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BEYOND THE BENCHMARK: A STUDY OF ACTIVE PORTFOLIO MANAGEMENT COMPARED TO PASSIVE STRATEGIES

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ABSTRACT

Nowadays, many investors decide to invest in a passive strategy, such as an ETF, which replicates exactly an index, without incurring in any cost related to management fees.

The objective of this study is to verify whether an active portfolio outperforms a passive strategy and if it is reasonable to invest in a portfolio which is characterized by active allocation.

Active portfolio management involves two areas: market timing and stock selection. While market timing and fundamental analysis skills can be assessed, evaluating stock selection ability remains challenging. In this paper we propose a framework to estimate the value of security analysis.

The primary aim of asset managers is to maximize returns while minimizing risks, safeguarding against inflation. With an array of strategies and techniques available for constructing and managing investment portfolios, the challenge of selecting the optimal investment remains pertinent for investors.

Additionally, a common goal for portfolio investors is to attain a greater risk–adjusted return compared to investing in a single asset. Diversifying assets into a portfolio presents the opportunity for reducing risk while also potentially increasing returns compared to investing in individual assets.

In the early 1950s, Harry Markowitz introduced a groundbreaking model emphasizing the importance of diversification in risk reduction.

In our study, we use Sharpe's Index Model to create an optimal portfolio. We favor this model over the Markowitz Model due to its simplicity, requiring fewer inputs and offering easier computation. Through this model, we determine the allocation of investment proportions for each stock within the optimal portfolio.

However, because of its simplifying assumptions, the index model may fail to capture important nuances and complexities of the market, including the impact of industry-specific factors and other sources of risk. This can limit its effectiveness in portfolio allocation, especially in environments where these factors play a significant role in asset pricing. To overcome this limit, we extend the analysis to the case in which the error terms of the securities are correlated, using the so-called SURE methodology.

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I. INTRODUCTION

In the realm of investment management, the pursuit of optimal portfolio performance has been a central theme for practitioners and academics.

Within the investment context, the Index Model has emerged as a prominent tool for portfolio optimization, offering a streamlined approach to portfolio construction. This study seeks to delve deeper into the empirical application of this framework and its efficacy in constructing optimal portfolios.

The primary objective of this analysis is to assess whether an optimal active portfolio outperforms a passive strategy that invests in a market index.

In the first part of this study, we present the literature review of portfolio allocation, discussing the main theories on which this study is based. In particular, in subsection II.A, the Efficient Market Hypothesis is described, with its implications for portfolio management and the ongoing debate surrounding market efficiency. This is followed by an overview of the Capital Asset Pricing Model (II.B), explaining its assumptions, applications, and limitations in predicting expected returns.

Next, we illustrate the Security Market Line (II.C), with a focus on its role in the CAPM framework and its utility in determining the relationship between expected return and systematic risk. In subsection II.D, the Index Model is examined, detailing its methodology and significance in the context of portfolio optimization and performance evaluation.

The discussion then moves to "Performance Analysis" (II.E), where we delve into the methodologies used to assess the performance of investment portfolios, offering an overview of key performance metrics and their relevance. The subsection on Risk-Adjusted Measures of Performance (II.F) explains the various measures used to evaluate investment performance by adjusting for risk, such as the Sharpe Ratio, Treynor Ratio, and Jensen's Alpha.

The literature review ends with an overview on the costs of active management (II.G) and a comparison between passive and active strategies (II.H).

The second, and central part, of this study is dedicated to the data methodology, where we analyze the descriptive statistical profiles of twenty stocks selected from the NYSE and the NASDAQ (III.A). We explore three different approaches to portfolio allocation and their implications for investment outcomes. In subsection III.B, our first aim is to identify the optimal allocation strategy that maximizes returns while minimizing risks, considering an optimal portfolio with fixed weights. Then, we illustrate the estimation of the model applied to our selection of 20 stocks (III.C).

Building upon this strategy, we describe the rolling-window model, assuming that the active portfolio can be rebalanced as new data becomes available to the potential investor (III.D).

Lastly, since the Index Model presents some limitations in considering the risks of real financial markets, we implement the model using the SURE methodology, assuming that the error terms of the selected assets could be correlated (III.E).

Furthermore, in subsection III.F, we describe the multivariate regression model, through which we have derived the formulas of the assets 'weights for the SURE methodology extension. Then, in the following section (III.G), we estimate the optimal allocation of the portfolio considering possible correlation between the error terms.

For each estimation model, we compare various performance measures to assess whether the optimal active portfolio is more efficient compared to a passive strategy that invests in the S&P 500 index.

II. LITERATURE REVIEW

Modern portfolio analysis is concerned with grouping individual securities into an efficient set of portfolios[1]. A portfolio is defined *efficient* if it provides the largest expected return than any other portfolio with comparable risk. Return realization and expected return from portfolio are a weighted average of individual securities returns. The portfolio's return can be calculated using the following formula[2]:

$$r_p = \sum_{i=1}^{n} (w_i \, x \, r_i)$$

Where w_i is the weight or proportion of the portfolio's total value that is invested in the *i*-th asset and r_i is the return of the *i*-th individual asset.

[1] Farrell, J. L. (1976). The multi-index model and practical portfolio analysis. *The Financial Analysis Research Foundation*, Charlottesville, Virginia. https://www.cfainstitute.org/-/media/documents/book/rf-publication/1976/rf-v1976-n3-4732-pdf.pdf

[2] Gustyana, T., & Wijayangka, C. (2021). Optimal portfolio using single-index model and capital asset pricing model (CAPM) in COVID-19 pandemic era. *IEOM Society International*.

The basic framework for modern portfolio analysis was established by Markovitz[3], whose Modern Portfolio Theory laid the groundwork for portfolio optimization techniques. His work emphasized the importance of considering both risk and return when constructing portfolios. However, the size and complexity of the model makes it inapplicable for practical use.

William Sharpe[4] started from the Markovitz model to circumvent the difficulties of dealing with a great number of covariances among assets and introduced the *single-index model*. The fundamental concept of this simplified approach is that the only form of co-movement between securities comes from a common response to a general market index, such as the S&P 500. In particular, Sharpe's model assumes that the return (r_i) on any stock is given by random factors and a linear relationship with the market index (r_m).

Moreover, Sharpe and Lintner[5] developed the Capital Asset Pricing Model[6] (CAPM), which laid the foundation for understanding how systematic risk influences asset pricing and portfolio construction[7]. Lintner extended the work of Sharpe by introducing the concept of beta, which measures an asset's sensitivity to market movements within the framework of the single-index model. His research provided empirical evidence supporting the CAPM and the single-index model's validity. Michael Jensen further developed the single-index model by introducing the idea of the Jensen's alpha, which measures the performance of an investment relative to its expected return as predicted by the CAPM. His work emphasized the importance of portfolio managers' ability to generate excess returns (alpha) after adjusting for market risk.

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[3] Markowitz, H. M. (1959). Portfolio Selection: Efficient Diversification of Investments. *Yale University Press*. http://www.jstor.org/stable/j.ctt1bh4c8h

[4] Sharpe, W. F. (1963). A simplified model for portfolio selection. *Management Science*, 9(1), 277-293.

[5] Lintner, J. (1965). The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *The Review of Economics and Statistics*, 47(1), 13–37. https://doi.org/10.2307/1924119

[6] Sharpe, W. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance*, 19(3), 425-442.

[7] Yang, Z. (2021). Analysis on CAPM and Sharpe Ratio in Market Investment. *Proceedings of the 6th International Conference on Financial Innovation and Economic Development (ICFIED 2021)*. https://doi.org/10.2991/aebmr.k.210319.002

The Capital Asset Pricing Model (CAPM) and the Index model share a foundational relationship in asset pricing and portfolio theory.

The CAPM is a fundamental framework that links the expected return of an asset to its systematic risk, measured by beta. It suggests that the expected return of an asset is equal to the risk-free rate plus a risk premium proportional to the asset's beta, representing the asset's sensitivity to market risk. On the other hand, the index model is a specific form of asset pricing model that decomposes the returns of individual assets into systematic and firm-specific components. It posits that the return on a particular asset can be explained by its sensitivity to a common market factor (the index) and its unique, idiosyncratic risk[8].

A. Efficient Market Hypothesis

According to Fama's theory[9], although an investor may experience temporary success by purchasing a stock that yields significant short-term profits, in the long run, it is unlikely for them to attain returns on investment significantly exceeding the market average.

Fama's theory – which carries the same implications for investors as the Random Walk Theory[10] – is based on the concept that all pertinent information influencing stock prices is widely accessible since it is "universally shared" among all investors. According to the efficient market hypothesis, stock prices always trade at their *fair value*, therefore it is unfeasible to purchase undervalued stocks or sell overvalued stocks to generate additional profit. If this concept holds, then investors can only achieve superior returns by assuming significantly higher levels of risk.

The Efficient Markets Hypothesis (EMH) comes in three variations[11] – weak, semi-strong, and strong forms – representing different levels of market efficiency.

1. Weak Form

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[8] "Idiosyncratic risk" is an investment risk that is related to an individual asset. This type of risk is also referred to as a "specific risk" or "unsystematic risk".

[9] Keown, A. J., Martin, J. D., & Petty, J. W. (2014). Foundations of finance. Global Edition, 8th Edition. *Pearson.*

[10] The Random Walk theory assumes that single-period log returns are independent and follow a normal distribution.

[11] Corporate Finance Institute, Efficient Market Hypothesis: it is not possible to outperform the market by skill alone. *CFI.* https://corporatefinanceinstitute.com/resources/career-map/sell-side/capital-markets/efficient-markets-hypothesis/

The weak form of the EMH states that securities prices incorporate all publicly available market information but may not encompass new information that is not yet public. It also assumes that past data regarding prices, trading volume, and returns do not predict future prices.

Accordingly, the weak form suggests that technical trading strategies cannot consistently yield excess returns because past price trends cannot anticipate future price movements driven by new information. Despite discounting technical analysis, the weak form allows for the possibility that superior fundamental analysis might enable outperformance relative to the overall market average return on investment.

2. Semi-strong Form

The semi-strong form of the theory rejects the utility of both technical and fundamental analysis. Building upon the weak form assumptions, the semi-strong form asserts that prices adjust rapidly to any new public information, therefore fundamental analysis becomes unable to predict future price movements.

3. Strong Form

The strong form of the EMH asserts that prices reflect all available information, both public and private. This encompasses all publicly available information, historical as well as current, along with insider information. Even data not accessible to investors, such as confidential information known only to a company's CEO, is presumed to already be incorporated into the company's current stock price. Thus, according to the strong form of the EMH, even insider knowledge cannot provide investors with an edge that consistently outperforms the overall market average.

B. Capital Asset Pricing Model

The Capital Asset Pricing Model is based on the portfolio theory developed by Markowitz, which assumes that investors will choose the optimal portfolio based on investor preferences for risk and return[12].

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[12] Gustyana, T., & Wijayangka, C. (2021). Optimal portfolio using single-index model and capital asset pricing model (CAPM) in COVID-19 pandemic era. *IEOM Society International.*

The relationship between risk and return could be explained using the capital asset pricing model (CAPM), which includes the beta (systematic risk), the expected return of the asset, the risk-free asset (in this case Treasury bill rate) and the equity risk premium (the market return minus the risk-free asset).

The formula of the CAPM is the following:

$$E(r_i) = r_f + \beta_i[E(r_m) - r_f]$$

Where $E(r_i)$ is the expected return of asset i, r_f is the risk-free rate, r_m is the expected return of the market and β_i is the sensitivity of the investment's return to the return of the market. $[E(r_m) - r_f]$ is the market risk premium, which represents the additional return over the risk-free rate that investors require for taking on the higher risk of investing in the market rather than in a risk-free asset.

According to this equation, investors expect to be compensated more as the risk increases. The beta of an investment represents the extent to which the investment may increase the risk with respect to the market portfolio. If a stock poses higher risk than the market, its beta will exceed one. Conversely, a beta below one suggests that the stock could potentially mitigate portfolio risk.

The CAPM asks what would happen if all investors utilized the same input list to draw their efficient frontiers and had access to the same investable universe[13]. Their efficient frontiers would be the same. They would then draw an identity tangent CAL, having the same risk-free rate, and naturally, they would all arrive at the same risky portfolio. As a result, for every risky asset, all investors would select the same set of weights. According to the CAPM, since the market portfolio is the aggregation of all the identical portfolios, it will also have the same weight for all investors. Therefore, if all investors choose the same risky portfolio, such portfolio must be the market portfolio. It follows that the capital market line, as shown in Figure 1, will represent the capital allocation line based on each investor's optimal risky portfolio.

The CAPM aims to assist investors in risk management. If investors want to optimize a portfolio's return in perfect alignment with its risk, they should pick the

[13] Bodie, Z., Kane, A. and Marcus, A. (2014) Investments, Global Edition, 10th Edition. *McGraw Hill Higher Education,* New York.

portfolio that is tangent with the capital asset line, illustrated in the subsequent graph[14].

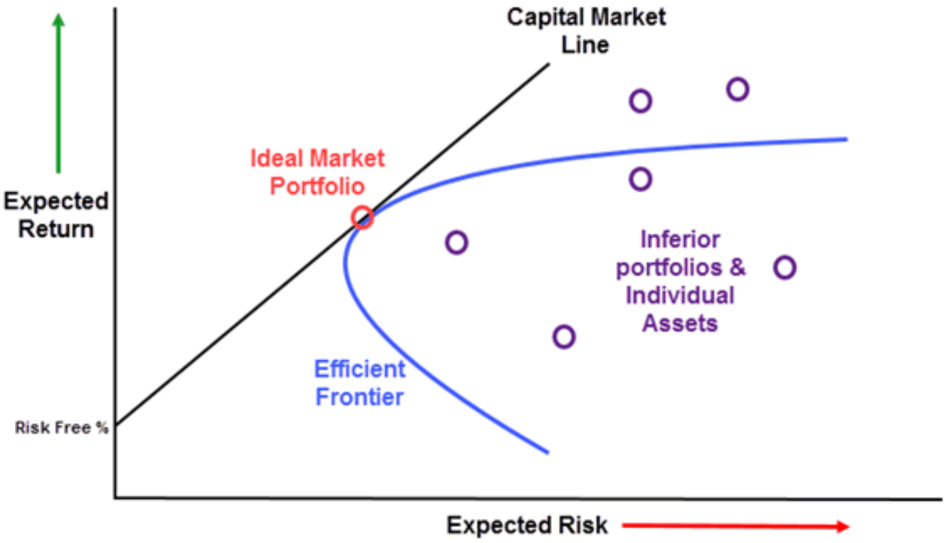

Figure 1: Capital Market Line and Efficient Frontier

The CAPM suggests the following implications:

- The market portfolio represents the efficient frontier portfolio. The efficient frontier consists of portfolios of the market portfolio and the risk-free asset.
- The systematic risk of any asset is determined by its covariance with the market, expressed as $\sigma_{i,m} = \mathrm{Cov}\,[r_i, r_m]$ for asset *i*.

The efficient frontier operates under the same assumptions as the CAPM and remains a theoretical concept. If a portfolio were situated on the efficient frontier, it would yield the maximum return corresponding to its level of risk. However, determining whether a portfolio resides on the efficient frontier is unattainable due to the unpredictability of future returns.

This risk-return trade-off is inherent in the CAPM, and the graphical representation of the efficient frontier can be reconfigured to depict the trade-off for individual assets. In the subsequent chart, the Capital Market Line (CML) is transformed into the Security Market Line (SML). Instead of expected risk on the x-axis, the stock's beta is utilized. The figure below reveals that as beta escalates from A to B, the expected return also increases.

[14] Image by Sultan Saif Al Maiyahi.

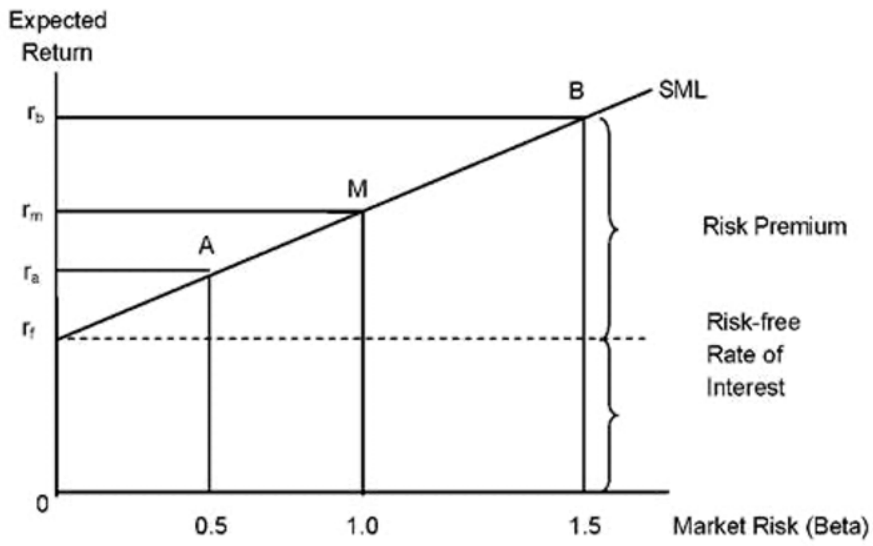

Figure 2: Security Market Line

C. Security Market Line

The anticipated return-beta correlation can be conceptualized as a balance between reward and risk. As mentioned earlier, this connection is often depicted by the Security Market Line (SML), which is a representation of the Capital Asset Pricing Model. Here, beta denotes the systematic risk, positioned on the x-axis, while expected return is plotted on the y-axis[15].

The market's beta is 1, and the market earns an expected return equal to $E(r_m)$. Using the SML it is possible to calculate the expected return on an asset, considering that Beta is given by:

$$\beta_i = \frac{\rho_{i,m}\sigma_i}{\sigma_m}$$

Where $\rho_{i,m}$ is the correlation between the asset and the market, while σ_i and σ_m are respectively the standard deviations of the asset and the market.

The Security market line applies to every asset, but what happens if in our portfolio we have more than one asset?

In this case we need to calculate the return on the two securities (r_p), which is a weighted average of the returns r_1 and r_2, as shown below.

$$E(r_1) = r_f + \beta_1[E(r_m) - r_f]$$

[15] CFA Institute. (2023). Fixed income, derivatives, alternative investments, and portfolio management. *CFA Institute Investment Series*, Level 1, Vol. 5, pp. 640-641.

$$E(r_2) = r_f + \beta_2[E(r_m) - r_f]$$
$$E(r_p) = w_1E(r_1) + w_2E(r_2) = r_f + (w_1\beta_1 + w_2\beta_2)\,[E(r_m) - r_f]$$

From the last equation is evident that also the portfolio beta is a weighted average between the beta of the two assets:

$$\beta_p = \sum_{i=1}^{n} w_i\,\beta_i$$

Therefore:

$$E(r_p) = r_f + \beta_p\,[E(r_m) - r_f]$$

The security market line is helpful for investor to understand whether an asset is overvalued or undervalued. All the securities that reflect the consensus market view are points directly on the SML (they are properly valued).

