LUISS

Department of Business And Management
Master’s Degree in Corporate Finance

Chair of Risk Management

Comparison between risk metrics used in Internal Models for capital requirement of banks; VaR vs Expected Shortfall : An empirical analysis

Prof. Daniele Penza
Supervisor

Prof. Giacomo Morelli
Co-supervisor

Antonio Piccolo
759851
Candidate

Accademic Year 2023/2024

Alla mia famiglia,

a mamma, papà e Gaia.

Ai miei nonni Maddalena,

Giulia e Antonio.

A chi c'era in un oggi,

e ha continuato a starmi a fianco in ogni domani.

A chi mi vuole bene.

Ad Antonio.

INDEX

INTRODUCTION

The market risk in the current economic and financial enviroment, and if we consider the events of the last decades, is a crucial component for the banking istritutions and the investors.

It was born the urgency, after the Bretton Woods collapse and in particular after the GFC, to develop models that would have estimated with the minimum error the potential loss in a financial investment.

Two of risk models most used to estimate the potential loss are the VaR (Value at Risk) and ES (Expected Shortfall), two very similar metrics but with differences in the approaches and pratical implications.

The VaR estimates, on the basis of a confidence interval and a time horizon, the loss threshold in which the financial investment should not incurr in the 1-a% (Confidence Interval) of the cases.

If we choose a Confidence Interval of 99%, the result of the VaR99% will represent the loss threshold in which the financial investment should not incurr in the 99% of the cases.

The main problem of the VaR is the not consideration of the gravity of the potential losses over the threshold chosen and this can take to several mistakes and wrong estimations in particular in extreme market conditions.

For this reason, in particular after the GFC, the ES or C-VaR (Conditional VaR) gained prominence because considers the average of the losses over the VaR threshold. It affirms the ES as a stronger and more coherent risk measure than VaR in particular in extreme market conditions where the uncertainty of potential returns is higher.

In this thesis, we'll make a comparison between the two risk metrics used by banks, applying them to different Portfolios with different level of risk and so different characteristics in the returns distribution (one Porfolio will have the returns distribution more similar to a normal distribution while the other will have a returns distribution strongly non-normal).

In this way we could observe the differences between the two risk metrics in different market risk conditions.

In particular we could notice the VaR limits, the more coherence of the ES in the risk measurement and their application in different market contexts.

The Chapter 1 will give an overview on the two metrics and on the historical process that made them the most used metrics in the estimation of the market risk for the banking istitutions and investors underlying the theoretical concepts.

In the Chapter 2 we'll start the empirical analysis making an in-depth analysis of the collected data, more specific of the log-returns distributions of the assets that will compose the Portfolios for two different time series (2 and 5 years).

The Chapter 3 we'll show the different Porfolios built with the assets analyzed in the previous chapter and we'll be applied to each of them the two risk metrics (VaR and ES) showing the results.

To compute the VaR and ES we'll be used two different: The Historical Method, based on the past data, and the Montecarlo Method, based on a random scenario simulated.

The following thesis aims to be part of the practical debate on the effectiveness of risk measures, highlighting why ES is necessarily preferable to VaR.

We'll be also shown how the shape of a distribution could influence the VaR and ES results and so the risk of the potential losses.

CHAPTER 1

THE RISK: EVOLUTION OF THE REGULATION AND THE RISK MEASURES

1.1 Origins of risk management

When it comes the topic of financial markets, is important to highlight the concept of risk, so when the return of investment can be different from expectations of a investor. The risk is perceived by the savers when it manifests as a loss.

The urgence for the adoption of models for measurement and management of exchange rate risk started in 1973, with the collapse of Bretton Woods system based on fixed exchange rate, the transition to a fluctuating rate system and publication of the Black-Scholes options valuation formula: in that year there was a strong volatility in exchange rates and the development of instruments derivatives useful for risk management of foreign currency prices and interest rates. The number and complexity of liquidity and derivative instruments makes difficult to calculate the risky exposure of company portfolios.

The main risks to which financial institutions are subjected are:

- **Market risk:** market risk arises from unwanted movements in prices, interest rates, exchange rates, volatilities of options. An important extension of modern portfolio theory is the value at risk (VaR) techniques which historically represent the first step of risk management systems that estimates the probability of monetary loss.
- **Credit risk:** credit risk refers to the potential inability of a counterparty to meet its contractual commitments, is the the main risk factor for a bank.
- **Liquidity risk:** refers to situations where an owner of financial instrument has difficult to transfer the instrument promptly.
- **Operational risk:** is the risk that improper system processing or management operations or daily activities result in monetary losses. It includes losses that may occur in the event of failure of the control system,, inexperience of personnel, unstable or inadequate IT systems.
- **Settlement risk:** is the risk resulting from the failure of payment systems.

1.2 Market Risk Amendement (MRA)

In 1988 the Basel Committee agreed on the minum capital requirements for banks to address credit risk and market risk. The latter is an international consultative body, established in 1974 by the central banks of the G10 countries with the aim of defining a system of rules for Banking Supervision, to guarantee stability to the global financial system by formulating proposals so that they are accepted by states.

In 1980s a package of supervisory proposal was introduced for applying capital charges to market risks incurred by banks defined as risk of losses in on-and-off-balance-sheet positions arising from movements in market prices.

The market risk is formed by 5 categories not independent:

- **interest rates risk:** is referred to the impact of the interest rates fluctuations on the price of a financial instrument. The sensitivity of a financial instrument to the interest rates fluctuations is defined by the duration (financial life of the investment)
- **credit risk:** can be defined as default risk, so the unability for an institution to face its financial commitments or spread risk that depends on changes of probability of default or belonging to rating class
- **exchange rate risk:** is referred to the exposition of exchange rates movements of the financial instruments owned in a porfolio and denominated in foreign currency
- **equity prices risk:** is referred to potential adverse price fluctuations of financial instruments exchanged on the market
- **commodity prices risk**

The set of proposal was a planned supplement to the basel capital accord of July 1988 and in 1996 became and amendment to the capital accord **(Basel I Market Risk Amendment).**

The Market Risk Amendment refers to changes made to **the Basel II framework** to enhance the treatment of market risk.

The main feature of the amendment was the request to allow banks to use proprietary in-house models for measuring market risks, so called **Internal Models.**

So to compute the market risk capital requirments was possible use the Standardized approach (SA) or the Internal Model Approach (IMA).

In the amendement was also introduced the trading book that involves equity and interest rate risk.

1.3 Basel II

The financial institutions that use the Standardized Approach compute their market risk regulatory capiral summing risk capital requirements across risk categories with a building block approach.

Thanks to Basel II (implementation in 2007), banks could choose if use or not proprietary models but had to respect some quantitative and qualitative criteria proposed by the **Basel Committee on Banking Supervision (BCBS)**, and relied on bank's own internal risk management model trying to improve the standardized approach.

The **qualitative criteria** to meet to use a model based approach were:

- independent risk control unit
- unit has to conduct a regular back-testing program
- involvment of board of directors and senior management in the process, review of daily reports
- bank's internal risk management model integrated into day to day risk management process of the bank
- rigourous program stress testing
- risk measurement system in conjuction with internal trading and documented through a risk management manual

If these criteria were meet, the bank could choose to use a proprietary model for the computation of the capital charge.

Other important innovations of Basel II regard the introduction of required capital for operational risk in addition to credit and market risk.

Two new pillars about supervision were also added to the previous one about minimum capital requirments:

Pillar 2: capital and risk management, is concerned with supervisory review process (SREP) that covers aspects of how risk can be managed within a bank. The Pillar 2 describes how SREP can be used to determine that banks meet minimum capital requirments.

Pillar 2 covers all risk of Pillar 1 and add some considerations, in particular the creation of a governance structure within the bank to ensure internal supervision, Internal capital adequacy and assessment processes (ICAAP) so the evaluation by banking supervisors of the bank's own risk profile.

Pillar 3: required public disclosures(qualitative and quantitative), it represents an increase in publicly available information by banks around its capital structure and risk management.

1.4 VaR: the Internal Model used to compute the regulatory capital

The model used by financial institutions to compute the regulatory capital was the **V**aR with a certain confidence level in a defined time horizon (usually the Confidence Interval is 99% computed daily considered also a schock of 10 days movement in prices(holding period)).
Banks using this approach to determine their regulatory capital requirments would generally have lower regulatory minimum capital than those using Standardized Approach.
J.P. Morgan was the bank that used VaR to compute its portfolio risk for first time in 1990, so is considered first bank that made VaR a known measure.
It provides a single numerical estimate of the maximum potential loss that a portfolio or institution could suffer over a specified time horizon, at a certain level of confidence.

$$VaR_{(1-a)}(X) = MPL_{(1-a)}(X) - E(L)$$

Where MPL is the Maximum Potential Loss at a given Confidence Interval and E(L) is the expected loss of a portfolio in a given time horizon.
The value 1-a, represents the probability that a financial institution can occur in a higher loss than the loss defined by the VaR (figure 1).

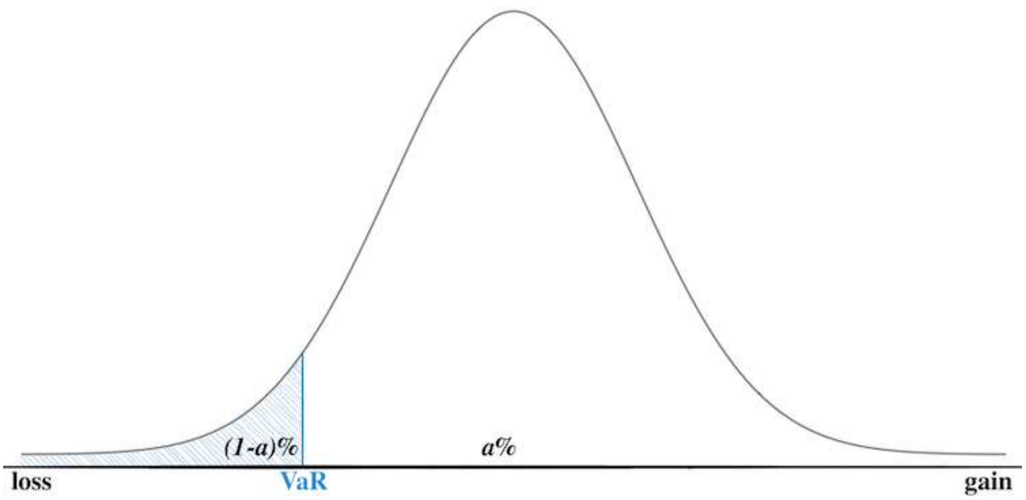


Figure 1: Var in a probability density function

So considering a distribution function Fx of losses in a 10 days period and 1-a the confidence level:

