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An application of the PageRank algorithm to the measurement of individual contributions in teams

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1) Introduction

The aim of this presentation is to apply Google's Page Rank algorithm to the measurement of individual contributions in teams. This will be possible following the model of the study of Müller J. and Uppmann T.[1] Then, as an exercise, the result will be applied to the PRIN 2022 PNRR final ranking, in order to assess the performance of Italian universities when working together in research projects.

In the first part, Google's Page Rank algorithm will be described and analyzed both from a theoretical and a mathematical point of view. Topics that will be covered are: the functioning of a search engine and why Google's one of the most well-renowned ones, the random surfing model, how to answer the problem of assigning a value for each page in a database and how to use this data to create a link matrix that can be later utilized to find a final ranking for the network considered. The paper will also show how some problems involving these calculations can be overcome, i.e. webs with dangling nodes and webs with nonunique rankings.

In addition, elements of linear algebra like matrices, vector spaces, eigenvectors, eigenvalues, Markov chains, the Power method and the Perron-Frobenius theorem will be introduced to provide a better understanding of the model and the calculations involved.

Then, the application provided by Müller J. and Uppmann T. in the article *Eigenvalue productivity: Measurement of individual contributions in teams* (2022) is shown, analyzing not only the assumptions and formulas involved, but also the basic ideas behind how an individual can actually "increase" the productivity of his teammates and what are the variables implicated in this process. To conclude this part of the presentation, an easy and artificial example is given, to provide the reader with a practical and "visual" idea of the concepts just described.

Finally, the EPV model will be applied to the Prin 2022 for the ranking of Italian universities and public research entities. After a presentation of the idea behind the PRIN and its basic rules, the analysis will proceed taking into consideration the results of the sector SH1 for the year 2022. Universities will be ranked with some modifications of the original model: the pair-success ratios for universities working in the same team will be evaluated with a different method based on the arithmetic mean and a value of 0 assigned to all the other pairwise interactions. In conclusion, the results obtained will be described and analyzed.

[1] Müller J., Uppmann T., Eigenvalue productivity: Measurement of individual contributions in teams, "PLOS ONE", vol.17, n.1, September 2022

2) Google's PageRank algorithm

Google's prominence as a search engine is by now well-recognized worldwide. It's functioning and ranking system had a significant influence on the development and structure of the internet as we know it today.

A key element in Google's success is its algorithm PageRank, created by Sergey Brin and Lawrence Page. The algorithm ranks the importance of each page on the web quantitively, according to an eigenvector of a weighted link matrix. The consequent result is that the user gets the most relevant and helpful pages first, instead of having to go through screens and screens of irrelevant links before finding the one in line with his original search. Some data[2] that helps understanding the enormous size of the web and the big challenge of finding a good quality page of the required information: 8% (in web terms) is the rate of creation of new pages per week and, according to some observations, only 20% of the new pages will be accessed by users after 1 year.

A model explaining the functioning of PageRank is the random surfing model, which gives the probability that a random user visits a web page. This is represented thought a random walk on the web in which the user clicks link after link that is presented to him. The probability of visiting a certain page is affected by a variety of factors like: the number of links leading to the page, the frequency with which the surfer arrives on pages containing those links, the order of outgoing links on those pages. In general, links from a long list of web pages and links from unpopular websites have a low value, while links from popular pages which don't link many other pages are valuable and will greatly increase the probability that a random survey arrives on a certain page.

More in general, the functioning of a search engine (including Google's) can be summarized in this process: software called crawlers (in Google's case "Googlebot"), crawl all public web pages and locate them. They go from link to link and bring data about those pages back to Google's servers. Usually there is a list of web addresses from past crawls with a similar objective, which is the starting point for the new search. Crawlers also consider new sites, changes to existing sites and dead links in their search process.

After crawling, information is organized by indexing. This step includes storing all the information about words and locations of the fetched web pages into a database that can later be retrieved. Therefore, if a user searches the same keyword, results will be retrieved faster.

Finally, thanks to an algorithm, the importance of each page in the database is rated, so that the more important pages will be presented first, showing the results in descending order of relevance.

In addition, we must consider that, when we make a search on Google, Google tries to determine the highest quality results, which have many factors including things like users' location, language, device (desktop or phone) and previous queries. Google doesn't accept payment to rank pages in a higher position, the ranking is done only algorithmically.

The focus of this presentation is on the step of rating the importance of each page in the database. A problem arises: what's the right criterion to meaningfully define and quantify the "importance" of a page? We will try to show how Google's PageRank algorithm answers the question and the many possible applications of the algorithm.

[2] Statistics from Singh P., Vidyarty A., *Power rank: An interactive web page ranking algorithm*, chapter from *Principles of Big Graph: In-depth Insight*, Biswas A., Deka G., Patgiri R., Elsevier (January 2022), 1^st edition

3) Mathematical background for a better understanding of the model

In the next paragraphs, we will describe the Page Rank algorithm. To provide a better understanding, we will go through the basic definitions of concepts like matrices, linear spaces, eigenvectors, eigenvalues, Markov chains and the Perron-Frobenius theorem.

-Matrices:

A matrix is a rectangular array, in which each entry a_{ij} is a number. An $m \times n$ matrix (with m and n positive integers), has m rows and n colums. $m \times n$ is read as "m by n".

According to the standard mathematical convention, it is possible to represent a matrix in any one of three ways: an uppercase letter such as A, B or C; a representative element enclosed in brackets like $[a_{ij}]$, $[b_{ij}]$ or $[c_{ij}]$; a rectangular array of numbers, like the one presented below:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The entry a_{ij} is in the ith row and the jth column. The index i is called the "row subscript" because it identifies the row in which the entry lies, and the index j is called the "column subscript" because it identifies the column in which the entry lies.

A matrix that has only one column, is a "column matrix" or "column vector". Similarly, a matrix that has only one row, is a "row matrix" or "row vector". Boldface lowercase letters often designate column matrices and row matrices.

When $m = n$, the matrix is square of order n and the entries a_{11}, a_{22}, a_{33} ... a_{nn} are the main diagonal entries. One relevant square matrix in mathematics is the "identity matrix" I, in which all the elements of the principal diagonal are one and all the other ones are zero. In the next paragraphs we will use the identity matrix as a tool to find the eigenvalues and eigenvectors of a matrix.

Furthermore, a diagonal matrix is a square matrix in which all off-diagonal entries are zero. Entries on the main diagonal may or may not be zero. Examples of diagonal matrices are:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

It is worth mentioning that the identity matrix I is a particular type of diagonal matrix: one with all ones in its diagonal entries.

Two matrices are equal when their corresponding entries are equal. A formal definition is *two matrices* $A = [a_{ij}]$ *and* $B = [b_{ij}]$ *are equal when they have the same size* $(m \times n)$ *and* $a_{ij} = b_{ij}$ *for* $1 \le i \le m$ *and* $1 \le j \le n$.

One common use of matrices is to represent systems of linear equations. The matrix derived from the coefficients and constant terms of a system of linear equations called the "augmented matrix" of the system. The matrix containing only the coefficients is the "coefficient matrix" of the system.

Relevant operations concerning matrices are matrix addition, subtraction, and scalar multiplication. First, two matrices can be added only if they have the same size (the sum of two matrices of different sizes is undefined). To perform matrix addition, we just have to add corresponding entries or, in other words *If* $A = [a_{ij}]$ *and* $B = [b_{ij}]$ *are matrices of size* $m \times n$*, then their sum is the* $m \times n$ *matrix* $A + B = [a_{ij} + b_{ij}]$.

Moreover, referring to real numbers as "scalars", to multiply a matrix A by a scalar c, we multiply each entry in A by c: *If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, then the scalar multiple of A by c is the $m \times n$ matrix $cA = [ca_{ij}]$.*

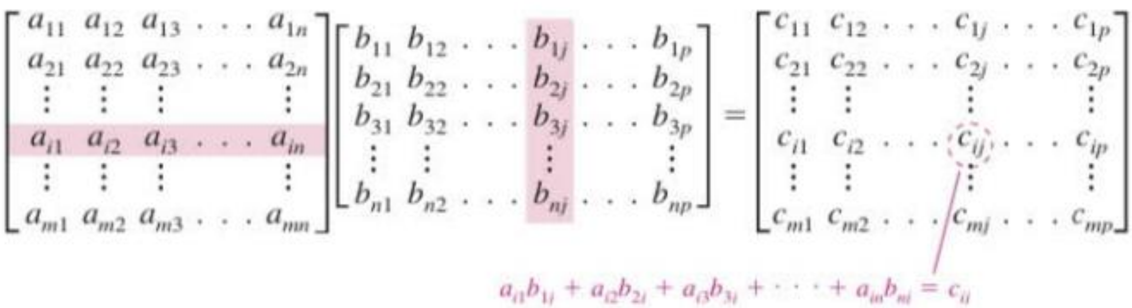
Considering two matrices of the same size A and B, the operation $A - B$ represents the sum of A and $(-1)B$. That is, $A - B = A + (-1)B$.

Finally, as far as matrix multiplication is concerned, for the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix. To find the entry in the ith row and the jth column of the product AB, multiply the entries in the ith row of A by the corresponding entries in the jth column of B and then add the results. A more formal definition of matrix multiplication is: *If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the product AB is an $m \times p$ matrix $AB = [c_{ij}]$ where*:

$$c = \sum_{k=1}^{n} a_{ik}b_{ik} = a_{i1}b_{1i} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$$

In general, for the properties listed above, matrix multiplication is not commutative. It is usually not true that the product AB is equal to the product BA.

Below is shown the general pattern for matrix multiplication:



[3]

A practical example of matrix multiplication is given by the formal definition of the general properties of the identity matrix I : *If A is a matrix of size $m \times n$, then the properties $AI_n = A$ and $I_m A = A$ are true*.

Considering the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$, where A is the coefficient matrix of the system, and $\boldsymbol{x}$ and $\boldsymbol{b}$ are column matrices, the system can be represented in a more convenient way by partitioning the matrices A and $\boldsymbol{x}$ in the manner shown below.

$$A\boldsymbol{x} = \boldsymbol{x_1 a_1} + \boldsymbol{x_2 a_2} + \cdots + \boldsymbol{x_n a_n} = \boldsymbol{b}$$

$$\boldsymbol{x_1}\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + \boldsymbol{x_2}\begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + \boldsymbol{x_n}\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$\boldsymbol{a_1}, \boldsymbol{a_2}...\boldsymbol{a_n}$ are the columns of the matrix A. The expression is called "linear combination" of the column matrices $\boldsymbol{a_1}, \boldsymbol{a_2}...\boldsymbol{a_n}$ with coefficients $\boldsymbol{x_1}, \boldsymbol{x_2}...\boldsymbol{x_n}$. Furthermore, *the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ can be expressed as such a linear combination, where the coefficients of the linear combination are a solution of the system*.

Another worth-mentioning feature of matrices is the inverse of a matrix. According to the general definition: *an $n \times n$ matrix A is "invertible" (or "nonsingular") where there exist an $n \times n$ matrix B such that $AB = BA = I_n$,*

[3] Image from Larson R., *Elementary linear Algebra*, Cengage Learning (2017), 8^th edition

where I_n is the identity matrix of order n. The matrix B is the multiplicative inverse of A. A matrix that doesn't have an inverse is called "noninvertible" (or "singular").

Not all square matrices have inverses and nonsquare matrices do not have them. If A is of size $m \times n$ and B is of size $n \times m$, then the products AB and BA are of different sizes and can be different. However, if a matrix has an inverse, then its inverse is unique. The inverse of A is denoted as A^{-1}.

Another relevant operation concerning matrices is the determinant of a matrix, a single numerical value used to calculate the inverse of a matrix or to solve systems of linear equations. The determinant of a matrix A can be referred to as $\det(A)$ or $|A|$. It is possible to define a determinant only for square matrices. Let A be a 2×2 matrix such that $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the general formula for $\det(A)$ is:

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Instead, for matrices size 3×3, the general approach is to break down the matrix into smaller 2×2 matrices as shown below:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

For each element a_{1i} in the top row, block out the row and column it belongs to, and calculate the determinant of the remaining uncovered 2×2 matrix, then multiply that by a_{1i}. The determinant is the sum of those values, alternating addition and subtraction.

Furthermore, the general definition of the determinant of a square matrix is: *If A is a square matrix of order $n \geq 2$, then the determinant of A is the sum of the entries in the first row of A multiplied by their respective cofactors. That is:*

$$\det(A) = |A| = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

For more clarifications on this topic see: Larson R. *Elementary linear Algebra*, Cengage Learning (2017), 8^{th}edition, chapter 3, page 112.

Moreover, a matrix is diagonalizable when it is similar to a diagonal matrix. In other words: *An $n \times n$ matrix A is diagonalizable when there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.* Another condition used to establish if a $n \times n$ matrix is diagonalizable or not is to check if it has n linearly independent eigenvectors[4]. It's diagonalizable only if it does.

-Vector Spaces:

The term vector derives from the Latin word *vectus*, meaning "carrying". The idea is that if a person were to carry something from the origin to the point (x_1, x_2), then the trip could be represented by the directed line segment from $(0, 0)$ to (x_1, x_2). The ordered pair used to represent the terminal point also represents the vector: $\boldsymbol{x} = (x_1, x_2)$, (vectors are represented by lowercase letters set in boldface type such as **u**, **v**, **w**, and **x**). The components of the vector $\boldsymbol{x}$ are its coordinates x_1 and x_2.

Basic vector operations are vector addition and scalar multiplication. In the first case, vectors are added by adding their respective components. In the second, each component of a vector is multiplied by a scalar c.

These two operations share ten properties that are summarized below:

[4] The concept of eigenvector will be explained in the next paragraphs.

Let **u**, **v**, and **w** be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in the plane.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property of addition
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive identity property
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Additive inverse property
6. $c\mathbf{u}$ is a vector in the plane.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property of multiplication
10. $1(\mathbf{u}) = \mathbf{u}$	Multiplicative identity property [5]

In the list, the zero vector $\mathbf{0}$ is the "additive identity" of the vector $\boldsymbol{u}$ while the vector $-\boldsymbol{u}$ is its "additive inverse".

The discussion of vectors in the plane can be extended to a discussion of vectors in n-space. An n-tuple $(x_1, x_2, x_3 \dots, x_n)$ can be considered as a point in R^n with the x_i as its coordinates or as a vector $\boldsymbol{x}$, with the x_i as its components.

More in general, an ordered n-tuple represents a vector in n-space, and the set of all n-tuples is n-space and is denoted by R^n.

As for vectors in the plane, the sum of two vectors in R^n and the scalar multiple of a vector in R^n are the standard operations in R^n. Their ten properties are the same as the ones listed above.

The notions just presented were necessary to properly define what a vector space is.

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the listed axioms are satisfied for every $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ in V and every scalar (real number) c and d, then V is a vector space.

Addition:

1. $\mathbf{u} + \mathbf{v}$ is in V.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	Associative property
4. V has a **zero vector** $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.	Additive identity
5. For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.	Additive inverse

Scalar Multiplication:

6. $c\mathbf{u}$ is in V.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property
10. $1(\mathbf{u}) = \mathbf{u}$	Scalar identity [6]

From this definition we can infer that a vector space consists of four entities: a set of vectors, a set of scalars, and two operations. Examples of vector spaces are: R^2 with the standard operations, the set of all continuous functions defined on the real number line and the set of all $n \times n$ square matrices.

Vector spaces are efficient in mathematics because, once a theorem has been proved for an abstract vector space, it can then be applied to the vector space in analysis without the need of further proof.

[5] Image from Larson R., *Elementary linear Algebra*, Cengage Learning (2017), 8^th edition
[6] Image from Larson R., *Elementary linear Algebra*, Cengage Learning (2017), 8^th edition

Furthermore, in many applications of linear algebra, vector spaces occur as "subspaces" of larger spaces. In general, a nonempty subset of a vector space is a subspace when it is a vector space with the same operations defined in the original vector space. In order to test if a subset of a vector space is a subspace of it, it is sufficient to check two closure conditions: I*f W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the two closure conditions listed below hold*:

1. *If **u** and **v** are in W, then **u+v** is in W.*
2. *If **u** is in W and c is a scalar, then c ×**u** is in W.*

It is worth mentioning that, after W is verified being a subspace of V, all ten properties of V are directly inherited by W, so there is no need of further verification.

