Luiss Guido Carli

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Disentangling the Yield Curve: a Global Dynamic Nelson-Siegel Approach

Supervisor: Prof. P.Santucci De Magistris
Co-supervisor: Prof. G.Traficante

Candidate: Emanuele Mellara
ID Number: 764471

Contents

Abstract

This study explores the extraction of an international yield curve from single-country yield curves in order to enhance and understand the interdependence among the global financial markets. For quantifying these relations, we extract the three country-factors proposed by Nelson and Siegel [1987] that characterise the yield curve shape through the linear regression model by using the Ordinary Least Square (OLS) estimator, then we bundle the single-country factors and we apply the Principal Component Analysis (PCA) for each class of factors in order to obtain the common component that explains the majority of factors' variance. Furthermore, we add several macro economic variables which explain the idiosyncratic movements of the single-country yield curves. We conclude that the global factors lead the movements and the shape of the yield curve while the macro factors affect the yield curve differently depending on the yield maturities.

1 Introduction

The yield curve, which shows the connection between bond yields and their maturity dates is highly significant for both researchers and professionals in the industry and the literature is unified by the assumption that the curve depends on a number of latent factors (e.g., Litterman and Scheinkman (1991)).Studying yield curve models has been a focus of research especially when it comes to government bond yield structures since this relationship has important economic implications such the market's expectations of future rate changes, the bond risk premia and the bias related to the yield curve convexity(Ilmanen (1995)). Regarding the second interpretation, the shape of the yield curve depends significantly on both what the market predicts about interest rate changes and the additional yield investors seek for holding longer term bonds to account for uncertainties and risks, over time. These risk premia, which represent the extra compensation investors demand for bearing risks associated with longer term bonds, represents an empirical evidence against the Pure Expectations Hypothesis (PEH)(Fisher (1930)) that tried to explain the shape of the yield curve in a risk-neutral world. As a result the yield curve does not reflect expected rate trends but also incorporates varying levels of risk premiums demanded by investors at different maturity levels. Therefore it is essential to consider both market expectations and risk premia when analyzing the slope of the yield curve as they collectively influence it beyond what's implied by pure expectations alone.

Many studies have looked into modeling yield curves within countries with models based on hidden factors such as level, which is related to the overall level of interest rates, slope, that represents the market's expectation of high or low rates and the bond risk premia, and curvature representing the non-linear relationship between the Price and the Yield of the bonds which causes the Humped Shape (e.g.,Andersen and Lund (1997)). The Nelson Siegel model, first introduced by Nelson and Siegel (1987) has become a framework for this purpose. Its dynamic extension by

Diebold and Li (2006) has further improved its effectiveness in capturing the changing nature of yield curves by introducing the Vector Autoregressive (VAR) models for these factors.Therefore the three factors follow a VAR(1) process which explains their dynamics by creating a model where each factor is expressed by a linear function of both its past value and the past values of the other factors along with an error term. This model generates good forecast of single-country yields by applying the Nelson-Siegel equation to the estimated factors.

While previous research has mostly centered on the dynamics of yield curves within countries, the increasing interconnectedness of financial markets(e.g., Diebold and Yilmaz (2015)) advocates that yield curves across different countries could be influenced by common global factors, especially extensive interconnections in the market can result in transmission of shocks and crises across markets potentially escalating local issues to global financial instability. This phenomenon was evident during events such as the Euro-Zone debt crisis and the 2008 financial crisis, where troubles in markets swiftly impacted global financial systems. However on a note these connections also offer opportunities for improved risk sharing and diversification as investors can distribute their investments across markets ultimately enhancing overall market efficiency and resilience. Furthermore the interconnectedness of bond markets plays a role in influencing interest rate movements, which holds implications, for monetary policies and economic stability. Therefore, understanding these influences on yield factors is crucial for gaining insights into the overall dynamics of the international bond market. However there is still exploration, into how yield curves interact across multiple countries. In this thesis we expand the Dynamic Nelson Siegel model (Diebold et al. (2008a)) to a perspective by including yield curve data, from four economies; the United States, the United Kingdom, Germany and Canada. Our aim is to recognize and grasp the shared influences that shape yield curve movements across these markets. Moreover we introduce several factors—Section 7—to create a European and a British yield curve model that considers both worldwide impacts and regional economic circumstances. The ratio behind this idea is that other papers have already shown that latent country-factors are linked to and interact to domestic macroeconomic factors(e.g., Ang and Piazzesi (2003)).

Our research offers proof of yield factors that significantly impact the yield curves of these studied nations. By breaking down the variability in country specific yield curves into unique elements: the level of the country yields, the slope of the yield curve and the curvature of the curve.Therefore, we can measure the effects of these factors, we can analyze the dynamics of these factors and, especially, we can identify periods of major influence due to macroeconomic factors. The first main results is the homogeneous relation between the three factors during the two decades period. This outcome provides strong evidences of global factors which would explain this commonality among local factors. Furthermore, incorporating single country factors, we found common dependence on global yield level, slope and curvature. Our model allows for a detailed comprehension of how global financial trends interact with local economic environments, how the common factors move over time and the amount of the yield curve explained by the idiosyncratic country factors. Thus we retrieve a sort of global bond yield curve analogue to the global real-side work of Lumsdaine and Prasad (2003), Gregory and Head (1999) and Kose et al. (2003) and we estimate the components that drive the idiosyncratic elements of the single-country yield curve dynamics. The results of this investigation contribute to knowledge by showcasing the significance of factors, in yield curve analysis and emphasizing the advantages of integrating macroeconomic variables into such models. This study carries implications, for policymakers, investors and academics aiming to gain insights, into the intricacies of worldwide bond markets and the drivers that impact them.

The paper is structured as follows. First, Section 2 and 3 describe, respectively, the model we use for estimate the country factors and the historical data analysis and the used yield curve visualisation. Then, in section 4 we describe the estimation of the single-country factors while,in section 5, we present the co.integrating and the commonality evidences and the successive extraction of the global factors and, in section 6, we analyze the impact of these factors on the single-country ones. The section 7 is about the idiosyncratic factor explained by the macroeconomic variables and how can we incorporate them into the model. Last, we present concluding remarks, results and economic interpretations of the research. At the end of the paper the is the Appendix where we explain the theory behind the PCA analysis and we report the tables representing the $Pvalue$, $T-statistic$ and the standard error of the SUR estimation.

2 Modeling Framework

In this part we expand the Dynamic Nelson Siegel model to a country context by introducing the Global Dynamic Nelson Siegel model. Our approach enables us to capture the dynamics of yield curves, across countries by including factors that impact the yield curves of the United States, the United Kingdom, Germany and Canada. Along with the level, slope and curvature factors we incorporate three macroeconomic variables—the USD/EUR exchange rate, Euribor, the European Industrial Production Index, the European Consumer Price Index, the Maastricht Debt to Gdp ratio and the European Economic Sentiment Index—into the model to address macroeconomic influences, on the European yield curve.

2.1 Single-Country Model

We begin by revisiting the single-country Dynamic Nelson-Siegel (DNS) model (Diebold and Li (2006)). The DNS model for a single country at a particular point in time can be expressed as

$$y_i(\tau) = l_i + s_i \left(\frac{1 - e^{-\lambda_i \tau}}{\lambda_i \tau} \right) + c_i \left(\frac{1 - e^{-\lambda_i \tau}}{\lambda_i \tau} - e^{-\lambda_i \tau} \right) + \nu_i(\tau) \quad \text{for} \quad \text{i} = 1, \dots, \text{N} \tag{1}$$

where $y_i(\tau) \quad \tau = 1, 2, \dots, 12$ denotes the continuously-compounded zero-coupon nominal yield on a bond with maturity τ months, and l_i, s_i, and c_i represent the overall level of yields that affect the long-term curve shape, the slope which represents the market's expectations of future yields and the risk premium related to the markets risk, and the curvature factor that explains the non linear-relation between price and yield, respectively. The parameter λ_i determines the decay rate of the factor loading and it varies across maturities and over time, and $\nu_i(\tau)$ is the disturbance term, thus a stochastic disturbance, and we assume it follows a normal distribution, therefore $v_i(\tau) \sim N(0, \sigma_i^2(\tau))$ with standard deviation $\sigma_i(\tau)$. Then, this term captures the deviations between the yields predicted by model and the actual yields. This is a generalized Nelson-Siegel model (1987) that can generate several time-varying yield curve shapes.

$$y_{i,t}(\tau) = l_{i,t} + s_{i,t}\left(\frac{1-e^{-\lambda_{i,t}\tau}}{\lambda_{i,t}\tau}\right) + c_{i,t}\left(\frac{1-e^{-\lambda_{i,t}\tau}}{\lambda_{i,t}\tau} - e^{-\lambda_{i,t}\tau}\right) + \nu_{i,t}(\tau) \quad \text{for} \quad \mathrm{t} = 1, \dots, \mathrm{T} \tag{2}$$

In the dynamic version of the model proposed by Diebold and Li [2006], the latent factors $l_{i,t}$, $s_{i,t}$, and $c_{i,t}$ vary over time, capturing the evolution of the yield curve and being a decreasing and a concave function of τ while the decay parameter λ_t is, initially, assumed to vary over time but we assume it to be constant over time and across countries. Therefore, we assume a simplified version of the yield curve and, recalling that λ_t determines the maturity at which the curvature factor (medium term) reaches its maximum, as Diebold and Li [2006] proposed , we fix a value for $\lambda_t = 0.0609$. By doing this, we can compute the values of the factor loading and use OLS to estimate the betas for each period (month). The ordinary least squares estimation enhances the simplicity of the model, the convenience and the numerical trustworthiness due to the replacement of several numerical optimizations with OLS regressions. Thus, the dynamic model can be expressed as

$$y_{i,t}(\tau) = l_{i,t} + s_{i,t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + c_{i,t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) + \nu_{i,t}(\tau) \quad \text{for} \quad \mathrm{t} = 1, \dots, \mathrm{T} \tag{3}$$

This equation is the measurement equation of a State-Space system with ($l_{i,t}$, $s_{i,t}$, and $c_{i,t}$)' as state vector, thus the Nelson-Siegel model is a State-Space model. Therefore, the OLS regression for each country at each time step is performed by regressing the observed yields $y_{i,t}(\tau)$ on the factor loadings

$$\mathbf{y_{i,t}} = \mathbf{X}\boldsymbol{\beta_{i,t}} + \boldsymbol{\epsilon_{i,t}} \quad \text{for} \quad \mathrm{t} = 1, \dots, \mathrm{T} \quad \text{and} \quad \mathrm{i} = 1, \dots, \mathrm{N} \tag{4}$$

and this is feasible due to the deterministic form of the Nelson-Siegel factors represented by the matrix $\mathbf{X}$. The equation (4) is built from the equation (3), where the matrix $\mathbf{X}$ contains the loadings of each factor, thus 1 for the level, $\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right)$ for the slope and $\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right)$ for the curvature. Then, the matrix $\boldsymbol{\beta_{i,t}}$ contains the level, slope and curvature values of the four countries at each time t. In section 4 we deep dive the OLS estimation of the single-country factors for Germany, the US, the UK and Canada.

2.2 State-Space Model

Given the dynamic nature of yield curves and their dependence on dynamic factors, a more sophisticated modeling approach is required and, also, tested : The State-Space Model. A State-Space Model (SSM) is particularly well-suited for this purpose due to its ability to handle latent variables, time-varying parameters, and multivariate time series data. In the following subsections, by using the MathWorks (2021) State-Space Model Framework, we analyze the inputs of the State-Space Model and, then in Section 4, we compare the results obtained with the ones of the two steps procedure (OLS). State Space Models, known as SSMs are a tool, for analyzing time series data by assuming that observed data is influenced by hidden variables (states) that change over time. A State Space Model consists of two components: the state equation and the observation equation. In this study we are focusing on estimating the parameters of a state space model where individual country factors follow an AR(1) process. The equations defining the state space model are as follows: The state equation describes how the latent state vector $\mathbf{S}_t$ $(S_{1,t},\ S_{2,t},\ S_{3,t})$ evolves over time where, in our case, it represents the latent factors obtained by regressing the observed single country yields on the factors loading matrix $\mathbf{X}$ at time t. It is typically modeled as a first-order Markov process, where the current state depends on the previous state and some process noise. The state equation is given by

$$\begin{pmatrix} S_{1,t} \\ S_{2,t} \\ S_{3,t} \end{pmatrix} = \mathbf{A} \begin{pmatrix} S_{1,t-1} \\ S_{2,t-1} \\ S_{3,t-1} \end{pmatrix} + \mathbf{w}_t \quad \text{for} \quad \mathrm{t} = 1, \ldots, \mathrm{T} \tag{5}$$

where

$$\mathbf{A} = \begin{pmatrix} \phi_{11} & 0 & 0 \\ 0 & \phi_{21} & 0 \\ 0 & 0 & \phi_{31} \end{pmatrix} \quad \text{and} \quad \mathbf{w}_t = \begin{pmatrix} w_{1,t} \\ w_{2,t} \\ w_{3,t} \end{pmatrix} \sim \mathcal{N}(0, \mathbf{Q}) \tag{6}$$

where $\mathbf{S_t}$ is 3 x 1, the matrix $\mathbf{A}$ is 3 x 3.

Then, the disturbance term variance matrix $\mathbf{Q} = \mathbf{B}\mathbf{B}^\top$ and $\mathbf{B}$ is a lower triangular matrix obtained from the Cholesky factor of Q. The process noise covariance matrix Q is defined as

$$\mathbf{Q} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix} \tag{7}$$

where σ_i^2 represents the variance of the process noise for the i-th state variable and σ_{ij} represents the covariance between the process noise of the i-th and j-th state variables.

If the noise terms are assumed to be uncorrelated, the Q matrix simplifies to a diagonal matrix

$$\mathbf{Q} = \begin{pmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{pmatrix} \tag{8}$$

The observation equation links the latent state vector $\mathbf{S}_t$ to the observed data $\mathbf{y}_t$. It specifies how the observations are generated from the states, typically including some measurement noise. The observation equation is defined as

$$y_t = \mathbf{C}\mathbf{S}_t + v_t \quad \text{for} \quad \mathrm{t} = 1, \dots, \mathrm{T} \tag{9}$$

where Y_t represents the observed yields, $\mathbf{C}$ is the loading matrix determined by the Nelson-Siegel model, and v_t is the observation noise, assumed to be normally distributed with mean zero and covariance matrix $\mathbf{R}$. The loading matrix C in the context of the Nelson-Siegel model, for example, can be represented as

$$\mathbf{C} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \left(\frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1}\right) \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \left(\frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2}\right) \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_{12}}}{\lambda\tau_{12}} & \left(\frac{1-e^{-\lambda\tau_{12}}}{\lambda\tau_{12}} - e^{-\lambda\tau_{12}}\right) \end{pmatrix} \tag{10}$$

whereλ is the decay factor in the Nelson-Siegel model,τ_i represents the time to maturity for the i-th observation and the first column corresponds to the level factor, the third column corresponds to the slope factor, and the fourth column corresponds to the curvature factor.

The observation noise covariance matrix R is defined as

$$\mathbf{R} = \begin{pmatrix} \sigma_{v1}^2 & \sigma_{v1,2} & \cdots & \sigma_{v1,m} \\ \sigma_{v1,2} & \sigma_{v2}^2 & \cdots & \sigma_{v2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{v1,m} & \sigma_{v2,m} & \cdots & \sigma_{vm}^2 \end{pmatrix} \quad \text{for} \quad \mathrm{v,m} = 1, \dots, 12 \tag{11}$$

where m is the number of maturities, thus it is 12, σ_{vi}^2 represents the variance of the observation noise for the i-th observed variable and σ_{vij} represents the covariance between the observation noise of the i-th and j-th observed variables. If the observation noise terms are assumed to be independent, the R matrix simplifies to a diagonal matrix

$$
\mathbf{R} = \begin{pmatrix} \sigma_{v1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{v2}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{vm}^2 \end{pmatrix} \tag{12}
$$

In this case, there is no correlation between the observation noise terms, and each diagonal element represents the variance of the noise for the corresponding observed variable. The observation noise covariance matrix R is defined as

$$
\mathbf{R} = \mathbf{D}\mathbf{D}^\top \tag{13}
$$

where $\mathbf{D}$ is a lower triangular matrix obtained through the Cholesky decomposition of $\mathbf{R}$, and it can be written as

$$
\mathbf{D} = \begin{pmatrix} d_{1,1} & 0 & \cdots & 0 \\ d_{2,1} & d_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ d_{12,1} & d_{12,2} & \cdots & d_{12,12} \end{pmatrix} \tag{14}
$$

In this case, $\mathbf{D}$ is a diagonal matrix with the elements d_{ii} representing the standard deviations of the observation noise for the i-th observed variable. The parameters of the state-space model, which include the elements of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, are estimated using Maximum Likelihood Estimation (MLE). The likelihood function is maximized with respect to these parameters and, in order to do that, e introduce $\mathbf{H_t}$ as the innovation covariance which measures the uncertainty in the prediction of y_t and it is defines as $CP_{t|t-1}C^T + R$ where $P_{t|t-1}$ is the covariance prediction matrix which is given by $AP_{t-1|t-1}A^T + Q$ indicating the predicted (A Priori) estimate of the covariance matrix of S_t at t given information at time t-1. At each time t, the log-likelihood contribution is given by

$$
\ell_t = -\frac{1}{2}\left(N \ln 2\pi + \ln|\mathbf{H}_t| + v_t^\top \mathbf{H}_t^{-1} v_t\right) \tag{15}
$$

where N is the number of observed yields at time t, v_t is the innovation vector, $\mathbf{H}_t$ is the innovation covariance matrix and $|\mathbf{H}_t|$ denotes the determinant of $\mathbf{H}_t$. The total log-likelihood over all T observations is

$$
\log \mathcal{L}(\theta) = -\frac{1}{2}\sum_{t=1}^{T}\left(N \ln 2\pi + \ln|\mathbf{H}_t| + v_t^\top \mathbf{H}_t^{-1} v_t\right) \tag{16}
$$

The process of maximizing θ can be summarized as follows

$$
\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\, \mathcal{L}(\theta \mid y_t) \tag{17}
$$

where $\mathcal{L}(\theta \mid y_t)$ is the log-likelihood function given the observed historical yields for each country y_t.

2.3 Global Dynamic Nelson-Siegel Model

In this analysis, we use Principal Component Analysis (PCA) to uncover the underlying structure of the yield curves across multiple countries. Thus, the global factors L_t, S_t, and C_t are extracted using (PCA) from the respective matrices of factors estimated by using OLS in the single-country framework. The PCA helps us extract the most important patterns (or factors) driving the variance in the data by analyzing the variance-covariance matrix. This matrix is symmetric, meaning the entries along its main diagonal represent the variances of each country's factors, while the off-diagonal entries represent the covariance between the factors of different countries.