$$VaR_{(a)}(X) = Fx^{-1}(1-a)$$

So the density function *fx* of losses will be represented in this way

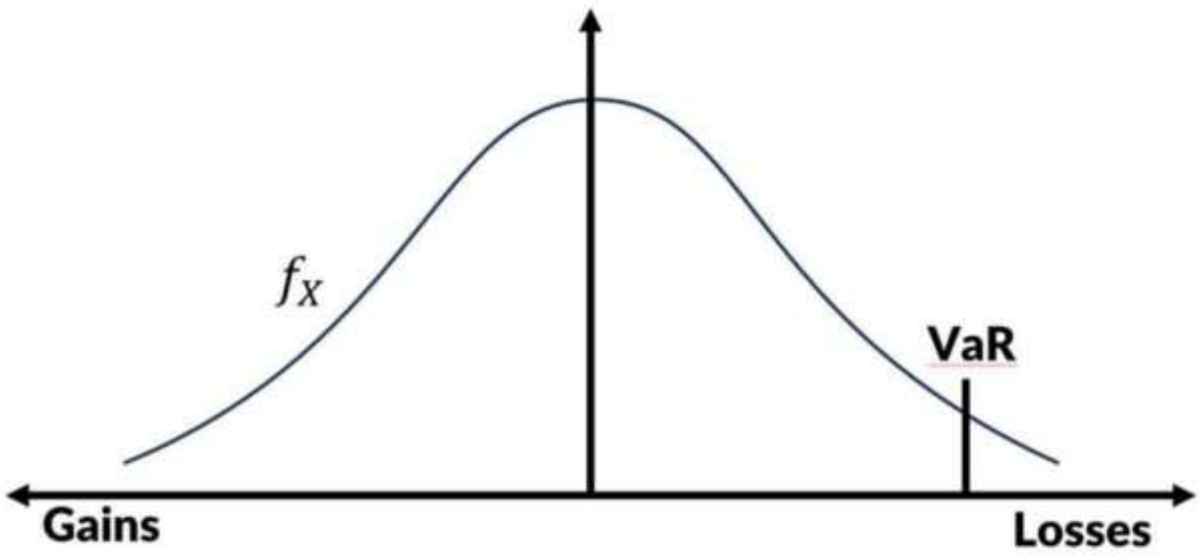


Figure 2: Visualization of the VaR

The Capital Charge was the highest between the previous day's VaR and the daily average of VaR preceeding sixty business days multiplicated for a factor (it was equal minimum 3 + value between 0 and 1 (so maximum 4)). The definition of the value between 0 and 1 is based on ex-post performance of the model (necessary to mantain a good predictive quality of the model, if model works the multiplicator remains 3).

$$Max\ (VaR(t-1)\ ;\ (3-4) * avg.VaR\ last\ 60\ days)$$

VaR is a measure which aggregates different components of market risk into a single number like interest rate risk , stock price risk , currency risk, exchange rate risk.

In the risk management has represented a significant step forward compared to more traditional measures based mainly on sensitivity to market variables.

The VaR computation on financial portfolios meets problems in the following cases:

- include in the analysis the entire set of risk variables that influence a portfolio
- estimate the probabilities of future market events

The calculation of the VaR can't be split into separate sub-calculations due to double non-additivity of the VaR, infact the non-additivity shows its ability to show diversification advantages of financial instruments and risk factors: is sensitive to hedging of different positions and to effect of risk factors correlation.

VaR can be calculated using various methodologies, including **Historical simulation**, **Parametric**, and **Monte Carlo simulation**.

- **Historical simulation:** involves estimating VaR based on past market movements, using historical data to simulate potential future outcomes.

- **Parametric methods**: rely on statistical models to estimate VaR, such as the normal distribution assumption for asset returns. These methods are computationally efficient but may be sensitive to model assumptions and may not capture non-normalities in asset returns.VaR is computed as multiples of standard deviation of future losses.
- **Monte Carlo simulation:** involves generating numerous random scenarios based on specified probability distributions for asset returns and estimating VaR based on the distribution of simulated outcomes. While more computationally intensive, Monte Carlo simulation offers flexibility in modeling complex risk factors and dependencies. VaR is estimated after have analyzed all the distribution of future losses and excluding a percentage of worst events.

1.5 Basel II.5, the ES and the weakness of the VaR

The fact that VaR is a not coherent risk measure and with weaknesses in the risk estimation, was noticed since previous the 2000: risk professionals have been looking for a coherent alternative for four years since the appearance in 1997 of « Thinking Coherently » and « Coherent measure of risk » of Artzner: the gap between market practice and theoretical progress had widened greatly. These articles generated a istintive question so, which was the properties of a model that should have to be considered a sensible risk measure.

The definition of the axioms that characterized a "coherent risk measure", highlighted how the VaR, the measure adopted by all the banks and regulators, didn't respected all the axioms to be considered a coherent risk measure.

This sentence summarizes why despite the evident weaknesses of the VaR was not paid the proper attention to this problem: *"The fact that for years the class of coherent measures didn't exhibit any known specimen that shared with VaR its formidable advantages (simplicity, wide applicability, universality,...) led many practitioners to think that coherence might be some sort of optional property that a risk measure can or cannot display."*[1]

Market prices of financial assets fell sharply during 2007-2009, so capital charges under MRA were inadequate for the trading book risks revealed during financial crisis. Most banks computed VaR using historical simulations but in 2007 when volatility rose for many assets, VaR from historical simulations was slow to follow because most historical observations were from a low volatility period before the GFC.

In July **2009,** the Committee introduced the **Basel II.5** framework to improve the framework's risk coverage in certain areas and increase the overall level of capital requirements, with a particular focus on trading instruments exposed to credit risk.

For this reason **VaR** calculations were expanded to include a **stressed-VaR** component, was added **capital for incremental risk(IRC)** and **comprehensive risk capital requirements** were added for securitizations and related instruments.

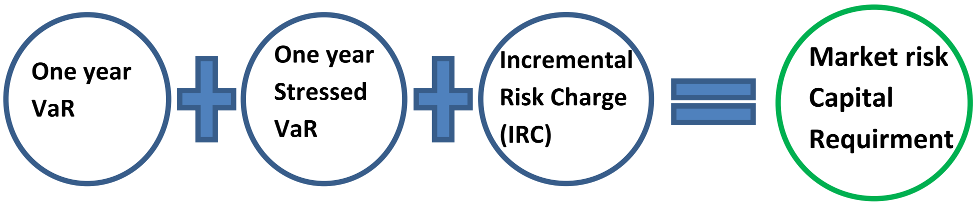
So was required to banks to compute two market risk VaRs and sum them to have the total market risk charge. The stressed VaR is computed using a 250-day period of stressed market conditions, so banks must identify a one year period of the last seven years where their portfolios performed poorly.

Incremental Risk Charge (IRC) attempts to capture the effects of low probabilities event occurring over long horizons in the trading book. IRC supplements existing Value-at-Risk (VaR) and captures the loss due to default and changes in credit quality at a 99.9% confidence level over a one year capital horizon.

For the banks holding a correlation trading portfolio has been introduced an additional metric so the **Comprehensive risk measure**, measure accounts for risks in correlation book including to securitization, re-securitizations and derivatives written on securitizations.

[1] Acerbi C. and Tasche D., « Expected Shortfall: a natural coherent alternative to Value at Risk », 2001

These changes were implemented by the end of 2011 **(Basel II.5).**



VaR (CI 99%, 10 days holding period)

$$Max\ (VaR(t-1)\ ;\ (3-4) * avg.VaR\ last\ 60\ days)$$

SVaR (CI 99%, 10 days holding period)

$$Max\ (SVaR(t-1)\ ;\ (3-4) * avg.SVaR\ last\ 60\ days)$$

ICR (CI 99,9%, 1 year holding period)

$$Max\ (ICR(t-1)\ ; avg.ICR\ last\ 60\ days)$$

Following the 2008 financial crisis, **Expected Shortfall (ES)**, also known as Conditional VaR or CVaR, gained prominence. The Fundamental Review of Trading Book (FRTB) represents an important change in how capital for market risk is computed, infact regulators decided to switch after many years of VaR 99% and 10 day time horizon to ES of 97,5% confidence level and varying time horizons. VaR99% and ES97.5% are considered two alternative risk metrics according to the capital adequacy bank regulation, as suggested by Basel III: moving from VaR99% to ES97.5% will affect the capital reqiuired to banks in terms of the quantity and quality.[2]

"*The Committee (Basel Commitee on Banking Supervision) believes that moving to a confidence level of 97.5% (relative to the* 99th *percentile confidence level for the current VaR measure) is appropriate.*"

While VaR provides a single-point estimate of potential losses, ES goes a step further by quantifying the average loss that would occur beyond the VaR threshold.
It means that **ES takes into account the severity of losses in extremity market conditions.**
Infact ES tried to respond to the following question:

"Which is the expected loss in the a% of worst cases of the portfolio?"

This makes ES a more comprehensive measure of downside risk, particularly in capturing the tail end of the distribution where extreme events occur, and a more coherent risk measure for internal model approach. ES

[2] Chia-Lin Changa, Juan-Angel Jimenez-Martinb, Esfandiar Maasoumic, Michael McAleerb, Teodosio Perez-Amaralb « Choosing expected shortfall over VaR in Basel III using stochastic dominance » , International Review of Economics and Finance, 2019

is able to provide a greater number of information regarding the potential loss that you will have to bear and respects the axioms following axioms:

- **Monotonicity:** if an asset A has better future returns than asset B, asset A is less riskier than asset B
- **Positive homogeneity:** if the investment on an asset increases, also the risk will increase so risk of an investment is proportional to its size.
- **Traslation invariance:** if is added to the investment X a certain component “a” (an investment in a risk free asset), the risk amount will reduce exactly of a.
- **Sub-additivity:** the risk of a portfolio with two assets is never higher than the sum of the risks of the two single assets

The fourth condition states that diversification helps to reduce risks, and the VaR doesn’t always respect it.

So ES is a better risk measure than VaR, infact VaR doesn’t respect the subadditivity axiom. Following the determination of VaR, the Expected Shortfall is computed by averaging losses that exceed the VaR providing a better assessment of potential lossess beyond a certain confidence level: for this reason is commonly used in finance and risk management, particularly in extreme market conditions. It helps investors and portfolio managers to better assess and quantify the downside risk, having a core role in the decision-making.The other essential inputs are:

- **The returns distribution shape**
- **Data periodicity**
- **Cut-off level**
- **Assumptions about the volatility of the securities**

The expected shortfall formula is:

$$ES = \frac{1}{a}\int_{-1}^{a} xp(x)dx$$

Where

- **p(x)dx** is the probability density of getting “x” return;
- **a** is the VaR cut-off point or breakpoint;
- VaR is the evaluated Value at the Risk level

So let's summarize in the table below the main differences between the two risk metrics of Value at Risk (VaR) and Expected Shortfall (ES).

	Expected Shortfall (ES)	**Value at Risk (VaR)**
Definition	Calculates magnitude of portfolio risk with loss that exceeds the VaR threshold	Represents highest potential loss within a specified confidence level
Interpretation	A comprehensive measure of risk that considers the worst possible conditions	Calculates the risk of portfolio without considering the magnitude of losses over the threshold
Sensitivity to extreme values	Computed with average of the losses in the tail	Doesn't give information about severity of losses over the VaR threshold
Properties	Is a coherent risk measure, is a convex and continuos function	No respect of subadditivity axiom, is a discontinous function
Risk management	Portfolio managers uses it to assess and report an investment portfolio's potential tail loss and risk.	Investors uses it to assess minimum potential loss

Figure 3: VaR and ES main differences

CHAPTER 2
DATA ANALYSIS

2.1 Data collection

In this chapter will be highlighted the difference between the two market risk models of VaR and ES with a practical analysis: first of all it'll be done a collection data of daily prices of ten assets listed in stock exchange of the last 2 years and 5 years with the target to compose some portfolios with different weights and see the different results of the two models taken into account.
The assets chosen for this analysis are the following ones:

Indexes

- S&P 500
- Dow Jones

ETF

- VTI (Vanguard Total Stock Market ETF)

BTP

- Btp1mz25 (IT451364)

Companies' stocks

- Apple
- Tesla
- Danone
- Bayer
- BNP Paribas

Commodities

- XAU/USD (Gold)

We have different investment instruments in this analysis also to analyze the different behaviour of varyous asset classes and to avoid potential risk concentrations, infact also the 5 companies belong to different sectors.