An interesting property of vector spaces with finite dimension[7] is that each vector in it can be represented as a "linear combination" of a selected number of vectors in the space. For example, setting a vector $\boldsymbol{v}$ in a vector space V, it can be written in the form: $\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k$, with $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ as vectors and $c_1, c_2, \ldots, c_k$ as scalars.

If every vector in a vector space can be written as a linear combination of vectors in a set, then the set is a "spanning set" of the vector space. The general definition is: *If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is a set of vectors in a vector space V, then the span of S is the set of all linear combinations of the vectors in S,*

$$span(S) = \{c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k : c_1, c_2, \ldots, c_k \ are\ real\ numbers\}$$

When $span(S) = V$, it is said that S spans V or that V is spanned by $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$.

In addition, a set of vectors in a vector space can be "linear dependent" or "linear independent". For the same set S of the vector space V as before: if the vector equation $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$ has only the trivial solution $c_1 = 0, c_2 = 0, \ldots, c_k = 0$, then the vectors in S are linearly independent. On the other hand, if there are nontrivial solutions other than the trivial one, then the vectors in S are linearly dependent.

As far as spanning sets are concerned, if a spanning set in a vector space has both linearly independent vectors and spans the entire space, then it is a "basis" for the vector space. *A set of vectors $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ in a vector space V is a basis for V when the conditions below are true:*

1. *S spans V*
2. *S is linearly independent*.

If a vector space has a basis with a finite number of vectors, then it is "finite dimensional". Otherwise, it's "infinite dimensional".

The notion of basis of a vector space is essential in order to set its dimension. *If a vector space V has a basis consisting of n vectors, then the number n is the dimension of V, denoted by $dim(V) = n$. When V consists of the zero vector alone, the dimension of V is defined as zero*. For example, the dimension of R^n with the standard operations is n.

-Eigenvalues and Eigenvectors:

The origins of the terms *eigenvalue* and *eigenvector* are from the German word *Eigenwert,* meaning "proper value".

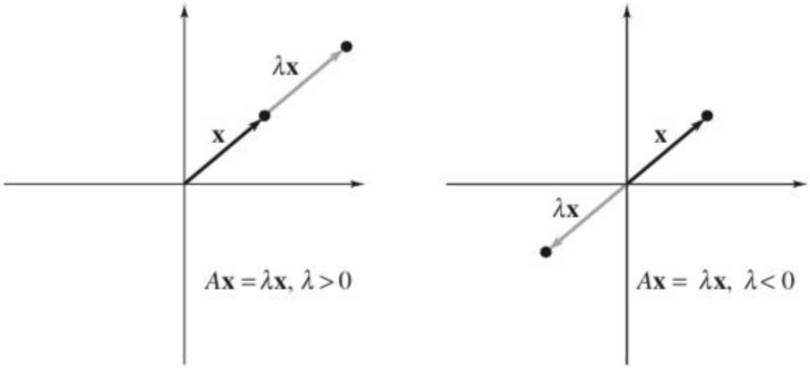
Eigenvalues and eigenvectors have many important applications, one of which is the topic of our presentation.

[7] For the formal definition of what is the dimension of a vector space see the following part.

One of the central problems in mathematics is the so-called "eigenvalue problem": when A is an $n \times n$ matrix (a square matrix), do non-zero vectors $\boldsymbol{x}$ in R^n exist such that $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$? Formally, does $A\boldsymbol{x} = \lambda\boldsymbol{x}$ apply?

The scalar, denoted by the Greek letter lambda (λ), is called eigenvalue of the matrix A, and the nonzero vector $\boldsymbol{x}$ is called an eigenvector of A corresponding to λ.

Considering a two-dimensional space, if λ is an eigenvalue of a matrix A and $\boldsymbol{x}$ is an eigenvector of A corresponding to λ, then multiplication of $\boldsymbol{x}$ by the matrix A produces a vector $\lambda\boldsymbol{x}$ that is parallel to $\boldsymbol{x}$, as is shown geometrically below.



[8]

From the previous considerations, we can derive a general definition of eigenvalue and eigenvector: *let A be a $n \times n$ matrix. The scalar λ is an eigenvalue of A when there is a nonzero vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$. The vector $\boldsymbol{x}$ is an eigenvector of A corresponding to λ.*

From this definition we can say that an eigenvector can't be zero. (If $\boldsymbol{x}$ is the zero vector, then $A\mathbf{0} = \lambda\mathbf{0}$, which is true for any real value of λ, so this can't be a solution.) On the other hand, it is possible to have $\lambda = 0$.

A matrix can have more than one eigenvector. In fact, *if the matrix A is an $n \times n$ matrix with an eigenvalue λ, and a corresponding eigenvector $\boldsymbol{x}$, then every nonzero scalar multiple of $\boldsymbol{x}$ is also an eigenvector of A*. Considering c, a nonzero scalar and $A\boldsymbol{x} = \lambda\boldsymbol{x}$: $A(c\boldsymbol{x}) = c(A\boldsymbol{x}) = c(\lambda\boldsymbol{x}) = \lambda(c\boldsymbol{x})$.

In addition, if $\boldsymbol{x_1}$ and $\boldsymbol{x}_2$ are eigenvectors corresponding to the same eigenvalue λ, then their sum is also an eigenvector corresponding to λ : $A(\boldsymbol{x_1} + \boldsymbol{x_2}) = A\boldsymbol{x_1} + A\boldsymbol{x_2} = \lambda\boldsymbol{x_1} + \lambda\boldsymbol{x_2} = \lambda(\boldsymbol{x_1} + \boldsymbol{x_2})$.

The set of all the eigenvectors of an eigenvalue λ, together with the zero vector, is a special subspace of R^n called the *eigenspace* of λ. The general definition is: *if A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ, together with the zero vector is a subspace of R^n. This subspace is the eigenspace of λ.* Mathematically, we can write the eigenspace as $\{\boldsymbol{x}: \boldsymbol{x}$ is an eigenvector of $\lambda\} \cup \{\mathbf{0}\}$.

Determining the eigenvalues and corresponding eigenspaces of a matrix can involve algebraic manipulation. In the next paragraphs we will describe the series of step that are necessary to find them.

To find the eigenvalues and eigenvectors of an $n \times n$ matrix A, we set I , the $n \times n$ identity matrix. Rewriting $A\boldsymbol{x} = \lambda\boldsymbol{x}$ as $\lambda I\boldsymbol{x} = A\boldsymbol{x}$ and rearranging gives $(\lambda I - A)\boldsymbol{x} = 0$.

This homogeneous system of equations has nonzero solutions if and only if the coefficient matrix $(\lambda I - A)$ is not invertible (if and only if its determinant is zero). Which is represented mathematically by the equation

[8] Image from Larson R., *Elementary linear Algebra*, Cengage Learning (2017), 8^th edition

$\det(\lambda I - A) = 0$, also called the "characteristic equation" of A. Moreover, when expanded to polynomial form, the polynomial $|\lambda I - A|$ is the "characteristic polynomial" of A.

$$|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_2\lambda^2 + c_1\lambda + c_0$$

So, the eigenvalues of an $n \times n$ matrix A correspond to the roots of the characteristic polynomial of A.

The next theorem formally states this solution: *Let A be an $n \times n$ matrix. An eigenvalue of A is a scalar λ such that $det(\lambda I - A) = 0$. The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A) = 0$.*

A brief and more informal description of the steps required to find eigenvalues and eigenvectors can be divided into three parts. In the first, taking the $n \times n$ matrix A, we form the characteristic equation $|\lambda I - A| = 0$, (it will be a polynomial equation of degree n in the variable λ, like in the formula written above). In the second, we find the real roots of the characteristic equation, the solutions are the eigenvalues of A. In the third, for each eigenvalue λ_i, we find the eigenvectors corresponding to it, by solving the homogeneous system $(\lambda_i I - A)\boldsymbol{x} = 0$.

If an eigenvalue λ_i occurs as a multiple root of the characteristic polynomial (it occurs k times), then we say that λ_i has multiplicity k. This implies that $(\lambda - \lambda_i)k$ is a factor of the characteristic polynomial.

Furthermore, the set of all the eigenvectors $\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3} \ldots \boldsymbol{x_n}$ that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ of the $n \times n$ matrix A is linearly independent.

-Markov Chains:

Many types of applications involve a finite set of states $\{S_1, S_2, \ldots, S_n\}$ of a population and a mechanism to move from one state to the other. We assume the movement occurs at a specified moment of time.

If it is possible to go from state S_i to state S_j, we say that S_j is "accessible" from S_i. Furthermore, two states are said to "communicate" if they are accessible to each other. We can write $S_i \leftrightarrow S_j$. From the graphical point of view, communication between two states is represented by two directed paths from S_i to S_j and from S_j to S_j. Following this reasoning, the communication relation is an equivalence relation with the properties of symmetry, reflexivity and transitivity.

The probability that a member of a population will change from the jth state to the ith state is represented by a number p_{ij}, where $0 \leq p_{ij} \leq 1$. A probability of $p_{ij} = 0$ means that the member is almost sure[9] not to change from the jth state to the ith state, whereas a probability of $p_{ij} = 1$ means that the member is almost sure[10] to change from the jth state to the ith state. The set of all these probabilities can be put into P, the "matrix of transition probabilities", which gives the probabilities of each possible type of transition (or change) within the population. At each transition, each member in a given state must either stay in that state or change to another state. From the point of view of probabilities, this means that the sum of the entries in any column of P is 1.

For instance, in the figure below, in the first column $p_{11} + p_{21} + \cdots + p_{n1} = 1$.

[9] The notation *almost sure* instead of *sure* referred to the probability takes into consideration the fact that the set of outcomes on which the event doesn't occur has probability equal to 0, even thought the set might not be empty. This is the case when the sample space taken into consideration is an infinite set.

[10] As in 8, events with probability 1 not necessarily include all possible outcomes.

$$\begin{array}{cc} & \overbrace{\begin{array}{cccc} S_1 & S_2 & \cdots & S_n \end{array}}^{\text{From}} \\ P = & \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \left.\begin{matrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{matrix}\right\} \text{To} \end{array}$$ [11]

P is a "stochastic matrix", (the term "stochastic" means "regarding conjecture"), because is a square matrix with all positive entries and the sum of the entries in each column is equal to 1. The general definition for a stochastic matrix is *an $n \times n$ matrix P is a stochastic matrix when each entry is a number between 0 and 1 inclusive, and the sum of the entries in each column of P is* 1.

Providing some mathematical examples, the matrix A is stochastic since is a square matrix with all the entries between 0 and 1 and the sum of its columns is 1. On the other hand, matrix B is not stochastic. Even if B is a square matrix with entries between 0 and 1, the sum of the entries in each column is not 1.

$$A = \begin{bmatrix} 0.9 & 0.8 \\ 0.1 & 0.2 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} \end{bmatrix}$$

An important remark to make is that, in our discussion, we will consider only "column-stochastic matrices", in which the sum of each column sums up to 1. In reality, also "row-stochastic matrices" exist (matrices where the sum of the entries of each row sums up to 1, but the other characteristic properties for stochastic matrices remain the same: the matrix is a square matrix, and the entries are nonnegative numbers between zero and one). For the reason just stated, in the following paragraphs, we will consider as synonyms "stochastic matrix" and "column-stochastic matrix", without taking into consideration the row-stochastic ones.

A stochastic matrix, representing a matrix of transitional probabilities can be referred to as "state matrix", a matrix in which the entries represent portions of the whole.

The concept of stochastic matrix is essential to define what is a Markov Chain.

A "Markov chain", named after Russian mathematician Andrey Andreyevich Markov (1856–1922), is a sequence $\{X_n\}$ of state matrices that are related by the equation $X_{k+1} = PX_k$, with P as a stochastic matrix.

In a Markov chain, the future depends only upon the present and not upon the past: only the most recent point in time affects what happens next. This means that X_{t+1} depends upon X_t, but not upon $X_{t-1}, \dots, X_1, X_0$; if X_t is known.

In his first application of the model, A. Markov studied the sequence of 20,000 letters in A.S. Pushkin's poem "Eugeny Onegin" discovering the stationary vowel probability of a vowel following another vowel and of a vowel following a consonant. This experiment was relevant not for the result in itself, but for the consideration Markov gave to the *temporal aspect*: his calculations were based on the assumption that a random event can depend only on its most recent past. This is called the *Markov Property*.

In general, the nth state matrix of a Markov chain for which P is the matrix of transition probabilities and X_0 is the initial state matrix is $X_n = P^n X_0$. It is relevant to empathize that, in a Markov chain, we always have to assume that the matrix P of transition probabilities remains constant between states.

[11] Image from Larson R., *Elementary linear Algebra*, Cengage Learning (2017), 8^th edition

A real-life case for a better understanding of this concept could be the process of analyzing a population finding the state matrix representing the portions of that population in each state in three years. This is computed by repeated multiplication of the initial state matrix X_0 by the matrix of transition probabilities P. Consequently: $X_1 = PX_0$, $X_2 = P^2X_0$ and $X_3 = P^3X_0$.

Continuing the process of calculating states year after year, the state matrix X_n eventually reaches a "steady state". That is, as long as the matrix P does not change, the matrix product P^nX approaches a limit $\overline{X}$. This limit is the "steady state matrix".

Furthermore, a stochastic matrix P can be "regular" or "not regular". In the first case, some power of P has only positive entries. In the second, every power of P has zeros in its entries. For a regular stochastic matrix P, the sequence of successive powers, $P^2, P^3, \dots, P^n$ approaches a stable matrix P. The entries in each column of P are equal to the corresponding entries in the steady state matrix $\bar{X}$. If P is not regular, then the corresponding Markov chain may or may not have a unique steady state matrix.

A summary for finding the steady state matrix $\bar{X}$ of a Markov chain is: check that the matrix of transition probabilities P is a regular matrix, solve the system of linear equations obtained from the matrix equation $PX = X$ along with the equation $x_1 + x_2 + \cdots + x_n = 1$, and finally check the solution found in the matrix equation $PX = X$.

A Markov Chain with n different States $\{S_1, S_2, \dots, S_n\}$ and a matrix P of transition probabilities can be represented by a directed graph in which edges are given by transitions with nonzero probabilities connecting the states. In the paragraphs below we will describe different types of Markov chain, providing also a graphical representation.

Markov chains can be used to model real-life situations, one of these is the so called "absorbing" one. Considering a Markov chain with n different states $\{S_1, S_2, \dots, S_n\}$, the ith state S_i is an absorbing state when, in the matrix of transition probabilities P, $p_{ij} = 1$. That is, the entry on the main diagonal of P is 1 and all other entries on the ith column of P are 0.

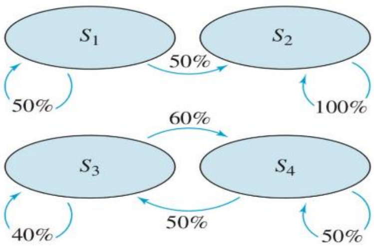
For a general-purpose classification of the states, a state is said to be "recurrent" if, any time that we leave that state, we will return to that state in the future with probability one. On the other hand, if the probability of returning is less than one, the state is called "transient".

An absorbing Markov chain has two properties: the Markov Chain has at least one absorbing state and it is possible for a member of the population to move from any nonabsorbing state to an absorbing state in a finite number of transitions.

A relevant remark to make is that, if the matrix of transition probabilities P of a Markov chain is absorbing, it is not granted that also the Markov chain is so. We will provide two examples for a better understanding of this concept.

Considering the matrix below, the second state, represented by the second column, is absorbing, but the corresponding Markov Chain is not. The reason is that it is not possible to move from the states S_3 and S_4 to the state S_2.

