot so they tend to slaw play by calling and then maybe re-raising (of course this could be a good strategy by it is again very easily exploitable). On the other hand, AK and AQ are usually played very aggressively since they are not *made hand* which means that most of the time in order to win the hand you will need to hit the flop, causing players to be uncomfortable when they do not. They are tricky hands that need a lot of practice to be played correctly. In short, our raising strategy before the flop aims to carefully manage the investment in the pot, considering both the potential challenge from strong hands and the opportunity to win against the big blind, all while keeping our play patterns concealed.

4.3 Play on the Flop

Now, the pre-flop action is over and the flop is dealt. The real challenge starts. We need to be able to analyze those three cards in order to understand the likelihood of our hand to be the best.

When evaluating how the flop influences the rest of the game, we primarily consider two key factors. First, we look at who the **previous aggressor** was. Typically, in a game where players are closely matched in skill, this player is expected to have a range of hands that, on average, are stronger than those of their opponent. However, this range also includes some weaker hands to maintain a strategic balance. The second factor is the **"texture"** of the flop, which broadly refers to how the cards on the flop interact with the potential hands each player might have. This includes evaluating the chances that the flop has given either player potential hands with draws and assessing the potential strength of those draws.

For example a board like J♡10♡9♠ contains a very large number of draws or good hands that players will likely have (sets, two pairs, flush draws). We call this type of flops **dynamic**. By contrast, a board like K♡7♣2♠ contains very few draws or combinations of good hands. This type of board is called instead **static**.

4.3.1 Original Raiser vs Big Blind

Consider the most classic situation where an early position raiser confronts a caller in the big blind. The blind will hold a significantly weaker distribution on the flop than the raiser, because he will have called with a much looser set of hands given his reduced price for entering the pot.

Assuming the strength of the early raiser's hand range, they are often in a position where betting with all of their hands when it is checked to them is a strong move. In response, the player in the big blind should ideally check all their hands. The idea is that if the early position raiser is likely to bet with any hand, the big blind's best response is to check, recognizing the raiser's advantage.

This strategy essentially respects the initial aggressor's strong position. There might be concerns that the early raiser could exploit this by occasionally checking instead of betting. However, because their hand range is so strong, they generally stand to gain more by betting to either increase the pot's value with strong hands or win the pot directly, rather than trying to be tricky by checking. This scenario is somewhat similar to one where a player doesn't bluff much, leading their opponent to fold more often since the risk of being bluffed is lower.

Anyway there are some tricky aspects to consider, one of the most interesting is perhaps the one of low cards flop such as 6♡5♡2♠.

In this scenario the original raiser needs to be very careful since this kind of flops deeply favour the big blind's range given that he is the one who could have a very strong hand. The original raiser will likely have two over cards such as

AKs,AQs,KQs and all the overpairs (77,88+)[4]. However he will never have nutted hands [5] in this situation (unless he raises 66 or 34, which is almost impossible assuming he his a nearly-optimal player), meaning that he should consider a check back instead of the standard auto-bet because he will benefit from a free card. Moreover he does not take the risk of being raised and forced to fold his hand in the case he has not a made hand such as the strong overpairs. To wrap it up, while there are general strategies for playing against the big blind, it's crucial to adjust the play based on the specific flop. Understanding which flops are good for the raiser's range and which are better for the big blind's range is key to making smart decisions.

4.3.2 Play on the Turn and River

Consequently the concepts explained for the play on the flop can be applied to both the turn and the river since basically nothing changes. The only thing to take in consideration is the realizations of draws which most of the time happen in these stages. It is very important to keep in consideration this possibility especially when the flop is a dynamic one. Most of the players will chase their draw until the river given their implied odds. It is up to our ability to understand when effectively such draws come in because it could be that if a "scary" card comes on the river our opponent might decide to bluff with a dead hand, as we considered in example 3.3 where Bob decides to bluff with a dead if a heart comes out on the river completing the flush. Anyway we will not discuss in detail how to approach this type of situations because we will need to take in considerations all the psychological

[4]AKs stands for AK suited, meaning that both the two cards are of the same suite. This is perceived as a great thing since the probability of hitting a flush is higher with respect to AK with two different suite

[5]In poker terminology the term "nuts" refers to the best possible hand to have in that stage of the game, "nutted hands" indicates those hands that are almost the nuts.

aspect that governs the player decision while actually playing the hand.

