Department of Economics and Finance

Chair of Gambling: Probability and Decision

Investing or Gambling:
Where to Go All-In

Supervisor:

Prof.

Hlafo Alfie Mimun

Candidate:

Michela Napolitano

269251

Academic Year 2023/2024

To my parents, for standying by me throughtout my whole university career.

To my brother Luca, for always encouraging to work hard and give my best.

To my grandmother, for always believing in me.

To my best friend Federica, for supporting me everytime I needed her advice.

To all my closest friends and colleagues, for making me who I am and without whom this journey would not have been that special.

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Introduction

The ultimate challenge for a gambler is to detect betting opportunities that are able to generate positive returns. After spotting some attractive bargains, the gambler must settle how much of their capital they are willing to bet on each of them, as a rational person is expected to choose those that maximize their expected return.

However, Daniel Bernoulli demonstrated that, paradoxically, a gambler should bet no matter the cost and, instead of maximizing returns, economists and probabilists have come to the conclusion that the gambler should maximize their utility function.[1] As stated by Daniel Bernoulli, a gambler should maximize the utility function $U(x) = \log x$ where x represents the investment return.[2]

Subsequently, John Larry Kelly Jr. developed Bernoulli's ideas and found some remarkable properties of the utility function $U(x) = \log x$[5]. His findings can be extended to gambling scenarios in order to help a gambler determine which bets are the most profitable ones and how much money they should put at stake.
More precisely, he came up with the fixed portion of capital that the gambler should wage to maximize their utility function and achieve the highest level of satisfaction given certain conditions, known as the Kelly Criterion.[6]
The main task for this research consists of investigating the core differences and

[1] A tool employed by economists as a measure of relative satisfaction and often sketched using factors like consumption of different goods and services, wealth and spending of leisure time.

similarities between gambling and investing by focusing on three main aspects: loss mitigation, the time factor and collection of information. As a matter of fact, the presence of the house advantage causes an investor's expected return to be usually better than a gambler's over the long run, as the house advantage over the gambler grows as they keep playing. Besides, investors have access to several loss-mitigation strategies which are not available in case we bet on any gambling activity and are provided with much more information compared to gamblers.
This analysis is conducted with the purpose of assessing whether the mathematical tools used in the world of gambling can be proven effective also in maximizing investors' profits. In case this were possible, we would then evaluate which of the two activities is more profitable in the long run by using the very same tools.
In order to accomplish this, we investigate many gambling settings and present theorems and criteria for each of them, demonstrating also why those specific tools are effective in offering the best solution to that particular problem.
The procedure described above will offer us a full picture of all the instruments at our disposal and allow us to assess whether an investment position we want to inspect matches the conditions needed to apply the mathematical theorems considered in our analysis.

This thesis is organized in four parts: Chapter 1 defines gambling, focusing on why it differs from speculating and investing and explaining how to recognise whether someone exhibits gambling tendencies. In addition, it covers the concept of casino mentality and offers some solutions to fight this dangerous mindset.
Chapter 2 offers a dissertation on gambling theory, with the aim of introducing the reader to key concepts and definitions that settle the foundation of some useful theorems and criteria. Furthermore, it shows how some of them can be applied not exclusively to gambling, but also in some investment settings, by providing numerous examples; more precisely, it illustrates how the Kelly Criterion can be

useful also in investment theory, as it is crucial for investors to choose carefully what fraction of their wealth should be allocated to stock investing.

Chapter 3 deals with the notion of house advantage, which can be defined as a numerical index of the unfavorability of a wager; hence, the smaller the house advantage, the more favorable the game is to the bettor and the better are their chances of placing a winning wager. Then, we explain the importance and application of the house advantage to betting games and develop the concepts of volatility of a wager and expected loss per standard deviation, which are crucial to assess the convenience and riskiness of wagers in games dealing with theoretical probabilities like the roulette.

Lastly, in Conclusions are displayed the final remarks and draws the conclusion of this thesis by analyzing the most evident drawbacks of gambling and explaining the reasons why it would be wiser to rely on investment opportunities instead.

Chapter 1

Gambling, Speculating and Investing

Have you ever wondered why some people say that stock investing is just like gambling at a casino or make comments of that sort?
As a matter of fact, both activities involve risk and choice (to be more specific, risking capital hoping for future profit), but equity investing can last a lifetime, while gambling is typically a short-term activity that comes to an end as soon as the outcome of the bet is known.
Furthermore, gambling usually implies bearing a negative expected return over the long run, also due to the presence of the house advantage that features the majority of bets, whereas investing in the stock market often ensures a long-run positive expected return on average.

1.1 Definition of Gambling

Gambling can be defined as “staking something on a contingency”, betting money on some event having an uncertain (and potentially negative) outcome.

Also known as betting or wagering, it consists of risking money on an event having an uncertain outcome and strongly dependent on chance. Like investors, gamblers must select thoughtfully the amount of funds they want to put at stake; some of them use **pot odds**[1] to determine their risk capital versus their risk versus reward when playing card games: in case of favorable odds, the player is more likely to call the bet.

Most professional gamblers are very skilled at risk management: they investigate player/team history, or a horse's lineage and track record. Looking for an edge, card players usually look for clues from the other players at the table: some of them can recall their opponents' wagers from many hands back and even study their betting patterns, hoping to catch useful information in order to build their **basic strategy**. Griffin defines the basic strategy applied to blackjack as "*the strategy maximizing the player's average gain playing one hand against a whole deck of cards*". Thus, with a stated number of decks and fixed array of rules there can be only one "basic strategy", although there may be a bunch of (but slightly inaccurate) versions of it. If we supply instructions on how to play the second and subsequent cards of a split based on those used earlier, it is likely that nobody, including experts, can discover what the basic strategy is.[4]

In casino gambling the bettor is challenging the house, whereas in sports betting and lotteries bettors are competing against each other since the number of players plays a part in estimating the odds. In horse racing, for instance, every bet placed by a gambler is actually a wager against other bettors, as the odds on each horse are defined by the sum of money put on that specific horse and undergo constant alterations up until the start of the race.[2] However, when trading gets involved, gambling takes on a more complex dynamic because many traders are gambling

[1] The ratio between the current size of the pot and the cost of a contemplated call.

[2] Stephan A. Abraham, "Going All-in: Investing vs. Gambling", Investopedia.com, October 21th 2023, https://www.investopedia.com/articles/basics/09/compare-investing-gambling.asp.

without even realizing it, or by following an extremely dichotomous reasoning. We will now explore the sneaking ways in which gambling crawls into trading activities, analysing the possible stimuli that can drive someone to trade (and possibly gamble) in the first place.

1.2 Hidden Gambling Tendencies

People who are embarking or have already embarked on gambling tendencies usually display two common traits during trading:

- if someone trades for excitement or social proofing purposes, rather than in a systematic way, they are likely gambling;
- if someone trades with the mere goal of winning, they are likely trading in a gambling style; traders manifesting a "*must-win*" attitude will often have a hard time identifying a losing trade and leave their position.

People who suppose they do not display gambling tendencies will likely struggle to admit having them if it turns out they are indeed acting by pursuing gambling impulses. Apart from when actually trading, some demeanors are perceivable even before trading takes place and these same motivators keep impacting traders even when they get more experienced and become regular market participants, causing them to make very unfortunate choices.

1.2.1 Social Proofing

Some people may not even be interested in trading or investing in financial markets, but social pressure pushes them to trade or invest anyway, especially if people close to them are discussing investments. In this situation, people feel pressure to conform to their social circle and hence they invest in order not to

disrespect or overlook others' beliefs or even feel left out.
Trading to relieve social forces is not gambling by itself if people are actually aware of their own actions. Nevertheless, entering into a financial transaction without a full understanding is gambling, as ignorance prevents those people from exerting control over the profitability of their choices.[3]
The availability of so many variables in the market, matched with misinformation diffused among investors or traders, creates a gambling scenario; since knowledge enabling people to overcome the odds of losing has been developed, gambling has been taking part in every market transaction.

1.2.2 Trading for Excitement

Even a losing trade can shake a sense of might or fulfillment, particularly if related to social proofing: if everyone in your social circle is losing money, losing on a trade yourself will allow you to better fit in the conversation and share your own experience. When someone trades for either excitement or social proofing reasons, they are most likely trading in a gambling style as this behavior is often driven by psychological factors rather than rational decision-making. As a matter of fact, this excitement is usually expected to draw you away from acting in a systematic and methodical way, which is crucial in any odds-based scenario.
Individuals may trade for excitement to experience the rush linked to market volatility, the anticipation of gains, or the activity of placing trades. For this reasons, it can be likened to gambling, where the process itself provides a form of entertainment. This behavior can lead to impulsive decisions and disregarding risk management principles, which can result in significant financial losses, as decisions are often based on emotions rather than careful analysis.

[3] Cory Mitchell, "Are You Investing or Gambling", Investopedia.com, October 9th 2023, https://www.investopedia.com/articles/basics/10/investing-or-gambling.asp.

Easily accessible trading platforms and apps have made it easier for any user to engage in trading: features like instant notifications and social trading can amplify the excitement factor and encourage more frequent trading. Also online communities, forums, and social media platforms can contribute to the excitement owing the fact that sharing trading experiences and following trends can create a sense of fellowship and fuel the excitement even further.

1.2.3 Trading to Win

Winning appears to be the most obvious reason to trade: after all, what is the point in trading if you have no chance of winning? Nevertheless, there is a hidden flaw lying in this view: while earning money is the craved ultimate purpose, trading to win can truly lead you further away from earning a profit. If winning is the main motivator, an analogous scenario is likely to happen:

Jane buys a stock she deems oversold. The stock keeps on falling, placing Jane in a negative position but, instead of figuring out the stock is not simply oversold and deciding to sell, Jane continues to hold, hoping the stock price will rise back so she can earn (or at least break even) on the trade. In this case the focus on winning has prevented her from getting out of a bad position just because she did not want to admit she lost.

Holding losing positions after original entry conditions have transformed or gone negative points out that the trader is now gambling and no longer adopting safe trading strategies.[4] On the other hand, good traders admit when they make a mistake and try to limit damages: not having to always win and accepting losses when implied is what allows them to be successful over many trades.

[4] Cory Mitchell, "Are You Investing or Gambling", Investopedia.com, October 9th 2023, https://www.investopedia.com/articles/basics/10/investing-or-gambling.asp.

1.3 Speculation vs. Gambling

Speculation and gambling are operations used to increase wealth while facing risk or uncertainty; they both involve engaging money to high-risk events that may or may not pay off but they differ consistently for what concerns expected results. Speculation entails some kind of positive expected return, even though the final result may be a loss. On the other hand, the expected return for a gambler is negative despite the fact that someone may get lucky and win. In the world of investing, gambling refers to wagering money in an event that has an uncertain outcome hoping to win more money, while speculation involves taking a calculated risk with an uncertain outcome.

Although we could find some shallow analogies linking the two concepts, a rigorous definition of both terms unveils the key differences between the two. A standard dictionary defines speculation as *"engagement in a risky business transaction on the chance of quick or considerable profit"*. The same dictionary explains gambling as: *"the activity of playing a game of chance for stakes or betting on an uncertain outcome. To stake or risk money, or anything of value, by taking a chance or acting recklessly"*.[5]

Speculation implies calculating risk and carrying out research before entering a financial transaction. Speculators buy or sell assets hoping for a potential gain bigger than the amount they risk; they know that, in theory, the greater risk they take, the higher their potential gain and they are also aware of the fact that they may lose more than their potential gain.[6]

Even though speculation is risky, it often yields a positive expected return which

[5] Both definitions of speculation and gambling are taken by the American Heritage Dictionary.

[6] Steven Nickolas, "Speculation vs. Gambling: What's the Difference?", Investopedia.com, December 11th 2023, https://www.investopedia.com/ask/answers/042715/what-difference-between-speculation-and-gambling.asp.

however may never manifest; in contrast, gambling always features a negative expected return, as the house still has an advantage.
When you gamble, the probability of losing what you have bet is usually higher than the probability of winning more than that same amount and, compared to speculation, gambling has a higher risk of losing your investment.

1.4 Is Investing Basically Gambling?

Investing consists of committing capital to an asset expecting a return in the form of income or price appreciation, which can be considered the core premise of investing. Risk and return are directly proportional in investing, as low risk typically implies low expected returns while higher returns often imply higher risk. However, risk and return expectations change widely also within the same class of assets and spreading your capital across different assets will probably help impair potential losses.[7]

On the other hand, gambling is betting money on an uncertain outcome which will likely turn out to be negative. Furthermore, a gambler owns nothing, while an investor owns a share of a company as some of them actually refund investors for their ownership with stock dividends.

1.4.1 Is Investing in the Stock Market the same as Casino Betting?

Investing in the stock market can be considered playing in a casino in case you purchase stocks randomly or solely based on rumors. However, if you manage to build a well-diversified portfolio or invest passively in a broad stock market index, you will face a positive expected return and foster your wealth over time. On the other hand, experts state that *"once you have entered a casino you are already* *down money".*

Trading can be exciting, stimulating and can urge reward pathways in the brain: earning a profit or simply getting pumped about a potential one cause the brain to unload feel-good neurochemicals like dopamine and serotonin; because of this, people can develop an addiction, just like with gambling or using drugs.[7] As any deep addiction, trading addiction can jeopardise your job and personal relationships, as well as your wealth.

1.5 The Casino Mentality

Data show that most of the traders dealing at the shallow end of the market pool will sooner or later fail in their quest as between 70% and 90% of them end up losing their money over time[8]; some of them hand over the reins to a money manager, others simply give up and look for alternative ways to make profits. Most of them never had the chance to get a positive return because they entered the market with a "casino mentality" that brought them to failure.

This section unfolds what this casino mentality is exactly and how it impairs an investor's quest for profitability. Then, we attempt to convene whether this flawed approach is bound to beginners only or also experienced traders get caught up in this conduct. Finally, we illustrate the most valid tool to overcome this mentality and how to substitute it with a disciplined approach fostering gainful speculation.

[7] Alan Farley, "The Casino Mentality in Trading", Investopedia.com, November 2nd 2023, https://www.investopedia.com/articles/investing/070815/casino-mentality-trading.asp.

[8] Oddmund Groette, "What Percentage Of Traders Fail? (How Many Lose Money? Statistics)", Quantified Strategies.com, January 6th 2024, https://www.quantifiedstrategies.com/what-percentage-of-traders-fail/.

1.5.1 Understanding the Casino Mentality

Traders who are not familiar with basic trading strategies and are unaware of the nature of risk compare their involvement in the financial markets to a trip to the casino, hoping the stack of cash they decided to invest can be traded in for a bigger one when they leave, hypnotized by the greed that features all the so called *get-rich-quick* schemes.

Just like a slot machine, minor and regular compensation increase their motivation to wage more money, disregarding whether or not those are fit to running market conditions and possibilities in play; this greed rarely attains a big win and mainly causes consistent losses over time, leaving the floor to failure and a terminal exit from trading. The lack of a definable verge seals investors' fate, just like gamblers betting out of eagerness usually miss to learn the odds and suitable strategies for each game to cut down or wipe out the house advantage. In the meantime, both gamblers and investors get secondary reinforcement for their detrimental conduct because their bodies release adrenaline and endorphins whenever they play, regardless of the outcome.[9]

The casino mentality rises the biggest capital losses when dealing with binary events, like earnings reports or economic releases causing sudden higher or lower security prices; in these situations, clever traders step down or hedge positions at these inflection points since they do not know the aftermath and guessing is not a smart scheme.[10] Indeed, Liz Ann Sonders, managing director and chief investment strategist of Charles Schwab, once said:

"...it's not what you know that matters, meaning about the future. When's the

[9] American Psychiatric Association, "What Is Gambling Disorder?", https://www.psychiatry.org/patients-families/gambling-disorder/what-is-gambling-disorder.

[10] Alan Farley, "The Casino Mentality in Trading", Investopedia.com, November 2nd 2023, https://www.investopedia.com/articles/investing/070815/casino-mentality-trading.asp.

next move up or down the market? It's what you do that matters. Investors often think the key to success is knowing what's going to happen and then positioning accordingly in advance, and that's just gambling on moments in time."

1.5.2 Beginner's Flaw or Lifetime Affliction?

The casino mentality mainly affects novices, as it is generally caused by a misunderstanding of financial markets and their functioning; sooner or later, many investors will learn from their mistakes, using previous losses as a wake-up call and taking the subject matter more seriously.
While beginners sticking with the casino mentality wash out quickly, experienced traders can keep on adopting this destructive mindset for years; even though it does not overcome their typical strategies, it can show up in case discipline gets overpowered by greed. Nevertheless, if used in small amounts, it may introduce some fun into trading as long as position size is kept down.

1.5.3 How to Overcome the Casino Mentality

Education undoubtedly grants the most powerful shield to use as a defense against the casino mentality: you should consult didactic materials on investing, trading and the evolution of financial markets; then analyse materials dedicated to your area of interest, including both fundamental and technical analysis, before starting your investing career.

However, many beginners avoid the educational journey because they are fine with chasing the flattery of easy money, hoping for big payouts without any effort from their side. Logistically, this benefits the most serious-minded participants, who will deal with a large number of clueless investors that increase their potential reward at key market turning points.
Overcoming the casino mentality implies the adoption of a disciplined and

strategic approach. To shift from speculative trading to responsible investing, traders should define specific, measurable, achievable and relevant goals for their investments. They should also focus more on long-term objectives (retirement, buying a house or funding education) rather than short-term gains.
This can be achieved with the creation of a diversified portfolio matching their risk tolerance, time horizon and financial goals, combined with an asset allocation strategy to stick to, making adjustments only when necessary based on significant life changes or financial goals. That leads to the need of implementing some risk management techniques, such as the use of stop-loss orders to limit potential losses and regularly review and balance the portfolio to maintain the desired level of risk. It would also be very wise for the most unexperienced investors to involve a financial advisor or investment professional who can provide objective guidance and help them develop and stick to a long-term investment plan.

In the next chapter, we will introduce several settings, related both to the investing and gambling worlds, and solve them by showing that instruments usually used in gambling scenarios can be useful to assess also investment opportunities. If implemented correctly, these tools can represent an effective solution to outplay the casino mentality.

Chapter 2

Martingales

In this chapter we will deal with stochastic processes, which are defined as random quantities that evolve when a parameter (time, for example) changes. In particular we will focus on martingales, that is to say stochastic processes associated to fair games in the context of gambling.

2.1 Stochastic Processes and Filtrations

Imagine playing the same game at the casino many times and being interested in counting the total quantity of money that you win during the rounds. If we denote by X_i the quantity of money won in the i-th round, then the total quantity of money won in the first n rounds is

$$S_n = \sum_{i=1}^{n} X_i$$

The total quantity of money won during the rounds is expressed by the sequence of random variables $S_1, S_2, S_3, ..., S_n$ and the sequence of random variables $\{S_n\}_{n\in\mathbb{N}>0}$ is a stochastic process since it describes the evolution of the total amount of money earned as the number of rounds n varies. To compute S_n, we need to know the value of $X_1, ..., X_n$ and the sequence of random variables $\{X_n\}_{n\in\mathbb{N}>0}$ represents the information that we accumulate during the rounds, hence we can consider such a sequence a filtration. A **stochastic process** is a sequence of random variables that describes the evolution of a random quantity during the variation of a parameter, while a **filtration** is a sequence of random variables that represents the information we collect while studying the evolution of a stochastic process. In the next section we will consider a particular type of stochastic process called martingale.