If a point representing an asset's estimated return lies above the SML, it suggests that the asset carries a relatively low level of risk compared to its expected return, making it an attractive investment option. Conversely, if the point representing an asset falls below the SML, it indicates that the stock is overvalued. In this scenario, the return fails to adequately compensate for the associated risk. In such instances, a profitable strategy may involve short selling.

In conclusion, assets positioned above the security market line are deemed undervalued as they offer higher expected returns for a given level of risk. Assets positioned below the security market line are considered overvalued since they provide lower expected returns for a given level of risk.

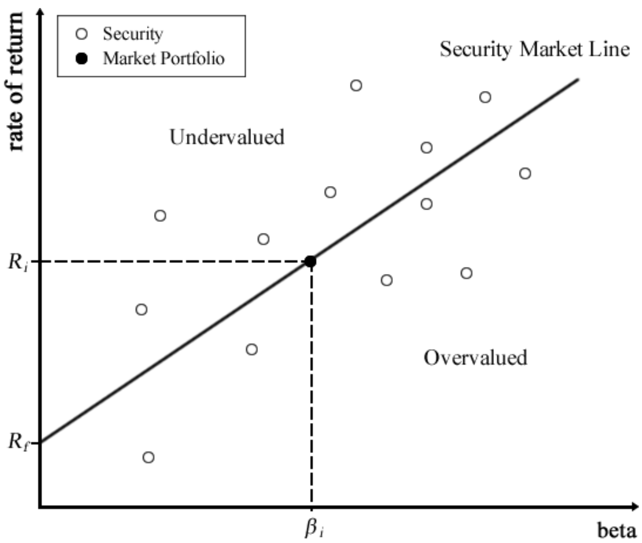

Figure 3: Undervalued and Overvalued assets

D. Index Model

Differently from Markowitz's approach, the index model suggests that by comparing the returns of individual securities with a single index, such as the "Market Index", the relationship between each pair of securities can be indirectly determined. This model significantly reduces the need for large data inputs and complex calculations.

$$r_i - r_f = \alpha_i + \beta_i(r_m - r_f) + \varepsilon_i$$

Where r_i is the return on asset i, and r_m is the return on the market.

Note that the model is a regression equation where β_i represents the slope coefficient and α_i represents the intercept of the regression. In the single-index model, the slope measures the responsiveness of the asset's return to fluctuations in the market, while the intercept is the component of the asset's return that is independent of the market return. As in standard regression analysis, the ε_i (or residual term) is assumed to be equal to zero on average. Most importantly, the residuals are assumed to be uncorrelated across securities: $(\varepsilon_i, \varepsilon_j) = 0$. This is in line with the single-index model assumption that the sole source of movement between securities is due to the general market; so that once this influence has been removed, the expectation is for no correlation between the residuals of different securities.

In this section, we present the index model which streamlines the estimation of covariance matrices and significantly improves the assessment of security risk premiums. By enabling us to separate risk into systematic and firm-specific components, this model provides valuable insights about the importance of diversification. Moreover, it facilitates the quantification of these risk components for individual securities and portfolios.

The index model operates under several assumptions:

1. All investors have homogeneous expectations.
2. The risk and return of every security are evaluated over a consistent holding time.

3. The price fluctuations of stocks are influenced by prevailing economic and business conditions.
4. The systemic risk affecting all stock returns is represented by only one macroeconomic factor, which is the rate of return on a market index, such as the S&P 500[16].

For instance, consider an equally weighted portfolio comprising *n* securities. Let us denote the excess return on the portfolio as $R_p = r_p - r_f$, and the excess return on the market as $R_M = r_m - r_f$.

The excess rate of return on the portfolio can be expressed as:

$$R_p = \alpha_p + \beta_p R_M + \varepsilon_i$$

As the number of stocks in the portfolio grows, the proportion of portfolio risk associated with non-market factors diminishes progressively. This portion of the risk is effectively diversified away. However, market risk persists regardless of the number of firms included in the portfolio. This means that no matter how many companies are combined into the portfolio, market risk is unchanged. Additionally, according to this model, as more securities are added into our portfolio, the portfolio's variance diminishes due to the diversification of firm-specific risk. Nonetheless, the effectiveness of diversification has its constraints. The curve in figure 4 shows how total risk decreases as the number of securities increases, but it asymptotically approaches a limit represented by systematic risk. This demonstrates that no matter how well-diversified a portfolio is, it will still be subject to systematic risk. Therefore, diversification can reduce risk but cannot eliminate it completely. Even with a substantial number of securities (denoted as "n"), a portion of the risk persists owing to the correlation of virtually all assets with the common market factor. Consequently, this systematic risk is deemed non-diversifiable.

[16] Single-Index Model for Security Returns - thismatter.com, https://thismatter.com/money/investments/single-index-model.htm.

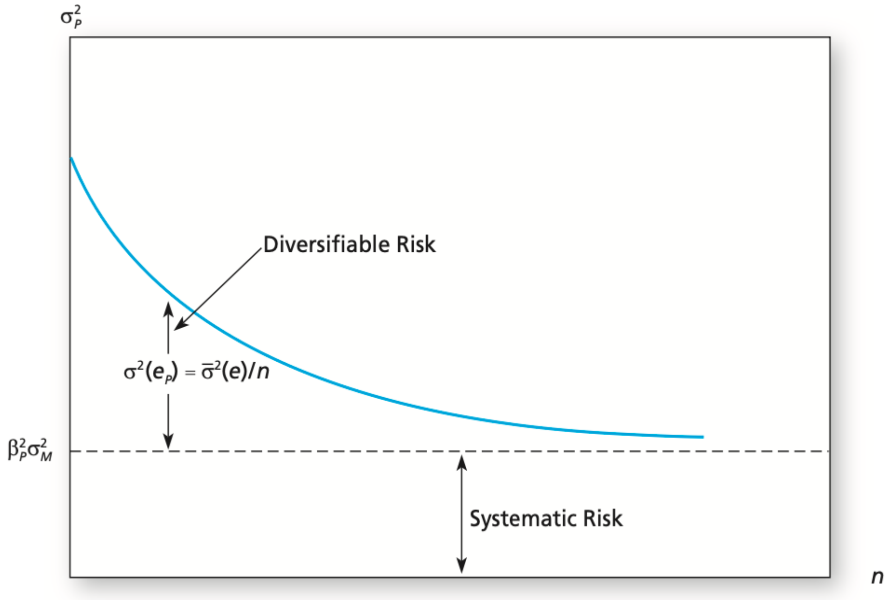

Figure 4: Risk Breakdown[17]

E. Performance Analysis

The analysis of performance is crucial for asset managers to make informed investment decisions and optimize strategies over time. In the evaluation of performance, it is fundamental to consider two key factors: the efficiency of financial markets and the ability to generate excess returns for investors[18].

Different strategies can be employed in managing a portfolio of assets, primarily categorized as passive or active. A passive strategy involves replicating a benchmark index (e.g., S&P 500, MSCI World, US Dollar Index) without requiring any rebalancing action. This approach typically minimizes fees due to the reduced need for active management. Employing a buy-and-hold strategy ensures that the portfolio's return is derived solely from the systematic risk[19] it carries, reflecting the performance of the overall market or index.

Conversely, an active strategy aims to generate abnormal returns, or alpha, by outperforming the benchmark. This requires the portfolio manager to engage in ongoing evaluation of assets to determine if they are undervalued or overvalued,

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[17] Bodie, Z., Kane, A. and Marcus, A. (2014) Investments, Global Edition, 10th Edition. *McGraw Hill Higher Education,* New York.

[18] Vitale, P. (2017-2018). Performance analysis, course of *Equity markets, and alternative investments.*

[19] "Systematic risk" refers to the risk inherent to the market. Systematic risk, also known as undiversifiable risk, affects the overall market, not just a particular stock or industry and it cannot be eliminated through diversification.

leading to buying or selling decisions based on this assessment. Active management assumes that financial markets are not fully efficient, meaning that not all available information is immediately reflected in asset prices. Therefore, portfolio managers seek to exploit these inefficiencies by leveraging superior information or analytical techniques to achieve higher returns than those of the benchmark.

In the context of performance analysis, several metrics and models are employed to assess how well a portfolio is managed. These include absolute return measures, such as the total return, and relative return measures, like alpha and beta, which compare the portfolio's performance against a benchmark. Additionally, risk-adjusted performance metrics, such as the Sharpe Ratio, Treynor Ratio, and Jensen's Alpha, are used to evaluate the returns generated per unit of risk taken. These metrics provide a more comprehensive understanding of the trade-offs between risk and return in the portfolio management process.

Furthermore, performance analysis involves a deep dive into the sources of returns, distinguishing between those derived from market movements (systematic risk) and those resulting from the manager's specific actions (unsystematic risk). This differentiation helps in understanding the true value added by active management.

F. Risk-Adjusted Measures of Performance

We've observed the existence of a trade-off between risk and return, which indicates that when assessing the performance of a portfolio, adjustments for risk are necessary. In finance, there are several risk-adjusted performance metrics[20]:

1. *Sharpe ratio:* $\frac{r_p - r_f}{\sigma_p}$

The **Sharpe's ratio** divides average portfolio excess return over the sample period by the standard deviation of returns over that period. It measures the reward to (total) volatility trade-off.

2. *Sortino Ratio:* $\frac{r_p - r_f}{\sigma_d}$, where $\sigma_d = \sqrt{\frac{1}{n}\sum_{i=1}^{n} min\left(r_{p,i} - r_f, 0\right)^2}$

[20] Bodie, Z., Kane, A. and Marcus, A. (2014) Investments, Global Edition, 10th Edition. *McGraw Hill Higher Education*, New York, pp 839-840.

The **Sortino ratio** measures the risk-adjusted return of an investment or portfolio, focusing specifically on downside risk. Unlike the Sharpe Ratio, which considers total volatility (standard deviation), the Sortino Ratio only considers the downside deviation (or semi-variance), which provides a more accurate assessment of the risk related to negative returns. The numerator of the Sortino ratio represents the excess return of the portfolio over the risk-free rate. The denominator is the semi-variance of the portfolio and captures the portfolio's downside risk by considering only the negative deviations from the risk-free rate.

3. *Treynor measure:* $\frac{r_p - r_f}{\beta_p}$, where β_p is the beta of the portfolio, which measures its sensitivity to market movements.

Like the Sharpe ratio, **Treynor's measure** gives excess return per unit of risk, but it uses systematic risk instead of total risk.

Jensen's alpha: $r_p - \left(r_f + \beta_p\left(r_m - r_f\right)\right)$

Jensen's alpha is the average return on the portfolio over and above that predicted by the CAPM, given the portfolio's beta and the average market return.

4. *Information ratio:* $\frac{\alpha_p}{TE}$, where $TE = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} r_p - r_b}$ and r_b is the return of the benchmark.

The **information ratio** divides the alpha of the portfolio by the nonsystematic risk of the portfolio, called "tracking error" in the industry. It measures abnormal return per unit of risk that in principle could be diversified away by holding a market index portfolio.

G. Costs of Active Management

A passive investment strategy would simply invest in a market index fund, such as the S&P 500 and some the risk-free asset. In such case, the derived utility of

wealth would be *Sharpe ratio:* $\mathrm{rf} + \frac{S_M^2}{2A}$, where S_M is the Sharpe measure of the market-index portfolio and, equivalently, the slope of the capital market line.

Therefore, if an investor can identify non-zero alphas, a fund with a larger Sharpe ratio can be constructed and would be more attractive to investors. According to this, it is reasonable to charge some fees on active allocation and for the service of selection. This charge may be either an upfront fee, or an ongoing annual management charge.

To derive the one-time fee that investors would be willing to pay to buy into the fund, we will use the formula derived by Kane, Marcus, and Trippi[21]:

$$f = \frac{S_P^2 - S_M^2}{2A}$$

Where *f* is the percentage fee that investors would be willing to pay for active services, S_P is the Sharpe ratio of the optimal portfolio and S_M is the Sharpe ratio of the market portfolio and *A* is the coefficient that represents the investor's risk aversion. Risk aversion measures an investor's reluctance to accept risk. Higher values of *A* indicate greater risk aversion.

Common methods to estimate risk aversion include surveys, historical data analysis, and modeling investor behavior in different market conditions.

The power of the active portfolio is given by the additive value of the squared information ratios $\left(\frac{\alpha_i}{\sigma(\varepsilon_i)}\right)$ and precision of individual analysts.

Recall that $S_P^2 = S_M^2 + \left[\frac{\alpha_i}{\sigma(\varepsilon_i)}\right]^2$, therefore we derive:

$$f = \frac{1}{2A} \sum_{i=1}^{n} \left[\left(\frac{\alpha_i}{\sigma(\varepsilon_i)}\right)\right]^2$$

As a result, the maximum fee, *f*, is determined by three variables: (1) the risk aversion coefficient; (2) the squared information ratio distribution throughout the securities universe; and (3) the accuracy of the security analysts. It is evident that this fee is more than what an index fund would impose. The active management may impose additional costs over the 20-basis point threshold[22].

H. Active vs. Passive Strategy

Whether it is better a passive strategy, or an active strategy is a debate in the financial industry. The decision between passive and active investing extends

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[21] Kane, A., Marcus, A., & Trippi, R. R. (1999). The valuation of security analysis. *Journal of Portfolio Management*, 25(3), 12-25.

[22] Bodie, Z., Kane, A. and Marcus, A. (2014) Investments, Global Edition, 10th Edition. *McGraw Hill Higher Education*, New York, pp 970-971.

beyond a broad overview. Different market conditions favor each strategy differently. In times of volatility or economic downturns, active strategies often perform better, while passive strategies tend to shine when market movements are synchronized, or equity valuations are uniform. Combining both approaches, known as a hybrid strategy, can harness the strengths of each depending on sector opportunities. However, since market dynamics constantly shift, determining the optimal balance between passive and active investments requires ongoing evaluation and a keen understanding of market trends[23].

When adopting a "best of both worlds" strategy, it is important to recognize the challenges associated with consistently achieving successful active management, especially within certain asset classes which are very volatile. Consequently, it may be reasonable to invest in a passive strategy in those areas and rely more active investing in asset classes where is has been historically proven to be profitable, for example in smaller U.S. companies and in emerging markets.

Moreover, investing in volatile assets exposes to the risk of high drawdowns, therefore an investor adopting an active strategy would incur in big losses, but also in high management fees for holding the actively managed portfolio. Therefore, it would be reasonable to adopt active strategies for investments that perform low volatility and, consequently, low risk.

When comparing active and passive management it is crucial to carefully assess the economic environment. Generally, when interest rates decline, equity markets typically thrive. Conversely, as rates increase, the discrepancy between the top and bottom performing stocks tends to widen. In such conditions, active managers often excel, as demonstrated by historical data. For instance, a Nomura Securities' analysis revealed that during the period from 1962 to 1968, when the 10-year treasury yield surged from 3.85% to 15.8%, larger company mutual funds achieved a median cumulative return more than 62% higher than the S&P 500 Index[24]. A similar scenario unfolded in 2013 when Federal Reserve Chairman Ben Bernanke signaled the end of quantitative easing, triggering the 'taper tantrum'. Active managers fared well in comparison during this period.

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[23] Hunt, D. A New Take on the Active vs. Passive Investing Debate. *Morgan Stanley*, https://www.morganstanley.com/articles/active-vs-passive-investing

[24] Active vs. passive investing – the Great Investment Debate, *Rathbone Investment Management Limited.* https://www.rathbones.com/sites/rathbones.com/files/literature/pdfs/rathbones_active_vs_passive_investing_james_pettit_investment_report_full_website.pdf

Unfortunately, in some macroeconomic environments, active management appeared to be less effective in terms of performance.

For example, in 2022, after the aggressive increase of interest rates by the Federal reserve, the performance of funds actively managed has been unfavorable for investors. Blame is being attributed to unexpectedly high inflation and the Federal Reserve's response, characterized by an aggressive policy of interest rate hikes[25].

III. DATA METHODOLOGY

For our study, let us consider a portfolio comprising 20 assets across various sectors. We gather monthly historical closing price[26] data from January 2008 to December 2022. Additionally, we downloaded historical information for a risk-free asset (one-month T-bills) and a market index (the S&P 500).

We divided the dataset in two windows:

- The *estimation window* contains data from 2008 to 2017,
- The *testing window* spans from 2018 to 2022.

Recall that prices of an asset recorded over times are often non-stationary due to long upward trends, and short-run upward or downward trends (i.e. the increase of productivity, the financial crisis). In fact, as we run the sample autocorrelation function, we notice that the prices are highly correlated with past prices (Figure 5).

Financial time series frequently exhibit non-stationarity. Consequently, conducting statistical analyses directly on prices, denoted as P_t, can be challenging. For various reasons, it is preferable to examine the relative changes in prices[27]. Therefore, we calculate the monthly return for each stock using the following formula:

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

r_t is defined as the simple return of the asset with price series P_t. However, in statistical analysis of financial data, it is preferable to consider log-returns, which are defined as:

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[25] Brown, G. (2022, December 9). Why are investments down in 2022? It's all about "real rates". *Kenan Institute of Private Enterprise*. https://kenaninstitute.unc.edu/commentary/why-are-all-investments-down-in-2022-its-all-about-real-rates/

[26] Yahoo Finance, www.yahoofinance.com

[27] Dettling, M. Statistical analysis of financial data. *Zurich University*. https://ethz.ch/content/dam/ethz/special–inter–est/math/statistics/sfs/Education/Advanced%20Studies%20in%20Applied%20Statistics/course–material–1921/FinancialData/Script_v210113.pdf

$$\log(r_t) = \log\left(\frac{P_t}{P_{t-1}}\right) = \log(P_t) - \log(P_{t-1})$$

Prices (and log-prices) are considered non-stationary, while returns are stationary. A weakly stationary time series is characterized by consistent mean, variance, and covariances over different time intervals[28]. A time series is deemed "stationary" if its statistical properties, such as sample mean, variance, and covariance, remain constant over time. Stationarity implies the absence of consistent upward or downward trends, consistent variability around the mean, and the absence of predictable patterns or cycles repeating at regular intervals.