	Asset Class	Listed Market
S&P	Index	NYSE
DJ	Index	NYSE
BTP1mz25	BTP	MOT
VTI	ETF	NYSE
XAU/USD	Commodity	Gold spot market
Apple	Company's stock	NASDAQ
Tesla	Company's stock	NASDAQ
Danone	Company's stock	Euronext Paris
Bayer	Company's stock	FWB
BNP	Company's stock	Euronext Paris

Figure 4: Asset classes and listed markets

Apple	Technology
Tesla	Automotive and Energy
Danone	Consumer Goods
Bayer	Healthcare and Agriculture
BNP	Financial Services

Figure 5: Sectors of company's stocks

In first phase of the analysis have been **collected the daily prices of the last 2 years (04/04/2022-02/04/2024) and 5 years (02/04/2019-02/04/2024) of every asset.** The number of data collected for every asset, in the last 2 and 5 years, **are the same** as shown by the following tables;

S&P	501
DJ	501
BTP1mz25	501
VTI	501
XAU/USD	501
Apple	501
Tesla	501
Danone	501
Bayer	501
BNP	501

Figure 6: Data collected of every asset in the last 2 years

S&P	1259
DJ	1259
BTP1mz25	1259
VTI	1259
XAU/USD	1259
Apple	1259
Tesla	1259
Danone	1259
Bayer	1259
BNP	1259

Figure 7: Data collected of every asset in the last 5 years

2.2 Price Trend

After the data collection has been possible build some graphs and show the price trend of the last five years of every asset.

Figure 8: Price trend of S&P500 Index

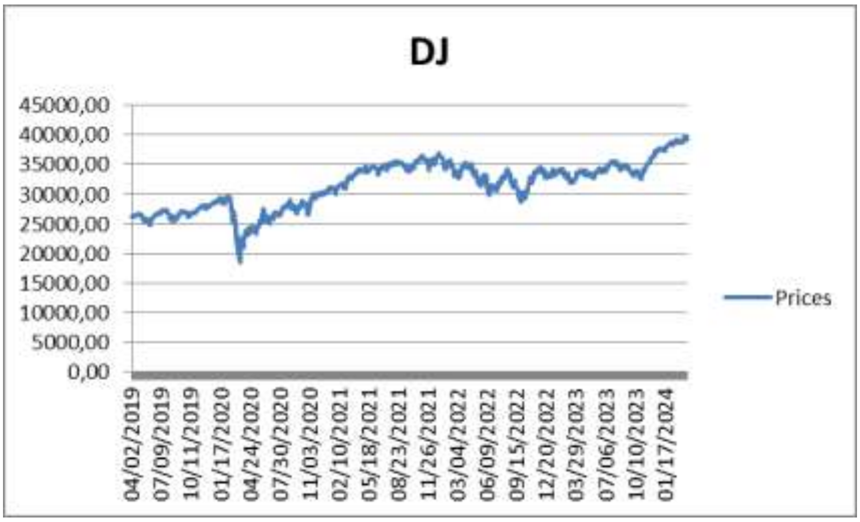


Figure 9: Price trend of DJ Index

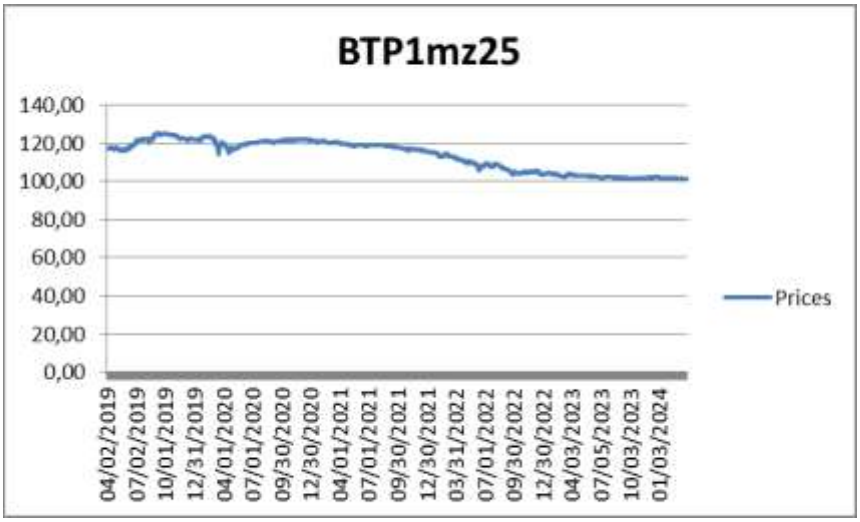


Figure 10: Price trend of BTP1mz25

Figure 11: Price trend of VTI ETF

Figure 12: Price trend of XAU/USD

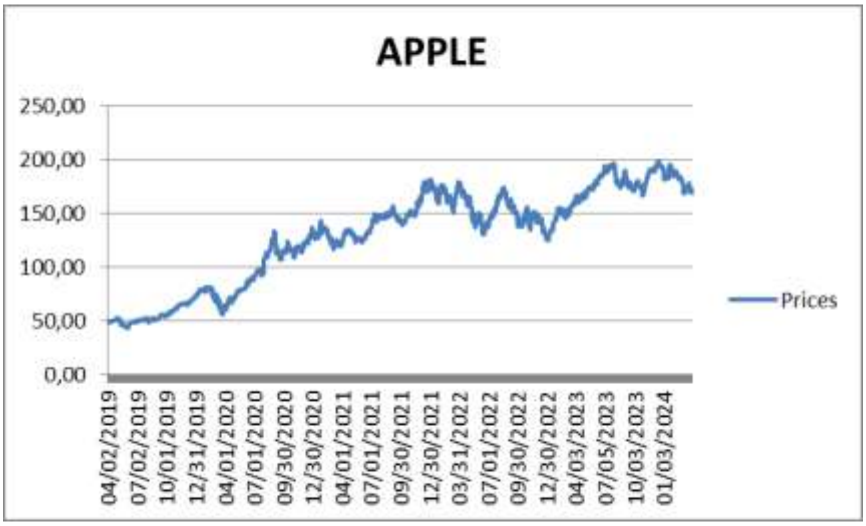


Figure 13: Price trend of APPLE stocks

Figure 14: Price trend of TESLA stocks

Figure 15: Price trend of DANONE stocks

Figure 16: Price trend of BAYER stocks

Figure 17: Price trend of BNP stocks

As we can see from the graphs, there are assets like the Italian BTP with a more stationary behaviour, some with an overall positive trend but not great stability during the last 5 years like TSLA and BNP and others with a overall negative trend like BAYER and DANONE.

The market indexes S&P and DJ and the ETF show a less sensitive growth in the price in the last 5 years but more stability than company’s stocks; so their price volatility is higher.

In the period of March 2020, due the pandemia of COVID-19, is clear a price decrease of the every asset despite the BTP.

2.3 Analysis of log-returns

Once we have collected the prices of the last 5 years, we compute the daily returns of the assets. For the analysis is very important to verify the normality of distribution of the prices of the assets chosen for the analysis;

Considering that the prices have a distribution with values from 0 to infinite, the most accurate way to estimate the normality of the prices is an analysis of the distribution of the log daily returns.

The log daily returns differently from the daily returns are better to use for long term analysis and can be summed linearly to have the return on a long period.

2.3.1 Log returns distribution compared to normal one

We'll represent the distribution of log returns of every asset in the next graphs doing a comparison between the log returns dsitribution of the last 2 and 5 years. For every distribution of log returns we'll be superimposed a normal curve with same mean and standard deviation of the distribution.