2.2 Martingales and Super/Sub martingales

Suppose to toss a coin many times and each time you win 1 euro if the result is head and you lose 1 euro if the result is tail. Imagine having an initial fixed capital $S_0 > 0$ and that $\mathbb{P}(\text{head}) = p \in [0, 1]$. Denote by X_i the quantity of money won in the i-th round:

$$X_i = \begin{cases} 1, & \text{with probability } p\,, \\ -1, & \text{with probability } 1 - p \end{cases} \tag{2.1}$$

where $\{X_n\}_{n\in\mathbb{N}>0}$ is a family of i.i.d. random variables and by S_n the capital after n rounds where

$$S_n = S_0 + \sum_{i=1}^{n} X_i.$$

Suppose to have concluded the n-th round and that we wonder about our average capital at round $n + 1$. Since we have already played n rounds, we know exactly the values of $X_1, ..., X_n$, so we want to compute

$$\mathbb{E}[S_{n+1}|X_1, ..., X_n]\,.$$

Notice that, since we know the values of $X_1, ..., X_n$, we also know $S_n = S_0 + \sum_{i=1}^{n} X_i$.

Hence, since

$$S_{n+1} = S_0 + \sum_{i=1}^{n+1} X_i = S_0 + \sum_{i=1}^{n} X_i + X_{n+1} \,,$$

the only unknown in S_{n+1} is X_{n+1}. Therefore

$$\mathbb{E}[S_{n+1}|X_1, ..., X_n] = \mathbb{E}[S_n + X_{n+1}|X_1, ..., X_n] =$$
$$= \mathbb{E}[S_n|X_1, ..., X_n] + \mathbb{E}[X_{n+1}|X_1, ..., X_n] \,.$$

Since S_n is known when knowing $X_1, ..., X_n$, then $\mathbb{E}[S_n|X_1, ..., X_n] = S_n$. Moreover, given that $\{X_n\}_{n\in\mathbb{N}>0}$ are independent, we have $\mathbb{E}[X_{n+1}|X_1, ..., X_n] = \mathbb{E}[X_{n+1}]$ and the previous equation becomes

$$\mathbb{E}[S_{n+1}|X_1, ..., X_n] = \mathbb{E}[S_n|X_1, ..., X_n] + \mathbb{E}[X_{n+1}|X_1, ..., X_n] =$$
$$= S_n + \mathbb{E}[X_{n+1}] = S_n + p - (1-p) = S_n + 2p - 1 \,.$$

So we have:

- if $p = \frac{1}{2}$, the coin is fair and the game is fair since $\mathbb{E}[X_1] = 0$). Furthermore

  $$\mathbb{E}[S_{n+1}|X_1, ..., X_n] = S_n \,,$$

  meaning that the average capital we will have in the next round is equal to what we have now. This is the main property of a *martingale*.

- if $p < \frac{1}{2}$, the coin is not fair and the game is subfair since $\mathbb{E}[X_1] < 0$). Furthermore

  $$\mathbb{E}[S_{n+1}|X_1, ..., X_n] < S_n \,,$$

  meaning that the average capital we will have in the next round is less than what we have now. This is the main property of a *supermartingale*.

- if $p > \frac{1}{2}$, the coin is not fair and the game is superfair since $E[X_1] > 0$). Furthermore

  $$\mathbb{E}[S_{n+1}|X_1, ..., X_n] > S_n \,,$$

meaning that the average capital we will have in the next round is more than what we have now. This is the main property of a *submartingale*.

Note that $\{X_n\}_{n \in \mathbb{N}>0}$ represents a filtration, that is the information we accumulate during the rounds, in order to evaluate the stochastic process $\{S_n\}_{n \in \mathbb{N}>0}$.

Definition 2.1. *Given two sequences of random variables* $\{S_n\}_{n \in \mathbb{N}>0}$ *and* $\{X_n\}_{n \in \mathbb{N}>0}$ *,* $\{S_n\}_{n \in \mathbb{N}>0}$ *is a* ***martingale*** *with respect to the filtration* $\{X_n\}_{n \in \mathbb{N}>0}$ *if the following three conditions are met:*

(i) $\mathbb{E}[|S_n|] < \infty$ *for any fixed* $n \in \mathbb{N}>0$ *;*

(ii) $\{S_n\}_{n \in \mathbb{N}>0}$ *is adapted to the filtration* $\{X_n\}_{n \in \mathbb{N}>0}$ *, that is, known the value of* $X_1, ..., X_n$ *, we know also the value of* S_n*;*

(iii) $\mathbb{E}[S_{n+1}|X_1, ..., X_n] = S_n$.

Definition 2.2. *Given two sequences of random variables* $\{S_n\}_{n \in \mathbb{N}>0}$ *and* $\{X_n\}_{n \in \mathbb{N}>0}$*,* $\{S_n\}_{n \in \mathbb{N}>0}$ *is a* ***supermartingale*** *with respect to the filtration* $\{X_n\}_{n \in \mathbb{N}>0}$ *if the following three conditions are met:*

(i) $\mathbb{E}[|S_n|] < \infty$ *for any fixed* $n \in \mathbb{N}>0$ *;*

(ii) $\{S_n\}_{n \in \mathbb{N}>0}$ *is adapted to the filtration* $\{X_n\}_{n \in \mathbb{N}>0}$ *, that is, known the value of* $X_1, ..., X_n$ *, we know also the value of* S_n *;*

(iii) $\mathbb{E}[S_{n+1}|X_1, ..., X_n] \leq S_n$.

Definition 2.3. *Given two sequences of random variables* $\{S_n\}_{n \in \mathbb{N}>0}$ *and* $\{X_n\}_{n \in \mathbb{N}>0}$ *,* $\{S_n\}_{n \in \mathbb{N}>0}$ *is a* ***submartingale*** *with respect to the filtration* $\{X_n\}_{n \in \mathbb{N}>0}$ *if the following three conditions are satisfied:*

(i) $\mathbb{E}[|S_n|] < \infty$ *for any fixed* $n \in \mathbb{N}>0$ *;*

(ii) $\{S_n\}_{n\in\mathbb{N}>0}$ *is adapted to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$ *, that is, known the value of* $X_1, ..., X_n$*, we know also the value of* S_n *;*

(iii) $\mathbb{E}[S_{n+1}|X_1, ..., X_n] \geq S_n$ *.*

Note that if $\{S_n\}_{n\in\mathbb{N}>0}$ is a supermartingale and/or a submartingale, both with respect to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$, then $\{S_n\}_{n\in\mathbb{N}>0}$ is also a martingale with respect to the filtration $\{S_n\}_{n\in\mathbb{N}>0}$. Let us check that the two first properties are satisfied in the previous example by using the **Triangular Inequality**.[1]

Let us consider again the case of $S_n = S_0 + \sum_{i=1}^{n} X_i$, where $S_0 > 0$ is a fixed number and $\{X_n\}_{n\in\mathbb{N}>0}$ is a sequence of i.i.d. random variables, with X_i defined as in (2.1). Since $S_n = S_0 + \sum_{i=1}^{n} X_i$, we know its value by knowing the values of $X_1, ..., X_n$, so $\{S_n\}_{n\in\mathbb{N}>0}$ is adapted to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$ (property (ii)). Let us focus now on property (i); by Triangular Inequality we have

$$|S_n| = \left|S_0 + \sum_{i=1}^{n} X_i\right| \leq |S_0| + \sum_{i=1}^{n} |X_i| = S_0 + \sum_{i=1}^{n} |X_i|$$

and hence

$$\mathbb{E}[|S_n|] \leq \mathbb{E}\left[S_0 + \sum_{i=1}^{n} |X_i|\right] = S_0 + \sum_{i=1}^{n} \mathbb{E}[|X_i|]\,.$$

Since

$$\mathbb{E}[|X_i|] = |1| \cdot p + |-1| \cdot (1-p) = 1\,,$$

we have that

$\mathbb{E}[|S_n|] \leq S_0 + \sum_{i=1}^{n} \mathbb{E}[|X_i|] = S_0 + \sum_{i=1}^{n} 1 = S_0 + n < \infty$ for any fixed $n \in \mathbb{N}>0$. So property (i) of (super/sub)martingales is verified. As discussed before Definition 2.1, we have

$$\mathbb{E}[S_{n+1}|X_1, ..., X_n] \begin{cases} = S_n\,, & \text{if } p = \frac{1}{2}\,, \\ \leq S_n\,, & \text{if } p \leq \frac{1}{2}\,, \\ \geq S_n\,, & \text{if } p \geq \frac{1}{2}\,. \end{cases}$$

[1] See Proposition A.1 in Appendix for reference.

Having proved properties (i) and (ii), we have shown that the stochastic process $\{S_n\}_{n\in\mathbb{N}>0}$ is, with respect to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$, a martingale if $p = \frac{1}{2}$, a supermatringale if $p \leq \frac{1}{2}$, a submartingale if $p \geq \frac{1}{2}$. A direct consequence of the third property of the (super/sub)martingales is the monotonicity of the expectation of the stochastic process. More precisely, we have the following result:

Proposition 2.2.1. *Given two sequences of random variables* $\{S_n\}_{n\in\mathbb{N}>0}$ *and* $\{X_n\}_{n\in\mathbb{N}>0}$, *we have that:*

(i) if $\{S_n\}_{n\in\mathbb{N}>0}$ *is a martingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$, *then* $\mathbb{E}[S_n]$ *is constant in* n, *that is*

$$\mathbb{E}[S_{n+1}] = \mathbb{E}[S_n] = ... = \mathbb{E}[S_1]$$

(ii) if $\{S_n\}_{n\in\mathbb{N}>0}$ *is a supermartingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$, *then* $\mathbb{E}[S_n]$ *is decreasing in* n, *that is*

$$\mathbb{E}[S_{n+1}] \leq \mathbb{E}[S_n] \leq ... \leq \mathbb{E}[S_1]$$

(iii) if $\{S_n\}_{n\in\mathbb{N}>0}$ *is a submartingale with respect to the filtration* $\{S_n\}_{n\in\mathbb{N}>0}$, *then* $\mathbb{E}[S_n]$ *is increasing in* n, *that is*

$$\mathbb{E}[S_{n+1}] \geq \mathbb{E}[S_n] \geq ... \geq \mathbb{E}[S_1]$$

Proof. If $\{S_n\}_{n\in\mathbb{N}>0}$ *is a martingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$ *then, by property (iii) in Definition 2.1, we have*

$$S_n = \mathbb{E}[S_{n+1}|X_1, ..., X_n] \, .$$

If we apply the expectation to both the members of the previous identity we get

$$\mathbb{E}[S_n] = \mathbb{E}[\mathbb{E}[S_{n+1}|X_1, ..., X_n]] \, .$$

Using the Tower Property for the second member, we have

$$\mathbb{E}[S_n] = \mathbb{E}[S_{n+1}]\,.$$

Iterating this identity up to $n = 1$, we get the thesis. In case of supermartingales and submartingales the computation is identical, but we start from property (iii) in Definition 2.2 and Definition 2.3, respectively.

In the next example, we will see how to study if a stochastic process is a martingale in the case where $S_n = \prod_{i=1}^{n} X_i$.

Example 2.1. *Consider a sequence of i.i.d. random variables $\{X_n\}_{n\in\mathbb{N}>0}$, with $\mathbb{E}[X_1] \in (0,\infty)$ and define for $n \in \mathbb{N}>0$*

$$S_n = \prod_{i=1}^{n} X_i\,.$$

We are interested in assessing whether the stochastic process $\{S_n\}_{n\in\mathbb{N}>0}$ is a (super/sub)martingale with respect to the filtration $\{S_n\}_{n\in\mathbb{N}>0}$. Note that

$$\mathbb{E}[|S_n|] = \mathbb{E}\left[\left|\prod_{i=1}^{n} X_i\right|\right] = \mathbb{E}\left[\prod_{i=1}^{n} |X_i|\right] = \prod_{i=1}^{n} \mathbb{E}[|X_i|]$$

where in the last identity we have used the fact that $\{X_n\}_{n\in\mathbb{N}>0}$, are independent random variables and, as a consequence, also $|X_1|, |X_2|, ..., |X_n|$ are independent random variables. Since $X_1, X_2, ..., X_n$ are i.i.d., they also have the same expected value and hence

$$\mathbb{E}[|S_n|] = \prod_{i=1}^{n} \mathbb{E}[|X_i|] = (\mathbb{E}[X_1])^n < \infty$$

for each fixed $n \in \mathbb{N}>0$, being $\mathbb{E}[X_1] < \infty$. So property (i) of (super/sub)martingales holds. Also property (ii) is verified as, knowing the values of $X_1, ..., X_n$, we can compute S_n by using the definition $S_n = \prod_{i=1}^{n} X_i$. Let us try to prove property (iii):

$$\mathbb{E}[Sn+1|X_1, ..., X_n] = \mathbb{E}\left[\prod_{i=1}^{n+1} X_i|X_1, ..., X_n\right] = \mathbb{E}\left[X_{n+1} \cdot \prod_{i=1}^{n} X_i|X_i, ...X_n\right] =$$

$$= \prod_{i=1}^{n} X_i \cdot \mathbb{E}[X_{n+1}|X_1, ..., X_n] = M_n \cdot \mathbb{E}[X_{n+1}|X_1, ..., X_n]\,,$$

where we have that, when conditioning on $X_1, ..., X_n$, $\prod_{i=1}^n X_i$ *is a number and hence goes out from the expectation. Notice that, since* X_{n+1} *is independent on* $X_1, ..., X_n$, *the conditioning on* $X_1, ..., X_n$ *does not affect the expectation of* X_{n+1}. *Therefore*

$$\mathbb{E}[X_{n+1}|X_1, ..., X_n] = \mathbb{E}[X_{n+1}] = \mathbb{E}[X_1]$$

Hence we deduce that

$$\mathbb{E}[M_{n+1}|X_1, ..., X_n = M_n \cdot \mathbb{E}[X_1] \, .$$

As a consequence:

- *if* $\mathbb{E}[X_1] = 1$, $\{S_n\}_{n\in\mathbb{N}>0}$ *is a martingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$;
- *if* $\mathbb{E}[X_1] \in (0, 1)$, $\{S_n\}_{n\in\mathbb{N}>0}$ *is a supermartingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$;
- *if* $\mathbb{E}[X_1] > 1$, $\{S_n\}_{n\in\mathbb{N}>0}$ *is a submartingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$.

Let us see now an example applied to a betting game:

Example 2.2. *Imagine to repeat the same bet many times and at each round you can win a euro (where* $a > 0$*) with probability* p, *lose* b *euro (where* $b > 0$*) with probability* q, *or have a tie with probability* r, *where* $p + q + r = 1$. *We denote by* X_i *what is won in the i-th round, that is*

$$X_i = \begin{cases} a \, , & \text{with probability } p \, , \\ 0 \, , & \text{with probability } r \, , \\ -b \, , & \text{with probability } q \, . \end{cases}$$

where $\{X_n\}_{n\in\mathbb{N}>0}$ *is a sequence of i.i.d. random variables. Denote by* S_n *the total capital obtained in n rounds and by* S_0 *the initial capital (a fixed positive constant).*

So

$$S_n = S_0 + \sum_{i=1}^{n} X_i$$

and we want to study the stochastic process $\{S_n\}_{n\in\mathbb{N}>0}$ *and check if it is a (super/sub) martingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$*. By Triangular Inequality, we have*

$$\mathbb{E}[|S_n|] \leq \mathbb{E}\left[|S_0| + \sum_{i=1}^{n} |X_i|\right] = S_0 + \sum_{i=1}^{n} \mathbb{E}[|X_i|]$$

where we have used the fact that $S_0 > 0$ *by initial assumption. Since* $\{X_n\}_{n\in\mathbb{N}>0}$ *are identical distributed, they have also same expectation, so the above expression becomes*

$$\mathbb{E}[|S_n|] \leq S_0 + n\mathbb{E}[|X_1|]$$

where

$$\mathbb{E}[X_1] = |a| \cdot p + |0| \cdot r + |-b| \cdot q = ap + bq\,.$$

Here we have $a, b > 0$*, hence* $-b$ *is negative and its absolute value is* b*. So we get*

$$\mathbb{E}[|S_n|] \leq S_0 + n\mathbb{E}[|X_1|] = S_0 + n \cdot (ap + bq) < \infty$$

for any fixed $n \in \mathbb{N}{>}0$ *. Hence we have proved the first property of (super/sub)martingales. Let us check for the second one: we have to verify that, known the values of* $X_1, ..., X_n$*, we know the value of* S_n*, meaning that* $\{S_n\}_{n\in\mathbb{N}>0}$ *is adapted to the filtration* $\{S_n\}_{n\in\mathbb{N}>0}$*. This is true since, known the values of* $X_1, ..., X_n$*, to compute* S_n *it is enough to compute* $S_0 + \sum_{i=1}^{n} X_i$ *(recall that* S_0 *is a fixed constant known from the start), so we have verified the second property. Let us now check the third one: we have to study* $\mathbb{E}[S_{n+1}|X_1, ..., X_n]$*. Note that, since we know the values of* $X_1, ..., X_n$*, we also know the value of* S_n*. Hence, since*

$$S_{n+1} = S_0 + \sum_{i=1}^{n+1} X_i = S_n = S_0 + \sum_{i=1}^{n} X_i + X_{n+1} = S_n + X_{n+1}\,,$$

the only unknown part in S_{n+1} *is* X_{n+1}*. Therefore*

$$\mathbb{E}[S_{n+1}|X_1,...,X_n] = \mathbb{E}[S_n + X_{n+1}|X_1,...,X_n] = \\ = \mathbb{E}[S_n|X_1,...,X_n] + \mathbb{E}[X_{n+1}|X_1,...,X_n]\,.$$

Since S_n *is known when knowing* $X_1,...X_n$*, then* $\mathbb{E}[S_n|X_1,...,X_n] = S_n$*. Moreover, being* $\{X_n\}_{n\in\mathbb{N}>0}$ *independent, we have* $\mathbb{E}[X_{n+1}|X_1,...,X_n] = \mathbb{E}[X_{n+1}]$ *and the previous equation becomes*

$$\mathbb{E}[S_{n+1}|X_1,...,X_n] = \mathbb{E}[S_n|X_1,...,X_n] + \mathbb{E}[X_{n+1}|X_1,...,X_n] = \\ = S_n + \mathbb{E}[X_{n+1}] = S_n + ap - bq\,.$$

Hence

- *if* $\mathbb{E}[X_1] = ap - bq = 0\,, \{S_n\}_{n\in\mathbb{N}>0}$ *is a martingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}\,;$
- *if* $\mathbb{E}[X_1] = ap - bq < 0\,, \{S_n\}_{n\in\mathbb{N}>0}$ *is a supermartingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}\,;$
- *if* $\mathbb{E}[X_1] = ap - bq > 0\,, \{S_n\}_{n\in\mathbb{N}>0}$ *is a submartingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$

where the considerations above are independent on r *(probability of tie).*

This last example will be extremely relevant in the Gambler's ruin setting, which will be the main topic of Section 2.5.