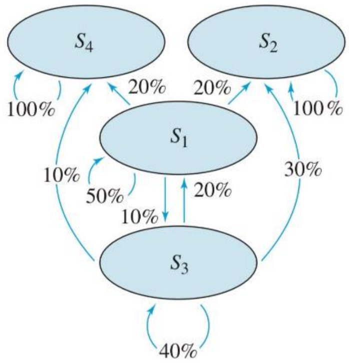
$$P = \begin{array}{c} \overbrace{\begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array}}^{\text{From}} \\ \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0.4 & 0.5 \\ 0 & 0 & 0.6 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ \left.\begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \end{array}\right\} \text{To} \end{array}$$ [12]



[13]

On the other hand, in the second example, the matrix P has two absorbing states S_2 and S_4. The corresponding Markov chain is absorbing because it is possible to move from either the nonabsorbing states (S_1 and S_3) to either of the absorbing states (S_2 and S_4) in one step.

$$P = \begin{array}{c} \overbrace{\begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array}}^{\text{From}} \\ \begin{bmatrix} 0.5 & 0 & 0.2 & 0 \\ 0.2 & 1 & 0.3 & 0 \\ 0.1 & 0 & 0.4 & 0 \\ 0.2 & 0 & 0.1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \left.\begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \end{array}\right\} \text{To} \end{array}$$ [14]



[15]

The steady state matrix for an absorbing Markov chain has nonzero values only in the absorbing states, since these states “absorb” the population. In general, an absorbing Markov chain with one absorbing state has a unique steady state matrix regardless of the initial state matrix. Furthermore, an absorbing Markov chain with two or more absorbing states has an infinite number of steady state matrices, which depend on the initial state matrix.

The opposite of an absorbing Markov chain is the “irreducible” one: a Markov Chain in which all states communicate with each other. Graphically, an irreducible Markov chain is represented by a strongly connected graph.

Another relevant property of Markov chains is periodicity, which measures if the chain returns or not to state i at regular times. Periodicity uses the variable $d(i)$ (for S_i) as a reference frame.

If $d(i) \neq 1$, the chain returns to state i at regular times. In this case we say that state i is “periodic”. On the other hand, if $d(i) = 1$ the chain returns at state i can occur at irregular times (or it can also never occur). Under such conditions, state i is “aperiodic”.

States that communicate have the same period: if $S_i \leftrightarrow S_j$, then $d(i) = d(j)$. If states don’t communicate, then they are aperiodic.

[12] Image from Larson R., *Elementary linear Algebra*, Cengage Learning (2017), 8^th edition
[13] Image from Larson R., *Elementary linear Algebra*, Cengage Learning (2017), 8^th edition
[14] Image from Larson R., *Elementary linear Algebra*, Cengage Learning (2017), 8^th edition
[15] Image from Larson R., *Elementary linear Algebra*, Cengage Learning (2017), 8^th edition

A Markov chain can be "periodic" or "aperiodic", depending on their states. In general, an irreducible Markov chain is aperiodic if and only if all its states are aperiodic, otherwise it is periodic.

A useful method to check if a Markov Chain is aperiodic or not is to use the greatest common divisor (gcd) between two numbers, for example m and l. If gcd (m, l)=1, then m and l are said to be *co-prime*. If we can find two co-prime numbers l and m such that $p_{ii}^{(l)} > 0$ and $p_{ii}^{(m)} > 0$, then we can conclude that state i is aperiodic. That is, we can go from state i to itself in l steps, and also in m steps. If we have an irreducible Markov chain, this means that the chain is aperiodic. Since the number 1 is co-prime to every integer, any state with a self-transition is aperiodic.

-The Power Method:

As stated before, eigenvalues of an $n \times n$ matrix are found by solving the characteristic equation: $|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_2\lambda^2 + c_1\lambda + c_0$. The problem with this method is that, for large values of n, it is complicated and time consuming to solve.

The alternative approach that is presented below is the Power Method: an iterative method with the aim of approximating "dominant eigenvalues" (the eigenvalues of a square matrix A that are largest in absolute value). This is possible if we consider a square matrix A with n linearly independent eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and the corresponding eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$. As the eigenvalues are scalars, we can rank them such that: $|\lambda_1| > |\lambda_2| \ldots > |\lambda_n|$. In addition, eigenvectors can be considered as a basis for the vector space since they are linear independent and span it. Consequently, it is possible to write: $\boldsymbol{x}_0 = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$.

By multiplying both sides by A we get: $A\boldsymbol{x}_0 = c_1A\boldsymbol{v}_1 + c_2A\boldsymbol{v}_2 + \cdots + c_nA\boldsymbol{v}_n$.

Considering also $A\boldsymbol{x}_0 = \lambda\boldsymbol{x}_0$: $A\boldsymbol{x}_0$=$c_1\lambda_1\boldsymbol{v}_1 + c_2\lambda_2\boldsymbol{v}_2 + \cdots + c_n\lambda_n\boldsymbol{v}_n$.

Formally, the general definition of a dominant eigenvalue is "*Let $\lambda_1, \lambda_2 \ldots \lambda_n$ be the eigenvalues of an $n \times n$ matrix A, λ_1 is called the dominant eigenvalue of A if $|\lambda_1| > |\lambda_i|$, $i = 2, \ldots n$. The eigenvectors corresponding to λ_1 are called the dominant eigenvectors of A".* Where $|a|$ is the absolute value of the real number a.

A relevant remark to make is that not every matrix has a dominant eigenvector.

To apply the Power method, we have to assume that the square matrix A has a dominant eigenvalue with corresponding dominant eigenvectors. Then we set the initial approssimation $\boldsymbol{x}_0$ of one of the dominant eigenvectors of A. $\boldsymbol{x}_0$ must be a nonzero vector in R^n. With the first iteration we get:

$A\boldsymbol{x}_0 = c_1\lambda_1\left[\boldsymbol{v}_1 + \frac{c_2}{c_1}\frac{\lambda_2}{\lambda_1}\boldsymbol{v}_2 + \cdots + \frac{c_n}{c_1}\frac{\lambda_n}{\lambda_1}\boldsymbol{v}_n\right] = c_1\lambda_1\boldsymbol{x}_1$, where $\boldsymbol{x}_1 = \boldsymbol{v}_1 + \frac{c_2}{c_1}\frac{\lambda_2}{\lambda_1}\boldsymbol{v}_2 + \cdots + \frac{c_n}{c_1}\frac{\lambda_n}{\lambda_1}\boldsymbol{v}_n$ is a new vector. This result ends the first iteration. Then, it is possible to multiply A to $\boldsymbol{x}_1$ to start the second iteration, obtaining:

$A\boldsymbol{x}_1 = \lambda_1\left[\boldsymbol{v}_1 \frac{c_2}{c_1}\frac{\lambda_2^2}{\lambda_1^2}\boldsymbol{v}_2 + \cdots + \frac{c_n}{c_1}\frac{\lambda_n^2}{\lambda_1^2}\boldsymbol{v}_n\right] = \lambda_1\boldsymbol{x}_2$ and $\boldsymbol{x}_2 = \boldsymbol{v}_1 + \frac{c_2}{c_1}\frac{\lambda_2^2}{\lambda_1^2}\boldsymbol{v}_2 + \cdots + \frac{c_n}{c_1}\frac{\lambda_n^2}{\lambda_1^2}\boldsymbol{v}_n$.

We can continue multiply A with the new vector we get from the iteration k times:

$$A\boldsymbol{x}_{k-1} = \lambda_1\left[\boldsymbol{v}_1 \frac{c_2}{c_1}\frac{\lambda_2^k}{\lambda_1^k}\boldsymbol{v}_2 + \cdots + \frac{c_n}{c_1}\frac{\lambda_n^k}{\lambda_1^k}\boldsymbol{v}_n\right] = \lambda_1\boldsymbol{x}_k$$

Since λ_1 is the dominant eigenvalue, the ratio $\frac{\lambda_i}{\lambda_1} < 1$ for all $i > 1$. So, when k is increased to a sufficient large, the ratio $\left(\frac{\lambda_n}{\lambda_1}\right)^k$ will be close to zero.

For large powers of k, and by properly scaling this sequence, we obtain a good approximation of the dominant eigenvector of A. The term "scaling" refers to the fact that it is best to "scale down" each approximation before proceeding with the next iteration: determine the component of $A\boldsymbol{x}_i$ that has the largest absolute value and multiply the vector by the reciprocal of this component, for a resulting vector with components of value less than or equal to one.

The power method with scaling converges to a dominant eigenvector. A sufficient condition for convergence is that the matrix A is diagonalizable and has a dominant eigenvalue. *If A is an $n \times n$ diagonalizable matrix with a dominant eigenvalue, then there exists a nonzero vector $\boldsymbol{x_0}$ such that the sequence of vectors given by $A\boldsymbol{x_0}, A^2\boldsymbol{x_0}, A^3\boldsymbol{x_0}, \dots, A^k\boldsymbol{x_0}, \dots$ approaches a multiple of the dominant eigenvector of A.*

-the Perron-Frobenius theorem:

The Perron-Frobenius theorem was developed by Oskar Perron in 1907 and Georg Frobenius in 1912. This theory has important applications in probability theory, economics, social networks, and demography. The general definition is: *if all entries of an $n \times n$ real matrix A are positive, then it has a unique maximal eigenvalue. Its eigenvector has strictly positive entries*.

This implies that, if we have a matrix $A \geq 0$, then the dominant eigenvalue of A, $r(A)$, is real-valued and non-negative. For any other eigenvalue λ of A, $|\lambda| \leq r(A)$. Finally, we can find a nonnegative and nonzero eigenvector $\boldsymbol{x}$ such that $A\boldsymbol{x} = r(A)\boldsymbol{x}$.

Moreover, if A is also irreducible then the eigenvector $\boldsymbol{x}$ associated with the eigenvalue $r(A)$ is strictly positive and there exists no other positive eigenvector $\boldsymbol{x}$ (except scalar multiples of $\boldsymbol{x}$), associated with $r(A)$.

For example, given the irreducible nonnegative matrix A:

$$A = \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.2 & 0.4 & 0.3 \\ 0.2 & 0.5 & 0.1 \end{bmatrix}$$

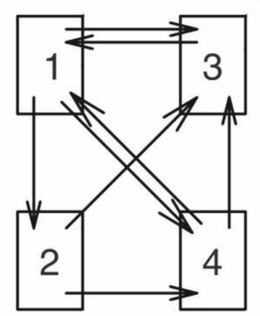
Thanks to the Perron-Frobenius theorem we get the value of its dominant eigenvalue $\lambda = 0.8444$.

4) Mathematical explanation of the model

The web is formed of millions and millions of pages which are interconnected with each other through a series of links. At each web page can be assigned a number called “score” or “importance score”, rating quantitatively the webpage’s importance. Such value must be a nonnegative real number and is delivered from the links made to that page from other web pages. The web thus becomes a democracy where pages vote for the importance of other pages by linking on them.

This concept can be represented graphically through a *directed graph* with a set of vertices (web pages) and a set of edges (links) which join a pair of vertices. An arrow starting from a page and pointing another indicates a link. The graph is directed since each edge has a direction, a starting and an ending vertex.

Below we can see a graphical representation:



16

On various websites it is possible to find an approximation of a page's PageRank. This reported value is based on a scale of 10. For instance, the home page of "The American Mathematical Society" has a PageRank of 8. It is important to underline that it is only an approximation, since Google declines to publish actual PageRank in the attempt to frustrate those who would manipulate the ranking in their favor.

Denoting as k (with k as a real number) the "importance score" of a web page, we can set $k \leq n$, where n represents the total number of pages in the web of interest. Comparing consequently, if $k = 0$, the page has the lowest possible importance score. Each link to page k becomes a vote for page k's importance and adds to its final value. Comparing page i and page j, the first is more important than the latter if $k_i > k_j$.

Another relevant feature to consider in our analysis is that each page has its own relative importance, influencing our study in assessing a value for k. Therefore, a link to page i from an important page should boost k_i's value more than a link from an unimportant page. Even if this principle is correct in itself, it could be used to gain extra influence by a web page, by simply linking to lots of other pages, resulting in a too self-referential scheme. Therefore, in our model, for each link the web page gets a total of one vote, weighted by that web page's score, which is divided up among all its outgoing links. In addition, a link from a page to itself is not counted. Mathematically, k_i's value can be expressed as $k_i = \sum_{j=0}^{n} \frac{k_j}{n_j}$, where k_j is the importance score of page j and n_j the number of outgoing links from page j. All the numbers are assumed to be positive.

Thanks to the formula to find a value for k_i, the web ranking problem can be transformed into a problem of finding an eigenvector for a square matrix. Calling A the "link matrix" of a given web, we search for an eigenvector $\boldsymbol{x}$ with eigenvalue λ equal to 1 for the matrix A. We also call $\boldsymbol{x}$ a "stationary vector" of A. (By definition, $A\boldsymbol{x} = \lambda\boldsymbol{x}$ with $\boldsymbol{x} \neq 0$, for eigenvalues λ and eigenvectors $\boldsymbol{x}$ of a matrix A). Using this formula, we obtain a ranking of pages which is more accurate and different from the one we could obtain by simply counting back links.

Taking as an example the previous graph:

$A = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}$, with eigenvectors equal to 1 all the multiples of the vector $\boldsymbol{x} = \begin{bmatrix} 12 \\ 4 \\ 9 \\ 6 \end{bmatrix}$. The subsequent results are the importance scores: $x_1 = \frac{12}{31} \approx 0.378, x_2 = \frac{4}{31} \approx 0.129, x_3 = \frac{9}{31} \approx 0.290, x_4 = \frac{6}{31} \approx 0.194.$

In general, the link matrix A has 1 as an eigenvalue if the web has no dangling nodes (pages without outgoing links). This result is used in the study of Markov chains, in which the principle applies: *A square matrix is a column stochastic matrix if all of its entries are nonnegative and the entries in each column sum to 1*. The matrix A for

[16] Image taken from Brian K., Leise T., *The $25,000,000,000 Eigenvector: The Linear Algebra behind Google*, "Siam REVIEW", vol.48 n.3, September 2006

the web in analysis has no dangling nodes and the sum of the elements present in each column is equal to 1, so it is column stochastic.

From this example we can derive the general proposition: *every column stochastic matrix has 1 as an eigenvalue*.

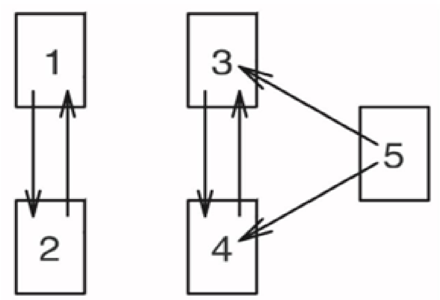
Formally we can write: $\dim(V_1(A)) = 1$, where $V_1(A)$ represents the eigenspace for the eigenvalue 1 of a column-stochastic matrix A. Its dimension is equal to 1 so that there is a unique eigenvector $\boldsymbol{x}$, such that $\Sigma_i x_i = 1$.

However, using the previously explained formula may bring some issues in the process of ranking the elements of a web. Worth mentioning examples are webs with nonunique rankings and webs with dangling nodes.

As far as dangling nodes are concerned, the link matrix A contains one or more columns of zeros. A is column-substochastic: the column sums of A are less than or equal to 1 and all its eigenvectors are less or equal to 1, but 1 need not actually be an eigenvalue for A.

Later, in our discussion, we will present a solution to this problem provided thanks to the Power method.

On the other hand, webs with nonunique rankings consist of r disconnected sub-webs $W_1, W_2 \ldots W_r$, resulting in a difficulty to find a common reference frame for comparing the scores of pages in one sub-web with those into another. In fact, the link matrix A doesn't yield a unique ranking for all the webs, instead provides more eigenvectors that can be considered as a solution to the problem, and it is not clear which should be used for the final ranking. This situation arises since $\dim(V_1(A)) \geq r$, and there is no unique importance score vector $\boldsymbol{x}$ with $\Sigma_i x_i = 1$; which are values different from the ones of the previous example. We summarize the problem with the following graph and corresponding link matrix A.