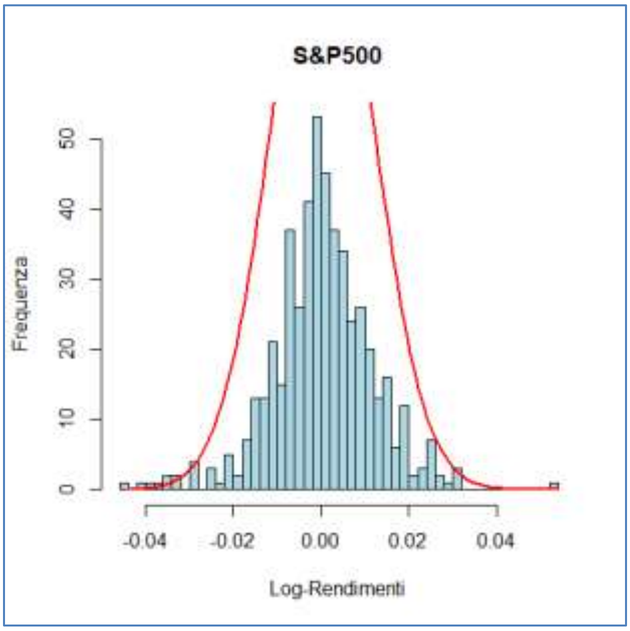


Figure 18: Log-returns distribution of the last 2 years of S&P

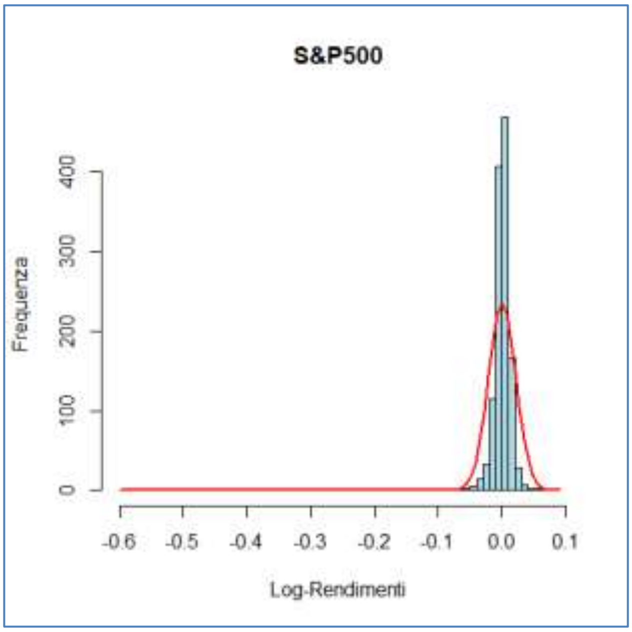


Figure 19: Log-returns distribution of the last 5 years of S&P

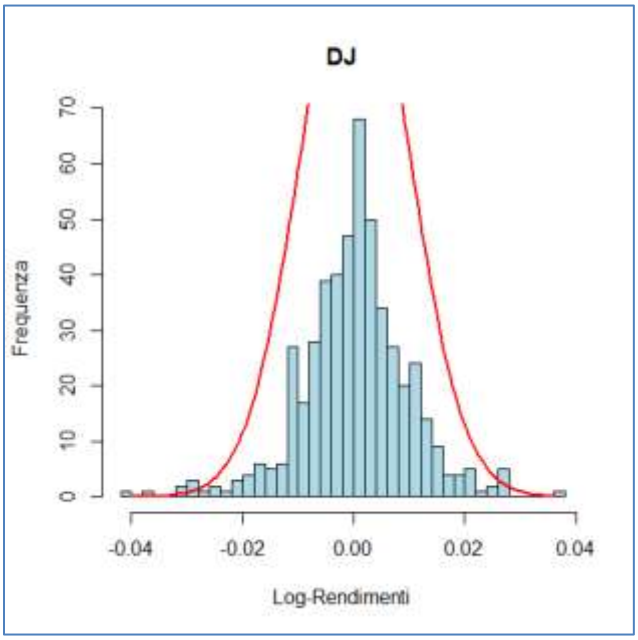


Figure 20: Log-returns distribution of the last 2 years of DJ

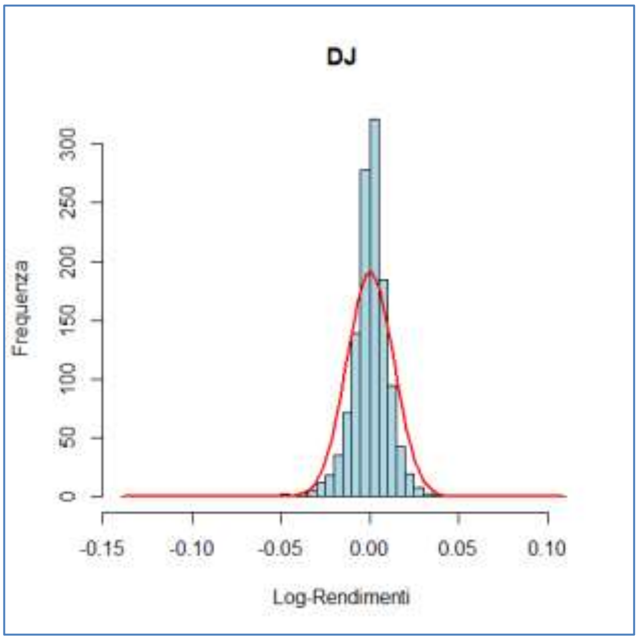


Figure 21: Log-returns distribution of the last 5 years of DJ

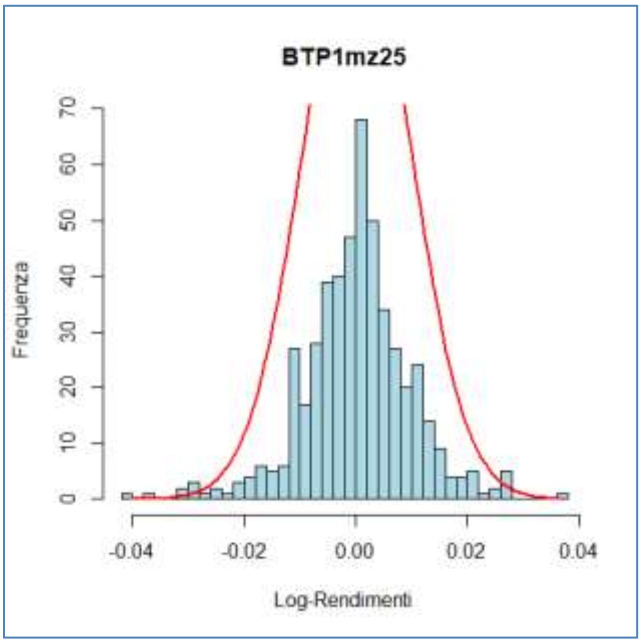


Figure 22: Log-returns distribution of the last 2 years of BTP

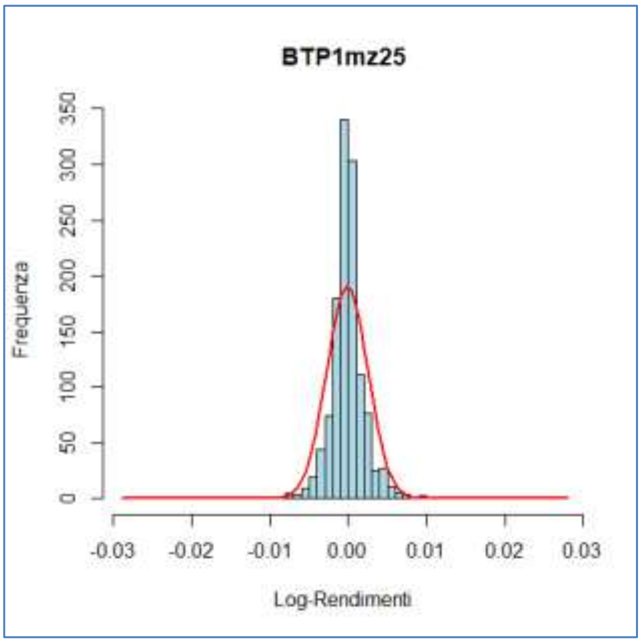
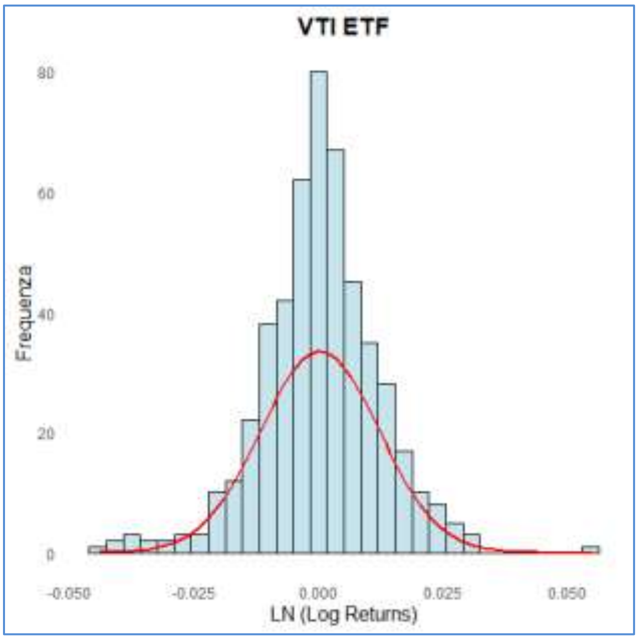


Figure 23: Log-returns distribution of the last 5 years of BTP



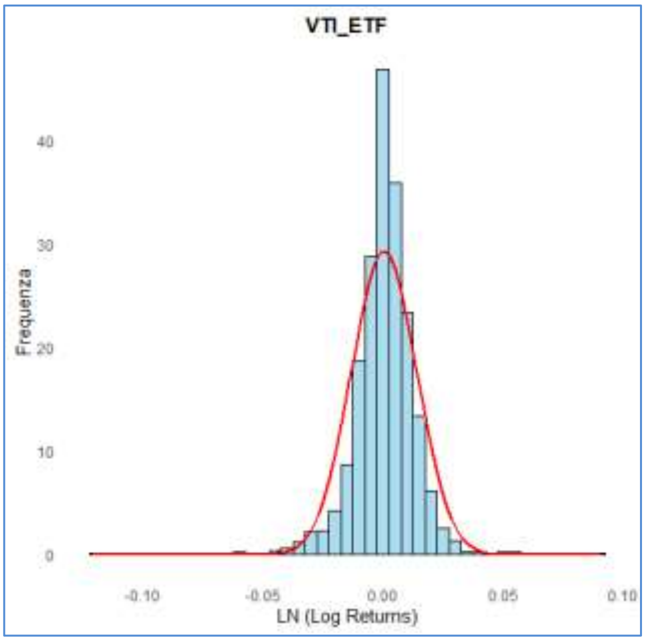
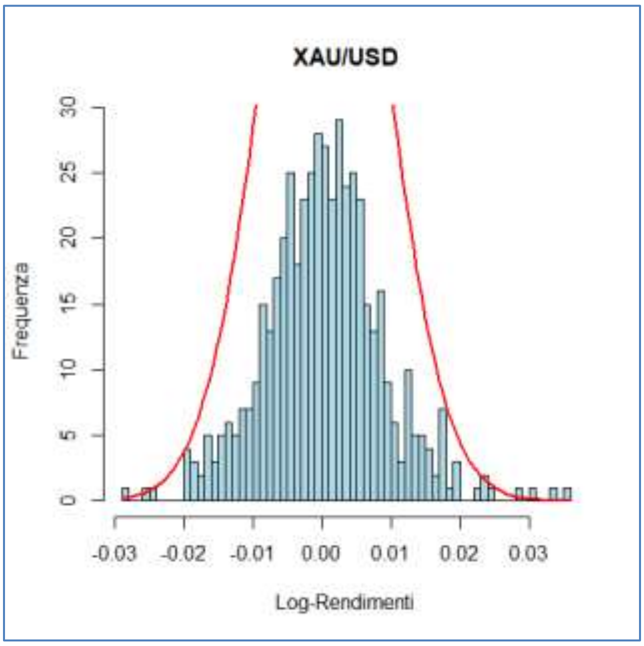


Figure 24: Log-returns distribution of the last 2 years of VTI Figure 25: Log-returns distribution of the last 5 years of VTI



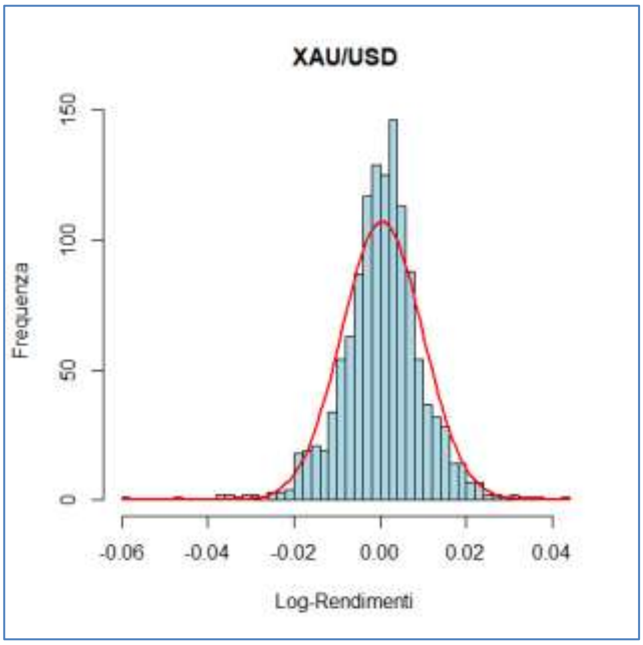


Figure 26: Log-returns distribution of the last 2 years of XAU/USD Figure 27: Log-returns distribution of the last 5 years of XAU/USD