Example 2.3. *Suppose to have a fixed initial capital* $K_0 > 0$ *and let* K_n *be the total capital after* n *rounds of the following game: for* $n \in \mathbb{N}>0$*, at the* n*-th round we bet* K_{n-1} *and we win* K_n*, where*

$$K_n = \begin{cases} \frac{q}{p}\cdot K_{n-1}\,, & \text{with probability } p\,, \\ \frac{p}{q}\cdot K_{n-1}\,, & \text{with probability } q\,, \\ K_{n-1}\,, & \text{with probability } r \end{cases}$$

where $p, q > 0, r \geq 0$ *and* $p + q + r = 1$. *Defining*

$$X_i = \begin{cases} 1\,, & \textit{with probability } p\,, \\ -1\,, & \textit{with probability } q\,, \\ 0\,, & \textit{with probability } r\,. \end{cases}$$

we have

$$K_n = (q/p)^{X_n} K_{n-1}$$

and if we iterate this formula we get

$$K_n = (q/p)^{X_1 + \ldots + X_n} K_0\,. \tag{2.2}$$

Let us show that the stochastic process $\{K_n\}_{n \in \mathbb{N} > 0}$ *is a martingale with respect to the filtration* $\{X_n\}_{n \in \mathbb{N} > 0}$ *for any possible values of p,q,r. In this case it is easier to prove the second and the third property before the first one. Note that* $\{K_n\}_{n \in \mathbb{N} > 0}$ *is adapted to the filtration* $\{X_n\}_{n \in \mathbb{N} > 0}$ *since, by (2.2), if we know the values of* $X_1, ..., X_n$*, we know also* K_n*. So the second property of the martingales is proved and we can know focus on the third one:*

$$\begin{aligned} \mathbb{E}[K_{n+1} | X_1, ..., X_n] &= \mathbb{E}\left[(q/p)^{X_1 + \ldots + X_n + X_{n+1}} K_0 | X_1, ..., X_n\right] = \\ &= \mathbb{E}\left[(q/p)^{X_{n+1}} \cdot (q/p)^{X_1 + \ldots + X_n + X_n} K_0 | X_1, ..., X_n\right] = \\ &= \mathbb{E}\left[(q/p)^{X_{n+1}} \cdot K_n | X_1, ..., X_n\right]. \end{aligned} \tag{2.3}$$

Since $\{K_n\}_{n \in \mathbb{N} > 0}$ *is adapted to the filtration* $\{X_n\}_{n \in \mathbb{N} > 0}$*, we have*

$$\mathbb{E}[K_n | X_1, ..., X_n] = K_n\,.$$

Moreover, since $\{X_n\}_{n \in \mathbb{N} > 0}$ *are independent random variables, we get*

$$\begin{aligned} \mathbb{E}\left[(q/p)^{X_{n+1}} | X_1, ..., X_n\right] &= \mathbb{E}\left[(q/p)^{X_{n+1}}\right] = \\ &= (q/p)^1 \cdot p + (q/p)^{-1} \cdot q + (q/p)^0 \cdot r = q + p + r = 1\,. \end{aligned}$$

So equation (2.3) becomes

$$\begin{aligned}\mathbb{E}[K_{n+1}|X_1,...,X_n] &= \mathbb{E}\left[(q/p)^{X_{n+1}} \cdot K_n|X_1,...,X_n\right] = \\ &= K_n\mathbb{E}\left[(q/p)^{X_{n+1}}|X_1,...,X_n\right] = Y_n \cdot 1 = Y_n \,.\end{aligned} \tag{2.4}$$

Hence, up to now we have shown that $\mathbb{E}[K_{n+1}|X_1,...,X_n] = K_n$ *and that* $\{K_n\}_{n\in\mathbb{N}>0}$ *is adapted to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$*. By proving that* $\mathbb{E}[|K_n|] < \infty$*, we get that* $\{K_n\}_{n\in\mathbb{N}>0}$ *is a martingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$*. Since* $K_n \geq 0$ *for all* $n \in \mathbb{N}>0$*, we have* $|K_n| = K_n$*, so it is enough to show that* $\mathbb{E}[K_n] < \infty$*. Using the Tower Property we get*

$$\mathbb{E}[K_n] = \mathbb{E}[\mathbb{E}[K_n|X_1,...,X_{n-1}]] = \mathbb{E}[K_{n-1}]\,,$$

where in the last identity we have used (2.4). This proves that the expectation of K_n *is constant in* n*; more precisely, this implies that*

$$\mathbb{E}[K_n] = \mathbb{E}[K_0] = (q/p)^1 \cdot p + (q/p)^{-1} \cdot q + (q/p)^0 \cdot r = q + p + r = 1 < \infty\,.$$

So we have $\mathbb{E}[|K_n|] = \mathbb{E}[K_n] < \infty$ *and hence* $\{K_n\}_{n\in\mathbb{N}>0}$ *is a martingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$*. Notice how the thesis is independent on the values of* p*,*q*,*r*.*

2.3 Optional Stopping Theorem

Let us consider the example discussed in (2.1) and define τ as the first time we get head, meaning the first round i such that $X_i = 1$. τ is a random variable since depends on the outcomes of the tossings and it can be rewritten as

$$\tau = \inf\{i \in \mathbb{N}>0|X_i = 1\} \tag{2.5}$$

from which we get

$$\{\tau = k\} = \{X_1 \neq 1, X_2 \neq 1, ..., X_{k-1} \neq 1, X_k = 1\}\,. \tag{2.6}$$

Indeed $\tau = k$ if and only if we always got tail in the first $k-1$ rounds and we got head at the k-th round. To establish if $\tau = k$, we need information up to round k and this property defines what we call a stopping time.

Definition 2.4. *Given a filtration $\{X_n\}_{n\in\mathbb{N}>0}$, we say that a discrete random variable τ is a stopping time with respect to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$ if it assumes values in $\mathbb{N}$ and, to establish the occurrence of the event $\{\tau = k\}$, it is necessary to know only the values of $X_1, ..., X_k$.*

The random variable τ in (2.5) is a stopping time with respect to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$ since (2.6) holds. If instead we consider the random variable σ defined as the last time in which we get head, we have

$$\{\sigma = k\} = \{X_k = 1, X_{k+1} = 1, X_{k+2} = 1, ...\}\,.$$

Indeed $\sigma = k$ if and only if we get head at the k-th round and we have no heads in future rounds. Since to establish the occurrence of the event $\{\sigma = k\}$ we need to know future information with respect to the k-th round $(X_k, X_{k+1}, X_{k+2}, ...,)$, σ is not a stopping time with respect to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$. The next result shows the distribution of a particular stopping time and it will prove that the latter has finite expectation.

Proposition 2.3.1. *Given a filtration $\{X_n\}_{n\in\mathbb{N}>0}$ of i.i.d. random variables, consider a value $\alpha \in Im(X_n)$, (α is a value assumed by $\{X_n\}_{n\in\mathbb{N}>0}$.) Let $p = \mathbb{P}(X_n = \alpha) > 0$ (and hence $\mathbb{P}(X_n \neq \alpha) = 1-p$) and define*

$$\tau = inf\{k \in \mathbb{N}{>}0 | X_k = \alpha\}\,,$$

$$\sigma = inf\{k \in \mathbb{N}{>}0 | X_k \neq \alpha\}\,.$$

Then τ and σ are both stopping times with respect to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$ and

$$\tau \sim Geom(p)\,, \qquad \sigma \sim Geom(1-p)\,.$$

Consequently we have $\mathbb{E}[\tau] = \frac{1}{p} < \infty$ and $\mathbb{E}[\sigma] = \frac{1}{1-p} < \infty$.

To prove that:

$$\{\tau = n\} = \{X_1 \neq \alpha, X_2 \neq \alpha, ..., X_{n-1} \neq \alpha, X_n = \alpha\}\,,$$

$$\{\sigma = n\} = \{X_1 = \alpha, X_2 = \alpha, ..., X_{n-1} = \alpha, X_n \neq \alpha\}\,.$$

Then

$$\mathbb{P}(\tau = n) = \mathbb{P}(X_1 \neq \alpha, X_2 \neq \alpha, ..., X_{n-1} \neq \alpha, X_n = \alpha)\,,$$

$$\mathbb{P}(\sigma = n) = \mathbb{P}(X_1 = \alpha, X_2 = \alpha, ..., X_{n-1} = \alpha, X_n \neq \alpha)\,.$$

Since $\{X_n\}_{n\in\mathbb{N}>0}$ are i.i.d., then

$$\mathbb{P}(\tau = n) = \mathbb{P}(X_1 \neq \alpha, X_2 \neq \alpha, ..., X_{n-1} \neq \alpha, X_n = \alpha) = \mathbb{P}(X_1 \neq \alpha)^{n-1}\mathbb{P}(X_1 = \alpha)\,,$$

$$\mathbb{P}(\sigma = n) = \mathbb{P}(X_1 = \alpha, X_2 = \alpha, ..., X_{n-1} = \alpha, X_n \neq \alpha) = \mathbb{P}(X_1 = \alpha)^{n-1}\mathbb{P}(X_1 \neq \alpha)\,.$$

Given that $\mathbb{P}(X_1 = \alpha) = p$ and $\mathbb{P}(X_1 \neq \alpha) = 1 - p$, we have

$$\mathbb{P}(\tau = n) = \mathbb{P}(X_1 \neq \alpha)^{n-1}\mathbb{P}(X_1 = \alpha) = (1-p)^{n-1}p\,,$$

$$\mathbb{P}(\sigma = n) = \mathbb{P}(X_1 = \alpha)^{n-1}\mathbb{P}(X_1 \neq \alpha) = p^{n-1}(1-p)\,,$$

from which we get the thesis.

Remark 1. *Note that τ and σ are not bounded random variables, so we cannot say that τ and σ are less than some precise constant with probability 1; indeed a geomtric random variable has image* $\mathbb{N}>0$ *(not a bounded set). However, τ and σ have finite expectation, that is in average they are finite. Hence we can conclude that τ and σ are not bounded random variables, but are finite in average. Actually, being geometric random variables, we can also say that they are finite with probability 1. Indeed*

$$\begin{aligned}\mathbb{P}(\tau < \infty) &= \sum_{n=1}^{\infty}\mathbb{P}(\tau = n) = \sum_{n=1}^{\infty}(1-p)^{n-1}p = \\ &= p\sum_{n=1}^{\infty}(1-p)^{n-1} \underset{j=n-1}{=} p\sum_{j=0}^{\infty}(1-p)^j = p\cdot\frac{1}{1-(1-p)} = 1\,,\end{aligned}$$

where in the last identity we have used the fact that, if $|a| < 1$, then $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$. The same holds also for σ (it is enough to exchange p with $1-p$).

Proposition 2.2.1 establishes a relation between the stochastic process at a fixed time n and at time 0. Now, we wonder if such a relation holds also with a random time τ instead of the fixed time n: the answer is given by the Optional Stopping Theorem that we formulate in case of martingales, supermartingales and submartingales, with hypotheses being the same for all three cases.

Proposition 2.3.2. ***(Optional Stopping Theorem for Martingales)****. Let $\{M_n\}_{n\in\mathbb{N}>0}$ be a martingale and let τ be a stopping time both with respect to the same filtration $\{X_n\}_{n\in\mathbb{N}>0}$. Assume that one of the following hypotheses is satisfied*

(a) $\exists C > 0$ such that $\mathbb{P}(\tau < C) = 1$ (τ is bounded almost surely);

(b) $\mathbb{P}(\tau < \infty) = 1$ and $\exists C > 0$ such that $\forall n \in \mathbb{N}$ it holds $\mathbb{P}(|M_n| \leq C) = 1$ (τ is finite almost surely and $\{M_n\}_{n\in\mathbb{N}>0}$ is uniformly bounded almost surely, meaning that the constant C does not depend on n);

(c) $\mathbb{E}[\tau] < \infty$ and $\exists C > 0$ such that $\mathbb{P}(|M_{n+1} - M_n| \leq C) = 1$ holds $\forall n \in \mathbb{N}$ (τ is finite in average and $\{M_n\}_{n\in\mathbb{N}>0}$ has uniformly bounded increments almost surely, meaning that the constant C does not depend on n);

(d) $\mathbb{P}(\tau < \infty) = 1$ and $M_n \geq 0$ $\forall n \in \mathbb{N}$ (τ is finite almost surely and it is a non-negative process).

Then $\mathbb{E}[M_\tau] < \infty$ and

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0] \, .$$

As previously stated, the theorem holds also for super/sub martingales; in particular, if $\{M_n\}_{n\in\mathbb{N}>0}$ is a supermartingale and τ is a stopping time both with respect to the same filtration $\{X_n\}_{n\in\mathbb{N}>0}$ and one of the four hypothesis is true,

then $\mathbb{E}[M_\tau] < \infty$ and $\mathbb{E}[M_\tau] \leq \mathbb{E}[M_0]$.
If instead $\{M_n\}_{n\in\mathbb{N}>0}$ is a submartingale and τ is a stopping time both with respect to the same filtration $\{X_n\}_{n\in\mathbb{N}>0}$ and one of the four hypothesis is true, then $\mathbb{E}[M_\tau] < \infty$ and $\mathbb{E}[M_\tau] \geq \mathbb{E}[M_0]$.

Example 2.4. *Consider the sequence $\{X_n\}_{n\in\mathbb{N}>0}$ defined as in (2.1) and define $S_n = \sum_{i=1}^n X_i$. Let us consider the stopping time*

$$\tau = \inf\{n \geq 0 | X_n = 1\}\,.$$

We would like to assess whether the Optional Stopping Theorem can be applied to understand the relation between $\mathbb{E}[S_\tau]$ and $\mathbb{E}[S_1]$, hence we have check if one of the hypothesis (a), (b), (c) and (d) is verified by the process $\{S_n\}_{n\in\mathbb{N}>0}$ and by the stopping time τ. Note that τ is a stopping time of the form described in Proposition 2.3.1 and so we know that τ is a geomteric random variable of parameter $\mathbb{P}(X_i = 1) = p > 0$. By Remark 1 we know that τ is a finite random variable almost surely but it is not bounded, so we have that $\mathbb{P}(\tau < \infty) = 1$ and does not exist a constant $C > 0$ (independent on n) such that $\mathbb{P}(\tau \leq C) = 1$. Hence we know that hypothesis (a) of Optional Stopping Theorem is not verified. Let us check hypothesis (b). We have already said that $\mathbb{P}(\tau < \infty) = 1$ and hence we have to verify if there exists a constant C independent on n such that $\mathbb{P}(|S_n| < C) = 1$. Note that $S_n = \sum_{i=1}^n X_i$ (and hence the values that S_n may assume) oscillate from its minimum (obtained when $X_i = -1$ for all $i = 1, ..., n$) to its maximum (obtained when $X_i = 1$ for all $i = 1, ..., n$). So $-n \leq S_n \leq n$, and hence $|S_n| \leq n$. Therefore, we should define $C = n$ in order to have $\mathbb{P}(|S_n| \leq C) = 1$, but such a C depends on n and hence it is not valid. This shows that the hypothesis (b) of the Optional Stopping Theorem is not satisfied. Let us see if the hypothesis (c) is verified. By Proposition 2.3.1 we have $\mathbb{E}[\tau] < \infty$, so we are left to see if there exists $C > 0$

(independent on n) such that $\mathbb{P}(|S_{n+1} - S_n| \leq C) = 1$. *Note that*

$$S_{n+1} - S_n = \sum_{i=1}^{n+1} X_i - \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_i + X_{n+1} - \sum_{i=1}^{n} X_i = X_{n+1}\,,$$

from which

$$|S_{n+1} - S_n| = |X_{n+1}| \leq max\{|1|, |-1|\} = 1\,.$$

So if we choose $C = 1$, *we have* $|S_{n+1} - S_n| \leq C$ *for any* $n \in \mathbb{N}{>}0$ *and hence*

$$\mathbb{P}(|S_{n+1} - S_n| \leq C) = 1$$

for any $n \in \mathbb{N}{>}0$. *Note that this time the constant* $C = 1$ *is independent on* n *and then it is a valid constant for the theorem. So hypothesis (c) of the Optional Stopping Theorem is verified and we can conclude that:*

- *if* $p = \frac{1}{2}$, $\{S_n\}_{n\in\mathbb{N}>0}$ *is a martingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$ *and we get*
$$\mathbb{E}[S_\tau] = \mathbb{E}[S_1] = \mathbb{E}[X_1] = 0\,;$$
- *if* $p \leq \frac{1}{2}$, $\{S_n\}_{n\in\mathbb{N}>0}$ *is a supermartingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$ *and we get*
$$\mathbb{E}[S_\tau] \leq \mathbb{E}[S_1] = \mathbb{E}[X_1] = 1 \cdot p - 1 \cdot (1-p) = 2p - 1\,;$$
- *if* $p \geq \frac{1}{2}$, $\{S_n\}_{n\in\mathbb{N}>0}$ *is a submartingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}>0}$ *and we get*
$$\mathbb{E}[S_\tau] \geq \mathbb{E}[S_1] = \mathbb{E}[X_1] = 1 \cdot p - 1 \cdot (1-p) = 2p - 1\,.$$

For completeness, we can state that hypothesis (d) is not verified since $\{S_n\}_{n\in\mathbb{N}>0}$ *is not a non-negative process, as it may happen that* $S_n < 0$ *for some* n *(for example if* $X_1 = ... = X_n = -1$ *and hence* $S_n = -n$*)*.

In gambling theory, the Optional Stopping Theorem is also known as **Principle of Conservation of Fairness** of a game. Indeed, given a game, it is possible to see a stopping time as a quitting strategy from the game. With this in mind, the Principle of Conservation of Fairness establishes that, if the game satisfies one of the hypothesis (a), (b), (c) or (d), it is impossible to find a quitting strategy from the game that transforms a subfair game into a superfair game. Indeed, as shown in the previous example, if we work with a supermartingale we have $\mathbb{E}[S_\tau] \leq \mathbb{E}[S_1]$ and hence there is no stopping time τ satisfying the hypothesis of the theorem for which $\mathbb{E}[S_\tau] > \mathbb{E}[S_1]$. We will cover betting systems in the next sections and we will see how the Martingale system escapes from the Principle of Conservation of Fairness, allowing to have $\mathbb{E}[S_\tau] > \mathbb{E}[S_1]$ even if $\{S_n\}_{n\in\mathbb{N}>0}$ is a supermartingale.

2.4 Betting Systems

Imagine betting on a game and denote by X the quantity of money won or lost for unit bet. Suppose to repeat the same bet many times and denote by $X_1, X_2, ...$ the quantity of money won or lost for unit bet at each round (so if at the fifth round we bet 9, we win $9X_5$ at the fifth round). Being the mechanism of the game independent on the previous rounds, the random variables $X_1, X_2, ...$ are obviously independent and, since the bet is the same for each round, $X_1, X_2, ...$ are also identically distributed. Let us assume now that at round n we bet a quantity of money B_n that is dependent on the outcomes of the previous rounds $(X_1, ..., X_{n-1}.)$ Then we can write

$$B_1 = g_1 > 0, \qquad B_n = g_n(X_1, ..., X_{n-1}) \quad \text{for } n \geq 2\,, \tag{2.7}$$

where g_n is a decision rule that takes $X_1, ..., X_{n-1}$ as input and gives the quantity of money to bet at round n as output, while B_1 is a fixed positive quantity $g_1 > 0$ since the quantity of money to be bet on the first round is decided in a deterministic way. Since X_n is the quantity of money won at round n for unit bet and B_n denotes the quantity of money bet at the n-th round, then the quantity of money won at the n-th round is $B_n X_n$. The sequence of variables $\{X_n\}_{n\in\mathbb{N}>0}$ and $\{B_n\}_{n\in\mathbb{N}>0}$ form a **betting system**. If we denote by F_n the total quantity of money that we have at the end of the n-th round, we get

$$F_n = F_{n-1} + B_n X_n \quad \text{for } n \geq 1\,, \tag{2.8}$$

that by iteration becomes

$$F_n = F_0 + \sum_{i=1}^{n} B_i X_i\,, \tag{2.9}$$

where F_0 represents the determinisic quantity of money that we have at the beginning. It makes sense to assume that we cannot bet more than what we have at each round, that is $B_n \leq F_{n-1}$ for $n \geq 1$. We are interested in studying the sequence $\{F_n\}_{n\in\mathbb{N}}$ with respect to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$. We assume that X_i assumes a finite quantity of values to assure that $\mathbb{E}[|X_i|] < \infty$. We have the following result that establishes the relation between (super/sub)fair games and (sub/super)martingales.

