Dipartimento di Economia e Finanza

Indirizzo Banche ed Intermediari Finanziari

Rigorous Mathematical Derivations for Option Pricing and Monte Carlo Analysis: from Black-Scholes to Local Volatility Model

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Ai miei Genitori

Per aspera ad astra

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Introduction

The valuation of options stands as fundamental tasks in the sphere of financial derivatives, demanding a sophisticated understanding of both theoretical and practical aspects. This thesis undertakes a comprehensive examination of options pricing, focusing on the application of Monte Carlo simulations to derive option prices and calculate the Greeks. These Greeks, Delta, Gamma, Vega, Theta, and Rho, are critical measures of the sensitivity of an option to various market parameters and are essential for informed risk management and trading strategies.

Financial markets have seen substantial evolution, becoming more intricate and volatile over the decades. The advent of the Black-Scholes model in the early 1970s was a groundbreaking development, offering a significant solution for pricing European options. Despite its impact, the assumptions of the model of constant volatility and risk-free interest rates have been subject to critique for their oversimplification of the dynamic and stochastic nature of real-world markets.

Recognizing these limitations, this thesis explores more flexible and realistic approaches. Monte Carlo simulations emerge as a powerful alternative, capable of modeling the complex and stochastic behavior of market variables more accurately. These simulations use the concept of randomness to generate numerous potential paths for an asset price, thereby facilitating a thorough analysis of options pricing and the associated risks.

Among the goals of this thesis is to develop a Monte Carlo simulation algorithm that not only prices options but also computes the Greeks using both Black-Scholes differential equations and pathwise estimators. These methodologies originate from rigorous mathematical foundations for the analysis, ensuring robustness and precision. By putting together the results from these methods with data obtained from the Bloomberg Terminal, the thesis aims to validate the simulation model and elucidate any variances observed.

Among the goals of this thesis is to develop a Monte Carlo simulation algorithm that not only prices options but also computes the Greeks using both Black-Scholes differential equations and pathwise estimators. These methodologies originate from rigorous mathematical foundations for the analysis, ensuring robustness and precision. A significant part of this thesis involved an in-depth punctual derivation of the Black-Scholes model. This comprehensive derivation provided a solid theoretical foundation, enabling a deeper understanding of the assumptions of the model and limitations. By putting together the results from these methods with data obtained from the Bloomberg Terminal, the thesis aims to validate the simulation model and elucidate any variances observed.

Additionally, this thesis presents a more refined approach by incorporating the local volatility model through the Dupire formula, offering a sophisticated alternative to the traditional Black-Scholes framework. This model adeptly captures the dynamic and evolving nature of implied volatility, which is often observed to be non-constant and varies with time. By considering the variability of the implied volatility surface, the local volatility model provides a more detailed and accurate reflection of market conditions, thereby enhancing the precision and adaptability of options pricing methodologies.

The Monte Carlo simulation algorithm developed in this thesis is designed to simulate the stochastic price paths of the underlying assets, allowing for the estimation of option prices and Greeks under diverse market conditions. Pathwise estimators are employed to compute the Greeks, providing an alternative sensitivity analysis method that complements the traditional differential equation approach. This dual methodology ensures a comprehensive assessment, grounded in rigorous mathematical principles.

The accuracy and reliability of the simulation model are benchmarked against real-world data sourced from the Bloomberg Terminal. This comparison is crucial for validating the model's performance and ensuring its practical applicability. The empirical data from Bloomberg serve not only as a benchmark but also to understand the intrinsic dynamics of market behavior that may not be fully captured by theoretical models alone.

This thesis is structured to reflect the rigorous mathematical and analytical processes involved in the study. The methodological rigor extends to every aspect of the thesis, from the development of the Monte Carlo simulation algorithm to the application of pathwise estimators and the subsequent comparison with Bloomberg data.

The significance of this study is various. By advancing the application of Monte Carlo simulations in the context of options pricing and risk management, the thesis provides valuable insights into the strengths and limitations of different methodological approaches. The detailed comparison between the results derived from differential equations and those obtained through pathwise estimators offers a complete understanding of the practical implications for risk management and investment strategies.

This thesis aims to contribute to the field of financial derivatives by enhancing the application of Monte Carlo simulations in options pricing. The rigorous mathematical foundation underlying the thesis ensures that the findings are both theoretically and practically relevant. By integrating real world data from the Bloomberg Terminal, the study bridges the gap between theoretical models and market realities, offering insights that are both academically enriching and practically significant.

Chapter 1 provides an overview of financial derivatives, including forward contracts, futures, swaps, and options. This chapter explains the instruments and their applications, as well as their impact on the market.

In Chapter 2 we have constructed the mathematical foundations of stock price modeling and derivatives pricing. We have derived the lognormal model for stock prices, the Black-Scholes model, and discussed the principles of risk neutrality and no-arbitrage.

Chapter 3 is focused on the Greeks, detailing their significance and how they are computed for both call and put options. This chapter also explores the practical applications and challenges associated with managing these sensitivities.

In Chapter 4 we have derived pathwise estimators methods for calculating the Greeks.

Chapter 5 describes the implementation of Monte Carlo simulations for option pricing and the computation of Greeks. It includes the extraction of data from the Bloomberg Terminal and provides detailed results for both call and put options, validating the simulation model against real-world data.

Chapter 6 introduces advanced option pricing models, such as the Constant Elasticity of Variance (CEV) model and the Local Volatility model via the Dupire formula. This chapter demonstrates how these models capture the dynamics of implied volatility more accurately than the Black-Scholes model.

Chapter 1

The Dynamics of Financial Derivatives: Instruments, Applications, and Market Impact

Over the past four decades, derivatives have grown to play a critical role in the financial sector. Financial instruments such as futures and options are now extensively traded across global exchanges, while a variety of derivatives including forwards, swaps, and others are frequently managed in the over-the-counter (OTC[1]) markets by financial institutions, fund managers, and corporate treasurers. But let us clarify further. A derivative is a financial security whose value is derived from an underlying asset or group of assets. The underlying asset can be anything from stocks, bonds, commodities, currencies, interest rates, or market indexes. Derivatives are essentially contracts between two or more parties whose value is determined by fluctuations in the underlying asset. The applications of derivatives are diverse, including:

[1]Decentralized market where securities, such as stocks, bonds, commodities, or currencies, are traded directly between two parties without the supervision of an exchange.

1. Hedging: Derivatives are used to reduce risk by providing a way to insure against price movements in an asset that could result in a financial loss.

2. Speculation: Investors use derivatives to profit from the price movements of the underlying asset without actually owning it. This involves predicting the direction in which the prices of assets will move.

3. Arbitrage: This is the practice of exploiting price differences of the same or similar financial instruments on different markets or in different forms. Traders use derivatives to profit from discrepancies in prices.

We have now reached a stage where it is necessary for all financial professionals to understand how these markets work, how they can be used, and what determines their prices. Derivatives, whether viewed positively or negatively, are indispensable in the financial market. The scale of this market is immense, surpassing the stock market in the value of underlying assets and vastly exceeding global gross domestic product (GDP).

As we said, derivatives are contract in which two parties agree on future transaction, their value depends from one or more underlying assets or, as we will see, from any variables, such as interest rates, weather conditions, or even the outcome of specific events. These variables can affect the terms and the payoff of the contract, making derivatives a flexible and powerful tool for financial and risk management strategies, allowing parties to tailor the contracts to meet specific investment goals or risk exposure. The markets for exchange-traded derivatives and OTC derivatives are enormous. The number of OTC contracts traded in a year is fewer than the number of exchange-traded contracts, but the unit size is much larger, often estimated at over 1 quadrillion dollar, some market analysts even place the size of the market at more than 10 times that of the total world GDP.

1.1 Forward Contracts

In this sections, we will dive into the diverse world of financial derivatives, beginning with an examination of forward contracts. These are relatively simple agreements that play a crucial role in the financial strategies of various institutions, by allowing to buy or sell an asset at a future date for a specific price. Unlike spot contracts, which require immediate settlement, forwards are tailored to future needs and are typically conducted directly between parties, often off-exchange.

Moving further, an understanding of "long" and "short" positions within these contracts helps clarify how investors and institutions manage their strategies. In a forward contract, taking a long position means agreeing to buy the underlying asset at a predetermined future date and price, betting that the asset's price will rise. This is an optimistic view hoping for potential price increases. Contrarily, taking a short position in a forward contract involves agreeing to sell the underlying asset at a future date and price, anticipating a decline in the asset's price. This strategy is often used when expectations are bearish and there's a forecast of falling prices.

Both long and short positions enable investors to hedge against potential losses, speculate on future price movements, and exploit price differentials between markets. Forward contracts can be used to hedge against exchange rate risks.

As previously discussed, a forward contract involves two distinct prices. The first is the forward price F, which is the agreed delivery price for the underlying asset at a specific future date T. This price represents what would be the delivery price if a forward contract was established today. The second price, represented as f, reflects the current value of the forward contract, influenced by variations in the underlying asset's price, prevailing interest rates, and other relevant factors. Let us calculate the theoretical forward price F for a contract initiated at time $t = 0$ with a delivery due at time T. Let us assume the absence of transaction

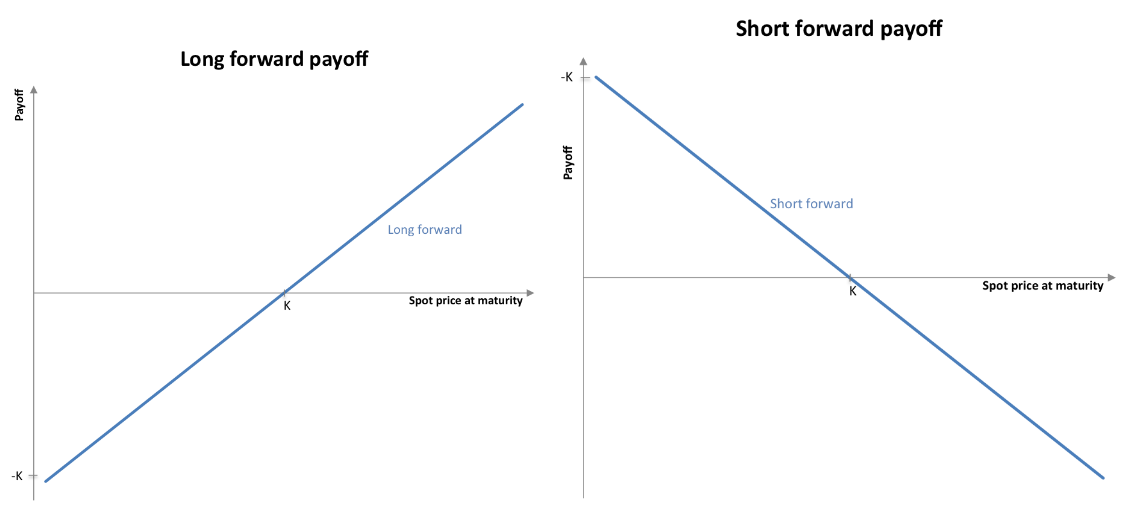

Fig. 1.1: Value of a Forward contract in case of a long position and short position.

Source: https://brilliant.org/wiki/forward-contract/

costs and the possibility to sell short.

Theorem 1.1.1. *We aim to derive the theoretical forward price F for a contract initiated at time $t = 0$ and set to deliver at time T assuming that there are no transaction costs. Consider that at $t = 0$, the asset has a spot price S and is contracted for future delivery at price F. The challenge lies in calculating the theoretical value of the forward contract by utilizing the connection with the spot market to facilitate borrowing or lending. The interest rate implicit in this transaction should be consistent with ordinary lending rates to prevent arbitrage opportunities.*

Specifically, if one purchases a unit of the commodity at the spot price S and concurrently enters into a forward contract to sell it at F, one engages in direct arbitrage by storing the commodity and fulfilling the forward contract at a later date. The cash flows associated with these market operations at $t = 0$ must align with the interest rate spanning from $t = 0$ to $t = T$, hence

$$S = d(0, T)F \,,$$

where $d(0,T)$ is the discount factor.

Proof. Suppose $F > \frac{S}{d(0,T)}$. Then, by borrowing S, purchasing the asset, and entering a short position in the forward market, the initial investment is zero. At T, delivering the stored asset yields F, and repaying $\frac{S}{d(0,T)}$ clears the loan, resulting in a profit of $F - S/d(0,T)$, indicating an arbitrage if such opportunities were deemed nonexistent.

If $F < \frac{S}{d(0,T)}$, the opposite transaction involves borrowing the asset, selling it at the current spot price, and taking a long position in the forward market. The profit from this reverse arbitrage would also indicate a market anomaly unless $F = \frac{S}{d(0,T)}$.

The relationship between the spot price S and the forward price F illustrates that the forward price at inception is essentially the spot price adjusted for the prevailing interest rate over the duration of the contract. □

Example 1.1. *If the interest rate r is compounded continuously, the forward rate formula adjusts to*

$$F = Se^{rT}\,.$$

The discount rate $d(0,T) = e^{rT}$ used should align with the market rates accessible to traders. In forward and futures markets, the repo rate commonly associated with repurchase agreements is used, which is marginally higher than the Treasury bill rate.

Another way to view the payoffs of a forward contract is by considering the Figure 1.1. Suppose that S_T is the spot price of the asset at maturity time of T and K is the agreed-upon delivery price. The investor with the long position is obligated to buy at price K an asset whose value is S_T, hence, the payoff is

$$S_T - K\,.$$

Similarly, the payoff of a short forward contract written on a unit quantity of the underlying asset is

$$K - S_T \,.$$

1.2 Futures Contracts

Futures contracts, like forwards, are agreements between two parties to buy or sell an asset at a future date for a specific price. Unlike forwards, futures are typically traded on regulated markets. To facilitate trading, the exchange specifies certain standard aspects of the contract. Since the two parties do not necessarily know each other, the exchange's clearinghouse[2] intervenes between the parties, thus ensuring that the contract will be honored.

1.3 Swaps Contracts

"Swaps are OTC contracts in which two companies agree to exchange future payments. The contract defines the dates when payments must be made and how they are calculated. Typically, the payments depend on the future value of an interest rate, an exchange rate, or some other market variable". The key attraction of swaps is the ability to transform one type of cash flow into another by enabling the exchange of one for another directly. This process, which can involve transactions worth hundreds of billions of dollars, is their ability to be customized to specific needs.

[2]A clearinghouse acts as an intermediary between buyers and sellers in financial markets, particularly for securities and derivatives transactions. Its primary role is to ensure the smooth and secure completion of trades, reducing the risk for all parties involved.

1.3.1 Interest Rate Swap

The most commonly used type is the interest rate swap, where one party exchanges a series of fixed interest payments for a series of variable interest payments from another party. This type of swap functions similarly to a series of forward contracts, making it possible to apply forward pricing concepts to them.

Example 1.2. *Take, for instance, a plain vanilla interest rate swap: Party A commits to making semiannual payments to Party B based on a fixed rate of interest on a notional principal amount, while Party B agrees to pay Party A based on a variable interest rate, such as the prevailing 6-month LIBOR rate, using the same notional principal. Notably, the term "notional principal" refers to the fact that there is no actual exchange of principal, it merely serves to calculate the payment amounts. Typically, the swap involves payments only on the net difference between the agreed amounts by the party owing more at each settlement. The swap between Party A and Party B effectively converts the floating rate payments into fixed payments, simplifying financial management for Party B.*

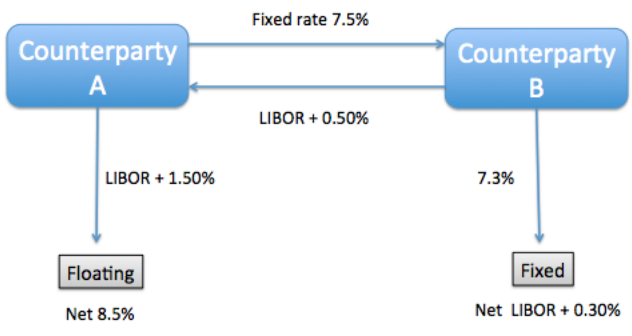

Fig. 1.2: Scheme of a plain vanilla swap

Source: https://financetrain.com/plain-vanilla-interest-rate-swap

1.3.2 Currency Swaps

Another commonly utilized type of swap is the "currency swap." In its most basic form, known as a "fixed-for-fixed" currency swap, the agreement involves exchanging the principal and interest payments of a fixed-rate loan in one currency with those of a fixed-rate loan in a different currency. At the start of the contract, the principal amounts are swapped based on the prevailing exchange rate, and these amounts are exchanged again when the contract concludes. Although the principal values are approximately equivalent at the beginning, they can fluctuate significantly by the time of the final exchange due to changes in rates over the contract period.

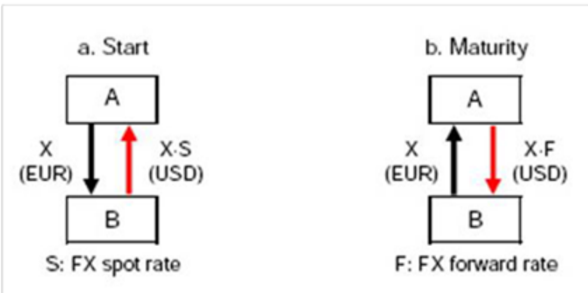

Fig. 1.3: Currency swap example

Source: `https://www.bis.org/publ/qtrpdf/r_qt0803z.htm`

1.3.3 Other Currency Swaps

Two other fairly common types of currency swaps are:

- Floating-for-fixed currency swaps, in which a variable rate denominated in one currency is exchanged for a fixed rate denominated in another currency
- Floating-for-floating currency swaps, in which a variable rate denominated

in one currency is exchanged for a variable rate denominated in another currency

1.3.4 Credit Default Swaps

Credit Default Swaps (CDS) are sophisticated financial instruments that function much like insurance policies against the risk of default by a borrower. A CDS protects investors, typically lenders or bondholders, by transferring the credit risk of a debt from the holder to the seller of the swap. The process involves a protection buyer who pays a premium to hedge against potential defaults and a protection seller who collects this premium and agrees to compensate the buyer if the referenced debtor defaults. The reference entity is the third party whose debt is being insured. The cost of a CDS is influenced by factors such as the creditworthiness of the reference entity, overall market conditions, and the balance of supply and demand in the market. CDSs are also utilized for speculation on changes in a debtor's credit status and for arbitrage opportunities to exploit price inefficiencies between the bond and CDS markets.

1.4 Option Contracts

Options are traded both on exchanges and in over-the-counter (OTC) markets. There are two fundamental types of options: calls and puts. A "call option" gives the holder the right to buy an asset by a certain date for a specified price. A "put option" gives the holder the right to sell an asset by a certain date for a specified price.

Options contracts are financial instruments that specify the terms under which certain assets can be bought (i.e. call) or sold (i.e. put) by a certain date (i.e. maturity) for a specified price, the strike price, in other words is the cost per share that the buyer agrees to pay if they choose to exercise the option.

Options are divided into two main styles: American and European. An American option provides the holder with the flexibility to exercise the option at any time up until the maturity. This ability to choose the timing of the exercise can be particularly valuable if the underlying asset's price moves favourably. On the other hand, a European option restricts exercise to the expiration date only, typically resulting in a lower premium due to this inflexibility.

The transactional structure of options involves two sides: the option writer and the option buyer. The writer, or seller, of the option grants the rights contained within the option to the buyer. In return, the buyer pays a premium to the writer. This premium compensates the writer for the risk they undertake, as the option writer is obligated to fulfill the terms of the contract if the buyer decides to exercise the option.

For instance, if the buyer exercises a call option, the writer must ensure the delivery of the asset at the agreed strike price. If the writer does not already own the asset, they must purchase it at the current market price to meet the contract's terms. This could result in a significant loss if the market price exceeds the strike price. Conversely, in the case of a put option, the writer might need to purchase the asset at the strike price, potentially much higher than the market price, if the buyer exercises their right to sell.

Options trading on exchanges involves rigorous regulations to ensure fair practices and risk management. The clearinghouse associated with the exchange oversees these trades and requires the option writer to post margin[3]. This margin acts as a form of security, ensuring that the writer can cover any losses incurred from fulfilling their obligations under the options contracts.

Consider we have a call option on a stock with a strike price of K. If at

[3] A security deposit

expiration, the stock's price is S, how would we determine the value of the option? If $S < K$, the option holds no value since exercising the option to buy the stock at K would incur in a loss compared to buying directly at the market price S. Conversely, if $S > K$, the option is valuable as it allows the holder to buy the stock at K and potentially sell it at S, yielding a profit of $S - K$, so we define the value of the call option at maturity as

$$C = \max(0, S - K)\,.$$

This equation indicates that the value of the call option is either zero (if exercising the option is not advantageous) or the difference between the stock price and the strike price (if beneficial).

For a put option, where the holder has the right, but not the obligation, to sell the stock at a strike price K, the situation reverses. If the stock price at expiration is S and it satisfies $S > K$, the put option is considered worthless because selling the stock in the open market at S would be more advantageous than exercising the option. However, if $S < K$, the option gains value as it permits the holder to sell the stock at a higher guaranteed price K, making a profit of $K - S$. The formula for calculating the value of a put option at expiration is

$$P = \max(0, K - S)\,.$$

This function shows that the value of a put option is bounded by the difference between the strike price and the stock price, whereas the potential gain for a call option is theoretically unlimited as the stock price could rise infinitely.

Let us define a call option in the money (ITM), at the money (ATM), and out of the money (OTM) based on their value relative to the current stock price S and strike price K. Calls are ITM when $S > K$, ATM when $S = K$, and OTM when $S < K$. Puts use the opposite terminology, in other words being ITM when $S < K$ and OTM when $S > K$.

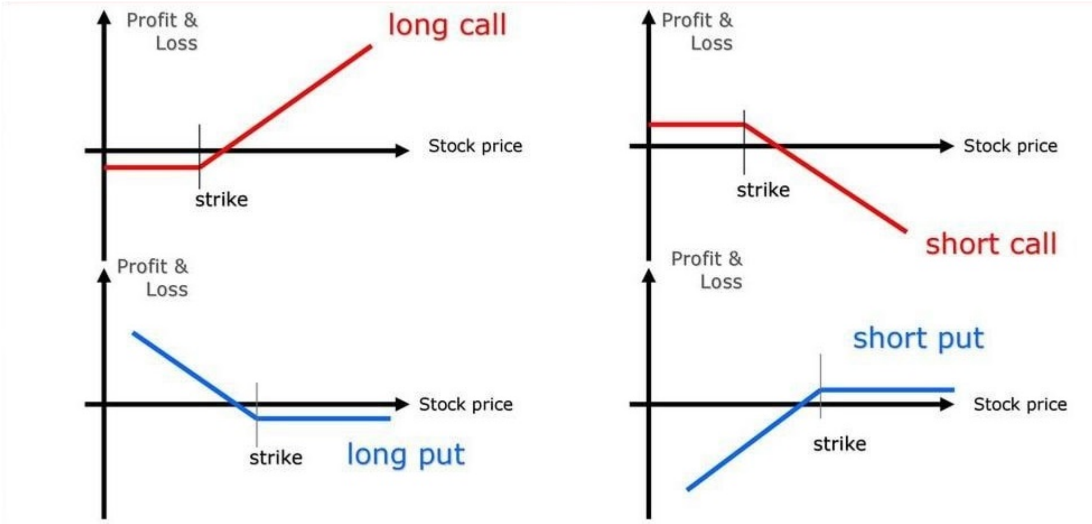

Fig. 1.4: Call and Put payoffs

Source: https://libertex.com/blog/put-vs-call-option

1.4.1 Options Time Value

The previous discussions have primarily addressed the value of an option at the time of its expiration. This valuation is based on the fundamental structure of an option. However, even European options, which are constrained to be exercised only at expiration, can possess intrinsic value well before this date, given their potential for eventual exercise. When there is significant time remaining until an option's expiration, the value of a call option can be modeled as a smooth curve rather than a fixed point, which better reflects the real-world dynamics of option pricing. This curve can be extrapolated from actual market data and illustrates how option prices tend to evolve over time until expiration. In Figure 1.5, the lowest curve represents an option with three months to expiration, with higher curves representing longer duration. As more time is available until expiration, the probability of a stock price increase enhances the potential payout from exercising the option. However, this potential diminishes significantly when the stock price

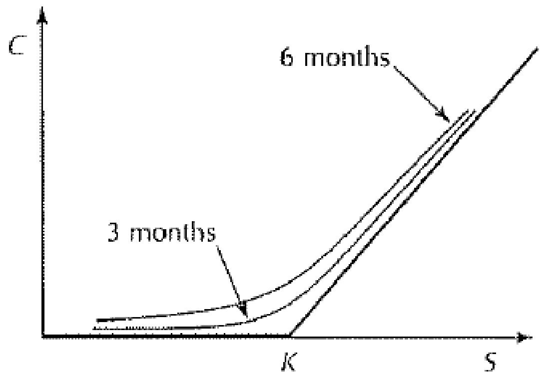

Fig. 1.5

Source: Investment Science, David G. Luenberger, p.324

is much below or far above the strike price K. Specifically, if S is much lower than K, the likelihood that S will surpass K is little, and the value of the option remains near zero. Conversely, if S is significantly higher than K, the benefit of holding the option over simply owning the stock becomes negligible.

