Department of Economics and Finance

Chair of Gambling: Probability and Decision

Title

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To my family and friends,

Thank you for always believing in me, supporting me, and encouraging me to pursue my dreams. Your unwavering faith and love have been my greatest strength.

With all my gratitude,
Riccardo

Contents

Introduction

This thesis explores the concept of martingales, a mathematical model used to describe certain types of betting strategies, particularly in gambling. A martingale system is a system where a player increases their bet after each loss with the belief that a single win will recover all previous losses and yield a profit. Although the simplicity of the system can make it appear appealing, it carries significant risks, especially because it assumes the player has unlimited capital and that a win is inevitable.

The martingale strategy has its roots in gambling, where it is commonly applied in games like roulette. A player, for instance, might bet on red, and if they lose, they would double their next bet on red, repeating this process until they win. While this may work in theory, the stakes quickly escalate during long losing streaks, making it financially unsustainable for most. The mathematical underpinnings of this strategy are grounded in the study of stochastic processes, specifically martingales, which model fair games where the expected value of future outcomes equals the present value.

Beyond gambling, the martingale strategy has been adopted in financial markets, especially in forex trading. Traders use the same doubling mechanism after losses, believing that the market will eventually move in their favor. However, financial markets introduce additional complexities, such as leverage and market volatility, which can amplify both the potential rewards and risks. A prolonged losing streak in such markets can lead to significant financial damage, often much faster than in gambling scenarios due to the use of borrowed capital.

This thesis begins by outlining the mathematical foundation of martingales, explaining their role in stochastic processes and how they differ from related concepts like supermartingales and submartingales. These variations account for situations where the system is biased either against or in favor of the player, and understanding these distinctions is key to grasping the broader implications of such strategies. We will also discuss the Optional Stopping Theorem, which shows why quitting strategies like martingales cannot turn an unfair game into a fair one.

In addition to their mathematical framework, this thesis examines the practical applications and limitations of martingale strategies. While they offer a theoretical promise of recovery, their real-world application is fraught with risks. We will explore the emotional and psychological impact of using these strategies, as the growing stakes after losses can create intense stress, leading to irrational decision-making. Many individuals fall into the Gambler's Fallacy, believing that after several losses, a win is more likely, which often leads to further financial strain.

Finally, we will consider variations of the martingale system, such as the anti-martingale approach, where bets are increased after wins, and fixed martingale systems, which attempt to control the rapid escalation of bets. While these alternatives offer some risk mitigation, they remain risky overall. By analyzing the theoretical advantages and real-world pitfalls of martingales, this thesis aims to provide a comprehensive understanding of why such systems may seem attractive, but in practice, they can lead to significant financial dangers if not carefully managed.

Chapter 1

Martingales

In this chapter, we will explore stochasic processes, which are collections of random variables that evolves with respect to a change in a parameter such as time. Specifically, we will focus on martingales, a special class of stochastic processes that model fair games in the context of gambling.

1.1 Stochastic Processes and Filtration

Suppose we are at the casino and repeat a game many times. We are interest in tracking the total amount of money won over several rounds. By denoting X_i as the amount of money won in the i-th round, the total money won in the first n rounds is given by:

$$S_n = \sum_{i=1}^{n} X_i. \tag{1.1}$$

Thus, the total amount of money won during the rounds is described by the sequence of random variables $S_1, S_2, S_3, \ldots, S_n, \ldots$. The sequence of random variables $\{S_n\}_{n \in \mathbb{N}>0}$ is an example of a stochastic process because it describes the evolution of a quantity as the rounds progress. To determine S_n, we need to evaluate the values of $X_1, \ldots, X_n$. The series of random variables $\{X_n\}_{n\in\mathbb{N}>0}$ captures the information we gather throughout the rounds. This series is known as a **filtration**.

Generally, a **stochastic process** consists of a sequence of random variables and typically describes the progression of a random quantity as a parameter changes, often representing time or the number of rounds. A filtration is a series of random variables that embodies the information collected while observing the development of a stochastic process. In the next section, we will focus on a specific type of stochastic process called a martingale.

1.2 Martingales, Supermartingales and Submartingales

Suppose we flip a coin many times and each time you win 1 euro if it lands on heads and lose 1 euro if it lands on tails. Assume you start with an initial fixed capital $S_0 > 0$ and that the probability of heads is $p \in [0, 1]$. Let X_i denote the amount of money won in the i-th round. Therefore,

$$X_i = \begin{cases} 1, & \text{with probability } p, \\ -1, & \text{with probability } 1-p, \end{cases} \tag{1.2}$$

and $\{X_n\}_{n\in\mathbb{N}}$ is a family of i.i.d. random variables. Let S_n be the capital after n rounds. Thus,

$$S_n = S_0 + \sum_{i=1}^{n} X_i.$$

Assume we have finished the n-th round. We want to determine our average capital at round $n + 1$. Since we have already played n rounds, we know the exact values of $X_1, \ldots, X_n$. Therefore, we aim to compute

$$\mathbb{E}[S_{n+1} \mid X_1, \ldots, X_n].$$

Note that, since we know the values of $X_1, \dots, X_n$, we also know the value of $S_n = S_0 + \sum_{i=1}^n X_i$. Therefore,

$$S_{n+1} = S_0 + \sum_{i=1}^{n+1} X_i = S_0 + \sum_{i=1}^{n} X_i + X_{n+1} = S_n + X_{n+1}.$$

The only unknown part in S_{n+1} is X_{n+1}. So

$$\mathbb{E}[S_{n+1} \mid X_1, \dots, X_n] = \mathbb{E}[S_n + X_{n+1} \mid X_1, \dots, X_n] =$$

$$= \mathbb{E}[S_n \mid X_1, \dots, X_n] + \mathbb{E}[X_{n+1} \mid X_1, \dots, X_n].$$

Since S_n is known when knowing $X_1, \dots, X_n$, then $\mathbb{E}[S_n \mid X_1, \dots, X_n] = S_n$. Moreover, since $\{X_n\}_{n\in\mathbb{N}} > 0$ are independent, we have $\mathbb{E}[X_{n+1} \mid X_1, \dots, X_n] = \mathbb{E}[X_{n+1}]$. So the previous equation becomes

$$\mathbb{E}[S_{n+1} \mid X_1, \dots, X_n] = \mathbb{E}[S_n \mid X_1, \dots, X_n] + \mathbb{E}[X_{n+1} \mid X_1, \dots, X_n] =$$

$$= S_n + \mathbb{E}[X_{n+1}] = S_n + p - (1-p) = S_n + 2p - 1.$$

So we have:

- if $p = \frac{1}{2}$, the coin is fair (hence the game is fair since $\mathbb{E}[X_i] = 0$) and

  $$\mathbb{E}[S_{n+1} \mid X_1, \dots, X_n] = S_n,$$

  that is, the average capital that we will have in the future round is what we have now. This is the main property of the martingale.

- if $p < \frac{1}{2}$, the coin is not fair and the game is subfair since $\mathbb{E}[X_i] < 0$. Moreover

  $$\mathbb{E}[S_{n+1} \mid X_1, \dots, X_n] < S_n,$$

  that is, the average capital that we will have in the future round is less than what we have now. This is the main property of the supermartingale.

So we have:

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Let us describe better these stochastic processes. Note that $\{X_n\}_{n\in\mathbb{N}_0}$ represents a filtration, that is the information that we accumulate during the rounds, in order to evaluate the stochastic process $\{S_n\}_{n\in\mathbb{N}_0}$.

Definition 1.1. *Given two sequences of random variables $\{M_n\}_{n\in\mathbb{N}_0}$ and $\{Y_n\}_{n\in\mathbb{N}_0}$, we say that $\{M_n\}_{n\in\mathbb{N}_0}$ is a* ***martingale*** *with respect to the filtration $\{Y_n\}_{n\in\mathbb{N}_0}$ if the following three conditions are satisfied:*

(i) $\mathbb{E}[|M_n|] < \infty$ *for any fixed $n \in \mathbb{N} > 0$;*

(ii) $\{M_n\}_{n\in\mathbb{N}_0}$ *is* ***adapted*** *to the filtration $\{Y_n\}_{n\in\mathbb{N}_0}$, that is, knowing the value of $Y_1, \ldots, Y_n$, we know also the value of M_n;*

(iii) $\mathbb{E}[M_{n+1} \mid Y_1, \ldots, Y_n] = M_n$.