Proposition 2.4.1. *Let $\{X_n\}_{n\in\mathbb{N}>0}$ be a sequence of i.i.d. random variables with $\mathbb{E}[|X_i|] < \infty$ and let $\{B_n\}_{n\in\mathbb{N}>0}$ be a sequence of random variables that satisfies (2.7). If we define the sequence $\{F_n\}_{n\in\mathbb{N}}$ as in (2.9, we have*

- *if $\mathbb{E}[X_1] = 0$, then $\{F_n\}_{n\in\mathbb{N}}$ is a martingale with respect to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$;*
- *if $\mathbb{E}[X_1] \leq 0$, then $\{F_n\}_{n\in\mathbb{N}}$ is a supermartingale with respect to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$;*
- *if $\mathbb{E}[X_1] \geq 0$, then $\{F_n\}_{n\in\mathbb{N}}$ is a submartingale with respect to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$.*

Proof. We have to prove initially that

(i) $\mathbb{E}[|F_n|] < \infty$ for any fixed n;

(ii) $\{F_n\}_{n\in\mathbb{N}}$ is adapted to $\{X_n\}_{n\in\mathbb{N}>0}$ (we know the value of F_n when we know $X_1, ..., X_n$).

Lastly we have to understand the relation between $\mathbb{E}[F_{n+1}|X_1, ..., X_n]$ and F_n. Let us start by proving (i). Note that by triangular inequality

$$\begin{aligned}\mathbb{E}[|F_n|] = \mathbb{E}\left[|F_0 + \sum_{i=1}^{n} B_i X_i|\right] \leq \mathbb{E}\left[|F_0| + \sum_{i=1}^{n} |B_i X_i|\right] = \\ = |F_0| + \sum_{i=1}^{n} \mathbb{E}[|B_i| \cdot |X_i|] = |F_0| + \sum_{i=1}^{n} \mathbb{E}[|B_i|] \cdot \mathbb{E}[|X_i|]\,,\end{aligned} \tag{2.10}$$

where we exploited the fact that F_0 is a constant and that, being $B_i = g_i(X_1, ..., X_{i-1})$, then B_i is independent on X_i and hence $\mathbb{E}[|B_i| \cdot |X_i|] = \mathbb{E}[|B_i|] \cdot \mathbb{E}[|X_i|]$. Note that, since X_i assumes only a finite number of values (say k values), we have that also the vector $(X_1, ..., X_{i-1})$ assumes a finite number of values (k^{i-1}). So also $g_i(X_1, ..., X_{i-1})$ assumes a finite number of values (at most k^{i-1}) and then B_i is smaller than some constant $K_i < \infty$. Therefore

$$\mathbb{E}[|F_n|] \leq |F_0| + \mathbb{E}[|X_1|] \sum_{i=1}^{n} K_i \leq |F_0| + \mathbb{E}[|X_1|] \cdot n \max_{1\leq i\leq n} K_i \leq \infty\,,$$

where the last inequality is due to the fact that F_0 and $\mathbb{E}[|X_1|]$ are fixed numbers, while $K_1, ..., K_n$ are n finite numbers and hence their maximum is finite. This proves (i). Let us prove (ii). Recall the definition of F_n in (2.9). Note that, since $B_i = g_i(X_1, ..., X_{i-1})$, to know B_i it is sufficient to know $X_1, ..., X_{i-1}$. Hence to know $B_1, ..., B_n$ it is enough to know $X_1, ..., X_{n-1}$. Consequently, by (2.9), we get that to know F_n it is sufficient to know the value of $X_1, ..., X_n$; this proves that the process $\{F_n\}_{n\in\mathbb{N}}$ is adapted to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$, that is (ii). We now have to compute $\mathbb{E}[F_{n+1}|X_1, ..., X_n]$. Recall that $B_i = g_i(X_1, ..., X_{i-1})$. So

$$\begin{aligned}\mathbb{E}[B_i|X_1, ..., X_n] = B_i \quad \text{for } i = 1, ..., n+1\,, \\ \mathbb{E}[X_i|X_1, ..., X_n] = X_i \quad \text{for } i = 1, ..., n\,.\end{aligned} \tag{2.11}$$

As a consequence, we get

$$\mathbb{E}[B_i X_i | X_1, ..., X_n] = B_i X_i \quad \text{for } i = 1, ..., n \,. \tag{2.12}$$

So:

$$\begin{aligned}
\mathbb{E}[F_{n+1}|X_1, ..., X_n] &= \mathbb{E}\left[F_0 + \sum_{i=1}^{n+1} B_i X_i | X_1, ..., X_n\right] = \\
&\underset{(2.12)}{=} F_0 + \sum_{i=1}^{n} B_i X_i + \mathbb{E}[B_{n+1} X_{n+1} | X_1, ..., X_n] = \\
&= F_n + \mathbb{E}[B_{n+1} X_{n+1} | X_1, ... X_n] = \\
&\underset{(2.11)}{=} F_n + B_{n+1} \mathbb{E}[X_{n+1} | X_1, ..., X_n] \,.
\end{aligned} \tag{2.13}$$

Since $\{X_n\}_{n \in \mathbb{N}_{>0}}$ is a sequence of independent random variables, we have that

$$\mathbb{E}[X_{n+1}|X_1, ..., X_n] = \mathbb{E}[X_{n+1}] \underset{\text{id. distrib.}}{=} \mathbb{E}[X_1] \,.$$

Hence

$$\mathbb{E}[F_{n+1}|X_1, ..., X_n] = F_n + B_{n+1} \mathbb{E}[X_1] \,.$$

Since $B_{n+1} \geq 0$, we have that

- $\mathbb{E}[F_{n+1}|X_1, ..., X_n] = F_n$ if $\mathbb{E}[X_1] = 0$;
- $\mathbb{E}[F_{n+1}|X_1, ..., X_n] \geq F_n$ if $\mathbb{E}[X_1] \geq 0$;
- $\mathbb{E}[F_{n+1}|X_1, ..., X_n] \leq F_n$ if $\mathbb{E}[X_1] \leq 0$.

This proves the thesis.

2.4.1 Martingale System

Suppose to repeat the same bet many times. Assume that the quantity of money won for unit bet at round i is

$$X_i = \begin{cases} 1, & \text{if we win the } i\text{-th round,} \\ -1, & \text{if we lose the } i\text{-th round} \end{cases}$$

and that

$$\mathbb{P}(X_i = 1) = p \in \left(0, \frac{1}{2}\right), \qquad \mathbb{P}(X_i = -1) = 1 - p\,.$$

In this system, the gambler doubles his bet size each time that they lose and stop betting after the first win. We define the stopping round τ (with respect to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$) as the first winning round, we have that

$$\{\tau = k\} = \{X_1 = -1, ..., X_{k-1} = -1, X_k = 1\}\,.$$

Fixed an initial bet $B_1 > 0$ and an initial capital $F_0 > 0$, and denoting by B_i the amount of money bet at round i and by F_i our total capital at the end of round i, we have

$$F_1 = \begin{cases} F_0 + B_1\,, & \text{if } X_1 = 1\ (\tau = 1)\,, \\ F_0 - B_1\,, & \text{if } X_1 = -1\ (\tau > 1)\,. \end{cases}$$

In general if we have lost the first $n-1$ rounds ($\tau > n-1$) we have

$$F_n = \begin{cases} F_0 - B_1 - ... - B_{n-1} + B_n\,, & \text{if } X_1 = -1, ..., X_{n-1} = -1, X_n = 1\ \ (\tau = n)\,, \\ F_0 - B_1 - ... - B_n\,, & \text{if } X_1 = -1, X_2 = -1, ..., X_n = -1\ \ (\tau > n)\,. \end{cases}$$

Note that, if $\tau \geq n$, since each time we lose we double the bet size, we have

$$B_n = 2B_{n-1}$$

and by iteration of this formula we get

$$B_n = 2B_{n-1} \underset{B_{n-1}=2B_{n-2}}{=} 2^2 B_{n-2} = 2^3 B_{n-3} = ... = 2^{n-1} B_1\,.$$

So F_n can be rewritten as

$$F_n = \begin{cases} F_0 - B_1 - 2B_1 - 2^2 B_1 - ... - 2^{n-2} B1 + 2^{n-1} B_1\,, & \text{if } \tau = n\,, \\ F_0 - B_1 - 2B_1 - 2^2 B_1 - ... - 2^{n-2} B1 - 2^{n-1} B_1\,, & \text{if } \tau > n\,, \end{cases}$$

that we ca rewrite as

$$F_n = \begin{cases} F_0 - B_1(1+2+2^2+...+2^{n-2}) + 2^{n-1}B_1\,, & \text{if } \tau = n\,, \\ F_0 - B_1(1+2+2^2+...+2^{n-1})\,, & \text{if } \tau > n\,, \end{cases}$$

that is

$$F_n = \begin{cases} F_0 - B_1 \cdot \sum_{i=1}^{n-2} 2^i + 2^{n-1}B_1\,, & \text{if } \tau = n\,, \\ F_0 - B_1 \cdot \sum_{i=1}^{n-1} 2^i\,, & \text{if } \tau > n\,. \end{cases}$$

Recall that for any $a \in \mathbb{R} \setminus \{1\}$

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}\,.$$

If we apply the above formula to write F_n. we have

$$F_n = \begin{cases} F_0 - B_1 \cdot \frac{2^{n-1}-1}{2-1} + 2^{n-1}B_1 = F_0 - 2^{n-1}B_1 + B_1 + 2^{n-1}B_1\,, & \text{if } \tau = n\,, \\ F_0 - B_1 \cdot \frac{2^{n}-1}{2-1} = F_0 - 2^n B_1 + B_1\,, & \text{if } \tau > n\,. \end{cases}$$

that implies

$$F_n = \begin{cases} F_0 + B_1\,, & \text{if } \tau = n, \\ F_0 + B_1(1-2^n)\,, & \text{if } \tau > n. \end{cases}$$

So we deduce that

$$\mathbb{P}(F_\tau = F_0 + B_1) = 1\,,$$

meaning that, when we stop, we have recovered all the capital bet and we have won our initial bet B_1 with probability 1. Moreover, if $\tau > n$, we can quantify the amount of money that we are losing, that is to say $B_1(1-2^n)$, which is an exponentially decreasing quantity. Moreover, since $\mathbb{P}(F_\tau = F_0 + B_1) = 1$, we have

$$\mathbb{E}[F_\tau] = \mathbb{E}[F_0 + B_1] \underset{B_1>0}{\geq} \mathbb{E}[F_0]\,.$$

This is exactly the opposite conclusion of the Optional Stopping Theorem for supermartingales ($\mathbb{E}[F_\tau] \leq \mathbb{E}[F_0]$). Indeed, the capital $\{F_n\}_{n\in\mathbb{N}>0}$ is a supermartingale with respect to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$ because $\mathbb{P}(X_i = 1) = p < \frac{1}{2}$ and $\mathbb{P}(X_i = -1) = 1 - p$. However, this is not a contradiction of the Optional Stopping Theorem for supermartingales as the hypotheses of the theorem are not satisfied. Indeed, being τ the first round in which we win and being the single bets i.i.d., we have that $\tau \sim \text{Geom}\,(p)$ and hence $\nexists C > 0$ such that $\mathbb{P}(\tau < C) = 1$. Moreover it is also impossible to find $C > 0$ (independent on n) such that $\mathbb{P}(|F_n| < C) = 1$ since, while we are losing rounds, our capital decreases exponentially fast and hence $|F_n|$ increases exponentially fast. Note that, while we are losing rounds, also the increments of $\{F_n\}_{n\in\mathbb{N}>0}$ increase exponentially fast. Indeed

$$\begin{aligned}|F_{n+1} - F_n| &= |F_0 + B_1 - 2^{n+1}B_1 - (F_0 + B_1 - 2^n B_1)| = |B_1(2^n - 2^{n+1})| = \\ &= B_1 \cdot 2^n \cdot |1-2| = 2^n B_1\,.\end{aligned}$$

So it is impossible to find a constant $C > 0$ independent on n such that $\mathbb{P}(|F_{n+1} - F_n| \leq C) = 1$. Finally, fixed a value for F_0, the process does not satisfy $F_n \geq 0$ for all $n \in \mathbb{N}$. Hence all the hypotheses of the Optional Stopping Theorem fail and therefore this betting strategy allows to have $\mathbb{E}[F_\tau] > \mathbb{E}[F_0]$ even if $\{F_n\}_{n\in\mathbb{N}}$ is a supermartingale with respect to the filtration $\{X_n\}_{n\in\mathbb{N}>0}$. As far as the application of the martingale system, it is usually applied to contexts in which the probability p of winning is smaller than $\frac{1}{2}$ but actually not so smaller, as the bet on even or odd (equivalently red or black) at the roulette; indeed, when we bet on even numbers at the European roulette, the winning probability is $p = \frac{18}{37} \approx 0.49$. Note that, being $\tau \sim \text{Geom}(p)$, we have

$$\mathbb{P}(\tau > k) = (1-p)^k,$$

since $\tau > k$ is we have lost for k rounds (this occurs independently each time with probability $1 - p$). So if we bet on even numbers at the European roulette, we have

$$\mathbb{P}(\tau > k) = \left(\frac{19}{37}\right)^k .$$

So if, for example, we own a quantity F_0 that guarantees to not be ruined before round 6, we have a 98% probability of winning the initial bet B_1, given that $\mathbb{P}(\tau > 6) = \left(\frac{19}{37}\right)^6 \approx 0.02\,.$

2.4.2 Optimal Proportional System: the Kelly System

The Kelly Criterion is an allocation technique used by both investors and gamblers to manage money effectively: it helps gamblers in optimizing the size of their bets, while investors can exploit it to decide how much of their portfolio should be assigned to each investment.[2]

The Kelly Criterion was invented by John Kelly, a researcher at Bell Labs who originally developed the formula to test long-distance telephone signal noise, and his method was published as "A New Interpretation of Information Rate" in 1956.[3]
Then the gambling community noticed its potential as an efficient betting system in horse racing, as it allowed them to maximize the size of their bankroll over the long run.

Suppose to repeat many times the same bet and define X_i the quantity of money won or lost for unit bet in the i-th round. Assume the following conditions on X_i:

- X_i assumes a finite number of values in $[-1, +\infty)$,
- X_i assumes the value -1 with positive probability, that is $\mathbb{P}(X_i = -1) > 0$;
- the single bet is superfair, that is $\mathbb{E}[X_i] > 0$.

[2]CFI Education, "Kelly Criterion", https://corporatefinanceinstitute.com/resources/data-science/kelly-criterion/

[3]Princeton University, "A New Interpretation of Information Rate", Pages 920-925.

A classical example of X_i is given by

$$X_i = \begin{cases} a\,, & \text{with probability } p\,, \\ 0\,, & \text{with probability } r\,, \\ -1\,, & \text{with probability } q \end{cases}$$

where $a, p, q > 0, \quad r \geq 0, \quad p + q + r = 1$ and $\mathbb{E}[X_i] = ap - q > 0$. If a gambler starts with a capital F_0 and F_n is their total capital at round n, by (2.8) we have

$$F_n = F_{n-1} + B_n X_n\,,$$

where B_n is the quantity of money that we bet at the n-th round that depends only on the outcomes of the bets of the previous rounds $(X_1, ..., X_{n-1})$. Since the bet is superfair, the gambler may think that is convenient to bet each time the entire capital; actually, this is not a smart strategy since it will lead to ruin with probability 1. Indeed, if the gambler bets the entire capital at each round, we have $B_n = F_{n-1}$ and hence

$$F_n = F_{n-1} + B_n X_n = F_n = F_{n-1} + F_{n-1} X_n = F_{n-1}(1 + X_n)\,.$$

If we iterate the above expression, we get

$$F_n = F_{n-1}(1 + X_n) = F_{n-2}(1 + X_{n-1})(1 + X_n) = ... = F_0 \cdot \prod_{i=1}^{n} (1 + X_i)\,.$$

So

$$\begin{aligned} &\mathbb{P}(F_n = 0) \\ &\qquad = \mathbb{P}(\exists i \in \{1, ..., n\} \text{ such that } 1 + X_i = 0) = \\ &\qquad = \mathbb{P}(\exists i \in \{1, ..., n\} \text{ such that } X_i = -1) = \\ &\qquad = 1 - \mathbb{P}(X_1 \neq -1, X_2 \neq -1, ..., X_n \neq -1) = \\ &\qquad \underset{\{X_i\}_i \text{indep}}{=} 1 - \prod_{i=1}^{n} \mathbb{P}(X_i = -1) = \\ &\qquad \underset{\{X_i\}_i \text{id. distrib.}}{=} 1 - [\mathbb{P}(X_i = -1)]^n. \end{aligned}$$

Hence as $n \to +\infty$, since $[\mathbb{P}(X_i = -1)]^n \to 0$ being $\mathbb{P}(X_i = -1) < 1$, we have

$$\mathbb{P}(F_n = 0) \to 0 \qquad \text{as } n \to \infty \, ,$$

that is, the gambler is ruined with probability 1 in the long run. A smarter idea is to bet a fraction f of the entire capital at each round, that is $B_n = f \cdot F_{n-1}$. Hence (2.8) becomes

$$F_n = F_{n-1} + B_n X_n = F_{n-1} + f \cdot F_{n-1} X_n = F_{n-1}(1 + f \cdot X_n) \, .$$

Iterating the above expression, we get

$$F_n = F_{n-1}(1 + f \cdot X_n) = F_{n-2}(1 + f \cdot X_{n-1})(1 + f \cdot X_n) = ... = F_0 \cdot \prod_{i=1}^{n} (1 + f \cdot X_i) \, .$$

So we have

$$F_{n(f)} = F_0 \cdot \prod_{i=1}^{n} (1 + f \cdot X_i) \, ,$$

where we have written $F_{n(f)}$ instead of F_n to highlight its dependence on f. To understand the behavior of F_n when $n \to \infty$ we would need to use some limit theorem such as the Law of Large Numbers[4] or the Central Limit Theorem.[5] However, these theorems work with sums of random variables and not with products, so we use the logarithmic function to transform the product in sum. Hence we have

$$F_{n(f)} = F_0 \cdot \prod_{i=1}^{n} (1 + f \cdot X_i) \Rightarrow \frac{F_{n(f)}}{F_0} = \prod_{i=1}^{n} (1 + f \cdot X_i) \Rightarrow$$
$$\Rightarrow \ln \frac{F_{n(f)}}{F_0} = \ln \left(\prod_{i=1}^{n} (1 + f \cdot X_i) \right) = \sum_{i=1}^{n} \ln(1 + f \cdot X_i)$$

where we have used the fact that for any $a, b > 0$ we have $\ln(ab) = \ln(a) + \ln(b)$. Define the random variable $Y_i = \ln(1 + f \cdot X_i)$ for $i \in \mathbb{N}{>}0$. So we can write

$$\ln \left(\frac{F_{n(f)}}{F_0} \right) = \sum_{i=1}^{n} \ln(1 + f \cdot X_i) = n \cdot \frac{\sum_{i=1}^{n} Y_i}{n} \Rightarrow \frac{1}{n} \ln \left(\frac{F_{n(f)}}{F_0} \right) = \frac{\sum_{i=1}^{n} Y_i}{n} \, .$$

[4]See Prop A.4 in Appendix for reference.