1.4.2 More Factors Influencing Option Prices

As we saw in the previous section, duration of an option can significantly influence the price itself, but there are even more variables that can modify the option value such as:

- the current price of the stock S_0
- the strike price K
- the volatility of the stock price σ: if the volatility increases, the likelihood that the stock's performance will be either very good or very poor increases. Holding a call benefits from price increases and has limited downside risk because, in case of a price drop, they cannot lose more than the premium

paid. Similarly, someone holding a put benefits from price declines but has limited downside risk in case of a price rise. Therefore, the value of both calls and puts increases as volatility increases.

- the risk-free interest rate r: if interest rates increase, expected stock price growth rates also tend to increase. Additionally, from the perspective of option holders, higher interest rates decrease the present value of future cash flows, so that the price of call options tends to increase, while the price of put options tends to decrease.

- dividends[4]: dividends decrease the stock price on the ex-dividend date, increasing the value of puts and decreasing the value of calls.

In summary, options are complex financial instruments with specific characteristics and rules that govern their pricing and execution. They offer strategic opportunities for investors to hedge, speculate, or increase their investment income.

1.5 Put-Call Parity

Let us now derive an important relationship between P and C in the case where the underlying asset is a security that does not pay dividends.

Theorem 1.5.1. *Let C and P be the prices of an European call and an European put, both with strike price K and both defined on the same stock with price S. Then*

$$C + dK = P + S\,, \tag{1.1}$$

[4] A dividend is a payment made by a corporation to its shareholders, usually as a distribution of profits. When a company earns a profit or surplus, it can reinvest the profit into the business and pay a portion of the profit as a dividend to shareholders.

where $d = e^{-rT}$ *is the discount factor to the expiration date. Such a relation is called "put-call parity".*

Equation (1.1) shows us how the value of a European call with a certain exercise price and a certain expiration can be deduced from the value of a European put with the same exercise price and the same expiration, and vice versa. If the put-call parity was not respected, there would have been arbitrage opportunities.

Chapter 2

Mathematical Foundations of Stock Price Modeling and Derivatives Pricing

2.1 The Lognormal Model for the Stock Price

The log-normal model is commonly used to describe the behaviour of stock prices over time, providing a theoretical framework for understanding price changes. The main assumption of this model is that prices cannot be negative over time, in the worst case scenario they converge to zero.

It could be used a more simplistic normal-model to describe price changes because many random quantities are them-self normally distributed, but it is not suitable for stock prices since a normal random quantity X satisfies $\mathbb{P}(X < 0) > 0$. So, for this reason a better model for understanding stock prices movements is the log-normal one.

Consider a random variable $S(t)$ that denote the stock price a time t. We say that $S(t)$ has lognormal if the natural logarithm of $S(t)$, i.e. $\log(S(t))$, follows a normal distribution.

The following proposition shows that the hypothesis that $\log(S(t))$, follows a normal distribution is satisfied under some assumption.

Proposition 2.1.1. *Given the stock price $S(t)$ at time t, for any time $u > t$ define the (random) accumulation factor for an investment in stock between time t and u as*

$$A(t,u) = \frac{S(u)}{S(t)}\,.$$

Note that if $A(t,u) > 1$, then the stock price increased between time t and u, while if $A(t,u) < 1$ it decreased.
Suppose that the following conditions are satisfied:

- *for any $h > 0$ we have that $A(t,t+h)$ is influenced by h but remains unaffected by the specific starting time t;*
- *the probability distribution of $A(t,t+h)$, i.e. $\mathbb{P}[A(t,u) \leq x]$ for any $x \in \mathbb{R}$, is not influenced by the current stock price $S(t)$, neither by its preceding values $S(s)$ for $s \leq t$, meaning that our distribution $A(t,t+h)$ is independent of the stock price $S(t)$.*

Then $\log(\frac{S(u)}{S(t)}) = \log(A(t,u))$ has a normal distribution. More precisely,

$$\log\frac{S(u)}{S(t)} = \log(A(t,u)) \sim \mathcal{N}(c(u-t), \sigma^2(u-t))\,,$$

where $c \in \mathbb{R}$ and $\sigma > 0$ are two constants.

Proof. Fix two time points, t and u, so that $t < u$, this define the time frame in which we want to analyze the evolution of the stock price. We divide the interval $[t,u]$ in n subintervals of same length, using the sequence of temporal points

$$t_j = t + \frac{j}{n}(u-t)$$

for $j = 0, 1, 2..., n$, meaning that each t_j represents a temporal point within the interval, with t_0 corresponding to the starting point of the interval (i.e. t), and t_n corresponding to the ending point of the interval (i.e. u). With the above notation, the term $\frac{u-t}{n}$ is essentially the length of each interval.

Then:

$$\frac{S(u)}{S(t)} = \frac{S(t_n)}{S(t_0)} = \frac{S(t_1)}{S(t_0)} \cdot \frac{S(t_2)}{S(t_1)} \cdot \frac{S(t_3)}{S(t_2)} \cdot \ldots \cdot \frac{S(t_n)}{S(t_{n-1})}$$

This equation demonstrates that the total change in the stock price from t to u can be represented as the product of changes over n smaller intervals. Recalling the definition of $A(t, u)$, we can rewrite the above expression as

$$A(t, u) = A(t_0, t_1)A(t_1, t_2)A(t_2, t_3) \ldots A(t_{n-1}, t_n) \,.$$

Setting $L(t, u) = \log(A(t, u))$, we have

$$L(t, u) = L(t_0, t_1) + L(t_1, t_2) + L(t_2, t_3) + \ldots + L(t_{n-1}, t_n) \,.$$

This step transform the multiplicative process of stock price changes into an additive process. By assumptions the random quantities $L(t_{j-1}, t_j)$ are i.i.d. and, taking n large enough, we can use the Central Limit Theorem A.2 to demonstrate that $L(t, u)$ is normally distributed. Since we have assumed that $A(s, s + t)$ depends only on t (and not on the starting point s), such a property extends to $L(s, s + t)$. So we can deduce that $L(s, s + t)$ follows a normal distribution, with mean $m(t)$ and variance $v(t)$, where $m(t)$ and $v(t)$ are two functions of t. In order to understand the expressions of these two functions note that

$$L(0, t + u) = L(0, t) + L(t, t + u) \,.$$

This is a sum of independent normal random variable, so by taking mean

$$m(t + u) = m(t) + m(u)$$

This equation implies that the logarithmic change over an interval $t+u$ is the sum of the changes over the individual intervals t and u.
Following the same reasoning for the variance, we get

$$v(t+u) = v(t) + v(u)$$

From these two equations we can deduce that both $m(t)$ and $v(t)$ are proportional to t, so there is a constant $c \in \mathbb{R}$ and a constant $\sigma > 0$ such that:

$$m(t) = ct, \qquad v(t) = \sigma^2 t\,.$$

This statement suggests that, for each unit of time, the average $m(t)$ increases linearly with a constant of proportionality c, while the variance $v(t)$ increases linearly with a proportionality constant σ^2. These linear relationships indicate that the mean of logarithmic change of the price of a stock is directly proportional to the length of the time interval, with c which represents the average growth rate per unit of time.
The variance of the logarithmic change in a stock's price is also proportional to the length of the time interval, reflecting how the uncertainty or volatility of the stock's price increases over time.
So, for any $t < u$, according to the statement done before, the logarithmic accumulation factor follows a normal distribution with mean $c(u-t)$ and variance $\sigma^2(u-t)$, that is

$$\log \frac{S(u)}{S(t)} = \log A(t,u) = L(t,u) \sim \mathcal{N}(c(u-t), \sigma^2(u-t))\,,$$

that concludes the proof. □

Suppose now that we know $S(0) = 1$. We are interested in determining its expected value after a certain period t, i.e. $\mathbb{E}[S(t)]$. As before, let us assume that the price changes follows a lognormal distribution (that is the assumptions in Proposition 2.1.1 are verified). Since

$$\log S(t) = \log A(0,t) \sim \mathcal{N}(ct, \sigma^2 t)\,,$$

we have that

$$\log S(t) = ct + \sigma\sqrt{t}Z\,,$$

where $Z \sim \mathcal{N}(0,1)$. So

$$S(t) = A(0,t) = e^{ct+\sigma\sqrt{t}Z} = e^{ct}e^{\sigma\sqrt{t}Z}\,.$$

As a consequence

$$\mathbb{E}[S(t)] = e^{ct}\mathbb{E}[e^{\sigma\sqrt{t}Z}] = e^{ct}e^{\sigma^2 t/2} = e^{\left(c+\frac{\sigma^2}{2}\right)t}\,.$$

Defining $\mu = c + \frac{\sigma^2}{2}$, we can express the expected value as

$$\mathbb{E}[S(t)] = e^{\mu t}\,,$$

where μ represents the growth rate for the expected stock price. This shows that the stock price follows a geometric Brownian motion, that we will define better later, with μ representing the growth rate and σ the volatility.

Definition 2.1. *Let $\sigma > 0$ and let μ be constants, a* ***lognormal process****, is a family of random variable $\{S(t)\}_{t\geq 0}$ with the following properties*

- *for all $t \geq 0$ and $u \geq 0$*

$$\log \frac{S(t+u)}{S(t)} \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)u, \sigma^2 u\right)\,.$$

- *for any $t \geq 0$, $u \geq 0$ and any sequence $0 \leq s_1 < s_2 < s_3 < \ldots < s_n \leq t$ we have that $\frac{S(t+u)}{S(t)}$ is independent on $S(s_1), S(s_2), \ldots, S(s_n)$, meaning that the change in the asset's price form t to $t+u$ is independent of the asset's prices at previous times $s_1, s_2, \ldots, s_n$.*

The value μ is called infinitesimal drift.

Example 2.1. *Suppose we want to calculate $\mathbb{E}[S(t)]$ when $S(0) = 1$, knowing that $\{S(t)\}_{t\geq 1}$, follows a lognormal process with infinitesimal drift μ and volatility σ. Suppose also that for some particular time $t = T > 0$, the value of $S(t)$ is known. Then for $u \geq T$ we have*

$$\mathbb{E}[S(u)] = S(T)\mathbb{E}[e^{(\mu - \frac{1}{2}\sigma^2)(u-T)+\sigma\sqrt{u-T}Z}] =$$
$$= S(T)e^{(\mu-\frac{1}{2}\sigma^2)(u-T)+\frac{1}{2}\sigma^2(u-T)} = S(T)e^{\mu(u-T)} .$$

So

$$\mathbb{E}[S(u)] = S(T)e^{\mu(u-T)} .$$

If we want to comprehend the behaviour of a lognormal process, we can study it in terms of the change of the stock price over a millesimal time interval.
Let us suppose that $S(t)$ follows a lognormal process, again, with expected rate return μ and volatility σ. For $t \geq 0$ and $h \geq 0$, using the definition of lognormal explained before we have

$$\log \frac{S(t+h)}{S(t)} \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)h, \sigma^2 h\right) ,$$

Defining Z as a standard normal random variable, we have

$$\log \frac{S(t+h)}{S(t)} = \left(\mu - \frac{\sigma^2}{2}\right)h + \sigma\sqrt{h}Z \Longrightarrow \frac{S(t+h)}{S(t)} = e^{(\mu-\frac{\sigma^2}{2})h+\sigma\sqrt{h}Z} .$$

We denote with Λ the proportionate increase in the stock price from time t to $t+h$

$$\Lambda(t,h) = \frac{S(t+h)-S(t)}{S(t)} = \frac{S(t+h)}{S(t)} - 1 \Longrightarrow \Lambda(t,h) = e^{((\mu-\frac{\sigma^2}{2})h+\sigma\sqrt{h}Z)} - 1 .$$

We can use the MacLaurin expansion of the exponential function, that for small x approximates the function

$$e^x - 1 \approx x + \frac{x^2}{2} .$$

We can apply this expansion since we considered an infinitesimal time interval, that is h is sufficiently small. Taking care only of terms of grade lower than two, we get

$$\begin{aligned}\Lambda(t,h) &\approx \left(\mu - \frac{\sigma^2}{2}\right) h + \sigma\sqrt{h}Z + \frac{1}{2}\sigma^2 h Z^2 \\ &= \left(\mu h - \frac{\sigma^2}{2} h\right) + \sigma\sqrt{h}Z + \frac{1}{2}\sigma^2 h Z^2 \\ &= \mu h + \sigma\sqrt{h}Z + \left(\frac{1}{2}\sigma^2 h Z^2 - \frac{1}{2}\sigma^2 h\right) \\ &= \mu h + \sigma\sqrt{h}Z + \frac{1}{2}\sigma^2 h \left(Z^2 - 1\right) .\end{aligned}$$

Analyzing what we have above, we can conclude that the first term of the right hand side is deterministic, meanwhile the second term has zero mean and variance $\sigma^2 h$. The third term instead can be considered negligible since its mean is zero and its variance is of smaller order with respect to the other terms. So

$$\Lambda(t,h) \approx \mu h + \sigma\sqrt{h}Z ,$$

that is

$$\Lambda(t,h) \sim \mathcal{N}(\mu h, \sigma^2 h) .$$

From now on we use dt instead of h to denote a small increment in time, $dS(t)$ to represent the increase in stock price over the brief interval from t to $t+dt$, and dZ for a normal variable with mean 0 and variance dt, independent of prior events. Within this framework, we have

$$\frac{dS(t)}{S(t)} = \Lambda(t,dt) = \mu dt + \sigma dZ ,$$

which we may write as

$$\frac{dS}{S} = \mu dt + \sigma dZ ,$$

that is

$$dS = \mu S dt + \sigma S dZ . \tag{2.1}$$

This whole model, in the economic literature, is also referred to as the **geometric Brownian motion**, a milestone in the quantitative finance field. It forms the mathematical underpinning for the Black-Scholes option pricing model and has widespread applications in risk management and financial derivatives pricing. The strength of this model lies in its ability to capture the continuous-time stochastic behavior of stock prices, reflecting the unpredictability of markets and the compounded effect of small, random fluctuations over time. Despite its simplicity, the geometric Brownian model provides a surprisingly accurate description of real market behavior for certain types of assets and under certain market conditions.

2.2 Risk Neutrality and No-Arbitrage Principle

2.2.1 No-Arbitrage Principle

Before going in deep in the next section where we explain the Black-Scholes model, we have to clarify two concept, no-arbitrage principle and risk neutrality. The no-arbitrage principle asserts there should be no way to make a risk-free profit with zero net investment. It serves as a critical regulatory mechanism within financial markets, ensuring that asset prices remain fair and representative of underlying economic realities. When this principle is violated, arbitrage opportunities arise, allowing traders to make profits without exposure to risk by exploiting pricing inefficiencies across different markets. Practically it dictates that all current market prices must be aligned in a way that no combination of positions can lead to an arbitrage opportunity.

Definition 2.2. *The no-arbitrage condition can be expressed using the concept of a self-financing portfolio*[1]*. Consider two assets, A and B, priced at $p_A(t)$ and $p_B(t)$*

[1]In financial mathematics, a self-financing portfolio is a portfolio having the feature that, if

at time t. Let us suppose that the portfolio $V(t)$ starts with no initial investment at time t and holds s units of A and b units of B, then

$$V(t) = sp_A(t) + bp_B(t)$$

Following the no-arbitrage principle, the value change of the portfolio over time is describes as

$$dV(t) = sdp_A(t) + bdp_B(t) = 0$$

Any gains in one part of the portfolio are offset by losses in another, maintaining a zero net investment status throughout the portfolio's existence. In other words, if the initial value of an admissible portfolio is zero, $V(0) = 0$, then $V(1) = 0$ with probability one.
The no-arbitrage principle can also be represented in terms of stochastic processes. Let us consider a financial market that follows a set of price processes $P_1, P_2, P_3, ..., P_n$. The market satisfies the no-arbitrage condition if there are no investment strategies $\theta_1, \theta_2, \theta_3, ..., \theta_n$ such that

$$\sum_{i=1}^{n} \theta_n dP_i > 0,$$

with probability 1.

2.2.2 Risk Neutrality

Risk neutrality is a theoretical concept in financial mathematics that assumes investors are indifferent to risk, simplifying the modeling and valuation of financial instruments, providing a standardized approach for valuation. In a risk-neutral world, the expected return from any investment is adjusted to match the risk-free rate r. Risk neutrality is expressed by setting the expected utility of a portfolio's

there is no exogenous infusion or withdrawal of money, the purchase of a new asset must be financed by the sale of an old one return equal to the utility of its expected return. Denoting U as our utility function and R as our portfolio's return, under risk-neutrality we have

$$\mathbb{E}[U(R)] = U(\mathbb{E}[R])$$

allowing to use the expected return $\mathbb{E}[R]$ directly in the calculation, permitting to focus only on the expected future cash flows from the investment discounted at the risk-free rate, aligning with the no-arbitrage principle, ensuring that the investment does not offer risk-free profit opportunities.

2.3 The Black-Scholes Model

Let us consider a stock whose price $S(t)$ for $t \geq 0$ follows a geometric Brownian motion with expected rate of return μ and volatility σ. If we have at derivative that has a payoff $f(S(T))$ a time T, with $f(x)$ being a function defined for $x > 0$, how can we price this derivative? Let us also assume that $S(0)$ is a known quantity and that the interest rate r is constant and compounded. A plausible approach to price this derivative is to use the present value of the expected future payout a time T, that is

$$e^{-rT}\mathbb{E}[f(S(T))] . \tag{2.2}$$

However this method is based on the expected present value in (2.2) that diverges from the no-arbitrage pricing. In continuous time as in discrete time, this no-arbitrage pricing aligns with the expected present value under a risk-neutral probability measure, assuming that the expected return rate of the stock is not its actual rate, but instead the risk-free rate r, ensuring that the expected growth rate of the stock mirrors that of a risk-free investment. As we refine the discrete model to increasingly smaller time increments, it approximates the continuous model closely enough. So we can change the (2.2) to be suitable

$$e^{-rT}\widetilde{\mathbb{E}}[f(S(T))]\,,$$

where the expected value $\widetilde{\mathbb{E}}[\cdot]$ is the expectation with respect to the risk-neutral probability measure.
The expected payoff $e^{-rT}\widetilde{\mathbb{E}}[f(S(T))]$, can be computed as we saw earlier, getting

$$\log\frac{S(T)}{S(0)} \sim \mathcal{N}\left(\left(\mu-\frac{\sigma^2}{2}\right)T, \sigma^2 T\right)\,,$$

while, in the risk-neutral model we replace μ with r, that is

$$\log\frac{S(T)}{S(0)} \sim \mathcal{N}((r-\frac{\sigma^2}{2})T, \sigma^2 T)\,.$$

Denoting Z as a standard normal random variable and S for $S(0)$ we have:

$$\log\frac{S(T)}{S} = (r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z\,,$$

so that

$$S(T) = S\exp\left(\left(r-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right)\,.$$

Hence

$$\widetilde{\mathbb{E}}[f(S(T))] = \int_{-\infty}^{\infty} f\left(S\exp\left(\left(r-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x\right)\right)\phi(x)\,dx\,,$$ [2]

where $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ is the standard normal probability density function.

From the risk-neutral model, we move to the Black-Scholes model, marking a milestone of modern finance. The Black-Scholes model is revolutionary for using the concept of risk neutrality to determine a correct price for European options. This method minimize the uncertainty associated with asset return expectations

[2] Since we're in a continuous space for calculating the expected value we need to use the formula $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(t)f_x(t)\ dt$.

by substituting them with the risk-free interest rate, allowing for the valuation of options without considering the investors risk preferences.
This shift to risk-neutral probabilities is made possible by moving from a model that assumes an expected return rate μ to one that employs the risk-free interest rate r. In the Black-Scholes model, the price of a European put option with strike price K and maturity T is calculated as the present value of the expected payoff under the risk-neutral measure.
To connect these concepts, we might say that risk neutrality in option valuation moves away from investors subjective expectations and is based instead on the premise of a world where the asset's return rate is replaced by the risk-free interest rate.

2.3.1 Black-Scholes Formula for a Put Option

Let us start to analyze the Black-Scholes formula for a put option. Suppose that the stock price process $\{S(t)\}_{t\geq 0}$ follows a geometric Brownian motion with expected rate of return μ and volatility σ. To study the put option we define the function f as

$$f(x) = (K - x)^+ \,. \tag{2.3}$$

Defining $S = S(0)$, we want to compute

$$e^{-rt}\widetilde{\mathbb{E}}[f(S(T))] \,.$$

We have

$$\begin{aligned}\widetilde{\mathbb{E}}[f(S(T))] &= \int_{-\infty}^{\infty} f\left(S\exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x\right)\right)\phi(x)\,dx = \\ &= \int_{-\infty}^{\infty} f\left(S\exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x\right)\right)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx \,.\end{aligned} \tag{2.4}$$

Let us define y_0 as the value for which

$$Se^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}y_0)} = K \,.$$

So

$$f(S(T)) = (K - S(T))^{+} = K - Se^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}y_0)} = 0\,.$$

Let us now compute the value of y_0

$$\begin{aligned}
\frac{K}{S} &= e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}y_0} \\
\log\left(\frac{K}{S}\right) &= \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}y_0 \\
\sigma\sqrt{T}y_0 &= \log\left(\frac{K}{S}\right) - \left(r - \frac{\sigma^2}{2}\right)T
\end{aligned}$$

that implies

$$y_0 = \frac{log\left(\frac{K}{S}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\,.$$

Since

$$f\left(S\exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x\right)\right) = 0 \qquad \forall\, x \geq y_0$$

to compute $\widetilde{\mathbb{E}}[f(S(T))]$ we can restrict the integral in (2.4) to the interval $(-\infty, y_0)$

$$\begin{aligned}
\widetilde{\mathbb{E}}[f(S(T))] &= \int_{-\infty}^{y_0}\left(K - S\exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x\right)\right)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx = \\
&= \int_{-\infty}^{y_0} K\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx - S\int_{-\infty}^{y_0}\exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x\right)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx = \\
&= K\int_{-\infty}^{y_0}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx - \frac{Se^{rT}}{\sqrt{2\pi}}\int_{-\infty}^{y_0} e^{-\frac{\sigma^2}{2}T}e^{\sigma\sqrt{T}x}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx\,.
\end{aligned}$$

The term $\int_{-\infty}^{y_0}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx$ represents the cumulative distribution function of the standard normal distribution, evaluated at y_0.[3] Therefore this term can be written as

[3]The probability density function for a standard normal random variable Z is

$$f_Z(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)\,.$$

Integrating $f_Z(z)$ from $-\infty$ to y_0, we get $\Phi(y_0)$, where Φ is the cumulative distribution function.