Definition 1.2. *Given two sequences of random variables* $\{M_n\}_{n\in\mathbb{N}_0}$ *and* $\{Y_n\}_{n\in\mathbb{N}_0}$*, we say that* $\{M_n\}_{n\in\mathbb{N}_0}$ *is a* ***supermartingale*** *with respect to the filtration* $\{Y_n\}_{n\in\mathbb{N}_0}$ *if the following three conditions are satisfied:*

(i) $\mathbb{E}[|M_n|] < \infty$ *for any fixed* $n \in \mathbb{N} > 0$*;*

(ii) $\{M_n\}_{n\in\mathbb{N}_0}$ *is adapted to the filtration* $\{Y_n\}_{n\in\mathbb{N}_0}$*, that is, knowing the value of* $Y_1, \ldots, Y_n$*, we know also the value of* M_n*;*

(iii) $\mathbb{E}[M_{n+1} \mid Y_1, \ldots, Y_n] \leq M_n$.

Definition 1.3. *Given two sequences of random variables* $\{M_n\}_{n\in\mathbb{N}_0}$ *and* $\{Y_n\}_{n\in\mathbb{N}_0}$*, we say that* $\{M_n\}_{n\in\mathbb{N}_0}$ *is a* ***submartingale*** *with respect to the filtration* $\{Y_n\}_{n\in\mathbb{N}_0}$ *if the following three conditions are satisfied:*

(i) $\mathbb{E}[|M_n|] < \infty$ *for any fixed* $n \in \mathbb{N} > 0$*;*

(ii) $\{M_n\}_{n\in\mathbb{N}_0}$ *is adapted to the filtration* $\{Y_n\}_{n\in\mathbb{N}_0}$*, that is, knowing the value of* $Y_1, \ldots, Y_n$*, we know also the value of* M_n*;*

(iii) $\mathbb{E}[M_{n+1} \mid Y_1, \ldots, Y_n] \geq M_n$.

Note that if $\{M_n\}_{n\in\mathbb{N}>0}$ is a supermartingale and a submartingale, both with respect to the filtration $\{Y_n\}_{n\in\mathbb{N}>0}$, then $\{M_n\}_{n\in\mathbb{N}>0}$ is a martingale with respect to the filtration $\{Y_n\}_{n\in\mathbb{N}>0}$.

Proposition 1.2.1. *Given two sequences of random variables* $\{M_n\}_{n\in\mathbb{N}>0}$ *and* $\{Y_n\}_{n\in\mathbb{N}>0}$*, we can state the following:*

- *if* $\{M_n\}_{n\in\mathbb{N}>0}$ *is a martingale with respect to the filtration* $\{Y_n\}_{n\in\mathbb{N}>0}$*, then* $\mathbb{E}[M_n]$ *remains constant for all* n*, meaning*

$$\mathbb{E}[M_{n+1}] = \mathbb{E}[M_n] = \cdots = \mathbb{E}[M_1];$$

- *if* $\{M_n\}_{n\in\mathbb{N}>0}$ *is a supermartingale with respect to the filtration* $\{Y_n\}_{n\in\mathbb{N}>0}$*, then* $\mathbb{E}[M_n]$ *is non-increasing as* n *increases, meaning*

$$\mathbb{E}[M_{n+1}] \leq \mathbb{E}[M_n] \leq \cdots \leq \mathbb{E}[M_1];$$

- *if* $\{M_n\}_{n\in\mathbb{N}>0}$ *is a submartingale with respect to the filtration* $\{Y_n\}_{n\in\mathbb{N}>0}$*, then* $\mathbb{E}[M_n]$ *is non-decreasing as* n *increases, meaning*

$$\mathbb{E}[M_{n+1}] \geq \mathbb{E}[M_n] \geq \cdots \geq \mathbb{E}[M_1].$$

Proof. If $\{M_n\}_{n\in\mathbb{N}>0}$ is a martingale with respect to the filtration $\{Y_n\}_{n\in\mathbb{N}>0}$, then by condition (iii) in Definition 1.1, we have

$$M_n = \mathbb{E}[M_{n+1} \mid Y_1, \ldots, Y_n].$$

Taking the expectation of both sides of this equality gives us

$$\mathbb{E}[M_n] = \mathbb{E}[\mathbb{E}[M_{n+1} \mid Y_1, \ldots, Y_n]].$$

Using the Tower Property for the second member of the above identity, we get

$$\mathbb{E}[M_n] = \mathbb{E}[M_{n+1}].$$

Iterating this identity up to $n = 1$, we obtain the thesis. For the cases of a supermartingale and a submartingale, the computation is identical, but we start from property (iii) in Definition 1.2 and Definition 1.3, respectively. □

Example 1.1. *Suppose to repeat the same wager many times, and at each turn you can win* a *euros (where* $a > 0$*) with probability* p*, lose* b *euros (where* $b > 0$*) with probability* q*, or break even with probability* r*, where* $p+q+r = 1$*. We denote by* X_i *the amount won in the* i*-th round, that is*

$$X_i = \begin{cases} a, & \text{with probability } p, \\ 0, & \text{with probability } r, \\ -b, & \text{with probability } q. \end{cases}$$

and $\{X_n\}_{n\in\mathbb{N}_0}$ *is a sequence of independent and identically distributed random variables. Denote by* S_n *the total capital obtained in* n *rounds and by* S_0 *the initial capital (that we assume to be a fixed positive constant). Thus,*

$$S_n = S_0 + \sum_{i=1}^{n} X_i.$$

We want to investigate the stochastic process $\{S_n\}_{n\in\mathbb{N}_0}$ *in order to verify if it is a (super/sub)martingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}_0}$. *Note first that applying the Triangular Inequality (i.e., Proposition 3.2.4) we have*

$$\begin{aligned}\mathbb{E}[|S_n|] &\leq \mathbb{E}\left[\left|S_0 + \sum_{i=1}^{n} X_i\right|\right] \\ &= S_0 + \sum_{i=1}^{n} \mathbb{E}[|X_i|],\end{aligned} \tag{1.3}$$

where we have used the fact that $S_0 > 0$ *by initial assumption. Since* $\{X_n\}_{n\in\mathbb{N}_0}$ *are identically distributed, they also have the same expectation and hence the above expression becomes*

$$\mathbb{E}[|S_n|] \leq S_0 + n\mathbb{E}[|X_1|], \tag{1.4}$$

Note that

$$\mathbb{E}[|X_1|] = |a| \cdot p + |0| \cdot r + |-b| \cdot q = ap + bq, \tag{1.5}$$

where we have used the fact that $a, b > 0$ *and hence* $-b$ *is negative and its absolute value is* b. *So we have*

$$\mathbb{E}[|S_n|] \leq S_0 + n\mathbb{E}[|X_1|] = S_0 + n \cdot (ap + bq) < \infty, \tag{1.6}$$

for any fixed $n \in \mathbb{N}_0$. *So we have proved the first property of (super/sub)martingales. Let us investigate the second one: we have to verify that, given the values of* $X_1, \ldots, X_n$, *we know the value of* S_n, *that is* $\{S_n\}_{n\in\mathbb{N}_0}$ *is adapted to the filtration* $\{X_n\}_{n\in\mathbb{N}_0}$. *This is true since, given the values of* $X_1, \ldots, X_n$, *to compute* S_n *it is enough to compute* $S_0 + \sum_{i=1}^{n} X_i$ *(recall that* S_0 *is a fixed constant that we know* *from the start). So we have validated the second property. Let us now examine the third one, that is, we have to investigate* $\mathbb{E}[S_{n+1}|X_1,\ldots,X_n]$*. Note that, since we know the values of* $X_1,\ldots,X_n$*, we also know the value of* $S_n = S_0 + \sum_{i=1}^n X_i$*. Hence, since*

$$\begin{aligned} S_{n+1} &= S_0 + \sum_{i=1}^{n+1} X_i \\ &= S_0 + \sum_{i=1}^{n} X_i + X_{n+1} \\ &= S_n + X_{n+1}, \end{aligned} \tag{1.7}$$

the only unknown part in S_{n+1} *is* X_{n+1}*. So*

$$\begin{aligned} \mathbb{E}[S_{n+1}|X_1,\ldots,X_n] &= \mathbb{E}[S_n + X_{n+1}|X_1,\ldots,X_n] \\ &= \mathbb{E}[S_n|X_1,\ldots,X_n] + \mathbb{E}[X_{n+1}|X_1,\ldots,X_n]. \end{aligned} \tag{1.8}$$

Since S_n *is known when knowing* $X_1,\ldots,X_n$*, then* $\mathbb{E}[S_n|X_1,\ldots,X_n] = S_n$*. Moreover, since* $\{X_n\}_{n\in\mathbb{N}_0}$ *are independent, we have* $\mathbb{E}[X_{n+1}|X_1,\ldots,X_n] = \mathbb{E}[X_{n+1}]$*. So the previous equation becomes*

$$\begin{aligned} \mathbb{E}[S_{n+1}|X_1,\ldots,X_n] &= \mathbb{E}[S_n|X_1,\ldots,X_n] + \mathbb{E}[X_{n+1}|X_1,\ldots,X_n] \\ &= S_n + \mathbb{E}[X_{n+1}] \\ &= S_n + ap - bq. \end{aligned} \tag{1.9}$$

So

- *if* $\mathbb{E}[X_1] = ap - bq = 0$*,* $\{S_n\}_{n\in\mathbb{N}_0}$ *is a martingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}_0}$*;*
- *if* $\mathbb{E}[X_1] = ap - bq < 0$*,* $\{S_n\}_{n\in\mathbb{N}_0}$ *is a supermartingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}_0}$*;*
- *if* $\mathbb{E}[X_1] = ap - bq > 0$*,* $\{S_n\}_{n\in\mathbb{N}_0}$ *is a submartingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}_0}$*.*

Note that the above considerations are independent of the probability of a tie r.