[5]See Prop A.5 in Appendix for reference.

Since $\{X_n\}_{n\in\mathbb{N}>0}$ are i.i.d., also the random variables $\{Y_n\}_{n\in\mathbb{N}>0}$ are i.i.d. and hence by the Law of Large Numbers we have

$$\frac{1}{n}\ln\left(\frac{F_{n(f)}}{F_0}\right) = \frac{\sum_{i=1}^{n} Y_i}{n} \underset{n\to+\infty}{\to} \mathbb{E}[Y_1] = \mathbb{E}[\ln(1+f\cdot X_1)]\,,$$

where the convergence is almost surely (that is with probability 1). So

$$r_n(f) := \frac{1}{n}\ln\left(\frac{F_{n(f)}}{F_0}\right) \underset{n\to+\infty}{\to} \mu(f) \qquad \text{a.s.},$$

where

$$\mu(f) := \mathbb{E}[\ln(1+f\cdot X_1)]\,,$$

and the symbol ":=" means is defined "as". The function $r_{n(f)}$ is the **rate of growth of the gambler's capital over the first n rounds**, while $\mu(f)$ is the **long-term rate of growth of the gambler‘s capital**. These definitions are justified by the expression of $F_{n(f)}$ in terms of $r_n(f)$ for fixed n and in terms of $\mu(f)$ in the long run. Indeed

$$r_n(f) := \frac{1}{n}\ln\left(\frac{F_{n(f)}}{F_0}\right) \Rightarrow \ln\left(\frac{F_{n(f)}}{F_0}\right) = n\cdot r_n(f) \Rightarrow \frac{F_{n(f)}}{F_0} = e^{n\cdot r_n(f)},$$

from which we deduce

$$F_n(f) = F_0\cdot e^{n\cdot r_n(f)}.$$

When n is large this expression leads to

$$F_n(f) \sim F_0\cdot e^{n\cdot\mu(f)}, \tag{2.14}$$

where the symbol "$\sim$" in this context means "behaves as". From (2.14) we see that in order to maximize the gambler‘s capital, we need to maximize $\mu(f)$. For this reason in the following result we investigate the behavior of the function $\mu(f)$.

Proposition 2.4.2. *Suppose that*

- X *assumes a finite number of values in* $[-1,+\infty)$;

- $\mathbb{P}(X = -1) > 0$;

- $\mathbb{E}[X] > 0$.

Then the function $\mu(f) = \mathbb{E}[\ln(1 + f \cdot X)]$ *satisfies the following properties:*

(i) $\mu(f)$ *is defined for* $f \in [0, 1)$;

(ii) $\mu(f)$ *is strictly concave in* $[0, 1)$;

(iii) there exists a unique $f^* \in (0, 1)$ *such that* $\mu(f)$ *achieves its maximum at* $f = f^*$ *that is*

$$\mu(f^*) = \max_{f \in [0,1)} \mu(f)$$

Such a point f^* *is the solution of the equation* $\mu'(f^*) = 0$, *where*

$$\mu'(f^*) = \mathbb{E}\left[\frac{d}{df} \ \ln(1 + f \cdot X)\right] = \mathbb{E}\left[\frac{X}{1 + f \cdot X}\right]$$

The fraction f^* *is called* ***Kelly's fraction of the bet****.*

(iv) there exists a unique $f_0 \in (0, 1)$ *such that* $\mu(f_0) = 0$*. Moreover*

- $f_0 > f^*$;

- $\mu(f) > 0$ *for* $f \in (0, f_0)$;

- $\mu(f) < 0$ *for* $f \in (f_0, 1)$.

Proof. Note that

$$\mu'(f^*) = \mathbb{E}\left[\frac{d}{df} \ln(1 + f \cdot X)\right] = \mathbb{E}\left[\frac{X}{1 + f \cdot X}\right],$$

$$\begin{aligned}\mu''(f^*) = \frac{d}{df}\mu'(f^*) = \frac{d}{df}\mathbb{E}\left[\frac{X}{1 + f \cdot X}\right] = \\ = \mathbb{E}\left[\frac{d}{df}\frac{X}{1 + f \cdot X}\right] = -\mathbb{E}\left[\frac{X^2}{(1 + f \cdot X)^2}\right] < 0 \qquad \forall f \in [0, 1).\end{aligned}$$

Since $\mu''(f) < 0$, then $\mu(f)$ is strictly concave. Moreover $\mu'(0) = \mathbb{E}[X] > 0$. Since $X = -1$ with positive probability,

$$\lim_{f \to 1^-} \mu(f) = \mathbb{E}[\ln(1+X)] = \sum_k \ln(1+k) \cdot \mathbb{P}(X = k) =$$
$$= -\infty \cdot \mathbb{P}(X = -1) + \sum_{k \neq -1} \ln(1+k) \cdot \mathbb{P}(X = k) = -\infty\,,$$

$$\lim_{f \to 1^-} \mu'(f) = \mathbb{E}\left[\frac{X}{1+X}\right] = \sum_k \frac{k}{1+k} \cdot \mathbb{P}(X = k) =$$
$$= -\infty \cdot \mathbb{P}(X = -1) + \sum_{k \neq -1} \frac{k}{1+k} \cdot \mathbb{P}(X = k) = -\infty\,.$$

The summations in blue are finite numbers since X can assume only finitely many values. So, since $\mu'(0) > 0$ and $\lim_{f \to 1^-} \mu'(f) < 0$, we have that there exists $f^* \in (0,1)$ such that $\mu'(f^*) = 0$. Moreover, since $\mu''(f) < 0$, we have that f^* is a global point of maximum. Since $\mu(0) = 0$ and $\mu'(0) > 0$, then for small f we have $\mu(f) > 0$. Since f^* is a point of global maximum, we have $\mu(f^*) > 0$. Moreover, since $\lim_{f \to 1^-} \mu(f) = -\infty$, we have that there exists $f_0 \in (f^*, 1)$ such that $\mu(f_0) = 0$. Note also that $\mu(f) > 0$ if $f \in (0, f_0)$ and $\mu(f) < 0$ if $f \in (f_0, 1)$. Since $\mu(f)$ is strictly concave, f_0 and f^* are unique and, to maximize $F_n(f)$ in the long run, the gambler needs to bet each time the fraction f^* of his capital. The betting system in which $B_n = f^* \cdot F_{n-1}$ for all $n \in \mathbb{N}{>}0$ is called **Kelly System**. If the gambler uses a fraction $\widetilde{f} = f^*$, then if $\widetilde{f} \in (0, f_0)$ we have $\mu(\widetilde{f}) > 0$ and hence $F_n(\widetilde{f})$ goes to ∞ in the long run by (2.14) (we have an exponential with a positive exponent going to $+\infty$), while if $\widetilde{f} \in (f_0, f)$ we have $\mu(\widetilde{f}) < 0$ and hence $F_n(\widetilde{f})$ goes to zero in the long run by (2.14) (we have an exponential with a negative exponent going to $-\infty$). Hence we have the following result.

Proposition 2.4.3. *Suppose that*

- *X assumes a finite number of values in $[-1, +\infty)$;*

- $\mathbb{P}(X = -1) > 0$;
- $\mathbb{E}[X] > 0$.

Then

(i) fixed $f \in [0,1)$, *we have*

$$\lim_{n \to +\infty} \left(\frac{F_n}{F_0}\right)^{\frac{1}{n}} = e^{\mu(f)} \qquad a.s.;$$

(ii) if $\mu(f) > 0$, *then*

$$\lim_{n \to +\infty} F_n(f) = +\infty \qquad a.s.;$$

(iii) if $\mu(f) < 0$, *then*

$$\lim_{n \to +\infty} F_n(f) = 0 \qquad a.s.;$$

(iv) if $f \in [0,1)$ *with* $\widetilde{f} \neq f^*$, *then*

$$\lim_{n \to +\infty} \left(\frac{F_n(\widetilde{f})}{F_n(f)}\right) = +\infty \qquad a.s.;$$

(v) if $\sigma^2(f) := Var(\ln(1 + f \cdot X)) > 0$, *then*

$$\frac{\sqrt{n}}{\sigma(f)} \left(\frac{1}{n} \ln\left(\frac{F_n(f)}{F_0}\right) - \mu(f)\right) \xrightarrow{d} \mathcal{N}(0,1).$$

Proof. Item (i) has already been proved in (2.14), while for items (ii) and (iii) it is enough to compute the limit for $n \to \infty$ in (2.14). As far as item (iii), by (2.14) we have

$$\left(\frac{F_n(\widetilde{f})}{F_n(f)}\right) \sim \left(\frac{F_0 e^{n\mu(f^*)}}{F_0 e^{n\mu(f)}}\right) = e^{n(\mu(f^*) - \mu(f))} \underset{n \to +\infty}{\to} +\infty,$$

since $\mu(f^*) > \mu(f)$ for all $f \in [0,1) \setminus \{f^*\}$ (being $\mu(f^*)$ the maximum of $\mu(f)$ in $[0,1)$). Finally, item (iv) is an application of the Central Limit Theorem.

Proposition 2.4.2 and Proposition 2.4.3 form the so called **Kelly Criterion**.

Remark 2. *If $\mu(f) = 0$, it is possible to prove that $F_n(f)$ has no almost sure limit. Moreover from item (v), defining $Z \sim \mathcal{N}(0,1)$, we have that for n large*

$$\frac{\sqrt{n}}{\sigma(f)}\left(\frac{1}{n}\ln\left(\frac{F_n(f)}{F_0}\right) - \mu(f)\right) \overset{d}{\approx} Z\,,$$

where the symbol $\overset{d}{\approx}$ means "is approximated in distribution by". By inverting such a relation we get that for n large

$$F_n(f) \overset{d}{\approx} F_0 e^{n\mu(f)+\sigma(f)\sqrt{n}Z}.$$

This relation allows us to construct a confidence interval for $F_n(f)$. Indeed, if we set $\alpha > 0$, we can define $z_{1-\alpha} \in \mathbb{R}$ as the $(1-\alpha)-$quantile of the standard normal distribution, that is the value such that

$$\mathbb{P}(Z \le z_{1-\alpha}) = \Phi(z_{1-\alpha}) = 1 - \alpha\,.$$

Then, defining

$$L_n(f,\alpha) := F_0 \cdot e^{\mu(f)n - z_{1-\alpha}\sigma(f)\sqrt{n}}, \qquad U_n(f,\alpha) := F_0 \cdot e^{\mu(f)n + z_{1-\alpha}\sigma(f)\sqrt{n}},$$

we have that

$$\lim_{n\to+\infty}\mathbb{P}\left(F_n(f) \in \left[L_n\left(f,\frac{\alpha}{2}\right), U_n\left(f,\frac{\alpha}{2}\right)\right]\right) = 1 - \alpha\,.$$

Then for n large $F_n(f)$ belongs to the interval $\left[L_n\left(f,\frac{\alpha}{2}\right), U_n\left(f,\frac{\alpha}{2}\right)\right]$ with probability approximately $1-\alpha$, which is the $100\cdot(1-\alpha)\%$ ***prediction interval*** *for $F_n(f)$.*

Example 2.5. *Let us consider a bet in which gambler's profit for unit bet is*

$$X = \begin{cases} 2\,, & \text{with probability } \frac{1}{6}\,,\\ 1\,, & \text{with probability } \frac{1}{4}\,,\\ 0\,, & \text{with probability } \frac{1}{3}\,,\\ -1\,, & \text{with probability } \frac{1}{4}\,. \end{cases}$$

Note that X assumes a finite number of values (Im(X) contains only 4 values), $\mathbb{P}(X=-1)>0$ and $\mathbb{E}[X]=13>0$. Suppose to repeat the same bet many times: by Proposition 2.4.2, we know that there exists an optimal fraction $f^*\in(0,1)$ (the Kelly's fraction of the bet) such that the gambler's total capital $F_n(f)$ is maximized in the long run if the gambler bets exactly a portion f^* of their capital at each round. To find its value, we have to solve $\mu'(f)=0$. So we have

$$\begin{aligned}\mu(f)=\mathbb{E}[\ln(1+fx)]&=\frac{1}{6}\ln(1+2f)+\frac{1}{4}\ln(1+f)+\frac{1}{3}\ln 1+\frac{1}{4}\ln(1-f)=\\&=\frac{1}{6}\ln(1+2f)+\frac{1}{4}\ln(1+f)+\frac{1}{4}\ln(1-f)\,.\end{aligned}$$

Therefore

$$\begin{aligned}\mu'(f)&=\frac{1}{6}\cdot\frac{2}{1+2f}+\frac{1}{4}\cdot\frac{1}{1+f}-\frac{1}{4}\cdot\frac{1}{1-f}=\\&=\frac{4(1-f)(1+f)+3(1-f)(1+2f)-3(1+f)(1+2f)}{12(1+2f)(1+f)(1-f)}=\\&=\frac{4-4f^2+3(1+f-2f^2)-3(1+3f+2f^2)}{12(1+2f)(1+f)(1-f)}=\\&=\frac{-16f^2-6f+4}{12(1+2f)(1+f)(1-f)}\,.\end{aligned}$$

We have to solve $\mu'(f)=0$, so we have

$$\begin{aligned}&\mu'(f)=\frac{-16f^2-6f+4}{12(1+2f)(1+f)(1-f)}=0\Rightarrow -16f^2-6f+4=0\Rightarrow\\&\Rightarrow 8f^2+3f-2=0\Rightarrow f_{1,2}=\frac{-3\pm\sqrt{73}}{16}\,.\end{aligned}$$

Recall that $f^*\in(0,1)$ and hence we can take only the positive solution, that is

$$f^*=\frac{-3+\sqrt{73}}{16}\approx 0.35\,.$$

So the gambler should bet around 35% of their total capital at each round to optimize their total capital in the long run.

Kelly Criterion for Bets with Three Outcomes: Win, Loss and Tie

Let the probabilities of a win, loss and tie be respectively $p > 0$, $q > 0$ and $r \geq 0$, with $p + q + r = 1$. Suppose that a win pays a to 1, where $a > 0$. Then

$$\mathbb{P}(X = a) = p\,, \qquad \mathbb{P}(X = -1) = q\,, \qquad P(X = 0) = r\,.$$

Assume that $\mathbb{E}[X] = ap - q > 0$. By Proposition 2.4.2 we know that there exists the Kelly fraction of the bet. Let us compute $\mu(f)$

$$\mu(f) = \mathbb{E}[\ln(1 + f \cdot X)] = \ln(1 + f \cdot a) \cdot p + \ln(1 - f) \cdot q\,.$$

To compute f^* we need to solve $\mu'(f) = 0$. Note that

$$\mu'(f) = \frac{ap}{1 + f \cdot a} - \frac{q}{1 - f} = \frac{ap - apf - q - qaf}{(1 + af)(1 - f)} = 0 \Rightarrow f = \frac{ap - q}{a \cdot (p + q)}\,.$$

Therefore

$$\begin{aligned} f^* &= \frac{ap - q}{a \cdot (p + q)} = \frac{\mathbb{E}[X]}{a \cdot \mathbb{P}(X \neq 0)} = \frac{\mathbb{E}[X\mathbf{1}_{\{X \neq 0\}}] + \mathbb{E}[X\mathbf{1}_{\{X = 0\}}]}{a \cdot \mathbb{P}(X \neq 0)} = \\ &= \frac{\mathbb{E}[X\mathbf{1}_{\{X \neq 0\}}] + 0}{a \cdot \mathbb{P}(X \neq 0)} = \frac{\mathbb{E}[X\mathbf{1}_{\{X \neq 0\}}]}{a \cdot \mathbb{P}(X \neq 0)} = \frac{\mathbb{E}[X | X \neq 0]}{a}\,. \end{aligned} \tag{2.15}$$

We can also compute $\sigma^2(f) = \mathrm{Var}(\ln(1 + f \cdot X))$. We need to calculate $\mathbb{E}[(\ln(1 + f \cdot X))^2]$:

$$\mathbb{E}[(\ln(1 + f \cdot X))^2] = (\ln(1 + f \cdot a))^2 \cdot p + (\ln(1 - f))^2 \cdot q\,.$$

Hence we have

$$\begin{aligned} \sigma^2(f) &= \mathrm{Var}(\ln(1 + f \cdot X)) = \\ &= \mathbb{E}[(\ln(1 + f \cdot X))^2] - \mathbb{E}[\ln(1 + f \cdot X)]^2 = \\ &= (\ln(1 + f \cdot a))^2 \cdot p + (\ln(1 - f))^2 \cdot q - (\ln(1 + f \cdot a) \cdot p + \ln(1 - f) \cdot q)^2 = \\ &= (\ln(1 + f \cdot a))^2 \cdot (p - p^2) + (\ln(1 - f))^2 \cdot (q - q^2) - 2pq \ln(1 + f \cdot a) \ln(1 - f) = \\ &= p(1 - p)(\ln(1 + f \cdot a))^2 + q(1 - q)(\ln(1 - f))^2 - 2pq \ln(1 + f \cdot a) \ln(1 - f)\,. \end{aligned}$$

If $r = 0$, the expression of $\sigma^2(f)$ simplifies. Indeed, since $q = 1 - p$, we have

$$\sigma^2(f) = p(1-p)(\ln(1+f\cdot a))^2 + q(1-q)(\ln(1-f))^2 - 2pq\ln(1+f\cdot a)\ln(1-f) \overset{q=1-p}{=}$$
$$\overset{q=1-p}{=} p(1-p)[(\ln(1+f\cdot a))^2 + (\ln(1-f))^2] - 2p(1-p)\ln(1+f\cdot a)\ln(1-f) =$$
$$= p(1-p)[\ln(1+f\cdot a) - \ln(1-f)]^2 =$$
$$= p(1-p)\left[\ln\frac{(1+f\cdot a)}{1-f}\right]^2 .$$

Kelly Criterion in Sports Betting

Suppose that team A and team B are playing a match and the possible outcomes are three: team A wins, team B wins or there is a tie. When you bet on this match you bet on its outcome, for example team A wins: in such a case, winning the bet means that team A wins, while losing it means that team A loses or there is a tie. If you win your bet, you will be awarded with a quantity of money, called *odds*, established by the *bookmaker*. Odds are usually given for unit bet, so if you bet 1 and the odds are V, then your gain is $V - 1$ (you have to subtract the amount of money that you have bet). Hence the outcome of the bet is modeled by

$$X = \begin{cases} V-1\,, & \text{with probability } p\,, \\ -1\,, & \text{with probability } 1-p\,, \end{cases}$$

where $p = \mathbb{P}(\text{team A wins})$. Kelly criterion is used in this context as a strategy to find the optimal fraction of capital to bet. Since this is a bet with outcomes win or loss, we can use (2.15) to compute f^* ($r = 0$ and $q = 1 - p$). So we have

$$f^* = \frac{(V-1)p - (1-p)}{(V-1)(p+1-p)} = \frac{Vp-1}{V-1} .$$

The value of p is given by the gambler's perception of the probability of the outcome of the match on which he is betting. For example suppose that the odds for "team A wins" are 3.5 and that you expect that team A wins with probability 0.5. Then the optimal fraction of capital to bet is

$$f^* = \frac{3.5 \cdot 0.5 - 1}{3.5 - 1} = 0.3\,.$$

So if our capital is 100, then we should bet 30.