The country-specific factors were collected into a data matrix $\mathbf{F}$ of size $T \times N$, where T is the number of observations (290) and N (4) is the number of countries. Each column of $\mathbf{F}$ represents a time series of factors for one country. This PCA analysis returns the principal components, the explained variance, and the loadings (eigenvectors): The first few principal components were selected based on the amount of variance they explain (typically those that explain a significant portion of the total variance). These selected components were interpreted as global factors that drive the common dynamics in the country-specific factors. The global factors were analyzed to understand the underlying global influences on the yield curves. The loadings were examined to assess how each country's factors contributed to the global factors. Through PCA, we extract the eigenvalues and eigenvectors of this matrix where the eigenvalues reflect the magnitude of each principal component, indicating how much of the total variance is explained by that component, the eigenvectors, on the other hand, describe the direction of each principal component and how much each country's yield curve factor contributes to it. By examining the principal components, we can identify common global yield curve factors—level, slope, and curvature. The magnitudes of the principal components (given by the eigenvalues) tell us how much of the overall variance in the yield curves is captured by each of these factors, while the eigenvectors explain how strongly each country's yield curve factor contributes to these global patterns. One of the key benefits of using PCA is that the resulting factors are orthogonal, meaning they are statistically uncorrelated. This ensures that the global level, slope, and curvature factors are independent of each other, which simplifies their analysis. Since these factors are uncorrelated, we model them using separate autoregressive AR processes, instead of a more complex vector autoregressive VAR process where an AR process allows each factor to depend only on its own past values, reflecting the natural time series dynamics of each factor.

2.4 Global Factor Dynamics

Now we extend the DNS model to a global setting by adding the global factors that influence the yield curves across multiple countries. In the GDNS model, each country's yield curve is driven by both global factors and country-specific factors. The global factors include a global level factor L_t, a global slope factor S_t, and a global curvature factor C_t, which are shared across all countries. Therefore, we apply the Nelson-Siegel model to the global framework in order to obtain the yields driven and generated by the global latent factors. The global model can be written as

$$Y_t(\tau) = L_t + S_t\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + C_t\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) + \nu_{i,t}(\tau) \quad \text{for} \quad \text{t} = 1,\ldots,\text{T} \tag{18}$$

where $Y_t(\tau)$ has dimensions T x τ where $t = 1, 2..., T$ are the observations while τ represents the maturities vector. Therefore, $Y_t(\tau)$ represents the yields related and explained by common (global) factors among countries at time t for a bond with maturity τ, and $\nu_{i,t}(\tau) \sim N(0, \sigma_i^2(\tau))$ is the disturbance term. The global factors L_t, S_t, and C_t capture the common movements in yield curves across all countries.

The dynamics of the global factors can be expressed as

$$L_t = \phi_L L_{t-1} + \epsilon_t^L \quad \text{for} \quad \text{t} = 1,\ldots,\text{T} \tag{19}$$

$$S_t = \phi_S S_{t-1} + \epsilon_t^S \quad \text{for} \quad \text{t} = 1,\ldots,\text{T} \tag{20}$$

$$C_t = \phi_C C_{t-1} + \epsilon_t^C \quad \text{for} \quad \text{t} = 1,\ldots,\text{T} \tag{21}$$

where ϕ_L, ϕ_S, and ϕ_C are the autoregressive coefficients for the level, slope, and curvature factors, respectively, and ϵ_t^L, ϵ_t^S, and ϵ_t^C are the error terms, which are assumed to be normally distributed with mean zero and constant variance. This structure simplifies the modeling process while still capturing the key dynamics of the global yield curve.

2.5 Country-Specific Model

Each country's yield curve is influenced by the global factors as well as country-specific factors (Diebold et al. (2008b)). The country-specific level, slope, and curvature factors (l_{it}, s_{it}, c_{it}) are modeled as:

$$l_{i,t} = \alpha_i^L + \beta_i^L L_t + \epsilon_{i,t}^l \quad \text{for} \quad \text{t} = 1,\ldots,\text{T} \quad \text{and} \quad \text{i} = 1,\ldots,\text{N} \tag{22}$$

$$s_{i,t} = \alpha_i^S + \beta_i^S S_t + \epsilon_{i,t}^s \quad \text{for} \quad \text{t} = 1,\ldots,\text{T} \quad \text{and} \quad \text{i} = 1,\ldots,\text{N} \tag{23}$$

$$c_{i,t} = \alpha_i^C + \beta_i^C C_t + \epsilon_{i,t}^c \quad \text{for} \quad \text{t} = 1, \ldots, \text{T} \quad \text{and} \quad \text{i} = 1, \ldots, \text{N} \tag{24}$$

where α_i^L, α_i^S, and α_i^C are constant terms, β_i^L, β_i^S, and β_i^C are the loadings on the global factors, and ϵ_{it}^l, ϵ_{it}^s, and ϵ_{it}^c are the country-specific idiosyncratic factors. These idiosyncratic factors are assumed to follow a first order autoregressive process similar to the global factors but with their own dynamics.

$$\begin{pmatrix} \epsilon_{i,t}^l \\ \epsilon_{i,t}^s \\ \epsilon_{i,t}^c \end{pmatrix} = \begin{pmatrix} \phi_{i,11} & \phi_{i,12} & \phi_{i,13} \\ \phi_{i,21} & \phi_{i,22} & \phi_{i,23} \\ \phi_{i,31} & \phi_{i,32} & \phi_{i,33} \end{pmatrix} \begin{pmatrix} \epsilon_{i,t-1}^l \\ \epsilon_{i,t-1}^s \\ \epsilon_{i,t-1}^c \end{pmatrix} + \begin{pmatrix} u_{it}^l \\ u_{it}^s \\ u_{it}^c \end{pmatrix} \quad \text{for} \quad \text{t} = 1, \ldots, \text{T} \quad \text{and} \quad \text{i} = 1, \ldots, \text{N} \tag{25}$$

where the u_{it}^n are disturbances $= (\sigma_i^n)^2$ and n = l,s,c. An important assumption is that the shocks to the global factors and the shocks to the single-country factors are orthogonal. This structure allows the model to capture both the global influences on yield curves and the idiosyncratic behaviors specific to each country, providing a comprehensive framework for understanding the dynamics of yield curves in a global context.

Then, the final yield curve equation for country i at time t and maturity τ is:

$$\begin{aligned} y_{i,t}(\tau) = & (\alpha_L^i + \beta_L^i L_t + \epsilon_{l,i,t}) \\ & + (\alpha_S^i + \beta_S^i S_t + \epsilon_{s,i,t}) \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) \\ & + (\alpha_C^i + \beta_C^i C_t + \epsilon_{c,i,t}) \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) \\ & + v_{i,t}(\tau) \quad \text{for} \quad \text{t} = 1, \ldots, \text{T} \quad \text{and} \quad \text{i} = 1, \ldots, \text{N} \end{aligned} \tag{26}$$

where the first row represents the level factor that now it is expressed as function of the global level and the idiosyncratic component, the second row is the slope factor decomposed into the global and the single country driver, the third row is the curvature component expressed as global curvature and idiosyncratic one while the last row is the disturbance $v_{i,t}(\tau) \sim N(0, \sigma_i^2(\tau))$. Therefore, this model embeds the Nelson-Siegel factors and, then, it allows them to be dynamic over time assessing the AR process for both global and idiosyncratic components.

2.6 Data Analysis

The empirical analysis in this study is based on a dataset comprising yield curve data from four major economies: the United States, the United Kingdom, Germany, and Canada. The data cover 12 different maturities, ranging from 3 months to 10 years, and consist of 290 monthly observations from January 31, 2000, to February 29, 2024. The yield data for each country were obtained from the Federal Reserve Bank od St.Louis(USA)[1], the Bank of England (UK)[2], the Deutsche Bundesbank (Germany)[3], and the Bank of Canada(Canada)[4].

The chosen maturities are 3 months, 6 months, 1 year, 2 years, 3 years, 4 years, 5 years, 7 years, 8 years, 9 years, and 10 years. These maturities are selected to provide a comprehensive view of the term structure of interest rates across different time horizons, ensuring that both short-term and long-term segments of the yield curve are adequately captured. The sample period of 290 monthly observations allows for the analysis of yield curve dynamics over more than two decades, encompassing various economic cycles, monetary policy regimes, and global financial events. This period includes significant episodes such as the aftermath of the dot-com bubble, the global financial crisis of 2007-2008, the European sovereign debt crisis, and the COVID-19 pandemic, providing a rich context for studying the interactions between global and country-specific factors. Subsequently, we divide the sample period into four sub-samples with respect to the main financial crises: the fist sub-sample covers the period from 2000 to 2008, the second one form 2009 to 2012, the third period from 2013 to 2019 and the last one from 2020 to 2024. The core reason is to study the dynamic of the yield curve and of its drivers with respect to different economic conditions in order to analyze the various impacts of the three loadings and the macroeconomic variables during the two decades.

3 Preliminary Analysis

Prior to estimating the Global Dynamic Nelson Siegel (GDNS) model we conducted an examination of the data to grasp the characteristics of the yield curves in each country. We calculated statistics for yields at maturities including mean values, standard deviations, minimum and maximum figures, as well as autocorrelation coefficients at different time lags. Upon inspecting the yield curves over time we observed variations in levels among countries and across different periods. These differences reflect varying policies, economic situations and perceptions of risk. Furthermore we noted a persistence in the data with yields at maturities generally showing less volatility compared to shorter term yields. The preliminary analysis has confirmed the presence of trends across countries hinting at global factors influencing yield curve movements. These insights lay a groundwork for exploration into modeling both global and country specific elements, within the GDNS framework.

[1]https://fred.stlouisfed.org
[2]https://www.bankofengland.co.uk
[3]https://www.bundesbank.de/en
[4]https://www.bankofcanada.ca

3.1 Yield Curve Statistics

The average yield, in Germany shows an increase as bond maturity lengthens starting at 1.2496% for 3 month bonds and reaching 2.4023% for 10 year bonds. This mirrors the trend of returns being sought for longer term investments by investors. Similarly the United States displays a pattern with higher yields across different maturities ranging from 1.7398% for 3 month bonds to 3.2512% for 10 year bonds indicating a preference for long term debt with potentially increased risk or inflation expectations in the U.S. The United Kingdom also follows this trajectory in yields from 2.1789% for 3 month bonds to 3.1017% for 10 year bonds although the curve is less steep compared to Germany and the U.S. suggesting a more moderate progression. In Canada the yield curve has an slope starting at 2.0162% for 3 month bonds and peaking at 3.1514% for 10 year bonds reflecting trends akin, to those seen in the UK but beginning from a slightly lower point.

The consistency of yield fluctuations, in Germany remains quite stable regardless of the bond maturity with values hovering around 1.8 to 1.9. This indicates that the volatility of yields remains relatively uniform for both term and long term bonds. On the hand the United States demonstrates a decreasing trend in deviation as maturity lengthens, starting from 1.9204 for 3 month bonds and decreasing to 1.3118 for 10 year bonds. This suggests that shorter term U.S. Bonds exhibit volatility, due to their sensitivity to immediate economic conditions and policy adjustments. Similarly the United Kingdom also experiences a decline in deviation as maturity increases although the difference is not as prominent compared to the U.S. ranging from 2.132 for 3 month bonds to 1.577 for 10 year bonds indicating that short term UK bonds face fluctuation compared to longer term ones. Canada follows a pattern with a reduction in standard deviation from 1.6091 for 3 month bonds to 1.4534 for 10 year bonds implying that Canadian bonds display higher volatility in the short term. In Germany all minimum yields are negative across maturities reflecting periods where German bond yields dipped below zero due, to monetary policies. The returns, on bonds tend to rise as they mature starting at 5.1 for 3 month bonds and reaching 5.72 for 10 year bonds with the idea that longer term bonds offer returns. In the United States bond yields stay positive regardless of maturity unlike in Germany where negative yields have been observed. Similarly in the United Kingdom shorter term bond yields occasionally dip into territory while longer term bonds show maximum yields but with a slightly lower peak compared to Germanys 10 year bond. Canada also follows the trend of returns for maturities with minimum yields remaining positive and maximum yields increasing from 5.6852 for 3 month bonds to 6.4825 for 10 year bonds.

Table 1: Descriptive Statistics for Bond Yield

Country	Maturity (Months)	Mean	Std. Dev.	Min	Max	Rho(1)	Rho(12)	Rho(30)
Germany	3	1.2496	1.8381	-0.96	5.10	0.98741	0.71491	0.39018
	12	1.3239	1.8503	-0.92	5.17	0.98923	0.74850	0.43926
	60	1.8145	1.8971	-0.93	5.34	0.98819	0.84318	0.62812
	120	2.4023	1.8884	-0.71	5.72	0.98910	0.87275	0.67255
US	3	1.7398	1.9204	0.011429	6.3562	0.98148	0.50759	-0.071149
	12	1.9325	1.8671	0.0505	6.3264	0.98153	0.55032	-0.038105
	60	2.6796	1.4959	0.26667	6.6877	0.97518	0.64705	0.20737
	120	3.2512	1.3118	0.62364	6.661	0.97366	0.70289	0.38151
UK	3	2.1789	2.132	-0.073701	5.9739	0.98659	0.75233	0.47208
	12	2.2397	2.1238	-0.15036	6.3175	0.98606	0.75699	0.46115
	60	2.6825	1.838	-0.12769	6.2861	0.98359	0.81401	0.57440
	120	3.1017	1.577	0.12522	5.6139	0.98368	0.82882	0.60099
CAN	3	2.0162	1.6091	0.06062	5.6852	0.98076	0.54080	0.11748
	12	2.1896	1.5922	0.13451	6.0593	0.97745	0.56055	0.12065
	60	2.6962	1.4994	0.29378	6.5174	0.97712	0.77123	0.49580
	120	3.1514	1.4534	0.48203	6.4825	0.98053	0.82872	0.59820

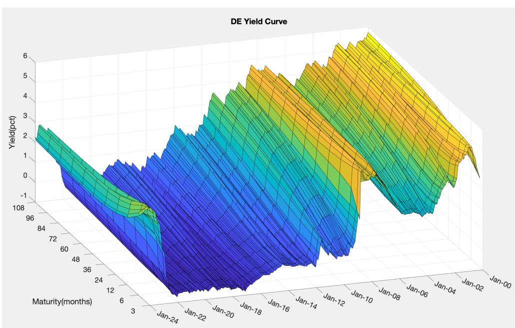
Regarding autocorrelation German bond yields display persistence in yield movements over a one month period, across all maturities showing autocorrelation levels exceeding 0.98. Over time the consistency decreases, as shown by the autocorrelation values declining after 12 and 30 months for bond maturities. In the United States a similar trend is observed, with autocorrelation, at a 1 month lag that lessens over time and even turns negative after 30 months for maturities hinting at possible shifts in yield trends. The United Kingdom follows a pattern with autocorrelation at the 1 month mark remaining relatively robust at 12 months especially for longer maturities but gradually weakening by the 30 month point. Canadas bond yields exhibit a pattern with autocorrelation at one month and a gradual decline over extended periods particularly for shorter maturities. However Canadian long term yields show autocorrelation over lags compared to other nations suggesting more consistent trends, in long term yield movements.

In conclusion, the yield time series of the four countries are non-stationary due to high values of autocorrelation even for large lags, as shown in the table above. Furthermore, we perform the $Augmented - Dickey - Fuller$ (ADF) proposed by Cheung and Lai (1995) test to confirm the presence of a unit root in the four time-series indicating the non-stationarity.

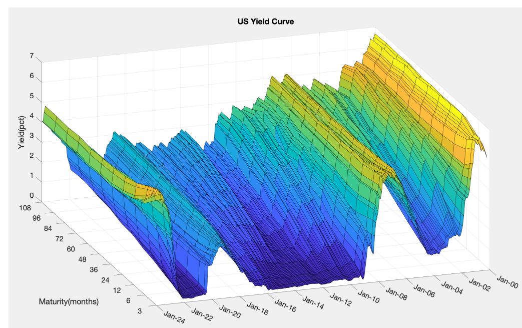
$$\Delta \mathbf{y_{i,t}} = \zeta + \eta_{\mathbf{i,t}} + \mu \mathbf{y_{i,t-1}} + \sum_{j=1}^{p} \delta_j \Delta \mathbf{y_{i,t-j}} + \varepsilon_{\mathbf{i,t}} \quad \text{for} \quad \mathrm{t} = 1, \ldots, \mathrm{T} \quad \text{and} \quad \mathrm{i} = 1, \ldots, \mathrm{N} \tag{27}$$

where the $\mathbf{y_t}$ is the single-country yield time-series being tested, the $\Delta \mathbf{y_{i,t}}$ represents the first difference which remove the trends, ζ is the intercept or a drift term which is constant and it allows the process to have non-zero mean that is consistent with our dataset, $\eta_{\mathbf{i,t}}$ is related to the trend term which represents the slope in the yield time series over time, thus it indicates a linear trend. Then, the parameter μ determines whether the yield time-series has a unit root through an Hypothesis test where H_0 concerns when μ is equal to zero which implies the presence of the unit root, therefore the series is non-stationary because $\mu \mathbf{y_{i,t-1}}$ disappears and $\Delta \mathbf{y_{i,t}}$ is driven by the drift and and the lagged differences, thus the process follows a random walk. H_1 regards when μ less than one, implying stationarity because the series exhibits mean reversion and, the more negative μ the faster the yield time series reaches the equilibrium after a shock. Last, the lagged differences $\sum_{j=1}^{p} \delta_j \Delta \mathbf{y_{i,t-j}}$ control for autocorrelation and they help ensure that $\varepsilon_{\mathbf{i,t}}$ is a White-Noise, the p is chosen in order to minimize the autocorrelation in the residuals. The results of the ADF imply the presence of the unit root in all yield time series confirming the non-stationarity.

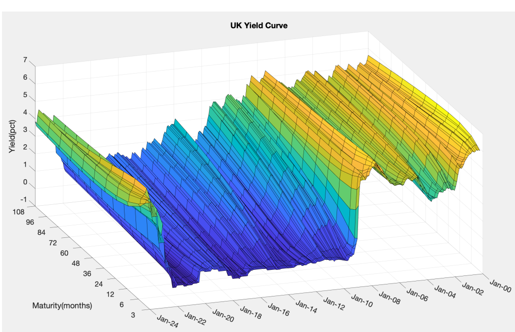
3.2 Yield Curve Analysis



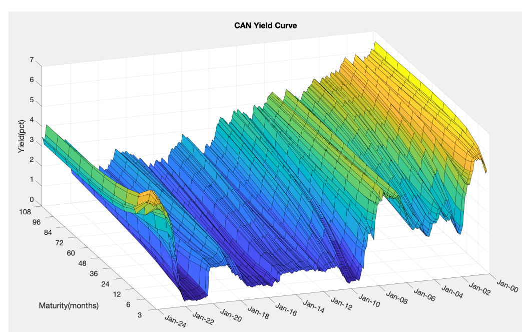
(a) Germany YC



(b) United States: YC



(c) United Kingdom: YC



(d) Canada: YC

Figure 1: Comparison of the Yield Curves over the sample period (2000-2024)

The charts represent the yield curves for Germany, the United States, the United Kingdom, and Canada over the sample period.