In fact, a time-series process is weakly stationary if and only if:

- $E[X_t] = \mu$ is finite and independent of t , $\forall t \in Z$
- $V[X_t] = \sigma^2$ is finite and independent of t, $\forall t \in Z$
- $Cov(X_t, X_{t-k}) = \gamma k$ is finite and independent of t, $\forall t \in Z$

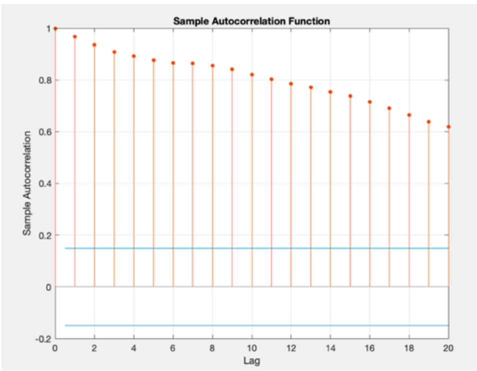

Figure 5: Sample Autocorrelation Function of Closing Prices (Asset 1[29])

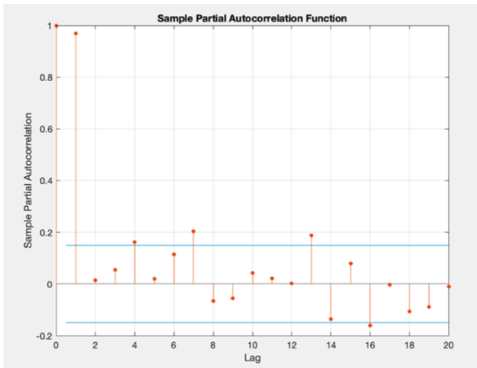

Figure 6: Sample Partial Autocorrelation Function of Closing Prices (Asset 1)

[28] Guidolin, M. (2018, February). Autoregressive moving average (ARMA) models and their practical applications, *Bocconi University.*

[29] We display only Asset 1 (SAP) as the same situation is observed for the other assets. Asset 1 is used for illustrative purposes.

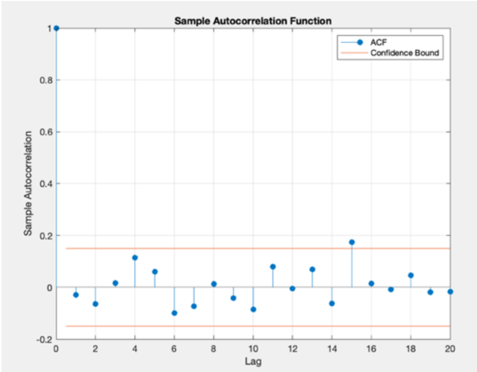

Figure 7: Sample Autocorrelation Function of Log-returns (Asset 1)

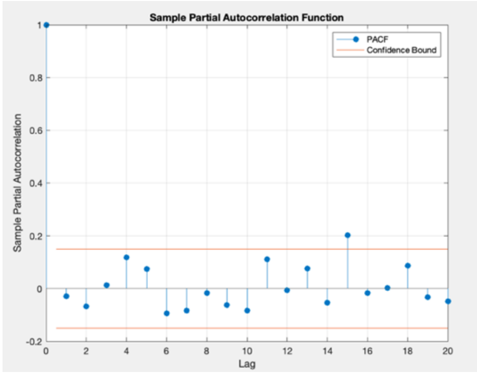

Figure 8: Sample Partial Autocorrelation Function of Log-returns (Asset 1)

In this study we are going to analyze log-returns since price series tend to be non-stationary. In fact, empirical evidence on time series analysis suggests that a typical approach involves log-transforming the series and taking first-order differences at lag 1, resulting in the transformed series denoted as r_t.

Logarithmic returns are extensively favored in quantitative analysis of financial time series over raw prices due to several advantages. These include the ability to normalize returns for comparison across different assets (which is challenging with raw prices), time-additivity properties, and other conveniences for classical statistical and mathematical analyses[30].

If shorter lags exhibit significant positive correlations, it means that a time series is characterized by a trend. In the case of closing prices (figure 5), there seems to be a pattern that relates the closing prices of subsequent lags. Instead, for log-returns (figure 7), this does not happen, meaning that returns are *stationary*.

Let us consider the partial autocorrelation function, which measures the unique correlation between a data point in a time series and its past values, while controlling for the influence of intervening observations at shorter lags. This allows to identify the specific relationship between a data point and its immediate historical values, independent of the influences from other time points. For example, the partial autocorrelation for lag 3 is only the correlation that lags 1 and 2 do not explain.

Moreover, considering the autocorrelation of the monthly prices (figure 5), we can see that previous closing prices appear to be a reliable indicator of future closing

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[30] Tsay, R. S. (2005). Analysis of financial time series (2nd Edition, pp. 7-10). *John Wiley & Sons, Inc*. https://cpb-us-w2.wpmucdn.com/blog.nus.edu.sg/dist/0/6796/files/2017/03/analysis-of-financial-time-series-copy-2ffgm3v.pdf

prices, because there is high correlation between monthly returns. Specifically, the autocorrelation function of prices is close to 1 and this means that there is dependence across prices over time.

Looking at the partial autocorrelation function there is only one statistically significant correlation at lag 1, which means that the asset is strongly correlated with the closing price of the previous month, but the other lags are nearly significant.

Instead, from figure 7-8 we can notice that the correlations between log returns are statistically irrelevant, which means that stock's returns are mostly independent from past returns.

A. Descriptive Analysis

In this section, we perform some descriptive analysis of the dataset, which is composed by 20 different stocks of the NYSE and the NASDAQ.

We have selected stocks from these two markets as they represent key segments of the global financial market, in fact they include a wide range of companies from various sectors and stages of development.

The NASDAQ is known for its technology and growth companies, while the NYSE includes more established and traditional companies.

From the NASDAQ, we have selected the most famous technological companies like Apple, Microsoft, Amazon, and Alphabet (Google). These companies are global leaders in innovation and technology, and studying their stocks can offer valuable insights into growth trends and emerging technologies.

On the other hand, the NYSE, with its long history and large-cap companies, is characterized by stability and less volatility compared to the NASDAQ, in fact most of the assets selected belong to the NYSE, in order to reduce the volatility of the optimal portfolio. Moreover, the NASDAQ and NYSE represent a significant portion of the global stock market. Many companies listed on these exchanges have operations and customers worldwide, making their stocks useful indicators of global economic trends.

The stocks taken into consideration for the allocation portfolio are:

- Asset 1: SAP - Technology
- Asset 2: KO (Coca-Cola) - Consumer Staples (Beverages)
- Asset 3: JPM (JPMorgan Chase) - Financials (Banking)
- Asset 4: BAC (Bank of America) - Financials (Banking)

- Asset 5: PSQ - Finance (Inverse ETF)
- Asset 6: DIS (Walt Disney Company) - Communication Services (Entertainment)
- Asset 7: XOM (Exxon Mobil Corporation) - Energy (Oil & Gas)
- Asset 8: JNJ (Johnson & Johnson) - Healthcare (Pharmaceuticals)
- Asset 9: LLY (Eli Lilly and Company) - Healthcare (Pharmaceuticals)
- Asset 10: MCD (McDonald's Corporation) - Consumer Discretionary (Fast Food)
- Asset 11: GOOGL (Alphabet Inc. Class A) - Technology (Internet)
- Asset 12: AMZN (Amazon.com Inc.) - Consumer Discretionary (E-Commerce)
- Asset 13: APPLE (Apple Inc.) - Technology (Consumer Electronics)
- Asset 14: PFE (Pfizer Inc.) - Healthcare (Pharmaceuticals)
- Asset 15: SHEL (Royal Dutch Shell) - Energy (Oil & Gas)
- Asset 16: E (ENI S.p.A.) - Energy (Oil & Gas)
- Asset 17: NFG (National Fuel Gas Company) - Utilities (Natural Gas)
- Asset 18: CPK (Chesapeake Utilities Corporation) - Utilities (Natural Gas)
- Asset 19: NJR (New Jersey Resources Corporation) - Utilities (Natural Gas)
- Asset 20: ATO (Atmos Energy Corporation) - Utilities (Natural Gas)

In addition, we have selected the S&P 500 as the benchmark for the portfolio. This market index includes 500 stocks from different companies listed in New York, accounting for approximately 80% of the market capitalization. All the stocks in the S&P 500 belong to U.S. companies with a market capitalization exceeding $6.1 billion, a float of at least 50%, monthly trading volume over the last 6 months of not less than 250,000 shares, and an average annual stock price exceeding $1[31].

[31] Wall Street: cos'è lo Standard & Poor’s 500. *Borsa Italiana*.

Asset	Mean	Standard Deviation	Skewness	Kurtosis
SAP	0.0048	0.1121	–0.9048	6.0647
KO	0.0048	0.0946	–0.1229	4.3838
JPM	0.0063	0.0932	–1.0753	8.0207
BAC	–0.0011	0.1440	–2.2430	19.9628
PSQ	–0.0152	0.1012	0.4692	3.9512
DIS	0.0065	0.0987	–0.8153	5.6438
XOM	0.0019	0.0898	–0.0641	4.8487
JNJ	0.0063	0.0901	–0.1235	3.8980
LLY	0.0115	0.1102	–0.1449	4.7848
MCD	0.0094	0.0930	–0.0196	4.6841
GOOGL	0.0108	0.1025	–0.0557	3.9760
AMZN	0.0177	0.1237	–0.0139	4.3077
APPLE	0.0189	0.1131	–0.3852	3.4726
PFE	0.0052	0.1058	–0.3548	5.6167
SHEL	–0.0008	0.0891	–0.1769	4.9009
E	–0.0040	0.0976	–0.1757	5.7671
NFG	0.0027	0.0914	–0.3712	3.8307
CPK	0.0104	0.1040	–0.2291	3.7495
NJR	0.0070	0.0964	–0.0652	3.6367
ATO	0.0081	0.1008	–0.1354	3.3340
S&P 500	0.0062	0.0874	–0.5573	5.4040

Table 1: Descriptive statistics of the 20 assets

Table 1 summarizes some statistical measures for each of the twenty assets, such as the mean, which represents the average return of each asset; the standard deviation, or the volatility of the assets' excess return; the skewness, which indicates the asymmetry of the distribution of returns; and the kurtosis, that represents the "tailedness" of the distribution of returns.

Apple, Amazon, and Eli Lilly have the highest mean returns, with values of 0.0189, 0.0177, and 0.0115 respectively. On the other hand, PSQ, E, and BAC are characterized by the lowest mean returns - note that these assets have negative mean of excess returns, therefore they have been not profitable for investors.

The most volatile assets are BAC (0.1440), Amazon (0.1237), and SAP (0.1121), meaning that they represent higher risk for investors. Conversely, the least volatile assets are Shell (0.0891), and ExxonMobil (0.0898), implying lower risk.

Almost all the assets exhibit negative values for the skewness, therefore they are characterized by a longer tail on the left side. A negatively skewed distribution suggests a lot of little gains and a few significant losses on the investment.

The only asset with positive skewness is PSQ, with a skewness of 0.4692, indicating a longer tail on the right side of the distribution. A positive skew in the return distribution indicates that investors should anticipate frequent modest losses and infrequent significant gains on their investments. The asset PSQ is an *inverse ETF*, which aims to provide inverse exposure to the NASDAQ-100 Index, therefore it is designed to move in the opposite direction of the index. The result of a positive skewness is reasonable, given that this investment strategy bets against the performance of the NASDAQ-100. Over the long term, such an investment is generally not favorable, as the NASDAQ-100 has historically tended to increase in value.

Finally, the kurtosis represents the "tailedness" of the distribution of returns, therefore it measures the heaviness of a distribution's tails with respect to a normal distribution. BAC, JPM, and SAP have the highest kurtosis values of 19.9628, 8.0207, and 6.0647, which indicate more extreme values or heavy tails in the distribution. In contrast, Atmos Energy, New Jersey Resources, and Chesapeake Utilities have the lowest kurtosis values, therefore their distributions tend to have light tails or lack of outliers.

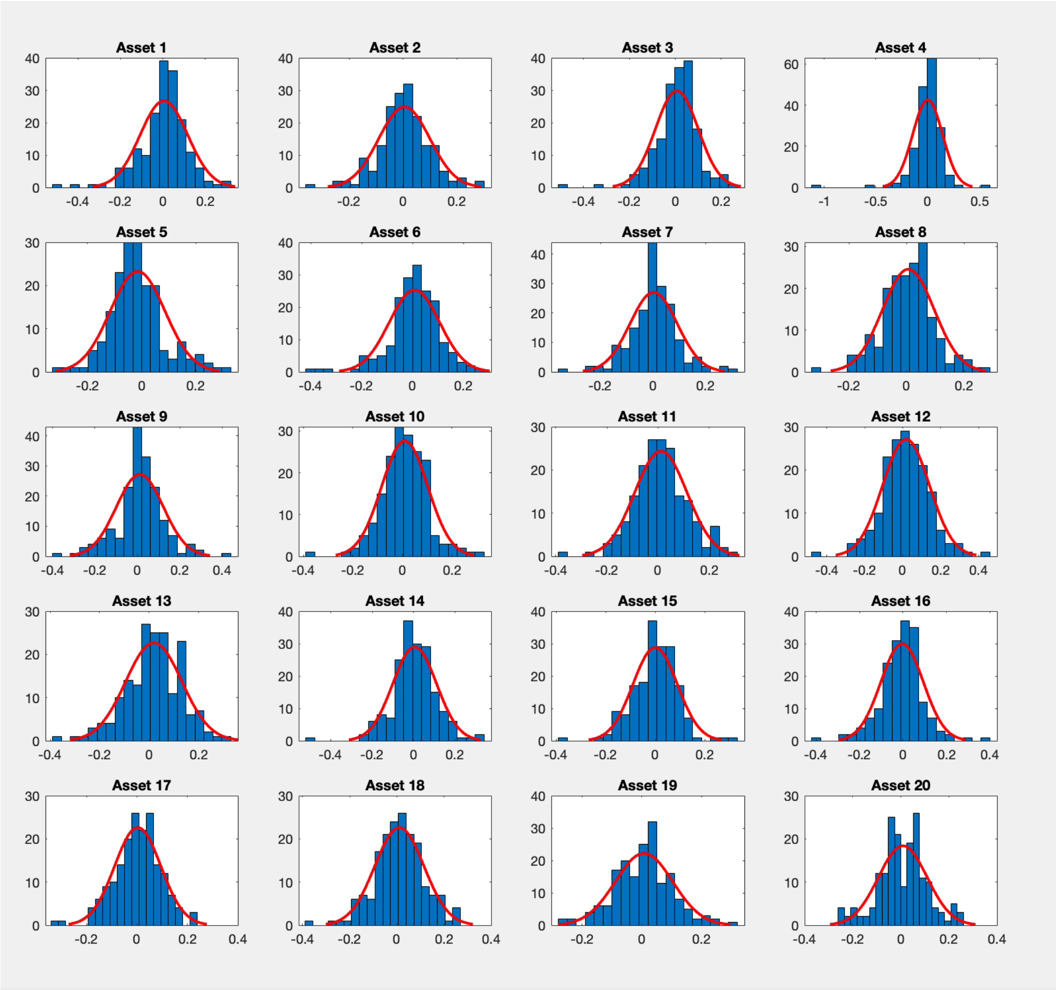

Figure 9: Plot of excess return for each asset i

Figure 9 illustrates the distribution of the excess return for each asset. Note that Asset 2 (Coca Cola), Asset 7 (Exxon), Asset 11 (Google), and Asset 14 (Pfizer) have histograms that closely match the normal distribution fit, showing a symmetrical distribution around the mean with tails that taper off evenly.

In contrast, Asset 4 (Bank of America) exhibits skewness in its distributions, in fact it shows a positive skew with a long right tail. Moreover, Bank of America has the highest kurtosis value (19.96), indicating a very peaked distribution with heavy tails, therefore a significant likelihood of extreme returns compared to a normal distribution. Additionally, some assets show potential outliers or extreme values that deviate significantly from the normal distribution fit. These outliers can be seen as isolated bars far from the central bar of the distribution, as observed in Asset 3 (JP Morgan) and Asset 20 (Atmos).

B. Optimal Allocation Portfolio

The primary advantage of the single-index model lies in its streamlined approach to portfolio construction, requiring fewer data inputs compared to Markowitz's model. Moreover, the Index model offers estimations for both individual security returns and index returns, becoming a valuable tool for determining optimal portfolio allocation.

According to the Index Model, we assume that $\tilde{\varepsilon}_i \perp \tilde{\varepsilon}_j \perp \tilde{R}_M$. Recall that we denote simple returns as *r* and excess returns as *R*.
Let $\sigma^2_{\varepsilon_i}$ denote the variance of the non-systematic error term $\tilde{\varepsilon}_i$, Var $[\tilde{\varepsilon}_i]$. Let us construct a portfolio *p* made of the *n* assets, the market portfolio, and the risk-free asset, with weights respectively w_i (with i = 1, 2, ..., n), w_M and $1 - \sum_{i=1}^{n+1} w_i - w_M$. The return of such a portfolio is:

$$\tilde{r}p = \sum_{i=1}^{n} w_i \tilde{r}_i + w_M \tilde{r}_M + (1 - \sum_{i=1}^{n} w_i - w_M) r_f$$

where r_f is the risk–free rate, $\tilde{r}_i$ the actual return on asset *i* (with *i* = 1, 2, ..., n) and $\tilde{r}_M$ the actual return on the market portfolio. Given that $\tilde{R} = \tilde{r} - r_f$, we get that

$$\tilde{R}_p = \sum_{i=1}^{n} w_i \tilde{R}_i + w_M \tilde{R}_M$$

Considering that $\tilde{R}_i = \alpha_i + \beta_i \tilde{R}_M + \tilde{\varepsilon}_i$ we conclude that

$$\tilde{R}_p = (w_M + \sum_{i=1}^{n} w_i \beta_i) \tilde{R}_M + \sum_{i=1}^{n} w_i \alpha_i + \sum_{i=1}^{n} w_i \tilde{\varepsilon}_i$$

For simplicity, let us introduce the following conventions:

$$w_{n+1} = w_M + \sum_{i=1}^{n} w_i \beta_i$$

$$\gamma_i = \alpha_i \text{ for } i = 1,2, \dots, \text{n and } \gamma_{n+1} = E_M$$

$$\sigma_i^2 = \sigma_{\varepsilon_i}^2 \text{ for } i = 1,2, \dots, \text{n and } \sigma_{n+1}^2 = \sigma_M^2$$

Then, we can write the mean and variance of $\tilde{R}_p$ in the following compact way (notice that we have exploited the orthogonality of $\tilde{R}_M$ and the error terms and the fact their expected value is zero).

The Sample Mean of the return:

$$E[\tilde{R}_p] = \sum_{i=1}^{n+1} w_i \gamma_i$$

The Variance of the return:

$$Var[\tilde{R}_p] = \sum_{i=1}^{n+1} w_i^2 \sigma_i^2$$

Suppose that we wish to minimize the variance of $\tilde{R}_p$ for a given expected value for the excess return on portfolio p, $E[\tilde{R}_p] = E_p$. It is worth emphasizing that because the standard deviation is a monotonic function of the variance, this optimization exercise is equivalent to maximizing the Sharpe ratio among all portfolios made of the market portfolio, the *n* assets and the risk-free asset with expected excess return E_p.