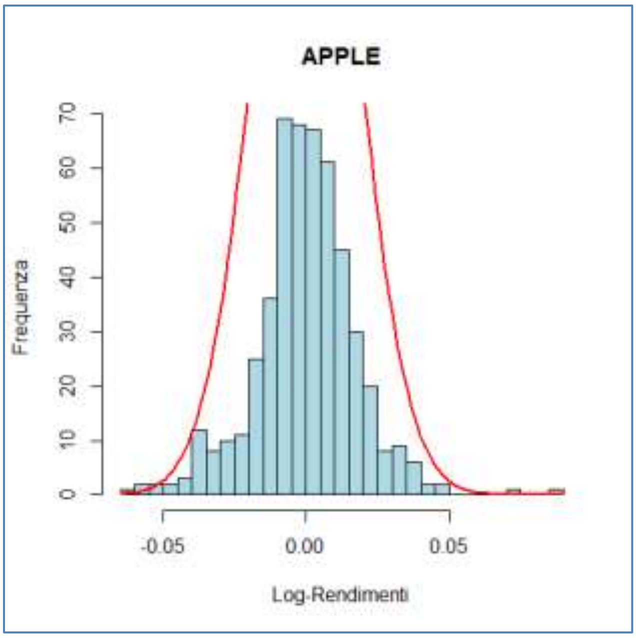


Figure 28: Log-returns distribution of the last 2 years of APPLE

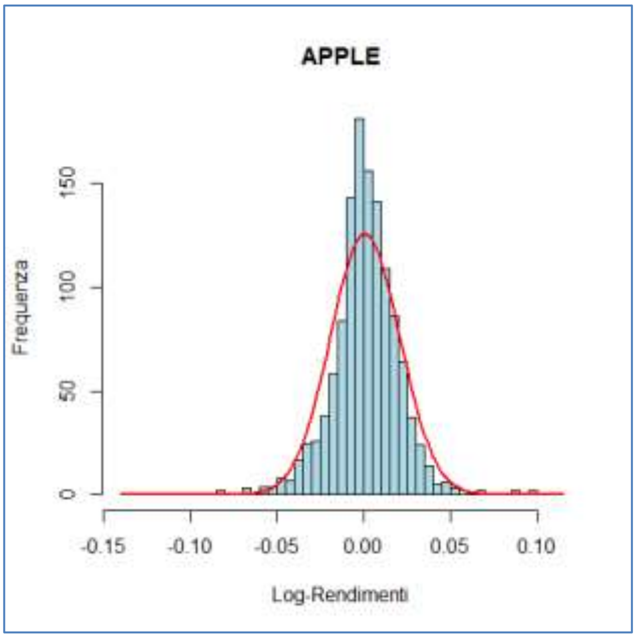


Figure 29: Log-returns distribution of the last 5 years of APPLE

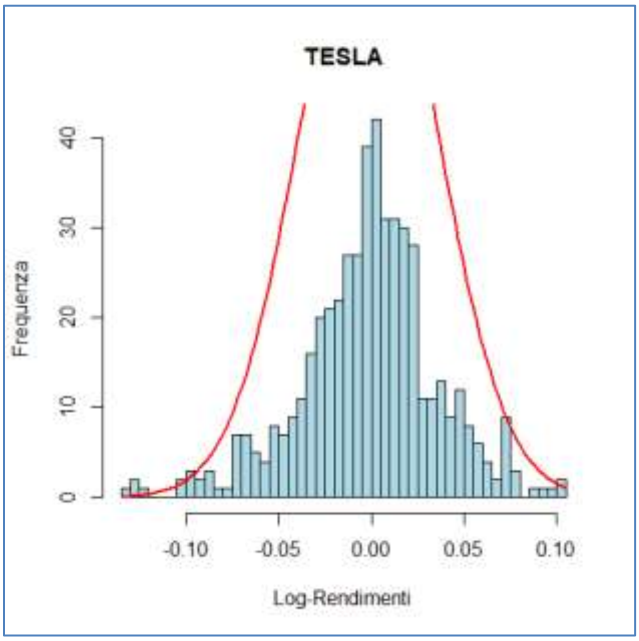


Figure 30: Log-returns distribution of the last 2 years of TESLA

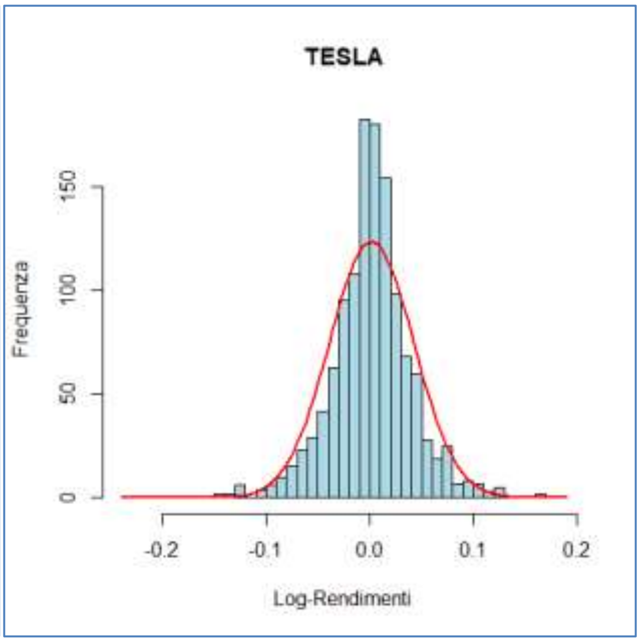
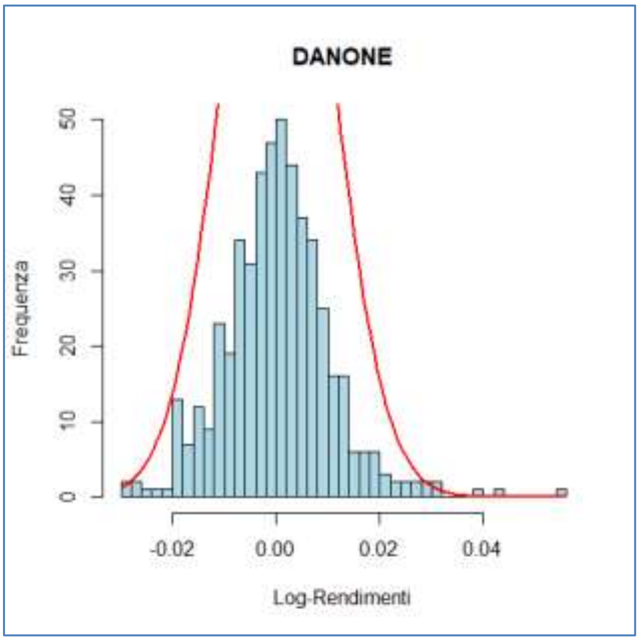


Figure 31: Log-returns distribution of the last 5 years of TESLA



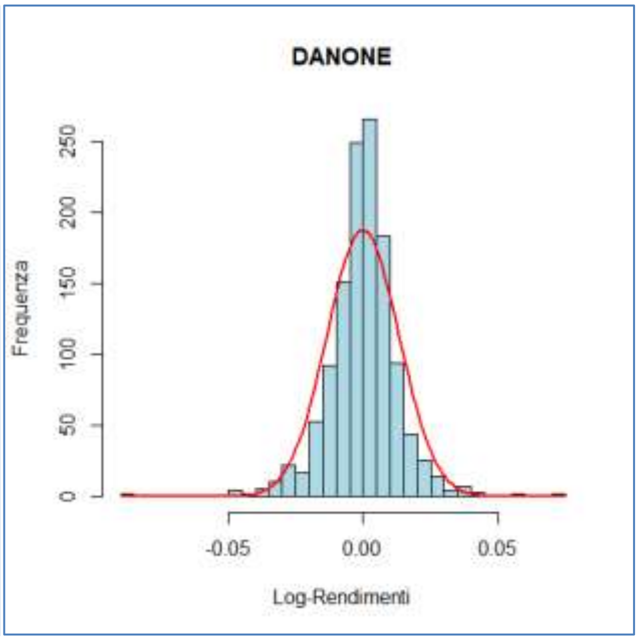
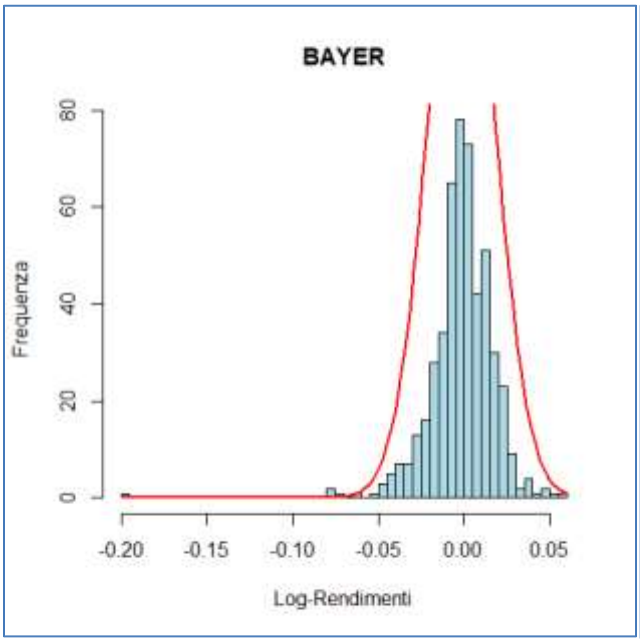


Figure 32: Log-returns distribution of the last 2 years of DANONE Figure 33: Log-returns distribution of the last 5 years of DANONE



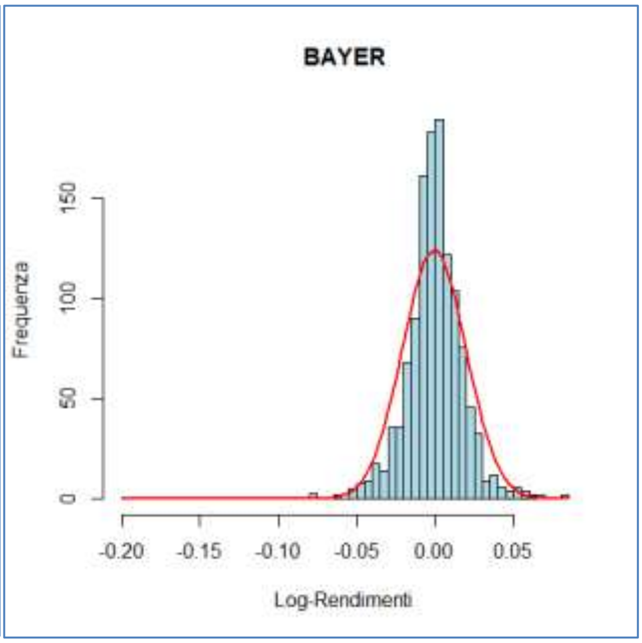


Figure 34: Log-returns distribution of the last 2 years of BAYER Figure 35: Log-returns distribution of the last 5 years of BAYER