The evolution of the yield curve over time in Germany, in the US, in the UK and in Canada has shown changes between January 31st, 2000 and February 29th, 2024. These changes reflect shifts in conditions and market expectations. The yield curves have alternated between upward sloping and inverted shapes during times of economic turmoil. Notably Germany and the UK experienced periods of yields for maturities due to loose monetary policies from their respective central banks (ECB[5] and BoE[6]). The US and Canada also saw yields during times of financial crisis and pandemic outbreaks. Instances of negative yields in countries highlight how monetary policy and economic crises impact short term interest rates. Recently leading up to February 2024 the German yield curve has shown a trend towards steepening with yields seen across all maturity periods. This indicates a shift in policy and a brighter economic outlook resulting in long term interest rates. Similarly the US yield curve is showing signs of steepening due to the Federal Reserves efforts to address inflation and normalize policy. The UK and Canadian yield curves also reflect a trend of steepening with yields increasing across maturity periods as part of measures to manage inflation and recover from economic challenges. This global pattern of yield curve steepening in these four countries suggests a move towards normalization following periods of low interest rates. The German yield curve demonstrates varying levels of volatility among maturity periods over time. Short term yields show fluctuations during uncertain economic conditions while long term yields are relatively stable but still exhibit significant changes, in trends over the past two decades. The trend can be seen in the yield curves of the US, UK and Canada where short term yields react more to pressures causing fluctuations. On the other hand longer term yields remain relatively stable reflecting the economic factors and market participants long range outlook. This consistent pattern of volatility, in these countries underscores how term and long term bonds respond differently to conditions. Examining the representations of yield curves for Germany, the US, UK and Canada from January 2000 to February 2024 offers a look at their evolution over more than two decades. These visualizations capture events like financial crises and periods of monetary policy adjustments showcasing how yield curves adjust based on changing economic landscapes. The recent steepening of these curves in all four countries indicates a shift, towards interest rates reflecting market sentiments and policy changes aimed at managing inflation and supporting rebound. This holistic viewpoint emphasizes the interconnected nature of bond markets. Stresses the importance of considering both structure and dynamics when studying yield curves.

4 Estimating the single country factors

In this chapter, we describe the methodology used to estimate the three key factors (level, slope, and curvature) for each country in the study. These factors were estimated using Ordinary Least

[5]European Central Bank
[6]Bank of England

Squares (OLS) for each time step.

4.1 Factor Model Representation

The yield curve for each country can be represented by a factor model, where the observed yields are modeled as a linear combination of three latent factors: the level, slope, and curvature. Mathematically, the model for the yield $Y_{i,t}(\tau)$ at maturity τ and time t is given by:

$$y_{i,t}(\tau) = l_{i,t} + s_{i,t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + c_{i,t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) + \nu_{i,t}(\tau) \tag{28}$$
$$\text{for} \quad \mathrm{t}=1,\ldots,\mathrm{T} \quad \text{and} \quad \mathrm{i}=1,\ldots,\mathrm{N}$$

For each time step t, the factors $l_{1,t}^{(i)}$, $s_{2,t}^{(i)}$, and $c_{3,t}^{(i)}$ are estimated for each country i using Ordinary Least Squares (OLS). The OLS procedure minimizes the sum of squared residuals, providing the best linear unbiased estimates (BLUE) of the factors. Furthermore, in Appendix B, we develop and analyze the output of the State-Space Model (SSM) used to smooth and estimate the three factors (level, slope and curvature). Then we compare the results of the OLS and of the SSM in terms of Root Mean Squared Errors(RMSE) of the yields with respect to the historical ones.

The OLS regression for each country at each time step is performed by regressing the observed yields $y_{i,t}(\tau)$ on the factor loadings

$$\mathbf{y_{i,t}} = \mathbf{X}\boldsymbol{\beta_{i,t}} + \boldsymbol{\epsilon_{i,t}} \quad \text{for} \quad \mathrm{t}=1,\ldots,\mathrm{T} \quad \text{and} \quad \mathrm{i}=1,\ldots,\mathrm{N} \tag{29}$$

where $\mathbf{y_{i,t}}$ (T x τ) is the vector of observed yields for country i at time t across different maturities, $\mathbf{X}$ (τ x 3) is the matrix of factor loadings with each row corresponding to a maturity τ. $\boldsymbol{\beta}_{i,t} = \begin{pmatrix} l_{i,t} & s_{i,t} & c_{i,t} \end{pmatrix}^\top$ is the vector of factors for country i at time t and $\boldsymbol{\epsilon}_{i,t}$ is the vector of residuals which follows a normal distribution, therefore $\epsilon_{i,t} \sim N(0, \sigma_{i,t}^2)$. The matrix $\mathbf{X}$ is defined as

$$\mathbf{X} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \left(\frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1}\right) \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \left(\frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2}\right) \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_{12}}}{\lambda\tau_{12}} & \left(\frac{1-e^{-\lambda\tau_{12}}}{\lambda\tau_{12}} - e^{-\lambda\tau_{12}}\right) \end{pmatrix} \tag{30}$$

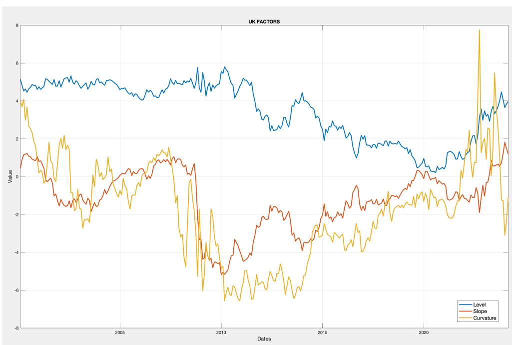
The OLS estimates for the factors are given by

$$\hat{\boldsymbol{\beta}}_{i,t} = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{y}_{i,t} \quad \text{for} \quad \mathrm{t}=1,\ldots,\mathrm{T} \quad \text{and} \quad \mathrm{i}=1,\ldots,\mathrm{N} \tag{31}$$

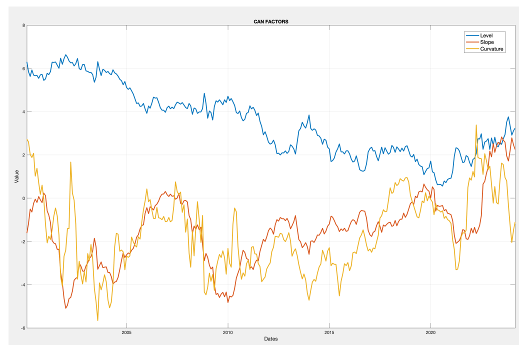
where $\hat{\boldsymbol{\beta}}_{i,t}$ are the estimated factors for country i at time t and it has dimensions (T x 3).

(a) Germany: Factors

(b) United States: Factors



(c) United Kingdom: Factors



(d) Canada: Factors

Figure 2: Single-country estimated factors over the sample period (2000-2024)

4.2 Statistical Meaning of the OLS Regressions

In order to assess the reliability of the estimated factor for each country we look at the R^2values, the $pvalue$ and the $Fstatistics$ calculated from T (290 months from 2000.01 to 2024.02) Least Squares (OLS) regressions for the four countries: Germany, the United States, the United Kingdom and Canada. The R^2 value represents the extent to which variations in the variable (yield curves) are accounted for by the variables (factors). The $pvalue$ helps determine if the results are statistically significant and if we obtain a $pvalue$ less than 0.05 (Significance level), there is a strong evidence that at least one independent variable $\mathbf{X}$ is related to the dependent variable $\mathbf{y_{i,t}}$. The $FStatistic$ is a statistical measure that helps us understand the existence of the regression by applying a restriction the the estimation of $\boldsymbol{\beta}_{it}$.

This test analyzes the amount of fit we would lose if we impose the restriction to our model and it sets

$$H_0 : RB = C \tag{32}$$

$$H_1 : RB \neq C \tag{33}$$

$$FStatistic = \frac{(\text{RSS}_r - \text{RSS}_u)/r}{\text{RSS}_u/(N-K)} \tag{34}$$

where r is the number of constraints equations, therefore R is the matrix r x K that relates the restrictions to the respective coefficient, B represents the vector β thus it has dimensions K x 1, C represents the value of the restriction, thus it has dimensions r x 1,N is the number of observations and K is the number of coefficients. The $FStatistics(r, N-K)$ is a χ^2 distribution and, therefore, we can compute its *pvalue* and, if we obtain an high value for the $FStatistic$ and the relative *pvalue* less than 0.05, we assess that the unrestricted model provides a great fit to the data by rejecting H_0.

Country	**Mean** R^2
Germany	0.9510
United States	0.9385
United Kingdom	0.9444
Canada	0.9567

Table 2: Mean R^2 values from the Regressions of the Single-Country Factors.

When we look at the Squares (OLS) regression models used in Germany, Canada, the United States and the United Kingdom it's clear that these models show a strong ability to explain outcomes as seen in the consistently high R squared values, above 0.93 in all cases. For instance the German model has an R value of 0.9510 indicating that 95.10 % of the variation in the variable is accounted for by the independent variables included in the regression analysis. This high R squared value along with a F statistic of 2204.5 highlights the models strength and suggests a significant relationship between predictors and outcomes. The statistical significance of the model is further supported by a p value of 0.0266 which is below the standard threshold of 0.05 indicating that these results are unlikely to have occurred by chance. Similar patterns can be seen in Canada, the United States and the United Kingdom models where high R^2 values, significant p values and large F statistics collectively confirm these OLS models reliability. The results indicate that the factors utilized in these models are very successful, in clarifying the differences in the outcomes making them dependable resources for comprehending the connections embedded in the data. The uniformity of these measures across country scenarios also underscores the relevance and resilience of the regression method, in environments with the German model showcasing this analytical prowess effectively.

In general the strong R^2 values, in all four countries indicate that the OLS model does a job of explaining the fluctuations in the yield curves showing that it captures a part of the underlying patterns in the data.

Table 3: F-Statistic and p-value from the Regressions of the four Single-Country Factors.

Country	F-statistic	p-value
Germany (DE)	2.2045×10^2	3.7616×10^{-4}
United States (US)	591.7392	0.0082
United Kingdom (UK)	1.7968×10^3	0.0039
Canada (CAN)	991.6275	0.0034

The high values for the F-Statistic, thus, imply that the explained variance is significantly greater than the unexplained variance while the p-value related to the F-Test are below the significance level (0.05) indicating that at least one of the predictors is significantly related to the dependent variables, therefore to the single-country yields. By using OLS estimation to determine the level, slope and curvature factors, for each country at points in time we can gain an insight into how the yield curve behaves. These identified factors are then used as inputs for analysis such as examining country relationships making predictions and assessing financial conditions, under stressful scenarios.

4.3 Comparing the LRM and SSM

In order to assess which model provides the best factor estimation, we compare the Linear regression model and the SSM in terms of Root Mean Square Errors (RMSE). The RMSE is a metric that evaluates the magnitude of the errors predicted by the model (LR or SSM) and the actual data. We used the factors estimated by the model, then we apply the Nelson-Siegel equation in order to obtain the predicted yields and, at the end, we compute the RMSE. The Root Mean Square Error is computed as:

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2} \tag{35}$$

Therefore, this measure gives more weight to large errors and the lower the RMSE the better the model.

Table 4: RMSE Comparison Between LRM and SSM Models Across Countries

Country	LRM	SSM
Germany	2.9656	6.4016
US	5.5009	26.6715
UK	6.4674	28.0688
Canada	5.1750	10.0363

Then, we can state that, according to the RMSE, we should use the OLS instead of the SSM because the values of the OLS are much lower than the values of the SSM.

Table 5: Comparison Between State-Space Model and Two-Step Model Across Different Maturities

(a) Germany

Maturity	State-Space Model		Two-Step	
	Mean (bps)	Std (bps)	Mean (bps)	Std (bps)
3.0000	-5.0391	17.2723	-5.6225	3.2971
6.0000	-2.8709	12.0369	0.7319	1.0643
12.0000	-0.0776	3.6298	7.0617	4.6351
24.0000	0.9472	4.1419	5.5366	3.2346
36.0000	0.7686	3.4392	-0.2117	1.1783
48.0000	0.4166	2.0853	-4.5463	2.7251
60.0000	0.1366	0.9185	-6.2627	3.5793
72.0000	0.0000	0.0000	-5.6100	3.1822
84.0000	-0.0295	0.5518	-3.2628	1.9499
96.0000	-0.0233	0.5139	0.1048	0.5811
108.0000	-0.0000	0.0000	4.0076	2.2942
120.0000	-0.0049	0.7890	8.0733	4.5200

(b) US

Maturity	State-Space Model		Two-Step	
	Mean (bps)	Std (bps)	Mean (bps)	Std (bps)
3.0000	-22.2023	56.5346	-7.1224	5.7336
6.0000	-14.9099	50.5253	3.6294	4.1615
12.0000	-13.9106	39.2577	6.3939	6.8862
24.0000	-9.4439	19.6035	4.0928	3.4106
36.0000	-5.7418	9.8157	-1.9777	3.4910
48.0000	0.1204	1.5813	-3.4969	1.7595
60.0000	1.2520	5.5856	-5.9018	6.0696
72.0000	2.2089	6.4287	-4.6851	3.4618
84.0000	0.9717	6.7641	-2.3873	6.4496
96.0000	0.2177	2.0227	3.0795	5.3334
108.0000	-2.1424	5.2441	9.1088	5.9988
120.0000	-22.1290	24.7638	-0.7332	9.1512

(c) UK

Maturity	State-Space Model		Two-Step	
	Mean (bps)	Std (bps)	Mean (bps)	Std (bps)
3.0000	16.1887	66.1449	-3.9945	11.7187
6.0000	15.7226	53.7508	1.8957	9.4572
12.0000	9.4510	40.5155	3.5939	9.9638
24.0000	2.9246	20.7000	2.2391	4.2140
36.0000	1.0976	9.2547	-0.0136	3.5722
48.0000	0.3846	3.1045	-2.0590	4.9825
60.0000	-0.0000	0.0000	-3.1296	5.3763
72.0000	-0.1833	1.2084	-3.0513	4.5537
84.0000	-0.1777	1.1004	-1.9763	2.7364
96.0000	-0.0000	0.0000	-0.1711	1.1576
108.0000	0.3135	1.8675	2.0912	3.6147
120.0000	0.7161	4.3100	4.5757	7.0007

(d) Canada

Maturity	State-Space Model		Two-Step	
	Mean (bps)	Std (bps)	Mean (bps)	Std (bps)
3.0000	-7.7114	28.7261	-4.8468	7.6820
6.0000	-3.7581	14.3752	0.7902	3.0025
12.0000	1.3596	7.4615	6.7306	11.0275
24.0000	0.4416	6.2140	2.7989	3.3107
36.0000	0.9601	5.4575	-0.5373	3.4075
48.0000	0.5902	4.1149	-3.2949	4.2248
60.0000	0.1862	1.9916	-4.3332	4.1122
72.0000	-0.0000	0.0000	-3.7580	3.8030
84.0000	-0.0395	0.8072	-2.1075	3.0839
96.0000	0.0000	0.0000	0.1532	1.8257
108.0000	0.1514	2.1854	2.7683	2.9265
120.0000	0.5049	5.1467	5.6365	6.2079

In order to understand and verify the huge differences in RSME, we analyze the mean and the standard deviations of the estimates for each maturity. In Germany specifically the Two step method offers estimates with fewer errors on average and notably lower variations compared to the State Space Model (SSM).We can see that the standard deviation of the SSM for short maturities (3 and 6 month) is much bigger than the OLS one. Regarding the mean, the State-Space Model provides values as good as the OLS estimation exception for the 6-month maturity (SSM) and 12 and 24 maturities (OLS). When the Two Step approach is used in the United States context a similar pattern appears, OLS displays average errors and standard deviations for shorter timeframes compared to SSM. On the hand, SSM shows greater variability and tends to overestimate during these periods. Especially, we can state that both for short and for long maturities (3-month and 120-month) the SSM provides greater mean in absolute values than the OLS. In the UK yield curve, the Two-Step method remains a choice delivering consistent estimates across various maturity levels. On the side SSM tends to show greater errors and deviations in mean especially for shorter maturities like in the German and in the US scenario. The findings from Canada also support the effectiveness of the Two step strategy, the absolute value of the mean is greater for almost all the maturities (especially shorter ones) and the standard deviation of the SSM is much bigger than the OLS one in the 3-month maturity.Concerning OLS, we can state that the standard deviation for 12-month and 120-month maturity are quite big with respect to the SSM ones, therefore the means are much greater than the SSM ones for those maturities.

In conclusion, we can state that the OLS framework works very well for short maturities and it provides lower RMSE values than SSM in the two decades sample period. Therefore, in order to study the macroeconomic variables that drive the unexplained Yield Curve by the global factors, we use the OLS instead of the SSM.

5 Estimating the Global Factors

In this section, we investigate the commonality among the country-specific factors by employing Principal Component Analysis (PCA). The objective is to identify and extract global factors that capture the common movements across the country-specific factors. These global factors represent underlying dynamics that influence all the countries under study, enabling a more comprehensive analysis of the global interest rate environment.

5.1 Co-integrated Yields

Now we have to handle with the non-stationary process of the four single-country yields which affects the reliability of the Principal Component Analysis due to the fact that the variance-covariance matrix of the country-factors is not defined as T increases. Then, we analyse the single-country yields in order to check the presence of co-integrated relationships among the yield indicating a long-run equilibrium among the yields at that specific maturity. We perform the $Johansen-cointegration$ test (proposed by Johansen (1995)) which is suitable for testing and analysing the four different yield time series and for determining the existence of co-integrated vectors among them. Co-integration refers to the presence of a stationary linear combination among two or more non-stationary time series. This suggests that while the individual series may drift over time, their relationship remains stable in the long term. The Johansen test is based on the following *Vector Error Correction Model (VECM)* representation of a VAR model

$$\Delta Y_t = \Pi Y_{t-1} + \sum_{i=1}^{p-1} \Upsilon_i \Delta Y_{t-i} + \epsilon_t \quad \text{for } t = 1, \dots, T \tag{36}$$

where Y_t is a vector of the n non-stationary variables (e.g., bond yields) ΔY_t represents the first differences of the variables, making them stationary, Π is an $n \times n$ matrix that contains information about the long-run relationships among the variables. Υ_i represents the short-run dynamics and ϵ_t is a white noise error term. The rank of the matrix Π determines the number of co-integrating relationships if rank$(\Pi) = 0$, no co-integrating relationship exists, and the system is non-stationary and if rank$(\Pi) = r$, there are r co-integrating vectors, indicating that there are r stationary linear combinations of the variables in Y_t. The matrix Π is not directly observable, thus it must be estimated. It can be estimated through covariance matrices and this method consists in two auxiliary regressions in order to isolate Π and to remove the short run dynamics. We regress Δy_t on lagged differences (R_{0t}) and we regress y_{t-1} on the same regressors to obtain residuals (R_{1t}), then we substitute these errors into the initial equation Δy_t and we obtain $R_0 = \Pi\ R_1 + E$ where R_0 and R_1 have dimensions nxT for the respective error terms and E is nxT for the ϵ_t. Then we compute the covariance matrices $S_{00} = \frac{1}{T}\sum_{t=1}^{T} R_{0t}R'_{0t}$, $S_{11} = \frac{1}{T}\sum_{t=1}^{T} R_{1t}R'_{1t}$ and $S_{01} = \frac{1}{T}\sum_{t=1}^{T} R_{0t}R'_{1t}$.