Example 1.2. *We consider a slight modification of the previous example: suppose that at each round you have to pay a fee that equals the expected gain in each round. More precisely, our gain after n rounds is*

$$S_n = S_0 + \sum_{i=1}^{n} X_i - n \cdot (ap - bq), \tag{1.10}$$

where $ap - bq$ is the fee paid in each round. We want to demonstrate that the process $\{S_n\}_{n\in\mathbb{N}_0}$ is a martingale with respect to the filtration $\{X_n\}_{n\in\mathbb{N}_0}$ for any possible value for a, b, p, q, r. Note first that by applying the Triangular Inequality we have

$$\begin{aligned} \mathbb{E}[|S_n|] &\leq \mathbb{E}\left[\left|S_0 + \sum_{i=1}^{n} X_i + |n \cdot (ap - bq)|\right|\right] \\ &= S_0 + \sum_{i=1}^{n} \mathbb{E}[|X_i|] + n \cdot |ap - bq|, \end{aligned} \tag{1.11}$$

where we have employed the fact that $S_0 > 0$ by initial assumption. Since $\{X_n\}_{n\in\mathbb{N}_0}$ are uniformly distributed, they also have the same expectation, and hence the above expression becomes

$$\mathbb{E}[|S_n|] \leq S_0 + n\mathbb{E}[|X_1|] + n \cdot |ap - bq|, \tag{1.12}$$

Note that

$$\mathbb{E}[|X_1|] = |a| \cdot p + |0| \cdot r + |-b| \cdot q = ap + bq, \tag{1.13}$$

where we have utilized the fact that $a, b > 0$ and hence $-b$ is negative and its magnitude is b. So we have

$$\mathbb{E}[|S_n|] \leq S_0 + n\mathbb{E}[|X_1|] = S_0 + n \cdot (ap + bq) + n \cdot |ap - bq| < \infty, \tag{1.14}$$

for any fixed $n \in \mathbb{N}_0$. So we have proven the first property of (super/sub)martingales. Let us search for the second one: we need to verify that, knowing the values of

$X_1, \ldots, X_n$, we know the value of S_n, that is $\{S_n\}_{n\in\mathbb{N}_0}$ is adapted to the filtration $\{X_n\}_{n\in\mathbb{N}_0}$. This is true since, given the values of $X_1, \ldots, X_n$, to compute S_n it is sufficient to compute $S_0 + \sum_{i=1}^n X_i - n\cdot(ap - bq)$ (recall that S_0 is a fixed constant that we know from the beginning). So we have validated the second property. Let us now examine the third one, that is, we need to study $\mathbb{E}[S_{n+1}|X_1, \ldots, X_n]$. Note that, since we know the values of $X_1, \ldots, X_n$, we also know the value of $S_n = S_0 + \sum_{i=1}^n X_i - n\cdot(ap - bq)$. Hence, since

$$\begin{aligned} S_{n+1} &= S_0 + \sum_{i=1}^{n+1} X_i - (n+1)(ap - bq) \\ &= S_0 + \sum_{i=1}^{n} X_i + X_{n+1} - (ap - bq) - n(ap - bq) \\ &= S_n + X_{n+1} - (ap - bq), \end{aligned} \tag{1.15}$$

the only unknown part in S_{n+1} is X_{n+1}. So

$$\begin{aligned} \mathbb{E}[S_{n+1}|X_1, \ldots, X_n] &= \mathbb{E}[S_n + X_{n+1} - (ap - bq)|X_1, \ldots, X_n] \\ &= \mathbb{E}[S_n|X_1, \ldots, X_n] + \mathbb{E}[X_{n+1}|X_1, \ldots, X_n] - (ap - bq). \end{aligned} \tag{1.16}$$

Since S_n is known when knowing $X_1, \ldots, X_n$, then $\mathbb{E}[S_n|X_1, \ldots, X_n] = S_n$. Furthermore, since $\{X_n\}_{n\in\mathbb{N}_0}$ are independent, we have $\mathbb{E}[X_{n+1}|X_1, \ldots, X_n] = \mathbb{E}[X_{n+1}]$. So the previous equation becomes

$$\begin{aligned} \mathbb{E}[S_{n+1}|X_1, \ldots, X_n] &= \mathbb{E}[S_n|X_1, \ldots, X_n] + \mathbb{E}[X_{n+1}|X_1, \ldots, X_n] - (ap - bq) \\ &= S_n + \mathbb{E}[X_{n+1}] - (ap - bq) \\ &= S_n + ap - bq - (ap - bq) \\ &= S_n. \end{aligned} \tag{1.17}$$

So we conclude that the stochastic process $\{S_n\}_{n\in\mathbb{N}_0}$ defined previously is a martingale with respect to the filtration $\{X_n\}_{n\in\mathbb{N}_0}$ for any possible value of a, b, p, q, r.

1.3 Optional Stopping Theorem

Let us consider the example discussed in (1.2) and define τ as the first time in which we get head, that is the first round i such that $X_i = 1$. Obviously τ is a random variable since depends on the outcomes of the tossings. Note that τ can be rewritten as

$$\tau = \inf\{i \in \mathbb{N}_{>0} | X_i = 1\} \tag{1.18}$$

from which we get

$$\{\tau = k\} = \{X_1 \neq 1, X_2 \neq 1, \ldots, X_{k-1} \neq 1, X_k = 1\}. \tag{1.19}$$

Indeed $\tau = k$ if and only if in the first $k-1$ tossings we have not got head, while at the k-th round we got head. We understand also that to establish if $\tau = k$, we need information up to round k. This last property defines what we call a stopping time.

Definition 1.4. *Given a filtration $\{X_n\}_{n\in\mathbb{N}_{>0}}$, we say that a discrete random variable τ is a **stopping time** with respect to the filtration $\{X_n\}_{n\in\mathbb{N}_{>0}}$ if it assumes values on $\mathbb{N}$ and to establish the occurrence of the event $\{\tau = k\}$ it is necessary to know only the values of $X_1, \ldots, X_k$. The random variable τ in 1.18 is a stopping time with respect to the filtration $\{X_n\}_{n\in\mathbb{N}_{>0}}$ since 1.19 holds. If we instead consider the random variable σ defined as the last time in which we get head, we get*

$$\{\sigma = k\} = \{X_k = 1, X_{k+1} \neq 1, X_{k+2} \neq 1, \ldots\}. \tag{1.20}$$

Indeed $\sigma = k$ if and only if at the k-th round we get head and we have no heads in the future rounds. Since to establish the occurrence of the event $\{\sigma = k\}$ we need to know future information (with respect to the k-th round), that is $X_k, X_{k+1}, X_{k+2}, \ldots$, we have that σ is not a stopping time with respect to the fil-

tration $\{X_n\}_{n\in\mathbb{N}_{>0}}$. *The next result shows the distribution of a particular stopping time. In particular this will prove that such a stopping time has finite expectation.*

Proposition 1.3.1. *Given a filtration* $\{X_n\}_{n\in\mathbb{N}_{>0}}$ *of i.i.d. random variables consider a value* $a \in Im(X_n)$*, that is* a *is a value assumed by* $\{X_n\}_{n\in\mathbb{N}_{>0}}$. *Let* $0 < p = P(X_n = a)$ *(and hence* $P(X_n \neq a) = 1 - p$*). Define*

$$\tau = \inf\{k \in \mathbb{N}_{>0} \mid X_k = a\},$$

$$\sigma = \inf\{k \in \mathbb{N}_{>0} \mid X_k \neq a\}.$$

Then τ *and* σ *are both stopping times with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}_{>0}}$ *and*

$$\tau \sim Geom(p), \quad \sigma \sim Geom(1-p).$$

Consequently, we have $\mathbb{E}[\tau] = \frac{1}{p} < \infty$ *and* $\mathbb{E}[\sigma] = \frac{1}{1-p} < \infty$.