$K\Phi(y_0)$. So we have

$$\widetilde{\mathbb{E}}[f(S(T))] = K\Phi(y_0) - \frac{Se^{rT}}{\sqrt{2\pi}} \int_{-\infty}^{y_0} e^{-\frac{\sigma^2}{2}T} e^{\sigma\sqrt{T}x} e^{-\frac{x^2}{2}}\, dx =$$

$$= K\Phi(y_0) - \frac{Se^{rT}}{\sqrt{2\pi}} \int_{-\infty}^{y_0} e^{\left(\left(\frac{\sigma\sqrt{T}}{\sqrt{2}}\right)^2 + \sigma\sqrt{T}x - \left(\frac{x}{\sqrt{2}}\right)^2\right)}\, dx\,,$$

that is

$$\widetilde{\mathbb{E}}[f(S(T))] = K\Phi(y_0) - Se^{rT} \int_{-\infty}^{y_0} \frac{1}{-}\sqrt{2\pi} e^{\left(\frac{-(x-\sigma\sqrt{T})^2}{2}\right)}\, dx\,. \tag{2.5}$$

Let us set $y_1 = y_0 - \sigma\sqrt{T}$. Such a choice will be justified by a change of variable that we will do inside the integral to get the integral of the density function of a standard normal random variable. Hence we have

$$\begin{aligned} y_1 &= \frac{\log\left(\frac{K}{S}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} - \sigma\sqrt{T} \\ &= \frac{\log\left(\frac{K}{S}\right) - \left(r - \frac{\sigma^2}{2}\right)T - \left(\sigma\sqrt{T}\right)\left(\sigma\sqrt{T}\right)}{\sigma\sqrt{T}} \\ &= \frac{\log\left(\frac{K}{S}\right) - \left(r - \frac{\sigma^2}{2}\right)T - \left(\sigma^2\sqrt{T}\right)}{\sigma\sqrt{T}} \\ &= \frac{\log\left(\frac{K}{S}\right) - rT + T\frac{\sigma^2}{2} - \sigma^2 T}{\sigma\sqrt{T}} \\ &= \frac{\log\left(\frac{K}{S}\right) - \left(r - \frac{\sigma^2}{2} + \sigma^2\right)T}{\sigma\sqrt{T}} \\ &= \frac{\log\left(\frac{K}{S}\right) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\,. \end{aligned}$$

Now we can solve the equation (2.5). We will solve the integral by substitution method. Let us impose

$$\begin{aligned} &s = x - \sigma\sqrt{T} \\ &dx = ds \\ &y_1 = y_0 - \sigma\sqrt{T}\,. \end{aligned}$$

We have

$$\widetilde{\mathbb{E}}[f(S(T))] = K\phi(y_0) - Se^{rT}\int_{-\infty}^{y_1} \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\, ds$$

It is easy to notice that $\int_{-\infty}^{y_1} \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} = \Phi(y_1)$ and hence

$$\widetilde{\mathbb{E}}[f(S(T))] = K\Phi(y_0) - Se^{rT}\Phi(y_1)\,.$$

Knowing that the price of the put option is given by

$$P = e^{-rT}\widetilde{\mathbb{E}}[f(S(T))]\,,$$

we have

$$\begin{aligned} P &= Ke^{-rT}\Phi(y_0) - Se^{-rT}e^{rT}\Phi(y_1) \\ &= Ke^{-rT}\Phi(y_0) - S\Phi(y_1)\,. \end{aligned}$$

The last expression is the **Black-Scholes formula for the no-arbitrage price of the European put option**.

2.3.2 Black-Scholes Formula for a Call Option

Building on our understanding of the Black-Scholes formula for a European Put option, we can derive the formula for a European Call option using the principle of put-call parity. Put-call parity establishes a relationship between the price of a call option, denoted by C, and the price of a put option, denoted by P, on the same underlying asset with the same strike price K and the same expiration time T. According to put-call parity, we have

$$C = P + S - Ke^{-rT}\,,$$

where:

- $S = S(0)$ is the stock price at time zero;
- C is the price of the call option;
- P is the price of the put option.

The Black-Scholes formula for the price of a European Put option, previously derived, is

$$P = Ke^{-rT}\Phi(y_0) - S\Phi(y_1)\,.$$

Substituting P into the put-call parity formula, we get

$$C = (Ke^{-rT}\Phi(y_0) - S\Phi(y_1)) + S - Ke^{-rT}\,.$$

This can be simplified further to

$$C = S(1 - \Phi(y_1)) - Ke^{-rT}(1 - \Phi(y_0))\,.$$

Using the symmetry property of the normal distribution, where $\Phi(-y) = 1 - \Phi(y)$, and setting

$$x_1 = -y_1 = \frac{\left(r + \frac{\sigma^2}{2}\right)T - \log\left(\frac{K}{S}\right)}{\sigma\sqrt{T}} \tag{2.6}$$

and

$$x_0 = -y_0 = x_1 - \sigma\sqrt{T}\,, \tag{2.7}$$

we can write

$$C = S\Phi(x_1) - Ke^{-rT}\Phi(x_0)\,. \tag{2.8}$$

This brings us to the Black-Scholes formula for a European Call option. The formula reflects the call price based on the risk-neutral principle, where the expected return rate of the underlying is replaced by the risk-free interest rate r. It is worth noting that this formula for C is also valid for American Call options because it is never optimal to exercise such options early when the underlying pays no dividends. However, for American Put options, the Black-Scholes formula for

P no longer holds as it might sometimes be optimal to exercise the American Put early. In summary, we have connected the concepts of risk neutrality and put-call parity to derive the Black-Scholes formula for both Put and Call options, highlighting the elegance and power of the Black-Scholes model in valuing European options.

2.3.3 Stochastic Ito Calculus

To get an idea of the fact that the equivalence between no-arbitrage and risk-neutral arbitrage pricing remains valid in continuous time, let us consider the Taylor Expansion for a twice continuously differentiable function $f(x)$. We have

$$f(x) = f(x_0) + f'(x_0) * (x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + o((x - x_0)^2)\,,$$

with

$$\lim_{x \to x_0} \frac{o((x - x_0))}{(x - x_0)^2}\,.$$

So, for any x the first-order Taylor expansion is

$$f(x + \delta x) = f(x) + \frac{df(x)}{dx}\delta x + O((\delta x)^2)\,.$$

Then if we want to have a more precise measurement we can use the second order Taylor expansion

$$\begin{aligned} f(x + \delta x) &= f(x) + \frac{df(x)}{dx}(x + \delta x - x) + \frac{d^2 f(x)}{2dx^2}(x + \delta x - x)^2 + o((x + \delta x - x)^2) \\ &= f(x) + \frac{df(x)}{dx}\delta x + \frac{1}{2}\frac{d^2 f(x)}{dx^2}(\delta x)^2 + o((\delta x)^2)\,. \end{aligned}$$

Let us define $f(t, s)$ as a function that is twice continuously differentiable in t and s. Here, t represents time and s represents stock price. This process helps in examining how changes in one variable affects the function while the other variable remains unchanged

$$\begin{aligned} f(t+\delta t, s+\delta s) =& f(t,s) + \frac{\partial f(t,s)}{\partial t}\delta t + \frac{\partial f(t,s)}{\partial s}\delta s \\ &+ \frac{1}{2}\frac{\partial^2 f(t,s)}{\partial t^2}(\delta t)^2 + \frac{1}{2}\frac{\partial^2 f(t,s)}{\partial s^2}(\delta s)^2 \\ &+ \frac{\partial^2 f(t,s)}{\partial t \partial s}\delta t \delta s + o((\delta t)^2) + o((\delta s)^2) = \\ &= f(t,s) + \frac{\partial f(t,s)}{\partial t}\delta t + \frac{\partial f(t,s)}{\partial s}\delta s \\ &+ \frac{1}{2}\frac{\partial^2 f(t,s)}{\partial s^2}(\delta s)^2 + o((\delta t)) + o((\delta s)^2) \end{aligned}$$

If δt and δs are sufficiently small, we can ignore terms of order higher than δt and $(\delta s)^2$, then

$$\begin{aligned} f(t+\delta t, s+\delta s) &\approx f(t,s) + \frac{\partial f(t,s)}{\partial t}\delta t + \frac{\partial f(t,s)}{\partial s}\delta s + \frac{1}{2}\frac{\partial^2 f(t,s)}{\partial s^2}(\delta s)^2. \\ f(t+\delta t, s+\delta s) - f(t,s) &\approx \frac{\partial f(t,s)}{\partial t}\delta t + \frac{\partial f(t,s)}{\partial s}\delta s + \frac{1}{2}\frac{\partial^2 f(t,s)}{\partial s^2}(\delta s)^2. \end{aligned}$$

Now let us consider a function $f(t, S_t)$ of two variables, where $S_t = S(t)$ is a geometric Brownian motion with expected rate return μ and volatility σ. We want to study how this function change over a short period of time interval, such as from t to $t+h$, where t is time. We will use the previous formula with $s = S_t$, writing h for δt and δS_t for $S_{t+h} - S_t$, so

$$f(t+h, S_{t+h}) - f(t, S_t) \approx \frac{\partial f(t,s)}{\partial t}h + \frac{\partial f(t,s)}{\partial s}\delta S_t + \frac{1}{2}\frac{\partial^2 f(t,s)}{\partial s^2}(\delta S_t)^2. \qquad (2.9)$$

All the partial derivatives are evaluated at $(t,s) = (t, S_t)$. Let us set $dt = h$ and $dS_t = S_{t+dt} - S_t$ and using (2.1) equation we have:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sigma dZ \\ \left(\frac{dS_t}{S_t}\right)^2 &= \mu^2 (dt)^2 + 2\sigma\mu dt dZ + \sigma^2 Z^2 dt = \\ &= \sigma^2 Z^2 dt\,, \end{aligned}$$

where $Z \sim \mathcal{N}(0,1)$ and we have ignored terms of order higher than dt. Since $\mathbb{E}[Z^2] = Var(Z) + \mathbb{E}[Z]^2 = 1 + 0 = 1$, we have

$$\mathbb{E}\left[\left(\frac{dS_t}{S_t}\right)^2\right] = \sigma^2 dt \mathbb{E}[Z^2] = \sigma^2 dt\,.$$

Recall that, since $Z \sim \mathcal{N}(0,1)$, then $Z^2 \sim \chi_1^2$ and hence $Var(Z^2) = 2$. So

$$\begin{aligned} Var\left(\left(\frac{dS_t}{S_t}\right)^2\right) &= (\sigma^2 dt)^2 Var(Z^2) \\ &= 2\sigma^4 (dt)^2\,. \end{aligned}$$

Since variance of $\left(\frac{dS_t}{S_t}\right)^2$ is of higher order than dt, for dt very small we can approximate such a variance to zero. Consequently, we can write approximate the random variable $\left(\frac{dS_t}{S_t}\right)^2$ with its expectation, that is

$$\begin{aligned} \left(\frac{dS_t}{S_t}\right)^2 &\approx \mathbb{E}\left[\left(\frac{dS_t}{S_t}\right)^2\right] = \sigma^2 dt \\ \left(\frac{dS_t}{S_t}\right)^2 &\approx \sigma^2 dt \\ (dS_t)^2 &\approx S_t^2 \sigma^2 dt\,. \end{aligned}$$

Putting this last equation with (2.9) and also substituting $\delta S_t = dS_t = S_{t+h} - S_t$ we have

$$f(t+h, S_{t+h}) - f(t, S_t) \approx \frac{\partial f(t,s)(S_{t+h} - S_t)}{\partial s} + \left(\frac{\partial f(t,s)}{\partial t} + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f(t,s)}{\partial s^2}\right) h\,,$$

with all the partial derivatives centered in $(t,s) = (t, S_t)$.

We can write the same differential equation in term of infinitesimal notation, substituting $df(t, S_t)$ for $f(t+h, S_{t+h}) - f(t, S_t)$

$$df(t, S_t) = \frac{\partial f(t, S_t)}{\partial s} dS_t + \left(\frac{\partial f(t, S_t)}{\partial t} + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f(t, S_t)}{\partial s^2}\right) dt\,. \tag{2.10}$$

Thus deriving **Ito's formula** when $\{S_t\}_{t\geq 0}$ follows a geometric Brownian motion.

2.3.4 The Black-Scholes Differential Equation

Theorem 2.3.1. *Consider a derivative with payoff $F(S_T)$ at time T, where the stock price $\{S_t\}_{t\geq 0}$ follows a geometric Brownian motion, with expected rate of return μ and volatility σ. Let us suppose the other possible investments, beside stocks, is a risk-free deposit account with interest rate r compounded continuously. For $s > 0$ and $0 \leq t \leq T$, define $f(t,s)$ the no arbitrage price of the derivative at time t if $s = S_t$. Then $f(t,s)$ is given by the unique solution in $[0,T] \times (0,+\infty)$ to the* ***Black-Scholes differential equation***

$$\frac{\partial f(t,s)}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f(t,s)}{\partial s^2} + rs\frac{\partial f(t,s)}{\partial s} = rf(t,s)\,, \tag{2.11}$$

with the boundary condition $f(T,s) = F(s)$ for all $s > 0$.

Proof. Assume that there is a unique solution f for this differential equation, for some (t,s), with $s = S_t$. Let us consider a portfolio at time t consisting of

- Δ units of the stock, where $\Delta = \frac{\partial f(t,s)}{\partial s}$;
- $f(t,s) - \Delta s$ is the amount of cash invested in a risk-free deposit.

If $\Delta < 0$ we are basically in a short position, if $f(t,s) - \Delta s < 0$ it can be interpreted in term of borrowing money.

Let W_t be the current value of the portfolio at time t, meaning that $W_t = f(t,s)$, we want to study what happen for an infinitesimal increment from time t to $t+h$.

It is easy to understand that the stock portion of the portfolio grows by a factor $\left(\frac{dS_t}{S_t}\right)$, setting $dS_t = S_{t+h} - S_t$, while the part invested in risk-free deposit grows by a factor rh.

Imposing that $dW_t = W_{t+h} - W_t$, we have

$$dW_t = \Delta(S_{t+h} - S_t) + r[f(t,s) - s\Delta]h\,. \tag{2.12}$$

Knowing that

$$\begin{aligned} dW_t &= W_{t+h} - W_t \\ &= f(t+h, S_{t+h}) \end{aligned}$$

Using Ito's differential equation (2.10), we have

$$f(t+h, S_{t+h}) \approx \frac{\partial f(t,S_t)}{\partial s}(S_{t+h} - S_t) + \left(\frac{\partial f(t,S_t)}{\partial t} + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f(t,S_t)}{\partial s^2} \right) h$$

that implies

$$\frac{f(t+h, S_{t+h})}{h} \approx \frac{\partial f(t,S_t)}{\partial s} \frac{S_{t+h} - S_t}{h} + \frac{\partial f(t,S_t)}{\partial t} + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f(t,S_t)}{\partial s^2}\,.$$

From this differential equation it is easily noticeable that the first member $\frac{f(t+h,S_{t+h})}{h}$ is $\frac{df(t,S_t)}{dt}$ and $(S_{t+h} - S_t)$ is dS_t. Setting again $dt = h$ we have

$$\frac{df(t,S_t)}{dt} = \frac{\partial f(t,S_t)}{\partial s} \frac{dS_t}{dt} + \frac{\partial f(t,S_t)}{\partial t} + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f(t,S_t)}{\partial s^2}$$

that implies

$$df(t,S_t) = \frac{\partial f(t,S_t)}{\partial s} dS_t + \left(\frac{\partial f(t,S_t)}{\partial t} + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f(t,S_t)}{\partial s^2} \right) dt\,.$$

Remembering that $W_t = f(t,s)$ and $S_t = s$, we can affirm $dW_t = df(t,S_t)$. Substituting this last differential equation in (2.12) we get

$$\begin{aligned}
&\frac{\partial f(t,S_t)}{\partial s} dS_t + \left(\frac{\partial f(t,S_t)}{\partial t} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 f(t,S_t)}{\partial s^2} \right) dt = \Delta(S_{t+h} - S_t) + r[f(t,S_t) - S_t \Delta]h \\
&\frac{\partial f(t,S_t)}{\partial s} dS_t + \left(\frac{\partial f(t,S_t)}{\partial t} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 f(t,S_t)}{\partial s^2} \right) dt = \frac{\partial f(t,S_t)}{\partial s} dS_t + r[f(t,S_t) - S_t \Delta]h \\
&\frac{\partial f(t,S_t)}{\partial t} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 f(t,S_t)}{\partial s^2} = r f(t,S_t) - r S_t \frac{\partial f(t,S_t)}{\partial s} \\
&\frac{\partial f(t,S_t)}{\partial t} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 f(t,S_t)}{\partial s^2} + r S_t \frac{\partial f(t,S_t)}{\partial s} = r f(t,S_t)
\end{aligned}$$

□

2.3.5 Justification of Risk-Neutrality

Let us deduce that the value of the derivative at time 0, $f(0, S_0)$ is given by the expected discounted payoff under the risk-neutral model.
We consider also a risk-neutral geometric Brownian motion $\widetilde{S}_t = \widetilde{S}(t)$ with expected rate return the risk-free r and volatility σ. Suppose we know $\widetilde{S}_t = s$.
Imposing that the payoff is $F(S_T)$ and $\widetilde{S}_t$ is a log normal process with infinitesimal drift r and volatility σ. For $t \leq T$ and $s > 0$, let $f(t, s)$ be a solution to the Black-Scholes differential equation with $f(T, s) = F(s)$. By the Ito formula (2.10), we have

$$df(t, \widetilde{S}_t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 \widetilde{S}_t^2 \frac{\partial^2 f}{\partial s^2} \right) dt + \frac{\partial f}{\partial s} d\widetilde{S}_t \, .$$

with all partial derivatives evaluated at $(t, s) = (t, \widetilde{S}_t)$. Note that

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sigma dZ \\ \frac{d\widetilde{S}_t}{\widetilde{S}_t} &= r dt + \sigma dZ \\ d\widetilde{S}_t &= \widetilde{S}_t r dt + \sigma dZ \widetilde{S}_t \, . \end{aligned}$$

Using the Ito's formula we get

$$\begin{aligned} df(t, \widetilde{S}_t) &= \frac{\partial f}{\partial t} dt + \frac{1}{2}\sigma^2 \widetilde{S}_t^2 \frac{\partial^2 f}{\partial s^2} dt + \frac{\partial f}{\partial s} d\widetilde{S}_t \\ df(t, \widetilde{S}_t) &= \frac{\partial f}{\partial t} dt + \frac{1}{2}\sigma^2 \widetilde{S}_t^2 \frac{\partial^2 f}{\partial s^2} dt + \frac{\partial f}{\partial s} \left(\widetilde{S}_t r dt + \widetilde{S}_t \sigma dZ \right) \\ df(t, \widetilde{S}_t) &= \frac{\partial f}{\partial t} dt + \frac{1}{2}\sigma^2 \widetilde{S}_t^2 \frac{\partial^2 f}{\partial s^2} dt + \frac{\partial f}{\partial s} \widetilde{S}_t r dt + \frac{\partial f}{\partial s} \sigma \widetilde{S}_t dZ \\ df(t, \widetilde{S}_t) &= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 \widetilde{S}_t^2 \frac{\partial^2 f}{\partial s^2} + \frac{\partial f}{\partial s} \widetilde{S}_t r \right) dt + \frac{\partial f}{\partial s} \widetilde{S}_t \sigma dZ \, , \end{aligned}$$

with all the partial derivatives evaluated at $(t, s) = (t, \widetilde{S}_t)$. Let us notice that using the Black-Scholes differential equation (2.11),

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 \widetilde{S}_t^2 \frac{\partial^2 f}{\partial s^2} + \frac{\partial f}{\partial s} \widetilde{S}_t r \right) = r f(t, \widetilde{S}_t) \, .$$

Substituting it in the previous differential equation we have

$$df(t, \widetilde{S}_t) = rf(t, \widetilde{S}_t)dt + \frac{\partial f}{\partial s}\widetilde{S}_t \sigma dZ\,.$$

Let us apply the expected value to both the members of this equation knowing that $\mathbb{E}[dZ] = 0$. Denoting by $g(t) = \mathbb{E}[f(t, \widetilde{S}_t)]$ we get

$$dg(t) = rg(t)dt\,.$$

The function $g(t) = g(0)e^{rt}$ is the solution to the ordinary differential equation $\frac{dg(t)}{dt} = rg(t)$. To verify that this function is indeed the solution, we differentiate $g(t)$ with respect to time t

$$\frac{d}{dt}g(t) = \frac{d}{dt}\left(g(0)e^{rt}\right)$$

Applying the rule for differentiating exponential functions, we obtain

$$\frac{d}{dt}g(t) = g(0)re^{rt}$$

Since $g(t) = g(0)e^{rt}$, we can rewrite the above expression as

$$\frac{d}{dt}g(t) = rg(t)\,.$$

This matches exactly the right side of the original differential equation, confirming that $g(t) = g(0)e^{rt}$ is indeed the solution to the differential equation $\frac{dg(t)}{dt} = rg(t)$. In mathematical terms, the equation $\frac{dg(t)}{dt} = rg(t)$ describes a process with an exponential growth, being the rate r positive.

Now assume that the stock price is $\widetilde{S}_0$ and is a known non-random value. Then

$$g(0) = \mathbb{E}[f(0, \widetilde{S}_0)] = f(0, \widetilde{S}_0)\,.$$

So

$$f(0, \widetilde{S}_0) = g(0) = e^{-rT}g(T) = e^{-rT}\mathbb{E}[f(0, \widetilde{S}_T)] = e^{-rT}\mathbb{E}[F(\widetilde{S}_T)]\,,$$

deriving the expected discounted payout under risk-neutrality.

Chapter 3

Greeks

Understanding the Greeks is indispensable for effective risk management and strategic trading. The Greeks Delta, Gamma, Vega, along with Theta and Rho, provide crucial insights into the sensitivity of options prices to various market factors. This chapter delves deeply into Delta, Gamma, and Vega, with a detailed explanation of each metric, strategies for their practical application, and their interrelationships.