[17]

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$V_1(A)$ is two-dimensional ($\dim(V_1(A)) > 1$) and so there is not a unique importance score vector $\boldsymbol{x}$ with $\Sigma_i x_i$=1. To solve the problem, we must assume that the matrix A has a block diagonal structure, where A_i represents the link matrix for W_i. This result is possible if we consider each sub-web W_i as a web on its own. Each matrix A_i is column-stochastic with some eigenvector $\boldsymbol{v_i}$ with value 1. In addition, for each i between 1 and r, we create a

[17] Image from Brian K., Leise T., *The $25,000,000,000 Eigenvector: The Linear Algebra behind Google*, "Siam REVIEW", vol.48 n.3, September 2006

vector $\boldsymbol{w}_i$, which has zero components for the elements corresponding to blocks other than block i, making them linearly independent eigenvectors for A with eigenvalue 1.

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \dots & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & 0 & 0 & A_r \end{bmatrix}, \boldsymbol{w}_1 = \begin{pmatrix} v_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } \boldsymbol{w}_2 = \begin{pmatrix} 0 \\ v_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \mathbf{w}_r = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v_r \end{pmatrix}$$

$$\text{so that} \qquad A\boldsymbol{w}_i = A \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \boldsymbol{v}_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \boldsymbol{w}_i$$

In addition, a possible remedy for $\dim(V_1(A)) > 1$ is given by a modification to the link matrix A. This solution applies to all the n-page webs without dangling nodes, including the case with multiple sub-webs.

Calling S an $n \times n$ column-stochastic matrix such that $\dim(V_1(S)) = 1$, we can calculate M, a matrix which is the weighted average of the link matrix A and S.

$$M = (1 - m)A + mS$$

m is a value between 0 and 1, (Google sets $m = 0.15$). The matrix M is column-stochastic and $\dim(V_1(M)) = 1$.

Two extreme cases are: $m = 0$ and $m = 1$. In the first one $M = A$, (we get the problem discussed in the above part). When this happens, M is sub-stochastic as A, but the formula doesn't provide a solution for dangling nodes. In the second case $M = S$: this is the most egalitarian result achievable since all web pages are equally important.

Considering again $M = A$, we can set the equation $\boldsymbol{x} = M\boldsymbol{x}$ as: $\boldsymbol{x} = (1 - m)A\boldsymbol{x} + m\boldsymbol{s}$, with $\boldsymbol{s}$ as a column vector with all entries $1/n$. This further step is really important in our reasoning for a variety of reasons that will be explained in the next paragraphs.

First, the advantage of using M instead of A is that it enables us to compare pages in different sub-webs. This is possible thanks to the properties of matrix M. We can say that if M is positive and column-stochastic, then any eigenvector in $V_1(M)$ has all positive or all negative components. Consequently, if each entry M_{ij} of M is strictly positive for all i and j, then $\dim(V_1(M)) = 1$.

A good approach to show this result is to use a contradiction: if we suppose that there are two linearly independent vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in the subspace $V_1(M)$; then, for any real number s the vector $\boldsymbol{x} = \boldsymbol{v} + s\boldsymbol{w}$ must be in $V_1(M)$, with all negative or all positive components. But it is a general proposition in linear algebra that the vector obtained by two linearly independent vectors and some choice of s, must contain components of mixed sign, a contradiction. We conclude that $V_1(M)$ can't contain two linearly independent vectors, and so it has dimension 1.

Second, the column vector $\boldsymbol{s}$ with all entries $1/n$ helps us solve the problem of dangling nodes, being an application of the Power Method, thanks to its general principle of "convergence".

The reasoning behind this is the random surfing model explained in the introduction of this text: if we surf randomly, at some point we will surely get stuck at a dangling node, a page with no links. To keep going we will choose the next page at random, pretending that a dangling node has a link to every other page in the web. This is a probabilistic interpretation of the link matrix. The direct effect is its modification: the columns of all zeros corresponding to a dangling node will be replaced with a column in which each entry has a value of $1/n$. The new matrix obtained is stochastic and, by property, always has a stationary vector.

A relevant example to show why the Power Method is so important is Google's real-life link matrix H: a square matrix with $n = 25\ billion$ columns and rows. Most of the entries in H are zero: studies show that web pages have on average about ten links, meaning that, on average, all but 10 entries in every column are 0. Furthermore, considering all $25{,}000{,}000{,}000$ Google's pages, it is more probable to find some dangling nodes than in simpler models like the ones presented above.

Following this theory and coming back to our reasoning with matrix M, we begin by choosing a vector $\boldsymbol{x}^0$as a candidate for the stationary vector $\boldsymbol{x}$, and then produce a sequence of vectors $\boldsymbol{x}^k$by $\boldsymbol{x}^{k+1} = M\boldsymbol{x}^k$. The general principle is that the sequence $\boldsymbol{x}^k$ will converge to the stationary vector $\boldsymbol{x}$, which will tell the relative importance of each page.

As the connectivity of the graph increases (a larger number of edges), convergence is usually achieved after fewer iterations, and the convergence curves for directed and undirected graph practically overlap.

Without the just explained reasoning, in applying $\boldsymbol{x}^{k+1} = M\boldsymbol{x}^k$, a page with no links would have taken all the importance from the other pages in each iterative step, without passing it to any other page. The final effect is that all the importance is drained off the web and the ranking of all the pages is zero.

For instance, considering a simple web of two pages in which only one is linked to the other while the other has no links, after only three iterations of the presented algorithm, the result would be that the importance score of both pages is zero.

Finally, after explaining all the reasoning behind Google's Page rank algorithm and the mathematic calculations involved, we can give a general and more schematic definition, providing the formula for the importance score k_i of page i.

$$k_i[A] = \frac{m}{N} + (1 - m)\left(\sum_{k=1}^{n} \frac{k_j}{n_j}\right)$$

Now, N is the total number of pages in the web in analysis. The number $d = (1 - m) = 0.85$ is Google's "dumping factor", which represents the probability that the random surfer will continue to click link after link iteration after iteration, without finding any dangling node. Instead, the probability that the random clicking will eventually stop is represented by $m = 0.15$. n represents the number of pages in the web containing at least a link towards the link matrix A. k_j is the importance score of each page linking page i, and n_j the number of outgoing links from page j.

5) Real-life application of the model

Google's algorithm PageRank has many possible real-life applications. The one we will describe and try to replicate in our analysis is the measurement of individual contributions in teams[18], taking as a unit of measure the coworker productivity also called *eigenvalue productivity* (EVP).

In general, when people are divided in teams and work together, the productivity of the team as a whole is evident and can be assessed in relation to a variety of observable and relatively easy to measure variables, for example: the quality of the final product, total sales, revenues, percentage of wins, points scored, number of orders, etc.

On the other hand, measuring the productivity of each team member is not readily identifiable, even if a fixed task is assigned to each of its components. This difficulty is inherently associated with the very nature of teamwork: a team is not just the sum of its members' abilities and the interaction between team members is "multifaceted". Coworkers interact, and the output is the result of the combination of their capacities and productivities. In addition, the interplay of workers also depends on their actual willingness to cooperate. A worker can work efficaciously only if he goes along with his endeavor. (Without the necessary cooperation and coordination, problems like free riding and moral hazard can arise in teams). This multiplicity of interactions raises the question of how one can consistently define and then calculate coworker productivities from the team's observed data.

A real-life example can be found in team sports. Throughout the years, a variety of units of measure have been used to infer the coworker productivity of a player: number of goals scored, assist provided, duels won, ball touches. However, these numbers wrongly attribute successful actions to an individual player, while they are the joint product of the player and his teammates. A player performs well only if the other components of the team are willing to lay the proper groundwork for the player 's success.

Consequently, we can state that the team productivity of a player depends on the productivities of all the teammates, and in particular on the ones of its "neighbors" in the field: empirical evidence is showing that some combinations of positions or some pairs of players are more complementary than others.

Summarizing: In a team, the team productivity of coworker i depends positively on the productivity of all other teammates and, in particular, of those who are "adjacent to" or "central for" that player. Specifically, the more a teammate of coworker i contributes to the team, the better will be the conditions for worker i to perform well and contribute to the success of the team. Since this is true for any team member, the productivities of all coworkers on a team must be determined simultaneously.

The aim of the model that will be presented here is to formalize this concept mathematically, in a general and flexible way that can be applied to distinct economic contests and with many different available databases. Moreover, since EVP provides the ranking of team members, it can also be related to the literature on ranking.

Before the mathematical explanation, further clarifications on basic assumptions are needed.

First, the general pattern is that groups arrive at more rational decisions than do individuals, suggesting that teamwork outperforms individual actors.

Moreover, the approach used is to calculate the coworker productivities of a given team assigned to a given task. The decision process of the manager on the optimal composition of the team or on how and why he/she selected a specific worker for a given project is not part of the model. This kind of decisions are considered as exogenously determined.

[18] In the next paragraphs we will make an in-depth description of the research article: Müller J., Uppmann T., *Eigenvalue productivity: Measurement of individual contributions in teams*, "PLOS ONE", vol.17, n.1, September 2022

In addition, certain team roles are more important for team performance than others. The recognition of a single member of a team will produce positive "spillover effects" on the performance of other team members, as well as the overall team performance, via social influences processes. This is particularly the case when the distinguished individual has a central position in the team. In fact, there are some individuals that, when added to a team, consistently lead to an outperformance of the team over its predicted one. Beyond they're task-specific skills, these people have higher social skills and seem to motivate teammates to exert more individual effort.

As far as team structure is concerned, the more centralized one is considered as the most successful. The degree and type of centralization of a team is frequently analyzed in network analysis, which studies the interactions between nodes and how they work in case of externalities.

Considering a team $N = \{1, 2 \ldots, n\}$ of a fixed set of n workers, and assuming the productivity of each worker as nonnegative, we can write the formula for the productivity of worker i as:

$$p^i(N) = \frac{1}{\lambda} \sum_{j \epsilon N} g_{ij}(N) p^j(N), \ \forall i \epsilon N$$

Where $g_{ij}(N) \geq 0$ quantifies the extent to which worker i benefits from the coworker productivity of worker j. It can be interpreted as a measure of the productivity-enhancing effect worker j exerts on worker i's productivity, thanks to j's proficiencies, team skills, social competencies, etc. Following the model, the corresponding $g_{ii}(N) \geq 0$ represents the idiosyncratic productivity of worker i, which is the productivity he would have without the "positive effect" of his teammate's productivities. A remark to make is that, as previously stated, team members receptiveness for other member's constitutions is not symmetric. Mathematically, it is mirrored by the fact that usually g_{ij} and g_{ji} are not equal.

$\lambda > 0$ is a strictly positive normalization factor, used to adjust values in the model. By multiplying the productivities of the workers by $\frac{1}{\lambda}$, they are reduced to a value between 0 and 1. The sum of all the normalized $p^i(N)$ will be equal to 1. Graphically, this concept could be represented by a function with the area under the graph equal to 1. An example is the standard normal distribution of a normalized Gaussian function.

Since the relation just presented holds for all the workers in N, we can write the equation system:

$$\boldsymbol{p}(N) \equiv \frac{1}{\lambda} G(N) \boldsymbol{p}(N)$$

$\boldsymbol{p}(N) = (p^i, \ldots, p^n)(N)$ is the vector of coworker productivities. $G(N)$, instead, represents the matrix of the coefficients measuring the extent to which individual productivities of team members affect each other: $G(N) \equiv \left[g_{ij}(N)\right]_{i,j \in N} \geq \boldsymbol{0}$.

The matrix of pairwise productivities coefficients G is nonnegative, nonzero, and irreducible with main diagonal elements being normalized to unity, i. e. $g_{ii}(N) = 1$, so that $G \geq I$.

(For a more general definition: $g_{ii}(N) = c$, where c is a nonnegative constant. In this way it is possible to write $G \geq cI$, even if for the purpose of our model is sufficient $G \geq \boldsymbol{0}$).

The main diagonal elements of the productivity matrix G represent stand-alone productivities of the team members, which can be interpreted as fixed effects in economic terms. These productivities include intrinsic and idiosyncratic components that can be employed on a stand-alone basis, for example in solo projects. The team-dependent productivities, instead, are captured by the off-diagonal elements of G.

Since in our model we are interested in conceptualizing and measuring team productivities, any heterogeneity in stand-alone productivities is disregarded. Such assumption is mirrored by the fact that diagonal elements of G are homogeneous, for example $g_{ii}(N) = 1, \forall i \epsilon N$.

In order to be able to compute also stand-alone productivities and relaxing the assumption, the required team data must have a richer structure, including also statistics from projects carried on by a single person. With this information available, the EPV can be generalized by substituting the constant diagonal $(1, \dots, 1)$ with the non-homogeneous stand-alone productivities $(g_{11}, \dots, g_{nn})$.

Following the method presented above about how to find eigenvalues and eigenvectors of a matrix, with the help of the identity matrix I, we can rearrange the previous formula as:

$$\lambda \boldsymbol{p} = G\boldsymbol{p}$$

$$\lambda I\boldsymbol{p} = G\boldsymbol{p}$$

$$(G - \lambda I)\boldsymbol{p} = \mathbf{0}$$

This system has a solution in $\boldsymbol{p}$ if and only if $\det(G - \lambda I) = \mathbf{0}$, for $\boldsymbol{p} \neq \mathbf{0}$. This reasoning is equal to λ being an eigenvalue of G, and $\boldsymbol{p}$ being the corresponding eigenvector. Since $\boldsymbol{p}$ is the vector of individual productivities we want to determine, we refer to the concept of coworker productivities as *eigenvalue productivity* (EVP).

In addition, by definition of an eigenvector we can conclude that $\boldsymbol{p} > \mathbf{0}$. Consequently, it is possible to prove the uniqueness of the EVP: *for any nonnegative, irreducible $n \times n$ matrix of pairwise, directional production coefficients G, the eigenvalue productivity vector $\boldsymbol{p} > \mathbf{0}$, is unique up to a scalar.*

This result is possible thanks to the Perron-Frobenius theorem, according to which a real square matrix with positive entries (G in our case) has a unique real eigenvalue of largest magnitude, with a corresponding eigenvector having strictly positive components ($\boldsymbol{p}(N)$).

The EPV vector is uniquely defined (except for scaling), since the corresponding Markov Chain is irreducible by assumption.

Eigenvalue productivity provides a ranking of team members. To do so, EVP relies on the pairwise interactions of all team members. These interactions include both intrinsic and behavioral characteristics.

The EPV vector has convenient economic properties, namely *symmetry, permutation covariance, null player property, aggregate balance, differentiability, relative monotonicity, absolute monotonicity, duplication monotonicity.*

The property of symmetry, from the ancient Greek *συμμετρία,* states that a mathematical object remains unchanged under a set of operation transformations. For example, in linear algebra, a symmetric matrix is a matrix that is equal to its transpose. Formally, considering the matrix A, it is symmetric if $A = A^T$. In the context of social interactions symmetry is used in a variety of cases, for example in the assessment of reciprocity, empathy, sympathy and dialogue between individuals. In our model, it is related to the fact that players contributing equally to each team member should be treated identically.

On the other hand, permutation covariance is a measure of the variability of the act of ordering or changing the linear ordering of the members of a set (ordered or not). In the model in analysis, this property requires that a renumbering or renaming of the players should not affect their productivity measures. Therefore, upon renumbering the players, the productivity measures should change accordingly.

The properties of symmetry and permutation covariance represent natural properties for a productivity measure.

For the null player property, if a player is a *null player*, (a player i of N who is contributing nothing to the productivity of the other team members), then he should be assigned a productivity measure of zero. Another relevant feature of this property is that null players can be added or removed from the team without affecting the EPV values of the other players, including the other null players if any. Also, the so-called *nullifying players* exist: a player that makes every coalition in which is put earn zero worth. We will not consider them in our analysis.

On the other hand, the property of aggregate balance sets that if all players contribute equally in total terms, then the same productivity measure should be assigned to each team member, irrespective of the distribution of their pairwise productivities. It is important to underline that this property doesn't require that two players contributing in equal total terms have the same productivity measure, unless all other $n-2$ players also contribute to the same amount or unless the two players contribute equally to each player for symmetry.

In mathematics, the property of differentiability describes a function whose derivative exists in all its domain. The graph of a differentiable function has a non-vertical tangent line for each point in which it is defined. A differentiable function doesn't contain any break, angle or cusp. For the case in analysis, differentiability sets that small productivity changes do not bring sudden changes in the productivity measure: small changes in the productivity matrix lead to small changes in the productivity measure.