Proof. Note that

$$\{\tau = n\} = \{X_1 \neq a, X_2 \neq a, \ldots, X_{n-1} \neq a, X_n = a\},$$

$$\{\sigma = n\} = \{X_1 = a, X_2 = a, \ldots, X_{n-1} = a, X_n \neq a\}.$$

Then

$$P(\tau = n) = P(X_1 \neq a, X_2 \neq a, \ldots, X_{n-1} \neq a, X_n = a),$$

$$P(\sigma = n) = P(X_1 = a, X_2 = a, \ldots, X_{n-1} = a, X_n \neq a).$$

Since $\{X_n\}_{n\in\mathbb{N}_{>0}}$ are i.i.d., then

$$\begin{aligned} P(\tau = n) &= P(X_1 \neq a) \cdot P(X_2 \neq a) \cdot \ldots \cdot P(X_{n-1} \neq a) \cdot P(X_n = a) \\ &= (1-p)^{n-1} \cdot p, \end{aligned} \tag{1.21}$$

$$\begin{aligned} P(\sigma = n) &= P(X_1 = a) \cdot P(X_2 = a) \cdot \ldots \cdot P(X_{n-1} = a) \cdot P(X_n \neq a) \\ &= p^{n-1} \cdot (1-p). \end{aligned} \tag{1.22}$$

Since $P(X_i = a) = p$ and $P(X_i \neq a) = 1 - p$, we have

$$P(\tau = n) = (1-p)^{n-1} \cdot p = (1-p)^{n-1} \cdot p,$$

$$P(\sigma = n) = p^{n-1} \cdot (1-p) = p^{n-1} \cdot (1-p),$$

from which we get the thesis. □

Remark 1. *Note that τ and σ are not bounded random variables; that is, we cannot claim with certainty that τ or σ are less than some specific constant. Indeed, a geometric random variable has image $\mathbb{N}_{>0}$, which is not a bounded set. However, τ and σ have finite expectations, meaning they are finite on average. Hence, we can conclude that τ and σ are not bounded random variables, but they are finite on average. Being geometric random variables, we can also assert that they are finite with probability one. Indeed,*

$$\mathbb{P}(\tau < \infty) = \sum_{n=1}^{\infty} \mathbb{P}(\tau = n) = \sum_{n=1}^{\infty} (1-p)^{n-1} p = p \sum_{n=1}^{\infty} (1-p)^{n-1}.$$

We can rewrite the above sum as

$$p \sum_{n=1}^{\infty} (1-p)^{n-1} = p \sum_{j=0}^{\infty} (1-p)^j = p \cdot \frac{1}{1-(1-p)} = 1,$$

where in the last identity we used the fact that, if $|a| < 1$, then $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$. Similar considerations apply to σ; it is sufficient to swap p with $1-p$.

Refer back to Proposition 1.2.1. This proposition establishes a relationship between the stochastic process at a fixed time n and at time 0. It is natural to inquire whether such a relationship holds for a random time τ instead of the fixed time n. The answer is provided by the Optional Stopping Theorem, which we state in the context of martingales, supermartingales, and submartingales. Note that in all three cases, the assumptions are the same.

Proposition 1.3.2. *(Optional Stopping Theorem for Martingales). Let $\{M_n\}_{n\in\mathbb{N}}$ be a martingale and let τ be a stopping time both with respect to the same filtration $\{X_n\}_{n\in\mathbb{N}}$. Assume that one of the following hypotheses is satisfied:*

(a) $\exists C > 0$ *such that* $\mathbb{P}(\tau < C) = 1$*, that is* τ *is bounded almost surely;*

(b) $\mathbb{P}(\tau < \infty) = 1$ *and* $\exists C > 0$ *such that* $\forall n \in \mathbb{N}$ *it holds* $\mathbb{P}(|M_\tau| \leq C) = 1$*, that is* τ *is finite almost surely and* $\{M_n\}_{n\in\mathbb{N}}$ *is uniformly bounded almost surely. The word "uniformly" is referred to the fact that the constant* C *does not depend on* n*;*

(c) $\mathbb{E}[\tau] < \infty$ *and* $\exists C > 0$ *such that* $\forall n \in \mathbb{N}$ *it holds* $\mathbb{P}(|M_{\tau\wedge(n+1)} - M_{\tau\wedge n}| \leq C) = 1$*, that is* τ *is finite in average and* $\{M_n\}_{n\in\mathbb{N}}$ *has uniformly bounded increments almost surely. The word "uniformly" is referred to the fact that the constant* C *does not depend on* n*;*

(d) $\mathbb{P}(\tau < \infty) = 1$ *and* $\forall n \in \mathbb{N}, M_n \geq 0$*, that is* τ *is finite almost surely and* $\{M_n\}_{n\in\mathbb{N}}$ *is a non-negative process.*

Then $\mathbb{E}[M_\tau] < \infty$ *and*

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0].$$

Proposition 1.3.3. *Let* $\{M_n\}_{n\in\mathbb{N}}$ *be a supermartingale and let* τ *be a stopping time both with respect to the same filtration* $\{X_n\}_{n\in\mathbb{N}}$*. Assume that one of the following hypotheses is satisfied*

(a) $\exists C > 0$ *such that* $\mathbb{P}(\tau < C) = 1$*, that is* τ *is bounded almost surely;*

(b) $\mathbb{P}(T < \infty) = 1$ *and* $\exists C > 0$ *such that* $\forall n \in \mathbb{N}$ *it holds* $\mathbb{P}(|M_n| \leq C) = 1$*, that is* τ *is finite almost surely and* $\{M_n\}_{n\in\mathbb{N}}$ *is uniformly bounded almost surely. The word "uniformly" is referred to the fact that the constant* C *does not depend on* n*;*

(c) $\mathbb{E}[\tau] < \infty$ *and* $\exists C > 0$ *such that* $\forall n \in \mathbb{N}$ *it holds* $\mathbb{P}(|M_{n+1} - M_n| \leq C) = 1$*, that is* τ *is finite in average and* $\{M_n\}_{n\in\mathbb{N}}$ *has uniformly bounded increments*

almost surely. The word "uniformly" is referred to the fact that the constant C does not depend on n;

(d) $\mathbb{P}(\tau < \infty) = 1$ *and* $\forall n \in \mathbb{N} M_n \geq 0$*, that is* τ *is finite almost surely and* $\{M_n\}_{n\in\mathbb{N}}$ *is a non-negative process.*

Then $\mathbb{E}[M_\tau] < \infty$ *and*

$$\mathbb{E}[M_\tau] \leq \mathbb{E}[M_0].$$

Proposition 1.3.4. *Let* $\{M_n\}_{n\in\mathbb{N}}$ *be a submartingale and let* τ *be a stopping time both with respect to the same filtration* $\{X_n\}_{n\in\mathbb{N}}$*. Assume that one of the following hypotheses is satisfied*

(a) $\exists C > 0$ *such that* $\mathbb{P}(\tau < C) = 1$*, that is* τ *is bounded almost surely;*

(b) $\mathbb{P}(T < \infty) = 1$ *and* $\exists C > 0$ *such that* $\forall n \in \mathbb{N}$ *it holds* $\mathbb{P}(|M_n| \leq C) = 1$*, that is* τ *is finite almost surely and* $\{M_n\}_{n\in\mathbb{N}}$ *is uniformly bounded almost surely. The word "uniformly" is referred to the fact that the constant* C *does not depend on* n;

(c) $\mathbb{E}[\tau] < \infty$ *and* $\exists C > 0$ *such that* $\forall n \in \mathbb{N}$ *it holds* $\mathbb{P}(|M_{n+1} - M_n| \leq C) = 1$*, that is* τ *is finite in average and* $\{M_n\}_{n\in\mathbb{N}}$ *has uniformly bounded increments almost surely. The word ,Äúuniformly,Äù is referred to the fact that the constant* C *does not depend on* n;

(d) $\mathbb{P}(\tau < \infty) = 1$ *and* $\forall n \in \mathbb{N} M_n \geq 0$*, that is* τ *is finite almost surely and* $\{M_n\}_{n\in\mathbb{N}}$ *is a non-negative process.*

Then $\mathbb{E}[M_\tau] < \infty$ *and*

$$\mathbb{E}[M_\tau] \geq \mathbb{E}[M_0].$$

Example 1.3. *Consider the sequence* $\{X_n\}_{n\in\mathbb{N}_{>0}}$ *defined as in (3.1) and define* $S_n = \sum_{i=1}^{n} X_i$. *Let us consider the stopping time*

$$\tau = \inf\{n \geq 0 \mid X_n = 1\}.$$

We would like to understand if it is possible to apply the Optional Stopping Theorem to understand the relation between $\mathbb{E}[S_\tau]$ *and* $\mathbb{E}[S_1]$. *Hence, we have to verify if one of the hypotheses (a), (b), (c), or (d) is verified by the process* $\{S_n\}_{n\in\mathbb{N}_{>0}}$ *and by the stopping time* τ.