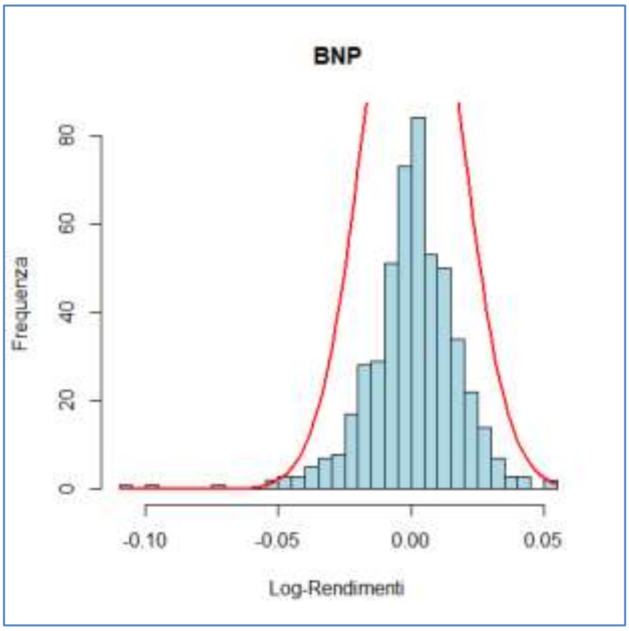


Figure 36: Log-returns distribution of the last 2 years of BNP

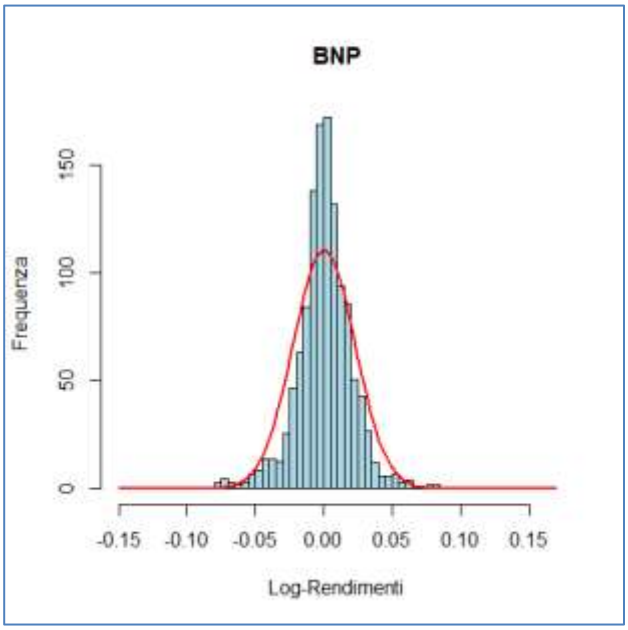


Figure 37: Log-returns distribution of the last 5 years of BNP

From the log returns distributions of the assets compared graphically with a normal curve with same mean and standard deviation of the distribution taken into account, we can observe some log returns distributions be more “similar” to a normal distribution like DJ, XAU/USD, APPLE, BAYER, BNP (in particular those of the log returns of the last 5 years, also if the most frequent values of the distribution are strongly more than a normal distribution), but by sight none of them really seems to come very close with a normal distribution.

2.3.2 QQ Plots

Before to check data that can be crucial indicator for the normality of a distribution , for greater accuracy we decided also to check the QQ-plots of the log-returns distributions of the last 2 and 5 years of the assets.

In the QQ-plots we'll see a red line that refers to the quantiles we would expect if the data followed a theoretical normal distribution; if the points of the distribution under analysis will be very near to the red line it means that the distribution would be a normal one.