In conclusion we obtain

$$\hat{\Pi} = S_{01}S_{11}^{-1} \tag{37}$$

The Johansen test provides two main test statistics to determine the rank r of Π, which indicates the number of co-integrating vectors: the *trace statistic* and the *maximum eigenvalue statistic*. The trace test statistic is given by

$$\text{Trace Statistic} = -T \sum_{i=r+1}^{n} \ln(1 - \psi_i) \tag{38}$$

where T is the sample size, ψ_i are the eigenvalues of the matrix Π and $\psi_i > \psi_j \quad \text{for} \quad i < j$. This test examines the null hypothesis that the number of co-integrating vectors is r against the alternative that there are more than r co-integrating vectors. The maximum eigenvalue test statistic is given by

$$\text{Max-Eigen Statistic} = -T \ln(1 - \psi_{r+1}) \tag{39}$$

This test examines the null hypothesis that the number of co-integrating vectors is r against the alternative of $r + 1$ co-integrating vectors.

5.1.1 Interpretation of Results

Both the trace statistic and the maximum eigenvalue statistic are compared against critical values at significance level (e.g., 5%). If the test statistic exceeds the critical value, the null hypothesis is rejected, indicating the presence of one or more co-integrating vectors. The number of co-integrating vectors r is chosen based on the point at which the null hypothesis of no co-integration is rejected, but the null hypothesis of $r + 1$ co-integrating vectors is not rejected. We test the four single-country yields at very short maturities (3-month and 6-month), at medium maturities (4-year and 5-year) and at long maturities (9-year and 10-year). The tables below are represented as follows, the columns $r_i(\quad \text{for} \quad i = 0, \ldots, 3)$ are the possible co-integrated vectors related to the rank of the

Π matrix, thus, if the matrix is not a full rank matrix having reduced rank r (where $0 < r < n$), it means that there are r co-integrated relationships; the Model $'H1'$ indicates an error-correction term[7] defined as $A(B'y_{t-1} + c_0) + c_1$ where A is the matrix of adjustment speeds, B is the co-integration matrix (either of them have dimensions nxr), c_0 (rx1) represents the constant terms in the co-integrated relations and c_1 (nx1) is the deterministic linear trend in Y_t, the Lags column represents the number of lagged differences q and the last column is the significance level. Then, h represents the test rejection decision where $true$ stands for a rejection of the null hypothesis of co-integration rank k and $false$ indicates failure to reject H_0. Last, the p-value and the T-Test are set with a significance level of 0.05, while c-Value represents the right-tale probability determined by the significance level.

Table 6: Estimation of the co-integrated vectors for 3-Month and 6-Month Yields

	r0	r1	r2	r3	Model	Lags	Test	Alpha
				3-Month Yields				
h								
	true	true	true	true	{'H1'}	0	{'trace'}	0.05
	true	true	false	true	{'H1'}	0	{'maxeig'}	0.05
***p*-value**								
	0.001	0.001	0.04196	0.0082911	{'H1'}	0	{'trace'}	0.05
	0.001	0.00649	0.33166	0.0082911	{'H1'}	0	{'maxeig'}	0.05
***T*-Stat**								
	101.99	43.215	16.005	7.0602	{'H1'}	0	{'trace'}	0.05
	58.78	27.21	8.9444	7.0602	{'H1'}	0	{'maxeig'}	0.05
***c*-Value**								
	47.856	29.798	15.495	3.8415	{'H1'}	0	{'trace'}	0.05
	27.586	21.132	14.264	3.8415	{'H1'}	0	{'maxeig'}	0.05
				6-Month Yields				
h								
	true	true	false	true	{'H1'}	0	{'trace'}	0.05
	true	true	false	true	{'H1'}	0	{'maxeig'}	0.05
***p*-value**								
	0.001	0.00395	0.10789	0.025831	{'H1'}	0	{'trace'}	0.05
	0.001	0.01028	0.40656	0.025831	{'H1'}	0	{'maxeig'}	0.05
***T*-Stat**								
	110.3	39.007	13.209	4.9716	{'H1'}	0	{'trace'}	0.05
	71.295	25.797	8.2378	4.9716	{'H1'}	0	{'maxeig'}	0.05
***c*-Value**								
	47.856	29.798	15.495	3.8415	{'H1'}	0	{'trace'}	0.05
	27.586	21.132	14.264	3.8415	{'H1'}	0	{'maxeig'}	0.05

For what concerns the short maturities, for the 3-month yields the trace test indicates the existence of 4 co-integrating relationships, as the null hypotheses for all ranks ($r0$, $r1$, $r2$, $r3$) were

[7] Linear function of the responses in levels used to stabilise the system

rejected while the max-eigenvalue test shows evidence of 2 co-integrating relationships, as the null hypothesis was rejected for $r0$ and $r1$, but not for higher ranks ($r2$ and $r3$). Therefore, there are at least 2 strong co-integrating relationships, with the trace test suggesting the possibility of up to 4. However, given the divergence between the trace and max-eigenvalue results, the co-integration is likely more robust up to 2 relationships. Regarding the 6-month yields, the trace test suggests 2 co-integrating relationships, rejecting the null hypothesis for $r0$ and $r1$, but not for higher ranks while the max-eigenvalue test also supports 2 co-integrating relationships, with rejection of the null for $r0$ and $r1$, but failure to reject beyond that, thus oth tests confirm the presence of 2 co-integrating relationships among the 6-month yields. In conclusion, the 3-month and 6-month yields (short-term) show evidence of 2 co-integrating relationships, with the 3-month yields possibly showing stronger or additional co-integration (up to 4 relationships).

Table 7: Estimation of the co-integrated vectors for 4-Year and 5-Year Yields

	r0	**r1**	**r2**	**r3**	**Model**	**Lags**	**Test**	**Alpha**
				4-Year Yields				
h								
	true	true	true	true	{'H1'}	0	{'trace'}	0.05
	true	false	false	true	{'H1'}	0	{'maxeig'}	0.05
***p*-value**								
	0.001	0.011964	0.025828	0.024668	{'H1'}	0	{'trace'}	0.05
	0.0012271	0.14743	0.099107	0.024668	{'H1'}	0	{'maxeig'}	0.05
***T*-Stat**								
	74.02	34.931	17.373	5.0497	{'H1'}	0	{'trace'}	0.05
	39.089	17.558	12.324	5.0497	{'H1'}	0	{'maxeig'}	0.05
***c*-Value**								
	47.856	29.798	15.495	3.8415	{'H1'}	0	{'trace'}	0.05
	27.586	21.132	14.264	3.8415	{'H1'}	0	{'maxeig'}	0.05
				5-Year Yields				
h								
	true	true	true	true	{'H1'}	0	{'trace'}	0.05
	true	false	false	true	{'H1'}	0	{'maxeig'}	0.05
***p*-value**								
	0.001	0.030983	0.045501	0.029393	{'H1'}	0	{'trace'}	0.05
	0.0041839	0.25455	0.15333	0.029393	{'H1'}	0	{'maxeig'}	0.05
***T*-Stat**								
	67.288	31.571	15.769	4.7475	{'H1'}	0	{'trace'}	0.05
	35.717	15.802	11.021	4.7475	{'H1'}	0	{'maxeig'}	0.05
***c*-Value**								
	47.856	29.798	15.495	3.8415	{'H1'}	0	{'trace'}	0.05
	27.586	21.132	14.264	3.8415	{'H1'}	0	{'maxeig'}	0.05

For the medium-maturity yields, for the 4-year ones, the trace test indicates 4 co-integrating relationships, rejecting the null hypotheses for all ranks ($r0$, $r1$, $r2$, $r3$) and the max-eigenvalue test suggests 1 co-integrating relationship, rejecting the null hypothesis for $r0$ but failing to reject it for higher ranks therefore, there is evidence of 1 co-integrating relationship, although the trace test suggests up to 4 while the max-eigenvalue test provides a more conservative estimate. For the 5-year yields, he trace test indicates 4 co-integrating relationships, rejecting the null hypotheses for all ranks. The max-eigenvalue test supports 1 co-integrating relationship, rejecting the null only for r0 and failing to reject for higher ranks, thus, similar to the 4-year yields, there is evidence of 1 co-integrating relationship, with the possibility of additional relationships suggested by the trace test (up to 4). In conclusion, the 4-year and 5-year yields show consistent evidence of 1 strong co-integrating relationship, with the possibility of more (up to 4) as suggested by the trace test.

Table 8: Estimation of the co-integrated vectors for 9-year and 10-year Yields

	r0	**r1**	**r2**	**r3**	**Model**	**Lags**	**Test**	**Alpha**
9-Year Yields								
h								
	true	false	false	false	{'H1'}	0	{'trace'}	0.05
	true	false	false	false	{'H1'}	0	{'maxeig'}	0.05
***p*-value**								
	0.001	0.32327	0.18062	0.067563	{'H1'}	0	{'trace'}	0.05
	0.001	0.70072	0.41085	0.067563	{'H1'}	0	{'maxeig'}	0.05
***T*-Stat**								
	71.164	22.095	11.54	3.3425	{'H1'}	0	{'trace'}	0.05
	49.069	10.555	8.1974	3.3425	{'H1'}	0	{'maxeig'}	0.05
***c*-Value**								
	47.856	29.798	15.495	3.8415	{'H1'}	0	{'trace'}	0.05
	27.586	21.132	14.264	3.8415	{'H1'}	0	{'maxeig'}	0.05
10-Year Yields								
h								
	true	false	false	false	{'H1'}	0	{'trace'}	0.05
	true	false	false	false	{'H1'}	0	{'maxeig'}	0.05
***p*-value**								
	0.001	0.43072	0.36827	0.071163	{'H1'}	0	{'trace'}	0.05
	0.001	0.66743	0.62027	0.071163	{'H1'}	0	{'maxeig'}	0.05
***T*-Stat**								
	63.137	20.425	9.4787	3.2567	{'H1'}	0	{'trace'}	0.05
	42.712	10.947	6.2219	3.2567	{'H1'}	0	{'maxeig'}	0.05
***c*-Value**								
	47.856	29.798	15.495	3.8415	{'H1'}	0	{'trace'}	0.05
	27.586	21.132	14.264	3.8415	{'H1'}	0	{'maxeig'}	0.05

Regarding the long-term maturity yields, the Johansen co-integration test indicates the presence of one co integrating relationship in both cases. For the 9-year yields, the trace test rejects the null hypothesis for $r0$ but fails to reject it for higher ranks (r1, r2, r3), suggesting one co-integrating relationship. Similarly, the max-eigenvalue test supports this conclusion by rejecting the null only for $r0$, confirming a single long-term equilibrium relationship. The results for the 10-year yields are consistent with those of the 9-year yields. Both the trace test and max-eigenvalue test indicate one co-integrating relationship, as they reject the null hypothesis for $r0$ but not for higher ranks. In summary, both the 9-year and 10-year yields exhibit one strong long-term equilibrium relationship, implying a level of co-movement between variables over these maturities. This suggests that while short-term fluctuations may occur, the variables tend to revert to a common long-term trend, offering insight into their long-term predictability and stability.

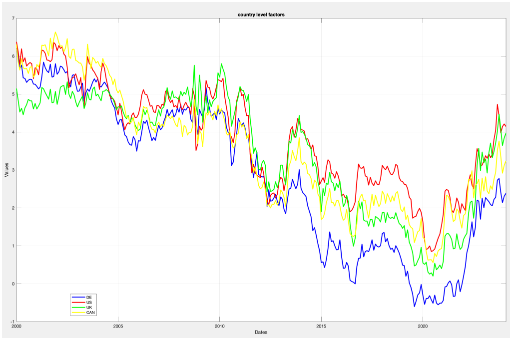
In conclusion, the co-integration tests reveal distinct patterns across different maturities. For short-term maturities (3-month and 6-month yields), there is evidence of up to 2 strong co-integrating relationships, with the possibility of additional relationships suggested by the trace test. In the medium-term (4-year and 5-year yields), both tests indicate one robust co-integrating relationship, though the trace test hints at the possibility of more. For long-term maturities (9-year and 10-year yields), both the trace and max-eigenvalue tests consistently identify a single co-integrating relationship. Overall, the results suggest that while the number of co-integrating relationships may vary by maturity, there is a clear long-term equilibrium relationship across all maturity ranges.

5.2 Factors' Commonality

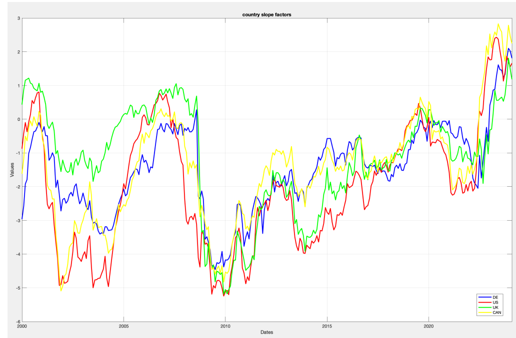
Now we show the behavior of the single factors extracted in order to underline and advocate the common trend among them (Figure 3). We prove that the yields, even if the time-series are not stationary, are characterised by long-term equilibrium and by several stationary components (the co-integrated vectors), therefore we are able to perform the PCA avoiding the no-definition issue of the variance-covariance matrix as T increases and, due to the common trend among the factors, through the Principal Component Analysis, we extract the common factors that explain the majority of the single-country variance. We can easily understand and see that the movement of the country factors is similar across these two decades. By plotting the time series of the three factors, we can state that they are driven by common components just by looking at the trends.

5.3 Application of PCA to Extract Global Factors

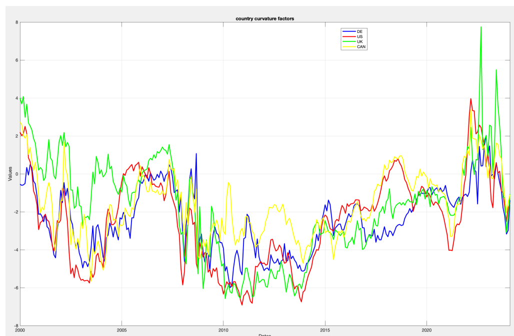
In this study, we applied PCA to the country-specific factors to extract global factors that represent common trends across countries. The country-specific factors were collected into a data matrix $\mathbf{F}$ of size $T \times N$, where T is the number of observations (290) and N (4) is the number of countries. Each column of $\mathbf{F}$ represents a time series of factors for one country. This PCA analysis returns the principal components, the explained variance, and the loadings (eigenvectors): The first few principal components were selected based on the amount of variance they explain (typically those that explain a significant portion of the total variance). These selected components were interpreted



(a) Level Factors



(b) Slope Factors



(c) Curvature Factors

Figure 3: Comparison of the country-factors over the sample period (2000-2024)

as global factors that drive the common dynamics in the country-specific factors. The global factors were analyzed to understand the underlying global influences on the yield curves. The loadings were examined to assess how each country's factors contributed to the global factors. The PCA analysis showed that the main components primarily capture most of the variations, in the factors of each country. These components were seen as factors suggesting a similarity among the countries in the dataset. By examining the loadings we gained insights into how each country contributes to these factors shedding light on which countries have an impact on shaping shared patterns.

The findings indicate that universal factors have an impact, on how yield curves behave in countries emphasizing the importance of taking these common influences into account when analyzing and modeling financial data. The Principal Component Analysis (PCA) results for the level factor indicate that the first principal component (PC1) explains the vast majority of the variance in the data, the eigenvalue corresponding to the first principal component is 10.372, which is significantly larger than the eigenvalues of the subsequent components (0.27745, 0.12942, and 0.045082). This suggests that the first principal component captures most of the variance in the data. The first

EIGENVALUES	VARIANCE	CUMULATIVE VARIANCE
10.372	95.825	95.825
0.27745	2.5633	98.388
0.12942	1.1957	99.583
0.045082	0.41651	100

Table 9: Principal Component Analysis Results for the Level Factor.

principal component alone explains 95.825% of the total variance. The second component explains an additional 2.5633%, the third component explains 1.1957%, and the fourth component explains only 0.41651%. The cumulative variance shows that the first two principal components together explain 98.388% of the total variance. The inclusion of the third component increases this to 99.583%, and with the fourth component, the explained variance reaches 100%. The PCA results suggest that the level factor is predominantly driven by a single underlying component, as the first principal component alone accounts for nearly all the variance (95.825%). The diminishing contribution of the subsequent components indicates that they capture only minor variations, which may not be significant for understanding the primary dynamics of the level factor. Given this distribution of variance, it would be reasonable to focus on the first principal component in any further analysis or modeling, as it encapsulates the most critical information regarding the level factor. The high cumulative variance explained by the first two components (98.388%) further supports the idea that these components can effectively summarize the underlying structure of the data.

In practical terms, this finding implies that the level factor across the dataset is highly homogeneous and influenced by a dominant global trend, with only minor deviations captured by the remaining components.

EIGENVALUES	VARIANCE	CUMULATIVE VARIANCE
3.358	83.95	83.95
0.39937	9.9841	93.934
0.19832	4.9579	98.892
0.044332	1.1083	100

Table 10: Principal Component Analysis Results for the Slope Factor.

The Principal Component Analysis (PCA) results for the slope factor show that the first principal component (PC1) explains a significant majority of the variance in the data, the eigenvalue corresponding to the first principal component is 3.358, which is substantially larger than the eigenvalues of the subsequent components (0.39937, 0.19832, and 0.044332). This indicates that the first principal component captures most of the variance in the data. The first principal component alone explains 83.95% of the total variance. The second component explains an additional 9.9841%, the third component explains 4.9579%, and the fourth component explains only 1.1083%. The cumulative variance shows that the first two principal components together explain 93.934% of the total variance. The inclusion of the third component increases this to 98.892%, and with the fourth component, the explained variance reaches 100%. The results, from the PCA analysis show that the first principal component plays a role in capturing the variations in the slope factor explaining 83.95% of the variability. This indicates that there is a trend underlying the slope factor, which is well represented by this component. The second and third components explain variances of 9.9841% and 4.9579% respectively suggesting that while the first component captures most of the trend there are patterns in the slope factor worth exploring. Although these secondary patterns are less prominent they contribute to our understanding of how the slope factor behaves. With a variance explanation of 93.934% by the two components combined they provide a comprehensive summary of the main behavior exhibited by the slope factor. Including more components only marginally increases power indicating that a few key influences primarily drive trends in this factor.