Note that τ *is a stopping time of the form described in Proposition 1.3.1, and hence we know that* τ *is a geometric random variable of parameter* $\mathbb{P}(X_i = 1) = p > 0$. *By Remark 1 we have that* τ *is a finite random variable almost surely, but it is not bounded. So we have that* $\mathbb{P}(\tau < \infty) = 1$ *and there does not exist a constant* $C > 0$ *(independent of* n*) such that* $\mathbb{P}(\tau \leq C) = 1$. *Hence, we know that hypothesis (a) of the Optional Stopping Theorem is not verified.*

Let us try to verify hypothesis (b). We have already said that $\mathbb{P}(\tau < \infty) = 1$ *and hence we have to verify if there exists a constant* C *independent of* n *such that* $\mathbb{P}(|S_n| < C) = 1$. *Note that* $S_n = \sum_{i=1}^{n} X_i$ *and hence the values that* S_n *may assume oscillate from its minimum (obtained when* $X_i = -1$ *for all* $i = 1, \ldots, n$*) and its maximum (obtained when* $X_i = 1$ *for all* $i = 1, \ldots, n$*). So* $-n \leq S_n \leq n$, *and hence* $|S_n| \leq n$. *So we should define* $C = n$, *in order to have* $\mathbb{P}(|S_n| \leq C) = 1$, *but such a* C *depends on* n *and hence it is not valid. This shows that the hypothesis (b) of the Optional Stopping Theorem is not satisfied.*

Let us see if hypothesis (c) is verified. By Proposition 1.3.1 we know $\mathbb{E}[\tau] < \infty$. *So we are left to see if there exists* $C > 0$ *(independent of* n*) such that* $\mathbb{P}(|S_{n+1} - S_n| \leq C) = 1$. *Note that*

$$S_{n+1} - S_n = \sum_{i=1}^{n+1} X_i - \sum_{i=1}^{n} X_i = X_{n+1} + \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} X_i = X_{n+1},$$

from which

$$|S_{n+1} - S_n| = |X_{n+1}| \leq \max\{|1|, |-1|\} = 1.$$

So if we choose $C = 1$*, we have* $|S_{n+1} - S_n| \leq C$ *for any* $n \in \mathbb{N}_{>0}$ *and hence*

$$\mathbb{P}(|S_{n+1} - S_n| \leq C) = 1$$

for any $n \in \mathbb{N}_{>0}$*. Note that this time the constant* $C = 1$ *is independent of* n *and then it is a valid constant for the theorem. So the hypothesis (c) of the Optional Stopping Theorem is verified and we can conclude that*

- *if* $p = \frac{1}{2}$ *(that implies that* $\{S_n\}_{n\in\mathbb{N}_{>0}}$ *is a martingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}_{>0}}$*), we get*

$$\mathbb{E}[S_\tau] = \mathbb{E}[S_1] = \mathbb{E}[X_1] = 0;$$

- *if* $p \leq \frac{1}{2}$ *(that implies that* $\{S_n\}_{n\in\mathbb{N}_{>0}}$ *is a supermartingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}_{>0}}$*), we get*

$$\mathbb{E}[S_\tau] \leq \mathbb{E}[S_1] = \mathbb{E}[X_1] = 1 \cdot p - 1 \cdot (1-p) = 2p - 1;$$

- *if* $p \geq \frac{1}{2}$ *(that implies that* $\{S_n\}_{n\in\mathbb{N}_{>0}}$ *is a submartingale with respect to the filtration* $\{X_n\}_{n\in\mathbb{N}_{>0}}$*), we get*

$$\mathbb{E}[S_\tau] \geq \mathbb{E}[S_1] = \mathbb{E}[X_1] = 1 \cdot p - 1 \cdot (1-p) = 2p - 1.$$

In particular, it can be noted for example that, if $p \leq \frac{b}{a+b}$*, it is not possible to have* $\mathbb{E}[S_\tau] > (a+b)p - b = \mathbb{E}[X_1]$*.*

For completeness, note that hypothesis (d) of the Optional Stopping Theorem is not verified by the process since $\{S_n\}_{n\in\mathbb{N}_{>0}}$ *is not a non-negative process, that is it may happen that* $S_n < 0$ *for some* n *(for example if* $X_1 = \ldots = X_n = -1$ *and hence* $S_n = -n$*).*

In the context of gambling theory, the Optional Stopping Theorem is also referred to as the **Principle of Conservation of Fairness** in a game. Essentially, when given a game, a stopping time can be viewed as a quitting strategy. According to this interpretation, the Principle of Conservation of Fairness asserts that if the game meets any of the conditions (a), (b), (c), or (d), no quitting strategy exists that can turn an unfair game into a fair one. For example, in the case of a supermartingale, we observe that $\mathbb{E}[S_\tau] \leq \mathbb{E}[S_1]$, which implies that no stopping time τ satisfying the theorem's conditions can result in $\mathbb{E}[S_\tau] > \mathbb{E}[S_1]$. In upcoming sections, we will explore betting strategies and how the Martingale system avoids the Principle of Conservation of Fairness, allowing for the possibility of $\mathbb{E}[S_\tau] > \mathbb{E}[S_1]$ even when $\{S_n\}_{n\in\mathbb{N}_0}$ is a supermartingale.

Chapter 2

Betting System

Imagine placing a bet on a game and let X represent the amount of money gained or lost per unit bet. Suppose we continue to place the same bet multiple times, and let $X_1, X_2, \ldots$ represent the amount of money won or lost per unit bet in each round (for instance, if we place a bet of 7 in the third round, we would win $7X_3$). It is clear that the random variables $X_1, X_2, \ldots$ are independent, as the mechanics of the game do not depend on the previous rounds. Let's consider the scenario where, at round n, a certain amount of money B_n is bet, which naturally depends on the outcomes of the previous rounds, specifically $X_1, \ldots, X_{n-1}$. In this case, we can express:

$$B_1 = g_1 > 0, \tag{2.1}$$

$$B_n = g_n(X_1, \ldots, X_{n-1}) \quad \text{for } n \geq 2, \tag{2.2}$$

where g_n represents a decision rule that takes $X_1, \ldots, X_{n-1}$ as inputs and determines the amount of money to be bet in round n. B_1 is a fixed positive amount $g_1 > 0$ since the amount to bet in the first round is determined in a deterministic way. Since X_n represents the amount of money won in round n per unit bet, and B_n denotes the amount bet in the n-th round, the total amount of money won in the n-th round is B_nX_n. The sequences of variables $\{X_n\}_{n\in\mathbb{N}_0}$ and $\{B_n\}_{n\in\mathbb{N}_0}$ constitute a **betting system**. If we denote by F_n the total amount of money we have at the end of the n-th round, then we have:

$$F_n = F_{n-1} + B_nX_n \quad \text{for } n \geq 1, \tag{2.3}$$

which, by iteration, becomes:

$$F_n = F_0 + \sum_{i=1}^{n} B_iX_i, \tag{2.4}$$

where F_0 represents the initial amount of money (and thus is deterministic). It is reasonable to assume that we cannot bet more than what we have in each round, that is, $B_n \leq F_{n-1}$ for $n \geq 1$. We are interested in studying the sequence $\{F_n\}_{n\in\mathbb{N}}$ with respect to the filtration $\{X_n\}_{n\in\mathbb{N}}$. We assume that X_i takes on a finite set of values to ensure that $\mathbb{E}[|X_i|] < \infty$. The following result establishes the relationship between (super/sub) fair games and (sub/super) martingales.

Proposition 2.0.1. *Let $\{X_n\}_{n\in\mathbb{N}_{>0}}$ be a sequence of i.i.d. random variables with $\mathbb{E}[|X_i|] < \infty$. Let $\{B_n\}_{n\in\mathbb{N}_{>0}}$ be a sequence of random variables that satisfies 2.1. If we define the sequence $\{F_n\}_{n\in\mathbb{N}}$ as in 2.4, then:*

- *If $\mathbb{E}[X_1] = 0$, then $\{F_n\}_{n\in\mathbb{N}}$ is a martingale with respect to the filtration $\{X_n\}_{n\in\mathbb{N}_{>0}}$.*
- *If $\mathbb{E}[X_1] \geq 0$, then $\{F_n\}_{n\in\mathbb{N}}$ is a submartingale with respect to the filtration $\{X_n\}_{n\in\mathbb{N}_{>0}}$.*
- *If $\mathbb{E}[X_1] \leq 0$, then $\{F_n\}_{n\in\mathbb{N}}$ is a supermartingale with respect to the filtration $\{X_n\}_{n\in\mathbb{N}_{>0}}$.*

Proof. We begin by showing that:

(i) $\mathbb{E}[|F_n|] < \infty$ for any fixed n;

(ii) The sequence $\{F_n\}_{n\in\mathbb{N}}$ is adapted to $\{X_n\}_{n\in\mathbb{N}_{>0}}$ (that is, F_n is measurable with respect to $X_1, \ldots, X_n$).