Then, we can solve the following constrained optimization, where we do not need to impose the restriction that the sum of the weights $\sum_{i=1}^{n} w_i + w_M$ is equal to 1 because we can use the position in the risk-free asset to accommodate the restriction that we hold a proper portfolio[32],

$$\min_{w_1,\ldots,w_{n+1}} \mathrm{Var}[\tilde{R}_p]$$

$$\text{s.t.} \quad E[\tilde{R}_p] = E_p.$$

In this analysis we are going to observe historical values of different assets, from which we are going to derive realized returns for the selected periods. Specifically, our aim is to establish risk-adjusted metrics for evaluating performance. To achieve this, we must outline summary statistics that capture the balance between risk and return in the portfolio.

Based on the CAPM, investors should hold a combination of the risk-free asset and the market portfolio. The market portfolio consists of a high number of securities, and an investor would have to own all of them in order to have a composition of assets completely diversified. Since holding all existing securities is not practical, in this session we will consider an alternative method of constructing a portfolio

[32] Vitale, P. Optimal Active Portfolio Management.

that may not require a great number of securities. As previously stated, adding more and more securities will result in a reduction of the non-systemic risk thanks to diversification.

In this section we are going to illustrate the steps followed to find the composition of portfolio $p*$ – consisting of n assets and the market portfolio – which exhibits the highest Sharpe Ratio.

The objective of a portfolio manager is to combine optimally a certain number of assets in order to obtain a return that outperforms the benchmark.

Such model allows fund managers to select a mix of active and passive portfolio that maximizes the (active) Sharpe ratio performance indicator[33].

When returns follow the index model, the optimal portfolio can be derived explicitly, and the solution for the optimal portfolio provides insight into the efficient use of security analysis in portfolio construction[34]. Explaining the logical progression of the solution is enlightening. We won't delve into every algebraic detail, instead, we'll focus on presenting the key outcomes to interpret the methodology.

Before showing the results, it is important to consider the fundamental trade-off that the model incorporates. If our sole focus were diversification, we would simply invest in the market index. However, through security analysis, we gain the opportunity to identify securities with a nonzero alpha and to strategically invest in them. Yet, this strategic positioning comes with a cost: it deviates from efficient diversification, implying an assumption of unnecessary firm-specific risk.

In brief, this model entails that the optimal risky portfolio trades off the search for alpha against efficient diversification.

The optimal risky portfolio is a mixed strategy of two component portfolios: (1) an active portfolio, denoted by A, of the n selected securities and (2) the market-index portfolio, the $(n+1)^{th}$ asset that we included to aid in diversification, which we call the passive portfolio (denote by M).

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[33] Violi, R. (2011). Optimal active portfolio management and relative performance drivers: Theory and evidence. (Vol. 58, pp. 187-209). *Bank for International Settlements*.

[34] Bodie, Z., Kane, A. and Marcus, A. (2014) Investments, Global Edition, 10th Edition. *McGraw Hill Higher Education*, New York.

The construction of the optimal risky portfolio involves several steps based on the index model estimates of security and market index parameters. These steps include using the following formulas:

1. Initial position of security *i* in the active portfolio:
$w_i^0 = \frac{\alpha_i}{\sigma^2(\varepsilon_i)}$
2. Scaled initial positions:
$w_i = \frac{w_i^0}{\sum_{i=1}^{n} \frac{\alpha_i}{\sigma^2(\varepsilon_i)}}$
3. Alpha of the active portfolio:
$\alpha_A = \sum_{i=1}^{n} w_i \alpha_i$
4. Residual variance of the active portfolio:
$\sigma^2(e_A) = \sum_{i=1}^{n} w_i^2 \sigma^2(\varepsilon_i)$
5. Initial position in the active portfolio:
$w_A^0 = \frac{\alpha_A}{\sigma^2(\varepsilon_A)} \cdot \frac{\sigma_M^2}{E(R_M)}$
6. Beta of the active portfolio:
$\beta_A = \sum_{i=1}^{n} w_i \beta_i$
7. Adjusted (for beta) position in the active portfolio:
$w_A^* = \frac{w_A^0}{1 + (1 - \beta_A) w_A^0}$
8. Final weights in passive portfolio and in security *i*:
$w_M^* = 1 - w_A^*$ $w_i^* = w_A^* w_i$
9. The beta of the optimal risky portfolio and its risk premium:
$\beta_P = w_M^* + w_A^* \beta_A = 1 - w_A^*(1 - \beta_A)$ $E(R_P) = \beta_P E(R_M) + w_A^* \alpha_A$
10. The variance of the optimal risky portfolio
$\sigma_P^2 = \beta_P^2 \sigma_M^2 + [w_A^* \sigma(\varepsilon_A)]^2$
11. Sharpe ratio of the risky portfolio

$$S_P^2 = S_M^2 + \sum_{i=1}^{n} \left(\frac{\alpha_i}{\sigma(\varepsilon_i)}\right)^2$$

Table 2[35]: Construction and properties of the optimal risky portfolio

We need to distinguish two components in the optimal portfolio $p*$: the former is a passive component invested entirely in the market portfolio; the latter is an active portfolio invested in the individual assets. We see from Table 2 – point 1 that the contribution of each asset is proportional to its ratio $\frac{\alpha_i}{\sigma_{\varepsilon_i}^2}$.

We denote the overall investment in the active portfolio, A, as w_A^* and we derived the following formulas to find the weights of the optimal active portfolio:

$$w_A^* = \frac{w_A^0}{1 + (1 - \beta_A)w_A^0}$$

$$w_M^* = \frac{1 - \beta_A w_A^0}{1 + (1 - \beta_A)w_A^0}$$

where w_A^0 is the sum of the assets' normalized alphas.

The Equations above (w_M^*, w_A^*) provide insights for determining the optimal allocation within the active portfolio, contingent upon its alpha, beta, and residual variance.

By allocating w_A^* to the active portfolio and 1 – w_A^* to the index portfolio, we can calculate the expected return, standard deviation, and Sharpe ratio of the optimal risky portfolio. It is noteworthy that the Sharpe ratio of a skillfully constructed risky portfolio is expected to surpass that of the index portfolio (representing the passive strategy). The relationship of the two Sharpe Ratios is:

$$S_P^2 = S_M^2 + \left[\frac{\alpha_A}{\sigma_{(\varepsilon_A)}}\right]^2$$

The equation above illustrates how the active portfolio's contribution to the Sharpe ratio of the total risky portfolio depends on the ratio of its alpha to its residual standard deviation. This ratio is known as the *information ratio*, and it

[35] Bodie, Z., Kane, A. and Marcus, A. (2014) Investments, Global Edition, 10th Edition, *McGraw Hill Higher Education*, New York.

represents the additional return achievable through security analysis in contrast to the firm-specific risk we incur when we overweight or underweight securities relative to the passive market index[36].

Note that to maximize the value of the portfolio's squared Sharpe ratio, we must maximize the information ratio of the active portfolio.

C. Estimation of the Optimal Active Portfolio

Let us take into consideration the data for the estimation window going from January 2008 to December 2017, while for the testing window we consider data from January 2019 to December 2022. In this section, our goal is to calculate the Sharpe ratio for the optimal portfolio and compare it with that of a passive strategy invested in the S&P 500 market index. To achieve this objective, we apply the index model to the excess returns of each stock using Ordinary Least Squares (OLS) regression.

Using Matlab, we have performed the index model individually for each stock, treating them as separate regressions. This approach is suitable under the assumption that the error terms associated with the stocks in the index model are uncorrelated, i.e., $\tilde{\varepsilon}_i \perp \tilde{\varepsilon}_j$. We have obtained estimates for each stock *i*, including alpha ($\hat{\alpha}$), beta ($\hat{\beta}$), and the standard error of the residual (interpreted as the standard deviation of the error term in the index model), $\hat{\sigma}_{\varepsilon i}$ (see table 3).

We have calculated estimates for each stock *i*, including alpha ($\hat{\alpha}_i$), beta ($\hat{\beta}_i$), and the standard error of the residual ($\hat{\sigma}_{\varepsilon i}$). Afterwards, we have estimated the weights assigned to each of the *n* stocks within this portfolio with the derived alphas and betas.

For these estimations, we have assumed that these values are time-invariant, therefore they do not change overtime.

$$\mu_i \equiv E\left[R_{i,t}\right] \qquad \mu_M \equiv E\left[R_{M,t}\right],$$

$$\sigma_i^2 \equiv Var\left[R_{i,t}\right] \qquad \sigma_M^2 \equiv Var\left[R_{M,t}\right],$$

$$\text{and} \qquad \sigma_{i,M}^2 \equiv Cov\left[R_{i,t}, R_{M,t}\right].$$

[36] Bodie, Z., Kane, A. and Marcus, A. (2014) Investments, Global Edition, 10th Edition, *McGraw Hill Higher Education,* New York.

Under this set of assumptions, it could be shown that, in the linear model, the following restrictions are respected by the following coefficients:

$$\alpha_i = \mu_i - \beta_i \mu_M, \qquad \beta_i = \frac{\sigma_{i,M}}{\sigma_M^2}$$

Straightforward econometric results suggest that if we estimate the coefficients α_i and β_i using the OLS method, we found that:

$$\hat{\alpha}_i = \bar{R}_i - \hat{\beta}_i \bar{R}_M,$$

$$\hat{\beta}_i = \frac{\hat{\sigma}_{i,M}}{\hat{\sigma}_M^2}$$

where $\bar{R}_i$ and $\bar{R}_M$ are the sample means for the excess returns of asset i and the market portfolio M, while $\hat{\sigma}_{i,M}$ is the sample covariance between the two excess returns, $(1/T)\sum_{t=1}^{T}(R_{i,t} - \bar{R}_i)(R_{M,t} - \bar{R}_M)$, and σ_M^2 the sample variance of the excess return on the market portfolio, $(1/T)\ \sum_{t=1}^{T}(R_{M,t} - \bar{R}_M)^2$. In addition, it can be shown that:

$$\hat{s}_{\varepsilon_i}^2 = \frac{1}{T-2}\sum_{t=1}^{T}\hat{\varepsilon}_{i,t}^2 \ \ with\ \hat{\varepsilon}_{i,t} \equiv R_{i,t} - \hat{\alpha}_i - \hat{\beta}_i R_{M,t}$$

is an *unbiased* estimator of the variance of the error term, $\varepsilon_{i,t}$. Importantly, the estimates of the beta coefficients, $\hat{\beta}_i$'s, and those of the variance of the error terms, $\hat{s}_{\varepsilon_i}^2$'s, are the inputs of the procedure to identify the optimal active portfolio in the baseline formulation.

Obs=119					
Asset	**Alpha**	**Beta**	**SD of the error term**	**Sharpe ratio**	**R^2**
SAP	0.0015	1.0191	0.0542	0.1190	0.668
KO	-0.0021	1.0258	0.0421	0.0871	0.772
JPM	0.0033	0.7812	0.0655	0.1229	0.448
BAC	-0.0112	1.2338	0.1286	0.0033	0.344
PSQ	-0.0185	0.6589	0.0859	-0.1250	0.251
DIS	0.0055	0.9647	0.0404	0.1778	0.764
XOM	-0.0048	0.8983	0.0434	0.0463	0.709

JNJ	0.0010	1.0165	0.0361	0.1258	0.819
LLY	-0.0033	1.2006	0.0488	0.0791	0.775
MCD	0.0044	0.9810	0.0428	0.1617	0.749
GOOGL	0.0069	0.8528	0.0629	0.1617	0.511
AMZN	0.0179	0.9308	0.0793	0.2541	0.439
APPLE	-0.0029	0.9669	0.0702	0.2205	0.519
PFE	-0.0053	1.1118	0.0424	0.0814	0.797
SHEL	-0.0101	0.9164	0.0502	0.0392	0.655
E	-0.0020	0.8906	0.0560	-0.0198	0.590
NFG	0.0057	0.8386	0.0593	0.0690	0.532
CPK	0.0057	1.0245	0.0608	0.1575	0.617
NJR	0.0027	0.9618	0.0593	0.1275	0.599
ATO	0.0034	1.0204	0.0593	0.1444	0.708

Table 3[37]: OLS estimates for the 20 assets

Table 3 summarizes the OLS (Ordinary Least Squares) estimation results for the 20 assets, presenting key metrics such as Alpha, Beta, Standard Deviation of the error term, Sharpe ratio, and R-squared (R^2).

Looking at the Alpha, which measures performance relative to market expectations, AMZN has the highest value (0.0179), suggesting that it outperformed market expectations. In contrast, BAC has a negative alpha (-0.0112), meaning that it underperformed market expectations.

Furthermore, Bank of America (1.2338) and LLY (1.2006) exhibit high betas, implying higher volatility compared to the market, while JPM (0.7812) and PSQ (0.6589) show lower betas, indicating they are less volatile than the market.

The Standard Deviation of the error term reflects the unexplained volatility in the asset's excess returns. BAC has the highest standard deviation (0.1286), indicating high unexplained volatility, while JNJ has the lowest (0.0361), indicating stable excess returns.

[37] Table 3 shows OLS estimation results for 20 assets, including Alpha (performance relative to market expectations), Beta (market sensitivity), Standard Deviation of the error term (unexplained volatility), Sharpe ratio (risk-adjusted return), and R-squared (proportion of return variability explained by the market). Each estimation was based on the 119 observations corresponding to the estimation window going from January 2008 to December 2017.

For what concerns the Sharpe ratio, AMZN has the highest Sharpe ratio (0.2541), suggesting it provides the best return per unit of risk. Instead, PSQ, with a negative Sharpe ratio (-0.1250), displays poor risk-adjusted performance.

R-squared (R^2) represents the proportion of excess return variability explained by the market. JNJ has the highest R^2 (0.819), therefore its excess returns are well-explained by market movements, whereas BAC has a relatively low R^2 (0.344), so a portion of its excess returns are not explained by the market.

The optimal portfolio p^* has the following weights for each asset:

- Asset 1: SAP = 0.35
- Asset 2: KO = –0.83
- Asset 3: JPM = 0.55
- Asset 4: BAC = –0.48
- Asset 5: PSQ = –1.77
- Asset 6: DIS = 2.39
- Asset 7: XOM = –1.80
- Asset 8: JNJ = 0.51
- Asset 9: LLY = –0.97
- Asset 10: MCD = 1.70
- Asset 11: GOOGL = 1.22
- Asset 12: AMZN = 2.00
- Asset 13: APPLE = 1.86
- Asset 14: PFE = –1.53
- Asset 15: SHEL = –1.50
- Asset 16: E = –2.28
- Asset 17: NFG = –0.40
- Asset 18: CPK =1.08
- Asset 19: NJR = 0.55
- Asset 20: ATO = 0.99
- Market Index: S&P 500 = –1.02

Note that the weights for the optimal portfolio suggest a strategy that heavily leverages certain assets while shorting others.

DIS (2.39), AMZN (2.00), APPLE (1.86), and MCD (1.70) are heavily and positively weighted, therefore the strategy implies a strong belief in their future performances.

Other assets with significant positive weights include GOOGL (1.22), CPK (1.08), and JPM (0.55). These assets are expected to outperform and contribute positively to the portfolio returns.

Conversely, PSQ (-1.77), XOM (-1.80), PFE (-1.53), SHEL (-1.50), and E (-2.28) are significantly shorted, indicating an expectation of poor performance.

KO (-0.83), BAC (-0.48), LLY (-0.97), NFG (-0.40), and S&P 500 (-1.02) also have negative weights, therefore they are expected to underperform relatively to the market or other assets in the portfolio.

If the selected long positions outperform and the short positions underperform as expected, these outcomes could be very profitable. However, this strategy also introduces substantial risk, as these positions are highly leveraged.

In addition, the negative weight on the S&P 500 suggests a hedge against broad market declines, but it also requires the specific asset choices to outperform the general market trends. In the provided optimal portfolio $p*$, the market index (S&P 500) has a weight of -1.02. This indicates a bearish outlook on the overall market performance.

In brief, we have obtained an intuitive characterization of the optimal portfolio, $p*$, and the optimal active portfolio. Specifically, the overall investment in the active portfolio is a function of the sum of the normalized individual alphas, α_A, and of the weighted average of the corresponding betas, β_A.

Furthermore, notice that long and short positions in the *n* assets are combined in the active portfolio to the extent that they exhibit both positive and negative alphas. This makes sense given that the ideal active portfolio takes a long (short) position in assets that are underperforming or overperforming.

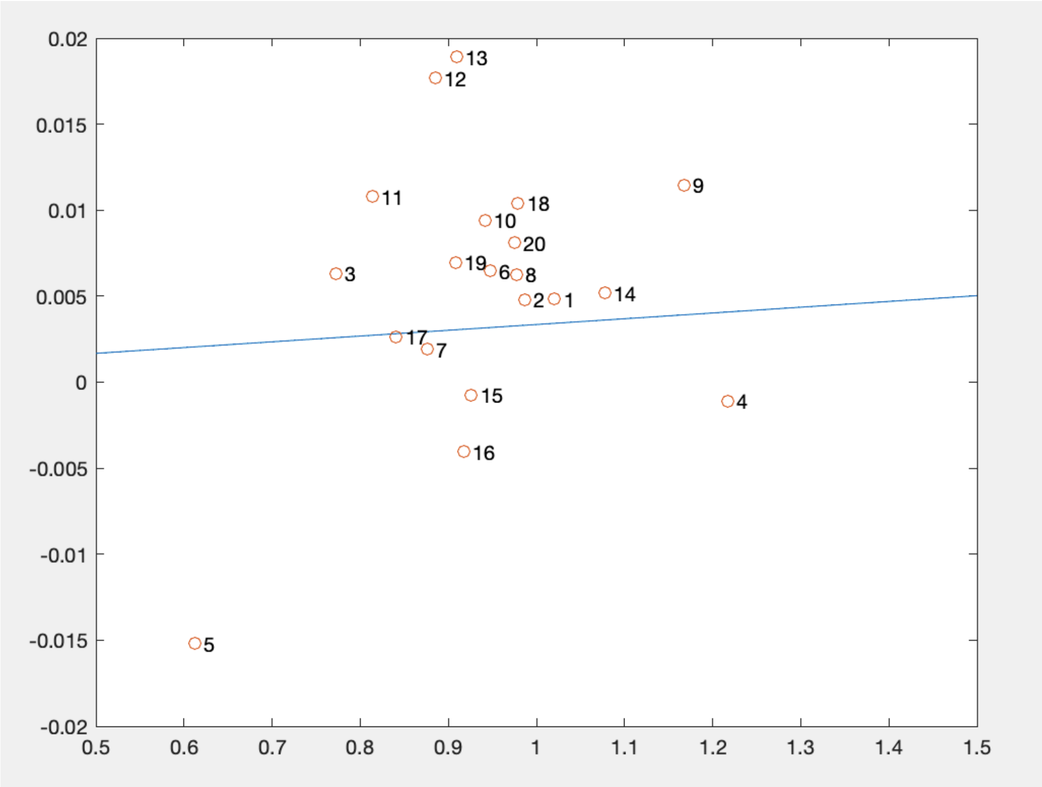

Figure 10: Security Market Line of the optimal portfolio

Looking at figure 10, we can see that the assets lying above the security market line are the ones that present a positive weight - as they are undervalued - therefore, according to the index model, it is reasonable to buy them. Instead, since the assets disposed below the security market line are overvalued, their weights are negative.