Let us consider a portfolio of financial derivatives that depends just on a single underlying asset that follows a geometric Brownian motion. Let us also assume that the returns r and the volatility σ are constant, the portfolio value is

$$f = f(t, S),$$

as a function depending on time and stock price $S = S(t)$, we can define the following key sensitivities, often called the Greeks:

- **Delta** Δ: Represents the rate of change of the portfolio's value with respect to changes in the underlying asset price.

$$\Delta = \frac{\partial f}{\partial S}$$

- **Gamma** Γ: Measures the rate of change in Delta with respect to changes in the underlying asset price, offering insight into the curvature of the portfolio's value relative to the stock price

$$\Gamma = \frac{\partial^2 f}{\partial S^2}$$

- **Vega** V: Shows the sensitivity of the portfolio's value to changes in volatility of the underlying asset

$$V = \frac{\partial f}{\partial \sigma}$$

- **Theta** Θ: Indicates the rate of change of the portfolio's value with respect to the passage of time

$$\Theta = \frac{\partial f}{\partial T}$$

- **Rho** ρ: Indicates the rate of change of the portfolio's value with respect to changes in the risk-free interest rate

$$\rho = \frac{\partial f}{\partial r}$$

These derivatives provide the foundational understanding of how small changes in each variable affect the portfolio's value, assuming all other variables are held constant. Using a Taylor expansion, the change in the portfolio's value for small movements in time, stock price can be approximated as:

$$\begin{aligned} \delta f &\approx f(t + \delta t, S + \delta S) - f(t, S) = \\ &= \Theta \delta t + \Delta \delta S + \frac{1}{2} \Gamma (\delta S)^2 . \end{aligned}$$

Furthermore, we can not assume that in a real world scenario volatility σ is constant, in this case, the value of our portfolio is

$$f = f(t, S, \sigma)$$

a function depending also on the stock price volatility σ. In this case, we have a more general Taylor expansion for the value of our portfolio such as

$$\begin{aligned}\delta f &\approx f(t+\delta t, S+\delta S, \sigma+\delta\sigma) - f(t, S, \sigma) = \\ &= \Theta\delta t + \Delta\delta S + V\delta\sigma + \frac{1}{2}\Gamma(\delta S)^2\,.\end{aligned}$$

This expansion provides a versatile framework for anticipating changes in the portfolio under different market conditions, especially when the volatility σ is not constant, emphasizing the dynamic nature of financial markets.

3.1 Delta Δ

Delta (Δ) measures the rate of change in the price of an option relative to a one-unit change in the price of the underlying asset. It is mathematically defined as or

$$\Delta = \frac{\partial f(t, S, \sigma)}{\partial S}$$

where $f(t, S, \sigma)$ represents the option price as a function of time t, stock price S, and volatility σ. Specifically:

- Δ represents a measure of risk associated with changes in the price of the underlying asset.
- if $\Delta > 0$, an increase in the price of the underlying asset will typically lead to a corresponding increase in the price of the option.
- if $\Delta < 0$, an increase in the price of the underlying asset usually results in a decrease in the price of the option;
- Δ of a European call option ranges from 0 to +1. This means the option's price moves in the same direction as the price of the underlying asset;

- Δ of a European put option ranges from 0 to -1, indicating that the option's price moves inversely relative to the price of the underlying asset;
- Δ of the underlying asset itself is always 1, as any change in the asset's price directly translates into an equivalent change in itself [1].

3.1.1 The Delta Δ of an European Call Option

Theorem 3.1.1. *The delta of a European call option is given by*

$$\Delta = \Phi(x_1)\,,$$

where Φ is the standard normal distribution function and x_1 is the value obtained in (2.6)*.*

Proof. To demonstrate this theorem let us start from the Black-Scholes formula (2.8) for a call.

Using the Black-Scholes formula, we can calculate the Delta (Δ) of a European call option, which measures how the price of the call option changes with respect to changes in the underlying stock price.

The price of a European call option can be represented by the Black-Scholes formula as

$$C = S\Phi(x_1) - Ke^{-rT}\Phi(x_0)$$

remembering that

- C is the price of the call option,
- S is the current stock price,
- Φ is the cumulative distribution function of the standard normal distribution,

[1] since $\frac{\partial S}{\partial S} = 1$

- x_1 and x_0 are given by:

$$x_0 = \frac{\log(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}, \quad x_1 = x_0 + \sigma\sqrt{T}$$

- K is the strike price,

- r is the risk-free interest rate,

- T is the time to expiration,

- σ is the volatility of the stock.

To find Δ_{call}, the change in the option price with respect to the stock price, differentiate the Black-Scholes formula with respect to S, so

$$\Delta_{call} = \frac{\partial C}{\partial S} \tag{3.1}$$

$$= \frac{\partial S\Phi(x_1)}{\partial S} - \frac{\partial K e^{-rT}\Phi(x_o)}{\partial S} \tag{3.2}$$

Let us start deriving the first member of the equation (3.2) using the chain rule [2]

$$\begin{aligned}\frac{\partial S\Phi(x_1)}{\partial S} &= \frac{\partial S}{\partial S}\Phi(x_1) + S\frac{\partial\Phi(x_1)}{\partial S}\\ &= \Phi(x_1) + S\frac{\partial\Phi(x_1)}{\partial S}\\ &= \Phi(x_1) + S\frac{\partial\Phi(x_1)}{\partial x_1}\frac{\partial x_1}{\partial S},\end{aligned}$$

where $\frac{\partial\Phi}{\partial x_1} = \phi(x_1)$ and

$$\frac{\partial x_1}{\partial S} = \frac{\partial}{\partial S}\left(\frac{\log(S/K)}{\sigma\sqrt{T}}\right) = \frac{1}{\sigma\sqrt{T}S},$$

[2] $f(x) = g(h(x))$ where g and h are differentiable functions, for $\frac{df}{dx}$ we have

$$\frac{df}{dx} = \frac{dg(h(x))}{dh(x)}\frac{dh(x)}{dx}$$

hence

$$S\frac{\partial\Phi(x_1)}{\partial S} = \phi(x_1)\frac{1}{\sigma\sqrt{T}}\,.$$

So the first member of the differential equation equals to

$$\frac{\partial S\Phi(x_1)}{\partial S} = \Phi(x_1) + \frac{\phi(x_1)}{\sigma\sqrt{T}}$$

Let us continue with deriving the second member of the equation (3.2)

$$\begin{aligned}\frac{\partial(-Ke^{-rT}\Phi(x_0))}{\partial S} &= -Ke^{-rT}\frac{\partial\Phi(x_0)}{\partial S}\\ &= -Ke^{-rT}\frac{\partial\Phi(x_0)}{\partial x_0}\frac{\partial x_0}{\partial S}\\ &= -Ke^{-rT}\frac{\phi(x_0)}{S\sigma\sqrt{T}}\end{aligned}$$

so

$$\begin{aligned}\Delta_{call} &= \Phi(x_1) + \frac{\phi(x_1)}{\sigma\sqrt{T}} - \frac{-Ke^{-rT}\phi(x_0)}{S\sigma\sqrt{T}}\\ &= \Phi(x_1) + \frac{S\phi(x_1) - Ke^{-rt}\phi(x_0)}{S\sigma\sqrt{T}}.\end{aligned}$$

Let us study $S\phi(x_1) - Ke^{-rt}\phi(x_0)$, knowing that

$$\begin{aligned}\frac{\phi(x_0)}{\phi(x_1)} = \frac{(2\pi)^{-\frac{1}{2}}\exp(-x_0^2/2)}{(2\pi)^{-\frac{1}{2}}exp(-x_1^2/2)} = \exp\left(\frac{-x_0^2}{2} + \frac{x_1^2}{2}\right) =\\ = \exp((x_1^2 - x_0^2)/2) = \exp((x_1 + x_0)(x_1 - x_0)/2)\end{aligned}$$

furthermore

$$\begin{aligned}(x_1 + x_0) &= \frac{\log(S/K)}{\sigma\sqrt{T}} + \frac{(r+\sigma^2/2)T}{\sigma\sqrt{T}} + \frac{\log(S/K)}{\sigma\sqrt{T}} + \frac{(r-\sigma^2/2)T}{\sigma\sqrt{T}}\\ &= 2\left(\frac{\log(S/K)}{\sigma\sqrt{T}}\right) + \left(\frac{2rT}{\sigma\sqrt{T}}\right)\\ &= 2\left(\frac{\log(S/K) + rT}{\sigma\sqrt{T}}\right)\end{aligned}$$

and

$$(x_1 - x_0) = \sigma\sqrt{T}$$

so

$$(x_1 + x_0)(x_1 - x_0) = 2\left(\frac{\log(S/K) + rT}{\sigma\sqrt{T}}\right)\sigma\sqrt{T} = 2(\log(S/K) + rT)$$

and

$$\begin{aligned}
\frac{\phi(x_0)}{\phi(x_1)} &= \exp(\log(S/K) + rT) \\
\frac{\phi(x_0)}{\phi(x_1)} &= \exp\left(\log\left(\frac{S}{K}\right)\right)\exp(rT) \\
\frac{\phi(x_0)}{\phi(x_1)} &= \frac{S}{K}e^{rT} \\
\phi(x_0) &= S\phi(x_1)\frac{1}{K}e^{rT} \\
Ke^{-rT}\phi(x_0) &= S\phi(x_1).
\end{aligned} \tag{3.3}$$

Remembering that

$$\Delta_{call} = \Phi(x_1) + \frac{S\phi(x_1) - Ke^{-rt}\phi(x_0)}{S\sigma\sqrt{T}}$$

it is easy to observe that the second member of this equation is equal to 0, so we have proven that

$$\Delta_{call} = \Phi(x_1)$$

The Delta for a European call option is primarily determined by $\Phi(x_1)$, the probability that the option will finish in-the-money under the log-normal distribution assumption for stock price returns. This captures the sensitivity of the option's price to movements in the underlying asset price, incorporating in the volatility and time to maturity.

□

3.1.2 The Delta Δ of an European Put Option

Deriving the delta Δ of an European Put Option is a much easier job, let us start from the put-call parity

$$P = C + Ke^{-rT} - S$$

and deriving respectively to S, ∂S we have

$$\frac{\partial P}{\partial S} = \frac{\partial C}{\partial S} + 0 - 1 = \Delta_{call} - 1$$

so

$$\Delta_{put} = \Phi(x_1) - 1\,.$$

3.1.3 Exploring Delta in Depth

For instance, a Delta of 0.5 suggests that the option's price will move approximately \$0.50 for every \$1 increase in the underlying asset. If an option is "at-the-money," its Delta will be approximately 0.5, reflecting a 50% chance it will end up in the money. As the option moves deeper into the money, the Delta approaches 1, indicating a higher likelihood of price movement in conjunction with the underlying asset.

3.1.4 Delta-Hedging

Given a derivative with a payoff at maturity $\phi(S_T)$, whose value over time is given by the function $F(t, S_t)$, the hedging strategy involves constructing a portfolio consisting of the underlying asset and a risk-free asset that replicates the derivative's value at all times. This portfolio eliminates the risk by accurately replicating the derivative.

Let us construct the replicating portfolio for the derivative, denoted by $\{F'_t\}_{t\geq 0}$, composed of u_t^S units of the underlying asset and u_t^B units of the risk-free asset. The balance constraint in this case is given by

$$F'_t = u_t^S S_t + u_t^B e^{rt},$$

while the dynamics of the replicating portfolio are

$$dF'_t = u_t^S dS_t + r u_t^B e^{rt} dt.$$

By setting $dF'_t = dF_t$, where equating the terms in dW_t, we obtain

$$u_t^S = \frac{\partial F}{\partial S}.$$

The meaning of equation is clear: to hedge the risk of a derivative whose value at time t is given by $F(t, S_t)$, one needs to purchase $\frac{\partial F}{\partial S}$ units of the underlying asset. The quantity $\frac{\partial F}{\partial S}$ is known as the delta, and the described hedging procedure is referred to as delta hedging.

3.2 Gamma Γ

Gamma (Γ) describes the rate of change of Delta and is a measure of curvature. Mathematically, it is expressed as

$$\Gamma = \frac{\partial^2 f}{\partial S^2}$$

3.2.1 The Gamma Γ of an European Call Option

Theorem 3.2.1. *The Gamma of a European call option is given by*

$$\Gamma = \frac{e^{-x_1^2/2}}{\sqrt{2\pi}} \frac{1}{S\sigma\sqrt{T}}$$

Proof. Let us start with the proof, remembering that

$$\Gamma = \frac{\partial^2 f}{\partial S^2} = \frac{\partial \Delta}{\partial S} = \frac{\partial \Phi(x_1)}{\partial S}$$

Using the fundamental theorem of calculus and the chain rule we have[3]

$$\frac{\partial^2 f}{\partial S^2} = \frac{\partial \Phi(x_1)}{\partial x_1}\frac{\partial x_1}{\partial S} = \phi(x_1)\frac{\partial x_1}{\partial S}\,.$$

Tackling the second term of the equation we have

$$\frac{\partial x_1}{\partial S} = \frac{\partial}{\partial S}\frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{1}{S\sigma\sqrt{T}}$$

and so

$$\Gamma_{call} = \phi(x_1)\frac{1}{S\sigma\sqrt{T}} = \frac{e^{-x_1^2/2}}{\sqrt{2\pi}}\frac{1}{S\sigma\sqrt{T}}\,.$$

□

3.2.2 The Gamma Γ of an European Put Option

Similarly to the Delta, the Gamma of a put option is

$$\Gamma_{put} = \frac{\partial}{\partial S}\Delta_{put} = \frac{\partial}{\partial S}(\Delta_{call} - 1) = \Gamma_{call}.$$

This measure is particularly important for assessing the stability of Delta. A high Gamma indicates that Delta is very sensitive to changes in the underlying price, which can lead to larger-than-expected changes in the option's price, necessitating more frequent rebalancing of a Delta-hedged portfolio.

3.2.3 The Implications of Gamma in Trading

High Gamma can be advantageous when expecting significant price movements as it allows for quicker adjustments to the hedge. However, it also implies higher

[3]Notice that $\frac{\partial \Phi(x_1)}{\partial x_1} = \phi(x_1) = \frac{1}{\sqrt{2\pi}}e^{-x_1^2/2}$.

risk as the price of the option is more sensitive to changes in the underlying asset. For traders, managing Gamma effectively is crucial, especially near the expiry of options where Gamma tends to increase significantly.

3.3 Vega $\mathcal{V}$

Vega ($\mathcal{V}$) quantifies an option's price sensitivity to changes in the volatility of the underlying asset. It is represented as:

$$\mathcal{V} = \frac{\partial f}{\partial \sigma}$$

Vega indicates the dollar change in an option's price for a 1% change in implied volatility. It is the highest for at-the-money options and decreases as the option becomes either deeply in-the-money or out-of-the-money.

3.3.1 The Vega $\mathcal{V}$ of an European Call Option

Theorem 3.3.1. *The Vega of a European call option is given by*

$$\mathcal{V} = S\phi(x_1)\sqrt{T}$$

Proof. Let us start again from the Black-Scholes formula for a call

$$C = S\Phi(x_1) - Ke^{-rT}\Phi(x_0)$$

as we said, Vega is the partial derivative of the function f, in this case C with respect to σ, so

$$\begin{aligned}
\mathcal{V} &= \frac{\partial C}{\partial \sigma} \\
&= S\frac{\partial \Phi(x_1)}{\partial (x_1)}\frac{\partial (x_1)}{\partial \sigma} - Ke^{-rT}\frac{\partial \Phi(x_0)}{\partial (x_0)}\frac{\partial (x_0)}{\partial \sigma} \\
&= S\phi(x_1)\frac{\partial (x_1)}{\partial \sigma} - Ke^{-rT}\phi(x_0)\frac{\partial x_0}{\partial \sigma}
\end{aligned}$$

Recall (3.3), that is

$$Ke^{-rT}\phi(x_0) = S\phi(x_1)\,.$$

We have

$$\mathcal{V} = S\phi(x_1)\left(\frac{\partial x_1}{\partial \sigma} - \frac{\partial x_0}{\partial \sigma}\right)$$

Let us study the two derivative

$$\frac{\partial x_1}{\partial \sigma} = -\frac{\log(S/K) + rT}{\sigma^2\sqrt{T}} + \frac{\sqrt{T}}{2}$$

and

$$\frac{\partial x_0}{\partial \sigma} = -\frac{\log(S/K) + rT}{\sigma^2\sqrt{T}} - \frac{\sqrt{T}}{2}$$

hence

$$\left(\frac{\partial x_1}{\partial \sigma} - \frac{\partial x_0}{\partial \sigma}\right) = -\frac{\log(S/K) + rT}{\sigma^2\sqrt{T}} + \frac{\sqrt{T}}{2} + \frac{\log(S/K) + rT}{\sigma^2\sqrt{T}} + \frac{\sqrt{T}}{2} = \sqrt{T}$$

so that

$$\mathcal{V}_{call} = S\phi(x_1)\sqrt{T}.$$

□

3.3.2 The Vega $\mathcal{V}$ of an European Call Option

Theorem 3.3.2. *The Vega of a European put option is*

$$\mathcal{V}_{put} = \mathcal{V}_{call}$$

Proof. To demonstrate this theorem, we use again the Put-Call Parity $P = C + Ke^{-rT} - S$, so

$$\frac{\partial P}{\partial \sigma} = \frac{\partial C}{\partial \sigma} + \frac{Ke^{-rT}}{\partial \sigma} - \frac{\partial S}{\partial \sigma}$$

since $Ke^{-rT} - S$ does not depend on σ, their partial derivative with respect to σ is zero.

So

$$\mathcal{V}_{put} = \mathcal{V}_{call}$$

□

3.3.3 Vega and Volatility Trading

Vega is crucial for traders who speculate on future volatility. For example, if a trader expects an increase in volatility, they might purchase options with high Vega to profit from the expected increase in option premiums. Conversely, selling options with high Vega can be beneficial if the volatility is expected to decrease.

3.4 Theta Θ

Theta (Θ) measures the rate of change of the option price with respect to time. It is a measure of the time decay of the option's value. In symbols, it is expressed as

$$\Theta = \frac{\partial f}{\partial T}.$$

3.4.1 The Theta Θ of a European Call Option

Theorem 3.4.1. *The Theta of an European call option is given by*

$$\Theta_{call} = S\phi(x_1)\sigma\frac{1}{2\sqrt{T}} + Kre^{-rT}\Phi(x_0)\,.$$

Proof. Let us start from the Black-Scholes formula for a call option

$$C = S\Phi(x_1) - Ke^{-rT}\Phi(x_0).$$

As we said, Theta is the partial derivative of the function C with respect to T, so

$$\begin{aligned}\Theta_{call} &= \frac{\partial C}{\partial T} \\ &= \frac{\partial}{\partial T}\Big(S\Phi(x_1) - Ke^{-rT}\Phi(x_0)\Big)\,.\end{aligned}$$

We differentiate each term separately. Starting with the first term

$$\begin{aligned}\frac{\partial}{\partial T}\left(S\Phi(x_1)\right) &= S\frac{\partial\Phi(x_1)}{\partial x_1}\frac{\partial x_1}{\partial T}\\ &= S\phi(x_1)\frac{\partial}{\partial T}\left(\frac{\log(S/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)\\ &= S\phi(x_1)\left(\frac{T(2r+\sigma^2/2)-2\log(S/K)}{2\sigma T^{3/2}}\right).\end{aligned}$$

For the second term

$$\begin{aligned}\frac{\partial}{\partial T}\left(Ke^{-rT}\Phi(x_0)\right) &= -Kre^{-rT}\Phi(x_0)+Ke^{-rT}\frac{\partial\Phi(x_0)}{\partial x_0}\frac{\partial x_0}{\partial T} =\\ &= -Kre^{-rT}\Phi(x_0)+Ke^{-rT}\phi(x_0)\frac{\partial x_0}{\partial T} =\\ &= -Kre^{-rT}\Phi(x_0)+Ke^{-rT}\phi(x_0)\left(\frac{T(2r-\sigma^2/2)-2\log(S/K)}{2\sigma T^{3/2}}\right)\end{aligned}$$

Combining these results, we get

$$\begin{aligned}\Theta_{call} = S\phi(x_1)&\left(\frac{T(2r+\sigma^2/2)-2\log(S/K)}{2\sigma T^{3/2}}\right)+Kre^{-rT}\Phi(x_0)+\\ &-Ke^{-rT}\phi(x_0)\left(\frac{T(2r-\sigma^2/2)-2\log(S/K)}{2\sigma T^{3/2}}\right)\end{aligned}$$

Using (3.3) we get

$$\begin{aligned}\Theta_{call} &= S\phi(x_1)\left(\frac{T(2r+\sigma^2/2)-2\log(S/K)}{2\sigma T^{3/2}}\right)+Kre^{-rT}\Phi(x_0)+\\ &- S\phi(x_1)\left(\frac{T(2r-\sigma^2/2)-2\log(S/K)}{2\sigma T^{3/2}}\right) = S\phi(x_1)\sigma\frac{1}{2\sqrt{T}}+Kre^{-rT}\Phi(x_0).\end{aligned}$$

Hence the final expression for the Theta of a European call option

$$\Theta_{call} = S\phi(x_1)\sigma\frac{1}{2\sqrt{T}}+Kre^{-rT}\Phi(x_0).$$

□

3.4.2 The Theta Θ of an European Put Option

Theorem 3.4.2. *The Theta of an European put option is given by*

$$\Theta_{put} = S\phi(x_1)\sigma\frac{1}{2\sqrt{T}}+Kre^{-rT}(\Phi(x_0)-1).$$

Proof. Let us start from the Put-Call parity

$$P = C + Ke^{-rT} - S$$

As we said, Theta is the partial derivative of the function P with respect to T, so

$$\begin{aligned}\Theta_{put} &= \frac{\partial P}{\partial T} \\ &= \frac{\partial C}{\partial T} + \frac{\partial Ke^{-rT}}{\partial T} + 0 \\ &= S\phi(x_1)\sigma\frac{1}{2\sqrt{T}} + Kre^{-rT}\Phi(x_0) - Kre^{-rT}.\end{aligned}$$

□

3.5 Rho ρ

Rho (ρ) measures the rate of change of the option price with respect to changes in the risk-free interest rate. It reflects the sensitivity of the option's value to the interest rate. In symbols, it is expressed as

$$\rho = \frac{\partial f}{\partial r}$$

3.5.1 The Rho ρ of an European Call Option

Theorem 3.5.1. *The Rho of an European call option is given by*

$$\rho_{call} = KTe^{-rT}\Phi(x_0)$$

Proof. Let us start from the Black-Scholes formula for a call option

$$C = S\Phi(x_1) - Ke^{-rT}\Phi(x_0)$$

As we said, Rho is the partial derivative of the function C with respect to r, so

$$\begin{aligned}\rho_{call} &= \frac{\partial C}{\partial r} \\ &= \frac{\partial}{\partial r}\left(S\Phi(x_1) - Ke^{-rT}\Phi(x_0)\right)\end{aligned}$$

We differentiate each term separately. Starting with the first term

$$\frac{\partial}{\partial r}(S\Phi(x_1)) = S\frac{\partial\Phi(x_1)}{\partial x_1}\frac{\partial x_1}{\partial r}$$

Knowing that

$$\frac{\partial x_1}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right) = \frac{T}{\sigma\sqrt{T}} = \frac{\sqrt{T}}{\sigma}$$

we have

$$\frac{\partial}{\partial r}(S\Phi(x_1)) = S\phi(x_1)\frac{\sqrt{T}}{\sigma}$$

For the second term

$$\begin{aligned}\frac{\partial}{\partial r}(Ke^{-rT}\Phi(x_0)) &= -TKe^{-rT}\Phi(x_0) + Ke^{-rt}\frac{\partial\Phi(x_0)}{\partial x_0}\frac{\partial x_0}{\partial r} \\ &= -KTe^{-rT}\Phi(x_0) + Ke^{-rT}\phi(x_0)\frac{\partial x_0}{\partial r} \\ &= Ke^{-rT}\left(-T\Phi(x_0) + \phi(x_0)\frac{\partial x_0}{\partial r}\right).\end{aligned}$$

Knowing that

$$\frac{\partial x_0}{\partial r} = \frac{\sqrt{T}}{\sigma},$$

we have

$$\frac{\partial}{\partial r}(Ke^{-rT}\Phi(x_0)) = Ke^{-rT}\left(-T\Phi(x_0) + \phi(x_0)\frac{\sqrt{T}}{\sigma}\right).$$

Combining these results, we get

$$\rho_{call} = KTe^{-rT}\Phi(x_0)\,.$$

□

3.5.2 The Rho ρ of an European Put Option

Theorem 3.5.2. *The Rho of an European put option is given by*

$$\rho_{put} = Ke^{-rT}T(\Phi(x_0) - 1)$$

Proof. Let us start from the Put-Call Parity formula

$$P = C + Ke^{-rT} - S.$$

As we said, Rho is the partial derivative of the function P with respect to r, so

$$\begin{aligned}\rho_{put} = \frac{\partial P}{\partial r} &= \frac{\partial C}{\partial r} + \frac{\partial}{\partial r}(Ke^{-rT}) - 0 \\ &= Ke^{-rT}T\Phi(x_0) - Ke^{-rT}T \\ &= Ke^{-rT}T(\Phi(x_0) - 1).\end{aligned}$$

□

3.6 Use of Delta, Gamma and Vega

Understanding the interplay between Delta, Gamma, and Vega is essential for sophisticated options strategies. These Greeks are not static and change as market conditions fluctuate, requiring continuous adjustments and monitoring by traders. The management of these risks shapes the strategies employed in options trading, affecting decisions on when to enter or exit positions based on predictions of directional moves, volatility shifts, and time decay.