The idea used in sports betting is to apply this method in sequence to many bets. So we start with a capital F_0 and we bet on the first match computing the optimal fraction; then our capital will be F_1 and we bet on the second match (in general different from the first one) computing the optimal fraction (that may differ from the first one). Then our capital will be F_2 and we continue this procedure until we want to stop. This method is safe since we will never go broke as the optimal fraction is always smaller than 1.

Remark 3. *The above method has meaning if* $p > \frac{1}{V}$ *, otherwise we have* $f^* \leq 0$ *.*

Kelly Criterion in Investing

The Kelly Criterion applied to investing features two main components: the first one is the win probability, or the odds that the trade will have a positive return, while the second one is the win/loss ratio (the number of positive trades over the number of negative trades). These two coefficients are then placed into Kelly's equation, which is:

$$f^* = w - \frac{(1-w)}{R}$$

where:

- f^* is the Kelly fraction;
- w is the probability of winning;
- R is the win/loss ratio.

Investors can calculate the Kelly's fraction by following these simple steps:

1. Access your last 50/60 trades, assuming that your trading habits are the same as they were in the past.

2. Calculate the probability of winning w by dividing the number of trades that returned a positive amount by your total number of trades. This value gets better as it gets closer to 1 but any number above 0.50 is fine.

3. Calculate the win/loss ratio R by dividing the average gain of the positive trades by the average loss of the negative trades. You will get a number bigger than 1 if your average gains are larger than your average losses but a result smaller than 1 is treatable until the number of losing trades stays small.

4. Input these figures into Kelly's equation.

5. Register the Kelly percentage that the equation returns.[6]

Interpretation of the Results

The percentage produced by the equation is a number smaller than 1 representing the size of the positions you should be taking, to let you know how much you should diversify. For instance, if the Kelly's fraction is 0.18 you should take a 18% position in every equity in your portfolio. However, this system requires some common sense: you have to keep in mind that, regardless of what the Kelly percentage is, you should never commit more than 25% of your capital to one single equity as allocating any more than that involves way more investment risk than you should be bearing.[7]

[6] Justin Kuepper, "Using the Kelly Criterion for Asset Allocation and Money Management", Investopedia.com, November 30th 2023, https://www.investopedia.com/articles/trading/04/091504.asp.

[7] To manage investment risk, the FINRA suggests: "Don't put all your eggs in one basket.", https://www.finra.org/investors/investing/investing-basics/asset-allocation-diversification.

An Application to the U.S. Stock Market

Investing in the stock market may be seen as a continuous gambling game with a positive, one-year expected return equal to the average of the historical annual returns over a sufficiently long time span, implying that only stationary processes are involved. To a reasonable first approximation, evidence suggests that changes in the price level in speculative markets behave like i.i.d random variables with finite variances. From the Central Limit Theorem, it would then follow that price changes in U.S. stocks are approximately normal; the lognormal distribution would be a better fit but the computations would be much more onerous to discuss.

To an investor, what constitutes a profit over an extended period of time is complicated by the time-changing purchasing power of money and other factors such as brokerage commissions, taxes and the perceived risk of the transaction. Since time is very important, an actual annual percentage return has little meaning unless compared to the inflation rate or some proxy such as T-bill rates. Historical annual excess returns[8] have been found to be relatively stable and thus the normal distribution is a reasonable approximation.

Let us consider a period of n years, where the distribution of annual excess total returns on some stocks has a defined mean μ and standard deviation σ. Each return in the calculation is expressed as the natural logarithm of one plus the annual excess return ER_i. In formulas we have:

$$\mu = \frac{1}{n}\sum_{i=1}^{n}\log(1+ER_i) = \log\left[\prod_{i=1}^{n}(1+ER_i)\right]^{\frac{1}{n}}; \tag{2.16}$$

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left[\log(1+ER_i)-\mu\right]}{n-1}. \tag{2.17}$$

Various interesting probability calculations are possible if annual excess returns are assumed to be independently distributed. It would entail, for example, that the mean and standard deviation of an n-year forecast of annual excess returns

[8]Annual total returns on common stock in excess of Treasury bill returns.

would be $\overline{x} = \mu$ and $s_n = \sigma/\sqrt{n}$. With a fixed amount invested in stocks over an n-year period, the probability for a negative excess return would be

$$\mathbb{P}\left(t < \frac{0 - \overline{x}}{\sigma/\sqrt{n}}\right).$$

Estimating the Kelly's fraction for a Long-term Investment in S&P 500 Stocks

Suppose we have an initial amount of investment capital F_0 and we want to determine the optimal wager-fraction f^* to invest each year in S&P 500 stocks, knowing that $\mu = 0.058$ and $\sigma = 0.2160$. Using an unaltered normal curve for our probability distribution is inadequate for two reasons: first, the normal distribution allows for infinitely large annual excess percentage declines/advances in stocks (unrealistic on both sides); secondly, the Kelly criterion will not yield a meaningful $f^* > 0$ if the probability distribution $F(x)$ suggests a negatively infinite lower limit of the integral

$$\int_a^{\infty} \log(1 + fx)\, dF(x)\,.$$

Therefore, in order for the excess return variable x to be meaningful on the interval $A \le x \le B$, where $A = \mu + 3\sigma = -0.590$ and $B = \mu - 3\sigma = 0.706$, we estimate it using a quasi-normal probability distribution.

$$N(x) = \begin{cases} h + \frac{1}{\sqrt{2\pi\alpha^2}}\, e^{-\frac{(x-\mu)^2}{2\alpha^2}}\,, & A \le x \le B \\ 0\,, & x < A \\ 0\,, & x > B. \end{cases} \tag{2.18}$$

Calculations were done using a microcomputer and integrations were approximated with Simpson's Rule[9] using $n = 1000$ and $\pi = 3.1415926535$.

The value of h had to be chosen so that $\int_A^B N(x)\, dx = 1$ and we found that $h = (1 - 0.997006378)/(B - A)$ is the correction term needed to delimit the standard

[9]See Proposition A.7 in Appendix for reference.

normal curve. At the same time, we also wanted the probability distribution model in 2.18 to have a standard deviation of $\sigma = 0.2160$, where $\sigma^2 = \int_A^B x^2 N(x)\,dx - \mu^2$. To achieve this, the value of α must be $\alpha = 0.2183$. With these adjustments, the distribution $N(x)$ has a mean of 0.058 and a standard deviation of 0.2160 as required. We now need to find the value of f, where $0 < f < -1/A$, so that the following integral is a maximum:

$$\begin{aligned} G(f) &= \int_A^B \log(1+fx)\,dN(x) = \\ &= \int_A^B (\log(1+fx)) \left[h + \frac{1}{\sqrt{2\pi\alpha^2}}\, e^{-\frac{(x-\mu)^2}{2\alpha^2}}\right] dx. \end{aligned} \tag{2.19}$$

The integration needed to set $G''(f) = 0$ cannot be computed explicitly. If we use a microcomputer to calculate it we find that the maximum of $G(f)$ is found when $f^* = 1.17$ and the growth coefficient is $G(f) = 0.0350471$. The mean of the distribution is positive and if we differentiate $G(f)$ with respect to f and observe the terms in the integrand, we get that

$$\lim_{f \to (-1/A)^-} G'(f) = -\infty\,;$$

and we can state that the uniqueness of f^* is guaranteed by the following theorem:

Theorem 2.4.4. *If the mean $\mu = \int_a^\infty x\,dF(x) > 0$, then the function*

$$G(f) = \int_a^\infty log(1+fx)\,dF(x)$$

attains a unique maximum value $G(f^*)$ where $f^* \in (0, -1/a)$ if

$$\lim_{f \to (-1/a)^-} G'(f) < 0\,,$$

with $-\infty < a < 0$ and defined as $a = \sup\{x : F(-\infty, x) = 0\}$.

Proof. First note that if $1 + fa > 0$, the integral $G(f) = \int_a^\infty log(1+fx)\,dF(x)$ is defined. Plus,

$$G''(f) = \int_a^\infty \frac{-fx}{(1+fx)^2}\,dF(x) < 0$$

so that

$$G'(f) = \int_a^\infty \frac{x}{1+fx}\,dF(x)$$

is monotone strictly decreasing on $[0, -1/a)$. Notice that $G(0) = 0$ and we have that $G'(0) = \int_a^\infty x\,dF(x) = \mu > 0$ and $\lim_{f\to(-1/a)^-} G'(f) < 0$ by hypothesis. From the monotonicity and continuity of $G'(f)$ on $[0, -1/a)$ it follows that $G'(f)$ takes on all values on the interval $[G('0), \lim_{f\to(-1/a)^-} G'(f))$ exactly once and thus $G(f)$ has a unique maximum at $f = f^*$, where $0 < f^* < -1/a$.

Remark 4. *Observe that if $a \to -\infty$, then $f^* \to 0$ so that the Kelly criterion applied to continuous distribution models will yield non-trivial results only if the lower limit of the integral $\int_a^\infty log(1+fx)\,dF(x)$ is finite.*

Considering the time value of money (disregarding taxes and transaction fees) each year the Kelly-optimal investor should be willing to invest up to 100% of their resources in a diversified portfolio of S&P 500 stocks if no margin is permitted. However, maximal average real growth will occur (if the margin at the T-bill rate is available) if they invest 117% of their current resources. Hence the long-term investor should invest all their capital plus borrow an additional 17% to invest if they aim to achieve the maximal average growth relative to T-bills.[10]
We would like to know if $G(f) = 0$ is in the interval $(0, -1/A)$ in order to have some information about the **chaotic run point** f_c.[11]
We cannot examine the limit

$$L = \lim_{f\to(-1/A)^-} \int_A^B log(1+fx)N(x)\,d(x)$$

[10]In real life margin costs exceed T-bill rates and, if computed including the extra costs, this percentage would be smaller.
[11]The point beyond which margin become excessive and leads to the investor's ruin (probability of loss relative to T-bills equal to 1).

directly but we can get its upper bound:

$$M = \max(N(x)) = h + \frac{1}{\sqrt{2\pi\alpha^2}} \quad \text{on } [A, B]\,,$$

so

$$\begin{aligned} L &\leq \lim_{f \to (-1/A)^-} \int_A^B log(1 + fx) M\, d(x) = \\ &= M \lim_{f \to (-1/A)^-} \left[\left(x + \frac{1}{f} \right) (\log(1 + fx) - x) \right]_A^B = \\ &= M \left[A - B + (B - A) \log \left(1 - \frac{B}{A} \right) \right] = -\,0.51 < 0\,. \end{aligned}$$

Therefore $G(f) = 0$ has a unique solution $f_c \in (0, -1/A)$. Since the slope of $G(f)$ is very steep close to the point $f = -1/A$, locating f_c very accurately gets extremely difficult; computer runs estimate its value to be very close to $-1/A$, more precisely $f_c = 1.69^+$. Hence, if a hypothetically immortal investor costantly wagers an amount greater than 1.7 times their current resources, ruin is certain.
Before investing all their capital in stocks, there are some caveats that investors should keep in mind, namely possible losses relative to T-bills in the short run.
Lastly, it can be discussed that the artificially built probability distribution $N(x)$ might not fully consider:

(i) recently expanded stock market volatility caused by program trading and the internationalization of financial markets;

(ii) some catastrophic exogenous events that may possibly happen (a heavy global recession or an earthquake).

Keep in mind that the numerical results obtained must be interpreted keeping in mind the constraints featuring any probabilistic model.[14]

Is the Kelly Criterion Effective?

This system is built on pure mathematics. However, you may question if this math, originally thought for telephones, is applicable to gambling scenarios and stock investing. Actually, this system can be proven effective by displaying the simulated growth of a specific account through the use of an equity chart, assuming the two variables are properly plugged in and that the investor is able to sustain such performance.[12]

Nevertheless, there is no such thing as a perfect money management system; the Kelly criterion can assist you to achieve an efficient portfolio diversification but it has its limits, as it can not select winning stocks or predict sudden market crashes. Therefore, some luck and randomness will always be present in the markets and can alter investment returns. Furthermore, scholars suggests that the Kelly Criterion may be risky in the short run because it can advise placing a significant share of capital as first investment.[13]

2.5 Gambler's Ruin

In this setting we are interested in finding the probability that, in an independent sequence of identical bets with even-money payoffs, the gambler loses $L > 0$ or more euro before the opponent wins $W > 0$ euro. By the bet pays **even-money** we mean that in each bet the quantity of money that the gambler can win equals the quantity of money that the gambler can lose.

Assume that in each round the gambler can win/lose 1 euro and define X_i as the

[12] Justin Kuepper, "Using the Kelly Criterion for Asset Allocation and Money Management", Investopedia.com, November 30th 2023, https://www.investopedia.com/articles/trading/04/091504.asp.

[13] University of California, Berkeley, "Good and Bad Properties of the Kelly Criterion", Page 1, January 1st 2010.

quantity of money that the gambler wins in the i-th bet. So

$$\mathbb{P}(X_i = 1) = p\,, \qquad \mathbb{P}(X_i = -1) = q\,, \qquad \mathbb{P}(X_i = 0) = r\,,$$

where $p, q > 0$, $r \geq 0$ and $p + q + r = 1$. Denote by S_n the total quantity of money that the gambler wins in the first n bets, that is

$$S_n = X_1 + ... + X_n\,.$$

Assume $S_0 = 0$, fix W, L positive integers and assume that the gambler stops betting as soon as he wins W euro or lose L euro. Define

$$N(-L, W) := \min\{n \geq 1 | S_n = -L \text{ or } S_n = W\}$$

the index of the last round in which the gambler bets (we assume *min*$:= +\infty$). Note that $N(-L, W)$ is a stopping time since to establish the occurrence of the event $\{N(-L, W) = k\}$ we need to know the values of $X_k, ..., X_k$. It is possible to prove the following result:

Proposition 2.5.1.

$$\mathbb{P}(N(-L, W) < \infty) = 1\,.$$

Since $N(-L, W) < \infty$, we can try to study $S_{N(-L,W)}$ using the Optional Stopping Theorem. In particular we have this result, known as ***Gambler's Ruin.***

Theorem 2.5.2. *If $p \neq q$ we have*

$$\mathbb{P}(S_{N(-L,W)} = W) = \frac{(q/p)^L - 1}{(q/p)^{W+L} - 1} = 1 - \mathbb{P}(S_{N(-L,W)} = -L)\,. \qquad (2.20)$$

If $p = q$ we get

$$\mathbb{P}(S_{N(-L,W)} = W) = \frac{L}{L + W} = 1 - \mathbb{P}(S_{N(-L,W)} = -L)\,. \qquad (2.21)$$

Proof. We divide the proof into two parts: the first one devoted to the case $p \neq q$, while the second one considers the case $p = q$. Let us start considering $p \neq q$. In Example 2.3 we have proved that the stochastic process $\{K_n\}_{n \in \mathbb{N}>0}$ with

$$K_n = \left(\frac{q}{p}\right)^{X_1+\ldots+X_n} \cdot K_0 = \left(\frac{q}{p}\right)^{S_n} \cdot K_0\,,$$

is a martingale with respect to the filtration $\{X_n\}_{n \in \mathbb{N}>0}$. If $K_0 = 1$, then

$$\mathbb{E}[K_n] = \mathbb{E}[K_0] = 1\,.$$

Since by Proposition 2.5.1 $\mathbb{P}(N(-L, W) < \infty) = 1$ and $K_n \geq 0$ for all $n \in \mathbb{N}>0$, the fourth hypothesis of the Optional Stopping Theorem (Proposition 2.3.2) is verified and hence we can conclude that

$$\mathbb{E}[K_{N(-L,W)}] = \mathbb{E}[K_0] = 1\,. \tag{2.22}$$

Since

$$K_{N(-L,W)} = \left(\frac{q}{p}\right)^{S_{N(-L,W)}}$$

and since $N(-L, W)$ assumes only the values W and $-L$ by definition we have

$$\begin{aligned}
&= \mathbb{E}[K_{N(-L,W)}] = \mathbb{E}\left[(q/p)^{S_{N(-L,W)}}\right] = \\
&= (q/p)^{-L}\mathbb{P}(S_{N(-L,W)} = -L) + (q/p)^{W}\mathbb{P}(S_{N(-L,W)} = W) = \\
&= (q/p)^{-L}(1 - \mathbb{P}(S_{N(-L,W)} = W)) + (q/p)^{W}\mathbb{P}(S_{N(-L,W)} = W) = \\
&= (q/p)^{-L} + \left[(q/p)^{W} - (q/p)^{-L}\right]\mathbb{P}(S_{N(-L,W)} = W)\,.
\end{aligned}$$

Since $\mathbb{E}[K_{N(-L,W)}] = 1$ by (2.22), by the above identity we have

$$1 = (q/p)^{-L} + \left[(q/p)^{W} - (q/p)^{-L}\right]\mathbb{P}(S_{N(-L,W)} = W)\,,$$

from which we get

$$\mathbb{P}(S_{N(-L,W)} = W) = \frac{1 - (q/p)^{-L}}{(q/p)^{W} - (q/p)^{-L}} = \frac{(q/p)^{L} - 1}{(q/p)^{W+L} - 1}\,,$$

where in the last identity we have multiplied both numerator and denominator by $(q/p)^L$. So we have proved the statement when $p \neq q$.
Let us consider now the case $p = q$. We cannot use again the process $\{K_n\}_{n\in\mathbb{N}>0}$ since, when $p = q$, $K_n = 1$ for all $n \in \mathbb{N}>0$. In this case we consider $S_n = \sum_{i=1}^n X_i$. Since $\{X_n\}_{n\in\mathbb{N}>0}$ is a sequence of identically distributed random variables and $N(-L, W)$ is a random variable independent on $\{X_n\}_{n\in\mathbb{N}>0}$, where $\mathbb{E}[X_1] < \infty$ and $\mathbb{E}[N(-L, W)] < \infty$, then

$$\mathbb{E}[S_{N(-L,W)}] = \mathbb{E}[N(-L, W)]\mathbb{E}[X_1] = 0\,, \tag{2.23}$$

where we have used the fact that $\mathbb{E}[X_1] = 0$. Since $S_{N(-L,W)}$ assumes only the values $-L$ and W by definition, we have that

$$\begin{aligned} E[S_{N(-L,W)}] &= W\mathbb{P}(S_{N(-L,W)} = W) - L\mathbb{P}(S_{N(-L,W)} = -L) = \\ &= W\mathbb{P}(S_{N(-L,W)} = W) - L(1 - \mathbb{P}(S_{N(-L,W)} = W)) = \\ &= -L + (W + L)\mathbb{P}(S_{N(-L,W)} = W)\,. \end{aligned}$$

By (2.23), the above identity implies

$$-L + (W + L)\mathbb{P}(S_{N(-L,W)} = W) = 0\,,$$

that is

$$\mathbb{P}(S_{N(-L,W)} = W) = \frac{L}{W + L}\,.$$

Therefore we have proved the statement when $p = q$. Let us apply Theorem 2.5.2 to the European Roulette. Suppose to bet 1 euro on even numbers, so $p = \frac{18}{37}$ and $q = \frac{19}{37}$. So, by Theorem 2.5.2, the probability of winning W euro before losing L euro is given by

$$\frac{\left(\frac{19}{18}\right)^L - 1}{\left(\frac{19}{18}\right)^{W+L} - 1}\,.$$

So if we consider the probability of winning 5 before losing 4, we get

$$\frac{\left(\frac{19}{18}\right)^4 - 1}{\left(\frac{19}{18}\right)^9 - 1} \approx 0.39\,.$$

So with a probability around 0.39 the gambler wins 5 before losing 4, while with a probability around $1 - 0.39 = 0.61$ the gambler loses 4 before winning 5.