For instance, looking at asset 13 (Apple), our model suggests a long position of 186%, since it is highly undervalued.

Using the weights that we found above, it is possible to calculate the return on the optimal portfolio.

Let us consider the testing window (Jan 2018-Dec 2022) to analyze the performance of the optimal active portfolio. Recall that the excess return on the portfolio is given by the following formula:

$$R_p = \sum_{i=1}^{n} w_i R_i + w_m R_m$$

As previously pointed out, it is relevant to estimate the risk of the portfolio to determine its profitability. Therefore, we have calculated the Sharpe Ratio of the optimal portfolio:

$$S_p = \frac{\bar{R}_p}{\sigma_p}\sqrt{12} = 0.28$$

Since we have downloaded monthly historical data, it is important to annualize the result of all the measures of performance by multiplying the result by the square root of 12.

Next, we have compared the realized Sharpe ratio with the theoretical one for the optimal portfolio. The realized Sharpe ratio is based on historical data from the testing window, while the theoretical Sharpe ratio is derived from the expressions for $E[\widetilde{R}_{p*}]$ and $\mathrm{Var}[\widetilde{R}_{p*}]$, estimated with the data from the estimation window. We have derived the square value of the Sharpe ratio for the optimal portfolio p*:

$$S_p^2 = \frac{E[\tilde{R}_{p*}]^2}{\mathrm{Var}[\tilde{R}_{p*}]} = \left(\frac{\lambda \sum_{i=1}^{n+1} \frac{\gamma_i^2}{\sigma_i^2}}{\lambda^2 \sum_{i=1}^{n+1} \frac{\gamma_i^2}{\sigma_i^2}} \right)^2 = \sum_{i=1}^{n+1} \frac{\gamma_i^2}{\sigma_i^2} = S_M^2 + \sum_{i=1}^{n} \frac{\alpha_i^2}{\sigma_{\varepsilon_i}^2}$$

This formula shows the improvement in the Sharpe ratio that we can obtain combining optimally the market portfolio and the assets with non-zero alphas.

	Theoretical Sharpe Ratio	**Sharpe Ratio**	**Sortino Ratio**	**Treynor Ratio**	**Jensen Alpha**
Portfolio	1.8511	0.2822	0.3813	0.0479	0.1941
Benchmark	0.4365	-0.0056	-0.0079	-0.0006	0

Table 4: Performance measures of the optimal portfolio and the benchmark

From the results in table 4, we conclude that both the realized Sharpe Ratios of the active and the passive strategies, which are based on data from 2018 to 2022, are significantly lower than the theoretical Sharpe Ratio. The theoretical Sharpe Ratio is derived from the estimation window (which considers the previous ten years of data with respect to the realized Sharpe ratio), therefore it entails a calculation based on past performances. This suggests that the portfolio's performance was not as strong as expected.

The explanation for these results may lie in the differing market conditions during the period of 2018-2022 compared to those assumed in the theoretical model.

For example, unexpected events, such as the pandemic and the lockdown, have significantly impacted financial markets and portfolio performance in years 2021 and 2022.

In addition, it is possible to compare the Sharpe Ratio of such portfolio with the one of the market index. We have estimated the Sharpe ratio of the market portfolio dividing the mean of the excess returns of the S&P 500 with its standard deviation:

$$S_M = \frac{\bar{R}_M}{\sigma_M} = -0.0056$$

Since the realized Sharpe Ratio of the optimal portfolio over the period 2018-2022 is 0.28, it is significantly higher than the Sharpe Ratio of the S&P 500, which stands at -0.0056. The negative Sharpe Ratio for the market index indicates that an investment solely concentrated in the S&P 500 would result in a loss over this period. We conclude that the strategy of the optimal portfolio outperformed an investment exclusively concentrated in the market index (S&P 500).

Note that the superior performance of the optimal portfolio is not surprising. In fact, in the previous section, we have obtained a negative weight for the market index. Specifically, the short position in the S&P 500 (-1.02) represents a hedging strategy against market risk, suggesting a bearish outlook on the overall market.

If we consider also other measures of performance, such as the Sortino ratio and the Treynor ratio, the outcome is still favorable for the optimal active portfolio, which overcomes the benchmark for every ratio (see table 4).

However, even if the market index displays a lower return with respect to the optimal portfolio, the volatility of the market portfolio is lower, therefore the latter is less risky. For instance, the standard deviation of the returns in the testing window is 2.35 for the optimal portfolio and 0.38 for the market index.

D. Rebalancing Dynamics

The primary objective of constructing optimal portfolios is to minimize risk while maximizing investment returns regardless of market conditions.

In this section, the aim is to maximize portfolio returns through the practice of dynamic portfolio rebalancing.

As new data become available to investors, it is possible to rebalance the weights of the active portfolio, including more recent data in the *estimation window*.

We have started by considering the estimation window covering from January 2008 to December 2017, and the testing window spanning from January 2018 to December 2018. In this analysis, the testing window becomes shorter than the one used previously, enhancing a more precise estimation.

By updating the estimation window every year, we have re-estimated the alphas, betas, and standard deviations of the error term for the *n* stocks using data from January 2009 to December 2018.

For instance, shifting the estimation window one year ahead, we could derive the optimal portfolio for the updated testing window (from January 1st, 2019 to December 1st, 2019).

Of course, the expected value and standard deviation of the excess return on the market proxy must also be recalculated using the updated estimation window.

With the new estimates, we have estimated the weights of the *n* stocks in the optimal active portfolio, and we have derived the realized returns of the rebalanced optimal active portfolio for the updated period: January 2019 - December 2019. Thus, once again, we have only calculated the realized returns on the rebalanced optimal active portfolio over one-year interval.

Replicating this procedure four times, the optimal active portfolio is rebalanced on January 1st, 2019, January 1st, 2020, January 1st, 2021, and January 1st, 2022.

Notice that, thanks to this procedure, we can visualize the realized return over each of the five years of our optimal portfolio, and we can identify which years are more profitable.

Lastly, let us compare the return on the rebalanced portfolio with the initial one, using the formula for the final return recalculated every year (with t=2018,2019,2020,2021,2022):

$$R_{p,t} = \sum_{i=1}^{n} w_{i,t} R_{i,t} + w_{M,t} R_{M,t}$$

		Theoretical Sharpe Ratio	Sharpe Ratio	Sortino Ratio	Treynor Ratio	Jensen Alpha
2018	Portfolio	1.8511	0.5635	0.8502	0.0729	0.3748
	Benchmark	0.4365	-0.7788	-0.9897	-0.0468	0
2019	Portfolio	2.2042	3.2400	4.5941	3.0917	1.2545
	Benchmark	0.6357	1.7725	2.9679	0.1409	0
2020	Portfolio	2.3224	4.0081	6.5480	0.7774	0.8099
	Benchmark	0.8491	1.2263	1.6952	0.1510	0
2021	Portfolio	2.5981	0.1107	0.1479	0.0139	-0.0531
	Benchmark	0.8693	0.2565	0.3453	0.0266	0
2022	Portfolio	2.5041	-2.1665	-3.3113	-0.3794	-1.0921
	Benchmark	0.6657	-2.3870	-3.3149	-0.2747	0
Mean	Portfolio	2.2960	1.1512	1.7658	0.7153	0.2588
	Benchmark	0.6913	0.0179	0.1408	-0.0006	0

Table 5: Performance measures using the rolling-window estimation

According to Table 5, the measures of performance for the optimal active portfolio are higher than the passive strategy for years 2018, 2019 and 2020. Instead, the trend is inverted for years 2021 and 2022. We suppose that this inversion of performance could be explained through the big shock on the market caused by the spread of Covid-19, and the subsequent lockdown.

Notice that the weights of assets' allocation are calculated on the basis of the ten years prior to the testing window. Thus, the evaluation of the weights is based on different market conditions with respect to the two years affected by the pandemic.

Moreover, let us compare the effective Sharpe ratio and the theoretical one.

According to the results, the effective Sharpe ratio overcomes the theoretical one for years 2019 and 2020, while it is underperforming with respect to the expectations for the rest of the periods. In fact, in these two specific years the stock American markets were particularly prosperous. For example, in 2019, the Nasdaq celebrated its strongest performance since 2009. The rebound in 2019 was particularly notable, since it was characterized by robust performances across various sectors and market capitalizations. This growth was not limited to the Nasdaq alone; global exchanges also experienced a notable recovery following a challenging 2018, bolstered by a renewed wave of central bank easing[38]. Instead, in 2020, despite the global lockdown due to the pandemic, equities demonstrated remarkable strength, reflecting the underlying resilience of the financial system amidst turbulent times. Both the S&P and the Dow Jones Industrial Average achieved record highs for 2020, boasting annual gains of 16.3% and 7.2% correspondingly. The Nasdaq also surged significantly with a remarkable 43.6% year-on-year increase, marking its largest gain since 2009 for the tech-heavy index[39].

Despite the low performances of years 2021 and 2022, a hypothetical investor who employed the strategy of rebalancing weights every year from 2018 to 2022 would get an average Sharpe ratio of $S_p = 1.15$, that is higher than the Sharpe ratio of the optimal portfolio with fixed weights (0.28).

In conclusion, the rebalancing frequency can be adjusted with the aim of achieving a more precise weights' estimation. For instance, we could opt for rebalancing every six months instead of annually.

E. Correlated Residuals: SURE Methodology

Dynamic rebalancing models have long been used in solving optimal asset allocation problems, and a number of trading systems have been implemented in order to rebalance the optimal portfolio in order to align to market conditions[40].

In the previous estimation, we have assumed that the error terms were not correlated among themselves, following one of the main assumptions of the standard formulation of the index model.

However, in the index model, asset returns are assumed to be linearly related to the market return and to a unique risk component. This assumption imposes a

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[38] Market Intelligence Desk Team. (2020, January). US markets: Review and outlook. *Nasdaq*. https://www.nasdaq.com/articles/2019-review-and-outlook-2020-01-07

[39] Mikolajczak, C. (2021, January 1). U.S. stocks in 2020: A year for the history books. *Reuters*. https://www.reuters.com/article/idUSKBN2951LQ/

[40] Almahdi, S., & Yang, S. (2017). An adaptive portfolio trading system: A risk-return portfolio optimization using recurrent reinforcement learning with expected maximum drawdown. *Expert Systems with Applications*, 70, 1-15.

specific structure on the variances and covariances of asset returns, simplifying the estimation process.

By imposing this structure, the index model reduces the number of parameters that need to be estimated compared to more complex models. This can make the estimation more manageable, especially when dealing with a large number of assets.

Although the index model's streamlined structure can be convenient, it might not adequately represent the intricacies of actual financial markets. In fact, asset returns may be influenced by factors other than market and asset-specific risk, as market interactions are frequently nonlinear. According to the single-index model, there are two types of risk associated with assets: systematic risk, related to the market, and unsystematic risk, which is the asset-specific risk. However, the index model does not account for other risk factors like industry-specific risk, regulatory risk, or geopolitical risk. This happens because the index model assumes that the error terms are uncorrelated between the assets, but the error terms include all the external risk factors that can affect the performance of the excess returns of each asset, other than the market risk.

Because of its simplifying assumptions, this model may fail to capture important complexities of the market, such as the impact of industry-specific factors and other sources of risk[41]. Therefore, in the next section, we implement an extension of the index model, with the inclusion of possible correlation between the error terms of the assets.

F. Multivariate Regressions

Typically, time-series studies are conducted by employing data on a *group* of assets, such as securities and portfolios of securities. In this way, rather than conducting tests on individual assets, we have considered the group of assets as a whole. This is a superior econometric approach, since the resulting analysis possesses larger power[42].

Hence, let us construct a system of *N* regressions, one for any asset, security or portfolio:

[41] Aldrich, E. M. (2024, January). Index models - Security markets and financial institutions.
[42] Vitale, P. Optimal Active Portfolio Management.

$$R_{1,t} = \alpha_1 + \beta_1 R_{M,t} + \varepsilon_{1,t}$$
$$R_{2,t} = \alpha_2 + \beta_2 R_{M,t} + \varepsilon_{2,t}$$
$$\vdots \qquad \vdots$$
$$R_{n,t} = \alpha_n + \beta_n R_{M,t} + \varepsilon_{N,t}$$

This system could be written in a more compact form, through vector form representation, therefore:

$$\underset{n\times 1}{\boldsymbol{R}_t} = \underset{n\times 1}{\boldsymbol{\alpha}} + \underset{n\times 1}{\boldsymbol{\beta}} R_{M,t} + \underset{n\times 1}{\boldsymbol{\varepsilon}_t}$$

where $\mathbf{R}_t$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\varepsilon}_t$ represent respectively the excess returns on the *n* risky assets, their alphas, betas, and idiosyncratic errors.

Note that we have assumed that the error term is unpredictable and uncorrelated with the regressor. We also have assumed that it is *homoscedastic* and uncorrelated overtime.

$$\forall\, t, s \;\; E\left[\varepsilon_t \mid R_{M,t}\right] = 0$$

$$Cov\left[\varepsilon_t \mid R_{M,t}\right] = 0$$

$$E\left[\varepsilon_t \cdot \varepsilon_t^T \mid R_{M,t}\right] = \Omega, \quad E\left[\varepsilon_t \cdot \varepsilon_s^T \mid R_{M,t}\right] = 0$$

Recall that $\boldsymbol{\Omega}$ is a $n \times n$ covariance matrix, which is symmetric and positive definite, as in the univariate specification, the first condition entails that $E[\boldsymbol{\varepsilon_t}] = \mathbf{0}$, while the last implies that $E[\varepsilon_t\, \varepsilon_t^T] = \boldsymbol{\Omega}$.

In addition, we have assumed that the excess returns on the *n* assets and the market portfolio still possess time-invariant mean and variance,

$$\boldsymbol{\mu} \equiv E[\boldsymbol{R}_t], \qquad \mu_M \equiv E\left[R_{M,t}\right],$$
$$\boldsymbol{V} \equiv Var[\boldsymbol{R}_t], \qquad \sigma_M^2 \equiv Var[R_{M,t}],$$

And we have introduced the following notation:

$$\boldsymbol{V_{r\,M}} \equiv E[(\boldsymbol{R}_t - \boldsymbol{\mu})(R_{M,t} - \mu_M)]$$

to indicate the vector of covariances between the (excess) returns on the *N* assets and the (excess) return on the market portfolio.

Because the matrix $\boldsymbol{\Omega}$ could be non-diagonal, correlation between the contemporaneous values of the error terms is allowed. Indeed, when the index model does not capture all possible sources of risk which affect asset prices, these error terms are very likely to be correlated. Since the SURE methodology captures the possible effects of such correlation it may be favored.

Under this set of assumptions, as seen for the univariate case, it can be shown that in this system of seemingly unrelated regression equations (SURE) the following applies:

$$\boldsymbol{\alpha} = \boldsymbol{\mu} - \boldsymbol{\beta}\,\mu_M, \qquad \boldsymbol{\beta} = \frac{\mathbf{V}_{\boldsymbol{r}\,\boldsymbol{M}}}{\sigma_M^2}$$

To estimate the SURE components, we could rely on the OLS method.

The OLS estimators for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the following:

$$\widehat{\boldsymbol{\alpha}} = \overline{\mathbf{R}} - \widehat{\boldsymbol{\beta}}\bar{R}_M,$$

$$\widehat{\boldsymbol{\beta}} = \frac{\sum_{t=1}^{T}(\mathbf{R}_t - \overline{\mathbf{R}})\,(\mathrm{R}_{M,t} - \bar{R}_M)}{\sum_{t=1}^{T}(\mathrm{R}_{M,t} - \bar{R}_M)^2} = \frac{\widehat{\mathbf{V}}_{\boldsymbol{r}\,\boldsymbol{M}}}{\hat{\sigma}_M^2}$$

where $\overline{\mathbf{R}}$ and $\bar{R}_M$ are the sample means of the excess returns on the risky assets and the market portfolio,

$$\overline{\mathbf{R}} \equiv \frac{\mathbf{1}}{\mathbf{T}}\sum_{\mathbf{t=1}}^{\mathbf{T}}\mathbf{R}_{\mathrm{t}}\,, \qquad \text{and} \qquad \overline{\mathrm{R}}_{\mathrm{M}} \equiv \frac{1}{\mathrm{T}}\sum_{\mathrm{t=1}}^{\mathrm{T}}\mathrm{R}_{\mathrm{M,t}}$$

While $\widehat{\mathbf{V}}_{\boldsymbol{r}\,\boldsymbol{M}}$ is the vector of sample covariances between the excess returns of the individual assets and that of the market portfolios[43]. As in the univariate case, the OLS estimators of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ correspond to the sample counterparts of their theoretical values. Finally, notice that a consistent estimator for the covariance matrix $\boldsymbol{\Omega}$ is $\widehat{\boldsymbol{\Omega}}$

[43] These correspond to individual covariances we denoted with $\hat{\sigma}_{i,m}$ in the univariate regressions and hence the estimates of the betas from the SURE regressions correspond to the OLS ones obtained from the univariate ones.

$\equiv \frac{1}{(T-2)} \sum_{t=1}^{T} \hat{\varepsilon}_t \cdot \hat{\varepsilon}_t^T$, where $\hat{\varepsilon}_t^T \equiv \mathbf{R}_t - \hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\beta}} R_{M,t}$.[44] Clearly, the vector of estimates for the beta coefficients, $\hat{\boldsymbol{\beta}}$, and the estimated covariance matrix of the error terms, $\hat{\boldsymbol{\Omega}}$, were the inputs of the procedure to identify the optimal active portfolio in the formulation with correlated error terms.

G. Estimation of the Optimal Portfolio with SURE Methodology

Building upon the estimation of the optimal portfolio through the index model, an extension considers the potential correlation among error terms for the *n* stocks. In such case, the *n* regressions linked to the index model need to be estimated utilizing the Seemingly Unrelated Regression Equations (SURE)[45] methodology[46].