Moreover, the concept of monotonicity is generally used to indicate the property of a function or a sequence of being increasing or decreasing. For example, a function of an ordered set E is monotonic if for each couple of points x_1 and x_2 in E with $x_1 < x_2$, the function is $f(x_1) < f(x_2)$ (increasing monotonicity); or $f(x_2) < f(x_1)$ (decreasing monotonicity).

In our model, relative monotonicity and absolute monotonicity consider the effects of nonnegative and nonzero perturbations of the i^{th} row of G: player i becomes more productive than at least one other player, so the productivity measure should mirror this increase in productivity.

Finally, the property of duplication monotonicity considers the case in which a clone of the player i is added to the team N. The direct consequence is that the productivity of player i doesn't increase: adding clones decreases the productivity measure of all players whose characteristics are duplicated. The more clones of player i are part of the team, the less crucial this type of player becomes.

In order to compute the matrix G, the first step is to delete from the set of workers N, the ones that weren't in action during the period of our study. Then, for each pair of workers $\{i, j\}$ in N, we must consider all the projects in which i and j worked together: for each of them the ratio of the points in a "success measure" to the maximal numbers of points the team (including players i and j) could have achieved is computed. This point ratio, denoted as s_{ij}, measures the success performance of the pair over all team compositions.

Then, all the projects in which worker i was a member of the team are considered. In this case worker j may or may not be part of the team. The points these teams have achieved are divided by the maximal number of points they could have potentially gained. The result is the point ratio measuring the success performance of worker i, denoted as: s_{ii}.

Consequently, the pair-success ratio of workers that have been jointly included in team compositions during the period is set as: $s_{ij} = \sqrt{s_{ii}s_{jj}}$.

Collecting all the pair-success ratio gives the symmetric matrix $S \equiv (s_{ij})_{i,j \in N}$.

Then we compute the ratio $g_{ij}(N) \equiv \frac{s_{ij}}{s_{jj}}$, which is the relative performance of the pair $\{i, j\}$ compared to the overall performance of worker j, an effect attributable to the cooperation with worker i. This definition explains why the main diagonal elements equal unity, since $g_{ii}(N) \equiv \frac{s_{ii}}{s_{ii}} = 1, \forall i \in N$.

The elements of the i^{th} row of G represent the increase in productivity of each worker j, due to the contributions of worker i. Therefore, the i^{th} column of G shows how each of the team members contributes to the performance of i. Following this reasoning, the matrix G represents all relative normalized pairwise performance measures.

In the next paragraphs we will provide a simple and artificial example, showing how the matrix G and the EPV vector can be calculated and how the EPV is consistent with the differences in productivities of the team members that are observable after a close look at the data.

A team N of five workers is considered ($N = \{A, B, C, D, E\}$). Team members have worked over seventeen projects in six different compositions over a given period of time. The results of each specific team performance has been evaluated on a ratio of 26 out of 51 possible abstract points of success. The detailed results for the specific team composition are displayed below:

no.	compositions	projects	points	max. pts.	ratio
1	ACD	2	3	6	$\frac{1}{2}$
2	ACE	4	12	12	1
3	ADE	3	1	9	$\frac{1}{9}$
4	BCD	3	4	9	$\frac{4}{9}$
5	BCE	2	6	6	1
6	BDE	3	0	9	0
	sum	17	26	51	$\frac{26}{51}$

[19]

From the data provided, we can infer that the success of a team improved whenever C joined it, so C must have a relatively high coworker productivity. Comparing ACD with ADE, ADE with ACE, BDE with BCD, and BDE with BCE; the ratio of achieved points to maximal points has gone up by replacing D or E by C.

Another important observation to make is that A and B were never included in the same team composition: this situation can arise for workers with the same area of expertise, for example IT specialists or goalies in team sports. However, team performance has improved each time B has been replaced by A. In addition, team performance has declined each time D was included. The EPV should then assign a higher coworker productivity to A than B, and a particularity low value for D.

From the data provided in the previous table, it is possible to compute the individual results for each worker of N, by disregarding the projects in which he/she was not included in the team composition:

worker	incl. in composition	points	max. pts.	s_i
A	{1, 2, 3}	16	27	$\frac{16}{27}$
B	{4, 5, 6}	10	24	$\frac{5}{12}$
C	{1, 2, 4, 5}	25	33	$\frac{25}{33}$
D	{1, 3, 4, 6}	8	33	$\frac{8}{33}$
E	{2, 3, 5, 6}	19	36	$\frac{19}{36}$

[20]

Taking the data, we can write the matrix of pairwise success S and then compute the matrix G, following the steps described above.

[19] Image from Müller J., Uppmann T., *Eigenvalue productivity: Measurement of individual contributions in teams*, "PLOS ONE", vol.17, n.1, September 2022

[20] Image from Müller J., Uppmann T., *Eigenvalue productivity: Measurement of individual contributions in teams*, "PLOS ONE", vol.17, n.1, September 2022

$$S = \begin{pmatrix} \frac{16}{27} & \frac{2\sqrt{5}}{9} & \frac{5}{6} & \frac{4}{15} & \frac{13}{21} \\ \frac{2\sqrt{5}}{9} & \frac{5}{12} & \frac{2}{3} & \frac{2}{9} & \frac{2}{5} \\ \frac{5}{6} & \frac{2}{3} & \frac{25}{33} & \frac{7}{15} & 1 \\ \frac{4}{15} & \frac{2}{9} & \frac{7}{15} & \frac{8}{33} & \frac{1}{18} \\ \frac{13}{21} & \frac{2}{5} & 1 & \frac{1}{18} & \frac{19}{36} \end{pmatrix}, G = \begin{pmatrix} 1 & \frac{8}{3\sqrt{5}} & \frac{11}{10} & \frac{11}{10} & \frac{156}{133} \\ \frac{3\sqrt{5}}{8} & 1 & \frac{22}{25} & \frac{11}{12} & \frac{72}{95} \\ \frac{45}{32} & \frac{8}{5} & 1 & \frac{77}{40} & \frac{36}{19} \\ \frac{9}{20} & \frac{8}{15} & \frac{77}{125} & 1 & \frac{2}{19} \\ \frac{117}{112} & \frac{24}{25} & \frac{33}{25} & \frac{11}{48} & 1 \end{pmatrix}$$

Finally, we can calculate the dominant eigenvalue of G: $\lambda = 4.97$. The associated eigenvector (the EPV) is $p(N) = (1, 0.7849, 1.3276, 0.4492, 0.9203)$.

The EPV values of the workers mirror the observations made before the calculation, as workers A and C have high values of coworker productivity (1 and 1.3276 respectively), while B and D are low (0.7849 and 0.4492).

6) The EPV applied to the PRIN 2022

In the next paragraphs we will present an application of the EPV model[21] to some real-word data: the PRIN 2022 research program.

PRIN which stands for *Progetti di ricerca di Rilevante Interesse Nazionale* or Projects of research of relevant national interest, is a contest of the MUR (the Italian ministry of university and research) [22]. The aim of the PRIN is the promotion of the national system of research with the financing of project of public research to strengthen the interaction between universities and research institutions and enabling Italy's participation to the initiatives of the *Programma Quadro di ricerca ed innovazione dell'innovazione europea* (also called Horizon 2020[23]).

The PRIN finances three-year projects that are demanding both from the point of view of the number of professors and researchers involved and from the one of the funds needed, which are much more than what a single institution could provide, for example universities.

The candidates that can present this project are Italian universities (public and private), and all public research entities working inside the MUR located in Italy.

The principles of the program are three. According to the first one the scientific coordinator called PI (principal investigator) must have a high-quality scientific profile. In addition, it sets some conditions on the originality of the project, the adequacy of the applied method and its feasibility. The second principle states that it can be financed a project relative to any research field. Finally, the third one establishes that the MUR must guarantee adequate financing.

The project is divided in three macro-areas: Life sciences (LS category), Physical, Chemical, Engineering Sciences (PE category), Social and Human sciences (SH category).

[21] We will use the same notation of the EPV model to the PRIN analysis.

[22] Before called *Ministero dell'Istruzione e del Merito* or ministry of the education and merit.

[23] Horizon 2020 is an EU research and innovation program, with almost €77 billion of funding, created to achieve smart, sustainable and inclusive economic growth. The goal is to ensure Europe produces world-class science and technology, removes barriers to innovation and makes it easier for the public and private sectors to work together in delivering solutions to the big challenges our society faces.

The focus will be on the SH1 category, which includes as areas of study individuals, markets, organizations, economics, finance and management.

Taking the data from the documents: Bando PRIN 2022 PNRR: Allegato C Piano dei Costi e dei Contributi[24] and ALLEGATO A - GRADUATORIE SETTORE SH1[25], it is possible to apply the model developed by Müller J. and Uppmann T.[26], to measure the level of individual contributions of universities and research entities when working in teams. First, we will make a list of the required steps necessary to compute the ranking. Then, an application with some modifications of the EPV model is presented.

Universities were divided in teams of two to four members and each team was assigned a certain score. All the teams involved into the ranking provided by the PRIN 2022 and the final score they obtained were taken into consideration.

Then, for each university i the value s_{ii} is estimated, computing the arithmetic mean of all the projects in which i contributed. The values of the points obtained are expressed in a range from 0 to 100 and represent percentage points. Below there is a list of the Italian universities and public research entities involved, with the corresponding scores:

Ranking	University / Research Entity	Score
1	LA SAPIENZA	88
2	CATANIA	87
3	UNICUSANO	86
4	BASILICATA	86
5	FEDERICO II	85
6	ROMA TRE	85
7	UNINT	85
8	PIEMONTE ORIENTALE	85
9	BOCCONI	84,22222222
10	MILANO	83,625
11	GRAN SASSO	83
12	PARTHENOPE	82,5
13	CAMERINO	82,5
14	TORINO	82,42857143
15	TOR VERGATA	82,33333333
16	AQUILA	82
17	CA' FOSCARI	81,85714286
18	CATTOLICA	81,71428571
19	BOLOGNA	81,69230769
20	BOLZANO	81
21	TRENTO	81
22	PERUGIA	81
23	GIUSEPPE DE GENNARO	81
24	CALABRIA	81
25	TERAMO	81

[24] ALLEGATO C-Piano dei Costi e dei Contributi, “Ministero dell’Università e della Ricerca”, see Appendix
[25] ALLEGATO A - GRADUATORIE SETTORE SH1, “Ministero dell’Università e della Ricerca”, https://www.mur.gov.it/sites/default/files/2023-07/DD%20n.%201206%20SH1_Allegato%20A.pdf
[26] Müller J., Uppmann T., Eigenvalue productivity: Measurement of individual contributions in teams, “PLOS ONE”, vol.17, n.1, September 2022

26	LINK CAMPUS	81
27	LUISS	80,2
28	ALTI STUDI LUCCA	80
29	MEDITERRANEA	80
30	POLITECNICO DI TORINO	79,5
31	PAVIA	79,5
32	MILANO BICOCCA	79,4
33	PADOVA	79,3
34	CONSIGLIO NAZIONALE RICERCHE	79,2
35	SANT'ANNA	79
36	POLITECNICO DI MILANO	79
37	GENOVA	79
38	FERRARA	78,5
39	PALERMO	78,5
40	FIRENZE	78,33333333
41	PARMA	78
42	PISA	77,66666667
43	BERGAMO	77,6
44	MODENA REGGIO EMILIA	77
45	SIENA	77
46	POLITECNICO MARCHE	77
47	LUMSA	77
48	MESSINA	77
49	CAGLIARI	77
50	BARI ALDO MORO	77
51	VERONA	76
52	POLITECNICO BARI	76
53	CHIETI-PESCARA	75,5
54	UDINE	75
55	CARLO CATTANEO	75
56	TUSCIA	75
57	SANNIO DI BENEVENTO	75
58	FOGGIA	75
59	FORO ITALICO	75
60	BRESCIA	75

By following the EPV model, it is possible to compute the values $s_{ij} = \sqrt{s_{ii}s_{jj}}$, representing the pair-success ratio of the universities i and j working together in the same team. Consequently, we can construct the matrix $S_0 \equiv (s_{ij})_{i,j\in N}$ and calculate the relative performance of the pair of universities $\{i,j\}$ compared to the overall performance of university j, expressed by the ratio $g_{ij} \equiv \frac{s_{ij}}{s_{jj}}$. The matrix G_0 is formed by all these ratios. From G_0 we get a list of the corresponding eigenvalues: from the dominant eigenvalue of G_0, λ_0, we get the corresponding eigenvector $\boldsymbol{p}_0$ providing us with the ranking of the universities and the research entities involved.

In the next paragraphs we will show that, we can make a modification to our model by creating a matrix S_1 with the pair-success ratios s_{ij} expressed in the form of the arithmetic mean of the project in which the two universities or research entities i and j worked together. For the remaining pair-success ratios, representing the case in which the two entities didn't work together in any project, a value of 0 is assigned. From the resulting matrix S_1, we obtain the corresponding matrix G_1, by applying the previously explained formula $g_{ij} = \frac{s_{ij}}{s_{jj}}$.

The dominant eigenvalue of G_1 is $\lambda_1 = 8{,}574154006$. The ranking provided by the eigenvector $\boldsymbol{p_2}$, corresponding to λ_1, is shown below: for the name of each university there is the corresponding value assigned to it, listed from the largest to the smallest.

Ranking	University / Research Entity	Corresponding score from the eigenvector
1	PADOVA	0,318509885
2	BOLOGNA	0,306984043
3	TORINO	0,291476497
4	POLITECNICO DI MILANO	0,274690652
5	BOCCONI	0,24627073
6	MILANO	0,223790727
7	TRENTO	0,216685288
8	CA' FOSCARI	0,211551994
9	POLITECNICO DI TORINO	0,198143249
10	MILANO BICOCCA	0,193452274
11	LUISS	0,188772575
12	PAVIA	0,187111271
13	BERGAMO	0,179016929
14	CONSIGLIO NAZIONALE RICERCHE	0,168911782
15	CATTOLICA	0,161723724
16	LA SAPIENZA	0,155008195
17	BOLZANO	0,122628501
18	PIEMONTE ORIENTALE	0,115280359
19	SANT'ANNA	0,109351281
20	FIRENZE	0,107328055
21	VERONA	0,103455657
22	POLITECNICO MARCHE	0,09300768
23	PARTHENOPE	0,085624606
24	UDINE	0,084403041
25	CARLO CATTANEO	0,083110222
26	PALERMO	0,080768743
27	CHIETI-PESCARA	0,080699248
28	PISA	0,080160927
29	LUMSA	0,08003566
30	ROMA TRE	0,077788404
31	AQUILA	0,076797316
32	FEDERICO II	0,071782665
33	SIENA	0,066430756
34	MODENA REGGIO EMILIA	0,062931138
35	GENOVA	0,06256882
36	POLITECNICO BARI	0,062427289
37	CAMERINO	0,06131811
38	GRAN SASSO	0,058155144
39	BARI ALDO MORO	0,056720862
40	TOR VERGATA	0,050212014

41	PERUGIA	0,04748086
42	FERRARA	0,04705536
43	MEDITERRANEA	0,043676317
44	CATANIA	0,043045939
45	MESSINA	0,042052206
46	FORO ITALICO	0,040530473
47	ALTI STUDI LUCCA	0,039853544
48	UNICUSANO	0,030735657
49	TERAMO	0,027564674
50	PARMA	0,026596894
51	BASILICATA	0,024440546
52	FOGGIA	0,023136356
53	TUSCIA	0,022301076
54	SANNIO DI BENEVENTO	0,0212215
55	UNINT	0,01793423
56	GIUSEPPE DE GENNARO	0,017070105
57	CALABRIA	0,017070105
58	CAGLIARI	0,008770716
59	BRESCIA	0,008260833
60	LINK CAMPUS	0,006629389

From the just presented data, we can see a final ranking of the universities and research entities working in teams for the PRIN 2022 projects. The first ranked is the University of Padova, which obtained the highest score ($0{,}318509885$); while the last ranked is the Link Campus University which obtained the lowest one ($0{,}006629389$) compared to all the other participants involved.