Finally, we need to explore the relationship between $\mathbb{E}[F_{n+1}|X_1, \ldots, X_n]$ and F_n. Let's begin by proving (i). Notice that by the triangular inequality:

$$\begin{aligned}\mathbb{E}[|F_n|] &= \mathbb{E}\left[\left|F_0 + \sum_{i=1}^{n} B_i X_i\right|\right] \leq \mathbb{E}\left[|F_0| + \left|\sum_{i=1}^{n} B_i X_i\right|\right] \\ &= |F_0| + \sum_{i=1}^{n} \mathbb{E}[|B_i| \cdot |X_i|] = |F_0| + \sum_{i=1}^{n} \mathbb{E}[|B_i|] \cdot \mathbb{E}[|X_i|],\end{aligned} \tag{2.5}$$

where equation 2.5 is derived. In the final inequality, we utilized the fact that F_0 is a constant. Additionally, we leveraged the independence of B_i from X_i due to $B_i = g_i(X_1, \ldots, X_{i-1})$, which gives us the relationship:

$$\mathbb{E}[|B_i| \cdot |X_i|] = \mathbb{E}[|B_i|] \cdot \mathbb{E}[|X_i|].$$

Note that $B_i = g_i(X_1, \ldots, X_{i-1})$. Since X_i takes on only a finite number of possible values (let,Äôs say k values), the vector $(X_1, \ldots, X_{i-1})$ also takes on a finite number of values (specifically k^{i-1} values). Therefore, $g_i(X_1, \ldots, X_{i-1})$ also takes on a finite number of values (at most k^{i-1}), implying that B_i is bounded by some constant $K_i < \infty$. Consequently, we have:

$$\mathbb{E}[|F_n|] \leq |F_0| + \mathbb{E}[|X_1|] \sum_{i=1}^{n} K_i \leq |F_0| + \mathbb{E}[|X_1|] \cdot n \max_{1\leq i\leq n} K_i < \infty, \tag{2.6}$$

which completes the bound. The final inequality results from the fact that F_0 and $\mathbb{E}[|X_1|]$ are fixed values, whereas $K_1, \ldots, K_n$ are finite constants, making their maximum finite as well. This completes the proof of (i). Now, let's prove (ii). Recall the definition of F_n. Observe that since $B_i = g_i(X_1, \ldots, X_{i-1})$, to determine B_i, it suffices to know $X_1, \ldots, X_{i-1}$. Therefore, to determine $B_1, \ldots, B_n$, it suffices to know $X_1, \ldots, X_{n-1}$. Consequently, by (3.11), we see that to determine F_n, it suffices to know $X_1, \ldots, X_n$. This establishes that the process $\{F_n\}_{n\in\mathbb{N}}$ is adapted to the filtration $\{X_n\}_{n\in\mathbb{N}_{>0}}$, proving (ii). Next, we need to compute $\mathbb{E}[F_{n+1}|X_1, \ldots, X_n]$. Recall that $B_i = g_i(X_1, \ldots, X_{i-1})$. Therefore, we have:

$$\mathbb{E}[B_i|X_1, \ldots, X_n] = B_i \quad \text{for } i = 1, \ldots, n+1, \tag{2.7}$$

$$\mathbb{E}[X_i|X_1, \ldots, X_n] = X_i \quad \text{for } i = 1, \ldots, n. \tag{2.8}$$

This leads us to:

$$\mathbb{E}[B_i X_i|X_1, \ldots, X_n] = B_i X_i \quad \text{for } i = 1, \ldots, n. \tag{2.9}$$

Therefore, we have:

$$\begin{aligned} \mathbb{E}[F_{n+1}|X_1, \ldots, X_n] &= \mathbb{E}\left[F_0 + \sum_{i=1}^{n+1} B_i X_i \middle| X_1, \ldots, X_n\right] = \\ &= F_0 + \sum_{i=1}^{n} B_i X_i + \mathbb{E}[B_{n+1} X_{n+1}|X_1, \ldots, X_n]. \end{aligned} \tag{2.10}$$

Finally:

$$= F_n + \mathbb{E}[B_{n+1} X_{n+1}|X_1, \ldots, X_n] = F_n + B_{n+1}\mathbb{E}[X_{n+1}|X_1, \ldots, X_n]. \tag{2.11}$$

Since $\{X_n\}_{n\in\mathbb{N}_{>0}}$ is a sequence of independent random variables, we have:

$$\mathbb{E}[X_{n+1}|X_1, \ldots, X_n] = \mathbb{E}[X_{n+1}] = \mathbb{E}[X_1] \quad \text{(i.i.d. distribution)}.$$

Thus, it follows that:

$$\mathbb{E}[F_{n+1}|X_1, \ldots, X_n] = F_n + B_{n+1}\mathbb{E}[X_1].$$

Given that $B_{n+1} \geq 0$, we can conclude:

- $\mathbb{E}[F_{n+1}|X_1, \ldots, X_n] = F_n$ if $\mathbb{E}[X_1] = 0$;
- $\mathbb{E}[F_{n+1}|X_1, \ldots, X_n] \geq F_n$ if $\mathbb{E}[X_1] \geq 0$;
- $\mathbb{E}[F_{n+1}|X_1, \ldots, X_n] \leq F_n$ if $\mathbb{E}[X_1] \leq 0$.

This completes the proof. □

2.1 Martingale System

Suppose a gambler repeats the same bet multiple times. Assume that the amount of money won for a unit bet at round i is given by

$$X_i = \begin{cases} 1, & \text{if we win the } i\text{-th bet,} \\ -1, & \text{if we lose the } i\text{-th bet.} \end{cases}$$

and that

$$\mathbb{P}(X_i = 1) = p \in \left(0, \frac{1}{2}\right), \quad \mathbb{P}(X_i = -1) = 1 - p.$$

In this system, the gambler doubles their bet size after each loss and stops betting after the first win.

We define the stopping time τ (with respect to the filtration $\{X_n\}_{n \in \mathbb{N}_{>0}}$) as the first time a win occurs. Therefore, we have

$$\{\tau = k\} = \{X_1 = -1, \ldots, X_{k-1} = -1, X_k = 1\}.$$

Given an initial bet $B_1 > 0$ and an initial capital $F_0 > 0$, and denoting by B_i the amount of money bet in round i and by F_i our total capital at the end of round i, we have:

$$F_1 = \begin{cases} F_0 + B_1, & \text{if } X_1 = 1 \text{ (i.e., } \tau = 1), \\ F_0 - B_1, & \text{if } X_1 = -1 \text{ (i.e., } \tau > 1). \end{cases}$$

If we lose the first round (i.e., $\tau > 1$), then:

$$F_2 = \begin{cases} F_0 - B_1 + B_2, & \text{if } X_1 = -1, X_2 = 1 \text{ (i.e., } \tau = 2), \\ F_0 - B_1 - B_2, & \text{if } X_1 = -1, X_2 = -1 \text{ (i.e., } \tau > 2). \end{cases}$$

If we lose the first two rounds (i.e., $\tau > 2$), then:

$$F_3 = \begin{cases} F_0 - B_1 - B_2 + B_3, & \text{if } X_1 = -1, X_2 = -1, X_3 = 1 \text{ (i.e., } \tau = 3), \\ F_0 - B_1 - B_2 - B_3, & \text{if } X_1 = -1, X_2 = -1, X_3 = -1 \text{ (i.e., } \tau > 3). \end{cases}$$

In general, if we lose the first $n-1$ rounds (i.e., $\tau > n-1$), then:

$$F_n = \begin{cases} F_0 - B_1 - \cdots - B_{n-1} + B_n, & \text{if } X_1 = -1, \ldots, X_{n-1} = -1, X_n = 1 \text{ (i.e., } \tau = n), \\ F_0 - B_1 - \cdots - B_n, & \text{if } X_1 = -1, X_2 = -1, \ldots, X_n = -1 \text{ (i.e., } \tau > n). \end{cases}$$