In terms these findings suggest that a strong global trend significantly influences the slope factor with some variations captured by just a few principal components. Therefore focusing on modeling and forecasting efforts based on on the principal component while considering insights from both second and third components, for deeper analysis would be beneficial.

EIGENVALUES	VARIANCE	CUMULATIVE VARIANCE
3.1968	79.921	79.921
0.41114	10.279	90.2
0.2305	5.7626	95.962
0.1615	4.0376	100

Table 11: Principal Component Analysis Results for the Curvature Factor.

The Principal Component Analysis (PCA) results for the curvature factor show that the first principal component (PC1) explains a significant portion of the variance in the data, the eigenvalue corresponding to the first principal component is 3.1968, which is considerably larger than the eigenvalues of the subsequent components (0.41114, 0.2305, and 0.1615). This indicates that the first principal component captures most of the variance in the data. The first principal component alone explains 79.921% of the total variance. The second component explains an additional 10.279%, the third component explains 5.7626%, and the fourth component explains 4.0376%. The cumulative variance shows that the first two principal components together explain 90.2% of the total variance. The inclusion of the third component increases this to 95.962%, and with the fourth component, the explained variance reaches 100%. The PCA results for the curvature factor suggest that the first principal component is the most influential in capturing the variability in the curvature factor, explaining 79.921% of the variance. This indicates that there is a dominant underlying trend in the curvature factor that is well-represented by the first principal component. The additional variance explained by the second and third components (10.279% and 5.7626%, respectively) suggests that while the first component captures most of the trend, there are still important secondary patterns in the curvature factor that contribute to the overall variability. Given that the first two components together explain 90.2% of the variance, they can be considered sufficient for summarizing the main behavior of the curvature factor. The inclusion of further components adds marginal explanatory power, indicating that the curvature factor is primarily driven by a few key influences.

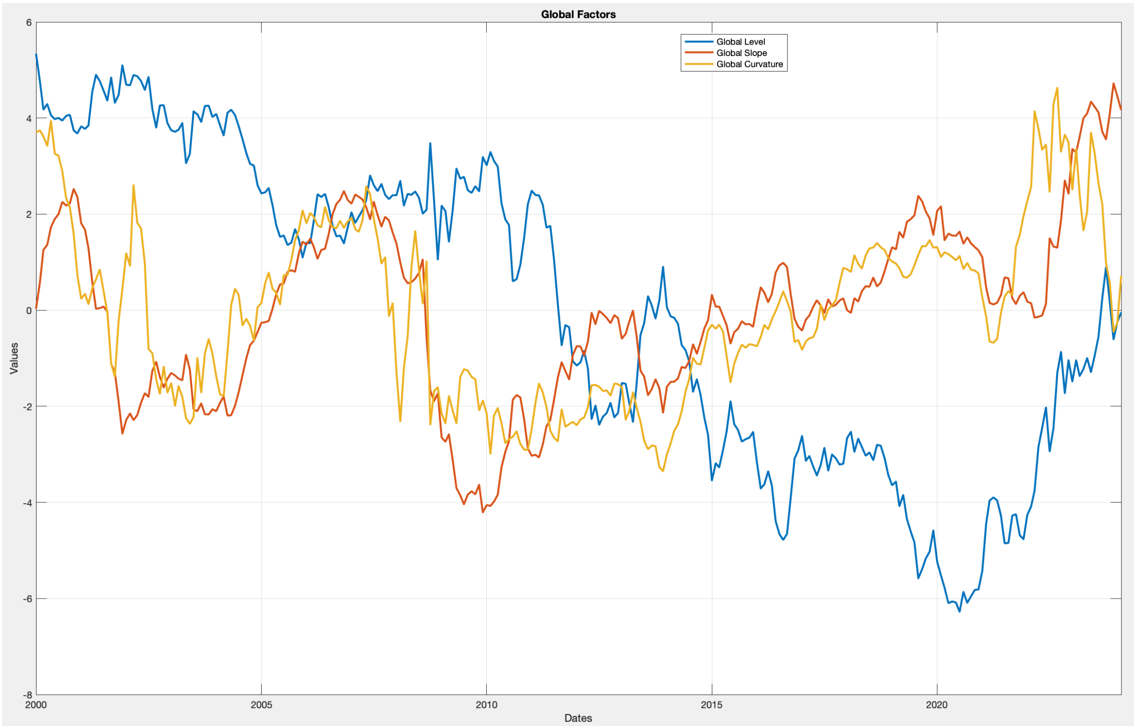
In practical terms, this finding implies that the curvature factor is influenced by a strong global trend, with some additional variation that can be captured by a small number of principal components. This suggests that modeling and forecasting efforts for the curvature factor should focus primarily on the first principal component.

5.4 Global Factors Chart

By applying PCA to the country-specific factors, we were able to extract global factors that capture the common dynamics across countries. These global factors provide valuable insights into the underlying drivers of yield curves in the international context, enabling a more nuanced understanding of global financial markets.

This chart represent the path followed by the three global factors during the last two decades and we can see some similarities with the single-country factors. The global level behave like the level factor of the single countries, starting approximately at 5% (slightly lower than the single-coutry values) and showing a downward path over the years. The main difference with the country-level factors is the amount of negative values of the global factor. Especially, we can state that after the 2011 crisis, the global level of interest rate has always been negative with one peak only(slightly positive) just before the 2015.The lowest global level value is around the 2020, thus during the Covid-19 pandemic and, after that, the series is upward-sloping reaching positive values. The global slope factor follows the same path as the country slope factors but it starts greater than zero while the single country factors start lower than zero. Than, before the 2005, the downward series reaches lower values for the single country slopes than for the global one (which reaches approximately -2%). Further, between the 2005 and 2008 we can see positive values both for global slope and single-country slopes but, after the 2008 crisis, the slope factor reaches the lowest value, approximately -4% for the global one and -5% for the single country ones.After that, the series is upward-sloping and it touches its peak in the recent years. The only exception, both for global and single-country slope, is around the 2022, thus this negative peak could be due to the Ukrainian-Russian war. Regarding the last factor, the global curvature, we can state that the core difference concerns the lower negative values of the global factor than the single-country curvatures. The

Figure 4: Global Factors



former reaches approximately -3% before 2015 while the latter touches huge negative values more than -6% both in around 2010 and before 2015. We can see that both in the country framework and in the global one, the slope of the series is positive until before 2020 (Pandemic Crisis) reaching the local minimum around 2021-2022 (-4% for US curvature and slightly negative for the global one). Another important insight is the reverse pattern followed by the global curvature and the global slope during the last four years; while the former increase, the latter decreases and vice-versa. This relationship is more noticeable in the global framework than in the single-country one and it could be interpreted as the global financial markets experienced significant shifts, reflected in the evolving shape of the yield curve. During this period, a noticeable trend was the decreasing curvature and increasing slope of the yield curve, which offers valuable insights into investor sentiment, economic expectations, and central bank policies.

In 2022 the world economy was on its way to recovering from the pandemics aftermath. There was an uptick in economic growth with businesses reopening and consumers spending more freely. However this recovery also brought about inflationary pressures due to supply chain disruptions lack of available labor and increased consumer demand. To address this issue the central banks (U.S Federal Reserve and ECB) started moving away from very lenient monetary policies towards measures aimed at controlling inflation. As a result of these adjustments short term interest rates began to climb. In 2023 central banks kept raising interest rates throughout the year to address worries about rising inflation.. These rate increases also raised worries about a potential economic slowdown or even a full fledged recession. The rise in borrowing expenses started affecting individuals and businesses in sectors such as estate that are sensitive to fluctuations, in interest rates. In times the yield curve sometimes showed an inverted shape that raised concerns about a possible economic downturn affecting short, to medium term investments. In 2024 as inflation started to level off and economic growth slowed down with the increase in interest rates the outlook, therefore investors were uncertain about the tightening policies and the possibility of a recession.

The decreasing curvature of the yield curve during 2022-2024 suggests that the difference between medium-term yields and short/long-term yields was shrinking. This flattening of the middle part of the yield curve created by these global factors could be attributed to several factors like investor uncertainty which leads to the balanced demand across bond maturities where investors may have equally favored bonds across the sample maturities which brings to reduce the difference between medium-term and short/long term bond yields. Simultaneously, the increasing of the global slope factor that long-term interest rates were rising faster than short-term rates. This steepening could reflect market expectations of future economic developments like the expectations of future rate cuts.

Given the economic uncertainties prevailing in the global environment short term bonds are being preferred due to their liquidity and safety features. Investors found long term bonds appealing because they anticipated that central banks might need to decrease interest rates in the future to boost activity and secure higher yields at present. Yields on medium term bonds showed variation than those on short term and long term bonds which indicates a relatively steady medium term outlook despite some uncertainty. During this time frame the yield curve functioned as an indicator of short term risk awareness and term strategic planning in a scenario characterized by worries about inflation trends and central bank actions alongside varied economic outlooks.

5.5 Global Factors Dynamics

We assume that the global factors follow an AR process during the sample period and, in this subsection, we analyse this assumption by performing the in-sample forecast of the global factors. The primary goal of the in-sample forecast is to evaluate the predictive power of global betas, which are extracted through Principal Component Analysis (PCA). These global betas represent the common factors influencing the yield curves of different countries. Understanding the dynamics of these global factors is crucial for forecasting future yield curves and for evaluating how these factors interact with local economic conditions.

Two in-sample forecasting models are considered: an $AR(1)$ Model where all three global factors follow a first-order autoregressive process $AR(1)$ and an $AR(2)$ Model where the first global factor (Level) follows a second-order autoregressive process $AR(2)$, while the other two factors follow an $AR(1)$ process. The objective is to estimate the autoregressive coefficients for each model, perform in-sample forecasts of the global betas, and subsequently calculate the Mean Squared Forecast Error (MSFE).

5.5.1 Estimation and Forecasting Process

The first step in the in-sample forecasting process is to estimate the autoregressive coefficients (ϕ, ϕ_1, and ϕ_2) for each global factor using the log-likelihood approach. Once the coefficients are estimated, in-sample forecasts are generated for each time period t within the sample. The matrix G_t has dimensions T x 3 and it represents the Global Level, Slope and Curvature extracted by the PCA. The forecasted values $\hat{G}_t$ are compared to the actual observed values G_t, and the difference between them is squared and averaged to obtain the Mean Squared Forecast Error (MSFE):

$$\text{MSFE} = \frac{1}{T}\sum_{t=1}^{T}(G_t - \hat{G}_t)^2, \tag{40}$$

where T is the total number of observations.

For the AR(1) model, the MSFE of the In-Sample Forecast is 0.1142 while fot the AR(2) model where the first global factor is modeled with two lags, while the other two factors remain as AR(1) processes, the MSFE is 0.1133 which is smaller than the first model's one. The in-sample forecast performance is evaluated by comparing the MSFE of the AR(1) model to that of the AR(2) model. A lower MSFE indicates a better fit of the model to the in-sample data, providing insight into which model more accurately captures the dynamics of the global factors.

6 Impact of the global factors to the single country factors

In this section we analyze the process of the three Global Factors, the impact of each factor on the single-country one and we estimate the process for the single country error terms. The equations for the decomposition of each single-country factor help us understand the persistence of the global factors and the relationship between the country factors and the global ones. Furthermore, these equations make possible the forecasting of the single country factors and, therefore, of the single country yields.

6.1 Influence on Single-Country Factors

Regarding the global factors, we state that the level factor (which represents the overall level of the global interest rates) follows and $AR(2)$ process, therefore it depends on its two previous observations. For what concerns the other two factors, the global slope and the global curvature, they follow an $AR(1)$. The country-factors are expressed in function of the global ones and, in order to estimate them, we run a OLS regression where the dependent variables are the single-country factor and the independent ones are the global factor. The most important part is the last parameter of the right hand side of the equations which represents the idiosyncratic factor that follows an $AR(1)$ process. We estimate these country-specific factors by regressing the residuals of the previous OLS, thus these factors represent the unexplained part of the single country factors.

Table 12: Estimates of the Global Yield Curve Model Parameters

Estimated Parameters
Global Level Factor
$L_t = 1.0669 L_{t-1} - 0.0801 L_{t-2} + U_l$
(0.1747) (0.1747)
Global Slope Factor
$S_t = 0.99224 S_{t-1} + U_s$
(0.0874)
Global Curvature Factor
$C_t = 0.95055 S_{t-1} + U_s$
(0.1944)
Country Level Factors
$l_{DE,t} = 2.8522 + 0.61205 L_t + \epsilon_{l,DE,t}$
(0.015) (0.0047)
$l_{US,t} = 3.8694 + 0.42234 L_t + \epsilon_{l,US,t}$
(0.017) (0.005)
$l_{UK,t} = 3.4622 + 0.4616 L_t + \epsilon_{l,UK,t}$
(0.024) (0.0075)
$l_{CAN,t} = 3.5112 + 0.4836 L_t + \epsilon_{l,CAN,t}$
(0.020) (0.006)
Country Slope Factors
$s_{DE,t} = -1.4781 + 0.72105 S_t + \epsilon_{s,DE,t}$
(0.028) (0.017)
$s_{US,t} = -2.035 + 1.0782 S_t + \epsilon_{s,US,t}$
(0.034) (0.021)
$s_{UK,t} = -1.203 + 0.8335 S_t + \epsilon_{s,UK,t}$
(0.047) (0.029)
$s_{CAN,t} = -1.44 + 0.9093 S_t + \epsilon_{s,CAN,t}$
(0.035) (0.021)
Country Curvature Factors
$c_{DE,t} = -2.4138 + 1.019 S_t + \epsilon_{c,DE,t}$
(0.043) (0.029)
$c_{US,t} = -2.4424 + 1.4957 S_t + \epsilon_{c,US,t}$
(0.049) (0.033)
$c_{UK,t} = -1.7767 + 1.4881 S_t + \epsilon_{c,UK,t}$
(0.083) (0.056)
$c_{CAN,t} = -2.4424 + 1.4957 S_t + \epsilon_{c,CAN,t}$
(0.049) (0.033)

Table 13: Estimates of the AR(1) Process for the Error Terms for Level, Slope, and Curvature

Country
Germany (DE)
$\epsilon_{l,DE,t} = 0.90742 \epsilon_{l,DE,t-1} + 0.2365\, u_{l,DE,t}$
(0.053469) (0.009837)
$\epsilon_{s,DE,t} = 0.91333 \epsilon_{s,DE,t-1} + 0.44765\, u_{s,DE,t}$
(0.053818) (0.01862)
$\epsilon_{c,DE,t} = 0.85151 \epsilon_{c,DE,t-1} + 0.63047\, u_{c,DE,t}$
(0.050176) (0.026224)
United States (US)
$\epsilon_{l,US,t} = 0.80728 \epsilon_{l,US,t-1} + 0.2467\, u_{l,US,t}$
(0.047569) (0.010261)
$\epsilon_{s,US,t} = 0.91099 \epsilon_{s,US,t-1} + 0.53451\, u_{s,US,t}$
(0.05368) (0.022233)
$\epsilon_{c,US,t} = 0.84464 \epsilon_{c,US,t-1} + 0.71877\, u_{c,US,t}$
(0.049771) (0.029897)
United Kingdom (UK)
$\epsilon_{l,UK,t} = 0.94106 \epsilon_{l,UK,t-1} + 0.38549\, u_{l,UK,t}$
(0.055452) (0.016034)
$\epsilon_{s,UK,t} = 0.96909 \epsilon_{s,UK,t-1} + 0.78522\, u_{s,UK,t}$
(0.057104) (0.032661)
$\epsilon_{c,UK,t} = 0.87095 \epsilon_{c,UK,t-1} + 1.2433\, u_{c,UK,t}$
(0.051321) (0.051713)
Canada (CAN)
$\epsilon_{l,CAN,t} = 0.94634 \epsilon_{l,CAN,t-1} + 0.33065\, u_{l,CAN,t}$
(0.055764) (0.013753)
$\epsilon_{s,CAN,t} = 0.96339 \epsilon_{s,CAN,t-1} + 0.58207\, u_{s,CAN,t}$
(0.056768) (0.024211)
$\epsilon_{c,CAN,t} = 0.84464 \epsilon_{c,CAN,t-1} + 0.71877\, u_{c,CAN,t}$
(0.049771) (0.029897)

Now we briefly anticipate the main results,if we consider first the results for the country level factors, both the intercepts and the loadings related to the Global Factors are estimated with high precision. The Germany level factor is the least persistent since the loading on the global factors is the highest of the four countries thus the dynamics of the German yield curve match those of the global level. Now consider the slope results, all the factors load positively on the global factor and, like before, the global factors are highly persistent especially for the US which has a loading on the Global Slope grater than one. Finally, consider the curvature factors, in this case all the countries have a loading on the global factor grater than one which can be interpreted as the completely dependence on global trends during the last twenty years.

Now we deep dive into the estimation results, the autoregressive models for the global yield curve factors suggest a high degree of persistence, consistent with the nature of yield curves over time. The Global Level Factor (L_t) shows a strong positive relationship with its first lag (L_{t-1}) with a coefficient of 1.0669, while the slight negative coefficient for the second lag (-0.0801) indicates a mean-reverting tendency, typical of long-term interest rates. The Global Slope Factor (S_t) behaves almost as a unit root process, with a coefficient of 0.99224, reflecting the persistent impact of changes in the yield curve slope over time. The Global Curvature Factor (C_t) also exhibits high persistence, with a coefficient of 0.95055, implying that the curvature of the global yield curve remains stable over time. These autoregressive (AR) models indicate that global yield curve factors are highly influenced by their recent history.

In country-specific regressions, the level factors ($L_{\text{country},t}$) for Germany, the US, the UK, and Canada show strong positive relationships with the global level factor (L_t), although the sensitivity varies across countries. For example, Germany's coefficient of 0.61205 suggests that global level factors significantly affect its yield curve, while the US and UK exhibit somewhat lower coefficients of 0.42234 and 0.4616, respectively. This shows that while global trends strongly drive national yield curves, local economic factors still play a role. The slope factors ($S_{\text{country},t}$) exhibit similar patterns, with Germany showing a significant coefficient of 0.72105, indicating that global slope movements substantially influence its yield curve slope, and similar relationships hold for the US and Canada, albeit with slightly weaker coefficients. For the curvature factors ($C_{\text{country},t}$), the results demonstrate strong co-movement with the global curvature factor, with Germany's coefficient at 1.0195 and other countries showing similar high values. This suggests that global changes in the yield curve's shape affect national curves, likely due to synchronized global economic cycles and monetary policies.

Finally, the idiosyncratic terms for these models exhibit significant autocorrelation, modeled as $AR(1)$ processes. For Germany, the $AR(1)$ coefficient for the level factor's error term ($\epsilon_{L,DE,t}$) is 0.90742, indicating high persistence in the error terms. Similar persistence is observed in the US, UK, and Canadian yield curve errors, with coefficients close to 0.9. This persistence suggests that shocks to country-specific yield curve factors (e.g., from domestic monetary policy) tend to have long-lasting effects, influencing yield curve dynamics for several periods before decaying.