Such results provide a different ranking from the one given by the simple arithmetic mean of the projects in which the universities took part[27]. The reason of this divergence is given by the operations made in the models.

In the classification provided by the arithmetic mean, the value of each university is "weighted" for the number of projects in which it was involved. Consequently, if a university worked in many of them, obtaining both high and low scores, the resulting value will be an average between all the scores earned. On the other hand, a university working in just one project, will obtain a score reflecting just the value gained in that single one.

For example, the University of Padova[28] took part in ten projects. For each of them, it obtained a different score (98,81,80,79,77,77,76,75,75,75), but the final value corresponding to the university was 79,3 in the arithmetic mean ranking. The University of Catania, instead, worked in only one project with a final value of 87. The resulting classification puts Catania before Padova (2^{nd} and 33^{rd} position respectively).

The problem with this reasoning is that it doesn't take into consideration the number of projects in which the universities collaborated, penalizing the ones that obtained a wide range of scores, e.g. the University of Padova.

In contrast, our modification of the EPV model, considers the number of projects in which the university or research entity took part by modifying the matrix formed by all the pair-success ratios: assigning a value corresponding to the arithmetic mean of the projects in which the two universities collaborated, otherwise

[27] See the table at page 26 and 27.

[28] For more clarifications on the data see the Appendix below.

setting the value equal to 0 if they didn't work together. Consequently, in the new ranking, The University of Padova is in the 1^{st} position, while the University of Catania is in the 44^{th} one.

In conclusion, by applying our modification of the EPV model, it is possible to obtain a more precise ranking for evaluating individual contributions in teams rather than the simple arithmetic mean.

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APPENDIX:

Segretariato Generale

Direzione Generale della Ricerca

Bando PRIN 2022 PNRR

Allegato C

Piano dei Costi e dei Contributi

Settore ERC: SH1

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
1. P20227XP7P - Costo progetto: 269.782				
1	Francesca GRASSETTI	Università della CALABRIA	80003950781	131.031
2	Giovanni VILLANI	Università degli Studi di BARI ALDO MORO	80002170720	81.138
3	Armando SACCO	Università degli Studi di NAPOLI "Parthenope"	80018240632	57.613
2. P2022L7T42 - Costo progetto: 289.160				
4	Marco BATTAGLINI	Università Commerciale "Luigi Bocconi" MILANO	80024610158	104.892
5	Francesco LANCIA	Università "Ca' Foscari" VENEZIA	80007720271	32.666
6	Alessia RUSSO	Università degli Studi di PADOVA	80006480281	34.708
7	Valerio LEONE SCIABOLAZZA	Università degli Studi di ROMA "La Sapienza"	80209930587	116.894
3. P2022C3XSS - Costo progetto: 157.694				
8	Paolo ROBERTI	Libera Università di BOLZANO	94060760215	54.194
9	Riccardo GHIDONI	Università degli Studi di BOLOGNA	80007010376	103.500
4. P2022KHP8L - Costo progetto: 258.458				
10	Monica LANGELLA	Università degli Studi di Napoli Federico II	00876220633	180.407
11	Marco Giovanni NIEDDU	Università degli Studi di CAGLIARI	80019600925	78.051
5. P20223THLS - Costo progetto: 284.876				
12	Elisabetta DE CAO	Università degli Studi di BOLOGNA	80007010376	120.257
13	Massimo ANELLI	Università Commerciale "Luigi Bocconi" MILANO	80024610158	117.381
14	Silvia MENDOLIA	Università degli Studi di TORINO	80088230018	47.238
6. P2022EN9PJ - Costo progetto: 299.291				

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
15	Pier Paolo MIGLIETTA	Università del SALENTO	80008870752	120.048
16	Raffaele IANNONE	Università degli Studi di SALERNO	80018670655	89.920
17	Alessandra CAPOLUPO	Politecnico di BARI	93051590722	89.323
7. P20224K95M - Costo progetto: 299.869				
18	Marco MODICA	Gran Sasso Science Institute - Scuola di dottorato internazionale	01984560662	165.169
19	Pietro PIZZUTO	Università degli Studi di PALERMO	80023730825	134.700
8. P2022LRLBH - Costo progetto: 282.815				
20	Julien SAUVAGNAT	Università Commerciale "Luigi Bocconi" MILANO	80024610158	141.407
21	Annalisa SCOGNAMIGLIO	Università degli Studi di Napoli Federico II	00876220633	141.408
9. P202223J4J - Costo progetto: 181.282				
22	Paolo LI DONNI	Università degli Studi di PALERMO	80023730825	154.294
23	Vincenzo CARRIERI	Università della CALABRIA	80003950781	26.988
10. P2022XJW4T - Costo progetto: 174.391				
24	Giovanni IMMORDINO	Università degli Studi di Napoli Federico II	00876220633	89.480
25	Fabrizio PANEBIANCO	Università Cattolica del Sacro Cuore	02133120150	84.911
11. P20227H2XW - Costo progetto: 206.750				
26	Agne KAJACKAITE	Università degli Studi di MILANO	80012650158	115.436
27	Stephanie HEGER	Università degli Studi di BOLOGNA	80007010376	91.314
12. P2022SRW8N - Costo progetto: 299.999				
28	Stefano DI BUCCHIANICO	Università degli Studi di SALERNO	80018670655	180.366
29	Matteo DELEIDI	Università degli Studi di BARI ALDO MORO	80002170720	119.633
13. P2022H483A - Costo progetto: 284.951				
30	Valerio DOTTI	Università "Ca' Foscari" VENEZIA	80007720271	147.786

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
31	Luca ROSSINI	Università degli Studi di MILANO	80012650158	137.165
14. P2022JTSFB - Costo progetto: 299.925				
32	Fabio Gaetano SANTERAMO	Università degli Studi di FOGGIA	94045260711	123.350
33	Giuseppe MAGGIO	Università degli Studi di PALERMO	80023730825	114.006
34	Teresa RANDAZZO	Università degli Studi di NAPOLI "Parthenope"	80018240632	62.569
15. P2022FT9ZK - Costo progetto: 266.436				
35	Giulia GIUPPONI	Università Commerciale "Luigi Bocconi" MILANO	80024610158	142.076
36	Vincenzo SCRUTINIO	Università degli Studi di BOLOGNA	80007010376	124.360
16. P2022WM82K - Costo progetto: 298.072				
37	Dario SALERNO	Università degli Studi di NAPOLI "Parthenope"	80018240632	108.422
38	Helen CHIAPPINI	Università degli Studi "G. d'Annunzio" CHIETI-PESCARA	93002750698	94.981
39	Stefano ZEDDA	Università degli Studi di CAGLIARI	80019600925	94.669
17. P202233ZTR - Costo progetto: 269.881				
40	Stefano SCHIAVO	Università degli Studi di TRENTO	00340520220	93.419
41	Mariagrazia ALABRESE	Scuola Superiore di Studi Universitari e Perfezionamento Sant'Anna	93008800505	88.166
42	Giuseppe MANGIONI	Università degli Studi di CATANIA	02772010878	88.296
18. P2022Z3TP8 - Costo progetto: 299.797				
43	Mattea Regina STEIN	Università degli Studi di Napoli Federico II	00876220633	229.688
44	Davide DEL PRETE	Università degli Studi di NAPOLI "Parthenope"	80018240632	70.109
19. P20227JN7R - Costo progetto: 245.251				

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
45	Maria Rosaria CARILLO	Università degli Studi di NAPOLI "Parthenope"	80018240632	121.251
46	Damiano Bruno SILIPO	Università della CALABRIA	80003950781	62.000
47	Tommaso OLIVIERO	Università degli Studi di Napoli Federico II	00876220633	62.000
20. P2022XT8C8 - Costo progetto: 269.696				
48	Elisa LUCIANO	Università degli Studi di TORINO	80088230018	127.938
49	Marco SCARSINI	Luiss Libera Università internazionale degli studi sociali Guido Carli	02508710585	117.588
50	Fabio FAGNANI	Politecnico di TORINO	00518460019	24.170
21. P20229YMBZ - Costo progetto: 246.000				
51	Marta MELEDDU	Università degli Studi di SASSARI	00196350904	158.377
52	Pierpaolo DUCE	Consiglio Nazionale delle Ricerche	80054330586	75.099
53	Fabrizio CESARONI	Università degli Studi di MESSINA	80004070837	12.524
22. P2022NK39A - Costo progetto: 255.457				
54	Andrea SCOZZARI	UNICUSANO Università degli Studi Niccolò Cusano - Telematica Roma	09073721004	115.887
55	Federica RICCA	Università degli Studi di ROMA "La Sapienza"	80209930587	72.076
56	Lorenzo LAMPARIELLO	Università degli Studi ROMA TRE	04400441004	67.494
23. P2022KTM7H - Costo progetto: 245.868				
57	Ruggiero SARDARO	Università degli Studi di FOGGIA	94045260711	88.330
58	Concetta NAZZARO	Università degli Studi del SANNIO di BENEVENTO	01114010620	79.580
59	Rosanna SALVIA	Università degli Studi della BASILICATA	96003410766	77.958
24. P2022R8ZTW - Costo progetto: 269.134				
60	Tiziano DISTEFANO	Università degli Studi di FIRENZE	01279680480	106.134

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
61	Mauro VICCARO	Università degli Studi della BASILICATA	96003410766	59.000
62	Luca SALVATICI	Università degli Studi ROMA TRE	04400441004	104.000
25. P20229CJRS - Costo progetto: 269.315				
63	Alessandra CARACENI	Scuola Normale Superiore di PISA	80005050507	54.345
64	Giuseppe BUCCHERI	Università degli Studi di VERONA	93009870234	63.163
65	Piero MAZZARISI	Università degli Studi di SIENA	80002070524	151.807
26. P2022ANZ72 - Costo progetto: 244.407				
66	Amelia MANUTI	Università degli Studi di BARI ALDO MORO	80002170720	104.450
67	Paola SPAGNOLI	Università degli Studi della Campania "Luigi Vanvitelli"	02044190615	112.450
68	Barbara BARBIERI	Università degli Studi di CAGLIARI	80019600925	27.507
27. P2022A9N8J - Costo progetto: 269.682				
69	Alessio D'AMATO	Università degli Studi di ROMA "Tor Vergata"	80213750583	97.892
70	Amedeo ARGENTIERO	Università degli Studi Internazionali di ROMA (UNINT)	97136680580	96.408
71	Elisabetta MARZANO	Università degli Studi di NAPOLI "Parthenope"	80018240632	75.382
28. P2022WP49F - Costo progetto: 241.217				
72	Olivier Karl BUTZBACH	Università degli Studi della Campania "Luigi Vanvitelli"	02044190615	171.217
73	Paolo COCCORESE	Università degli Studi di SALERNO	80018670655	35.000
74	Roberto Giovanni BASILE	Università degli Studi dell'AQUILA	01021630668	35.000
29. P20228CHNL - Costo progetto: 269.696				
75	Luca REGIS	Università degli Studi di TORINO	80088230018	140.550
76	Andrea FLORI	Politecnico di MILANO	80057930150	52.247

nº	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
77	Anna Maria GAMBARO	Università degli Studi del PIEMONTE ORIENTALE "Amedeo Avogadro"-Vercelli	94021400026	48.866
78	Luca TRAPIN	Università degli Studi di BOLOGNA	80007010376	28.033
30. P2022ZRPBL - Costo progetto: 240.465				
79	Domenico SALVATORE	Università degli Studi Suor Orsola Benincasa - NAPOLI	80040520639	107.452
80	Antonio RICCIARDI	Università della CALABRIA	80003950781	50.513
81	Ezio RIGGI	Consiglio Nazionale delle Ricerche	80054330586	82.500
31. P2022B5HR7 - Costo progetto: 262.656				
82	Matteo Carlo Maria SANDI	Università Cattolica del Sacro Cuore	02133120150	162.495
83	Gianmarco DANIELE	Università degli Studi di MILANO	80012650158	23.243
84	Maria Anna LEONE	Università degli Studi di PAVIA	80007270186	38.909
85	Andrea GUARISO	Università degli Studi di MILANO-BICOCCA	12621570154	38.009
32. P2022JH22T - Costo progetto: 245.349				
86	Katia CORSI	Università degli Studi di SASSARI	00196350904	140.349
87	Daniela MANCINI	Università degli Studi di TERAMO	92012890676	105.000
33. P2022HTXXM - Costo progetto: 203.446				
88	Hannes WAGNER	Università Commerciale "Luigi Bocconi" MILANO	80024610158	91.334
89	Monica BILLIO	Università "Ca' Foscari" VENEZIA	80007720271	112.112
34. P2022RF38Y - Costo progetto: 232.323				
90	Corrado CUCCURULLO	Università degli Studi della Campania "Luigi Vanvitelli"	02044190615	116.573
91	Massimo ARIA	Università degli Studi di Napoli Federico II	00876220633	115.750
35. P2022N2TPJ - Costo progetto: 269.851				

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
92	Daniele GIACHINI	Scuola Superiore di Studi Universitari e Perfezionamento Sant'Anna	93008800505	94.495
93	Alina SIRBU	Università di PISA	80003670504	41.171
94	Rosita CAPURRO	Università degli Studi di NAPOLI "Parthenope"	80018240632	47.683
95	Fabrizio FORNARI	Università degli Studi di CAMERINO	81001910439	86.502
36. P2022YY9P2 - Costo progetto: 174.899				
96	Francesco Flaviano RUSSO	Università degli Studi di Napoli Federico II	00876220633	130.000
97	Marcello D'AMATO	Università degli Studi Suor Orsola Benincasa - NAPOLI	80040520639	44.899
37. P2022FKLHH - Costo progetto: 269.545				
98	Andrea FILIPPETTI	Consiglio Nazionale delle Ricerche	80054330586	105.559
99	Mara GIUA	Università degli Studi ROMA TRE	04400441004	101.992
100	Roberto GABRIELE	Università degli Studi di TRENTO	00340520220	61.994
38. P2022LWZZX - Costo progetto: 242.386				
101	Diego Angelo Gaetano REFORGIATO RECUPERO	Università degli Studi di CAGLIARI	80019600925	89.175
102	Gianluigi DE PASCALE	Università degli Studi di FOGGIA	94045260711	76.967
103	Maria CIPOLLINA	Università degli Studi del MOLISE	92008370709	76.244
39. P20224K52W - Costo progetto: 238.732				
104	Andrea TOMO	Università degli Studi di Napoli Federico II	00876220633	77.719
105	Aizhan TURSUNBAYEVA	Università degli Studi di NAPOLI "Parthenope"	80018240632	69.705
106	Gilda ANTONELLI	Università degli Studi del SANNIO di BENEVENTO	01114010620	44.958

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
107	Mario PEZZILLO IACONO	Università degli Studi della Campania "Luigi Vanvitelli"	02044190615	46.350
40. P2022FLLPY - Costo progetto: 269.594				
108	Alessio TEI	Università degli Studi di GENOVA	00754150100	122.327
109	Cristiano CERVELLERA	Consiglio Nazionale delle Ricerche	80054330586	73.008
110	Giacomo BORACCHI	Politecnico di MILANO	80057930150	74.259
41. P20228FFHF - Costo progetto: 269.927				
111	Giacomo PALLANTE	Università degli Studi di TRENTO	00340520220	114.333
112	Daniele CURZI	Università degli Studi di MILANO	80012650158	57.561
113	Alessandro PALMA	Gran Sasso Science Institute - Scuola di dottorato internazionale	01984560662	98.033
42. P2022XLL94 - Costo progetto: 240.500				
114	Domenico LISI	Università degli Studi di CATANIA	02772010878	158.730
115	Massimo FINOCCHIARO CASTRO	Università degli Studi "Mediterranea" di REGGIO CALABRIA	80006510806	31.265
116	Concetta CASTIGLIONE	Università della CALABRIA	80003950781	50.505
43. P20225P48L - Costo progetto: 268.013				
117	Paolo BARBIERI	Università degli Studi di BOLOGNA	80007010376	112.388
118	Luciano FRATOCCHI	Università degli Studi dell'AQUILA	01021630668	51.550
119	Albachiara BOFFELLI	Università degli Studi di BERGAMO	80004350163	57.250
120	Antonella Maria MORETTO	Politecnico di MILANO	80057930150	46.825
44. P2022CW4HX - Costo progetto: 242.037				
121	Rocco ROMA	Università degli Studi di BARI ALDO MORO	80002170720	146.680
122	Valeria BORSELLINO	Università degli Studi di PALERMO	80023730825	47.570
123	Fabio Albino MADAU	Università degli Studi di SASSARI	00196350904	47.787