3.6.1 Practical Considerations and Challenges

In real-world applications, the theoretical constructs of the Greeks must be adjusted for market realities, including transaction costs, bid-ask spreads, and discrete rebalancing intervals. These factors can affect the efficacy of hedging strategies and must be carefully considered in the construction and maintenance of portfolios.

In conclusion, the Greeks serve as the backbone of risk management in the options market. They provide a robust framework for understanding how various factors impact the pricing of options and how these effects can be mitigated through strategic trading and hedging practices.

3.6.2 Strategic Considerations for Managing Greeks

Effective management of the Greeks requires not only a deep understanding of each Greek's implications but also a valuation for the market context in which they are applied. The following strategies can enhance the application of Greeks in managing options portfolios:

- **Dynamic rebalancing**: Given that the values of Delta, Gamma, and Vega change with market conditions, dynamic rebalancing is necessary. This involves continuously monitoring the Greeks and adjusting the portfolio to maintain the desired levels of risk exposure.

- **Advanced analytics**: Utilizing software tools that can calculate and forecast changes in the Greeks can provide traders with advanced insights into potential price movements, helping them make more informed decisions.

- **Diversification across Greeks**: Just as diversification is essential in traditional portfolios, diversifying the risk exposure among different Greeks can help in managing the overall risk. Combining positions with varying Deltas, Gammas, and Vegas can balance the portfolio's sensitivity to market movements.

- **Scenario analysis**: Conducting stress tests and scenario analyses to understand how extreme market movements could impact the portfolio based on its Greek exposures can prepare traders for unexpected market conditions.

Chapter 4

Pathwise Estimators and Analytical Methods for Greeks Calculation

4.1 Pathwise Method

Let $\alpha(\theta) := \mathbb{E}[Y(\theta)]$ represent the price of a specific derivative security when the parameter value is θ. The derivative of $\alpha(\theta)$, denoted as $\alpha'(\theta)$, indicates the sensitivity of the derivative price to changes in θ. For instance, if Y is the discounted payoff of a standard European call option within the Black-Scholes model and $\theta = S_0$, the initial price of the underlying security, then $\alpha'(\theta)$ corresponds to the delta of the option. However, in many cases, an explicit expression for $\alpha'(\theta)$ is not available, necessitating the use of Monte Carlo methods to estimate it using the pathwise estimator.

The pathwise estimator is obtained by interchanging the order of differentiation and integration, leading to

$$\alpha'(\theta) = \frac{\partial}{\partial\theta}\mathbb{E}[Y(\theta)] = \mathbb{E}\left[\frac{\partial Y(\theta)}{\partial\theta}\right]$$

providing an unbiased estimator of $\alpha'(\theta)$. To use this approach, we first need to clearly define the relationship between Y and θ. This is achieved by assuming a collection of random variables $\{Y(\theta) : \theta \in \Theta\}$ defined on a single probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a fixed $\omega \in \Omega$, $Y'(\theta) = \frac{\partial Y(\theta)}{\partial\theta} = Y'(\theta, \omega)$ represents the derivative of this random function with respect to θ, keeping ω constant. This is known as the pathwise derivative of Y at θ, assuming that the pathwise derivative exists with probability 1, the expectation in the above expression is well-defined.

In the next sections, we will calculate the estimator for the Greeks using the pathwise method described above. This will involve deriving the sensitivities of derivative prices with respect to various parameters, including the initial underlying security price S_0, volatility σ, time to maturity T, and interest rate r. Each Greek will be derived and expressed in terms of the pathwise derivatives as outlined, keeping in mind that this method is valid for any model of security prices where $S_t = S_0 e^{X_t}$ for any risk-neutral stochastic process X_t that does not depend on S_0.

4.2 Delta Δ Estimator

4.2.1 Delta Δ Estimator for a Call

Let us start from an European call option with strike K and maturity T in the Black-Scholes framework

$$Y = e^{-rt}(S_T - K)^+ , \qquad S_T = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z} , \tag{4.1}$$

where $Z \sim N(0,1)$.

Knowing that $\Delta_{call} = \frac{\partial Y}{\partial S_0}$ we have

$$\Delta_{call} = \frac{\partial Y}{\partial S_0} = \frac{\partial Y}{\partial S_T}\frac{\partial S_T}{\partial S_0}\,.$$

Starting deriving from the first factor we get

$$\frac{\partial Y}{\partial S_T} = e^{-rt}\mathbb{1}_{\{S_T>K\}}$$

and

$$\begin{aligned}\frac{\partial S_T}{\partial S_0} &= e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z}\\ &= \frac{S_T}{S_0}\,.\end{aligned}$$

Hence, our estimator is

$$\Delta_{call} = e^{-rT}\mathbb{1}_{\{S_T>K\}}\frac{S_T}{S_0}\,.$$

4.2.2 Delta Δ Estimator for a Put

Let us start from an European put option with strike K and maturity T in the Black-Scholes framework

$$Y = e^{-rt}(K-S_T)^+\,, \qquad S_T = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z}\,. \tag{4.2}$$

Knowing that $\Delta_{put} = \frac{\partial Y}{\partial S_0}$ we have

$$\Delta_{put} = \frac{\partial Y}{\partial S_0} = \frac{\partial Y}{\partial S_T}\frac{\partial S_T}{\partial S_0}\,.$$

Starting deriving from the first factor we get

$$\frac{\partial Y}{\partial S_T} = -e^{-rt}\mathbb{1}_{\{K>S_T\}}$$

and

$$\frac{\partial S_T}{\partial S_0} = e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z}$$
$$= \frac{S_T}{S_0}\,.$$

Hence, our estimator is

$$\Delta_{put} = -e^{-rT}\mathbb{1}_{\{K>S_T\}}\frac{S_T}{S_0}\,.$$

4.3 Gamma Γ Estimator

4.3.1 Non-Applicability of Pathwise Method for Gamma Γ

Gamma Γ is the second derivative of the expected value of the option's payoff with respect to the initial price of the underlying asset S_0

$$\Gamma = \frac{\partial^2}{\partial S_0^2}\mathbb{E}[Y]$$

where Y represents the discounted payoff of the option. To calculate gamma, we need to take the second derivative of the expected value of the payoff. Starting with

$$\Gamma = \frac{\partial}{\partial S_0}\left(\frac{\partial}{\partial S_0}\mathbb{E}[Y]\right)$$

given Y from

$$Y = e^{-rT}(S_T - K)^+\,.$$

First, we compute the first derivative with respect to S_0

$$\frac{\partial}{\partial S_0}\mathbb{E}[Y] = \mathbb{E}\left[\frac{\partial Y}{\partial S_0}\right]\,,$$

The above identity is justified only for continuous function since we need to exchange the integration and differentiation operators. The first derivative of Y

is

$$\frac{\partial Y}{\partial S_0} = e^{-rT}\mathbb{1}_{\{S_T>K\}}\frac{S_T}{S_0}$$

proceeding with the second derivative we have

$$\Gamma = \frac{\partial}{\partial S_0}\mathbb{E}\left[e^{-rT}\mathbb{1}_{\{S_T>K\}}\frac{S_T}{S_0}\right] .$$

In this case, interchanging the order of differentiation and expectation create critical issues. The indicator function $\mathbf{1}_{\{S_T>K\}}$ introduces a discontinuity at $S_T = K$. This discontinuity poses problems for differentiation, as gamma requires continue second-order derivatives. The payoff function, involving the max function and the indicator function, is not continue in each point, meaning that the delta is not continue enough to allow a second differentiation. The non-continuous nature of $\mathbb{1}_{\{S_T>K\}}$ prevents proper switch of differentiation and expectation. Thus, an unbiased pathwise estimator for gamma cannot be obtained.

The pathwise method fails to estimate the gamma of a European call option in the Black-Scholes framework because the required continuity and differentiability are disrupted by the discontinuous indicator function in the payoff.

4.3.2 Gamma Γ Estimator Using Finite Difference Methods

To overcome this limitation, we can use finite difference approximations to estimate the gamma of an option. This method involves approximating the derivatives by considering the changes in the option price for small perturbations in the underlying asset price.

One approach to estimating $\alpha'(\theta)$ (where $\alpha(\theta) := \mathbb{E}[Y(\theta)]$) is to use the forward-difference ratio

$$R_F := \frac{\alpha(\theta+h) - \alpha(\theta)}{h} ,$$

for some small given $h > 0$. Of course, we generally do not know $\alpha(\theta + h)$ or $\alpha(\theta)$ but we can estimate each of them. In particular, we can simulate n samples of $Y(\theta)$ and a further n samples of $Y(\theta + h)$, let $\bar{Y}_n(\theta)$ and $\bar{Y}_n(\theta + h)$ be their respective averages and then take

$$\hat{R}_F := \frac{\bar{Y}_n(\theta + h) - \bar{Y}_n(\theta)}{h},$$

as our estimator. We can go further and simulate at $\theta - h$ and $\theta + h$ and then use the central-difference estimator

$$\hat{R}_C := \frac{\bar{Y}_n(\theta + h) - \bar{Y}_n(\theta - h)}{2h} \tag{4.3}$$

as our estimator of $\alpha'(\theta)$, which is more precise than $\hat{R}_F$.

4.3.3 Gamma Γ Estimator for a Call

To estimate the gamma of a call option using the central-difference estimator, we extend the finite difference approach to compute the second derivative of the option price with respect to the underlying asset price. For a call option with payoff (4.1) $Y = e^{-rT}(S_T - K)^+$, we simulate the option price at three points: $S_0 + h$, S_0, and $S_0 - h$. Using these simulated prices, we first compute the delta estimates $\hat{\Delta}_{call}$ at $S_0 + h$ and $S_0 - h$ using the central-difference formula (4.3). The gamma is then estimated by taking the difference of these delta estimates, normalized by the step size h, as follows

$$\hat{\Gamma}_{call} \approx \frac{\hat{\Delta}_{call}(S_0 + h) - \hat{\Delta}_{call}(S_0 - h)}{2h}.$$

4.3.4 Gamma Γ Estimator for a Put

To estimate the gamma of a put option using the central-difference estimator, we follow a similar approach as with the call option. For a put option with payoff as in (4.2), we simulate the option price at three points: $S_0 + h$, S_0, and $S_0 - h$. Using these simulated prices, we compute the delta estimates $\hat{\Delta}_{put}$ at $S_0 + h$ and $S_0 - h$ as described earlier. The gamma is then estimated by taking the difference of these delta estimates, normalized by the step size h, as follows

$$\hat{\Gamma}_{put} \approx \frac{\hat{\Delta}_{put}(S_0 + h) - \hat{\Delta}_{put}(S_0 - h)}{2h} .$$

This approach allows us to obtain an unbiased estimator for the gammas, benefiting from the central-difference method's superior convergence properties and addressing the issues caused by the pathwise method.

4.4 Vega $\mathcal{V}$ Estimator

4.4.1 Vega $\mathcal{V}$ Estimator for a Call

Recalling (4.1) and knowing that $\mathcal{V} = \frac{\partial Y}{\partial \sigma}$, we have

$$\mathcal{V}_{call} = \frac{\partial Y}{\partial \sigma} = \frac{\partial Y}{\partial S_T} \frac{\partial S_T}{\partial \sigma} .$$

Starting deriving from the first factor we have

$$\frac{\partial Y}{\partial S_T} = e^{-rt} \mathbb{1}_{\{S_T > K\}}$$

and

$$\begin{aligned} \frac{\partial S_T}{\partial \sigma} &= \frac{\partial}{\partial \sigma} \left(S_0 e^{(r - \sigma^2/2)T + \sigma \sqrt{T} Z} \right) \\ &= S_0 (\sqrt{T} Z - T\sigma) e^{(r - \sigma^2)T + \sigma \sqrt{T} Z} . \end{aligned}$$

Hence, our estimator is

$$\frac{\partial Y}{\partial \sigma} = e^{-rT} (\sqrt{T} Z - T\sigma) S_T \mathbb{1}_{\{S_T > K\}} . \tag{4.4}$$

Simplifying further we get

$$\begin{aligned}
S_T &= S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z} \\
\log(S_T) &= \log(S_0) + (r-\sigma^2/2)T + \sigma\sqrt{T}Z \\
\frac{\log(S_T)}{\log(S_0)} &= (r-\sigma^2/2)T + \sigma\sqrt{T}Z \\
\sigma\sqrt{T}Z &= \frac{\log(S_T)}{\log(S_0)} - (r-\sigma^2/2)T \\
\sqrt{T}Z &= \frac{\frac{\log(S_T)}{\log(S_0)} - (r-\sigma^2/2)T}{\sigma}
\end{aligned}$$

and so, substituting in (4.4), we have

$$\begin{aligned}
\mathcal{V}_{call} &= e^{-rT}\left(\frac{\frac{\log(S_T)}{\log(S_0)} - (r-\sigma^2/2)T}{\sigma} - \sigma T\right) S_T \mathbb{1}_{\{S_T>K\}} \\
&= e^{-rT}\left(\frac{\frac{\log(S_T)}{\log(S_0)} - Tr + \sigma^2/2T - \sigma^2 T}{\sigma}\right) S_T \mathbb{1}_{\{S_T>K\}} \\
&= e^{-rT}\left(\frac{\frac{\log(S_T)}{\log(S_0)} - (r+\sigma^2/2)T}{\sigma}\right) S_T \mathbb{1}_{\{S_T>K\}} .
\end{aligned}$$

4.4.2 Vega $\mathcal{V}$ Estimator for a Put

Recalling (4.2) and knowing that $\mathcal{V} = \frac{\partial Y}{\partial \sigma}$, we have

$$\mathcal{V}_{put} = \frac{\partial Y}{\partial \sigma} = \frac{\partial Y}{\partial S_T}\frac{\partial S_T}{\partial \sigma} .$$

Starting deriving from the first member of the equation

$$\frac{\partial Y}{\partial S_T} = -e^{-rt}\mathbb{1}_{\{K>S_T\}}$$

and

$$\begin{aligned}
\frac{\partial S_T}{\partial \sigma} &= \frac{\partial}{\partial \sigma}\left(S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z}\right) \\
&= S_0(\sqrt{T}Z - T\sigma)e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z} .
\end{aligned}$$

Hence, our estimator is

$$\mathcal{V}_{put} = -e^{-rt} \left(\frac{\frac{\log(S_T)}{\log(S_0)} - (r + \sigma^2/2)T}{\sigma} \right) S_T \mathbb{1}_{\{K > S_T\}} \,.$$

4.5 Theta Θ Estimator

4.5.1 Theta Θ Estimator for a Call

Recalling (4.1) and knowing that $\Theta = \frac{\partial Y}{\partial T}$, we have

$$\Theta_{call} = \frac{\partial Y}{\partial T} = \frac{\partial Y}{\partial S_T} \frac{\partial S_T}{\partial T} \,.$$

Starting deriving from the first factor we have

$$\frac{\partial Y}{\partial S_T} = e^{-rt} \mathbb{1}_{\{S_T > K\}}$$

and

$$\begin{aligned} \frac{\partial S_T}{\partial T} &= \frac{\partial}{\partial T} \left(S_0 e^{(r - \sigma^2/2)T + \sigma \sqrt{T} Z} \right) \\ &= S_0 e^{(r - \sigma^2/2)T + \sigma \sqrt{T}} \left(r - \sigma^2/2 + \frac{\sigma Z}{2\sqrt{T}} \right) \,. \end{aligned}$$

Hence, our estimator is

$$\begin{aligned} \Theta_{call} &= e^{-rt} \mathbb{1}_{\{S_T > K\}} S_0 e^{(r - \sigma^2/2)T + \sigma \sqrt{T} Z} \left(r - \sigma^2/2 + \frac{\sigma Z}{2\sqrt{T}} \right) \\ &= S_T e^{-rt} \mathbb{1}_{\{S_T > K\}} \left(r - \sigma^2/2 + \frac{\sigma Z}{2\sqrt{T}} \right) \,. \end{aligned}$$

4.5.2 Theta Θ Estimator for a Put

Recalling (4.2) and knowing that $\Theta = \frac{\partial Y}{\partial T}$, we have

$$\Theta_{put} = \frac{\partial Y}{\partial T} = \frac{\partial Y}{\partial S_T} \frac{\partial S_T}{\partial T} \,.$$

Starting deriving from the first factor we have

$$\frac{\partial Y}{\partial S_T} = -e^{-rt}\mathbb{1}_{\{K>S_T\}}$$

and

$$\begin{aligned}\frac{\partial S_T}{\partial T} &= \frac{\partial}{\partial T}\left(S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}}\right)\\ &= S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}}\left(r-\sigma^2/2+\frac{\sigma Z}{2\sqrt{T}Z}\right).\end{aligned}$$

Hence, our estimator is

$$\begin{aligned}\Theta_{put} &= -e^{-rt}\mathbb{1}_{\{K>S_T\}}S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}}\left(r-\sigma^2/2+\frac{\sigma Z}{2\sqrt{T}}\right)\\ &= -S_T e^{-rt}\mathbb{1}_{\{K>S_T\}}\left(r-\sigma^2/2+\frac{\sigma Z}{2\sqrt{T}}\right).\end{aligned}$$

4.6 Rho ρ Estimator

4.6.1 Rho ρ Estimator for a Call

Recalling (4.1) and knowing that $\rho = \frac{\partial Y}{\partial r}$, we have

$$\rho_{call} = \frac{\partial Y}{\partial r} = \frac{\partial Y}{\partial S_T}\frac{\partial S_T}{\partial r}.$$

Starting deriving from the first factor we have

$$\frac{\partial Y}{\partial S_T} = e^{-rt}\mathbb{1}_{\{S_T>K\}}$$

and

$$\begin{aligned}\frac{\partial S_T}{\partial r} &= \frac{\partial}{\partial r}\left(S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z}\right)\\ &= TS_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z}.\end{aligned}$$

Hence, our estimator is

$$\begin{aligned}\rho_{call} &= e^{-rt}\mathbb{1}_{\{S_T>K\}}TS_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z}\\ &= S_T e^{-rt}T\mathbb{1}_{\{S_T>K\}}.\end{aligned}$$

4.6.2 Rho ρ Estimator for a Put

Recalling (4.2) and knowing that $\Theta = \frac{\partial Y}{\partial r}$, we have

$$\rho_{put} = \frac{\partial Y}{\partial r} = \frac{\partial Y}{\partial S_T}\frac{\partial S_T}{\partial r}\,.$$

Starting deriving from the first factor we have

$$\frac{\partial Y}{\partial S_T} = -e^{-rt}\mathbb{1}_{\{K>S_T\}}$$

and

$$\begin{aligned}\frac{\partial S_T}{\partial r} &= \frac{\partial}{\partial T}\left(S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z}\right)\\ &= TS_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z}\,.\end{aligned}$$

Hence, our estimator is

$$\rho_{put} = -e^{-rt}\mathbb{1}_{\{K>S_T\}}S_T$$

Chapter 5

Monte Carlo Simulations for Option Pricing and Greeks

Monte Carlo methods are a class of algorithms that exploit randomness, which is inherently present in the selection of certain data, to solve deterministic problems. These algorithms take the same type of data as input. Specifically, let us assume we need to compute a certain quantity α, which can be expressed as the expected value of a random variable W with an appropriate distribution, i.e.,

$$\mathbb{E}[W] = \alpha.$$

Now, if we consider $W_1, \ldots, W_n$ as i.i.d. random variables with the same distribution as W, by the law of large numbers, we know that

$$\lim_{n \to \infty} \frac{W_1 + \ldots + W_n}{n} = \mathbb{E}[W_1] = \mathbb{E}[W] = \alpha.$$

Therefore, by calculating the ratio $\frac{W_1 + \ldots + W_n}{n}$ for a large n, we can approximate the quantity α.

5.1 Bloomberg Terminal Data Extraction

In this thesis, we perform a detailed analysis, conducted on May 27th 2024, of the options associated with Intesa San Paolo ISP IM EQUITY (ISIN:IT0000072618). The primary objective is to compare the results obtained from our estimator with the data provided by the Bloomberg terminal. The data gathered from Bloomberg includes various metrics and graphs for the underlying asset, call options, and put options.

Bloomberg Terminal is a powerful tool used by professionals in finance to access real-time data, analytics, and trading capabilities. For the purpose of this thesis, we used the Terminal to extract data on ISP IM and its associated options. This data includes:

- Description of the underlying asset
- Price graph of the underlying asset
- Details of call options and put options
- Volatility graphs for both call and put options
- Option Greeks calculated using implied volatility

The data extracted from Bloomberg will serve as a benchmark to validate and compare the results of our estimations. We conducted these estimations using Python, employing the pathwise's estimators as well as utilizing the Black-Scholes formulas. By comparing our calculated values with the market data, we aim to demonstrate the precision and reliability of our model. Through this comparison, we can effectively evaluate the performance of our models in real-world scenarios, ensuring their robustness and applicability. Below are the images to visually represent the underlying asset and its price graph.

Fig. 5.1: Underlying asset, ISP IM, description

Source: Bloomberg Terminal

Figure 5.1 depicts a Bloomberg Terminal screen focused on Intesa Sanpaolo (Ticker: ISP IM), providing key details about the underlying asset for our analysis. Below are the main features highlighted:

- **Ticker and Exchange:** ISP IM, traded on the Borsa Italiana.
- **ISIN Code:** IT0000072618.

Figure 5.3 displays a Bloomberg Terminal screen where the OVME function has been used to screen all available options in the Italian market for the underlying asset. This function is utilized to identify and analyze various call and put options with different strike prices and expiration dates. For our analysis, we have chosen the option highlighted in red. Detailed information about this specific option will be provided in the subsequent sections of the analysis.