6.2 Variance decomposition

Now we conduct the variance decomposition of country level, slope and curvature into parts explained by the global factors and by the single country ones. These equations split the variance related to the single country factor into three components: the factor loading on the global factors G_i, the variance of the global factors and the part driven by the idiosyncratic factor. Therefore, from (20), (21) and (22), we decompose the variation of each single-country factor into parts driven by the global yield variation (left part of the right hand side of the equations), thus the respective loading multiplied by the variation of the specific global factor, and the single-country specific variation (right part of the right hand side of the equations).

$$\mathrm{Var}(l_{it}) = (G_i^l)^2 \cdot \mathrm{Var}(L_t) + \mathrm{Var}(\epsilon_{it}^l) \tag{41}$$

$$\mathrm{Var}(s_{it}) = (G_i^s)^2 \cdot \mathrm{Var}(S_t) + \mathrm{Var}(\epsilon_{it}^s) \tag{42}$$

$$\mathrm{Var}(c_{it}) = (G_i^c)^2 \cdot \mathrm{Var}(C_t) + \mathrm{Var}(\epsilon_{it}^c) \tag{43}$$

for $\mathrm{i} = 1, \ldots, N$ and for $\mathrm{t} = 1, \ldots, T$.

Then in the table we report the global and the idiosyncratic variation contribution to the single-country factors and the results are common for all the four countries thus with the majority of variance explained by the global factors.

Concerning Germany, the variance of the level factor is 1.6373 and it is explained at 98.27% by the global level factor and at 1.73% by the German-specific level factor and this is consistent with the results of the previous paragraph. The percentage of explained variance decreases with the slope and the curvature factor both for Germany and the other countries, this is also consistent with the previous results. We notice that the country with the highest part of unexplained variance by the global factors or the greatest side of country-specific contribution in explaining the single-country factors, is the UK. The variance of the country-level factor is explained by the country-specific factor at a 7.07% and this percentage increases for the slope factor (26%) and for the curvature one (29.20%). These results suggest that the UK yield curve is less dependent on the global factors than the other countries and, therefore, the UK financial market is driven by idiosyncratic factors. For what concerns the US and Canada, the results are similar to the German ones with the only exception in the curvature factor of the US which is the highest value among the economies that is explained by the global factor (87.32%). One interpretation of the huge persistence of the global factors into the US factors could be that the American economy and, especially, its market's participants,influence the Canadian and German yield curve shape (mostly) and, therefore, the drivers of the monetary policy, of the expectations of future rates and of the bond risk premia while the UK seems to be dependent on domestic factors.

Table 14: Variance Decomposition Results

Germany		
Factor	**Global Contribution (%)**	**Country-Specific Contribution (%)**
Level Factor	98.27	1.73
Slope Factor	85.30	14.70
Curvature Factor	80.86	19.14

US		
Factor	**Global Contribution (%)**	**Country-Specific Contribution (%)**
Level Factor	95.19	4.81
Slope Factor	90.09	9.91
Curvature Factor	87.32	12.68

UK		
Factor	**Global Contribution (%)**	**Country-Specific Contribution (%)**
Level Factor	92.93	7.07
Slope Factor	74.00	26.00
Curvature Factor	70.80	29.20

Canada		
Factor	**Global Contribution (%)**	**Country-Specific Contribution (%)**
Level Factor	95.22	4.78
Slope Factor	85.92	14.08
Curvature Factor	80.00	20.00

7 Idiosyncratic components

In this section we analyze the behavior of the selected macroeconomic variables and, also, we disentangle both the idiosyncratic German and the UK yield curve into possible drivers in order to complete the Global Model. Therefore, we analyse the residual yield curves for the purpose of capturing the idiosyncratic components of the UK and of Germany. The choice of studying the residual yield curves of these two country is connected to the fact that: Germany represents the Euro Area since it is the largest economy due to the industrial base, the exports and the robust SMEs, while the UK is the country whose yield curve is most driven by idiosyncratic factors, especially for the slope and the curvature factor.

$$
\begin{aligned}
y_{i,t}(\tau) = & (\alpha_L^i + \beta_L^i L_t + \epsilon_{l,i,t}) \\
& + (\alpha_S^i + \beta_S^i S_t + \epsilon_{s,i,t}) \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) \\
& + (\alpha_C^i + \beta_C^i C_t + \epsilon_{c,i,t}) \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) \\
& + v_{i,t}(\tau) \quad \text{for} \quad \text{t} = 1, \ldots, T
\end{aligned} \tag{44}
$$

for $\text{i} = 1, \ldots, N$.

Now we try to understand if the specific Macroeconomic factors are able to expressed the $\epsilon_{l,i,t}$, $\epsilon_{s,i,t}$ and $\epsilon_{c,i,t}$), thus if we can treat them as specific drivers of the idiosyncratic yield curves. In order to do it, we study the residual yield curves for Germany and for the UK and we treat them as dependent variables while the Macro variables are treated as independent factors with the aim of comprehending their impacts on the idiosyncratic movements of the yield curve.

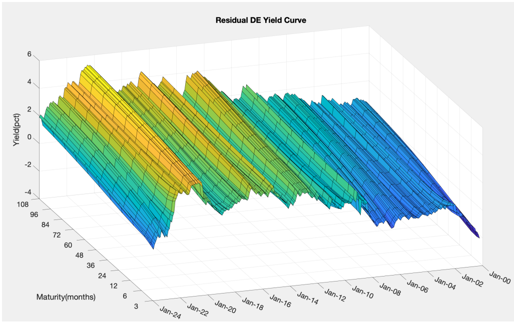
7.1 Residual Yield Curves

We define the Residual Yield Curve as

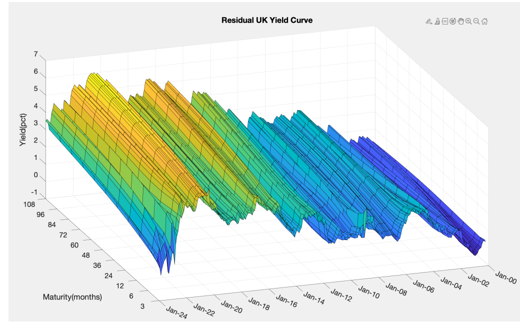
$$
\text{Residual YC}(i,t) = y(i,t) - g(t) \quad \text{for} \quad \text{t} = 1, \ldots, T \tag{45}
$$

for $\text{i} = 1, \ldots, N$, where i stands for Germany and the UK, $g(t)$ is the yield curve obtained by applying the Nelson-Siegel model to the three global factors. The residuals represent the portion of the yield curves that are not explained by the global factors extracted from the PCA, and we aim to quantify the influence of these macroeconomic variables on the residuals. where the German and the British factors represent $\epsilon_{l,DE,t}$, $\epsilon_{s,DE,t}$ and $\epsilon_{c,DE,t}$ and $\epsilon_{l,UK,t}$, $\epsilon_{s,UK,t}$ and $\epsilon_{c,UK,t}$. The ratio behind the selection of the macro-variables is related to the economic impact to both the short and the long maturity rates like the interest rates (Central Banks and Exchange rates) and the Macroeconomic Indicators(Inflation, Production, Economic Sentiment and Unemployment)[8]. The macroeconomic variables chosen that regard the Monetary policy interest rates are: the 3-month Euribor,the EONIA rate, the €STR and the European Central Bank rates[9] for the EuroArea while for the UK the SONIA rate and the BoE Base rate. Then we select the exchange rate (\$/€),the exchange rate (£/€),the exchange rate (\$/£) and the exchange rate (€/£) in order to study the currency impact between the two areas of interest(Euro-zone and the UK) and with the US dollars since we have defined the US as the country whose Yield Curve is the most explained by the

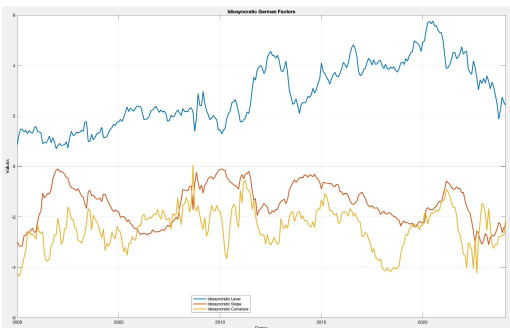
[8] All the times series are available on https://data.ecb.europa.eu and on https://www.ons.gov.uk
[9] Deposit Facility rate, Marginal Landing rate and Refinancing Operations rate



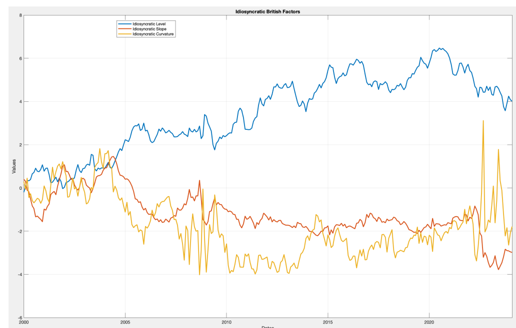
(a) German Residual Yield Curve



(b) British Residual Yield Curve



(c) German Idiosyncratic Factors



(d) British Idiosyncratic Factors

Figure 5: Euro-Zone Macroeconomic Factors over the sample period (2000-2024)

common factors. Further, Industrial Production Index (IPI), Consumer Price Index (CPI) , the Debt to GDP rate, the Economic Sentiment Index (ESI) and the Unemployment rate.

7.2 Analysis of the Macro-Variables

We analyse the macro-variables for the purpose of studying the correlation and the multi-collinearity among the macroeconomic variables since factors like interest rates, for instance the ECB[10] key interest rates, depend on common drivers and they could affect our analysis on the residuals yield curves. Therefore, we look at the correlation matrix for the Euro-Area and for the UK macroeconomic factors separately and, then, we compute the VIF. The VIF measures the degree of multi-collinearity among the macro variables and,for large values, it indicates high collinearity due to R^2 obtained by regressing the specific macro-factor on the other independent variables. If the R^2 is close to one (very large explanatory value)the variable is predictable from the other factors. We analyse this measure because the multi-collinearity leads to a difficult determination of single-

[10]European Central Bank

macro-variable effect on the residual yields, to unstable regression coefficients and, especially, to interpretation problems of the results.

$$\text{VIF}(M_i) = \frac{1}{1 - R_i^2} \quad \text{for} \quad \text{i} = 1, \dots, \text{N} \tag{46}$$

Where the matrix M is the matrix of the Macroeconomic variables, it has dimensions T x m_f, where f stands for the respective factors, either Interest rates (Monetary Policy or Exchange Rates) or Macroeconomic indicators and we define N as the total number of Macroeconomic variables.

7.2.1 Euro-Zone Macro-Variables

Table 15: Correlation Matrix for EU Monetary Policy Rates

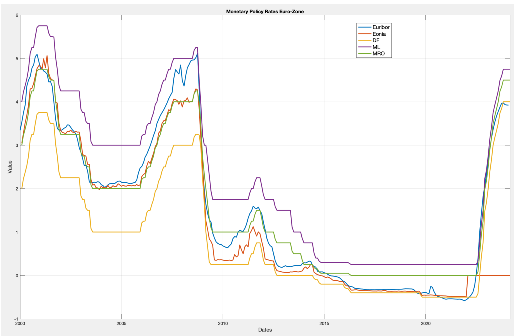
	Euribor	**Eonia**	**ESTR**	**DepFac**	**MarLen**	**MRO**
Euribor	1.0000	0.9011	0.3364	0.9743	0.9888	0.9888
Eonia	0.9011	1.0000	-0.0800	0.8149	0.9029	0.8635
ESTR	0.3364	-0.0800	1.0000	0.4857	0.3261	0.4108
DepFac	0.9743	0.8149	0.4857	1.0000	0.9658	0.9911
MarLen	0.9888	0.9029	0.3261	0.9658	1.0000	0.9910
MRO	0.9888	0.8635	0.4108	0.9911	0.9910	1.0000

The Monetary Policy rates of the Euro-Zone impact the shape of the Yield curve, especially the slope factor. A steep Yield curve is a sign of accessible borrowing costs and it represents an expectation for solid growth and inflation down, while an inverted yield curve leads toward a recession due to tight credit conditions and, therefore, these are signals of low growth and high inflation expectations. These interest rates are both connected to the control of inflation and growth expectations which typically affects the Long-Term Bonds and to short term yields because the lower the monetary policy interest rate, the lower the bond yield.

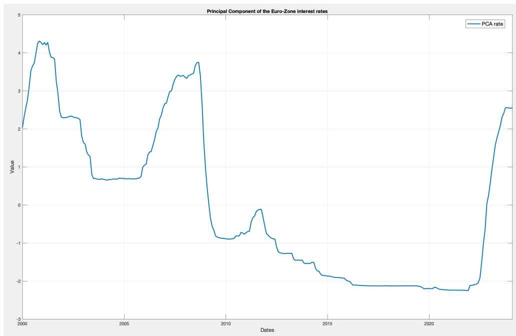
The correlation matrix shows a strong positive correlation among the rates which is a multicollinearity indicator. The positive correlation (close to one) is expected since ECB key rates (Deposit Facility rate,Marginal Lending Facility rate and MRO rate) are pure monatery policy instruments which have an impact on loans and inflation. The Deposit Facility rate is the rate banks receive when depositing excess funds with the ECB overnight, the Marginal Lending Facility rate is the interest rate at which banks can borrow from the ECB overnight and this costs them more than if they borrow for one week while the MRO rate is the interest rate banks pay when they borrow money from the ECB for one week and, when they do this, they have to provide collateral to guarantee that the money will be paid back.

The Euribor and Eonia rates are interbank rates, thus used by Europenan Banks to borrow and to lend funds. Euribor (the Euro Interbank Offer Rate) and Eonia (the Euro Overnight Index

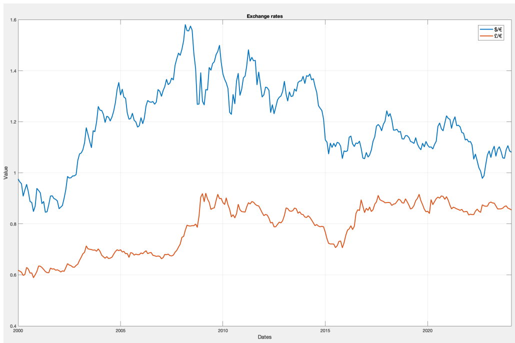
Average). The first is the interbank overnight rate, whereas the second is an estimate based on lengthier loan durations. Since the Eonia rate cannot be lower than the ECB deposit rate because, in that scenario, a bank with liquidity surplus would make an overnight deposit to the ECB rather than lending money to another bank, the Eonia rate fluctuates in accordance with the levels of the ECB overnight deposit rate and the ECB overnight marginal lending rate. Because a bank with a liquidity demand would borrow from the ECB rather than from another bank, the Eonia rate cannot be greater than the ECB's overnight marginal lending rate. Then, we introduce the €STR which is a rate which reflects the wholesale euro unsecured overnight borrowing costs of euro area banks. Since 2 October 2019, the current EONIA methodology has been modified to become €STR plus a fixed spread of 8.5 basis points where this spread is based on a simple average of the EONIA (Bank (2019))[11].



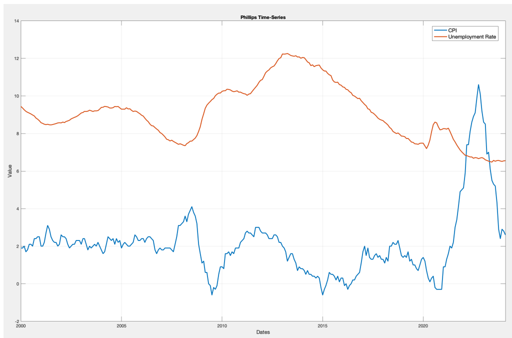
(a) European Interest Rates



(b) PCA Score of European Interest Rates



(c) Exchange Rates for the Euro-Zone



(d) Phillips Curve

Figure 6: Euro-Zone Macroeconomic Factors over the sample period (2000-2024)

The chart confirms the commonality component of the interest rates and, due to the high corre-

[11]European Central Bank

lation and the high values of VIF[12], these variables would alter the values and the interpretation of the coefficients. Then, we use the PCA in order to extract the common components that explain the majority of the variance (99%) and we plot the time series of the score. We can state that the path is similar,the score starts at 2% like the Deposit rate and then exceeds 4% (its maximum point), then equivalent to the global factor case, the score touches lower values than the interest rates especially after 2015. In recent years, we see a steepening of the interest curve which brings the score above the 2% (lower than the singular interest rates values). The PCA approach allows us to reduce the number of variables and to avoid the multi-collinearity issue but we lose the interpretation of the single-interest rate impact on the residual yield curve.

Eigenvalues	**Variance**	**Cumulative Variance**
4.7571	95.142	95.142
0.21518	4.3036	99.446
0.017349	0.34698	99.793
0.0098903	0.19781	99.990
0.00047733	0.0095467	100.000

Table 16: PCA European Interest Rates: Eigenvalues, Variance, and Cumulative Variance

The second category of macro-variables for the Euro-Zone concerns the exchange rates ($/€) and (£/€). We add the exchange rates with the US and the UK in order to understand the relationship between the € values and the residual yield curve, thus in order to capture the movements of the yield curve as the value of the European currency changes. The correlation matrix shows a low positive correlation (0.3637) while the VIF is close to 1 which means a very low level of collinearity then we can use both variables.

The third class of Macro-variables regards the Macro-Economic indicators like the Consumer Price Index which represents the monthly change in prices paid by the European consumers, the Industrial Production Index that is an economic indicator describing the real output in several sectors like manufacturing electric and gas industries. Then we select the Debt to GDP of the Euro-Zone which stands for the likely to repay the debt and, therefore, it is connected to the perception of defaulting. Subsequently, we choose the Economic Sentiment Indicator and the Unemployment rate, the former explains the GDP growth at Member states, EU and euro area levels, it is a weighted average of the balances of replies to selected questions addressed to firms in five sectors covered by the EU Business and Consumer Surveys and to consumers. The sectors covered are industry (weight 40 %), services (30 %), consumers (20 %), retail (5 %) and construction (5 %) while the latter lays out the share of the labour force without work. These Macro-Economic indicators are related to the shape of the European yield curve because they have a strong impact on interest rates, on monetary policy and, especially, on the economy growth of the Euro-Zone.