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
45. P2022XMYWW - Costo progetto: 269.999				
124	Maria SAVONA	Luiss Libera Università internazionale degli studi sociali Guido Carli	02508710585	197.999
125	Rinaldo EVANGELISTA	Università degli Studi di CAMERINO	81001910439	72.000
46. P2022YHPWZ - Costo progetto: 243.000				
126	Andrea ISONI	Università degli Studi di CAGLIARI	80019600925	121.500
127	Enrica CARBONE	Università degli Studi della Campania "Luigi Vanvitelli"	02044190615	121.500
47. P2022B22W2 - Costo progetto: 241.997				
128	Michele BATTISTI	Università degli Studi di PALERMO	80023730825	127.997
129	Massimo DEL GATTO	Università degli Studi "G. d'Annunzio" CHIETI-PESCARA	93002750698	114.000
48. P2022EJA8P - Costo progetto: 269.579				
130	Claudio SOREGAROLI	Università Cattolica del Sacro Cuore	02133120150	140.604
131	Stefanella STRANIERI	Università degli Studi di MILANO	80012650158	128.975
49. P2022LP43N - Costo progetto: 268.834				
132	Francesco VONA	Università degli Studi di MILANO	80012650158	146.578
133	Marco GRAZZI	Università Cattolica del Sacro Cuore	02133120150	122.256
50. P2022TALJF - Costo progetto: 269.753				
134	Sibilla DI GUIDA	Scuola IMT Alti Studi - LUCCA	92037570469	135.753
135	Luca POLONIO	Università degli Studi di MILANO-BICOCCA	12621570154	134.000
51. P2022NLZEB - Costo progetto: 262.170				
136	Andrea Pier Giovanni GALLICE	Università degli Studi di TORINO	80088230018	140.000
137	Edoardo GRILLO	Università degli Studi di PADOVA	80006480281	122.170
52. P2022A82P3 - Costo progetto: 265.807				

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
138	Carlo Andrea BOLLINO	Università degli Studi di PERUGIA	00448820548	87.056
139	Massimo GIANNINI	Università degli Studi di ROMA "Tor Vergata"	80213750583	80.831
140	Maria FERRARA	Università degli Studi di NAPOLI "Parthenope"	80018240632	60.131
141	Marzio Domenico GALEOTTI	Università degli Studi di MILANO	80012650158	37.789
53. P20224A38A - Costo progetto: 269.804				
142	Giustina SECUNDO	LUM "Giuseppe Degennaro"	93135780729	81.000
143	Renato PASSARO	Università degli Studi di NAPOLI "Parthenope"	80018240632	63.000
144	Alberto Michele FELICETTI	Università della CALABRIA	80003950781	62.899
145	Barbara BIGLIARDI	Università degli Studi di PARMA	00308780345	62.905
54. P2022C97X7 - Costo progetto: 267.829				
146	Fabio BARTOLINI	Università degli Studi di FERRARA	80007370382	99.101
147	Silvia CODERONI	Università degli Studi di TERAMO	92012890676	83.712
148	Paolo SCKOKAI	Università Cattolica del Sacro Cuore	02133120150	85.016
55. P2022ENNYP - Costo progetto: 171.540				
149	Marco NICOLOSI	LINK CAMPUS University	11933781004	140.000
150	Rocco CICIRETTI	Università degli Studi di ROMA "Tor Vergata"	80213750583	31.540
56. P20227AWSW - Costo progetto: 259.240				
151	Anna D'AMBROSIO	Politecnico di TORINO	00518460019	123.348
152	Alessandro MANELLO	Università degli Studi di TORINO	80088230018	57.281
153	Greta FALAVIGNA	Consiglio Nazionale delle Ricerche	80054330586	78.611
57. P2022EZBTE - Costo progetto: 269.966				
154	Carlo D'IPPOLITI	Università degli Studi di ROMA "La Sapienza"	80209930587	146.323

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
155	Luca FANTACCI	Università degli Studi di MILANO	80012650158	123.643
58. P20225MJW8 - Costo progetto: 265.703				
156	Silvia MUZZIOLI	Università degli Studi di MODENA e REGGIO EMILIA	00427620364	137.938
157	Andrea CIPOLLINI	Università degli Studi di PALERMO	80023730825	40.838
158	Massimiliano FERRARA	Università degli Studi "Mediterranea" di REGGIO CALABRIA	80006510806	40.988
159	Arianna AGOSTO	Università degli Studi di PAVIA	80007270186	45.939
59. P20228SXNF - Costo progetto: 269.366				
160	Paolo PIN	Università degli Studi di SIENA	80002070524	148.566
161	Marco MANTOVANI	Università degli Studi di MILANO-BICOCCA	12621570154	120.800
60. P2022X8LW5 - Costo progetto: 99.726				
162	Piera BELLO	Università degli Studi di BERGAMO	80004350163	40.600
163	Martina CELIDONI	Università degli Studi di PADOVA	80006480281	47.132
164	Vincenzo GALASSO	Università Commerciale "Luigi Bocconi" MILANO	80024610158	11.994
61. P2022MB44K - Costo progetto: 237.798				
165	Leonardo CORBO	Università degli Studi di BOLOGNA	80007010376	136.500
166	Federica BRUNETTA	Luiss Libera Università internazionale degli studi sociali Guido Carli	02508710585	101.298
62. P20227KSFW - Costo progetto: 221.154				
167	Annalisa CALOFFI	Università degli Studi di FIRENZE	01279680480	102.992
168	Silvia Rita SEDITA	Università degli Studi di PADOVA	80006480281	17.994
169	Diego D'ADDA	Università Politecnica delle MARCHE	00382520427	100.168
63. P2022NTXX5 - Costo progetto: 245.823				

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
170	Marco FABBRI	Università degli Studi di BOLOGNA	80007010376	69.431
171	Matteo RIZZOLLI	Libera Università degli Studi "Maria SS.Assunta" - LUMSA	02635620582	174.392
172	Andrea GERACI	Università degli Studi di PAVIA	80007270186	2.000
64. P2022TTPK7 - Costo progetto: 234.134				
173	Ennio BILANCINI	Scuola IMT Alti Studi - LUCCA	92037570469	125.033
174	Leonardo BONCINELLI	Università degli Studi di FIRENZE	01279680480	109.101
65. P2022RYFET - Costo progetto: 245.317				
175	Francesco QUATRARO	Università degli Studi di TORINO	80088230018	140.300
176	Marco VIVARELLI	Università Cattolica del Sacro Cuore	02133120150	105.017
66. P2022P5CHH - Costo progetto: 146.200				
177	Antonio NICOLO'	Università degli Studi di PADOVA	80006480281	119.568
178	Antonio MIRALLES ASENSIO	Università degli Studi di MESSINA	80004070837	26.632
67. P2022N3JTK - Costo progetto: 244.847				
179	Marco MINCIULLO	Università Cattolica del Sacro Cuore	02133120150	122.440
180	Alessandro ZATTONI	Luiss Libera Università internazionale degli studi sociali Guido Carli	02508710585	122.407
68. P2022ACFL7 - Costo progetto: 244.693				
181	Erica SANTINI	Università degli Studi di TRENTO	00340520220	42.283
182	Lorena Maria D'AGOSTINO	Università degli Studi di MILANO-BICOCCA	12621570154	100.259
183	Diletta PEGORARO	Politecnico di MILANO	80057930150	102.151
69. P2022FWHWM - Costo progetto: 245.967				
184	Eleonora MATTEAZZI	Università degli Studi di VERONA	93009870234	142.145
185	Ylenia BRILLI	Università "Ca' Foscari" VENEZIA	80007720271	95.532

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
186	Pamela GIUSTINELLI	Università Commerciale "Luigi Bocconi" MILANO	80024610158	8.290
70. P2022C4WE9 - Costo progetto: 233.570				
187	Daniela SORRENTINO	Università degli Studi di SIENA	80002070524	128.520
188	Davide ELTRUDIS	Università degli Studi di CAGLIARI	80019600925	105.050
71. P20222WWWA - Costo progetto: 245.783				
189	Antonio PARBONETTI	Università degli Studi di PADOVA	80006480281	130.662
190	Costantino VISCONTI	Università degli Studi di PALERMO	80023730825	115.121
72. P20227RKF9 - Costo progetto: 246.000				
191	Elias CARRONI	Università degli Studi di BOLOGNA	80007010376	97.000
192	Alessandro RUBINO	Università degli Studi di BARI ALDO MORO	80002170720	74.500
193	Carlo GALLIER	Libera Università di BOLZANO	94060760215	74.500
73. P2022S3KE9 - Costo progetto: 245.977				
194	Antonio GHEZZI	Politecnico di MILANO	80057930150	90.174
195	Antonio MESSENI PETRUZZELLI	Politecnico di BARI	93051590722	77.864
196	Federico CAVIGGIOLI	Politecnico di TORINO	00518460019	77.939
74. P2022BXP5X - Costo progetto: 242.886				
197	Silvia TIEZZI	Università degli Studi di SIENA	80002070524	121.443
198	Chiara RAPALLINI	Università degli Studi di FIRENZE	01279680480	121.443
75. P2022HXLBF - Costo progetto: 238.000				
199	Niloofar KAZEMARGI	Università degli Studi "G. d'Annunzio" CHIETI-PESCARA	93002750698	118.000
200	Simona LEONELLI	Università degli Studi di PADOVA	80006480281	86.250
201	Paolo SPAGNOLETTI	Luiss Libera Università internazionale degli studi sociali Guido Carli	02508710585	33.750
76. P2022JASLC - Costo progetto: 183.917				

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
202	Simone QUERCIA	Università degli Studi di VERONA	93009870234	136.921
203	Francesco FALLUCCHI	Università degli Studi di BERGAMO	80004350163	46.996
77. P2022B7PYX - Costo progetto: 246.001				
204	Laura GIRELLA	Università degli Studi di MODENA e REGGIO EMILIA	00427620364	94.355
205	Stefano ZAMBON	Università degli Studi di FERRARA	80007370382	75.816
206	Alessandro LAI	Università degli Studi di VERONA	93009870234	75.830
78. P2022HBE93 - Costo progetto: 230.730				
207	Roberta RABELLOTTI	Università degli Studi di PAVIA	80007270186	132.588
208	Stefano BRESCHI	Università Commerciale "Luigi Bocconi" MILANO	80024610158	98.142
79. P20229EL9W - Costo progetto: 230.312				
209	Laura ABRARDI	Politecnico di TORINO	00518460019	92.674
210	Fabio MANENTI	Università degli Studi di PADOVA	80006480281	44.806
211	Stefano COMINO	Università degli Studi di UDINE	80014550307	46.416
212	Mirco TONIN	Libera Università di BOLZANO	94060760215	46.416
80. P2022SHJJN - Costo progetto: 180.878				
213	Paola CANTARELLI	Scuola Superiore di Studi Universitari e Perfezionamento Sant'Anna	93008800505	96.471
214	Matilde MILANESI	Università degli Studi di FIRENZE	01279680480	84.407
81. P2022C44KE - Costo progetto: 230.488				
215	Arianna MARTINELLI	Scuola Superiore di Studi Universitari e Perfezionamento Sant'Anna	93008800505	95.023
216	Gianluca MURGIA	Università degli Studi di SIENA	80002070524	89.302
217	Andrea URBINATI	Università "Carlo Cattaneo" - LIUC	02015300128	46.163

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
82. P2022JZRWS - Costo progetto: 231.000				
218	Francesco Saverio PAVONE	Università degli Studi di FIRENZE	01279680480	138.600
219	Daniele VILONE	Consiglio Nazionale delle Ricerche	80054330586	92.400
83. P2022BNNEY - Costo progetto: 186.379				
220	Elisa TOSETTI	Università degli Studi di PADOVA	80006480281	119.950
221	Francesco MOSCONE	Università "Ca' Foscari" VENEZIA	80007720271	32.729
222	Veronica VINCIOTTI	Università degli Studi di TRENTO	00340520220	33.700
84. P2022LFR5M - Costo progetto: 230.504				
223	Raffaele CORTIGNANI	Università degli Studi della TUSCIA	80029030568	118.504
224	Giovanni CERULLI	Consiglio Nazionale delle Ricerche	80054330586	112.000
85. P2022XTLM2 - Costo progetto: 230.692				
225	Marco CORAZZA	Università "Ca' Foscari" VENEZIA	80007720271	115.346
226	Silvia ROMAGNOLI	Università degli Studi di BOLOGNA	80007010376	115.346
86. P2022KMR8A - Costo progetto: 210.134				
227	Lucia LEPORATTI	Università degli Studi di GENOVA	00754150100	163.904
228	Nicola PONTAROLLO	Università degli Studi di BRESCIA	98007650173	46.230
87. P202239XAE - Costo progetto: 230.529				
229	Fabiana PIROLA	Università degli Studi di BERGAMO	80004350163	98.010
230	Monica ROSSI	Politecnico di MILANO	80057930150	63.293
231	Rossella POZZI	Università "Carlo Cattaneo" - LIUC	02015300128	69.226
88. P20225M45W - Costo progetto: 225.615				
232	Laura BERARDI	Università degli Studi "G. d'Annunzio" CHIETI-PESCARA	93002750698	76.976
233	Filippo GIORDANO	Libera Università degli Studi "Maria SS.Assunta" - LUMSA	02635620582	74.430

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
234	Paolo ESPOSITO	Università degli Studi del SANNIO di BENEVENTO	01114010620	74.209
89. P2022NXPBB - Costo progetto: 227.708				
235	Kai ZHU	Università Commerciale "Luigi Bocconi" MILANO	80024610158	196.708
236	Toloue MIANDAR	Università degli Studi di BOLOGNA	80007010376	31.000
90. P2022MWXJY - Costo progetto: 229.850				
237	Pierpaolo MAGLIOCCA	Università degli Studi di FOGGIA	94045260711	84.740
238	Angelo BONFANTI	Università degli Studi di VERONA	93009870234	73.715
239	Francesco CAPUTO	Università degli Studi di Napoli Federico II	00876220633	71.395
91. P20222XM58 - Costo progetto: 230.584				
240	Alessandro STEFANINI	Università di PISA	80003670504	76.874
241	Mattia CATTANEO	Università degli Studi di BERGAMO	80004350163	76.855
242	Massimiliano DE LEONI	Università degli Studi di PADOVA	80006480281	76.855
92. P2022ZK7AK - Costo progetto: 230.484				
243	Donato IACOBUCCI	Università Politecnica delle MARCHE	00382520427	122.952
244	Ugo FRATESI	Politecnico di MILANO	80057930150	107.532
93. P2022R248L - Costo progetto: 211.101				
245	Valeria MAGGIAN	Università "Ca' Foscari" VENEZIA	80007720271	102.961
246	Natalia MONTINARI	Università degli Studi di BOLOGNA	80007010376	108.140
94. P2022LR3C5 - Costo progetto: 207.316				
247	Andrea TENUCCI	Scuola Superiore di Studi Universitari e Perfezionamento Sant'Anna	93008800505	108.520
248	Sara Giovanna MAURO	Università degli Studi di MODENA e REGGIO EMILIA	00427620364	98.796
95. P20227EJEE - Costo progetto: 230.882				

n°	Nome Responsabile dell'Unità	Ateneo/Ente	Codice Fiscale Ateneo/Ente	Contributo MUR / costo totale
249	Ambra POGGI	Università degli Studi di TORINO	80088230018	115.441
250	Elena Francesca MESCHI	Università degli Studi di MILANO-BICOCCA	12621570154	115.441
96. P2022HYP9M - Costo progetto: 230.993				
251	Giovanni SOGARI	Università degli Studi di PARMA	00308780345	140.217
252	Simone MANCINI	Università di PISA	80003670504	90.776
97. P20228W2Z2 - Costo progetto: 230.925				
253	Francesca VICENTINI	Università degli Studi di ROMA "Foro Italico"	80229010584	190.925
254	Giuseppe CAPPIELLO	Università degli Studi di BOLOGNA	80007010376	40.000