Note that if $\tau \geq n$, since each time we lose we double the bet size, we have:

$$B_n = 2B_{n-1}$$

and by iteration of this formula, we obtain:

$$B_n = 2B_{n-1} = 2^2 B_{n-2} = 2^3 B_{n-3} = \cdots = 2^{n-1} B_1.$$

So, we can express F_n as:

$$F_n = \begin{cases} F_0 - B_1 - 2B_1 - 2^2 B_1 - \cdots - 2^{n-2} B_1 + 2^{n-1} B_1, & \text{if } \tau = n, \\ F_0 - B_1 - 2B_1 - 2^2 B_1 - \cdots - 2^{n-2} B_1 - 2^{n-1} B_1, & \text{if } \tau > n, \end{cases}$$

which can be simplified as:

$$F_n = \begin{cases} F_0 - B_1(1 + 2 + 2^2 + \cdots + 2^{n-2}) + 2^{n-1} B_1, & \text{if } \tau = n, \\ F_0 - B_1(1 + 2 + 2^2 + \cdots + 2^{n-2} + 2^{n-1}), & \text{if } \tau > n, \end{cases}$$

That is:

$$F_n = \begin{cases} F_0 - B_1 \cdot \sum_{i=0}^{n-2} 2^i + 2^{n-1} B_1, & \text{if } \tau = n, \\ F_0 - B_1 \cdot \sum_{i=0}^{n-1} 2^i, & \text{if } \tau > n. \end{cases}$$

Recall that for any $a \in \mathbb{R} \setminus \{1\}$,

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}.$$

We apply the above formula to express F_n. Thus, we have:

$$F_n = \begin{cases} F_0 - B_1 \cdot \frac{2^{n-1}-1}{2-1} + 2^{n-1}B_1 = F_0 - 2^{n-1}B_1 + B_1 + 2^{n-1}B_1, & \text{if } \tau = n, \\ F_0 - B_1 \cdot \frac{2^{n-1}-1}{2-1} = F_0 - 2^n B_1 + B_1, & \text{if } \tau > n, \end{cases}$$

which implies:

$$F_n = \begin{cases} F_0 + B_1, & \text{if } \tau = n, \\ F_0 + B_1 - 2^n B_1, & \text{if } \tau > n. \end{cases}$$

So, we deduce that:

$$\mathbb{P}(F_\tau = F_0 + B_1) = 1,$$

which indicates that when we stop, with probability 1, we have recovered all the capital that we have bet during the rounds and won our initial bet B_1. Moreover, if $\tau > n$, we can quantify the amount of money that we are losing, which is $B_1 - 2^n B_1$, a quantity that decreases exponentially. Additionally, since $\mathbb{P}(F_\tau = F_0 + B_1) = 1$, we have:

$$\mathbb{E}[F_\tau] = \mathbb{E}[F_0 + B_1] \geq B_1 > 0 \quad \Rightarrow \quad \mathbb{E}[F_\tau] \geq \mathbb{E}[F_0].$$

This is precisely the opposite conclusion of the Optional Stopping Theorem for supermartingales. Indeed, the capital process $\{F_n\}_{n\in\mathbb{N}}$ forms a supermartingale with respect to the filtration $\{X_n\}_{n\in\mathbb{N}_{>0}}$ (because $\mathbb{P}(X_i = 1) = p < \frac{1}{2}$ and $\mathbb{P}(X_i = -1) = 1-p$). This conclusion does not contradict the Optional Stopping Theorem for supermartingales; rather, it arises because the assumptions of the theorem are not satisfied. Specifically, since τ represents the first round in which we win and given that the individual bets are i.i.d., we have $\tau \sim \text{Geom}(p)$, implying that there exists no $C > 0$ such that $\mathbb{P}(\tau < C) = 1$. Moreover, it is also impossible to find $C > 0$ (independent of n) such that $\mathbb{P}(|F_n| < C) = 1$ because while we are losing rounds, our capital decreases exponentially fast, and hence $|F_n|$ increases exponentially fast. Note also that while we are losing rounds, the increments of $\{F_n\}_{n\in\mathbb{N}}$ increase exponentially fast. Indeed,

$$\begin{aligned} |F_{n+1} - F_n| &= |F_0 + B_1 - 2^{n+1}B_1 - (F_0 + B_1 - 2^n B_1)| \\ &= |B_1(2^n - 2^{n+1})| \\ &= B_1 \cdot 2^n \cdot |1-2| = 2^n B_1. \end{aligned} \tag{2.12}$$

So, it is not possible to find a constant $C > 0$ such that $\mathbb{P}(|F_{n+1} - F_n| \leq C) = 1$. Finally, for any fixed value of F_0, the process obviously does not satisfy $F_n \geq 0$ for all $n \in \mathbb{N}$. Therefore, all the hypotheses of the Optional Stopping Theorem fail, and as a result, this betting strategy allows us to have $\mathbb{E}[F_\tau] > \mathbb{E}[F_0]$ even though $\{F_n\}_{n\in\mathbb{N}}$ is a supermartingale with respect to the filtration $\{X_n\}_{n\in\mathbb{N}_{>0}}$. Regarding the application of the martingale system, it is typically applied to situations such as betting on even or odd (or equivalently red or black) in roulette, where the probability p of winning is less than $\frac{1}{2}$, but not significantly so. For example, when betting on even numbers in European roulette, the probability of winning is $p = \frac{18}{37} \approx 0.49$. Note also that since $\tau \sim \mathrm{Geom}(p)$, we have:

$$\mathbb{P}(\tau > k) = (1-p)^k.$$

Since $\tau > k$ implies that we have lost for k consecutive rounds (and this occurs independently each time with probability $1-p$), if we bet on even numbers at the European roulette, we have:

$$\mathbb{P}(\tau > k) = \left(\frac{19}{37}\right)^k,$$

and hence:

$$\mathbb{P}(\tau > 1) \approx 0.51, \qquad \mathbb{P}(\tau > 2) \approx 0.26,$$
$$\mathbb{P}(\tau > 3) \approx 0.14, \qquad \mathbb{P}(\tau > 4) \approx 0.07,$$
$$\mathbb{P}(\tau > 5) \approx 0.04, \qquad \mathbb{P}(\tau > 6) \approx 0.02,$$
$$\mathbb{P}(\tau > 7) \approx 0.01, \qquad \mathbb{P}(\tau > 8) \approx 0.005.$$

So, for example, if we are able to maintain a capital F_0 that guarantees we avoid ruin before round 7, we have a 99% probability of winning the initial bet B_1.

2.2 Applications of the martingale system

The Martingale system was first conceptualized in the context of gambling, particularly in games like roulette. The core premise is straightforward: after each loss, the gambler doubles their bet, ensuring that a single win will recover all previous losses plus a profit equal to the original stake. For example, if a gambler starts with a $10 bet and loses, they would bet $20 on the next round. If they lose again, the next bet would be $40, and so on. The strategy hinges on the assumption that a win is inevitable and that the gambler has sufficient capital to withstand losing streaks.

Translating this strategy to financial markets, particularly in forex trading, involves increasing the position size after each loss in the hope that a subsequent trade will recover all previous losses and yield a profit. Unlike gambling, financial markets offer various instruments and leverage options, making the Martingale system both more versatile and riskier.

2.2.1 Forex trading

In forex trading, the Martingale strategy involves doubling the size of a trade after each loss. For instance, a trader might start with a position of 1 lot. If the trade results in a loss, the next trade would be 2 lots, followed by 4 lots after another loss, and so forth. The objective is to recover all previous losses with a single winning trade.

Example 2.1.

- ***Initial Trade:*** *A trader buys 1 lot of EUR/USD at 1.1000 with a stop loss at 1.0950 (a loss of 50 pips).*

- ***First Loss:*** *EUR/USD drops to 1.0950. The trader doubles the position to 2 lots, now risking 100 pips.*

- ***Second Loss:*** *EUR/USD drops to 1.0900. The trader doubles again to 4 lots, risking 200 pips.*

- ***Third Loss:*** *EUR/USD drops to 1.0700. At this point, the required position size would be 8 lots, risking 400 pips.*

If the trend continues downward, the trader may quickly face margin calls or account liquidation, resulting in substantial losses.

2.2.2 Investing

When applied to investing, the Martingale system can involve increasing the investment amount in a particular asset after each loss. For example, an investor might purchase shares of a stock, and if the stock price falls, they would purchase additional shares to lower the average cost per share, anticipating that the stock will eventually rebound.

Example 2.2.

- ***Initial Investment:*** *An investor buys 10 shares of a stock at $50 each.*

- ***First Loss:*** *The stock price falls to $45. The investor buys an additional 20 shares.*

- ***Second Loss:*** *The stock price drops to $40. The investor buys 40 more shares.*

- ***Third Loss:*** *The stock price falls to $35. The investor buys 80 shares.*

In this scenario, the investor has progressively doubled their holdings in a declining stock, exponentially increasing their exposure. If the stock continues to fall, the investor's losses can accumulate rapidly.

2.2.3 Theoretical advantages

One of the primary attractions of the Martingale system is its simplicity. The strategy does not require complex analysis or forecasting; it operates on a mechanical principle of increasing investment after losses.