4.4 Final Considerations

During the discussion of this chapter we have thoroughly examined the strategies involved in pre-flop play, offering a detailed exploration of the decisions that influence the later stages of a poker game. We have analyzed the critical choices of hand selection, positional awareness, and betting strategies, highlighting how these early decisions can set the tone for subsequent stages of the game.
The discussion emphasized that successful poker play begins well before the flop is dealt. Pre-flop strategic planning is essential to play with a criterion, as it sets up the possibilities for advantageous post-flop play. Through a series of practical examples, we demonstrated how varying pre-flop strategies could be adapted based on player position and the dynamic nature of the game, reinforcing the necessity of a serious reflection on how to approach poker.
Additionally, this chapter has shown that the tactical dimension of poker extends far beyond simple card strength; it involves a studied mixture of psychological insights, mathematical analysis, and strategic adjusments. Players must continually adapt their strategies based on the evolving game context and their opponents' responses.
In summary, the goal of this chapter was to provide novice player a easy way to approach the game having all the background knowledge needed to play with a criterion, in order to rely less on the pure "luck" aspects like many players do at the beginning.

Conclusions

The discussion of this thesis has made us embarking on a journey which has traversed a landscape where mathematics, psychology, and strategy converge, offering a complete view of a game that goes far beyond what we would have expected. Starting from the foundational theories underpinning game mechanics to the intricate strategies of play, this work has tried to discover the complexity of poker, illustrating it as a game that transcends mere chance and ventures into the realm of sophisticated strategic warfare.

Our journey actually started with an in-depth examination of the mathematical and theoretical foundations of poker. By analyzing the key mathematical aspects, such as game states and information sets, we demonstrate how poker aligns with, and often exemplifies, the principles of computational game theory. The discussions on Nash equilibrium and game theory optimal (GTO) strategies explained how players might navigate the game's inherent complexity and unpredictability with a precise set of skills required. From the mathematical one to the psychological.

The subsequent chapters build upon this theoretical foundation by addressing practical strategic dimensions. We explored the fundamental aspects of the game, from understanding pot odds and player equity to the strategic implications of various play stages, including pre-flop actions and decisions made on the flop, turn, and river. These sections not only detailed the strategic considerations a player must consider but also highlighted the significant impact of psychological factors and opponent behavior on decision-making processes.

As the thesis progressed, the focus shifted towards more advanced strategies, particularly the application of GTO play. This discussion underscored the importance of balancing exploitative play with strategies that safeguard against being exploited. Through illustrative examples and scenario analyses, we demonstrated how players could achieve a state of equilibrium where no single strategy consistently outperforms another, thus embracing the complexity and depth of poker strategy.

The practical chapters aimed to equip players with the tools necessary to apply theoretical knowledge in real game settings. By analyzing specific hands and situations, these sections provided a blueprint for navigating the multifaceted interactions and decisions that define well-played poker. Furthermore, these discussions emphasized the importance of adaptability, suggesting that the most successful players are those who can alter their strategies in response to dynamic game conditions and opponent behaviors.

In conclusion, the study of poker as presented in this thesis not only enriches our understanding of a popular cultural phenomenon but also contributes to the broader discourse on strategic decision-making and economic theory.

The idea behind the development of this thesis was born from the pure passion of an incredible game, which always leaves whoever plays it with an immensity of thoughts and possible strategies to adopt in the future. Playing poker involves a continuous personal evolution that makes the players question themselves about not only the way they approached the game but even general challenges in their lives (as a matter of fact, the more a player is experiencing a good period in his life, the better he will play). It is like embarking on a journey in which it is you and your thoughts against every single agent that influences the game, from the other players to yourself and even the cards. In order to succeed, you have to beat all of them and especially you need to exploit every single tool at your disposal, from the theoretical knowledge to the psychological one. Most importantly, you need to beat yourself by always pushing beyond your limits and never being afraid of making the wrong move. Poker is a mental battle in which your first enemy is yourself. Once you manage to not let yourself be influenced by all these factors, you are ready to play the actual game.

I hope this work will leave the reader with an idea of how to approach the game with all the pieces of the puzzle needed to fully enjoy it. And especially, I hope that whoever thought that poker was a mere game of luck may change their mind.

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