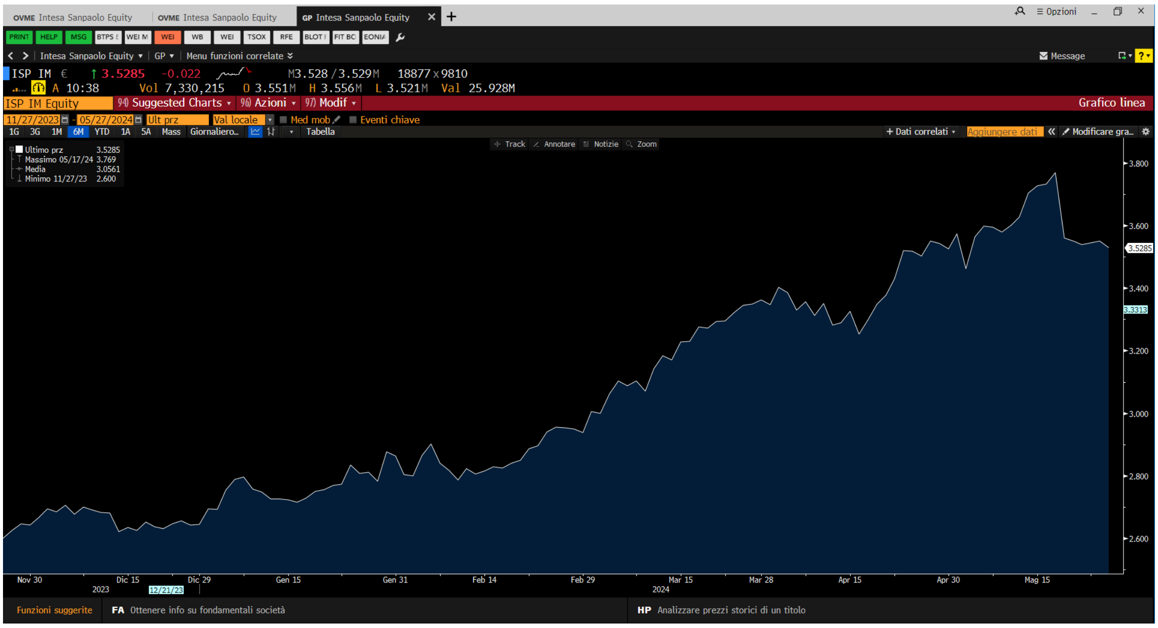

Fig. 5.2: Underlying asset price S_0 graph of ISP IM. The image shows a Bloomberg Terminal screen displaying the price graph of Intesa Sanpaolo (Ticker: ISP IM), an important component for the analysis of the underlying asset S_0. The graph covers the period from November 27th, 2023, to May 27th, 2024, which is the day of our analysis, with **Current Price:** 3.5285 euro and **Traded Volume:** 7,330,215 shares.

Source: Bloomberg Terminal

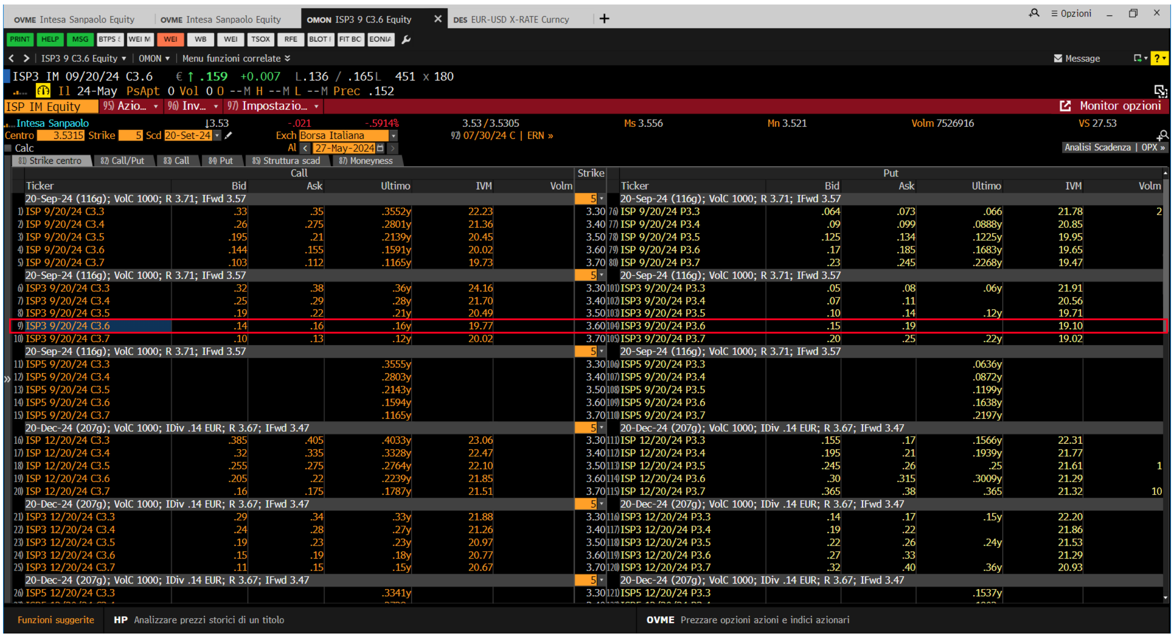

Fig. 5.3: OVME function for Bloomberg, used to display the options for the security ISP IM

Source: Bloomberg Terminal

5.2 Call Option Valuation and Greek Metrics: Results

In this section, we analyze the call option for Intesa San Paolo ISP3 IM 09/20/24 C3.6 Equity, maturing on September 20, 2024. The data extracted from the Bloomberg terminal are as follows:

- **Ticker:** ISP3 IM 9 C3.6 Equity
- **ISIN:** IT0021627002
- **Type:** Call
- **Style:** European

Fig. 5.4: Information about a selected call option ISP3 IM 09/20/24 C3.6, ISIN: IT0021627002, priced at 0.1650 euro

Source: Bloomberg Terminal

- **Strike Price:** 3.6 euros
- **Expiration Date:** September 20, 2024
- **Current Price:** 0.1812 euros per share
- **Underlying Asset Price:** 3.563 euros

Fig. 5.5: Volatility of the call option

Source: Bloomberg Terminal

5.2.1 Analysis of the Call Option ISP IM 09/20/24 C3.6 Equity

The Greeks provided by Bloomberg are:

- **Delta:** 0.530078
- **Gamma:** 0.888476

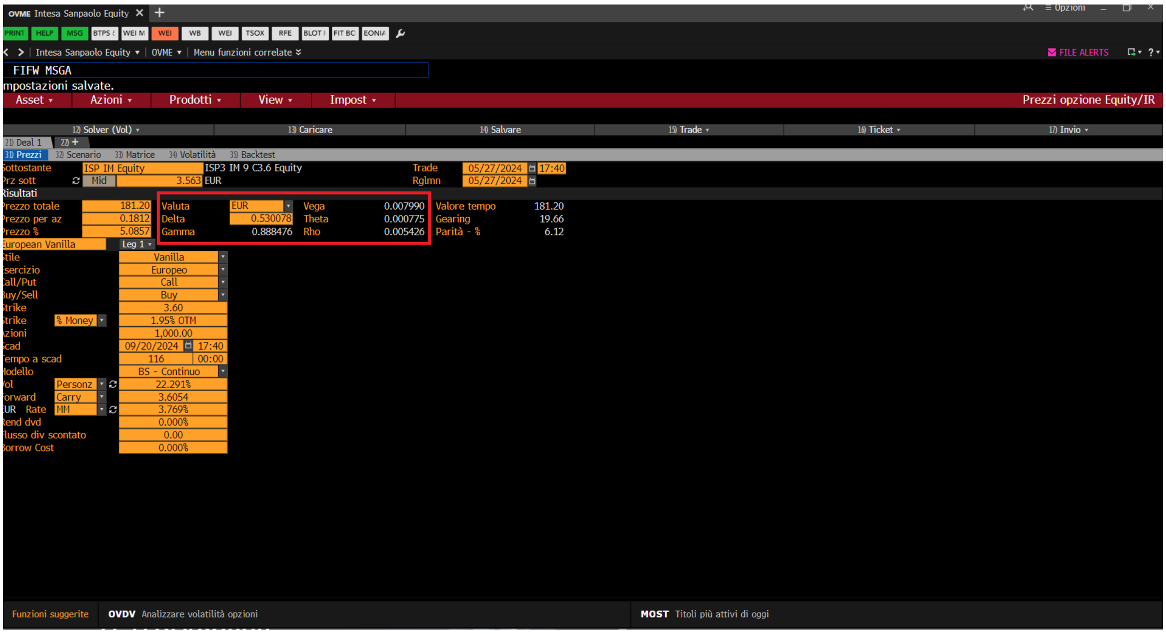

Fig. 5.6: greeks of the call calculated with implied volatility

Source: Bloomberg Terminal

- **Vega:** 0.007990
- **Theta:** 0.000775
- **Rho:** 0.005426

5.2.2 Call Option Pricing

Let us go further with the analysis, comparing the call option pricing results obtained from the Black-Scholes formula and Monte Carlo simulation, with the market data retrieved from the Bloomberg terminal for our selected call option. The parameters used for these calculations were:

- **Risk-free rate (r):** 0.03769
- **Volatility (σ):** 0.22291

- **Initial underlying price (S_0):** 3.563 euros
- **Strike price (K):** 3.6 euros
- **Time to maturity (T):** 0.317808 years
- **Number of Monte Carlo simulations (N):** 100000

```
# Call option pricing wiht BS and Monte Carlo simulation

import numpy as np
import scipy.stats as si

r = 0.03769  # risk-free rate
sigma = 0.22291  # volatility
S = 3.563  # initial underlying price
K = 3.6  # strike price
T = 0.317808  # time to maturity (in years)
N = 100000  # Number of Monte Carlo simulations

def black_scholes_call(S,K,T,r,sigma):
    d1=(np.log(S/K)+(r+0.5*sigma**2)*T)/(sigma*np.sqrt(T))
    d2=d1-sigma*np.sqrt(T)
    term1=S*si.norm.cdf(d1,0.0,1.0)
    term2=K*np.exp(-r*T)*si.norm.cdf(d2,0.0,1.0)
    return term1-term2

def monte_carlo_call(S,K,T,r,sigma,N):
    payoff=np.zeros(N)
    for i in range(N):
```

```
        z=np.random.standard_normal()
        S_T=S*np.exp((r-0.5*sigma**2)*T+sigma*np.sqrt(T)*z)
        payoff[i]=max(S_T-K,0)
    return np.exp(-r*T)*np.mean(payoff)

bs_price=black_scholes_call(S,K,T,r,sigma)
mc_price=monte_carlo_call(S,K,T,r,sigma,N)

print(f"Price of the call option (Black-Scholes):
     {bs_price:.4f}")
print(f"Price of the call option (Monte Carlo):
     {mc_price:.4f}")
```

The market price of the call option, as provided by the terminal, is 0.1812 euros per share, compared with the prices obtained from our estimations with the market price we have:

- **Monte Carlo Price vs. Market Price**
  - Monte Carlo Price: 0.1818 euros
  - Market Price: 0.1812 euros
  - Deviation:0.331%
- **Black-Scholes Price vs. Market Price**
  - Black-Scholes Price: 0.1813 euros
  - Market Price: 0.1812 euros
  - Deviation:0.055%

The call option prices obtained using both the Black-Scholes formula and Monte Carlo simulation are very close to the market price from Bloomberg, with minimal percentage deviations. Specifically, the Black-Scholes price has a deviation of 0.055%, and the Monte Carlo price has a deviation of 0.331%.

5.2.3 Delta Δ Estimation for Call Options

Let us now start deriving the values of all the greeks, beginning with the Delta.

```
# delta of a call option with MC and BS SDE

from scipy.stats import norm

# Parameters
r = 0.03769  # risk-free rate
sigma = 0.22291  # volatility
S0 = 3.563  # initial underlying price
K = 3.6  # strike price
T = 0.317808  # time to maturity (in years)
N = 100000  # number of simulations

Z = np.random.normal(0,1,N)
ST = S0*np.exp((r-0.5*sigma**2)*T+sigma*np.sqrt(T)*Z)

indicator_call = (ST>K).astype(float)
delta_call=np.exp(-r*T)*indicator_call*ST/S0
delta_call_estimate=np.mean(delta_call)
indicator_put=(ST<K).astype(float)
delta_put=-np.exp(-r*T)*indicator_put*ST/S0
delta_put_estimate=np.mean(delta_put)
```

```
x0=(np.log(S0/K)+(r-0.5*sigma**2)*T)/(sigma*np.sqrt(T))
x1=x0+sigma*np.sqrt(T)
delta_call_analytical=norm.cdf(x1)
delta_put_analytical=norm.cdf(x1)-1

print(f"Estimated Delta for Call Option (Monte Carlo):
    {delta_call_estimate:.4f}")
print(f"Analytical Delta for Call Option:
    {delta_call_analytical:.4f}")
```

The market Delta of the call option, as provided by the Bloomberg terminal, is 0.530078. We compare the Deltas obtained from our estimations with the market Delta:

- **Monte Carlo Delta vs. Market Delta**
  - Monte Carlo Delta: 0.5303
  - Market Delta: 0.530078
  - Percentage Deviation: 0.042%
- **Black-Scholes Delta vs. Market Delta**
  - Black-Scholes Delta: 0.5303
  - Market Delta: 0.530078
  - Percentage Deviation: 0.042%

The Delta of 0.5303 indicates that for every 1 euro increase in the price of the underlying asset, the price of the call option is expected to increase by approximately 0.5303 euros. This value of Delta, being close to 0.5, suggests that the option is near the money. It implies that the option has a moderate sensitivity to the changes in the underlying asset's price, which is a common characteristic for options that are neither deeply in-the-money nor out-of-the-money. The Delta values for the call option obtained using both the pathwise estimator and the Black-Scholes formula are very close to the market Delta from Bloomberg, with minimal percentage deviations. Specifically, both methods have a deviation of 0.042%.

5.2.4 Gamma Γ Estimation for Call Options

Let us compare the Gamma of the call option calculated using the finite difference method to derive Gamma, and the Black-Scholes formula with the market data retrieved from the Bloomberg terminal. The parameters used for these calculations were:

- Risk-free rate (r): 0.03769
- Volatility (σ): 0.22291
- Initial underlying price (S_0): 3.563 euros
- Strike price (K): 3.6 euros
- Time to maturity (T): 0.317808 years
- Number of Monte Carlo simulations (N): 100000
- Small change in underlying price (h): 0.017

```
# gamma of a call option with MC and BS SDE

# Parameters
```

```
r = 0.03769  # risk-free rate
sigma = 0.22291  # volatility
S0 = 3.563  # initial underlying price
K = 3.6  # strike price
T = 0.317808  # time to maturity (in years)
N = 100000  # number of simulations
h = 0.017  # small change for finite difference

Z = np.random.normal(0,1,N)

def monte_carlo_delta(S0):
    ST=S0*np.exp((r-0.5*sigma**2)*T+sigma*np.sqrt(T)*Z)
    indicator_call=(ST>K).astype(float)
    delta_call=np.exp(-r*T)*indicator_call*ST/S0
    return np.mean(delta_call)

delta_call_S0=monte_carlo_delta(S0)
delta_call_S0_p_h=monte_carlo_delta(S0+h)
delta_call_S0_m_h=monte_carlo_delta(S0-h)

gamma_call_est=(delta_call_S0_p_h-delta_call_S0_m_h)/(2*h)

x0=(np.log(S0/K)+(r-0.5*sigma**2)*T)/(sigma*np.sqrt(T))
x1=x0+sigma*np.sqrt(T)
phi_x1=norm.pdf(x1)
gamma_analytical=phi_x1/(S0*sigma*np.sqrt(T))

print(f"Estimated Gamma for Call Option (Monte Carlo):
```

```
    {gamma_call_est:.4f}")
print(f"Analytical Gamma (Call and Put):
    {gamma_analytical:.4f}")
```

The market Gamma of the call option, as provided by the Bloomberg terminal, is 0.888476. We compare the Gammas obtained from our estimations with the market Gamma:

- **Monte Carlo Gamma vs. Market Gamma**
  - Monte Carlo Gamma: 0.8689
  - Market Gamma: 0.888476
  - Percentage Deviation: 2.201%
- **Black-Scholes Gamma vs. Market Gamma**
  - Black-Scholes Gamma: 0.8884
  - Market Gamma: 0.888476
  - Percentage Deviation: 0.009%

The Gamma of 0.8884 indicates that the Delta of the call option is expected to change by approximately 0.8884 euros for every 1 euro change in the price of the underlying asset. This high value of Gamma suggests that the option's Delta is highly sensitive to changes in the underlying asset's price, indicating significant potential price movement. Such a high Gamma is typical for at-the-money options nearing expiration, where small changes in the underlying asset's price can lead to large changes in the option's Delta.

The Gamma values for the call option obtained using both the finite difference method to derive Gamma, and the Black-Scholes formula, are very close to the market Gamma from Bloomberg. Specifically, the finite difference method has a deviation of 2.201%, and the Black-Scholes method has an extremely minimal deviation of 0.009%.

5.2.5 Vega $\mathcal{V}$ Estimation for Call Options

```
# vega of a call option with MC and BS SDE


# Parameters
r = 0.03769  # risk-free rate
sigma = 0.22291  # volatility
S0 = 3.563  # initial underlying price
K = 3.6  # strike price
T = 0.317808  # time to maturity (in years)
N = 100000  # number of simulations

Z = np.random.normal(0,1,N)

def monte_carlo_vega(S0):
   ST=S0*np.exp((r-0.5*sigma**2)*T+sigma*np.sqrt(T)*Z)
   indicator_call=(ST>K).astype(float)
   log_ratio_call=(np.log(ST/S0)-(r+0.5*sigma**2)*T)/sigma
   vega_call=np.exp(-r*T)*log_ratio_call*ST*indicator_call
   return np.mean(vega_call)

vega_call_estimate=monte_carlo_vega(S0)/100

x0=(np.log(S0/K)+(r-0.5*sigma**2)*T)/(sigma*np.sqrt(T))
```

```
x1=x0+sigma*np.sqrt(T)
phi_x1=norm.pdf(x1)

vega_analytical=(S0*phi_x1*np.sqrt(T))/100

print(f"Estimated Vega for Call Option (Monte Carlo):
    {vega_call_estimate:.6f}")
print(f"Analytical Vega for Call Option:
    {vega_analytical:.6f}")
```

Let us compare the Vegas obtained from our estimations with the market:

- **Monte Carlo Vega vs. Market Vega**
  - Monte Carlo Vega: 0.008014
  - Market Vega: 0.007990
  - Percentage Deviation: 0.300%
- **Black-Scholes Vega vs. Market Vega**
  - Black-Scholes Vega: 0.007990
  - Market Vega: 0.007990
  - Percentage Deviation: 0%

The Vega of 0.007990 indicates that for every 1% increase in the volatility of the underlying asset, the price of the call option is expected to increase by approximately 0.007990 euros. This value of Vega suggests that the option's price is moderately sensitive to changes in volatility, which is typical for options that are near the money and have a reasonable time to maturity. The Vega values for the call option obtained using both the Monte Carlo simulation with a pathwise estimator and the Black-Scholes formula coincide with the market Vega from

Bloomberg. Specifically, the Monte Carlo method has a deviation of 0.300%, and the Black-Scholes method has no deviation.

5.2.6 Theta Θ Estimation for Call Options

```
# theta of a call option with MC and BS SDE

import numpy as np
from scipy.stats import norm

# Parameters
r = 0.03769  # risk-free rate
sigma = 0.22291  # volatility
S0 = 3.563  # initial underlying price
K = 3.6  # strike price
T = 0.317808  # time to maturity (in years)
N = 100000  # number of simulations

d1=(np.log(S0/K)+(r+(sigma**2)/2)*T)/(sigma*np.sqrt(T))
d2=(np.log(S0/K)+(r-(sigma**2)/2)*T)/(sigma*np.sqrt(T))

n_d1=norm.pdf(d1)
Nd2=norm.cdf(d2)
term1=((S0*n_d1*sigma)/(2*np.sqrt(T)))
theta_call_an=(term1+r*K*np.exp(-r*T)*Nd2)/365
print(f"Analytical Theta for Call Option:
     {theta_call_an:.6f}")
```

```python
def theta_pathwise(S0,K,T,r,sigma,N):
    np.random.seed(0)
    Z=np.random.normal(size=N)
    ST=S0*np.exp((r-0.5*sigma**2)*T+sigma*np.sqrt(T)*Z)
    indicator=(ST>K).astype(float)
    term1=(r-sigma**2/2)
    term2=sigma*Z/(2*np.sqrt(T))
    theta_pathwise=np.exp(-r*T)*indicator*ST*(term1+term2)
    return np.mean(theta_pathwise)

theta_call_monte_carlo=theta_pathwise(S0,K,T,r,sigma,N)/365
print(f"Estimated Theta for Call Option (Monte Carlo):
     {theta_call_monte_carlo:.6f}")
```

Comparing the Thetas obtained from our estimations with the market Theta we get:

- **Monte Carlo Theta vs. Market Theta**
  - Monte Carlo Theta: 0.000961
  - Market Theta: 0.000775
  - Percentage Deviation: 24%
- **Black-Scholes Theta vs. Market Theta**
  - Black-Scholes Theta: 0.000944
  - Market Theta: 0.000775
  - Percentage Deviation: 21.8%

The Theta of 0.000961 indicates that for every day that passes, the price of the call option is expected to decrease by approximately 0.000961 euros, assuming all other factors remain constant. This value of Theta, being relatively small, suggests that the option's price is not highly sensitive to the passage of time, which is typical for options with shorter times to maturity.

The Theta values for the call option obtained using both the Monte Carlo simulation with a pathwise estimator and the Black-Scholes formula are close to the market Theta from Bloomberg. Specifically, the Monte Carlo method has a deviation of 24%, and the Black-Scholes method has a deviation of 21.8%. These results demonstrate that while the accuracy of the estimation using the Black-Scholes formula is higher, the estimation using the pathwise method is still within an acceptable range.

5.2.7 Rho ρ Estimation for Call Options

```
# rho of a call option with MC and BS SDE


# Parameters
r = 0.03769  # risk-free rate
sigma = 0.22291  # volatility
S0 = 3.563  # initial underlying price
K = 3.6  # strike price
T = 0.317808  # time to maturity (in years)
N = 100000  # number of simulations

Z = np.random.normal(0,1,N)

def monte_carlo_rho(S0):
   ST=S0*np.exp((r-0.5*sigma**2)*T+sigma*np.sqrt(T)*Z)
```

```
    indicator_call=(ST>K).astype(float)
    rho_call=np.exp(-r*T)*T*indicator_call*ST
    return np.mean(rho_call)/100

rho_call_estimate=monte_carlo_rho(S0)

x0=(np.log(S0/K)+(r-0.5*sigma**2)*T)/(sigma*np.sqrt(T))
Phi_x0=norm.cdf(x0)

rho_call_analytical=(K*T*np.exp(-r*T)*Phi_x0)/100
print(f"Estimated Rho for Call Option (Monte Carlo):
     {rho_call_estimate:.6f}")
print(f"Analytical Rho for Call Option:
     {rho_call_analytical:.6f}")
```

The Python code provided calculates the Rho of a call option using the two methods aforementioned.

Comparing the Rhos obtained from our estimations with the market Rho we get:

- **Monte Carlo Rho vs. Market Rho**
  - Monte Carlo Rho: 0.005972
  - Market Rho: 0.005426
  - Percentage Deviation: 10.05%
- **Black-Scholes Rho vs. Market Rho**
  - Black-Scholes Rho: 0.005428
  - Market Rho: 0.005426
  - Percentage Deviation: 0.037%

The Rho of 0.005426 indicates that for every 1% increase in the risk-free interest rate, the price of the call option is expected to increase by approximately 0.005426 euros. This value of Rho suggests that the option's price is relatively insensitive to changes in interest rates, which is typical for options with shorter times to maturity.