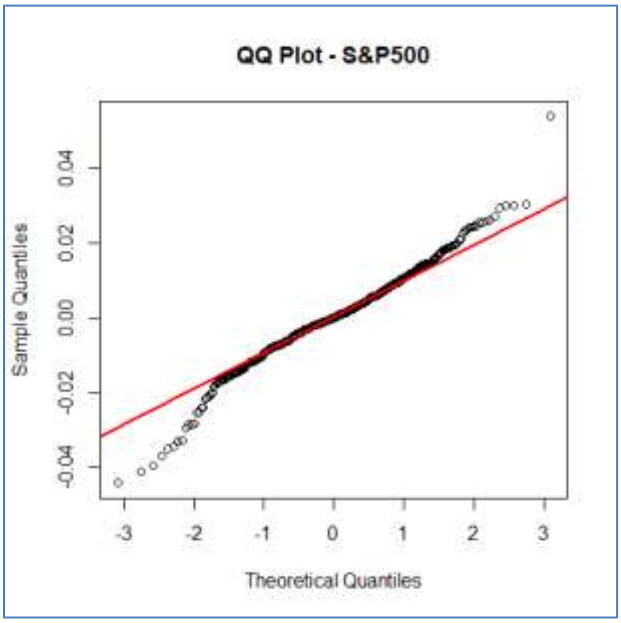


Figure 38: QQ-plot of log-returns of S&P of the last 2 years

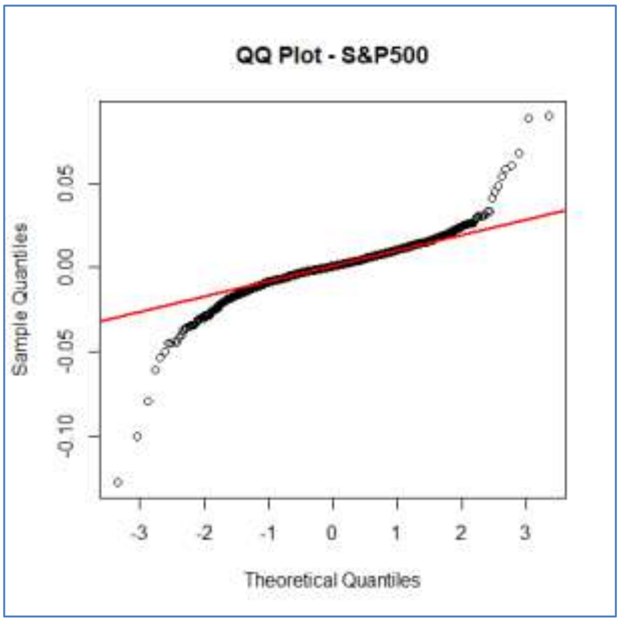


Figure 39: QQ-plot of log-returns of S&P of the last 5 years

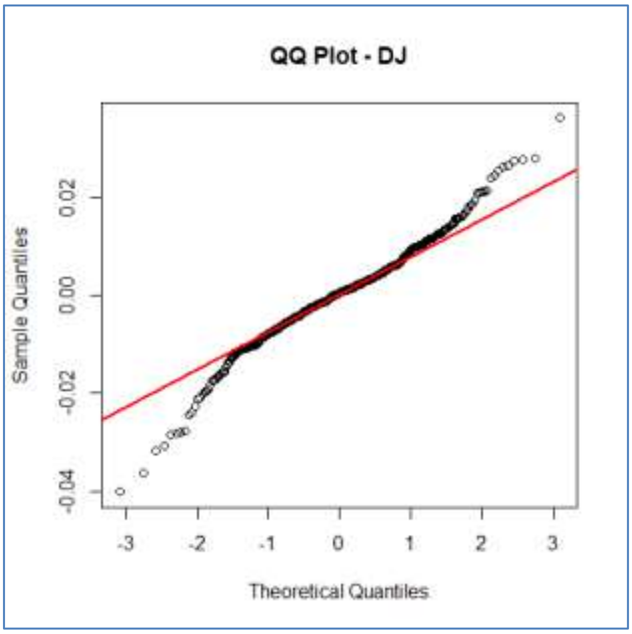


Figure 40: QQ-plot of log-returns of DJ of the last 2 years

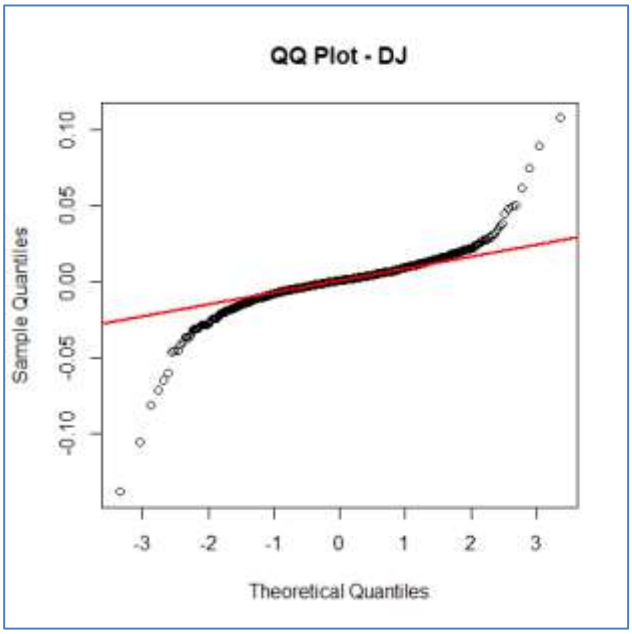


Figure 41: QQ-plot of log-returns of DJ of the last 5 years

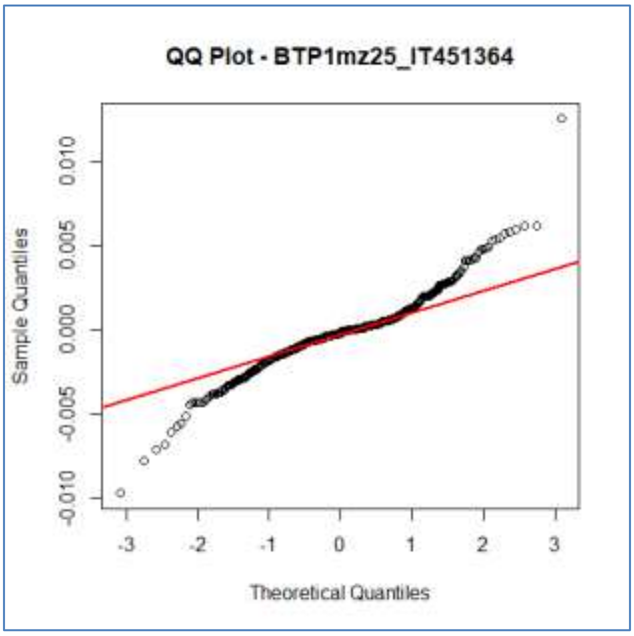


Figure 42: QQ-plot of log-returns of BTP of the last 2 years

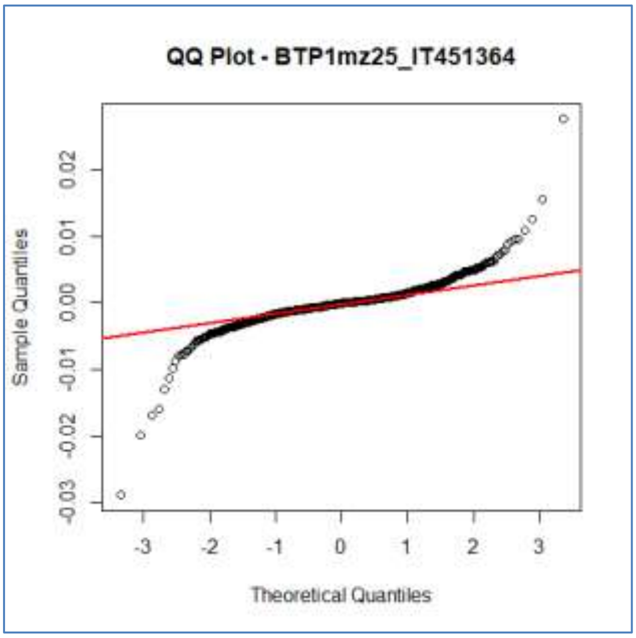


Figure 43: QQ-plot of log-returns of BTP of the last 5 years

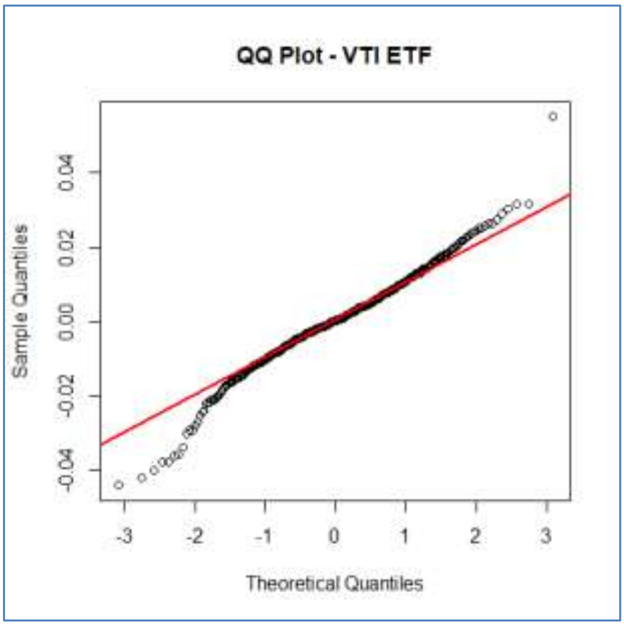


Figure 44: QQ-plot of log-returns of VTI of the last 2 years

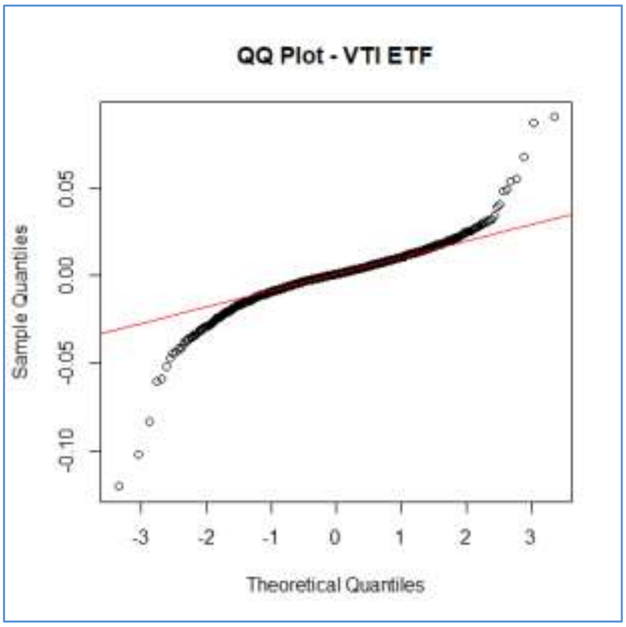


Figure 45: QQ-plot of log-returns of VTI of the last 5 years

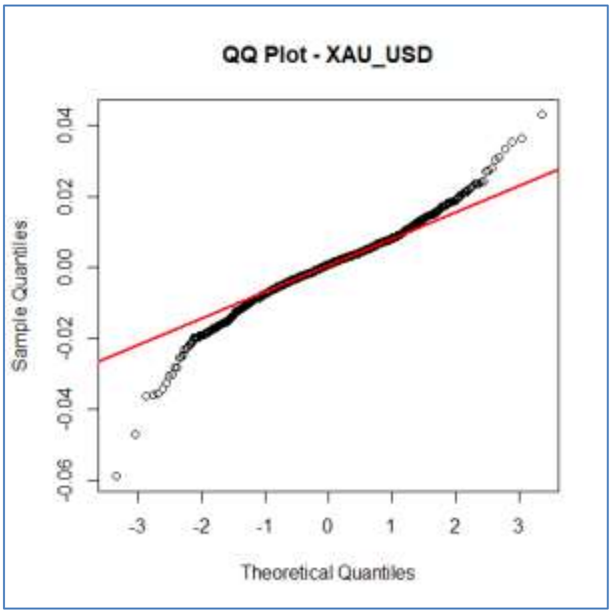


Figure 46: QQ-plot of log-returns of XAU of the last 2 years

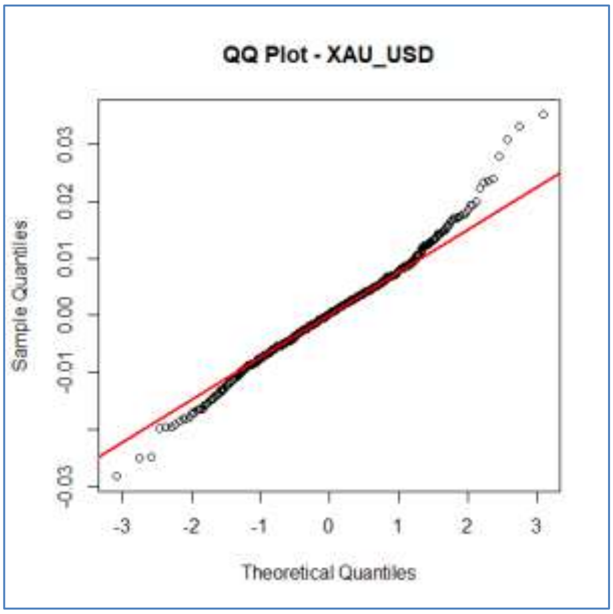


Figure 47: QQ-plot of log-returns of XAU of the last 5 years

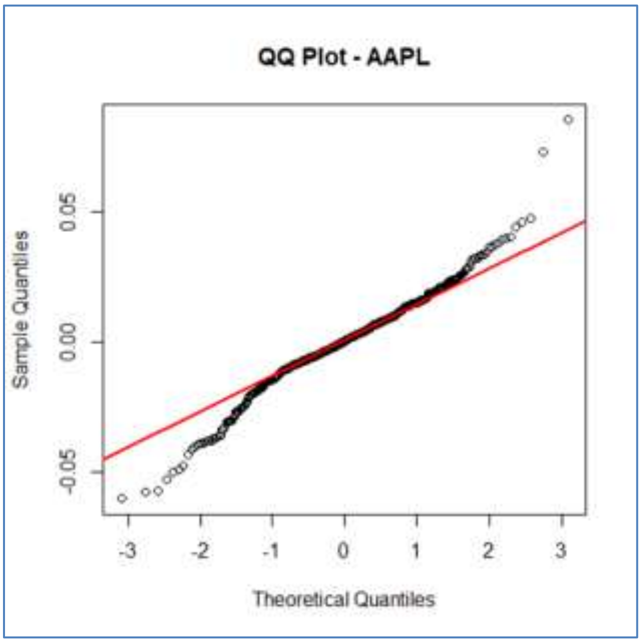


Figure 48: QQ-plot of log-returns of APPLE of the last 2 years

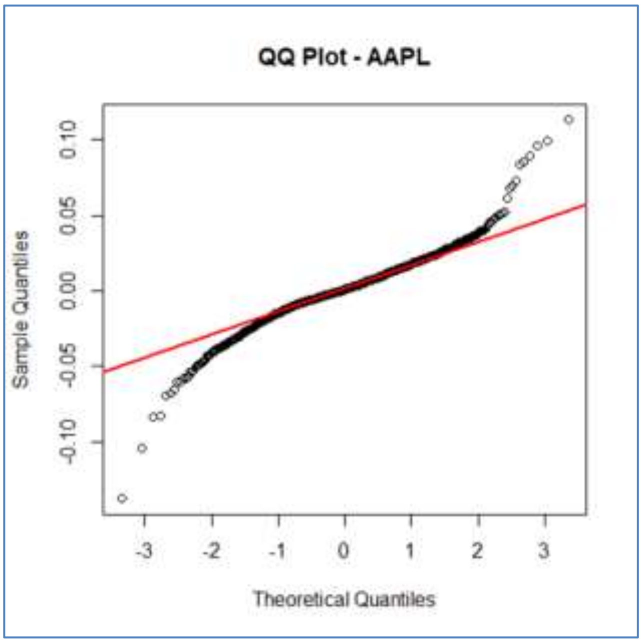


Figure 49: QQ-plot of log-returns of APPLE of the last 5 years

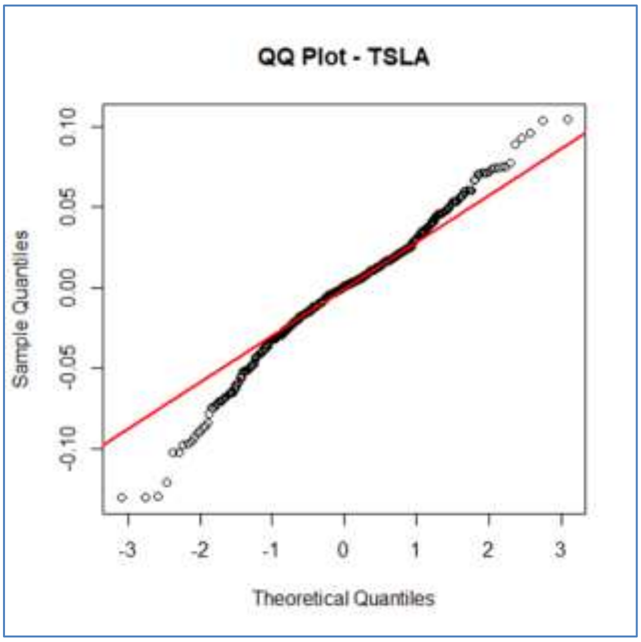


Figure 50: QQ-plot of log-returns of TESLA of the last 2 years

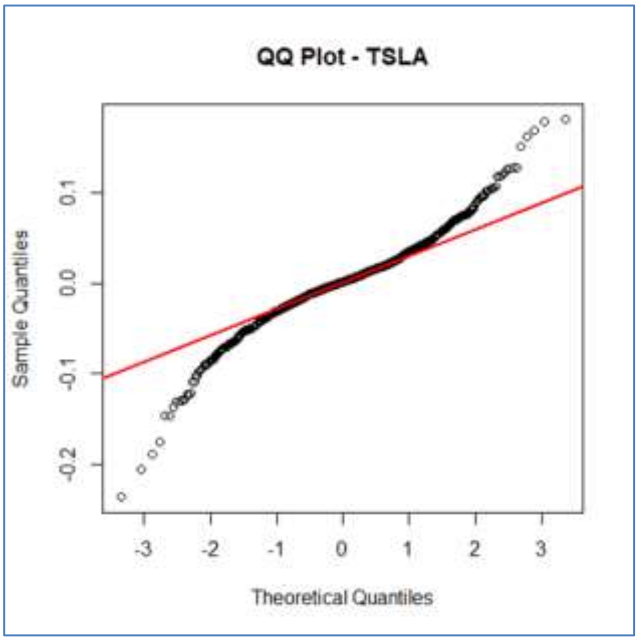


Figure 51: QQ-plot of log-returns of TESLA of the last 5 years

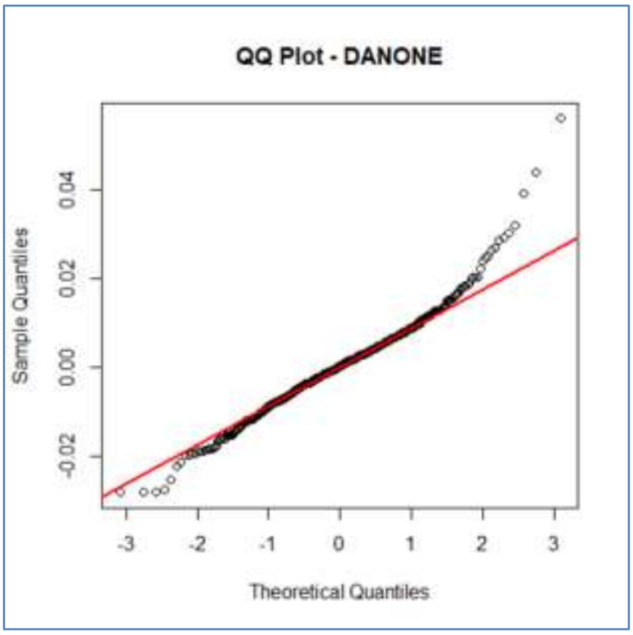


Figure 52: QQ-plot of log-returns of DANONE of the last 2 years

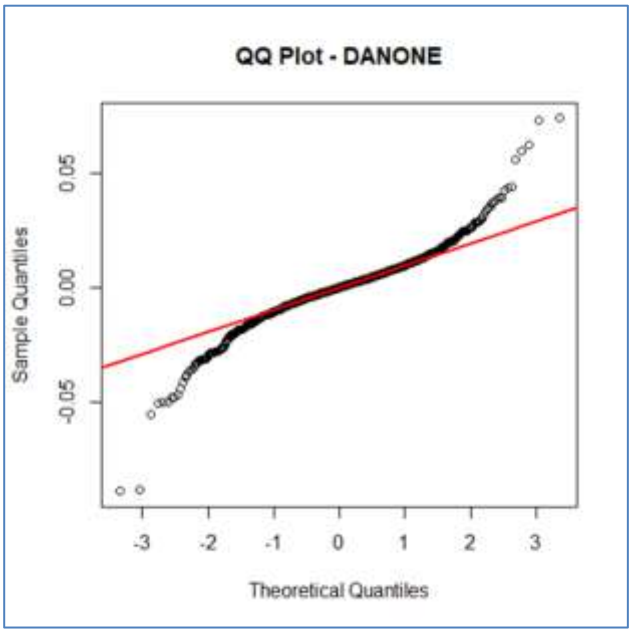


Figure 53: QQ-plot of log-returns of DANONE of the last 5 years

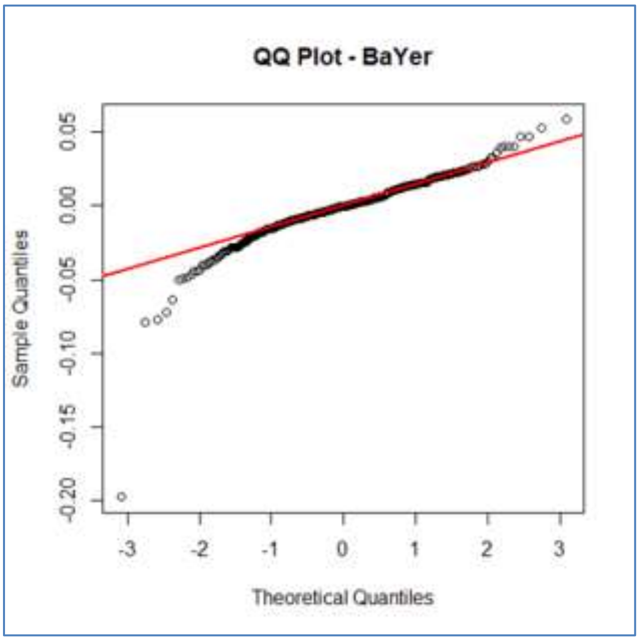


Figure 54: QQ-plot of log-returns of BAYER of the last 2 years

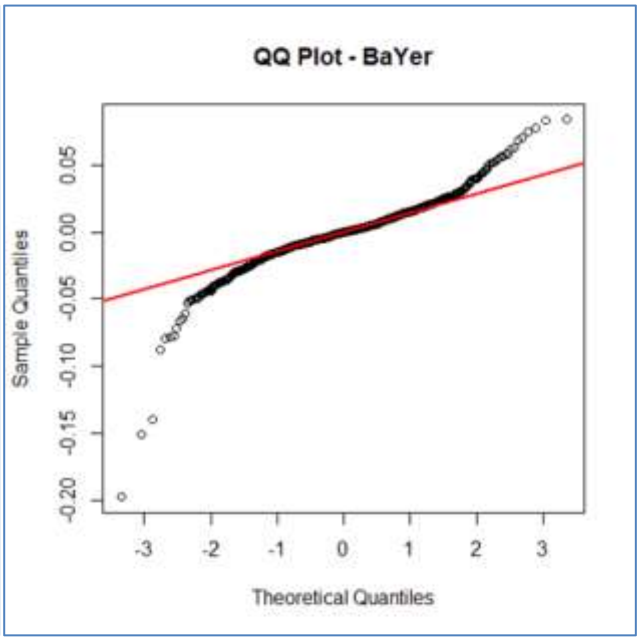


Figure 55: QQ-plot of log-returns of BAYER of the last 5 years

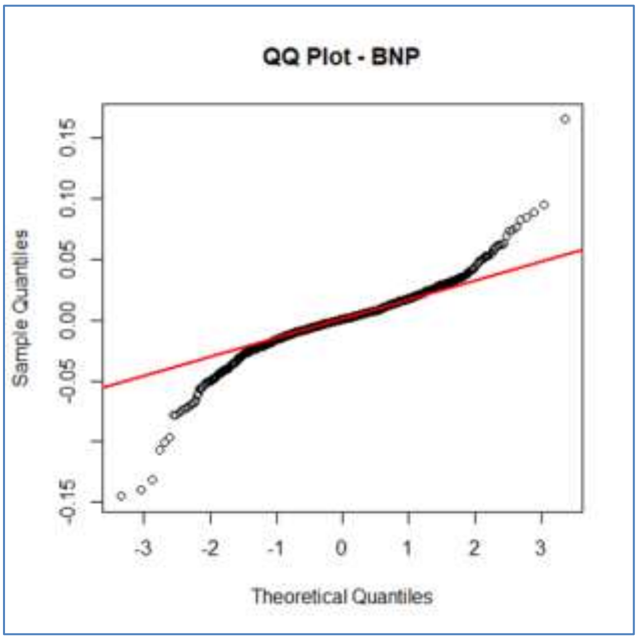


Figure 56: QQ-plot of log-returns of BNP of the last 2 years

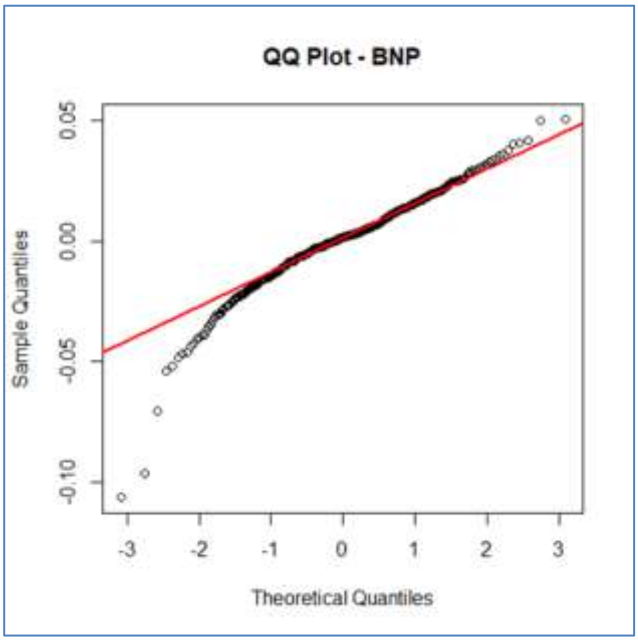


Figure 57: QQ-plot of log-returns of BNP of the last 5 years

The QQ-plots of log-returns distribution of assets under analysis indicate in some cases an evident discrepancy from the theoretical quantiles of the normal distribution; in both time series (2 and 5 years), is possible notice by sight the points far from the red line that represents the quantile of the theoretical normal distribution in particular in the upper and lower ends.

With this evidence we can confirm that the log-returns distribution of some assets under analysis can have heavier or lighters tails and higher or lower kurtosis than normal.

The log-returns distributions of the last 2 years of every asset seems to have lighter tails than 5 years one; TESLA,S&P and DANONE in particular show in the QQ-plots of the last 2 years a distribution with tails less heavy than QQ-plots of the other assets but is only a simple observation and could be contradicted by the following data.

Now we are going to show in a table the descriptive statistics about the log-returns distribution of the assets of the last 2 years.