[12]All the values are close to 100, thus the regressions have R^2 close to one

Table 17: Correlation Matrix of the Euro-Zone Macroindicators

	IPI	**CPI**	**Debt/GDP**	**ESI**	**Unemployment Rate**
IPI	1.0000	-0.0587	0.0420	0.2046	0.0753
CPI	-0.0587	1.0000	-0.0405	0.0925	-0.5280
Debt/GDP	0.0420	-0.0405	1.0000	-0.0237	0.2417
ESI	0.2046	0.0925	-0.0237	1.0000	-0.2325
Unemployment Rate	0.0753	-0.5280	0.2417	-0.2325	1.0000

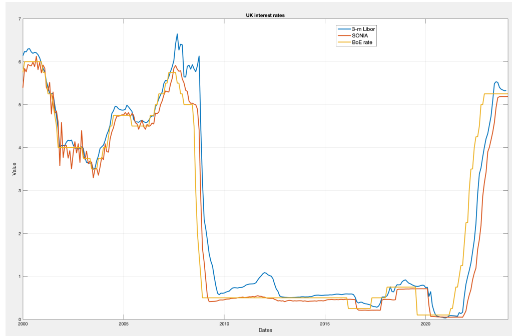
The correlation matrix shows very low linear dependence among these variable but for the Unemployment Rate and the Consumer Price Index (-0.5280), thus for high levels of inflation the Employment rate increases and vice-versa which is consistent to the Philipps CurvePhillips (1958), figure *d*. Plotting the relationship over time between the Consumer Price Index (CPI) and the unemployment rate may be interpreted as a reflection of some features of the Phillips curve, which suggests that unemployment and inflation are inversely related. Both variables show relative stability from 2000 to 2008, with certain times showing a slight adverse association. From 2000 to 2008, both macro-factors exhibit relative stability, with a light inverse relationship observed in different periods. However, in 2008, due to the financial crisis, there is a sharp increase in unemployment, alongside a brief drop in CPI, and then there is a slow recovery in both macroeconomic variables. Post-2010, the inverse relationship weakens, as the unemployment rate steadily declines while CPI remains more volatile. Later, in 2020, both CPI and unemployment rate experience significant spikes, likely due to the COVID-19 pandemic, illustrating a breakdown in the traditional Phillips curve dynamic as economic shocks led to simultaneous increases in both inflation and unemployment. This period highlights the complexities in macroeconomic relationships during times of crisis.

7.2.2 British Macro-Variables

For the UK, we select the same three categories as before, thus the class of Monetary policy interest rates which contains the 3-month Libor [13], the SONIA rate and the BoE Base Rate, the category of exchange rates regarding the exchange rate $/£ and the €/£ and, the last classification concerns the Macro-Economic Indicators IPI, CPI, Debt to GDP, ESI and Unemployment rate. The correlation matrix of the three interest rate shows high positive values (above 0.90)[14] like in the case of the European interest rates. In this case we use the SONIA rate as reference since it is the overnight rate that reflects the average of the interest rates that banks pay to borrow from both financial institutions and institutional investors, it is administrated by the BoE and, over the years, it has substituted the Libor due to its manipulation and scandalsSchrimpf and Sushko (2019).

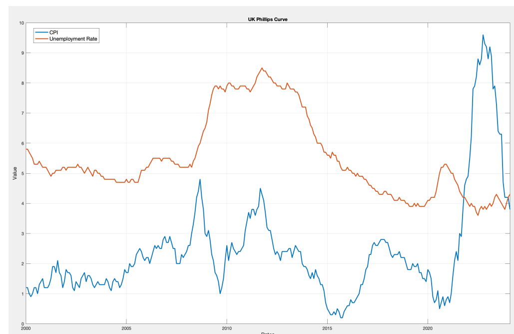
[13] London Inter-bank offered rate

[14] Very large VIF values



(a) British Interest Rates

(b) Exchange Rates for the UK



(c) Phillips Curve

Figure 7: British Macroeconomic Factors over the sample period (2000-2024)

Table 18: UK Interest Rates Correlation Matrix

	Libor	**Sonia**	**BoE Rate**
Libor	1.0000	0.9238	0.9341
Sonia	0.9238	1.0000	0.9544
Base Rate	0.9341	0.9544	1.0000

Regarding the exchange rates, the correlation matrix shows a positive correlation (0.4741) and a low VIF value (1.0292), the we can use both the rates in the regression since there is no collinearity.

Then, the correlation matrix of the Macro-Economic Indicators displays several strong dependencies like the IPI-Unemployment rate (-0.6091) indicating that if the Industrial Production increases the unemployment rate decreases, the CPI-ESI coefficient is -0.7034 which implyes that inflation erodes the Economic sentiment and the Debt to GDP-ESI correlatio is -0.5030 that stands for concerning over the sustainability of the Debt. The first main result is the lower correlation between the CPI and Unempoyment rate than the European one (-0.1431 vs -0.5280), it is always consistent with the Philipps Curve principle but the relationship is quite low.

	IPI	CPI	Debt/GDP	ESI	Unemployment Rate
IPI	1.0000	-0.1753	-0.3601	0.0587	-0.6091
CPI	-0.1753	1.0000	0.3327	-0.7034	-0.1431
Debt/GDP	-0.3601	0.3327	1.0000	-0.5030	-0.0886
ESI	0.0587	-0.7034	-0.5030	1.0000	0.2926
Unemployment Rate	-0.6091	-0.1431	-0.0886	0.2926	1.0000

Table 19: UK Macro-Indicators Correlation Matrix

Then, we can state that in the UK the correlations are stronger than in the Euro-Zone, for instance CPI, ESI, IPI and Unemployment rate show that the economic dynamics in the UK depend more on these variables than the European ones. The role of IPI in the UK has a strong impact on the unemployment rate which underlines the importance of employment while in the EU the relationship is lower and the highest correlation coefficient for the Unemployment rate is inflation which reflects a huge difference among the two zones.

7.3 Seemingly Uncorrelated Regressions

Now we have the Macro-Economic variables that do not have multi-collinearity problems and, then, we can use them as independent variables for studying the behavior and the drivers of the residual yield curves. To achieve this, we perform the Seemingly Uncorrelated Regressions (SUR) to obtain the coefficients for each maturity (from 3-month to 10 years), therefore, we create a system of linear regressions.

$$\begin{cases} y_{1i} = \mathbf{M}_{1i}^{\top}\boldsymbol{\gamma}_1 + \epsilon_{1i}, & i = 1, 2, \dots, n \\ y_{2i} = \mathbf{M}_{2i}^{\top}\boldsymbol{\gamma}_2 + \epsilon_{2i}, & i = 1, 2, \dots, n \\ \vdots \\ y_{mi} = \mathbf{M}_{mi}^{\top}\boldsymbol{\gamma}_m + \epsilon_{mi}, & i = 1, 2, \dots, n \end{cases} \tag{47}$$

where m is the number of equations and it corresponds to τ and n is the number of observations (T), therefore the vector y is (T x τ) x 1, thus observations x maturities, the matrix M is [(T x τ) x ($\sum_{i=1}^{m} f_i$)] where f is the Macro Factor and the number of Macro-Variables depends on the Euro-Zone and on the UK, the error vector ϵ has the same dimension as y and the γ coefficient vector is ($\sum_{i=1}^{M} f_i$) x 1. Last, the degrees of freedom are df $= T \times \tau - \sum_{i=1}^{m} f_i$ including the intercepts.

The ratio of the SUR model is that we want to study the impact of the Macro-Economic variables on the twelve different maturities in order to measure the influence of the Monetary Policy rates,the Exchange rates and the Macro-Indicators on the specific maturity.

7.4 Estimation of Gamma Coefficients

The gamma coefficients γ estimated by using the SUR model are no longer evaluated by the OLS but by the Generalized Least Squares estimator (GLS) because it takes into account the correlation of the error terms across different equations.

$$\hat{\gamma}_{GLS} = (\mathbf{X}^\top(\mathbf{I}_n \otimes \mathbf{\Sigma}^{-1})\mathbf{X})^{-1}\mathbf{X}^\top(\mathbf{I}_n \otimes \mathbf{\Sigma}^{-1})\mathbf{y} \tag{48}$$

The gamma coefficients measure the sensitivity of the residuals to each of the macroeconomic factors. A significant gamma coefficient indicates that the corresponding macroeconomic factor has a substantial influence on the residuals.

7.4.1 Interpretation of Gamma Coefficients in the Euro-Zone

Once the gamma coefficients are estimated, we interpret them to understand the relationship between the residuals and the macroeconomic factors and, in the table, the rows represent the twelve maturities while the columns the macro factors. We start the analysis by looking at the intercept values of the regressions. The intercept represents both the baseline level of the domestic yields for Germany taking into account the Euro-Zone factors and the amount of residual yields captured by these European macroeconomic variables. In the former case, the intercept indicates the average of German yields for each maturity that is not explained by global trends and, for shorter maturities the intercept is negative representing that (if the Euro-Zone Macro factors are all zero) the German yields are slightly negative with respect to the ones predicted by the global factors while for longer maturities is the opposite. The latter, instead, indicates whether the variables capture the behavior of the residual yield and, except for 12-month and 24-month maturities, the intercept values are below one representing the residual yields dependence on the selected macroeconomic factors [15].

Maturity	Intercept	Monetary Int Rates	$/€	£/€	IPI	CPI	Debt/GDP	ESI	Unemployment Rate
3-Month	-0.11024	-0.27959	1.1829	-1.9048	-0.007452	-0.18054	0.080414	-0.025682	-0.24298
6-Month	-0.64741	-0.25703	1.3284	-1.876	-0.0076634	-0.18168	0.084894	-0.025347	-0.24666
1-Year	-1.2786	-0.2297	1.5528	-1.9548	-0.0077466	-0.17439	0.091156	-0.024178	-0.26559
2-Year	-1.1602	-0.21802	1.6674	-2.2815	-0.0057114	-0.17341	0.094727	-0.022274	-0.30811
3-Year	-0.67256	-0.22012	1.6203	-2.4614	-0.0055507	-0.16924	0.093792	-0.02001	-0.34421
4-Year	-0.27144	-0.22385	1.5248	-2.4858	-0.0037095	-0.16208	0.091846	-0.017872	-0.36654
5-Year	-0.016349	-0.22645	1.4256	-2.4069	-0.0036876	-0.1538	0.089938	-0.016095	-0.37747
6-Year	0.095102	-0.22781	1.3347	-2.2573	-0.0039622	-0.14605	0.088361	-0.014712	-0.37861
7-Year	0.10258	-0.22743	1.2556	-2.0835	-0.0050626	-0.13934	0.087218	-0.013635	-0.37305
8-Year	0.04763	-0.22602	1.1947	-1.9125	-0.0058996	-0.13379	0.086519	-0.012912	-0.36364
9-Year	-0.0429	-0.22358	1.148	-1.772	-0.0072774	-0.1296	0.086404	-0.012397	-0.35338
10-Year	-0.15239	-0.22088	1.1132	-1.6494	-0.0080613	-0.12662	0.086539	-0.01201	-0.34278

Table 20: Gamma Coefficients for the Euro-Zone Macroeconomic Variables (DE) across Different Maturities

Then, we analyse the estimated coefficients of the Monetary Policy interest rate (extracted by

[15] Appendix B for the Standard Errors, T-statistic and P-values

PCA). We can state that, during the sample period, there is a negative relationship between the interest rates and the German residual yields and the shorter the maturity, the grater[16] the value of the coefficient. Therefore, when the ECB raises the interest rates the yields tend to decline relative to the yields predicted by the global factors indicating the strong impact of the Monetary Policy in the Euro-Zone like the presence of negative interest rates and strong Quantitative Easing) The $/€ coefficients are always positive and grater than one and they indicate that, when € appreciates against the $, the German yields rise with respect to the yields predicted by the global factors only because an appreciating € might make the German bonds less competitive globally since the currency is more costly for foreign investors and, therefore, it might lead to an increase in yields to attract foreign investments. The environment indicated by the £/€ is quite the opposite, all the coefficients are negative and, for middle maturities, lower the -2 implying a reverse relationship with respect to $/€, thus, when € appreciates against the £, the idiosyncratic yields tend to lower possibly due to a flight to Safety for the German Bonds which put downward the yields. The third class of macro variables begin with the Industrial Production Index whose coefficients are slightly negative indicating a very low impact on the residual yields which could be due to the fact that the IPI is captured by the global factor and, therefore, it has very small influence on the residual yield curve during the last two decades. The inflation, represented by the CPI, has a negative impact on the yields which specify that as Euro-Zone inflation rises the German yields decreases relative to the ones predicted by the global factors and it might be due to the strong ECB inflation-fighting thus with the solid dependence on rising monetary policy rates as investors expect inflation to be brought under control more quickly in Germany than in other countries. This finding brings a counter intuitive relationship with the bond yields since high inflation would lead to higher yield in order to demand for compensation for the erosion, at least from a global perspective while, for the Euro-Zone, this connection differs due to the weight given to the Monetary Policy interest rates, in particular for shorter maturities. Regarding the Debt to Gross Domestic Product ratio, the positive coefficients indicate that as government debt levels increase in Germany the bond yields go up compared to global averages. This trend is particularly noticeable for longer term bonds since investors are extra concerned about the fiscal stability of the government. The strong connection signifies worries in the market regarding stability when debt levels are on the rise (for example during the Euro-Zone Sovereign Debt Crisis market concerns about debt sustainability across Euro-Zone countries led to sharp increases in yields for several countries). This causes investors to seek returns for owning German bonds compared to the global standard. The residual German yield curve and the unemployment rate are inversely correlated, meaning that an increase in the unemployment rate lowers bond yields relative to expectations. Growing joblessness is an indication of worsening economic prospects, which increases demand for safer assets like German bonds and other European bonds. This lowers yields, and the effect is more pronounced for longer maturities because investors

[16] In absolute value

may view rising unemployment as a sign of persistent economic difficulties. On the other hand, residual yields and the economic sentiment index (ESI) show a positive coefficients that is clearly visible in the table. As confidence rises and future market expectation gets stronger, so do residual yields in comparison to global standards. This shift points to a decline in demand for safe-haven investments like German bonds as domestic economic optimism grows due to increased confidence in the economy. German bond yields are higher than those of other international bonds because investors are taking on greater risk in the context of betting on good future economic outlooks.

7.4.2 Interpretation of Gamma Coefficients in the UK

Now we explore the coefficients regarding the Macro Economic variables of the UK and we try to interpret the results and to compare them with the European ones. In the table below, the rows represent the maturities while the columns the macro factors.

Maturity	Intercept	Monetary Int Rates	$/£	€/£	IPI	CPI	Debt/GDP	ESI	Unemployment rate
3-Month	-0.16474	-0.42851	2.2309	-0.3388	0.02537	-0.18779	0.016608	-0.0036629	-0.43729
6-Month	-1.4916	-0.39703	2.2353	0.11419	0.02611	-0.11203	0.023754	-0.0028583	-0.44821
1-Year	-2.5545	-0.36401	2.1343	0.43711	0.028664	-0.092044	0.029604	-0.0027446	-0.43498
2-Year	-3.0457	-0.30967	1.8827	0.37422	0.033222	-0.11141	0.034619	-0.0026513	-0.39664
3-Year	-3.4989	-0.26199	1.7887	0.25214	0.035812	-0.1176	0.039539	-0.0018117	-0.36887
4-Year	-3.992	-0.22565	1.7559	0.18925	0.037153	-0.11258	0.044376	-0.00066508	-0.34776
5-Year	-4.3992	-0.19954	1.7462	0.13465	0.037729	-0.10365	0.048602	0.00044358	-0.32894
6-Year	-4.6749	-0.18109	1.7456	0.05922	0.037814	-0.094337	0.052111	0.0013921	-0.31112
7-Year	-4.8276	-0.16807	1.7472	-0.041976	0.037629	-0.085835	0.054986	0.0021626	-0.2944
8-Year	-4.8867	-0.15871	1.7461	-0.16219	0.037352	-0.078348	0.057366	0.0027792	-0.27924
9-Year	-4.8836	-0.15167	1.739	-0.29139	0.037099	-0.071795	0.059398	0.0032773	-0.26597
10-Year	-4.8448	-0.14605	1.7247	-0.42033	0.036928	-0.066097	0.061133	0.0036903	-0.25466

Table 21: Regression Coefficients for Macroeconomic Variables (UK)

The first main results concerns the intercept values because for the British framework, we obtain grater values in absolute value than the European ones. The intercepts represent the magnitude of unexplained residual yield by the British macroeconomic factors and the unknown amount increases with the maturity. For instance, the 3-month maturity residual yield has a very low intercept (-0.16474) thus the Macro variables explain most of the variation of the residual UK yields while the 96-month maturity has an intercept of -4.8867 that indicates the weakness of the model for longer maturities. Another interpretation may be that, as the maturity increases, the macroeconomic factors for the UK are already captured by the global factors, then they are no longer responsible for the idiosyncratic movements of the UK yield curve. Furthermore, there are other variables that play a crucial role in determining the residual yield curve, thus geopolitical factors, risk perceptions and future economic conditions are not fully captured by these variables, at least for long-term maturities.

The Monetary interest rates for the UK, SONIA rate, for the 3.month yield is negative and it represents an intense presence of the BoE in determining the residual shape of the yield curve, it has greater impact than the monetary policy interest rates of the ECB. Then, as the maturity increases, the amount of monetary policy impact decreases but, since as the maturity increases the inability of explaining the residual yields increases, for longer maturities we cannot determine the real impact of the SONIA rate on the shape of the yield curves.

The second class of Macro factors consists in the two exchange rates, $/£ and €/£. The former has all positive coefficients indicating that as the £ appreciates against the $ the UK residual yield rise with respect to the predicted global yield, this effect may be due to stronger economic conditions for the UK. The 3-month residual yield rise when the £ strengthens against $ (2.2309) and it reflects that a currency appreciation is a signal of economic strength and, for longer maturities, the relationship persists even if it decreases. Regarding the €/£, the coefficients are both positive and negative, when the coefficient is lower than zero, like in the 3-month case, when the € appreciates against the £, the residual yield of the UK is below the level predicted by the global factors. The relationship between € and £ affects the imports/exports costs and, when a foreign currency appreciates against a domestic currency, the exports becomes more expensive creating inflationary pressures which may lead to tighter monetary policy by the BoE and, then, to lower yields. Furthermore, for the mid-term maturities, the coefficient is positive which represents either a 'Capital Flows' effect which decreases the demand for UK bond and makes the yield higher in order to attract investors or a consequence of high intercept values and, therefore, weak explanatory power of the macro-economic variables.

The third category of the macro factors begins with the IPI coefficients that are all positive even if the amount is very low which indicates a weak impact on explaining the residual yield curve. The coefficient values greater than zero imply a relationship between strong industrial production and high residual yields. This effect may be due to a positive sentiment regarding the UK economic growth. However,the high intercept sizes suggests the poor explanatory power, especially for longer maturities while the IPI is likely to influence short- to medium-term residual yields.

The CPI has all negative coefficients indicating that as inflation increases, the market expects strong monetary policy interventions, therefore the yields usually grows as inflation increases but the effect of the SONIA rates, Base rates and any other monetary policy interest rate is greater. As the maturity increases, the CPI coefficients decrease and this may be due to the larger intercepts and, if we compare these results with the European ones, the inflation lose explanation power over the maturities while it stays constant (from -0.18 to -0.12) in the Euro-Zone framework.

The Debt to GDP ratio for the UK has positive coefficients like the European one, which implies that the country debt makes the residual yield increase relative to the yields generated by the global factors. Therefore, the debt level is connected to the sustainability and, then, to higher bond yields.