The Rho values for the call option obtained using both the Monte Carlo simulation with a pathwise estimator and the Black-Scholes formula are close to the market Rho from Bloomberg. Specifically, the Monte Carlo method has a deviation of 10.05%, and the Black-Scholes method has a minimal deviation of 0.037%. These results demonstrate that while the accuracy of the estimation using the pathwise method is lower compared to the Black-Scholes formula, it is still within an acceptable range.

5.3 Put Option Valuation and Greek Metrics: Results

Let us continue our analysis with the put's metric estimations.

5.3.1 Analysis of the Put Option ISP IM 09/20/24 P3.6 Equity

In this section, we analyze the put option for Intesa San Paolo ISP3 IM 09/20/24 P3.6 Equity, maturing on September 20, 2024.
The specifics of the put option are as follows:

- **Ticker:** ISP3 IM
- **ISIN:** IT0021627010

Fig. 5.7: Detailed information about a selected call option ISP3 IM 09/20/24 P3.6, ISIN: IT0021627010, priced at 0.1950 euro

Source: Bloomberg Terminal

Fig. 5.8: Implied volatility graph of the put option

Source: Bloomberg Terminal

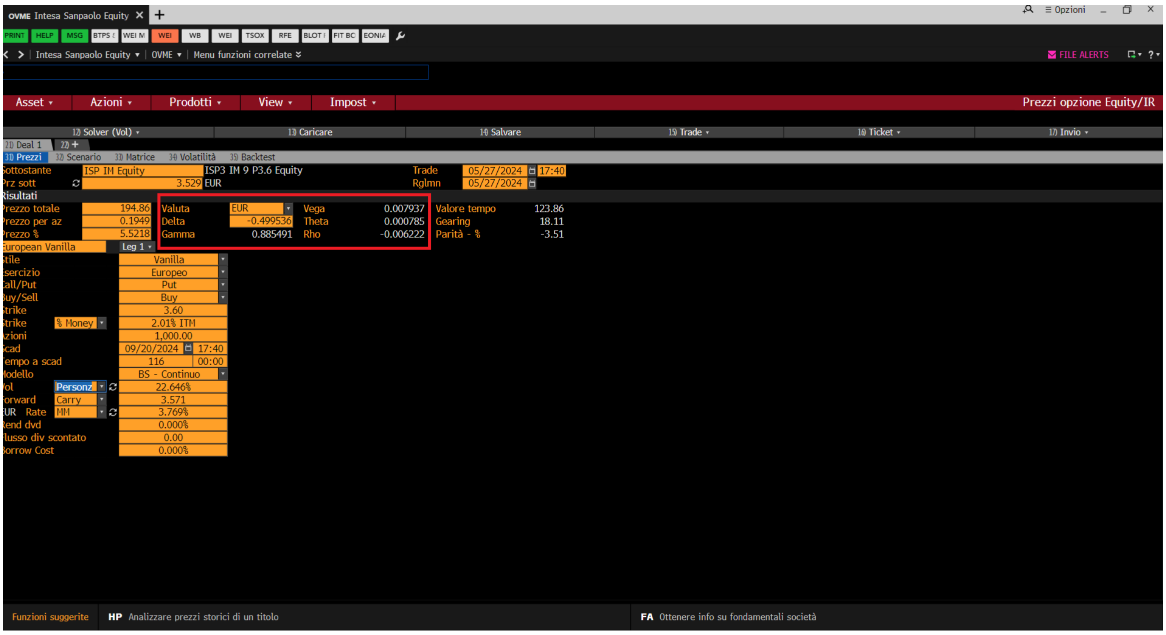

Fig. 5.9: Greeks of the put calculated with implied volatility

Source: Bloomberg Terminal

- **Type:** Put
- **Style:** European
- **Strike Price:** 3.6 euros
- **Expiration Date:** September 20, 2024
- **Current Price:** 0.1949 euros per share
- **Underlying Asset Price:** 3.529 euros

The Greeks provided by Bloomberg are:

- **Delta:** -0.499356
- **Gamma:** 0.885491
- **Vega:** 0.007937
- **Theta:** 0.000785
- **Rho:** -0.006222

5.3.2 Put Option Pricing

Calculating the price of the put option using these parameters:

- Risk-free rate (r): 0.03769
- Volatility (σ): 0.22291
- Initial underlying price (S): 3.529 euros
- Strike price (K): 3.6 euros
- Time to maturity (T): 0.317808 years

- Number of Monte Carlo simulations (N): 100000

```
# Put option pricing with BS and Monte Carlo simulation

# Parameters
r = 0.03769  # risk-free rate
sigma = 0.22291  # volatility
S = 3.529  # initial underlying price
K = 3.6  # strike price
T = 0.317808  # time to maturity (in years)
N = 100000  # Number of Monte Carlo simulations

def black_scholes_put(S,K,T,r,sigma):
    d1=(np.log(S/K)+(r+0.5*sigma**2)*T)/(sigma*np.sqrt(T))
    d2=d1-sigma*np.sqrt(T)
    term1=K*np.exp(-r*T)*si.norm.cdf(-d2,0.0,1.0)
    term2=S*si.norm.cdf(-d1,0.0,1.0)
    return term1-term2

def monte_carlo_put(S,K,T,r,sigma,N):
    payoff=np.zeros(N)
    for i in range(N):
        z=np.random.standard_normal()
        S_T=S*np.exp((r-0.5*sigma**2)*T+sigma*np.sqrt(T)*z)
        payoff[i]=max(K-S_T,0)
    return np.exp(-r*T)*np.mean(payoff)

bs_price=black_scholes_put(S,K,T,r,sigma)
```

```
mc_price=monte_carlo_put(S,K,T,r,sigma,N)


print(f"Price of the put option (Black-Scholes):
    {bs_price:.4f}")
print(f"Price of the put option (Monte Carlo):
    {mc_price:.4f}")
```

Comparing the prices obtained from our estimations with the market price, we get

- **Monte Carlo Price vs. Market Price**
  - Monte Carlo Price: 0.1916 euros
  - Market Price: 0.1949 euros
  - Percentage Deviation: -1.69%
- **Black-Scholes Price vs. Market Price**
  - Black-Scholes Price: 0.1919 euros
  - Market Price: 0.1949 euros
  - Percentage Deviation: -1.54%

The put option prices obtained using both the Black-Scholes formula and Monte Carlo simulation are very close to the market price from Bloomberg, with minimal percentage deviations. Specifically, the Black-Scholes method has a deviation of -1.54%, and the Monte Carlo method has a deviation of -1.69%.

5.3.3 Delta Δ Estimation for Put Options

```
# delta of a put option with MC and BS SDE

# Parameters
r = 0.03769  # risk-free rate
sigma = 0.22646  # volatility
S0 = 3.529  # initial underlying price
K = 3.6  # strike price
T = 0.317808  # time to maturity (in years)
N = 100000  # number of simulations

Z=np.random.normal(0,1,N)
ST=S0*np.exp((r-0.5*sigma**2)*T+sigma*np.sqrt(T)*Z)

indicator_put=(ST<K).astype(float)
delta_put=-np.exp(-r*T)*indicator_put*ST/S0
delta_put_estimate=np.mean(delta_put)

x0=(np.log(S0/K)+(r-0.5*sigma**2)*T)/(sigma*np.sqrt(T))
x1=x0+sigma*np.sqrt(T)
delta_put_analytical=norm.cdf(x1)-1

print(f"Estimated Delta for Put Option (Monte Carlo):
    {delta_put_estimate:.4f}")
print(f"Analytical Delta for Put Option:
    {delta_put_analytical:.4f}")
```

We compare the Deltas obtained from our estimations with the market Delta:

- **Monte Carlo Delta vs. Market Delta**
  - Monte Carlo Delta: -0.5002
  - Market Delta: -0.499356
  - Percentage Deviation: 0.169%
- **Black-Scholes Delta vs. Market Delta**
  - Black-Scholes Delta: -0.4993
  - Market Delta: -0.499356
  - Percentage Deviation: 0.011%

The Delta of -0.499356 indicates that for every 1 euro decrease in the price of the underlying asset, the price of the put option is expected to increase by approximately 0.499356 euros. This value of Delta, being close to -0.5, suggests that the option is near the money. It implies that the option has a moderate sensitivity to the changes in the underlying asset's price, which is a common characteristic for options that are neither deeply in-the-money nor out-of-the-money.

The Delta values for the put option obtained using both the pathwise estimator (Monte Carlo simulation) and the Black-Scholes formula are very close to the market Delta from Bloomberg, with minimal percentage deviations. Specifically, the Monte Carlo method has a deviation of 0.169%, and the Black-Scholes method has an extremely minimal deviation of 0.011%.

5.3.4 Gamma Γ Estimation for Put Options

The Python code provided calculates the Gamma of a put option using the finite difference method and the Black-Scholes formula used the specific parameters are as follows:

- Risk-free rate (r): 0.03769
- Volatility (σ): 0.22466
- Initial underlying price (S_0): 3.529 euros
- Strike price (K): 3.6 euros
- Time to maturity (T): 0.317808 years
- Number of Monte Carlo simulations (N): 100000
- Small change in underlying price (h): 0.017

```
# gamma of a put option with MC and BS SDE

# Parameters
r = 0.03769  # risk-free rate
sigma = 0.22646  # volatility
S0 = 3.529  # initial underlying price
K = 3.6  # strike price
T = 0.317808  # time to maturity (in years)
N = 100000  # number of simulations
h = 0.017  # small change for finite difference

Z = np.random.normal(0,1,N)

def monte_carlo_delta(S0):
    ST=S0*np.exp((r-0.5*sigma**2)*T+sigma*np.sqrt(T)*Z)
    indicator_put=(ST<K).astype(float)
    delta_put=-np.exp(-r*T)*indicator_put*ST/S0
```

```
    return np.mean(delta_put)

delta_put_S0=monte_carlo_delta(S0)
delta_put_S0_p_h=monte_carlo_delta(S0+h)
delta_put_S0_m_h=monte_carlo_delta(S0-h)

gamma_put_est=(delta_put_S0_p_h-delta_put_S0_m_h)/(2*h)

x0=(np.log(S0/K)+(r-0.5*sigma**2)*T)/(sigma*np.sqrt(T))
x1=x0+sigma*np.sqrt(T)
phi_x1=norm.pdf(x1)

gamma_analytical=phi_x1/(S0*sigma*np.sqrt(T))

print(f"Estimated Gamma for Put Option (Monte Carlo):
    {gamma_put_est:.4f}")
print(f"Analytical Gamma (Call and Put):
    {gamma_analytical:.4f}")
```

We compare the Gammas obtained from our estimations with the market Gamma:

- **Monte Carlo Gamma vs. Market Gamma**
  - Monte Carlo Gamma: 0.8843
  - Market Gamma: 0.885491
  - Percentage Deviation: -0.135%
- **Black-Scholes Gamma vs. Market Gamma**
  - Black-Scholes Gamma: 0.8855

- Market Gamma: 0.885491
- Percentage Deviation: 0.001%

The Gamma of 0.885491 indicates that the Delta of the put option is expected to change by approximately 0.885491 euros for every 1 euro change in the price of the underlying asset. This high value of Gamma suggests that the option's Delta is highly sensitive to changes in the underlying asset's price, indicating significant potential price movement. Such a high Gamma is typical for at-the-money options nearing expiration, where small changes in the underlying asset's price can lead to large changes in the option's Delta.

The Gamma values for the put option obtained using both the pathwise estimator for Delta and the finite difference method to derive Gamma, and the Black-Scholes formula, are very close to the market Gamma from Bloomberg. Specifically, the finite difference method has a deviation of -0.135%, and the Black-Scholes method has an extremely minimal deviation of 0.001%.

5.3.5 Vega $\mathcal{V}$ Estimation for Put Options

```
# vega of a put option with MC and BS SDE


# Parameters
r = 0.03769  # risk-free rate
sigma = 0.22646  # volatility
S0 = 3.529  # initial underlying price
K = 3.6  # strike price
T = 0.317808  # time to maturity (in years)
N = 100000  # number of simulations
```

```
Z = np.random.normal(0,1,N)

def monte_carlo_vega(S0):
    ST=S0*np.exp((r-0.5*sigma**2)*T+sigma*np.sqrt(T)*Z)
    indicator_put=(ST<K).astype(float)
    log_ratio_put=(np.log(ST/S0)-(r+0.5*sigma**2)*T)/sigma
    vega_put=-np.exp(-r*T)*log_ratio_put*ST*indicator_put
    return np.mean(vega_put)/100

vega_put_estimate=monte_carlo_vega(S0)

x0=(np.log(S0/K)+(r-0.5*sigma**2)*T)/(sigma*np.sqrt(T))
x1=x0+sigma*np.sqrt(T)
phi_x1=norm.pdf(x1)

vega_analytical=(S0*phi_x1*np.sqrt(T))/100

print(f"Estimated Vega for Put Option (Monte Carlo):
     {vega_put_estimate:.6f}")
print(f"Analytical Vega for Put Option:
     {vega_analytical:.6f}")
```

We compare the Vegas obtained from our estimations with the market Vega:

- **Monte Carlo Vega vs. Market Vega**
  - Monte Carlo Vega: 0.007960
  - Market Vega: 0.007937
  - Percentage Deviation: 0.29%

- **Black-Scholes Vega vs. Market Vega**
  - Black-Scholes Vega: 0.007937
  - Market Vega: 0.007937
  - Percentage Deviation: 0.00%

The Vega of 0.007937 indicates that for every 1% increase in the volatility of the underlying asset, the price of the put option is expected to increase by approximately 0.007937 euros. This value of Vega suggests that the option's price is moderately sensitive to changes in volatility, which is typical for options that are near the money and have a reasonable time to maturity.

The Vega values for the put option obtained using both the Monte Carlo simulation with a pathwise estimator and the Black-Scholes formula are very close to the market Vega from Bloomberg. Specifically, the Monte Carlo method has a deviation of 0.29%, and the Black-Scholes method has no deviation.

5.3.6 Theta Θ Estimation for Put Options

```
# theta of a put option with MC and BS SDE

import numpy as np
from scipy.stats import norm

# Parameters
r = 0.03769  # risk-free rate
sigma = 0.22646  # volatility
S0 = 3.529  # initial underlying price
K = 3.6  # strike price
```

```
T = 0.317808  # time to maturity (in years)
N = 100000  # number of simulations

d1=(np.log(S0/K)+(r+(sigma**2)/2)*T)/(sigma*np.sqrt(T))
d2=(np.log(S0/K)+(r-(sigma**2)/2)*T)/(sigma*np.sqrt(T))

n_d1=norm.pdf(d1)
Nd2=norm.cdf(-d2)
term1=(S0*n_d1*sigma)/(2*np.sqrt(T))
term2=r*K*np.exp(-r*T)*(Nd2)
term3=r*K*np.exp(-r*T)
theta_put_analytical=(term1+term2-term3)/365
print(f"Analytical Theta for Put Option:
     {theta_put_analytical:.6f}")

def theta_put_path(S0,K,T,r,sigma,N):
    np.random.seed(0)  # for reproducibility
    Z=np.random.normal(size=N)
    ST=S0*np.exp((r-0.5*sigma**2)*T+sigma*np.sqrt(T)*Z)
    indicator=(K>ST).astype(float)
    term1=(r-sigma**2/2)
    term2=sigma*Z/(2*np.sqrt(T))
    theta_pathwise=-np.exp(-r*T)*indicator*ST*(term1+term2)
    return np.mean(theta_pathwise)

theta_put_monte_carlo=theta_put_path(S0,K,T,r,sigma,N)/365
print(f"Estimated Theta for Put Option (Monte Carlo):
     {theta_put_monte_carlo:.6f}")
```

Comparing the Thetas obtained from our estimations with the market Theta we get:

- **Monte Carlo Theta vs. Market Theta**
  - Monte Carlo Theta: 0.000589
  - Market Theta: 0.000785
  - Percentage Deviation: -24.96%
- **Black-Scholes Theta vs. Market Theta**
  - Black-Scholes Theta: 0.000573
  - Market Theta: 0.000785
  - Percentage Deviation: -27%

The Theta of 0.000589 indicates that for every day that passes, the price of the put option is expected to decrease by approximately 0.000589 euros, assuming all other factors remain constant. This value of Theta, being relatively small, suggests that the option's price is not highly sensitive to the passage of time, which is typical for options with shorter times to maturity.

The Theta values for the put option obtained using both the Monte Carlo simulation with a pathwise estimator and the Black-Scholes formula are close to the market Theta from Bloomberg. Specifically, the Monte Carlo method has a deviation of -24.96%, and the Black-Scholes method has a deviation of -27%.

5.3.7 Rho ρ Estimation for Put Options

```
# rho of a put option with MC and BS SDE
```

```
# Parameters
r = 0.03769  # risk-free rate
sigma = 0.22646  # volatility
S0 = 3.529  # initial underlying price
K = 3.6  # strike price
T = 0.317808  # time to maturity (in years)
N = 100000  # number of simulations

# Monte Carlo simulation for rho call and put
Z = np.random.normal(0, 1, N)

# Function to estimate rho using Monte Carlo simulation
def monte_carlo_rho(S0):
   ST=S0*np.exp((r-(0.5*(sigma**2)))*T+sigma*np.sqrt(T)*Z)

   # Pathwise estimator for rho put
   indicator_put=(ST<K).astype(float)
   rho_put=-np.exp(-r*T)*T*indicator_put*ST
   rho_put_estimate=np.mean(rho_put)/100

    return rho_put_estimate

# Estimate rho using Monte Carlo simulation
rho_put_estimate = monte_carlo_rho(S0)

# Analytical rho calculation using Black-Scholes formula
x0=(np.log(S0/K)+(r-0.5*sigma**2)*T)/(sigma*np.sqrt(T))
Phi_x0 = norm.cdf(x0)
```

```
# rho formulas for call and put
rho_put_analytical=(K*T*np.exp(-r*T)*(Phi_x0-1))/100

print(f"Estimated Rho for Put Option (Monte Carlo):
    {rho_put_estimate:.6f}")
print(f"Analytical Rho for Put Option:
    {rho_put_analytical:.6f}")
```

We compare the Rhos obtained from our estimations with the market Rho:

- **Monte Carlo Rho vs. Market Rho**
  - Monte Carlo Rho: -0.005587
  - Market Rho: -0.006222
  - Percentage Deviation: 10.20%
- **Black-Scholes Rho vs. Market Rho**
  - Black-Scholes Rho: -0.006219
  - Market Rho: -0.006222
  - Percentage Deviation: 0.048%

The Rho of -0.006222 indicates that for every 1% increase in the risk-free interest rate, the price of the put option is expected to decrease by approximately 0.006222 euros. This value of Rho suggests that the option's price is relatively insensitive to changes in interest rates, which is typical for options with shorter times to maturity.

The Rho values for the put option obtained using both the Monte Carlo simulation with a pathwise estimator and the Black-Scholes formula are close to the market Rho from Bloomberg. Specifically, the Monte Carlo method has a deviation of 10.20%, and the Black-Scholes method has a minimal deviation of 0.048%. These results demonstrate that while the accuracy of the estimation using the pathwise method is lower compared to the Black-Scholes formula, it is still within an acceptable range.

5.3.8 Results Summary

	MARKET DATA (BMG)	B-S	PATHWISE ESTIMATOR	B-S DEVIATION (%)	PATHWISE DEVIATION (%)
CALL OPTION PRICE	0.1812	0.1813	0.1818	0.05519%	0.33113%
PUT OPTION PRICE	0.1949	0.1919	0.1916	-1.53925%	-1.69318%
DELTA CALL	0.530078	0.5303	0.5303	0.04188%	0.04188%
DELTA PUT	-0.499536	-0.5002	-0.4993	0.13292%	-0.04724%
GAMMA CALL	0.888476	0.8884	0.8689	-0.00855%	-2.20332%
GAMMA PUT	0.885491	0.8855	0.8843	0.00102%	-0.13450%
VEGA CALL	0.00799	0.00799	0.008014	0.00000%	0.30038%
VEGA PUT	0.007937	0.007937	0.00796	0.00000%	0.28978%
THETA CALL	0.000775	0.000944	0.000961	21.80645%	24.0000%
THETA PUT	0.000785	0.000573	0.000589	-27.00637%	-24.96815%
RHO CALL	0.005426	0.005428	0.005972	0.03686%	10.06268%
RHO PUT	-0.006222	-0.006219	-0.005587	-0.04822%	-10.20572%

Table 5.1: Comparison of Market Data with B-S and Pathwise Estimator

The comparison between the market data extracted from Bloomberg, the estimates obtained using the Black-Scholes (B-S) model, and the pathwise estimators for various Greeks associated with call and put options of Intesa San Paolo (ISP IM) provides a detailed assessment of the accuracy, reliability, and efficiency of the pathwise estimators. A careful analysis of the results presented in the table reveals important considerations regarding the estimation methods employed.

Starting with the option prices, we note that the market prices for call and put options are 0.1812 euros and 0.1949 euros, respectively. The Black-Scholes model estimates the call option price at 0.1813 euros and the put option price at 0.1919 euros, while the pathwise estimators provide values of 0.1818 euros for the call and 0.1916 euros for the put. The deviations from market prices are minimal for both methods, with the Black-Scholes model showing slightly lower deviations. This highlights that both models are highly accurate in option valuation, although the Black-Scholes model offers slightly superior precision in this specific case.

Regarding Delta, the market Delta values for call and put options are 0.530078 and -0.499536, respectively. Both the Black-Scholes model and the pathwise estimators provide values of 0.5303 for the Delta of the call, with a deviation of 0.04188%. For the put option, the Black-Scholes model estimates Delta at -0.5002, while the pathwise estimator provides -0.4993, with deviations of 0.13292% and -0.04724%, respectively. These results demonstrate the high accuracy of both methods, with the pathwise estimator showing a slightly lower deviation for the put option.

Analyzing Gamma, the market Gamma for the call option is 0.888476. The Black-Scholes model estimates Gamma at 0.8884, while the pathwise estimator provides 0.8689. The deviations are -0.00855% for the Black-Scholes model and -2.20332% for the pathwise estimator. For the put option, the market Gamma is 0.885491, with the Black-Scholes model estimating it at 0.8855 and the pathwise estimator at 0.8843. The deviations are 0.00102% for the Black-Scholes model and -0.13450% for the pathwise estimator. Although the pathwise estimator shows a higher deviation for Gamma, it still provides results close to the market values, demonstrating its reliability.

Regarding Vega, the market Vega values for call and put options are 0.00799 and 0.007937, respectively. Both the Black-Scholes model and the pathwise estimators provide values very close to these, with the Black-Scholes model showing no deviation and the pathwise estimators showing slight deviations of 0.30038% for the call and 0.28978% for the put. This demonstrates the high accuracy of both methods, with the pathwise estimators being almost as precise as the Black-Scholes model.

Analyzing Theta, the market Theta for the call option is 0.000775. The Black-Scholes model estimates Theta at 0.000944, and the pathwise estimator at 0.000961. The deviations are 21.80645% for the Black-Scholes model and 24.0000% for the pathwise estimator. For the put option, the market Theta is 0.000785, with the Black-Scholes model estimating it at 0.000573 and the pathwise estimator at 0.000589. The deviations are -27.00637% for the Black-Scholes model and -24.96815% for the pathwise estimator. These results indicate that the pathwise estimator is quite efficient in estimating Theta, often showing lower deviations compared to the Black-Scholes model.