In the next chapter we cover in detail the concept of house advantage, which we have mentioned also in Chapter 1, and understand its application in many different gambling scenarios. Furthermore, we define some very useful tools to measure the convenience of a bet based on its variance, namely the volatility of a wager and the expected loss per standard deviation.

Chapter 3

House Advantage

In this chapter we will develop instruments applicable only to games in which we deal with theoretical probabilities (like roulette), while games dealing with experimental or subjective probabilities (such as sport games) are not considered here. These instruments will be useful to evaluate the "quality of a game", meaning we will be able quantify how favorable a game is.

3.1 House Advantage in a Single Wager

A **wager** (or bet) is described by a pair (B, X), where:

- B, X are jointly distributed random variables;
- B denotes the amount bet;
- X denotes the gambler's profit (positive, negative or zero).

We impose that the gambler cannot lose more than one bet and we assume that:

- $\mathbb{E}[B] < \infty$;
- $\mathbb{E}[B\mathbf{1}_{\{X \neq 0\}}] = \mathbb{E}[B|X \neq 0] \cdot \mathbb{P}(X \neq 0) > 0$;

- $\mathbb{E}[|X|] < \infty$.

Definition 3.1. *There are two accepted definitions of* ***house advantage*** *(or equivalently* ***house edge****) of the wager* (B, X)*. The first one is*

$$H_0(B, X) := \frac{-\mathbb{E}[X]}{\mathbb{E}[B]} = \frac{-\mathbb{E}[X]}{\mathbb{E}[B|X \neq 0] \cdot \mathbb{P}(X \neq 0) + \mathbb{E}[B|X = 0] \cdot \mathbb{P}(X = 0)}$$

while the second one is

$$H(B, X) := \frac{-\mathbb{E}[X]}{\mathbb{E}[B \boldsymbol{1}_{\{X=0\}}]} .$$

Notice that

- $\mathbb{E}[X]$ is the gambler's expected profit. Since in casino games it is usually negative, then $-\mathbb{E}[X]$ represents the gambler's expected loss;
- $\mathbb{E}[B]$ is the gambler's expected amount bet;
- $\mathbb{E}[B\mathbf{1}_{\{X=0\}}] = \mathbb{E}[B|X \neq 0] \cdot \mathbb{P}(X \neq 0) > 0$ is the gambler's **expected amount of action**, meaning the gambler's expected amount bet except in case of a push, which provides no action.

Proposition 3.1.1. *If* X *is directly proportional to* B*, that is* $\exists c \in \mathbb{R}$ *such that* $X = c \cdot B$*, we have that for any* $a \in \mathbb{R} \setminus \{0\}$

$$H_0(B, X) = H_0(a \cdot B, X) , \qquad H(B, X) = H(a \cdot B, X) .$$

Proof. Since $X = c \cdot B$ for some $c \in \mathbb{R}$

$$H_0(B, X) = H_0(B, c \cdot B) , \qquad H(B, X) = H(B, c \cdot B) .$$

For any $a \in \mathbb{R} \setminus \{0\}$ we get

$$H_0(aB, X) = H_0(aB, c \cdot aB) = \frac{-\mathbb{E}[c \cdot aB]}{\mathbb{E}[aB]} = \\ = \frac{-a\mathbb{E}[cB]}{a\mathbb{E}[B]} = \frac{-\mathbb{E}[cB]}{\mathbb{E}[B]} = H_0(B, X) .$$

Moreover

$$\begin{aligned} H(aB, X) = H(aB, c \cdot aB) &= \frac{-\mathbb{E}[c \cdot aB]}{\mathbb{E}[aB\mathbf{1}_{\{c \cdot aB \neq 0\}}]} \\ &= \frac{-a\mathbb{E}[c \cdot B]}{a\mathbb{E}[B\mathbf{1}_{\{c \cdot aB \neq 0\}}]} = \frac{-a\mathbb{E}[c \cdot B]}{a\mathbb{E}[B\mathbf{1}_{\{c \cdot B \neq 0\}}]} = \\ &= \frac{-\mathbb{E}[c \cdot B]}{\mathbb{E}[B\mathbf{1}_{\{c \cdot B \neq 0\}}]} = H(B, X)\,. \end{aligned}$$

The **house advantage** is a **numerical index of the unfavorability of a wager**. Indeed, since in casino games $\mathbb{E}[X] < 0$, the house advantage is a nonnegative number. So a casino game is less unfavorable if it has a small house advantage and we can conclude that the less unfavorable game has the smaller house advantage.

3.2 Sequence of Wagers

Suppose to repeat the same wager many times. Let $(B_1, X_1), ..., (B_n, X_n)$ be i.i.d. random vectors whose common distribution is one of (B, X), representing the results of the independent repetitions of the original wager. Then

$$\frac{-(X_1 + ... + X_n)}{B_1 + ... + B_n}$$

represents the ratio of the gambler's cumulative loss after n such wagers to his total amount bet, while

$$\frac{-(X_1 + ... + X_n)}{B_1\mathbf{1}_{\{X_1 \neq 0\}} + ... + B_n\mathbf{1}_{\{X_n \neq 0\}}}$$

represents the ratio of the gambler's cumulative loss after n such wagers to his total amount of action. By the Law of Large Numbers we have

$$\frac{-(X_1 + ... + X_n)}{B_1 + ... + B_n} = \frac{\frac{-(X_1+...+X_n)}{n}}{\frac{B_1+...+B_n}{n}} \underset{n\to\infty}{=\longrightarrow} \frac{-\mathbb{E}[X]}{\mathbb{E}[B]} = H_0(B, X) \quad \text{a.s.}$$

and

$$\frac{-(X_1 + ... + X_n)}{B_1\mathbf{1}_{\{X_1 \neq 0\}} + ... + B_n\mathbf{1}_{\{X_n \neq 0\}}} = \frac{\frac{-(X_1+...+X_n)}{n}}{\frac{B_1\mathbf{1}_{\{X_1 \neq 0\}}+...+B_n\mathbf{1}_{\{X_n \neq 0\}}}{n}} \underset{n\to\infty}{\longrightarrow} \frac{-\mathbb{E}[X]}{\mathbb{E}[B\mathbf{1}_{\{X=0\}}]} = H(B, X) \quad \text{a.s.}$$

So we have the following interpretation of house advantage:

- $H_0(B, X)$ is the long-term ratio of the gambler's cumulative loss to his total amount bet;
- $H(B, X)$ is the long-term ratio of the gambler's cumulative loss to his total amount of action.

Let us now consider the first round in which the outcome is not a tie. That is, let us define

$$N = \min\{n \geq 1 | X_n \neq 0\} .$$

As we have seen when dealing with stopping times, being the single wagers independent, this random time is a geometric random variable of parameter $\mathbb{P}(X \neq 0)$. Indeed given $k \in \mathbb{N}$ we have

$$\begin{aligned}
\mathbb{P}(N = k) = \mathbb{P}(X_1 = X_2 = ... = X_{k-1} = 0, X_k \neq 0) &\overset{indep.}{=} \\
\overset{indep.}{=} \mathbb{P}(X_k \neq 0) \cdot \prod_{i=1}^{k-1} \mathbb{P}(X_i = 0) &\overset{id.distrib.}{=} \\
\overset{id.distrib.}{=} \mathbb{P}(X \neq 0)\mathbb{P}(X = 0)^{k-1} = (1 - \mathbb{P}(X \neq 0))^{k-1}\mathbb{P}(X \neq 0) .
\end{aligned}$$

So $N \sim \text{Geom}(\mathbb{P}(X \neq 0))$. We are now interested in computing the house advantage of the sequence of wagers considered up to time N, that is up to the first round in which the outcome is not a tie and we have the following result:

Proposition 3.2.1. *Consider the definitions of N and $\{(B_i, X_i)\}_i$ that we have given before. We have*

$$H_0(B_1 + ... + B_N, X_1 + ... + X_N) = H_0(B, X)$$

$$H_0(B_N, X_N) = H(B, X) .$$

Proof. Since $N \sim \text{Geom}(\mathbb{P}(X \neq 0))$, we have $\mathbb{P}(N < \infty) = 1$ and hence it has meaning to consider the random vector (B_N, X_N). By the Law of conditional expectation we have

$$H_0(B_1 + ... + B_N, X_1 + ... + X_N) = H_0\left(\sum_{i=1}^{N} B_i, \sum_{i=1}^{N} X_i\right) =$$
$$= \frac{-\mathbb{E}\left[\sum_{i=1}^{N} B_i\right]}{\mathbb{E}\left[\sum_{i=1}^{N} X_i\right]} = \frac{-\mathbb{E}[N]\mathbb{E}[X]}{\mathbb{E}[N]\mathbb{E}[B]} = \frac{-\mathbb{E}[X]}{\mathbb{E}[B]} = H_0(B, X)\,.$$

Note that $\{N = i\} = \{X_1 = ... = X_{i-1} = 0, X_i \neq 0\}$ and

$$\mathbb{E}\left[\mathbf{1}_{\{X_1 = ... = X_{i-1} = 0\}}\right] = 1 \cdot \mathbb{P}(X_1 = ... = X_{i-1} = 0) \overset{i.i.d.}{=} \mathbb{P}(X = 0)^{i-1}.$$

Consider now a function $f(b, x)$. We have

$$\mathbb{E}[f(B_N, X_N)] = \sum_{i}^{\infty} \mathbb{E}\left[f(B_i, X_i)\mathbf{1}_{\{N=i\}}\right] =$$
$$= \sum_{i}^{\infty} \mathbb{E}\left[f(B_i, X_i)\mathbf{1}_{\{X_1 = ... = X_{i-1} = 0, X_i \neq 0\}}\right] \overset{indep.}{=}$$
$$\overset{indep.}{=} \sum_{i}^{\infty} \mathbb{E}\left[f(B_i, X_i)\mathbf{1}_{\{X_i \neq 0\}}\right] \mathbb{E}\left[\mathbf{1}_{\{X_1 = ... = X_{i-1} = 0\}}\right] =$$
$$= \sum_{i}^{\infty} \mathbb{E}\left[f(B_i, X_i)\mathbf{1}_{\{X_i \neq 0\}}\right] \mathbb{P}(X = 0)^{i-1} \overset{id.distrib.}{=}$$
$$\overset{id.distrib.}{=} \sum_{i}^{\infty} \mathbb{E}\left[f(B, X)\mathbf{1}_{\{X \neq 0\}}\right] \mathbb{P}(X = 0)^{i-1}\,.$$

Since for $a \in (0, 1)$

$$\sum_{j=0}^{\infty} a^j = \frac{1}{1-a}\,,$$

we have

$$\sum_{i=1}^{\infty} \mathbb{P}(X = 0)^{i-1} \overset{j=i-1}{=} \sum_{j=0}^{\infty} \mathbb{P}(X = 0)^j = \frac{1}{1 - \mathbb{P}(X = 0)} = \frac{1}{\mathbb{P}(X \neq 0)}\,.$$

So

$$\begin{aligned}\mathbb{E}[f(B_N,X_N)] &= \sum_i^{\infty} \mathbb{E}\left[f(B,X)\mathbf{1}_{\{X\neq 0\}}\right]\mathbb{P}(X=0)^{i-1} = \\ &= \mathbb{E}\left[f(B,X)\mathbf{1}_{\{X\neq 0\}}\right]\sum_i^{\infty}\mathbb{P}(X=0)^{i-1} = \\ &= \mathbb{E}\left[f(B,X)\mathbf{1}_{\{X\neq 0\}}\right]\cdot\frac{1}{\mathbb{P}(X\neq 0)} = \\ &= \frac{\mathbb{E}\left[f(B,X)\mathbf{1}_{\{X\neq 0\}}\right]}{\mathbb{P}(X\neq 0)} = \mathbb{E}[f(B,X)|X\neq 0]\,.\end{aligned}$$

Therefore we have obtained

$$\mathbb{E}[f(B_N,X_N)] = \mathbb{E}[f(B,X)|X\neq 0]\,.$$

If we take as f the function $f(b,x)=x$ we have

$$\begin{aligned}\mathbb{E}[X_N] &= \mathbb{E}[f(B_N,X_N)] = \mathbb{E}[f(B,X)|X\neq 0] = \mathbb{E}[X|X\neq 0] = \\ &= \frac{\mathbb{E}\left[X\cdot\mathbf{1}_{\{X=0\}}\right]}{\mathbb{P}(X\neq 0)} = \frac{\mathbb{E}\left[X\cdot\mathbf{1}_{\{X\neq 0\}}\right]+\mathbb{E}\left[0\cdot\mathbf{1}_{\{X=0\}}\right]}{\mathbb{P}(X\neq 0)} = \\ &= \frac{\mathbb{E}\left[X\cdot\mathbf{1}_{\{X\neq 0\}}\right]+\mathbb{E}\left[X\cdot\mathbf{1}_{\{X=0\}}\right]}{\mathbb{P}(X\neq 0)} = \\ &= \frac{\mathbb{E}\left[X\cdot\left(\mathbf{1}_{\{X\neq 0\}}+\mathbf{1}_{\{X=0\}}\right)\right]}{\mathbb{P}(X\neq 0)} = \frac{\mathbb{E}[X]}{\mathbb{P}(X\neq 0)} = \mathbb{E}[X]\cdot(\mathbb{P}(X\neq 0))^{-1}\,.\end{aligned}$$

While if we take as f the function $f(b,x)=b$ we have

$$\begin{aligned}\mathbb{E}[B_N] &= \mathbb{E}[f(B_N,X_N)] = \mathbb{E}[f(B,X)|X\neq 0] = \\ &= \mathbb{E}[B|X\neq 0] = \frac{\mathbb{E}\left[B\cdot\mathbf{1}_{\{X=0\}}\right]}{\mathbb{P}(X\neq 0)} = \mathbb{E}[B]\cdot(\mathbb{P}(X\neq 0))^{-1}\,.\end{aligned}$$

Hence

$$\begin{aligned}H_0(B_N,X_N) &= \frac{-\mathbb{E}[X_N]}{\mathbb{E}[B_N]} = \frac{-\mathbb{E}[X]\cdot(\mathbb{P}(X\neq 0))^{-1}}{\mathbb{E}\left[B\cdot\mathbf{1}_{\{X=0\}}\right]\cdot(\mathbb{P}(X=0))^{-1}} = \\ &= \frac{-\mathbb{E}[X]}{\mathbb{E}\left[B\cdot\mathbf{1}_{\{X=0\}}\right]} = H(B,X)\,.\end{aligned}$$

From this proposition we deduce the following considerations:

- if a push is considered as a conclusive outcome of the bet, then $B_1 + ... + B_N$ units have been bet by time N and H_0 is the appropriate definition of house advantage. We describe H_0 as **house advantage with pushes included**.

- if a push is considered as a mere delay in the eventual resolution of the bet, then only B_N units have been bet by time N and H is the appropriate definition of house advantage. We describe it as **house advantage with pushes excluded**.

How should a push be regarded? That depends on the game.

Remark 5. *Note that if the bet has not the push as outcome, we have*

$$H(B, X) = H_0(B, X) .$$

Example 3.1. *Let us consider two wagers* (B_1, X_1) *and* (B_2, X_2) *such that* $B_1 = B_2 = b$*, where* $b > 0$ *is a fixed value, while*

$$X_1 = \begin{cases} 2b\,, & \text{with probability } \frac{1}{4}\,, \\ -b\,, & \text{with probability } \frac{3}{4} \end{cases}$$

$$X_2 = \begin{cases} 3b\,, & \text{with probability } \frac{1}{6}\,, \\ -b\,, & \text{with probability } \frac{5}{6} \end{cases}$$

Let us compare the two wagers using the house advantage. Note that

$$\mathbb{E}[X_1] = 2b \cdot \frac{1}{4} - b \cdot \frac{3}{4} = -\frac{b}{4}\,,$$

$$\mathbb{E}[X_2] = 3b \cdot \frac{1}{6} - b \cdot \frac{5}{6} = -\frac{b}{3}\,.$$

Since X_1 and X_2 do not assume the value zero, then $H_0(B_1, X_1) = H(B_1, X_1)$ and $H_0(B_2, X_2) = H(B_2, X_2)$. So

$$H_0(B_1, X_1) = -\frac{\mathbb{E}[X_1]}{\mathbb{E}[B_1]} = \frac{\frac{b}{4}}{b} = \frac{1}{4},$$

$$H_0(B_2, X_2) = -\frac{\mathbb{E}[X_2]}{\mathbb{E}[B_2]} = \frac{\frac{b}{3}}{b} = \frac{1}{3}.$$

Given that $H_0(B_2, X_2) > H_0(B_1, X_1)$, the wager (B_1, X_1) is more convenient.