Last, the ESI and the Unemployment rate for the UK have different impacts. The former implies a negative relationship with positive economic sentiment for the short-term maturities while the coefficient becomes positive with longer maturities but, for all twelve maturities, the amount of the coefficients is close to zero indicating a very low impact. The Unemployment rate has always a negative coefficient like the European context. Therefore, an high value of unemployment rate leads to lower yields with respect to the implied yields by the global factors. The unemployment rate measures the UK economy and, if the former increases, the economy is weakening and, thus, a slower economic growth then the investors increase the demand of Government Bonds lowering the short-term yields, especially.

7.5 Wald Test

Now we run a test on the $\hat{\gamma}$ coefficients estimated through the SUR model in order to assess the statistical significance of the macro economic factors on the residual yield curves. The Wald test allows us to compute the p value from a χ^2 distribution with as degrees of freedom as the number of parameters being tested. Then, we test each $\hat{\gamma}$ coefficient for Germany and for the UK, then eight macro factors for each country, therefore the degrees of freedom are 3372.

$$W = (\hat{\gamma}_{GLS} - \gamma_0)' \left(\text{Var}(\hat{\gamma}_{GLS})\right)^{-1} (\hat{\gamma}_{GLS} - \gamma_0) \tag{49}$$

where the null hypothesis H_0 is

$$H_0 : \gamma = \gamma_0$$

where γ_0 is imposed equal to zero.

Regarding both the Euro-Zone and the UK, we obtain very high values for the Wald Test implying a very low p value (below 0.05) which allow us to reject the hypothesis of meaningless coefficients. For both SURs, the only macro variable which results with no statistical significance is the IPI. This is consistent with the coefficients value obtained in the subsection before and it implies that the IPI is either captured by common factors extracted through PCA or unrelated to the yield curve shapes of the two macro-areas.

8 Conclusion

Our analysis of the entire sample period (2000.01 - 2024.02) provides important insights into the drivers and the dynamics of the yield curve across four of the world major economies. This Global Dynamic Nelson-Siegel model is capable to capture the common movements in the curve showing the presence of global factors such as global level of interest rates, slope of the 'global' curve and the common curvature that are primary responsible for the shape of the yield curves.

The dominance of the common/global factors suggests global economic trends such as financial crises, particularly the level factor which exhibited an high degree of commonality and a substantial amount of persistence, therefore we assume an $AR(2)$ process for its behavior. The level, with the global slope and the global curvature explain the most of the variance in the single country factors, confirming their dominance.

In parallel, we study the grade of dependence on the three global factors among the single countries and we notice that, for Germany, the US and Canada, the yield curve is mainly driven by the global factors while, for the UK, there is a strong presence of idiosyncratic components that impact the slope and the curvature factors.

Then, our analysis introduce specific macroeconomic variables in order to capture the dynamics and the drivers of the residual yield curve unexplained by the global factors for Germany and the UK. The macro factors include monetary interest rates for the Euro-Zone and the UK, inflation, industrial production and exchange rates between € and £, and between them and $. We find meaningful and varied effects on the two residual yield curves underlining the great impact of regional conditions on the yield curve that move up or down the yields obtained by the global factors. In the UK, inflation has a more pronounced impact while, in Germany, the residual yield curve is influenced by the interest rates providing the great impact of the monetary policy that is able to cancel the inflation effect and it be due to the sample period which is characterised by several crises which led both to negative interest rates and very high ones. Furthermore, for the UK, the specific macroeconomic factors are not capable to explain the total residual yield curve and it is given by the huge values of the intercept for the various yield maturities. This result highlights the presence of other idiosyncratic components related exclusively the to UK since the 'same' factors for the Euro-Zone are able to capture the residual drivers. Therefore, this output is consistent with the ones obtained by the residual variance that show a greater presence of country factor for the UK than for Germany.

In conclusion, this study underlines the interconnection between global and local factors in shaping the yield curve and the model provides a framework for capture the global dynamics and the residual dependence on the local factors where for Germany explain the majority part while for the UK highlight the presence of other local components.

A Appendix A: Theoretical Background of Principal Component Analysis

Principal Component Analysis (PCA) Jolliffe (2002) is a method used to simplify a dataset by reducing its dimensions while preserving most of the datas variability. It achieves this by transforming the variables into uncorrelated variables known as principal components. These components are arranged in such a way that the initial ones capture the majority of the variation found in the dataset. Mathematically, PCA involves the following steps: The data matrix $\mathbf{X}$ (of size $n \times p$, where n is the number of observations and p is the number of variables) is often standardized so that each variable has a mean of zero and a standard deviation of one. Compute the Variance-Covariance matrix $\mathbf{C} = \frac{1}{n-1}\mathbf{X}^\top\mathbf{X}$. The variance-covariance matrix captures the relationships between the the factor matrices. The covariance matrix $\mathbf{C}$ is then decomposed into its eigenvalues and eigenvectors:

$$\mathbf{C} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top \tag{50}$$

where $\mathbf{\Lambda}$ is the diagonal matrix of eigenvalues (representing the variance explained by each principal component) and $\mathbf{V}$ is the matrix of corresponding eigenvectors (representing the direction of each principal component). The principal components are obtained by projecting the original data onto the eigenvectors:

$$\mathbf{Z} = \mathbf{X}\mathbf{V} \tag{51}$$

Here, $\mathbf{Z}$ represents the principal components, with each column corresponding to a principal component. The principal components are ordered by the magnitude of their eigenvalues. The first principal component explains the most variance, followed by the second, and so on. A subset of the first few principal components is selected to represent the data with reduced dimensionality.

B Appendix B: Std, T-Stat and P-Values of SUR Model

Maturity	Parameter	Std Error	t-Stat	p-Value
3-Month	**Intercept**	0.8972	-0.1228	0.9023
	Monetary Int Rates	0.0345	-8.103	7.550×10^{-14}
	$/€	0.3065	3.8587	0.0001
	£/€	0.8558	-2.2256	0.0272
	IPI	0.0216	-0.3117	0.7556
	CPI	0.0232	-7.7751	5.391×10^{-13}
	Debt/GDP	0.0079	10.173	0
	ESI	0.0041	-6.1817	4.062×10^{-9}
	Unemployment Rate	0.0372	-6.5211	6.675×10^{-10}
6-Month	**Intercept**	0.8851	-0.7315	0.4654
	Monetary Int Rates	0.0340	-7.5518	2.007×10^{-12}
	$/€	0.3024	4.3930	1.895×10^{-5}
	£/€	0.8443	-2.2220	0.0275
	IPI	0.0213	-0.3590	0.7200
	CPI	0.0229	-7.9317	2.121×10^{-13}
	Debt/GDP	0.0078	10.887	0
	ESI	0.0041	-6.1847	3.998×10^{-9}
	Unemployment Rate	0.0368	-6.7108	2.375×10^{-10}
1-Year	**Intercept**	0.8726	-1.4653	0.1446
	Monetary Int Rates	0.0336	-6.8457	1.127×10^{-10}
	$/€	0.2981	5.2089	5.109×10^{-7}
	£/€	0.8323	-2.3486	0.0199
	IPI	0.0210	-0.3547	0.7232
	CPI	0.0226	-7.7229	7.343×10^{-13}
	Debt/GDP	0.0077	11.858	0
	ESI	0.0040	-5.9841	1.131×10^{-8}
	Unemployment Rate	0.0362	-7.3296	7.291×10^{-12}
2-Year	**Intercept**	0.8654	-1.3407	0.1817
	Monetary Int Rates	0.0333	-6.5517	5.658×10^{-10}
	$/€	0.2956	5.6397	6.412×10^{-8}
	£/€	0.8254	-2.7641	0.0063
	IPI	0.0209	-0.2737	0.7846
	CPI	0.0224	-7.7437	6.493×10^{-13}
	Debt/GDP	0.0076	12.425	0
	ESI	0.0040	-5.5591	9.534×10^{-8}
	Unemployment Rate	0.0359	-8.5739	4.219×10^{-15}
3-Year	**Intercept**	0.8606	-0.7815	0.4355
	Monetary Int Rates	0.0331	-6.6511	3.293×10^{-10}
	$/€	0.2940	5.5108	1.207×10^{-7}
	£/€	0.8209	-2.9984	0.0031
	IPI	0.0208	-0.2193	0.8267
	CPI	0.0223	-7.5988	1.525×10^{-12}
	Debt/GDP	0.0076	12.370	0
	ESI	0.0040	-5.0214	1.217×10^{-6}
	Unemployment Rate	0.0357	-9.6311	0
4-Year	**Intercept**	0.8579	-0.3164	0.7521
	Monetary Int Rates	0.0330	-6.7852	1.576×10^{-10}
	$/€	0.2931	5.2023	5.270×10^{-7}
	£/€	0.8183	-3.0376	0.0027
	IPI	0.0207	-0.1793	0.8579
	CPI	0.0222	-7.2999	8.648×10^{-12}
	Debt/GDP	0.0076	12.152	0
	ESI	0.0040	-4.4991	1.214×10^{-5}
	Unemployment Rate	0.0356	-10.288	0

Table 22: Standard Errors, t-Statistic, and p-Values for the Euro-Zone Macroeconomic Variables (3-Month to 4-Year)

Maturity	Parameter	Std Error	t-Stat	p-Value
5-Year	**Intercept**	0.8575	-0.0191	0.9848
	Monetary Int Rates	0.0330	-6.8673	9.998×10^{-11}
	$/€	0.2930	4.8661	2.457×10^{-6}
	£/€	0.8180	-2.9426	0.0036
	IPI	0.0207	-0.1783	0.8586
	CPI	0.0222	-6.9306	7.024×10^{-11}
	Debt/GDP	0.0076	11.9050	0
	ESI	0.0040	-4.0536	7.467×10^{-5}
	Unemployment Rate	0.0356	-10.6000	0
6-Year	**Intercept**	0.8602	0.1106	0.9121
	Monetary Int Rates	0.0331	-6.8870	8.957×10^{-11}
	$/€	0.2939	4.5416	1.014×10^{-5}
	£/€	0.8205	-2.7512	0.0065
	IPI	0.0207	-0.1910	0.8487
	CPI	0.0223	-6.5609	5.381×10^{-10}
	Debt/GDP	0.0076	11.6600	0
	ESI	0.0040	-3.6938	2.920×10^{-4}
	Unemployment Rate	0.0357	-10.5990	0
7-Year	**Intercept**	0.8662	0.1184	0.9059
	Monetary Int Rates	0.0333	-6.8277	1.246×10^{-10}
	$/€	0.2959	4.2428	3.511×10^{-5}
	£/€	0.8263	-2.5216	0.0125
	IPI	0.0209	-0.2424	0.8088
	CPI	0.0224	-6.2159	3.395×10^{-9}
	Debt/GDP	0.0076	11.4290	0
	ESI	0.0040	-3.3994	8.298×10^{-4}
	Unemployment Rate	0.0360	-10.3710	0
8-Year	**Intercept**	0.8735	0.0545	0.9566
	Monetary Int Rates	0.0336	-6.7289	2.150×10^{-10}
	$/€	0.2984	4.0032	9.089×10^{-5}
	£/€	0.8332	-2.2953	0.0229
	IPI	0.0211	-0.2801	0.7798
	CPI	0.0226	-5.9186	1.581×10^{-8}
	Debt/GDP	0.0077	11.2430	0
	ESI	0.0040	-3.1924	1.662×10^{-3}
	Unemployment Rate	0.0363	-10.0250	0
9-Year	**Intercept**	0.8821	-0.0486	0.9613
	Monetary Int Rates	0.0339	-6.5915	4.559×10^{-10}
	$/€	0.3014	3.8096	1.902×10^{-4}
	£/€	0.8414	-2.1061	0.0366
	IPI	0.0213	-0.3421	0.7327
	CPI	0.0228	-5.6776	5.317×10^{-8}
	Debt/GDP	0.0078	11.1190	0
	ESI	0.0041	-3.0352	2.756×10^{-3}
	Unemployment Rate	0.0366	-9.6474	0
10-Year	**Intercept**	0.8914	-0.1710	0.8645
	Monetary Int Rates	0.0343	-6.4438	1.012×10^{-9}
	$/€	0.3046	3.6553	3.359×10^{-4}
	£/€	0.8503	-1.9398	0.0540
	IPI	0.0215	-0.3750	0.7081
	CPI	0.0231	-5.4889	1.343×10^{-7}
	Debt/GDP	0.0079	11.0190	0
	ESI	0.0041	-2.9099	4.067×10^{-3}
	Unemployment Rate	0.0370	-9.2599	0

Table 23: Standard Errors, t-Statistic, and p-Values for the Euro-Zone Macroeconomic Variables (5-Year to 10-Year)

Maturity	Parameter	Std Error	t-Stat	p-Value
3-Month	**Intercept**	0.8034	-0.2051	0.8378
	E\$/£	0.0310	-13.817	0
	E€/£	0.2763	8.0746	8.993×10^{-14}
	IPI	0.7698	-0.4401	0.6604
	CPI	0.0195	1.3098	0.1919
	Debt/GDP	0.0209	-8.9744	4.441×10^{-16}
	ESI	0.0071	2.3311	0.02084
	Unemployment Rate	0.0037	-0.9816	0.3276
6-Month	**Intercept**	0.7530	-1.9808	0.04913
	E\$/£	0.0291	-13.6500	0
	E€/£	0.2593	8.6224	3.109×10^{-15}
	IPI	0.7220	0.1582	0.8745
	CPI	0.0183	1.4273	0.1552
	Debt/GDP	0.0196	-5.7061	4.614×10^{-8}
	ESI	0.0067	3.5534	4.845×10^{-4}
	Unemployment Rate	0.0035	-0.8168	0.4151
1-Year	**Intercept**	0.7296	-3.5015	5.819×10^{-4}
	E\$/£	0.0282	-12.9100	0
	E€/£	0.2514	8.4894	7.105×10^{-15}
	IPI	0.7000	0.6244	0.5331
	CPI	0.0177	1.6158	0.1079
	Debt/GDP	0.0190	-4.8343	2.832×10^{-6}
	ESI	0.0065	4.5667	9.108×10^{-6}
	Unemployment Rate	0.0034	-0.8092	0.4195
2-Year	**Intercept**	0.6813	-4.4702	1.372×10^{-5}
	E\$/£	0.0264	-11.7480	0
	E€/£	0.2352	8.0041	1.375×10^{-13}
	IPI	0.6546	0.5717	0.5682
	CPI	0.0166	2.0017	0.0468
	Debt/GDP	0.0178	-6.2539	2.780×10^{-9}
	ESI	0.0061	5.7077	4.578×10^{-8}
	Unemployment Rate	0.0032	-0.8362	0.4041
3-Year	**Intercept**	0.6694	-5.2272	4.688×10^{-7}
	E\$/£	0.0259	-10.1100	0
	E€/£	0.2313	7.7320	6.957×10^{-13}
	IPI	0.6435	0.3918	0.6957
	CPI	0.0163	2.1940	0.0295
	Debt/GDP	0.0175	-6.7124	2.354×10^{-10}
	ESI	0.0060	6.6280	3.737×10^{-10}
	Unemployment Rate	0.0031	-0.5813	0.5617
4-Year	**Intercept**	0.6743	-5.9201	1.569×10^{-8}
	E\$/£	0.0261	-8.6415	2.887×10^{-15}
	E€/£	0.2332	7.5302	2.277×10^{-12}
	IPI	0.6485	0.2918	0.7708
	CPI	0.0165	2.2583	0.02511
	Debt/GDP	0.0177	-6.3756	1.458×10^{-9}
	ESI	0.0060	7.3802	5.443×10^{-12}
	Unemployment Rate	0.0031	-0.2118	0.8325

Table 24: Standard Errors, t-Statistic, and p-Values for the British Macroeconomic Variables (3-Month to 4-Year)

Maturity	Parameter	Std Error	t-Stat	p-Value
5-Year	**Intercept**	0.6813	-4.4702	1.372×10^{-5}
	E$/£	0.0264	-11.7480	0
	E€/£	0.2352	8.0041	1.375×10^{-13}
	IPI	0.6546	0.5717	0.5682
	CPI	0.0166	2.0017	0.0468
	Debt/GDP	0.0178	-6.2539	2.780×10^{-9}
	ESI	0.0061	5.7077	4.578×10^{-8}
	Unemployment Rate	0.0032	-0.8362	0.4041
6-Year	**Intercept**	0.6694	-5.2272	4.688×10^{-7}
	E$/£	0.0259	-10.1100	0
	E€/£	0.2313	7.7320	6.957×10^{-13}
	IPI	0.6435	0.3918	0.6957
	CPI	0.0163	2.1940	0.0295
	Debt/GDP	0.0175	-6.7124	2.354×10^{-10}
	ESI	0.0060	6.6280	3.737×10^{-10}
	Unemployment Rate	0.0031	-0.5813	0.5617
7-Year	**Intercept**	0.6743	-5.9201	1.569×10^{-8}
	E$/£	0.0261	-8.6415	2.887×10^{-15}
	E€/£	0.2332	7.5302	2.277×10^{-12}
	IPI	0.6485	0.2918	0.7708
	CPI	0.0165	2.2583	0.02511
	Debt/GDP	0.0177	-6.3756	1.458×10^{-9}
	ESI	0.0060	7.3802	5.443×10^{-12}
	Unemployment Rate	0.0031	-0.2118	0.8325
8-Year	**Intercept**	0.6743	-5.9201	1.569×10^{-8}
	E$/£	0.0261	-8.6415	2.887×10^{-15}
	E€/£	0.2332	7.5302	2.277×10^{-12}
	IPI	0.6485	0.2918	0.7708
	CPI	0.0165	2.2583	0.02511
	Debt/GDP	0.0177	-6.3756	1.458×10^{-9}
	ESI	0.0060	7.3802	5.443×10^{-12}
	Unemployment Rate	0.0031	-0.2118	0.8325
9-Year	**Intercept**	0.6813	-4.4702	1.372×10^{-5}
	E$/£	0.0264	-11.7480	0
	E€/£	0.2352	8.0041	1.375×10^{-13}
	IPI	0.6546	0.5717	0.5682
	CPI	0.0166	2.0017	0.0468
	Debt/GDP	0.0178	-6.2539	2.780×10^{-9}
	ESI	0.0061	5.7077	4.578×10^{-8}
	Unemployment Rate	0.0032	-0.8362	0.4041
10-Year	**Intercept**	0.6743	-5.9201	1.569×10^{-8}
	E$/£	0.0261	-8.6415	2.887×10^{-15}
	E€/£	0.2332	7.5302	2.277×10^{-12}
	IPI	0.6485	0.2918	0.7708
	CPI	0.0165	2.2583	0.02511
	Debt/GDP	0.0177	-6.3756	1.458×10^{-9}
	ESI	0.0060	7.3802	5.443×10^{-12}
	Unemployment Rate	0.0031	-0.2118	0.8325

Table 25: Standard Errors, t-Statistic, and p-Values for the British Macroeconomic Variables (5-Year to 10-Year)

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