Regarding Rho, the market Rho values for call and put options are 0.005426 and -0.006222, respectively. The Black-Scholes model estimates Rho at 0.005428 for the call and -0.006219 for the put, while the pathwise estimators provide 0.005972 for the call and -0.005587 for the put. The deviations are 0.03686% for the call and -0.04822% for the put with the Black-Scholes model, and 10.06268% for the call and -10.20572% for the pathwise estimators. Although the pathwise estimators show greater deviations for Rho, they still provide results within a reasonable range.

The pathwise estimators demonstrate several significant advantages. Firstly, they do not heavily rely on the assumptions of the Black-Scholes model, making them adaptable to different underlying stochastic processes. This flexibility is crucial in markets where conditions deviate from Black-Scholes assumptions. Additionally, these estimators can be applied to a wide range of derivative instruments beyond standard options, including exotic instruments. This broad applicability makes them valuable tools for various types of financial instruments. Pathwise estimators can provide more accurate risk measures in models where market conditions do not follow these assumptions, which is essential for effective risk management and trading strategies.

Despite the larger deviations observed in some cases, pathwise estimators still provide reliable estimates that are close to market values. Their flexibility and adaptability make them valuable tools in modern financial practice, complementing traditional methods such as the Black-Scholes model. This comprehensive analysis underscores the effectiveness of pathwise estimators in real-world scenarios, highlighting their potential for broader applications in financial modeling and risk management.

Chapter 6

Advanced Option Pricing: The CEV and Local Volatility Models

6.1 The CEV Model

The constant elasticity of variance (CEV) model is a specific type of parametric local volatility model introduced by Cox in 1975. The risk-neutral dynamics of this model are described by the equation

$$dS_t = rS_t dt + \sigma S_t^{\gamma} dW_t \tag{6.1}$$

where r is the risk-free rate, σ and γ in $[0, 1]$ are the model parameters. Notably, the CEV model generalizes the Geometric Brownian Motion (GBM) model, which is a special case obtained by setting $\gamma = 1$.

By dividing by S_t we can rewrite (6.1) as

$$\frac{dS_t}{S_t} = rdt + \sigma S_t^{\gamma-1} dW_t.$$

This equation highlights that, when $\gamma < 1$, there is an inverse relationship between the asset price and its instantaneous volatility. Consequently, the CEV model is capable of capturing the volatility skew observed in empirical financial data. Additionally, when $\gamma < \frac{1}{2}$, there exists a positive probability that the asset price will hit zero, reflecting scenarios of significant downward price movements.

6.1.1 European Call and Put Option in the CEV Model

In the CEV model, the valuation formulas for European call and put options are

$$\begin{aligned} c &= S_0 \left[1 - \chi^2(a; b+2, c)\right] - Ke^{-rT}\chi^2(c; b, a) \\ p &= Ke^{-rT}\left[1 - \chi^2(c; b, a)\right] - S_0\chi^2(a; b+2, c)\,, \end{aligned}$$

for $0 < \gamma < 1$

$$\begin{aligned} c &= S_0 \left[1 - \chi^2(c; -b, a)\right] - Ke^{-rT}\chi^2(a; 2-b, c) \\ p &= Ke^{-rT}\left[1 - \chi^2(a; 2-b, c)\right] - S_0\chi^2(c; -b, a)\,. \end{aligned}$$

For $\gamma > 1$, the parameters are defined as follows

$$\begin{aligned} a &= \frac{Ke^{-(r-q)T}2^{(1-\gamma)}}{(1-\gamma)^2\nu} \\ b &= \frac{1}{1-\gamma} \\ c &= \frac{S_0^{2(1-\gamma)}}{(1-\gamma)^2\nu} \\ \nu &= \frac{\sigma^2\left[e^{2(r-q)(\gamma-1)T} - 1\right]}{2(r-q)(\gamma-1)} \end{aligned}$$

Here, $\chi^2(z, k, \nu)$ represents the probability that a non-central chi-square variable with k degrees of freedom and non-centrality parameter ν assumes a value less than z. The CEV model is particularly useful for valuing exotic options on stocks. The model parameters can be chosen to minimize the sum of squared deviations between theoretical prices and market prices of standard options.

6.2 The Volatility Surface

The Black-Scholes model, despite its effectiveness, has several limitations when applied in practice. For instance, stock prices often exhibit sudden jumps and do not always follow the smooth paths predicted by the Geometric Brownian Motion (GBM) model. Additionally, stock prices tend to have fatter tails than those predicted by GBM.

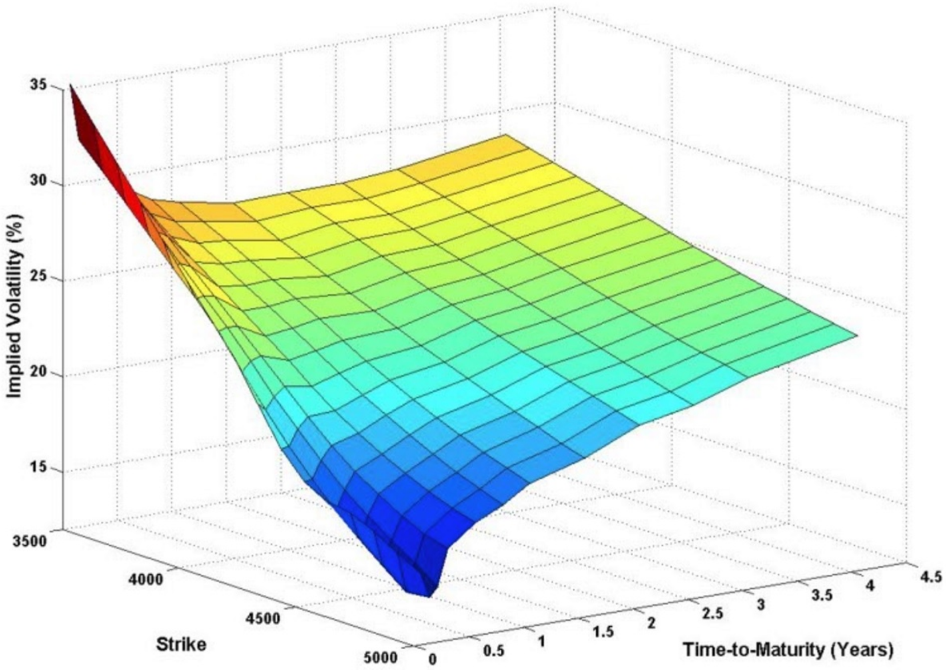

Fig. 6.1: Volatility Surface for the Eurostoxx 50 on 27/11/07.

Source: http://www.bsam.com/using-the-volatility-surface-to-estimate-expected-returns/

The volatility surface is a function of the strike price, K, and the time-to-maturity, T. It is implicitly defined by:

$$C(S, K, T) = BS(S, T, r, q, K, \sigma(K, T)),$$

where $C(S, K, T)$ represents the current market price of a call option with strike K and maturity T, and $BS(\cdot)$ is the Black-Scholes formula for pricing a call option. Here, $\sigma(K, T)$ is the implied volatility that, when substituted into the Black-Scholes formula, equates to the market price $C(S, K, T)$. Since the Black-Scholes formula is continuous and increases with σ, there will always be a unique implied volatility $\sigma(K, T)$.

If the Black-Scholes model were entirely accurate, the volatility surface would be flat with $\sigma(K, T) = \sigma$ for all K and T. In reality, the volatility surface is not flat and varies significantly over time as we can see from Figure 6.1 One of the

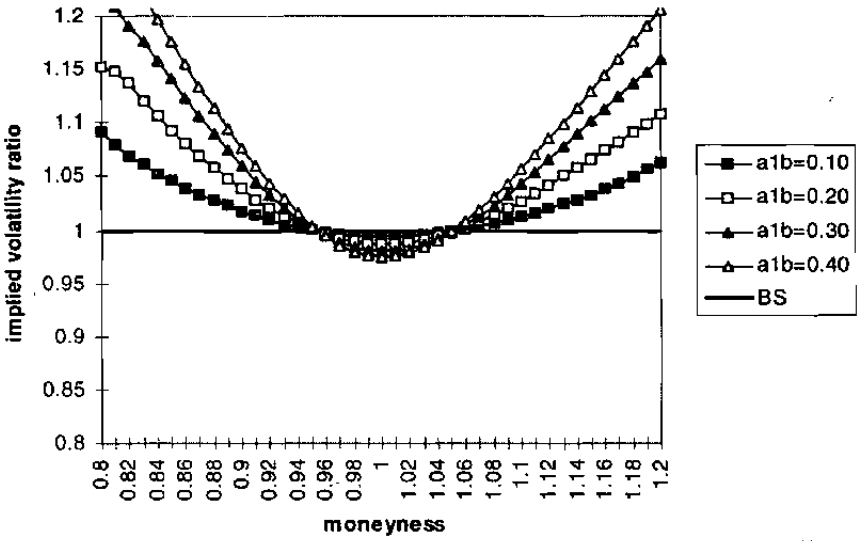

Fig. 6.2: Volatility Smile

Source: `https://www.researchgate.net/figure/Implied-volatility-ratios-Time-to-maturity-I-month-ala-0-Figures-2-and-3-show-the_fig2_45137175`

main characteristics of the volatility surface is that options with lower strike prices generally have higher implied volatilities. For a fixed maturity, T, this phenomenon is known as the volatility smile (see Figure 6.2). For a given strike price, K, the implied volatility can either increase or decrease with time-to-maturity. Generally, however, $\sigma(K, T)$ tends to converge to a constant as $T \to \infty$. In contrast, for short-term options, particularly during periods of market stress, we often observe an inverted volatility surface where short-term options exhibit much higher volatilities than those with longer maturities.

6.3 Local Volatility Model: the Dupire Formula

The Geometric Brownian Motion (GBM) model for stock prices is represented by the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

where r and σ are constants. The model has a single free parameter, σ, which can be calibrated to fit option prices or, equivalently, the volatility surface. However, this approach often fails because the volatility surface is typically not flat, making a constant σ insufficient to replicate market prices accurately. This became especially evident after the 1987 market crash, as it highlighted the necessity for modeling the skew, i.e., the observation that lower strike options are associated with higher implied volatilities. In response to this need, researchers proposed various alternative models to address the volatility skew. One of the simplest extensions of the Black-Scholes model is the local volatility model. This model assumes that the stock's risk-neutral dynamics are governed by

$$dS_t = rS_t dt + \sigma_l(t, S_t) S_t dW_t,$$

where the instantaneous volatility $\sigma_l(t, S_t)$ is a function of both time and the stock price. A key result within the local volatility model is the Dupire formula, which connects local volatilities $\sigma_l(t, S_t)$ to the implied volatility surface observed in the market.

Theorem 6.3.1 (Dupire Formula)**.** *Let C = C(K, T) be the price of a call option as a function of strike and time-to-maturity. Then the local volatility function satisfies*

$$\sigma_t^2(T,K) = \frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K} + C}{\frac{K^2}{2}\frac{\partial^2 C}{\partial K^2}}$$

Proof. Below a simple proof of the Dupire formula, where we will change the notation for derivatives, that is, given a function $f(x,t)$, we will express the partial derivatives like $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$, respectively, as f_x and f_{xx}, and so on.

Let us start from the Fokker-Plank equation which describes the evolution of the probability density function (PDF) of the state of a stochastic process. Consider a stock price S_t that follows a geometric Brownian motion given by the following SDE

$$dS_t = rS_t dt + \sigma(t,S_t)S_t dW_t\,.$$

The corresponding PDF $p(S_t,t)$ of the stock price a time t follows the Fokker-Planck equation

$$\frac{\partial p(S_t,t)}{\partial t} = -\frac{\partial}{\partial S}\left[rS_t p(S_t,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial S^2}\left[\sigma^2 S_t^2 p(S_t,t)\right]\,.$$

Evaluating the PDF at $S_t = y$, we get

$$-p_t(y,t) - r(yp(y,t))_y + \frac{1}{2}(\sigma^2(t,y)y^2p(y,t))_{yy} = 0 \quad \text{for } t>0 \qquad (6.2)$$

Imposing the initial condition $p(y,t) = \delta_{S_0}(y)$ at $t=0$, we can rewrite the call option price as

$$C(K,T) = e^{-rT}\mathbb{E}_0[(S_T-K)^+] = e^{-rT}\int_K^\infty (y-K)p(y,T)\,dy.$$

We want to differentiate twice this equation, let us start from the second term, so

$$\begin{aligned}\frac{\partial}{\partial K}\left(\int_K^\infty (y-K)p(y,T)\,dy\right) &= \frac{\partial}{\partial K}\left(\int_K^\infty yp(y,T)\,dy - K\int_K^\infty p(y,T)\,dy\right)\\ &= -Kp(K,T) - \left(\int_K^\infty p(y,T)\,dy - Kp(K,T)\right)\\ &= -\int_K^\infty p(y,T)\,dy.\end{aligned}$$

To compute this integral, we use the fundamental theorem of calculus.[1] Prosecuting on our calculus we have

$$\begin{aligned}\frac{\partial^2}{\partial K^2}\left(\int_K^\infty (y-K)p(y,T)\,dy\right) &= \frac{\partial}{\partial K}\left(-\int_K^\infty p(y,T)\,dy\right)\\ &= p(K,T),\end{aligned}$$

hence

$$C_{KK}(K,T) = e^{-rT}p(K,T).$$

Differentiating again $C_{KK}(K,T)$ with respect to T we have

$$\begin{aligned}C_{KKT}(K,T) &= -re^{-rT}p(K,T) + e^{-rT}p_T(K,T)\\ &= -rC_{KK}(K,T) + e^{-rT}p_T(K,T)\,.\end{aligned}$$

We can observe from (6.2) that $p_T(K,T) = \left(\frac{1}{2}(\sigma^2(T,K)K^2p(K,T))_{KK} - r(Kp(K,T))_K\right)$ and hence

$$\begin{aligned}C_{KKT}(K,T) + rC_{KK}(K,T) &= e^{-rT}p_T(K,T)\\ &= e^{-rT}\left(\frac{1}{2}(\sigma^2(T,K)K^2p_T(K,T))_{KK} - r(Kp_T(K,T))_K\right)\\ &= \frac{1}{2}e^{-rT}(\sigma^2(T,K)K^2p_T(K,T))_{KK} - re^{-rT}(Kp_T(K,T))_K\\ &= \frac{1}{2}(\sigma^2(T,K)K^2e^{-rT}p_T(K,T))_{KK} - r(Ke^{-rT}p_T(K,T))_K\,.\end{aligned}$$

Since $e^{-rT}p_T(K,T) = C_{KK}(K,T)$, we have

$$C_{KKT}(K,T) + rC_{KK}(K,T) = \frac{1}{2}(\sigma^2(T,K)K^2C_{KK}(K,T))_{KK} - r(KC_{KK}(K,T))_K.$$

Integrating the last equation with respect to K we get

$$C_{KT}(K,T) + rC_K(K,T) - \frac{1}{2}(\sigma^2(T,K)K^2C_{KK})_K + r(KC_{KK}) = h(T)\,,$$

[1]Given a continuous function f and given $a \in \mathbb{R}$, the function $\Phi(x) = \int_a^x f(t)dt$ is continuous and $\frac{d}{dx}\Phi(x) = f(x)$. As a consequence, if $\Phi(x) = \int_x^a f(t)dt$, we have $\frac{d}{dx}\Phi(x) = -f(x)$.

for some function $h(T)$. Since

$$(KC_K(K,T))_K = C_K(K,T) + KC_{KK}(K,T),$$

we get

$$KC_{KK}(K,T) = (KC_K(K,T))_K - C_K(K,T),$$

and hence

$$C_{KT}(K,T) - \frac{1}{2}(\sigma^2(T,K)K^2C_{KK}(K,T))_K + r(KC_K(K,T))_K = h(T).$$

Integrating again the last equation with respect to K we have

$$C_T(K,T) - \frac{1}{2}\sigma^2(T,K)K^2C_{KK}(K,T) + r(KC_K(K,T)) = h(T)K + g(T)\,,$$

for some function $g(T)$. As $K \to \infty$, the call option price C and its derivatives C_T, C_K, and C_{KK} all tend to zero because the option value becomes negligible for very high strike prices. This implies that the left-hand side of equation also tends to zero as $K \to \infty$ and the right-hand side also tend to zero. Given that $h(T)K+g(T)$ must tend to zero as $K \to \infty$, it follows that $h(T) = 0$ and $g(T) = 0$, and hence we have

$$C_T(K,T) - \frac{1}{2}\sigma^2(T,K)K^2C_{KK}(K,T) + r(KC_K(K,T)) = 0\,,$$

that implies

$$\frac{1}{2}\sigma^2(T,K)K^2C_{KK}(K,T) = C_T(K,T) + r(KC_K(K,T))\,.$$

So we can deduce the thesis

$$\sigma^2(T,K) = \frac{C_T(K,T) + r(KC_K(K,T))}{\frac{K^2}{2}C_{KK}(K,T)}\,.$$

□

6.4 Relationship between CEV and Local Volatility Models

The Constant Elasticity of Variance (CEV) model can be interpreted as a special case of the local volatility model. In the CEV model, the volatility of the underlying asset is not constant but instead depends on the level of the asset price itself. Specifically, the local volatility in the CEV model is expressed as

$$\sigma_{\text{loc}}(S) = \sigma S^{\gamma-1},$$

where $\sigma > 0$ is a scaling parameter and γ is the elasticity parameter. This formulation implies that the local volatility changes deterministically when the asset price is known, making the CEV model a particular representation of how local volatility can be modeled as a function of the current underlying price.

Conclusion

In this thesis, we have explored deeply into the complex dynamics of financial derivatives, with a primary focus on options and their valuation using advanced mathematical methods such as Monte Carlo simulations and pathwise estimators. The thesis provides a thorough examination and comparison of these sophisticated tools against the traditional Black-Scholes model, validated through empirical data obtained from the Bloomberg Terminal.

Monte Carlo methods, renowned for their flexibility and robustness, allow for the accurate modeling of complex stochastic processes by generating numerous potential paths for an asset's price. These simulations facilitate a comprehensive analysis of options pricing and the associated risks. The algorithm developed in this thesis demonstrates high precision in estimating option prices and Greeks under diverse market conditions, and its effectiveness is validated against real-world data.

Pathwise estimators, offer an alternative to differential equations for computing Greeks. Unlike the Black-Scholes model, which relies on several assumptions, pathwise estimators do not heavily depend on these assumptions, making them adaptable to different underlying stochastic processes. This adaptability is particularly advantageous in markets where conditions deviate from classical assumptions, thereby enhancing risk management and trading strategies.

The empirical analysis revealed that both Monte Carlo simulations and pathwise estimators provide accurate and reliable estimates for option prices and Greeks. Interestingly, pathwise estimators often showed lower deviations from market data, especially in scenarios where the Black-Scholes model's assumptions, such as constant volatility, do not hold. This underscores the potential of pathwise methods in more effectively capturing the dynamic nature of financial markets.

By providing precise estimates of Greeks, financial professionals can better manage the sensitivities of their portfolios to various market parameters. Moreover, the robustness of Monte Carlo simulations and pathwise estimators makes them suitable for integration into algorithmic trading systems, potentially enhancing the accuracy and performance of trading algorithms in dynamic and volatile market environments.

The rigorous benchmarking against Bloomberg data ensures that the models developed in this thesis are not only theoretically relevant but also practically effective. Furthermore, the comprehensive exploration of both classical and modern methods for options pricing and Greeks calculation, provide a solid foundation for further thesis and development in this field.

Additionally, the incorporation of the local volatility model through the Dupire formula added another layer of sophistication to the analysis. This model adeptly captures the variability of implied volatility, offering a more detailed and accurate reflection of market conditions. By considering the volatility surface, the local volatility model enhances the precision and adaptability of options pricing methodologies, providing significant practical benefits.

Despite the significant contributions of this thesis, several path for future research remain. Extending these methodologies to price and manage the risks of exotic options, which have more complex features than standard options, could further demonstrate the potential of Monte Carlo simulations and pathwise estimators. Further research could also explore more sophisticated volatility models, such as stochastic volatility and jump-diffusion models, to capture the full spectrum of market behaviors. Comparing these models with the local volatility approach could yield deeper insights into their relative strengths and weaknesses.

In conclusion, this thesis underscores the importance of advanced mathematical methods in the valuation of financial derivatives. By bridging the gap between theoretical models and market realities, the thesis provides valuable insights that are both academically enriching and practically significant. The Monte Carlo simulations and pathwise estimators developed here represent powerful tools for financial professionals, offering enhanced accuracy, flexibility, and robustness in a rapidly evolving market.

Appendix

In this appendix basic theorem and results used in the thesis are presented.

A.1 The Normal Random Variable

X is a normal (or Gaussian) variable with parameters μ and σ^2, and we write

$$X \sim \mathcal{N}(\mu, \sigma^2),$$

if X has the density function

$$f_X(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}},$$

for every $t \in \mathbb{R}$. It is not possible to explicitly calculate the distribution function, but we will soon see how to derive its values. The graph of f_X corresponds to a bell curve with a peak at the level of μ. The average width of the bell is σ. It can also be shown that

$$\mathbb{E}[X] = \mu, \quad \mathrm{Var}(X) = \sigma^2.$$

Now consider the variable

$$Z = \frac{X - \mu}{\sigma}.$$

We note that

$$\mathbb{E}[Z] = \frac{\mathbb{E}[X] - \mu}{\sigma} = 0,$$

$$\mathrm{Var}(Z) = \frac{1}{\sigma^2}\mathrm{Var}(X) = 1.$$

Let us calculate the distribution function of Z.

$$F_Z(t) = \mathbb{P}(Z \leq t) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \leq t\right) = \mathbb{P}(X \leq \sigma t + \mu) = F_X(\sigma t + \mu).$$

Consequently, regarding the density of Z, we have

$$f_Z(t) = \frac{d}{dt}F_Z(t) = \frac{d}{dt}F_X(\sigma t + \mu) = f_X(\sigma t + \mu) \cdot \sigma = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}},$$

where we have used the chain rule for differentiation, and the definition of the density of $X \sim \mathcal{N}(\mu, \sigma^2)$ evaluated at the point $\sigma t + \mu$. In particular, we have obtained that $Z \sim \mathcal{N}(0,1)$. The variable Z is called the standard normal variable (or standard Gaussian). Its distribution function is denoted by Φ (instead of F_Z) and is given by

$$\Phi(t) = \mathbb{P}(Z \leq t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\, ds.$$

The values of Φ are computed numerically and they are provided in a table (known as the Gaussian table).
So if $X \sim \mathcal{N}(\mu, \sigma^2)$, to compute $F_X(t)$ it is enough to compute $F_Z\left(\frac{t-\mu}{\sigma}\right)$.

A.2 Limit Theorems

Proposition A.1 (Law of Large Numbers)**.** *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, each one with a finite mean μ. Then*

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{X_1 + \ldots + X_n}{n} = \mu\right) = 1\,,$$

or equivalently, almost surely

$$\lim_{n\to\infty} \frac{X_1 + \ldots + X_n}{n} = \mu\,.$$

Proposition A.2 (Central Limit Theorem)**.** *Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables with mean $\mu < \infty$ and variance $\sigma^2 < \infty$. Defined $S_n = \sum_{i=1}^n X_i$ and $Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$ its normalization, we have for every $t \in \mathbb{R}$*

$$F_{Z_n}(t) := \mathbb{P}(Z_n \leq t) \to \mathbb{P}(Z \leq t) \quad \textit{for } n \to \infty\,,$$

where $Z \sim \mathcal{N}(0,1)$. Equivalently, Z_n converges in distribution to $Z \sim \mathcal{N}(0,1)$ as $n \to \infty$.

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