In order to implement this extension, we have verified if the error terms of the 20 assets were correlated, through the calculation of the following matrix:

$$\sum = \begin{pmatrix} \Omega & 0 \\ 0^T & \sigma_M^2 \end{pmatrix}$$

Where $\boldsymbol{\Omega}$ is a $n \times n$ covariance matrix for the assets' error terms, $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$..., $\tilde{\varepsilon}_n$, and $\mathbf{0}$ a n×1 vector of zeros. This implies that we have maintained the crucial assumption that such error terms are not correlated with the return on the market portfolio, so that they still represent non-systematic risk. On the other hand, as Ω could be non-diagonal they could also be correlated among themselves. In the case Ω is diagonal we would have returned to the baseline formulation we have discussed previously[47], without extending the index model with the SURE methodology.

Our covariance matrix is positive defined, but not diagonal, therefore, there exists correlation between the error terms. In light of this, we have used the SURE methodology to estimate simultaneously the *n* regressions related to the index model.

In Figure 11, the correlation between the error terms is represented.

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[44] The real difference with the univariate analysis is the estimation of the moments of the error terms.

[45] Seemingly Unrelated Regression Equations (SURE) is an estimator used in system regression, capable of simultaneously estimating multiple models. This facilitates the testing of hypotheses across models, as the covariance of parameters is resilient to residual correlation between models. This methodology can enhance the precision of parameter estimates, when certain residuals exhibit conditional homoskedasticity and regressors vary across equations.

[46] It is important to note that this process yields an estimation of the covariance matrix Ω, which may not necessarily be positive semi-definite. Should this occur, adjustments to the procedure must be considered.

[47] Vitale, P. Optimal Active Portfolio Management.

Note that there is high correlation between the error terms of assets 3 and 4 (JP Morgan and Bank of America), unsurprisingly, because both stocks belong to the banking sector. Moreover, there is also high correlation between the securities 18,19 and 20, that belong to the sector of natural gas. Instead, there is no correlation at all between assets 13 and 14, respectively Apple (Technology) and Pfizer (Pharmaceutical). This also holds for asset 4 with respect to assets 19 and 20.

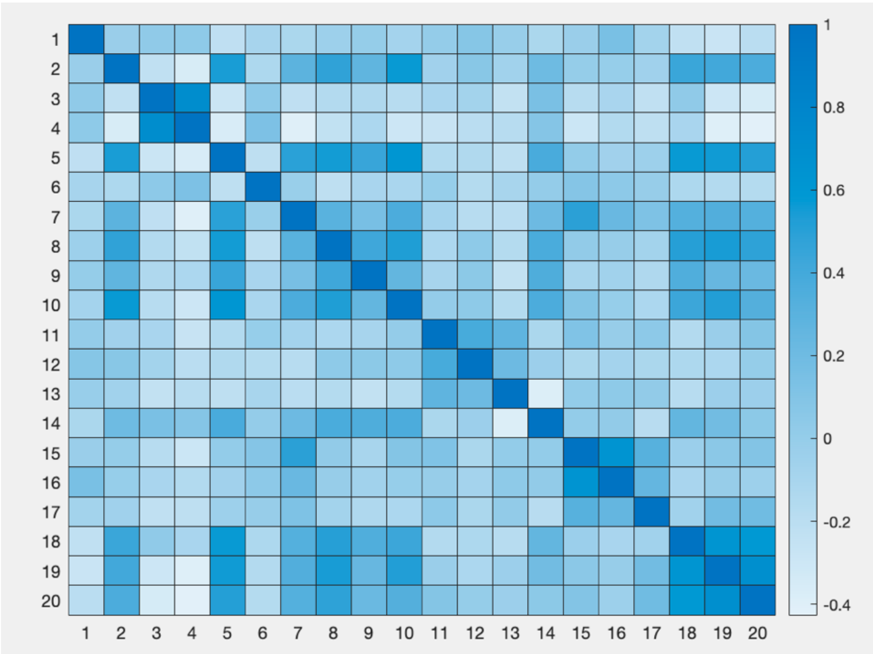

Figure 11[48]: Correlation between the error terms of the 20 assets

In this section, we aim to identify the Sharpe ratio and other performance measures of the optimal portfolio using the SURE methodology, and then we want to compare them with the ones related to the market[49].

Therefore, we have estimated the weights of the securities, extending the formulas with the SURE methodology:

[48] The image represents a heatmap of the correlation matrix for the error terms of 20 assets. Each cell in the heatmap corresponds to the correlation coefficient between the error terms of two different assets. The intensity of the color reflects the magnitude and direction of the correlation, with the color scale on the right ranging from -0.4 to 1.0. Darker blue hues indicate strong positive correlations, while lighter shades suggest weaker correlations. The diagonal elements, which are all equal to 1, are the darkest blue because each asset is perfectly correlated with itself.

[49] We have already estimated the performance measures of the market in section III.C, table 4.

$$w_M^* = \frac{\frac{E_M}{\sigma_M^2} - \beta T\Omega^{-1}\alpha}{\frac{E_M}{\sigma_M^2} + (1_F - \beta)T\Omega^{-1}\alpha} = \frac{1 - \frac{\beta T\Omega^{-1}\alpha}{\frac{E_M}{\sigma_M^2}}}{1 + \frac{(1_F - \beta)T\Omega^{-1}\alpha}{\frac{E_M}{\sigma_M^2}}}$$

$$w_F^* = \frac{\Omega^{-1}\alpha}{\frac{E_M}{\sigma_M^2} + (1_F - \beta)T\Omega^{-1}\alpha} = \frac{\frac{\Omega^{-1}\alpha}{\frac{E_M}{\sigma_M^2}}}{1 + \frac{(1_F - \beta)T\Omega^{-1}\alpha}{\frac{E_M}{\sigma_M^2}}}$$

We obtained the following results:

- Asset 1: SAP = 1.89
- Asset 2: KO = -12.28
- Asset 3: JPM = 9.33
- Asset 4: BAC = -7.94
- Asset 5: PSQ = -16.64
- Asset 6: DIS = 11.75
- Asset 7: XOM = -0.88
- Asset 8: JNJ = 3.78
- Asset 9: LLY = 1.53
- Asset 10: MCD = 24.27
- Asset 11: GOOGL = -5.35
- Asset 12: AMZN = 3.46
- Asset 13: APPLE = 8.00
- Asset 14: PFE = 2.98
- Asset 15: SHEL = -5.04
- Asset 16: E = -8.05
- Asset 17: NFG = 3.33
- Asset 18: CPK = 11.06
- Asset 19: NJR = -5.21
- Asset 20: ATO = 8.04
- Market Index = -27.00

As in the previous estimation, long and short positions in the *n* assets are combined in the active portfolio to the extent that they exhibit both positive and negative alphas. This makes sense given that the ideal active portfolio takes a long (short) position in assets that are underperforming (overperforming).

Yet, these findings highlight a significant issue with this model. The optimal portfolio requires highly leveraged long/short positions that could prove impractical for real-world portfolio management. For example, it suggests some long position in the active portfolio, such as 11.75 (1,175%) in the asset Disney or 24.27 in Mc Donald, which are primarily funded by a short position of 27.00 in the S&P 500 index. Additionally, the annual standard deviation of this optimal portfolio stands at 8.89, a risk level typically acceptable only to exceedingly aggressive hedge funds.

In conclusion, let us consider the Sharpe ratio for this system:

$$S_p^2 = \frac{E\left[\tilde{R}_{p*}\right]^2}{\mathrm{Var}\left[\tilde{R}_{p*}\right]} = \gamma^T \Sigma^{-1} \gamma = (\alpha^T \; E_M) \begin{pmatrix} \Omega^{-1} & 0 \\ 0^T & 1/\sigma_M^2 \end{pmatrix} \begin{pmatrix} \alpha \\ E_M \end{pmatrix} = \alpha^T \Omega^{-1} \alpha + \frac{E_M^2}{\sigma_M^2}$$

$$S_P = 0.1969 * \sqrt{12} = 0.6496$$

$$S_T = 2.1822$$

	Theoretical Sharpe Ratio	Sharpe Ratio	Sortino Ratio	Treynor Ratio	Jensen Alpha
Portfolio	2.1822	0.6496	0.8976	0.1386	1.6750
Benchmark	0.4365	-0.0056	-0.0079	-0.0006	0

Table 6: Performance measures using the SURE methodology

Also in this case, the optimal active portfolio exceeds the market portfolio investing in the S&P 500. Note that using the SURE extension, the Sharpe ratio has increased with respect to the basic formulation, from 0.28 to 0.65.

However, since the weights of the assets have increased for this estimation, we conclude that a higher Sharpe ratio is achieved at the cost of higher overall volatility. Specifically, the increased weights lead to higher overall volatility of the portfolio, because larger positions in individual assets amplify the portfolio's sensitivity to the assets 'specific risks.

IV. CONCLUDING REMARKS

This study investigates the performance of active portfolio management strategies compared to passive strategies. In particular, we have compared some performance measures of an active portfolio with those of a passive strategy investing solely in the market index S&P 500.

The analysis is based on the index model and considers a strategy investing in 20 assets from the NASDAQ and the NYSE, plus an investment in the market index S&P 500.

Firstly, we have estimated the weights of the optimal portfolio over a ten-years period spanning from 2008 to 2017. Through these weights, we have estimated some performance measures of the portfolio over a five-years period, going from 2018 to 2022, and we have obtained evidence that such portfolio outperforms the passive strategy investing in the S&P 500.

Building upon this basic formulation, we have estimated a portfolio characterized by a periodic update of the weights, where we have calculated the maximum Sharpe ratio every year. We have concluded that, since the mean of the Sharpe ratios of the rebalanced portfolios is higher than the one of the portfolio with fixed weights, the rebalancing strategy is more profitable and more accurate for investors.

Moreover, we have extended the analysis to the case in which the error terms of the assets may be correlated. For this formulation, the Sharpe ratio has resulted to be equal to 0.65, therefore higher than that of the basic formulation, but lower than that of the periodically rebalanced portfolio.

This analysis suggests that actively managed portfolios, especially those with a systematic rebalancing mechanism, can outperform passive strategies: the optimal choice consists in recalculating the portfolio's maximum Sharpe ratio periodically.

This suggests that periodic adjustments to the portfolio composition based on changing market conditions enhance the portfolio’s performance.

However, while the actively managed portfolios generally outperform, there are exceptions. In the years 2021 and 2022, the optimal active portfolio did not outperform the passive strategy. This indicates that there are periods where passive strategies may outshine active management, possibly due to market conditions or specific events impacting the chosen stocks.

The SURE methodology, which extends the analysis by verifying if the error terms for the n stocks are correlated, shows higher performance measures for the active portfolio compared to the case in which we assumed that the error terms of the n stocks are uncorrelated. However, these findings highlight a significant issue with this model, in fact the recommended portfolio requires highly leveraged long/short positions that could prove impractical for real-world portfolio management.

In conclusion, in the SURE methodology, the higher Sharpe ratio is obtained at the cost of higher overall volatility of the portfolio.

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VI. APPENDIX

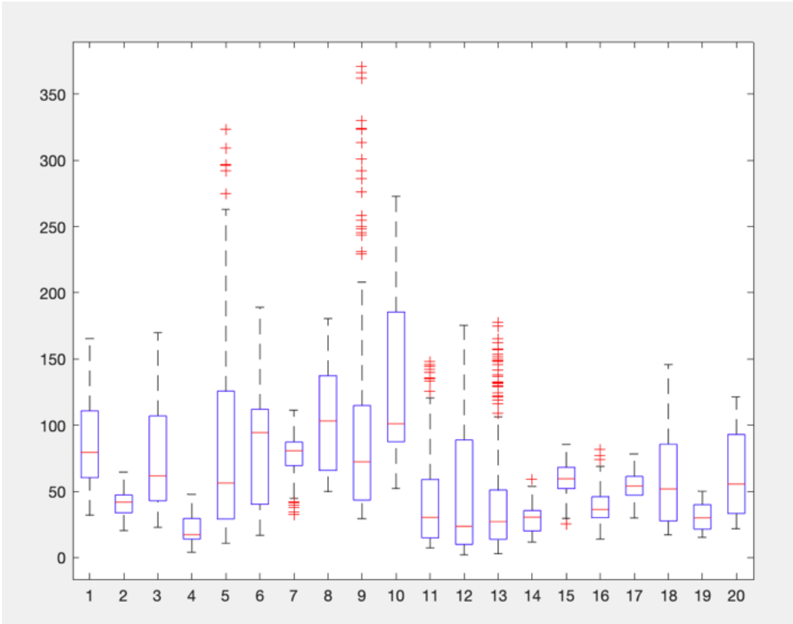

Figure a: Boxplot of closing prices of 20 assets

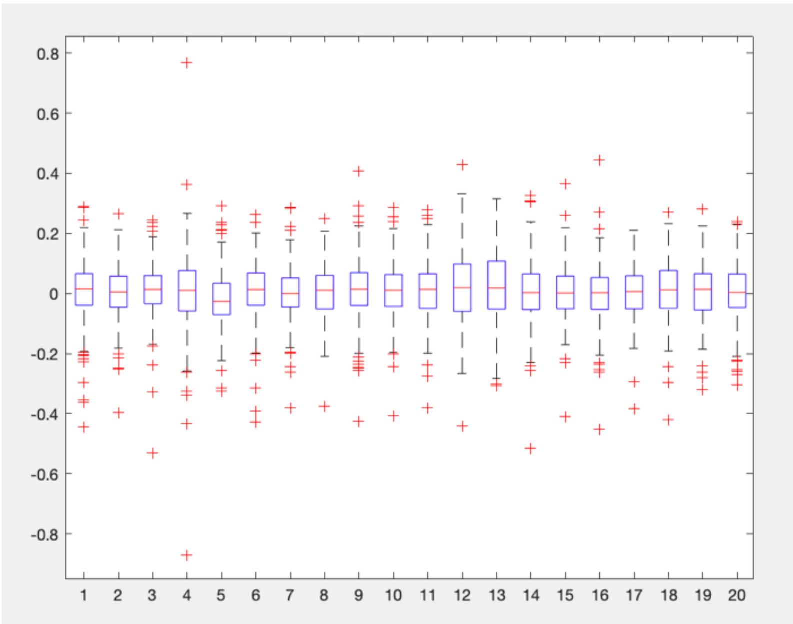

Figure b: Boxplot of log excess returns of 20 assets

We chose to analyze the excess returns for our analysis due to their stationarity. Unlike closing prices, which can exhibit trends and seasonality that complicate statistical analysis, excess returns are more likely to be stationary. This means their statistical properties, such as mean and variance, remain constant over time, making them more suitable for robust financial analysis and more reliable for evaluating the performance of investment strategies.

Figure *a* displays the distribution of closing prices. Asset 9 and Asset 12 show the highest variation in closing prices, as indicated by the height of their boxes and the length of their whiskers. Instead, Asset 2 and Asset 5 have the lowest variation, with shorter boxes and whiskers.

Moreover, we can see that assets 8, 9, 10, 11, 13, and 14, have numerous outliers. This indicates that these assets have experienced extreme closing prices that deviate significantly from their typical performance. The presence of many outliers suggests that these assets are subject to high volatility.

As we can see in Figure b, the average monthly excess return is extremely close to zero, indicating that stock excess returns tend to distribute with a normal distribution (average close to zero) in the long run.

Recall that if the excess return is positive the asset has outperformed the risk-free return, whereas when it is negative, it means that the asset has underperformed the risk-free return. The distribution of excess returns around zero reflects the randomness and inherent uncertainty in the financial market.[50]

[50] Marcel Dettling, "*Statistical Analysis of Financial Data",* ETH Swiss Federal Institute of Technology, Zurich, Jan 2021.

Matlab code

Descriptive Statistics of excess returns

```
A0=table2array(closingpricesnyse(1:180,2:23));

A= zeros(179,22);
for i=1:22
    for j=1:179
        A(j,i)=log(A0(j+1,i)/A0(j,i)); %logaritmic price varia-
tion
    end
end

for i=1:21
    A(:,i)=A(:,i)-A(:,22); %excess returns
end

m= mean(A(:,1:20));
mm= mean(A(:,21));
st=std(A);
sk=skewness(A);
k=kurtosis(A);
figure()
boxplot(A(:,1:20))
corr(A)
[acf,lags] = autocorr(A(:,1));
[acf lags]
[pacf,lags] = autocorr(A(:,1));
[pacf lags]
figure()
autocorr(A(:,1))
figure()
parcorr(A(:,1))

   figure()
num_assets = 20;
rows = 5;
cols = 4;

for i = 1:num_assets
    subplot(rows, cols, i);
    histfit(A(:, i), 20);
    title(['Asset ', num2str(i)]);
end
```