Theoretically, the Martingale system ensures that a single successful trade or investment will recover all previous losses and generate a profit equal to the initial investment. This can be particularly appealing in markets with mean-reverting behavior, where prices oscillate around an average value.

In forex trading, leveraging the Martingale system can offer the additional benefit of earning interest through carry trades. For example, if a trader holds a position in a currency pair where the bought currency has a higher interest rate than the sold currency, the trader can earn interest on the leveraged position, potentially offsetting some losses during unfavorable market movements.

2.2.4 Critical Drawbacks and Risks

The most significant drawback of the Martingale system is the exponential increase in risk. After each loss, the required investment doubles, leading to rapidly escalating exposure. For example, starting with a $100 bet:

- 1st loss: $100

- 2nd loss: \$200
- 3rd loss: \$400
- 4th loss: \$800
- 5th loss: \$1,600
- *And so on...*

This exponential growth means that a relatively short losing streak can deplete a substantial portion of a trader's or investor's capital.

The Martingale system assumes that the trader has access to unlimited capital, which is practically impossible. Most traders operate with finite resources, making it likely that a prolonged losing streak will exhaust their funds before a recovery occurs.

In leveraged markets like forex, the risks are magnified. Each time a position is doubled, the margin requirement increases, and brokers may issue margin calls if the trader's account balance falls below required levels. Forced liquidation of positions can occur before the trader has the chance to recover losses, leading to significant financial losses.

Financial markets can experience extended trends where prices move consistently in one direction for prolonged periods. In such scenarios, the Martingale strategy becomes perilous, as the assumption that losses will eventually reverse fails.

2.2.5 Variation of the Martingale System

Anti-Martingale Strategy: This strategy is also known as the "reverse Martingale," is an approach used in trading or gambling that focuses on increasing the size of the position after a win rather than after a loss. The core idea behind this strategy is to capitalize on winning streaks while minimizing exposure during losing streaks.

Unlike the traditional Martingale strategy, where the position size is doubled after each loss to recover previous losses, the Anti-Martingale strategy aims to maximize profits during favorable conditions. For example, a trader or gambler might start with an initial position size of $100. After each win, they would double the position size, moving to $200, then $400, and so on. This approach allows for substantial profit gains during a streak of consecutive wins. However, if a loss occurs, the position size is reset to the initial amount, thus limiting the potential for large losses during unfavorable conditions. The strategy's key advantage is that it leverages the "hot hand" phenomenon, but it requires discipline to avoid excessive risk when winning streaks do not materialize.

Fixed Martingale: It is a variation of the traditional Martingale strategy, where instead of doubling the position size after each loss, the trader or bettor increases their position size by a fixed increment. For instance, instead of increasing the size by 100% (doubling), the trader might increase it by a smaller percentage, such as 50%, after each loss.

The advantage of the Fixed Martingale approach is that it reduces the rate at which exposure to risk grows during a losing streak. For example, if the initial bet or position size is $100, a 50% increment would mean the next bet after a loss would be $150, then $225, and so on. This slower growth rate can help the trader or bettor withstand more extended periods of losses, as it requires less capital to continue participating in the market or game. The strategy aims to strike a balance between recovering losses and managing risk more conservatively than the traditional Martingale method.

Martingale with Stop-Loss Limits: The Martingale strategy with stop-loss limits is a modification designed to manage risk more effectively by imposing strict boundaries on how far the strategy can escalate. This variation incorporates stop-loss limits that cap the maximum position size or the total number of consecutive losses allowed.

For example, a trader might decide to limit their maximum position size to $1,000 or restrict themselves to a maximum of three consecutive losses before halting further increases in position size. By doing so, the trader protects against the risk of catastrophic losses that can occur with an unchecked Martingale strategy. The key advantage of this approach is that it enforces a disciplined exit strategy, ensuring that losses are limited and that the trader does not end up with an unsustainable level of exposure. This makes the strategy more practical for use in real-world scenarios, where market conditions can be unpredictable, and extended losing streaks can quickly lead to substantial financial damage.

2.2.6 Psychological and Behavioral Aspects

Emotional Strain: The Martingale system can induce significant emotional stress for traders and gamblers due to the increasing financial stakes associated with each successive loss. As the position size doubles after every loss, the pressure to recover accumulated losses intensifies, often leading to heightened anxiety and fear. This psychological burden can become overwhelming, especially during prolonged losing streaks, where the risk of substantial financial loss looms large.

The emotional toll of the Martingale strategy can cause even experienced traders to doubt their approach, resulting in impaired judgment. Under stress, they may become prone to deviating from their original strategy, abandoning it prematurely, or making impulsive decisions that further exacerbate losses. In extreme cases, the combination of high financial exposure and psychological strain can lead to a cycle of emotional distress and poor decision-making, diminishing the effectiveness of the strategy and potentially leading to severe financial consequences. This is why the Martingale strategy, despite its theoretical appeal, is often regarded as high-risk and requires not only substantial capital but also considerable emotional resilience and discipline to execute effectively.

Cognitive Biases:Traders who employ the Martingale strategy may be particularly susceptible to cognitive biases that distort their decision-making processes. One common bias is the Gambler's Fallacy, the erroneous belief that the probability of a win increases after a series of losses. This misconception arises from the assumption that random events are self-correcting, leading traders to believe that a loss streak is likely to be followed by a win.

This flawed reasoning can create a false sense of confidence, encouraging traders to continue with the Martingale strategy despite facing significant and mounting losses. As the financial stakes grow exponentially with each consecutive loss, the belief that a win is "due" can lead to escalating risk-taking behavior. Instead of reassessing the strategy based on objective market conditions or revisiting risk management principles, traders may irrationally persist in the hope that the next trade will recover all previous losses.

2.2.7 Considerations

The Martingale system presents an intriguing approach to recovering losses through the systematic doubling of investment positions. Its simplicity and the allure of guaranteed recovery make it appealing, especially to novice traders. However, the strategy's inherent risks,Äîexponential loss growth, capital requirements, margin constraints, and psychological stress,Äîrender it impractical and perilous for most traders and investors.

In financial markets characterized by volatility, extended trends, and leverage, the Martingale system's assumption of inevitable recovery is often unfounded. Prolonged losing streaks can lead to catastrophic losses, wiping out entire accounts and leaving traders financially devastated.

While variations like the anti-Martingale system offer some risk mitigation, they do not fully address the fundamental issues inherent in the Martingale approach. Effective risk management, capital allocation, and disciplined trading practices are essential for any strategy's success, and the Martingale system falls short in these critical areas.

Ultimately, the Martingale system should be approached with extreme caution. Traders and investors are advised to thoroughly understand its mechanics, evaluate their risk tolerance, and consider alternative strategies that offer more sustainable and controlled risk exposure. Emphasizing diversification, proper position sizing, and robust risk management frameworks can lead to more consistent and long-term success in financial markets.

Appendix

In this appendix there are some important results used in this thesis.

Proposition A.1 (Triangular Inequality)**.** *Given* $a, b \in \mathbb{R}$ *we have*

$$|a \pm b| \leq |a| + |b| \,.$$

Proposition A.2 (Tower Property)**.** *Let* X, Y *be jointly distributed two random variables. Then*

$$\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[X] \,.$$

Proposition A.3 (Law of Large Numbers)**.** *Let* $X_1, X_2, \ldots$ *be a sequence of i.i.d. random variables with* $\mathbb{E}[X_1] = \mu < \infty$*. Then*

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{X_1 + \ldots + X_n}{n} \to \mu\right) = 1 \,,$$

or equivalently almost surely we have

$$\lim_{n\to\infty} \frac{X_1 + \ldots + X_n}{n} = \mu \,.$$

Proposition A.4 (Central Limit Theorem)**.** *Let* $X_1, X_2, \ldots$ *be a sequence of i.i.d. random variables with* $\mathbb{E}[X_1] = \mu < \infty$ *and* $Var(X_1) = \sigma^2 < \infty$*. Define* $S_n = \sum i = 1^n X_i$ *and* $Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$*. Then for any* $t \in \mathbb{R}$ *we have*

$$F_{Z_n}(t) := \mathbb{P}(Z_n \leq t) \xrightarrow{q.c.} \mathbb{P}(Z \leq t) \quad \text{for } n \to \infty \,,$$

where $Z \sim \mathcal{N}(0, 1)$*. Equivalently* Z_n *converges in distribution to* $Z \sim \mathcal{N}(0, 1)$ *for* $n \to \infty$*.*

Proposition A.5 (Weierstrass' Theorem). *Let* $f : D \subset \mathbb{R}^n \to \mathbb{R}$ *be a continuous function and let* $D \subset \mathbb{R}^n$ *be a compact set in* $\mathbb{R}^n$ *(that is closed and bounded). Then* f *admits global maximum and minimum in* D.

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