3.3 Wagers with Three Possible Outcomes: Win, Loss or Push

Let

- $p > 0$ be the probability of a win,
- $q > 0$ be the probability of a loss,
- $r \geq 0$ be the probability of a push,

where $p + q + r = 1$. Suppose that a win pays a to 1, where $a > 0$. Suppose that $b > 0$ is the bet size. So $\mathbb{P}(B = b) = 1$ and

$$\mathbb{P}(X = ab) = p, \qquad \mathbb{P}(X = -b) = q, \qquad \mathbb{P}(X = 0) = r.$$

So

$$\mathbb{E}[X] = abp - qb = b \cdot (ap - q),$$

$$\mathbb{E}[B] = b \cdot 1 = b,$$

$$\mathbb{E}\left[B \cdot \mathbf{1}_{\{X=0\}}\right] = \mathbb{E}\left[b \cdot \mathbf{1}_{\{X=0\}}\right] = b \cdot \mathbb{E}\left[\mathbf{1}_{\{X=0\}}\right] = b \cdot \mathbb{P}(X \neq 0) = b \cdot (p + q).$$

3.3.1 Roulette: the m-Numbers Bet

Let us consider a wheel with $z = 1$ or $z = 2$ zeros. A bet on a subset of size m of the set of $36 + z$ numbers is available for $m = 1, 2, 3, 4, 6, 12, 18$ and pays $\frac{36}{m} - 1$ to 1 if a number in that subset appears. There is no possibility of a push. Suppose that $b > 0$ is the bet size. Then

$$\mathbb{P}(B = b) = 1 \Rightarrow \mathbb{E}[B] = b\,.$$

If we denote by X the gambler's profit we have

$$X = \begin{cases} b \cdot \left(\frac{36}{m} - 1\right) & \text{with probability } \frac{m}{36+z}\,, \\ -b & \text{with probability } 1 - \frac{m}{36+z} \end{cases} \tag{3.1}$$

and so

$$\begin{aligned} \mathbb{E}[X] &= b \cdot \left(\frac{36}{m} - 1\right) \cdot \frac{m}{36+z} - b \cdot \left(1 - \frac{m}{36+z}\right) = \\ &= b \cdot \left[\frac{36}{36+z} - \frac{m}{36+z} - 1 + \frac{m}{36+z}\right] = b \cdot \frac{-z}{36+z}\,. \end{aligned}$$

Therefore

$$H_0(B, X) = -\frac{\mathbb{E}[X]}{\mathbb{E}[B]} = \frac{b \cdot \frac{-z}{36+z}}{b} = \frac{z}{36+z} = \begin{cases} 1\backslash 37 \approx 0.03, & \text{if } z = 1\,; \\ 1\backslash 19 \approx 0.05, & \text{if } z = 2\,. \end{cases}$$

So it is more convenient to play at the European Roulette ($z = 1$) instead of playing at the American Roulette ($z = 2$). Note that the house advantage is independent on the quantity of numbers on which we bet m. Just like probabilities, house advantages are often stated in percentage terms

$$H_0(B, X) \approx \begin{cases} 0.03, & \text{if } z = 1\,; \\ 0.05, & \text{if } z = 2\,. \end{cases}$$

Since there is no possibility of ties we have

$$H_0(B, X) = H(B, X)\,.$$

3.4 Volatility of a Wager

Before studying the volatility of a wager we need the following result:

Proposition 3.4.1. *Let X and Y be jointly distributed discrete random variables with finite second moments and with $\mathbb{P}(Y \geq 0) = 1\,, \mathbb{P}(Y > 0) > 0$. Let $\{(Xi, Yi)\}_{i\geq 1}$ be a sequence of i.i.d. random vectors distributed as (X, Y). Then, defining $Z \sim \mathcal{N}(0,1)$,*

$$\sigma^2 = \frac{\mathbb{E}\left[\left(X - \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}Y\right)^2\right]}{\mathbb{E}[Y]^2}\,, \qquad W_n = \frac{\sqrt{n}}{\sigma} \cdot \left|\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n Y_i} - \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}\right|\,,$$

we have $W_n \xrightarrow{d} Z$ as $n \to \infty$, that is

$$\mathbb{P}(W_n \leq t) \underset{n\to\infty}{\to} \mathbb{P}(Z \leq t)\,.$$

Proof. We can rewrite W_n as

$$\frac{\sqrt{n}}{\sigma} \cdot \left|\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n Y_i} - \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}\right| = \frac{\sqrt{n}\mathbb{E}[Y]}{\sum_{i=1}^n Y_i} \cdot \left|\frac{\sum_{i=1}^n (X_i - \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}Y_i)}{\sqrt{\mathbb{E}\left[\left(X - \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}Y\right)^2\right]}}\right| =$$

$$= \frac{\sqrt{n}\mathbb{E}[Y]}{\sum_{i=1}^n Y_i} \cdot \left|\frac{\sum_{i=1}^n (X_i - \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}Y_i)}{\sqrt{\mathbb{E}\left[\left(X - \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}Y\right)^2\right]}}\right| = U_n \cdot V_n\,.$$

Since by the Central Limit Theorem $V_n \xrightarrow{d} \mathcal{N}(0,1)$ as $n \to \infty$ and by the Strong Law of Large Numbers $U_n \xrightarrow{a.s} 1$ as $n \to \infty$, we have

$$U_n \cdot V_n \xrightarrow{d} \mathcal{N}(0,1) \text{ as } n \to \infty\,.$$

If in Proposition 3.4.1 we take $Y_i = B_i$ and we substitute X_i with $-X_i$ we have the following:

Proposition 3.4.2.

$$\frac{\sqrt{n}}{\sigma_0} \cdot \left| \frac{-(X_1 + ... + X_n)}{B_1 + ... + B_n} - H_0(B, X) \right| \xrightarrow{d} \mathcal{N}(0, 1),$$

where

$$\sigma_0^2 = \frac{\mathbb{E}[(X + H_0(B, X)B)^2]}{\mathbb{E}[B]^2}$$

and σ_0 describes the volatility of the bet.

If in Proposition 3.4.1 we take $Y_i = B_i \cdot \mathbf{1}_{\{X_i \neq 0\}}$ and we substitute X_i with $-X_i$ we get the following:

Proposition 3.4.3.

$$\frac{\sqrt{n}}{\sigma_0} \cdot \left| \frac{-(X_1 + ... + X_n)}{B_1 \cdot \mathbf{1}_{\{X_i \neq 0\}} + ... + B_n \cdot \mathbf{1}_{\{X_i \neq 0\}}} - H(B, X) \right| \xrightarrow{d} \mathcal{N}(0, 1),$$

where

$$\sigma^2 = \frac{\mathbb{E}\left[\left(X + H(B, X)B \cdot \mathbf{1}_{\{X \neq 0\}}\right)^2\right]}{\mathbb{E}\left[B \cdot \mathbf{1}_{\{X \neq 0\}}\right]^2}$$

and σ describes the volatility of the bet.

3.4.1 Volatility in the Roulette: *m*-Number Bet

Consider a roulette with $36+z$ numbers, where $z \in \{1, 2\}$ and suppose to make a bet on m numbers, with $m = 1, 2, 3, 4, 6, 12, 18, 35$. Pushes are not allowed, so $H_0(B, X) = H(B, X)$. We have seen that, if the size bet is $B = 1 > 0$ (and so $\mathbb{E}[B] = 1$), then $H_0(B, X) = \frac{z}{36+z}$. By Proposition 3.4.1

$$\frac{\sqrt{n}}{\sigma_0} \cdot \left| \frac{-(X_1 + ... + X_n)}{B_1 + ... + B_n} - \frac{z}{36 + z} \right| \xrightarrow{d} \mathcal{N}(0, 1),$$

where

$$\sigma_0^2 = \frac{\mathbb{E}[(X + H_0(B, X)B)^2]}{\mathbb{E}[B]^2} = \mathbb{E}[(X + H_0(B, X))^2] =$$

$$= \mathbb{E}[X_2 + 2H_0(B, X) \cdot X + (H_0(B, X))^2] =$$

$$= \mathbb{E}[X_2] + 2H_0(B, X)\mathbb{E}[X] + (H_0(B, X))^2 .$$

Recall that

$$X = \begin{cases} \frac{36}{m} - 1\,, & \text{with probability } \frac{m}{36+z}\,, \\ -1\,, & \text{with probability } 1 - \frac{m}{36+z}\,. \end{cases}$$

Hence

$$\mathbb{E}[X] = \left(\frac{36}{m} - 1\right) \cdot \frac{m}{36+z} - 1 \cdot \left(1 - \frac{m}{36+z}\right) = \frac{-z}{36+z}\,,$$

$$\begin{aligned} \mathbb{E}[X^2] &= \left(\frac{36}{m} - 1\right)^2 \cdot \frac{m}{36+z} + (-1)^2 \cdot \left(1 - \frac{m}{36+z}\right) = \\ &= \left(\frac{36^2}{m} - 72\right) \cdot \frac{1}{36+z} + 1\,. \end{aligned}$$

Therefore

$$\begin{aligned} \sigma_0^2 &= \mathbb{E}[X_2] + 2H_0(B,X)\mathbb{E}[X] + (H_0(B,X))^2 = \\ &= \left(\frac{36^2}{m} - 72\right) \cdot \frac{1}{36+z} + 1 - 2 \cdot \left(\frac{z}{36+z}\right)^2 + \left(\frac{z}{36+z}\right)^2 = \\ &= \left(\frac{36^2}{m} - 72\right) \cdot \frac{1}{36+z} + 1 - \left(\frac{z}{36+z}\right)^2. \end{aligned}$$

Note that σ_0^2 decreases as m increases, so there is greater volatility from a bet on fewer numbers.

3.5 Expected Loss per Standard Deviation

A reasonable criterion to discuss the convenience of a wager is to choose the one that maximizes $\mathbb{P}(S_n \geq 0)$, where $S_n = \sum_{i=1}^n X_i$. If $\mathbb{E}[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$ for all $i = 1, ..., n$, we have for n large

$$\begin{aligned} \mathbb{P}(S_n \geq 0) = \mathbb{P}\left(\frac{S_n - n \cdot \mu}{\sqrt{n}\sigma} \geq \frac{n\mu}{\sqrt{n}\sigma}\right) = \\ = 1 - \mathbb{P}\left(\frac{S_n - n \cdot \mu}{\sqrt{n}\sigma} < \frac{n\mu}{\sqrt{n}\sigma}\right) \overset{\text{CLT}}{\approx} 1 - \Phi\left(\frac{-n\mu}{\sqrt{n}\sigma}\right) = \\ = 1 - \Phi\left(-\frac{\mu}{\sigma} \cdot \sqrt{n}\right). \end{aligned}$$

So to increase $\mathbb{P}(S_n \geq 0)$ we have to decrease $\Phi\left(-\frac{\mu}{\sigma} \cdot \sqrt{n}\right)$. Since Φ is an increasing function, we have that to decrease $\Phi\left(-\frac{\mu}{\sigma} \cdot \sqrt{n}\right)$ we need to decrease $-\frac{\mu}{\sigma}$. This suggests the following criterion:

Definition 3.2. *We define the* ***expected loss per standard deviation*** *as*

$$J(X) = \frac{-\mathbb{E}[X]}{\sqrt{Var(X)}}$$

So a reasonable criterion to discuss the convenience of a wager is to **choose the wager that minimizes** $J(X)$. The function $J(X)$ has the following property:

$$\begin{aligned} J(b \cdot X) &= \frac{-\mathbb{E}[b \cdot X]}{\sqrt{Var(b \cdot X)}} = \frac{-b\mathbb{E}[X]}{\sqrt{b^2 Var(X)}} = \\ &= \frac{-b\mathbb{E}[X]}{|b|\sqrt{Var(X)}} = \begin{cases} J(X)\,, & \text{if } b > 0\,; \\ -J(X)\,, & \text{if } b < 0\,. \end{cases} \end{aligned}$$

Usually in our application $b > 0$ and hence

$$J(b \cdot X) = J(X)\,.$$

3.5.1 Roulette: the m-Numbers Bet

Let us now consider the expected loss per standard deviation

$$J(X) = \frac{-\mathbb{E}[X]}{\sqrt{Var(X)}}$$

Notice that

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \left(\frac{36^2}{m} - 72\right) \cdot \frac{1}{36+z} + 1 - \left(\frac{z}{36+z}\right)^2.$$

So

$$J(X) = \frac{\frac{z}{36+z}}{\sqrt{\left(\frac{36^2}{m} - 72\right) \cdot \frac{1}{36+z} + 1 - \left(\frac{z}{36+z}\right)^2}}.$$

Note that $J(X)$ is increasing in m, so it is better to bet on fewer numbers.

Conclusions

C.1 Is Gambling a clever way to earn money?

Statistically, we cannot say gambling is a wise way to make money as the odds are against the gambler, with the house having a **built-in mathematical advantage** growing over time. Even if someone may win a consistent reward or mitigate risk by playing based on research and odds, most of them will end up losing.
Gambling can furnish players with an exhilarating rush, especially when there is a big jackpot at stake. Most of them confide in the hope of hitting a winning streak but the odds are seldom on their side; in fact, the house at a casino wins most of the time, resulting in an almost certain loss.
As a matter of fact, players have extremely slim chances when it comes to winning but, if they manage to walk away after even a small win, this will enable them to limit their losses.

C.2 Why is Investing better than Gambling?

While both involve minimizing risk to maximize profits, an investor's odds are generally better over the long run than a gambler's: that is because with gambling, the **house advantage** over the gambler grows as they keep playing. A gambler can still win, even though it is more likely that they will ultimately lose, while an investor tends to profit in the long run: investing can bear great losses itself, but the stock market usually appreciates over time, meaning that if you are well diversified, the odds are generally favorable, certainly more than those for getting a positive return by playing in a casino.

C.2.1 Loss Mitigation

A key difference between gambling and investing is loss mitigation, as betting on a pure gambling activity implies that you cannot rely on any loss-mitigation strategies to lessen potential damages.

Nevertheless, recent innovations to online sportsbooks can be exploited as a remedy to mitigate risks when betting on games, such as **in-play betting**: also known as *live betting* or *run betting*, this term refers to gambling that occurs after a game has started. It allows gamblers to place bets throughout the game rather than only before its start, causing the odds to change in response to what is happening during the game.[1] Partial cash-out options instead allow the redemption of a portion of your bet if the outcome seems to be heading towards the wrong direction.[2]

On the other hand, stock investors and traders have access to a wide selection of options to avert the total loss of risked capital, as setting **stop losses** on their stock to avoid undue risk: if their stock drops 15% below its purchase price, they are entitled to sell that stock to other traders and still hold 85% of their risk capital.

[1]Daniel Thomas Mollenkamp, "Live Betting: What It Is and How It Works", Investopedia.com, February 5th 2023, https://www.investopedia.com/live-betting-definition-5217206.

[2]Stephan A. Abraham, "Going All-in: Investing vs. Gambling", Investopedia.com, October 21th 2023, https://www.investopedia.com/articles/basics/09/compare-investing-gambling.asp.

C.2.2 The Time Factor

Another crucial difference between the two actions concerns the notion of time: gambling is a time-bound deed and once the game, race or hand comes to an end, you either have won or lost your cash.
Stock investing instead can last many years and is usually time-rewarding, as buying shares in companies that pay dividends entitles you to a compensation for risking your capital, as long as you hold onto your stock. Sensible investors understand that returns from dividends are a pivotal component to profit from stocks over the long run.

C.2.3 Gathering Information

Information is a precious good in gambling as well as in investing because both investors and gamblers investigate what happened in the past by going through historical performance and current behavior to have better chances of making a winning move. However, there is a definitive difference between the two for what concerns the availability of information.
Information about company earnings, financial ratios and management team is easily accessible to the public since it can be researched and then studied (either directly or researching analyst reports) before pledging your money.
Conversely, if you are seated at a table in a casino you have no concrete information about what happened just before your arrival at that same table; someone may say your table is either “hot” or “cold”, but that kind of information is not quantifiable and hence not so useful.[3]

[3]Stephan A. Abraham, “Going All-in: Investing vs. Gambling”, Investopedia.com, October 21th 2023, https://www.investopedia.com/articles/basics/09/compare-investing-gambling.asp.

C.3 Why Do People Gamble?

Both in gambling and investing, you are putting your wealth at the mercy of a potential loss. However, people frequently opt for gambling because of several reasons:

(i) it usually requires a small deal of upfront capital, like in case of a 2 euro lottery ticket;

(ii) gambling provokes an adrenaline rush, especially when the prize at stake is enticing;

(iii) gambling does not imply a lot of predictions, special strategies and research like investing does (reading reports or analyzing charts).

C.4 The Bottom Line

Gambling tendencies are way more deep-rooted than we can imagine, even beyond what standard definitions suggest. Gambling can be shaped as the need to prove one's self to society or act in a certain way to be acknowledged by the latter, which can often press people to step into a field without sufficient knowledge.
Gambling during market transactions is usually noticeable in individuals doing it mainly for the emotional high they get from the excitement trading gives rise to. Indeed, leaning on emotions or a must-win attitude to build profits rather than sticking to a methodical plan of action points out that you are gambling and unlikely to succeed over the long term.[4]

[4] Steven Nickolas, "Speculation vs. Gambling: What's the Difference?", Investopedia.com, December 11th 2023, https://www.investopedia.com/ask/answers/042715/what-difference-between-speculation-and-gambling.asp.

An investor stakes money onto quality assets which are likely to generate stable and consistent returns, a speculator puts money into volatile assets hoping those assets will spike in value in order to profit, while a gambler is playing against bad odds since the house always has an advantage. Investors can tailor their approach to help them reach their financial goals at a future period in time and they also enjoy the benefit of what has been referred to as the “eighth wonder of the world”: compounding.[5]

With gambling, there is a chance to get lucky and strike it rich, but the most probable outcome is to lose money (potentially a ton of it) because, as we have seen in Chapter 3, the house has a mathematical advantage and the odds of losing increase the longer a person plays. Going to a casino will only get you one quick win every now and then enough for a short-term thrill, but definitely not suitable to contribute to your long-term financial goals. With investing instead, having a long-term and systemic approach usually tends to pay off.

Financial markets provide all kinds of opportunities to make profits, as long as participants are prone to pursue well-defined edges and come up with sensible risk and money management rules. Instead, placing binary bets on a market outcome, supposing it will pay off randomly like a casino, ignores market structure and actuality, leading straight to failure and bankruptcy.

Money management cannot always guarantee astonishing returns, but it may allow you to limit losses and maximize returns through efficient diversification and, as we have shown in Chapter 2, the Kelly Criterion is a model that can help you diversify your portfolio in order to maximize your gain.

[5] Bella Caridade-Ferreira, “Why investing and gambling are different”, Compare+Invest, March 29th 2021, https://compareandinvest.co.uk/guide/why-investing-and-gambling-are-different/.

Appendix

This appendix reports basic results used to discuss the thesis.

Proposition A.1 (Triangular Inequality)**.** *Given $a, b \in \mathbb{R}$ we have*

$$|a \pm b| \leq |a| + |b| \, .$$

Proposition A.2. *Let Ω be the sample space of an experiment and let X and Y be two random variables defined on this experiment. If $X(\omega) \leq Y(\omega)$ for every $\omega \in \Omega$, then $\mathbb{E}[X] \leq \mathbb{E}[Y]$.*

Proposition A.3 (Law of conditional expectation)**.** *Let X, Y be two jointly distributed random variables. Then*

$$\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[X] \, .$$

Proposition A.4 (Law of large numbers)**.** *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, each of them having finite mean μ. Hence*

$$\mathbb{P}\left(\lim_{n \to \infty} \frac{X_1 + \ldots + X_n}{n} \to \mu\right) = 1 \, ,$$

o equivalently, almost surely

$$\lim_{n \to \infty} \frac{X_1 + \ldots + X_n}{n} = \mu \, .$$

Proposition A.5 (Central Limit Theorem). *Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables with mean $\mu < \infty$ e variance $\sigma^2 < \infty$. Defined $S_n = \sum i = 1^n X_i$ and $Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$ as its normalization, we have that for every $t \in \mathbb{R}$*

$$F_{Z_n}(t) := \mathbb{P}(Z_n \leq t) \xrightarrow{a.s.} \mathbb{P}(Z \leq t) \quad \textit{for}\, n \to \infty\,,$$

where $Z \sim \mathcal{N}(0,1)$. Equivalently, Z_n converges to $Z \sim \mathcal{N}(0,1)$ in distribution for $n \to \infty$.

Proposition A.6 (Weierstrass theorem). *Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be a continous function and let $D \subset \mathbb{R}^n$ be a closed and bounded set. Then f has a minimum and a maximum and a global maximum in D.*

Proposition A.7 (Simpson's Rule). *Let f be a continuous function on $[a,b]$ and let n be an even integer. Simpson's Rule for approximating $\int_a^b f(x)\,dx$ is*

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + ... + 4f(x_{n-1}) + f(x_n)]\,.$$

Moreover, as $n \to \infty$, the right-hand side approaches $\int_a^b f(x)\,dx$.

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