Basic formulation of the optimal active portfolio

```
A0=table2array(closingpricesnyse(1:180,2:23));

A= zeros(179,22);
for i=1:22
    for j=1:179
        A(j,i)=log(A0(j+1,i)/A0(j,i)); %logaritmic price varia-
tion
    end
end
```

```
for i=1:21
    A(:,i)=A(:,i)-A(:,22); %excess returns
end

R1=A(1:179,1)';
R2=A(1:179,2)';
R3=A(1:179,3)';
R4=A(1:179,4)';
R5=A(1:179,5)';
R6=A(1:179,6)';
R7=A(1:179,7)';
R8=A(1:179,8)';
R9=A(1:179,9)';
R10=A(1:179,10)';
R11=A(1:179,11)';
R12=A(1:179,12)';
R13=A(1:179,13)';
R14=A(1:179,14)';
R15=A(1:179,15)';
R16=A(1:179,16)';
R17=A(1:179,17)';
R18=A(1:179,18)';
R19=A(1:179,19)';
R20=A(1:179,20)';
Rm=A(1:179,21)';


%Simple Statistical Analysis

names=1:20;
names1=string(names)
m= mean(A(:,1:20));
mm= mean(A(:,21));
st=std(A);
sk=skewness(A);
k=kurtosis(A);
figure()
boxplot(A(:,1:20))
corr(A)
[acf,lags] = autocorr(A(:,1));
[acf lags]
[pacf,lags] = autocorr(A(:,1));
[pacf lags]
figure()
autocorr(A(:,1))
figure()
parcorr(A(:,1))

%3 - Estimation Window

R1=A(1:119,1)';
R2=A(1:119,2)';
R3=A(1:119,3)';
R4=A(1:119,4)';
R5=A(1:119,5)';
R6=A(1:119,6)';
R7=A(1:119,7)';
R8=A(1:119,8)';
R9=A(1:119,9)';
R10=A(1:119,10)';
```

```
R11=A(1:119,11)';
R12=A(1:119,12)';
R13=A(1:119,13)';
R14=A(1:119,14)';
R15=A(1:119,15)';
R16=A(1:119,16)';
R17=A(1:119,17)';
R18=A(1:119,18)';
R19=A(1:119,19)';
R20=A(1:119,20)';
Rm=A(1:119,21)';


R=A(1:119,1:20)'


alpha = zeros(20, 1);
beta = zeros(20, 1);
s = zeros(20,1);

R1=A(1:119,1)';
R2=A(1:119,2)';
R3=A(1:119,3)';
R4=A(1:119,4)';
R5=A(1:119,5)';
R6=A(1:119,6)';
R7=A(1:119,7)';
R8=A(1:119,8)';
R9=A(1:119,9)';
R10=A(1:119,10)';
R11=A(1:119,11)';
R12=A(1:119,12)';
R13=A(1:119,13)';
R14=A(1:119,14)';
R15=A(1:119,15)';
R16=A(1:119,16)';
R17=A(1:119,17)';
R18=A(1:119,18)';
R19=A(1:119,19)';
R20=A(1:119,20)';
Rm=A(1:119,21)';


R=A(1:119,1:20)'


alpha = zeros(20, 1);
beta = zeros(20, 1);
s = zeros(20,1);

res=zeros(119,20)

%mdl= fitlm(Rm,R17)


for i = 1:20
    % Select current column
    y = R(i, :);
    X = Rm;
    cov_0=cov(y,X,1);
```

```
    b=cov_0(1,2)/var(X,1);
    a=mean(y)-b*mean(X);
    residuals = y - a-b*X
    res(:,i)=residuals'

    % Save results in alpha and beta
    alpha(i) = a;
    beta(i) = b;
    s(i) = sqrt(sum(residuals.^2) / (length(residuals) - 2));

end

figure()
x=linspace(0,1.5,1000);
y=x*mm;  % The slope is given by the market treynor ratio
figure()
plot(x,y);
hold on
scatter(beta,m);
text(beta+0.01, m, names1);

em=mean(Rm); % market mean
varm=var(Rm,1); % variance market
s2=s.^2;   %variance epsilon
alpha2=alpha.^2;
num=alpha./s2;
den=em/varm;
w0=num/den;   %weights


somma=1+sum((1-beta).*w0);
w=w0/somma;
numwm=1-sum(beta.*w0);
wm=numwm/somma;
%wf=1-sum(w)-wm;

lambdainv=em/varm*somma
lambda=1/lambdainv % Lambda


w_n1=lambda*em/varm % weight w_n+1

%Optimal Portfolio

Rmf=wm*Rm;
Rf=R.*w;
Ro=sum(Rf)+Rmf;


sum(w.*alpha)+w_n1*em
mean(Ro)


sum(w.^2.*s2)+w_n1^2*varm
var(Ro)

Rmf=wm*Rm;
Rf=R.*w;
Ro=sum(Rf)+Rmf;
```

```
sm=em*em/varm;
alpha2=alpha.*alpha;
sum1=sum(alpha2./s2);

sp=sqrt(sm+sum1)

mean(Ro)/std(Ro,1)

figure()
heatmap(corr(res))

%Testing Window

RT1=A(120:179,1)';
RT2=A(120:179,2)';
RT3=A(120:179,3)';
RT4=A(120:179,4)';
RT5=A(120:179,5)';
RT6=A(120:179,6)';
RT7=A(120:179,7)';
RT8=A(120:179,8)';
RT9=A(120:179,9)';
RT10=A(120:179,10)';
RT11=A(120:179,11)';
RT12=A(120:179,12)';
RT13=A(120:179,13)';
RT14=A(120:179,14)';
RT15=A(120:179,15)';
RT16=A(120:179,16)';
RT17=A(120:179,17)';
RT18=A(120:179,18)';
RT19=A(120:179,19)';
RT20=A(120:179,20)';
RTm=A(120:179,21)';

RT=A(120:179,1:20)'

RTmf=wm*RTm;
RTf=RT.*w;
RTo=sum(RTf)+RTmf;


%SHARPE RATIO
sharpe = zeros(20, 1);

for i = 1:20
    % Select current column
    y = R(i, :);
    sh = mean(y)./std(y,1)

    % Save results in alpha and beta
    sharpe(i) = sh;

end

%Sharpe ratio annualized
sharpe_m= mean(RTm)/std(RTm)*sqrt(12)
sharpe_p = mean(RTo)/std(RTo)*sqrt(12)

%TRAYNOR RATIO
```

```
    yT = RTo;
    xT = RTm;
    t=cov(yT,xT,1);
    bT=t(1,2)/var(xT,1);

Traynor_p=(mean(RTo)/bT)*sqrt(12)
Traynor_m=mean(RTm)*sqrt(12)

%Downside Deviation

N=60;

  z = RTo;
  ddo = sqrt((1/(N-1)) * sum((min(0, z - mean(z))).^2));
  ddm  = sqrt((1/(N-1)) * sum((min(0, RTm - mean(RTm))).^2));

%Sortino ratio
mean(RTo)/ddo*sqrt(12)
mean(RTm)/ddm*sqrt(12)

%Jensen's alpha

jen= (mean(RTo)-bT.*mean(RTm))*sqrt(12)
```

Rolling window of the optimal active portfolio

```
A0=table2array(closingpricesnyse(1:180,2:23));

A= zeros(179,22);
for i=1:22
    for j=1:179
        A(j,i)=(A0(j+1,i)-A0(j,i))/A0(j,i);
    end
end

for i=1:21
    A(:,i)=A(:,i)-A(:,22); %excess returns
end

Rtott=A(120:179,1:20);
Rtotm=A(120:179,21);
RTtoto=zeros(60,1);
for j = 0:4
    R=A(1+j*12:119+j*12,1:20)'
    Rm=A(1+j*12:119+j*12,21)'
    alpha = zeros(20, 1);
    beta = zeros(20, 1);
    s = zeros(20,1);
    for i = 1:20
        % Select current column
        y = R(i, :);
        X = Rm;
        a=cov(y,X);
        b=a(1,2)/var(X);
        c=mean(y)-b*mean(X);
        residuals = y - c-b*X;
        % Save results in alpha and beta
        alpha(i) = c;
```

```
        beta(i) = b;
        s(i) = sqrt(sum(residuals.^2) / (length(residuals) - 2));

    end

    %Weights' calculation

    em=mean(Rm); % market mean
    varm=var(Rm); % market variance
    s2=s.^2      %epsilon variance
    num=alpha./s2;
    den=em/varm;
    w0=num/den;   %optimal weights
    somma=1+sum((1-beta).*w0);
    w=w0/somma;
    numwm=1-sum(beta.*w0);
    wm=numwm/somma;
    wf=1-sum(w)-wm;
    Rmf=wm*Rm;
    Rf=R.*w;
    Ro=sum(Rf)+Rmf;
    Rmf;

%Sharpe RATIO
    sm=em*em/varm; %Square value Sharpe ratio market
    alpha2=alpha.*alpha;
    sum1=sum(alpha2./s2);
    sp=sqrt(sm+sum1); %Sharpe Ratio optimal portfolio

    RT=A(120+j*12:131+j*12,1:20)'
    RTm=A(120+j*12:131+j*12,21)'
    RTmf=wm*RTm
    RTf=RT.*w
    RTtoto(1+j*12:12+j*12)=sum(RTf)+RTmf;
end

%SHARPE RATIO OPTIMAL PORTFOLIO TESTING WINDOW

mean(RTtoto)/std(RTtoto)
mean(Rtotm)/std(Rtotm)
```