ALLEGATO A - GRADUATORIE SETTORE SH1

Graduatoria Settore SH1 Linea A - Principale

I PI con meno di 40 anni alla data di pubblicazione del bando PRIN 2022 PNRR sono evidenziati in verde

L'indice di equità rappresenta l'equa distribuzione tra il numero di partecipanti di genere maschile e femminile di ciascun progetto. È un numero compreso tra 1 e 0 dove 1 rappresenta la massima equità e 0 la minima equità

N°	Codice progetto	Principal Investigator	Ente	Punteggio	Altri criteri
1	P2022L7T42	BATTAGLINI Marco	Università Commerciale "Luigi Bocconi" MILANO	98	
2	P2022C3XSS	ROBERTI Paolo	Libera Università di BOLZANO	91	Criterio 1: 37
3	P20223THLS	DE CAO Elisabetta	Università degli Studi di BOLOGNA	91	Criterio 1: 36
4	P2022LRLBH	SAUVAGNAT Julien	Università Commerciale "Luigi Bocconi" MILANO	90	Criterio 1: 37
5	P2022XJW4T	IMMORDINO Giovanni	Università degli Studi di Napoli Federico II	90	Criterio 1: 36 Criterio 3: 28
6	P20227H2XW	KAJACKAITE Agne	Università degli Studi di MILANO	90	Criterio 1: 36 Criterio 3: 27
7	P2022H483A	DOTTI Valerio	Università "Ca' Foscari" VENEZIA	89	
8	P2022FT9ZK	GIUPPONI Giulia	Università Commerciale "Luigi Bocconi" MILANO	88	
9	P202233ZTR	SCHIAVO Stefano	Università degli Studi di TRENTO	87	Criterio 1: 35 Criterio 3: 27 Equità: 1,00 (2 M; 2 F)
10	P2022XT8C8	LUCIANO Elisa	Università degli Studi di TORINO	87	Criterio 1: 35 Criterio 3: 27 Equità: 0,80 (3 M; 2 F)
11	P2022NK39A	SCOZZARI Andrea	UNICUSANO Università degli Studi Niccolò Cusano -Telematica Roma	86	Criterio 1: 36
12	P2022R8ZTW	DISTEFANO Tiziano	Università degli Studi di FIRENZE	86	Criterio 1: 35
13	P20229CJRS	LIVIERI Giulia	Scuola Normale Superiore di PISA	86	Criterio 1: 33
14	P2022A9N8J	D'AMATO Alessio	Università degli Studi di ROMA "Tor Vergata"	85	Criterio 1: 34
15	P20228CHNL	REGIS Luca	Università degli Studi di TORINO	85	Criterio 1: 33
16	P2022B5HR7	SANDI Matteo Carlo Maria	Università Cattolica del Sacro Cuore	84	Criterio 1: 36
17	P2022HTXXM	WAGNER Hannes	Università Commerciale "Luigi Bocconi" MILANO	84	Criterio 1: 33
18	P2022N2TPJ	GIACHINI Daniele	Scuola Superiore di Studi Universitari e Perfezionamento Sant'Anna	83	Criterio 1: 36
19	P2022FKLHH	FILIPPETTI Andrea	Consiglio Nazionale delle Ricerche	83	Criterio 1: 34 Criterio 3: 23 Equità: 1,00 (4 M; 4 F)
20	P2022FLLPY	TEI Alessio	Università degli Studi di GENOVA	83	Criterio 1: 34 Criterio 3: 23 Equità: 0,25 (7 M; 1 F)
21	P20228FFHF	PALLANTE Giacomo	Università degli Studi di TRENTO	83	Criterio 1: 32

ALLEGATO A - GRADUATORIE SETTORE SH1

N°	Codice progetto	Principal Investigator	Ente	Punteggio	Altri criteri
22	P20225P48L	BARBIERI Paolo	Università degli Studi di BOLOGNA	82	Criterio 1: 35
23	P2022XMYWW	SAVONA Maria	Luiss Libera Università internazionale degli studi sociali Guido Carli	82	Criterio 1: 34
24	P2022EJA8P	SOREGAROLI Claudio	Università Cattolica del Sacro Cuore	81	Criterio 1: 34 Criterio 3: 23
25	P2022LP43N	VONA Francesco	Università degli Studi di MILANO	81	Criterio 1: 34 Criterio 3: 22
26	P2022TALJF	DI GUIDA Sibilla	Scuola IMT Alti Studi - LUCCA	81	Criterio 1: 33 Criterio 3: 24 Equità: 1,00 (1 M; 1 F)
27	P2022NLZEB	GALLICE Andrea Pier Giovanni	Università degli Studi di TORINO	81	Criterio 1: 33 Criterio 3: 24 Equità: 0,00 (2 M; 0 F)
28	P2022A82P3	BOLLINO Carlo Andrea	Università degli Studi di PERUGIA	81	Criterio 1: 32 Criterio 3: 25 Equità: 1,00 (4 M; 4 F)
29	P20224A38A	SECUNDO Giustina	LUM "Giuseppe Degennaro"	81	Criterio 1: 32 Criterio 3: 25 Equità: 0,73 (7 M; 4 F)
30	P2022C97X7	BARTOLINI Fabio	Università degli Studi di FERRARA	81	Criterio 1: 32 Criterio 3: 25 Equità: 0,29 (6 M; 1 F)
31	P2022ENNYP	NICOLOSI Marco	LINK CAMPUS University	81	Criterio 1: 32 Criterio 3: 24
32	P20227AWSW	D'AMBROSIO Anna	Politecnico di TORINO	80	Criterio 1: 32 Criterio 3: 25
33	P2022EZBTE	D'IPPOLITI Carlo	Università degli Studi di ROMA "La Sapienza"	80	Criterio 1: 32 Criterio 3: 24 Equità: 0,80 (2 M; 3 F) Nascita PI: XX/XX/1981
34	P20225MJW8	MUZZIOLI Silvia	Università degli Studi di MODENA e REGGIO EMILIA	80	Criterio 1: 32 Criterio 3: 24 Equità: 0,80 (9 M; 6 F) Nascita PI: XX/XX/1974
35	P20228SXNF	PIN Paolo	Università degli Studi di SIENA	80	Criterio 1: 32 Criterio 3: 21
36	P2022X8LW5	BELLO Piera	Università degli Studi di BERGAMO	80	Criterio 1: 30
37	P2022MB44K	CORBO Leonardo	Università degli Studi di BOLOGNA	79	Criterio 1: 32

ALLEGATO A - GRADUATORIE SETTORE SH1

N°	Codice progetto	Principal Investigator	Ente	Punteggio	Altri criteri
38	P20227KSFW	CALOFFI Annalisa	Università degli Studi di FIRENZE	79	Criterio 1: 31 Criterio 3: 24 Equità: 1,00 (3 M; 3 F)
39	P2022NTXX5	FABBRI Marco	Università degli Studi di BOLOGNA	79	Criterio 1: 31 Criterio 3: 24 Equità: 0,73 (7 M; 4 F)
40	P2022TTPK7	BILANCINI Ennio	Scuola IMT Alti Studi - LUCCA	79	Criterio 1: 30
41	P2022RYFET	QUATRARO Francesco	Università degli Studi di TORINO	78	
42	P2022P5CHH	NICOLO' Antonio	Università degli Studi di PADOVA	77	Criterio 1: 36
43	P2022N3JTK	MINCIULLO Marco	Università Cattolica del Sacro Cuore	77	Criterio 1: 32
44	P2022ACFL7	SANTINI Erica	Università degli Studi di TRENTO	77	Criterio 1: 31 Criterio 3: 22
45	P2022FWHWM	MATTEAZZI Eleonora	Università degli Studi di VERONA	77	Criterio 1: 31 Criterio 3: 21
46	P2022C4WE9	SORRENTINO Daniela	Università degli Studi di SIENA	77	Criterio 1: 30 Criterio 3: 25
47	P20222WWWA	PARBONETTI Antonio	Università degli Studi di PADOVA	77	Criterio 1: 30 Criterio 3: 24
48	P20227RKF9	CARRONI Elias	Università degli Studi di BOLOGNA	77	Criterio 1: 30 Criterio 3: 23
49	P2022S3KE9	GHEZZI Antonio	Politecnico di MILANO	76	Criterio 1: 34
50	P2022BXP5X	TIEZZI Silvia	Università degli Studi di SIENA	76	Criterio 1: 32 Criterio 3: 20 Equità: 1,00 (2 M; 2 F)
51	P2022HXLBF	KAZEMARGI Niloofar	Università degli Studi "G. d'Annunzio" CHIETI-PESCARA	76	Criterio 1: 32 Criterio 3: 20 Equità: 0,60 (3 M; 7 F)
52	P2022JASLC	QUERCIA Simone	Università degli Studi di VERONA	76	Criterio 1: 31 Criterio 3: 22
53	P2022B7PYX	GIRELLA Laura	Università degli Studi di MODENA e REGGIO EMILIA	76	Criterio 1: 31 Criterio 3: 21
54	P2022HBE93	RABELLOTTI Roberta	Università degli Studi di PAVIA	75	Criterio 1: 35
55	P20229EL9W	ABRARDI Laura	Politecnico di TORINO	75	Criterio 1: 34 Criterio 3: 18
56	P2022SHJJN	CANTARELLI Paola	Scuola Superiore di Studi Universitari e Perfezionamento Sant'Anna	75	Criterio 1: 34 Criterio 3: 17 Equità: 1,00 (2 M; 2 F)

ALLEGATO A - GRADUATORIE SETTORE SH1

N°	Codice progetto	Principal Investigator	Ente	Punteggio	Altri criteri
57	P2022C44KE	MARTINELLI Arianna	Scuola Superiore di Studi Universitari e Perfezionamento Sant'Anna	75	Criterio 1: 34 Criterio 3: 17 Equità: 0,86 (3 M; 4 F)
58	P2022JZRWS	PAVONE Francesco Saverio	Università degli Studi di FIRENZE	75	Criterio 1: 33
59	P2022BNNEY	TOSETTI Elisa	Università degli Studi di PADOVA	75	Criterio 1: 31
60	P2022LFR5M	CORTIGNANI Raffaele	Università degli Studi della TUSCIA	75	Criterio 1: 30 Criterio 3: 25
61	P2022XTLM2	CORAZZA Marco	Università "Ca' Foscari" VENEZIA	75	Criterio 1: 30 Criterio 3: 23 Equità: 0,94 (9 M; 8 F)
62	P2022KMR8A	LEPORATTI Lucia	Università degli Studi di GENOVA	75	Criterio 1: 30 Criterio 3: 23 Equità: 0,80 (3 M; 2 F)
63	P202239XAE	PIROLA Fabiana	Università degli Studi di BERGAMO	75	Criterio 1: 30 Criterio 3: 23 Equità: 0,67 (3 M; 6 F)
64	P20225M45W	BERARDI Laura	Università degli Studi "G. d'Annunzio" CHIETI-PESCARA	75	Criterio 1: 30 Criterio 3: 22 Equità: 0,86 (4 M; 3 F)
65	P2022NXPBB	ZHU Kai	Università Commerciale "Luigi Bocconi" MILANO	75	Criterio 1: 30 Criterio 3: 22 Equità: 0,67 (2 M; 4 F)
66	P2022MWXJY	MAGLIOCCA Pierpaolo	Università degli Studi di FOGGIA	75	Criterio 1: 30 Criterio 3: 22 Equità: 0,59 (12 M; 5 F)
67	P20222XM58	STEFANINI Alessandro	Università di PISA	75	Criterio 1: 30 Criterio 3: 22 Equità: 0,50 (6 M; 2 F)
68	P2022ZK7AK	IACOBUCCI Donato	Università Politecnica delle MARCHE	75	Criterio 1: 30 Criterio 3: 22 Equità: 0,40 (4 M; 1 F)
69	P2022R248L	MAGGIAN Valeria	Università "Ca' Foscari" VENEZIA	75	Criterio 1: 30 Criterio 3: 21
70	P2022LR3C5	TENUCCI Andrea	Scuola Superiore di Studi Universitari e Perfezionamento Sant'Anna	75	Criterio 1: 29 Criterio 3: 25 Equità: 1,00 (2 M; 2 F)
71	P20227EJEE	POGGI Ambra	Università degli Studi di TORINO	75	Criterio 1: 29 Criterio 3: 25 Equità: 0,67 (2 M; 4 F)

ALLEGATO A - GRADUATORIE SETTORE SH1

N°	Codice progetto	Principal Investigator	Ente	Punteggio	Altri criteri
72	P2022HYP9M	SOGARI Giovanni	Università degli Studi di PARMA	75	Criterio 1: 29 Criterio 3: 22
73	P20228W2Z2	VICENTINI Francesca	Università degli Studi di ROMA "Foro Italico"	75	Criterio 1: 28

Graduatoria Settore SH1 Linea B - Sud

N°	Codice progetto	Principal Investigator	Ente	Punteggio	Altri criteri
1	P20227XP7P	GRASSETTI Francesca	Università della CALABRIA	90	
2	P2022KHP8L	LANGELLA Monica	Università degli Studi di Napoli	89	
3	P2022EN9PJ	MIGLIETTA Pier Paolo	Università del SALENTO	82	Criterio 1: 34
4	P20224K95M	MODICA Marco	Gran Sasso Science Institute - Scuola di dottorato internazionale	82	Criterio 1: 33 Criterio 3: 25
5	P202223J4J	LI DONNI Paolo	Università degli Studi di PALERMO	82	Criterio 1: 33 Criterio 3: 24
6	P2022SRW8N	DI BUCCHIANICO Stefano	Università degli Studi di SALERNO	82	Criterio 1: 32
7	P2022JTSFB	SANTERAMO Fabio Gaetano	Università degli Studi di FOGGIA	81	Criterio 1: 32 Criterio 3: 25
8	P2022WM82K	SALERNO Dario	Università degli Studi di NAPOLI "Parthenope"	81	Criterio 1: 32 Criterio 3: 24
9	P2022Z3TP8	STEIN Mattea Regina	Università degli Studi di Napoli	80	
10	P20227JN7R	CARILLO Maria Rosaria	Università degli Studi di NAPOLI "Parthenope"	79	
11	P20229YMBZ	MELEDDU Marta	Università degli Studi di SASSARI	78	Criterio 1: 32
12	P2022KTM7H	SARDARO Ruggiero	Università degli Studi di FOGGIA	78	Criterio 1: 29
13	P2022ANZ72	MANUTI Amelia	Università degli Studi di BARI ALDO	77	Criterio 1: 32
14	P2022WP49F	BUTZBACH Olivier Karl	Università degli Studi della Campania	77	Criterio 1: 31
15	P2022ZRPBL	SALVATORE Domenico	Università degli Studi Suor Orsola	76	Criterio 1: 30
16	P2022JH22T	CORSI Katia	Università degli Studi di SASSARI	76	Criterio 1: 29
17	P2022RF38Y	CUCCURULLO Corrado	Università degli Studi della Campania "Luigi Vanvitelli"	75	Criterio 1: 33
18	P2022YY9P2	RUSSO Francesco Flaviano	Università degli Studi di Napoli	75	Criterio 1: 32
19	P2022LWZZX	REFORGIATO RECUPERO Diego Angelo Gaetano	Università degli Studi di CAGLIARI	75	Criterio 1: 31
20	P20224K52W	TOMO Andrea	Università degli Studi di Napoli Federico II	75	Criterio 1: 30 Criterio 3: 23
21	P2022XLL94	LISI Domenico	Università degli Studi di CATANIA	75	Criterio 1: 30 Criterio 3: 22
22	P2022CW4HX	ROMA Rocco	Università degli Studi di BARI ALDO MORO	75	Criterio 1: 30 Criterio 3: 21
23	P2022YHPWZ	ISONI Andrea	Università degli Studi di CAGLIARI	75	Criterio 1: 30 Criterio 3: 20
24	P2022B22W2	BATTISTI Michele	Università degli Studi di PALERMO	75	Criterio 1: 29