Check on the rolling window

```
A0=table2array(closingpricesnyse(1:180,2:23));

A= zeros(179,22);
for i=1:22
    for j=1:179
        A(j,i)=log(A0(j+1,i)/A0(j,i)); % logaritmic price varia-
tion
    end
end


for i=1:21
    A(:,i)=A(:,i)-A(:,22);
end

R1=A(1:179,1)';
```

```
R2=A(1:179,2)';
R3=A(1:179,3)';
R4=A(1:179,4)';
R5=A(1:179,5)';
R6=A(1:179,6)';
R7=A(1:179,7)';
R8=A(1:179,8)';
R9=A(1:179,9)';
R10=A(1:179,10)';
R11=A(1:179,11)';
R12=A(1:179,12)';
R13=A(1:179,13)';
R14=A(1:179,14)';
R15=A(1:179,15)';
R16=A(1:179,16)';
R17=A(1:179,17)';
R18=A(1:179,18)';
R19=A(1:179,19)';
R20=A(1:179,20)';
Rm=A(1:179,21)';


%3 - Estimation Window 2008-2017

R1=A(1:119,1)';
R2=A(1:119,2)';
R3=A(1:119,3)';
R4=A(1:119,4)';
R5=A(1:119,5)';
R6=A(1:119,6)';
R7=A(1:119,7)';
R8=A(1:119,8)';
R9=A(1:119,9)';
R10=A(1:119,10)';
R11=A(1:119,11)';
R12=A(1:119,12)';
R13=A(1:119,13)';
R14=A(1:119,14)';
R15=A(1:119,15)';
R16=A(1:119,16)';
R17=A(1:119,17)';
R18=A(1:119,18)';
R19=A(1:119,19)';
R20=A(1:119,20)';
Rm=A(1:119,21)';


R=A(1:119,1:20)'


alpha = zeros(20, 1);
beta = zeros(20, 1);
s = zeros(20,1);

alpha = zeros(20, 1);
beta = zeros(20, 1);
s = zeros(20,1);

res=zeros(119,20)
```

```
%mdl= fitlm(Rm,R17)


for i = 1:20
    % Select current column
    y = R(i, :);
    X = Rm;
    cov_0=cov(y,X,1);
    b=cov_0(1,2)/var(X,1);
    a=mean(y)-b*mean(X);
    residuals = y - a-b*X
    res(:,i)=residuals'

    % Save results in alpha and beta
    alpha(i) = a;
    beta(i) = b;
    s(i) = sqrt(sum(residuals.^2) / (length(residuals) - 2));

end
em=mean(Rm); % market mean
varm=var(Rm,1); % market variance
s2=s.^2;   % epsilon variance
alpha2=alpha.^2;
num=alpha./s2;
den=em/varm;
w0=num/den;   %weights
somma=1+sum((1-beta).*w0);
w=w0/somma;
numwm=1-sum(beta.*w0);
wm=numwm/somma;
wf=1-sum(w)-wm;

lambdainv=em/varm*somma
lambda=1/lambdainv % Lambda


w_n1=lambda*em/varm % weight w_n+1

%Optimal portfolio

Rmf=wm*Rm;
Rf=R.*w;
Ro=sum(Rf)+Rmf;

sum(w.*alpha)+w_n1*em
mean(Ro)

sum(w.^2.*s2)+w_n1^2*varm
var(Ro)

Rmf=wm*Rm;
Rf=R.*w;
Ro=sum(Rf)+Rmf;

sm=em*em/varm;
alpha2=alpha.*alpha;
sum1=sum(alpha2./s2);

sp=sqrt(sm+sum1)
```

```
mean(Ro)/std(Ro,1)

%Testing Window 2018

RT1=A(120:131,1)';
RT2=A(120:131,2)';
RT3=A(120:131,3)';
RT4=A(120:131,4)';
RT5=A(120:131,5)';
RT6=A(120:131,6)';
RT7=A(120:131,7)';
RT8=A(120:131,8)';
RT9=A(120:131,9)';
RT10=A(120:131,10)';
RT11=A(120:131,11)';
RT12=A(120:131,12)';
RT13=A(120:131,13)';
RT14=A(120:131,14)';
RT15=A(120:131,15)';
RT16=A(120:131,16)';
RT17=A(120:131,17)';
RT18=A(120:131,18)';
RT19=A(120:131,19)';
RT20=A(120:131,20)';
RTm=A(120:131,21)';

RT=A(120:131,1:20)'

RTmf=wm*RTm;
RTf=RT.*w;
RTo=sum(RTf)+RTmf;

%SHARPE RATIO
sharpe = zeros(20, 1);

for i = 1:20
    % Select current column
    y = R(i, :);
    sh = mean(y)./std(y,1)

    % Save results in sharpe
    sharpe(i) = sh;
end

%check (sharpe)
mean(R1)/std(R1)

%Sharpe ratio annualized
mean(RTm)/std(RTm)*sqrt(12)
mean(RTo)/std(RTo)*sqrt(12)

%TRAYNOR RATIO
    yT = RTo;
    xT = RTm;
    t=cov(yT,xT,1);
    bT=t(1,2)/var(xT,1);

Traynor_p=(mean(RTo)/bT)*sqrt(12)
Traynor_m=mean(RTm)*sqrt(12)
```

```
%Downside Deviation

N=12;

  z = RTo;
  ddo = sqrt((1/(N-1)) * sum((min(0, z - mean(z))).^2));
  ddm  = sqrt((1/(N-1)) * sum((min(0, RTm - mean(RTm))).^2));

%Sortino ratio
mean(RTo)/ddo*sqrt(12)
mean(RTm)/ddm*sqrt(12)

Sharpe_m=mean(Rm)/std(Rm)*sqrt(12) %sharpe del mercato conside-
rando tutto il periodo

%Jensen's alpha

jen= (mean(RTo)-bT.*mean(RTm))*sqrt(12)



A0=table2array(closingpricesnyse(1:180,2:23));

A= zeros(179,22);
for i=1:22
    for j=1:179
        A(j,i)=log(A0(j+1,i)/A0(j,i));
    end
end


for i=1:21
    A(:,i)=A(:,i)-A(:,22);
end

R1=A(1:179,1)';
R2=A(1:179,2)';
R3=A(1:179,3)';
R4=A(1:179,4)';
R5=A(1:179,5)';
R6=A(1:179,6)';
R7=A(1:179,7)';
R8=A(1:179,8)';
R9=A(1:179,9)';
R10=A(1:179,10)';
R11=A(1:179,11)';
R12=A(1:179,12)';
R13=A(1:179,13)';
R14=A(1:179,14)';
R15=A(1:179,15)';
R16=A(1:179,16)';
R17=A(1:179,17)';
R18=A(1:179,18)';
R19=A(1:179,19)';
R20=A(1:179,20)';
Rm=A(1:179,21)';


%3 - Estimation Window 2009-2018
```

```
R1=A(12:131,1)';
R2=A(12:131,2)';
R3=A(12:131,3)';
R4=A(12:131,4)';
R5=A(12:131,5)';
R6=A(12:131,6)';
R7=A(12:131,7)';
R8=A(12:131,8)';
R9=A(12:131,9)';
R10=A(12:131,10)';
R11=A(12:131,11)';
R12=A(12:131,12)';
R13=A(12:131,13)';
R14=A(12:131,14)';
R15=A(12:131,15)';
R16=A(12:131,16)';
R17=A(12:131,17)';
R18=A(12:131,18)';
R19=A(12:131,19)';
R20=A(12:131,20)';
Rm=A(12:131,21)';


R=A(12:131,1:20)'

alpha = zeros(20, 1);
beta = zeros(20, 1);
s = zeros(20,1);

alpha = zeros(20, 1);
beta = zeros(20, 1);
s = zeros(20,1);

res=zeros(120,20)

%mdl= fitlm(Rm,R17)


for i = 1:20
    % Select current column
    y = R(i, :);
    X = Rm;
    cov_0=cov(y,X,1);
    b=cov_0(1,2)/var(X,1);
    a=mean(y)-b*mean(X);
    residuals = y - a-b*X
    res(:,i)=residuals'

    % Save results in alpha and beta
    alpha(i) = a;
    beta(i) = b;
    s(i) = sqrt(sum(residuals.^2) / (length(residuals) - 2));

end
em=mean(Rm); % market mean
varm=var(Rm,1); % market variance
s2=s.^2;   % epsilon variance
alpha2=alpha.^2;
num=alpha./s2;
den=em/varm;
```

```
w0=num/den;   %weights
somma=1+sum((1-beta).*w0);
w=w0/somma;
numwm=1-sum(beta.*w0);
wm=numwm/somma;
wf=1-sum(w)-wm;

lambdainv=em/varm*somma
lambda=1/lambdainv % Lambda


w_n1=lambda*em/varm % weight w_n+1

%Optimal portfolio

Rmf=wm*Rm;
Rf=R.*w;
Ro=sum(Rf)+Rmf;

sum(w.*alpha)+w_n1*em
mean(Ro)

sum(w.^2.*s2)+w_n1^2*varm
var(Ro)

Rmf=wm*Rm;
Rf=R.*w;
Ro=sum(Rf)+Rmf;

sm=em*em/varm;
alpha2=alpha.*alpha;
sum1=sum(alpha2./s2);

sp=sqrt(sm+sum1)

mean(Ro)/std(Ro,1)

%Testing Window 2019

RT1=A(132:143,1)';
RT2=A(132:143,2)';
RT3=A(132:143,3)';
RT4=A(132:143,4)';
RT5=A(132:143,5)';
RT6=A(132:143,6)';
RT7=A(132:143,7)';
RT8=A(132:143,8)';
RT9=A(132:143,9)';
RT10=A(132:143,10)';
RT11=A(132:143,11)';
RT12=A(132:143,12)';
RT13=A(132:143,13)';
RT14=A(132:143,14)';
RT15=A(132:143,15)';
RT16=A(132:143,16)';
RT17=A(132:143,17)';
RT18=A(132:143,18)';
RT19=A(132:143,19)';
RT20=A(132:143,20)';
RTm=A(132:143,21)';
```

```
RT=A(132:143,1:20)'

RTmf=wm*RTm;
RTf=RT.*w;
RTo=sum(RTf)+RTmf;

%SHARPE RATIO
sharpe = zeros(20, 1);

for i = 1:20
    % Select current column
    y = R(i, :);
    sh = mean(y)./std(y,1)

    % Save results in alpha and beta
    sharpe(i) = sh;

end

%check (sharpe)
mean(R1)/std(R1)


%Sharpe ratio annualized
Sharpe_m= mean(RTm)/std(RTm)*sqrt(12)
Sharpe_p= (mean(RTo)/std(RTo))*sqrt(12)

%TRAYNOR RATIO
for i = 1:20
    % Seleziona la colonna corrente
    yT = RT(i, :);
    xT = RTm;
    t=cov(yT,xT,1);
    bT=t(1,2)/var(xT,1);
end

Traynor_p=(mean(RTo)/bT)*sqrt(12)
Traynor_m=mean(RTm)*sqrt(12)

%Downside Deviation

for i = 1:20

    % Select current column
    z = RT(i,:);
    ddo = z-mean(z(z<0))
end

%SORTINO RATIO

N = 12; % number of excess returns during testing window

DDo = sqrt((1/N) * sum(ddo.^2)); % downside deviation
ddm = RTm-mean(RTm(RTm<0))
DDm= sqrt((1/N) * sum(ddm.^2))
Sortino_p= (mean(RTo)/DDo)*sqrt(12)
Sortino_m= (mean(RTm)/DDm)*sqrt(12)

%Jensen's alpha
```

```
jen= (mean(RTo)-bT.*mean(RTm))*sqrt(12)

A0=table2array(closingpricesnyse(1:180,2:23));

A= zeros(179,22);
for i=1:22
    for j=1:179
        A(j,i)=log(A0(j+1,i)/A0(j,i));
    end
end


for i=1:21
    A(:,i)=A(:,i)-A(:,22);
end

R1=A(1:179,1)';
R2=A(1:179,2)';
R3=A(1:179,3)';
R4=A(1:179,4)';
R5=A(1:179,5)';
R6=A(1:179,6)';
R7=A(1:179,7)';
R8=A(1:179,8)';
R9=A(1:179,9)';
R10=A(1:179,10)';
R11=A(1:179,11)';
R12=A(1:179,12)';
R13=A(1:179,13)';
R14=A(1:179,14)';
R15=A(1:179,15)';
R16=A(1:179,16)';
R17=A(1:179,17)';
R18=A(1:179,18)';
R19=A(1:179,19)';
R20=A(1:179,20)';
Rm=A(1:179,21)';


% Estimation Window years 2010-2019

R1=A(24:143,1)';
R2=A(24:143,2)';
R3=A(24:143,3)';
R4=A(24:143,4)';
R5=A(24:143,5)';
R6=A(24:143,6)';
R7=A(24:143,7)';
R8=A(24:143,8)';
R9=A(24:143,9)';
R10=A(24:143,10)';
R11=A(24:143,11)';
R12=A(24:143,12)';
R13=A(24:143,13)';
R14=A(24:143,14)';
R15=A(24:143,15)';
R16=A(24:143,16)';
R17=A(24:143,17)';
```

```
R18=A(24:143,18)';
R19=A(24:143,19)';
R20=A(24:143,20)';
Rm=A(24:143,21)';


R=A(24:143,1:20)'


alpha = zeros(20, 1);
beta = zeros(20, 1);
s = zeros(20,1);

alpha = zeros(20, 1);
beta = zeros(20, 1);
s = zeros(20,1);

res=zeros(120,20)

%mdl= fitlm(Rm,R17)


for i = 1:20
    % Select current column
    y = R(i, :);
    X = Rm;
    cov_0=cov(y,X,1);
    b=cov_0(1,2)/var(X,1);
    a=mean(y)-b*mean(X);
    residuals = y - a-b*X
    res(:,i)=residuals'

    % Save results in alpha and beta
    alpha(i) = a;
    beta(i) = b;
    s(i) = sqrt(sum(residuals.^2) / (length(residuals) - 2));

end
em=mean(Rm); % market mean
varm=var(Rm,1); % market variance
s2=s.^2;   % epsilon variance
alpha2=alpha.^2;
num=alpha./s2;
den=em/varm;
w0=num/den;   %weights
somma=1+sum((1-beta).*w0);
w=w0/somma;
numwm=1-sum(beta.*w0);
wm=numwm/somma;
wf=1-sum(w)-wm;

lambdainv=em/varm*somma
lambda=1/lambdainv % Lambda


w_n1=lambda*em/varm % weight w_n+1

%Optimal portfolio

Rmf=wm*Rm;
```

```
Rf=R.*w;
Ro=sum(Rf)+Rmf;

sum(w.*alpha)+w_n1*em
mean(Ro)

sum(w.^2.*s2)+w_n1^2*varm
var(Ro)

Rmf=wm*Rm;
Rf=R.*w;
Ro=sum(Rf)+Rmf;

sm=em*em/varm;
alpha2=alpha.*alpha;
sum1=sum(alpha2./s2);

sp=sqrt(sm+sum1)

mean(Ro)/std(Ro,1)

%Testing Window 2020

RT1=A(144:155,1)';
RT2=A(144:155,2)';
RT3=A(144:155,3)';
RT4=A(144:155,4)';
RT5=A(144:155,5)';
RT6=A(144:155,6)';
RT7=A(144:155,7)';
RT8=A(144:155,8)';
RT9=A(144:155,9)';
RT10=A(144:155,10)';
RT11=A(144:155,11)';
RT12=A(144:155,12)';
RT13=A(144:155,13)';
RT14=A(144:155,14)';
RT15=A(144:155,15)';
RT16=A(144:155,16)';
RT17=A(144:155,17)';
RT18=A(144:155,18)';
RT19=A(144:155,19)';
RT20=A(144:155,20)';
RTm=A(144:155,21)';

RT=A(144:155,1:20)'


RTmf=wm*RTm;
RTf=RT.*w;
RTo=sum(RTf)+RTmf;

%SHARPE RATIO
sharpe = zeros(20, 1);

for i = 1:20
    % Select current column
    y = R(i, :);
    sh = mean(y)./std(y,1)
```

```
    % Save results in alpha and beta
    sharpe(i) = sh;

end

%check (sharpe)
mean(R1)/std(R1)

%Sharpe ratio annualized

mean(RTm)/std(RTm)*sqrt(12)
mean(RTo)/std(RTo)*sqrt(12)


A= zeros(179,22);
for i=1:22
    for j=1:179
        A(j,i)=log(A0(j+1,i)/A0(j,i));
    end
end


for i=1:21
    A(:,i)=A(:,i)-A(:,22);
end

R1=A(1:179,1)';
R2=A(1:179,2)';
R3=A(1:179,3)';
R4=A(1:179,4)';
R5=A(1:179,5)';
R6=A(1:179,6)';
R7=A(1:179,7)';
R8=A(1:179,8)';
R9=A(1:179,9)';
R10=A(1:179,10)';
R11=A(1:179,11)';
R12=A(1:179,12)';
R13=A(1:179,13)';
R14=A(1:179,14)';
R15=A(1:179,15)';
R16=A(1:179,16)';
R17=A(1:179,17)';
R18=A(1:179,18)';
R19=A(1:179,19)';
R20=A(1:179,20)';
Rm=A(1:179,21)';


% Estimation Window years 2011-2020

R1=A(36:155,1)';
R2=A(36:155,2)';
R3=A(36:155,3)';
R4=A(36:155,4)';
R5=A(36:155,5)';
R6=A(36:155,6)';
R7=A(36:155,7)';
R8=A(36:155,8)';
```

```
R9=A(36:155,9)';
R10=A(36:155,10)';
R11=A(36:155,11)';
R12=A(36:155,12)';
R13=A(36:155,13)';
R14=A(36:155,14)';
R15=A(36:155,15)';
R16=A(36:155,16)';
R17=A(36:155,17)';
R18=A(36:155,18)';
R19=A(36:155,19)';
R20=A(36:155,20)';
Rm=A(36:155,21)';

R=A(36:155,1:20)'

alpha = zeros(20, 1);
beta = zeros(20, 1);
s = zeros(20,1);

alpha = zeros(20, 1);
beta = zeros(20, 1);
s = zeros(20,1);

res=zeros(120,20)

%mdl= fitlm(Rm,R17)


for i = 1:20
    % Select current column
    y = R(i, :);
    X = Rm;
    cov_0=cov(y,X,1);
    b=cov_0(1,2)/var(X,1);
    a=mean(y)-b*mean(X);
    residuals = y - a-b*X
    res(:,i)=residuals'

    % Save results in alpha and beta
    alpha(i) = a;
    beta(i) = b;
    s(i) = sqrt(sum(residuals.^2) / (length(residuals) - 2));

end
em=mean(Rm); % market mean
varm=var(Rm,1); % market variance
s2=s.^2;   % epsilon variance
alpha2=alpha.^2;
num=alpha./s2;
den=em/varm;
w0=num/den;   %weights
somma=1+sum((1-beta).*w0);
w=w0/somma;
numwm=1-sum(beta.*w0);
wm=numwm/somma;
wf=1-sum(w)-wm;

lambdainv=em/varm*somma
lambda=1/lambdainv % Lambda
```

```
w_n1=lambda*em/varm % weight w_n+1

%Optimal portfolio

Rmf=wm*Rm;
Rf=R.*w;
Ro=sum(Rf)+Rmf;

sum(w.*alpha)+w_n1*em
mean(Ro)

sum(w.^2.*s2)+w_n1^2*varm
var(Ro)

Rmf=wm*Rm;
Rf=R.*w;
Ro=sum(Rf)+Rmf;

sm=em*em/varm;
alpha2=alpha.*alpha;
sum1=sum(alpha2./s2);

sp=sqrt(sm+sum1)

mean(Ro)/std(Ro,1)

%Testing Window 2021

RT1=A(156:167,1)';
RT2=A(156:167,2)';
RT3=A(156:167,3)';
RT4=A(156:167,4)';
RT5=A(156:167,5)';
RT6=A(156:167,6)';
RT7=A(156:167,7)';
RT8=A(156:167,8)';
RT9=A(156:167,9)';
RT10=A(156:167,10)';
RT11=A(156:167,11)';
RT12=A(156:167,12)';
RT13=A(156:167,13)';
RT14=A(156:167,14)';
RT15=A(156:167,15)';
RT16=A(156:167,16)';
RT17=A(156:167,17)';
RT18=A(156:167,18)';
RT19=A(156:167,19)';
RT20=A(156:167,20)';
RTm=A(156:167,21)';

RT=A(156:167,1:20)'



RTmf=wm*RTm;
RTf=RT.*w;
RTo=sum(RTf)+RTmf;
```

```
%SHARPE RATIO
sharpe = zeros(20, 1);

for i = 1:20
    % Select current column
    y = R(i, :);
    sh = mean(y)./std(y,1)

    % Save results in alpha and beta
    sharpe(i) = sh;

end

%check (sharpe)
mean(R1)/std(R1)

%Sharpe ratio annualized

mean(RTm)/std(RTm)*sqrt(12)
mean(RTo)/std(RTo)*sqrt(12)

A0=table2array(closingpricesnyse(1:180,2:23));

A= zeros(179,22);
for i=1:22
    for j=1:179
        A(j,i)=log(A0(j+1,i)/A0(j,i));
    end
end


for i=1:21
    A(:,i)=A(:,i)-A(:,22);
end

R1=A(1:179,1)';
R2=A(1:179,2)';
R3=A(1:179,3)';
R4=A(1:179,4)';
R5=A(1:179,5)';
R6=A(1:179,6)';
R7=A(1:179,7)';
R8=A(1:179,8)';
R9=A(1:179,9)';
R10=A(1:179,10)';
R11=A(1:179,11)';
R12=A(1:179,12)';
R13=A(1:179,13)';
R14=A(1:179,14)';
R15=A(1:179,15)';
R16=A(1:179,16)';
R17=A(1:179,17)';
R18=A(1:179,18)';
R19=A(1:179,19)';
R20=A(1:179,20)';
Rm=A(1:179,21)';


% Estimation Window years 2012-2021
```

```
R1=A(49:167,1)';
R2=A(49:167,2)';
R3=A(49:167,3)';
R4=A(49:167,4)';
R5=A(49:167,5)';
R6=A(49:167,6)';
R7=A(49:167,7)';
R8=A(49:167,8)';
R9=A(49:167,9)';
R10=A(49:167,10)';
R11=A(49:167,11)';
R12=A(49:167,12)';
R13=A(49:167,13)';
R14=A(49:167,14)';
R15=A(49:167,15)';
R16=A(49:167,16)';
R17=A(49:167,17)';
R18=A(49:167,18)';
R19=A(49:167,19)';
R20=A(49:167,20)';
Rm=A(49:167,21)';

R=A(49:167,1:20)'


alpha = zeros(20, 1);
beta = zeros(20, 1);
s = zeros(20,1);

alpha = zeros(20, 1);
beta = zeros(20, 1);
s = zeros(20,1);

res=zeros(119,20)

%mdl= fitlm(Rm,R17)


for i = 1:20
    % Select current column
    y = R(i, :);
    X = Rm;
    cov_0=cov(y,X,1);
    b=cov_0(1,2)/var(X,1);
    a=mean(y)-b*mean(X);
    residuals = y - a-b*X
    res(:,i)=residuals'

    % Save results in alpha and beta
    alpha(i) = a;
    beta(i) = b;
    s(i) = sqrt(sum(residuals.^2) / (length(residuals) - 2));

end
em=mean(Rm); % market mean
varm=var(Rm,1); % market variance
s2=s.^2;   % epsilon variance
alpha2=alpha.^2;
num=alpha./s2;
```

```
den=em/varm;
w0=num/den;    %weights
somma=1+sum((1-beta).*w0);
w=w0/somma;
numwm=1-sum(beta.*w0);
wm=numwm/somma;
wf=1-sum(w)-wm;

lambdainv=em/varm*somma
lambda=1/lambdainv % Lambda


w_n1=lambda*em/varm % weight w_n+1

%Optimal portfolio

Rmf=wm*Rm;
Rf=R.*w;
Ro=sum(Rf)+Rmf;

sum(w.*alpha)+w_n1*em
mean(Ro)

sum(w.^2.*s2)+w_n1^2*varm
var(Ro)

Rmf=wm*Rm;
Rf=R.*w;
Ro=sum(Rf)+Rmf;

sm=em*em/varm;
alpha2=alpha.*alpha;
sum1=sum(alpha2./s2);

sp=sqrt(sm+sum1)

mean(Ro)/std(Ro,1)

%Testing Window 2022

RT1=A(168:179,1)';
RT2=A(168:179,2)';
RT3=A(168:179,3)';
RT4=A(168:179,4)';
RT5=A(168:179,5)';
RT6=A(168:179,6)';
RT7=A(168:179,7)';
RT8=A(168:179,8)';
RT9=A(168:179,9)';
RT10=A(168:179,10)';
RT11=A(168:179,11)';
RT12=A(168:179,12)';
RT13=A(168:179,13)';
RT14=A(168:179,14)';
RT15=A(168:179,15)';
RT16=A(168:179,16)';
RT17=A(168:179,17)';
RT18=A(168:179,18)';
RT19=A(168:179,19)';
RT20=A(168:179,20)';
```

```
RTm=A(168:179,21)';

RT=A(168:179,1:20)'


RTmf=wm*RTm;
RTf=RT.*w;
RTo=sum(RTf)+RTmf;

%SHARPE RATIO
sharpe = zeros(20, 1);

for i = 1:20
    % Select current column
    y = R(i, :);
    sh = mean(y)./std(y,1)

    % Save results in alpha and beta
    sharpe(i) = sh;

end

%Sharpe ratio annualized
mean(RTm)/std(RTm)*sqrt(12)
mean(RTo)/std(RTo)*sqrt(12)

%TRAYNOR RATIO
    yT = RTo;
    xT = RTm;
    t=cov(yT,xT,1);
    bT=t(1,2)/var(xT,1);

Traynor_p=(mean(RTo)/bT)*sqrt(12)
Traynor_m=mean(RTm)*sqrt(12)

%Downside Deviation

N=12;

  z = RTo;
  ddo = sqrt((1/(N-1)) * sum((min(0, z - mean(z))).^2));
  ddm  = sqrt((1/(N-1)) * sum((min(0, RTm - mean(RTm))).^2));

%Sortino ratio
mean(RTo)/ddo*sqrt(12)
mean(RTm)/ddm*sqrt(12)

%Jensen's alpha

jen= (mean(RTo)-bT.*mean(RTm))*sqrt(12)
```

SURE Methodology

```
A0=table2array(closingpricesnyse(1:180,2:23));

A= zeros(179,22);
for i=1:22
    for j=1:179
        A(j,i)=log(A0(j+1,i)/A0(j,i));
```

```
    end
end

for i=1:21
    A(:,i)=A(:,i)-A(:,22);
end

R=A(1:119,1:20);
Rm=A(1:119,21);
em=mean(Rm);
varm=var(Rm);

mm=mean(Rm)
m=mean(R)

beta=sum((R-m).*(Rm-mm))/(sum((Rm-mm).*(Rm-mm)))

alpha=m-mm*beta;

eps=R-alpha-Rm.*beta

omega=eps'*eps/119
stdDeviations = sqrt(diag(omega));

% Correlation matrix
omega1 = omega ./ (stdDeviations * stdDeviations');
figure()
heatmap(omega1)


try chol(omega1)
    disp('Matrix is symmetric positive definite.')
catch ME
    disp('Matrix is not symmetric positive definite')
end

omegainv=inv(omega);
p1=beta*omegainv*alpha'
p2=(1-beta)*omegainv*alpha'
sm=em/varm;

wm=(1-p1/sm)/(1+p2/sm)

p3=omegainv*alpha';

w=(p3/sm)/(1+p2/sm);


Rmf=wm*Rm;
Rf=R.*w';
Ro=sum(Rf,2)+Rmf;

mean(Ro)

mean(Ro)/std(Ro)

lambdainv=(sm+p2)
lambda=1/lambdainv

sp= sqrt(alpha*omegainv*alpha'+em*em/varm)*sqrt(12)
```

```
RT_sure=A(120:179,1:20)';
RTm_sure=A(120:179,21)';

RTmf_sure=wm*RTm_sure;
RTf_sure=RT_sure.*w;
RTo_sure=sum(RTf_sure)+RTmf_sure;


%TRAYNOR RATIO
    yTsure = RTo_sure;
    xTsure = RTm_sure;
    tsure=cov(yTsure,xTsure,1);
    bTsure=tsure(1,2)/var(xTsure,1);

Traynor_p=(mean(yTsure)/bTsure)*sqrt(12)
Traynor_m=mean(xTsure)*sqrt(12)

%Sharpe ratio annualized
mean(xTsure)/std(xTsure)*sqrt(12)
mean(yTsure)/std(yTsure)*sqrt(12)

%Downside Deviation metodo
N=60;

  zsure = yTsure;
  ddosure = sqrt((1/(N-1)) * sum((min(0, zsure -
mean(zsure))).^2));
  ddmsure  = sqrt((1/(N-1)) * sum((min(0, xTsure -
mean(xTsure))).^2));

%Sortino ratio
mean(yTsure)/ddosure*sqrt(12)
mean(xTsure)/ddmsure*sqrt(12)

%Jensen's alpha

jen= (mean(yTsure)-bTsure.*mean(xTsure))*sqrt(12)
```

Check on SURE methodology

```
A0=table2array(closingpricesnyse1(1:180,2:23));

A= zeros(179,22);
for i=1:22
    for j=1:179
        A(j,i)=(A0(j+1,i)-A0(j,i))/A0(j,i);
    end
end

for i=1:21
    A(:,i)=A(:,i)-A(:,22);
end


Y=A(1:119,1:20);
X=A(1:119,21);
n=20

Mdl1 = varm(n,0);
```

```
[EstMdl1,~,~,E] = estimate(Mdl1,Y,'X',X);
summarize(EstMdl1)


mat_cov=EstMdl1.Covariance;
stdDeviations = sqrt(diag(mat_cov));

% Correlation matrix
corrMatrix = mat_cov ./ (stdDeviations * stdDeviations');


try chol(corrMatrix)
    disp('Matrix is symmetric positive definite.')
catch ME
    disp('Matrix is not symmetric positive definite')
end
```