Asset	N°	Min	Max	Mean	Median	Std.dev.	Kurtosis	Skewness
S&P	501	-4,42%	5,40%	0,027%	0,008%	1,158%	1,973	-0,174
DJ	501	-4,02%	3,63%	0,024%	0,049%	0,968%	1,896	-0,224
BTP	501	-0,98%	1,26%	-0,020%	-0,020%	0,211%	4,388	0,346
VTI	501	-4,39%	5,50%	0,024%	0,039%	1,193%	1,825	-0,197
XAU	501	-2,83%	3,52%	0,038%	0,038%	0,869%	1,254	0,262
APPLE	501	-6,05%	8,52%	-0,006%	0,030%	1,769%	2,117	-0,04
TESLA	501	-13,06%	10,44%	-0,155%	0,082%	3,682%	1,052	-0,357
DANONE	501	-2,82%	5,60%	0,032%	0,032%	1,026%	2,646	0,582
BAYER	501	-19,78%	5,83%	-0,162%	-0,065%	1,942%	21,488	-2,447
BNP	501	-10,66%	5,05%	0,047%	0,167%	1,749%	4,954	-1,084

Figure 58: Descriptive statistics of log returns distributions over the last two years

The number of observations, as shown before, are the same for every asset; from the data in the table, is important to underline that four assets in this time series have an average negative return: BTP, APPLE, TESLA and BAYER.

In particular we can notice that the kurtosis of the most assets, precisely eight, in this time series is so far from the value of three(value of kurtosis of a normal distribution).

The distribution with more extreme values and so with heavier tails is BAYER that has a distribution with a kurtosis of 21,488 followed by the BNP with a kurtosis of 4,954.

Also their skewness are the less similar to that of a normal distribution, respectively -2,447 for BAYER and -1,084 for BNP, indicating a negatively skewed distribution.

In other words, both distributions, in particular BAYER, exhibit a significant asymmetry with a long tail extending to the left side.

Regarding the skewness of other assets, it is always in a range between -0,224 and 0,582, so it means that all distributions can be approximate to a symmetric distribution

The asset with kurtosis values more similar to a normal distribution is DANONE with 2,646.

All others distributions (except for BTP with a kurtosis of 4,388), have a kurtosis equal to 2,1 or lower, showing lighter tails than a normal distribution and also a flatter peak, so present less extreme values.

This indicates that the distribution of log-returns of these assets is platykurtic.

The standard deviation of five assets are almost equal to 1, showing a dispersion around the mean similar to a normal distribution but assets like TESLA, BAYER, APPLE and BNP have a distribution of log-returns riskier than a normal distribution.

After have commented the descriptive statistics of the log-returns distributions of the assets in the last two years, we'll make the same with the log-returns distributions of the same assets but in the last five years.
In the table below are shown all the data.

Asset	N°	Min	Max	Mean	Median	Std.dev.	Kurtosis	Skewness
S&P	1258	-12,76%	8,97%	0,047%	0,085%	1,343%	14,536	-0,836
DJ	1258	-13,84%	10,76%	0,032%	0,075%	1,315%	21,870	1,014
BTP	1258	-2,89%	2,76%	-0,012%	-0,008%	0,265%	27,411	-0,525
VTI	1258	-12,08%	9,07%	0,045%	0,082%	1,359%	12,842	-0,838
XAU	1258	-5,90%	4,30%	0,045%	0,084%	0,938%	3,104	-0,367
APPLE	1258	-13,77%	11,31%	0,099%	0,076%	2,001%	5,237	-0,177
TESLA	1258	-23,65%	18,15%	0,172%	0,194%	4,075%	3,613	-0,237
DANONE	1258	-8,89%	7,43%	-0,012%	0,032%	1,338%	6,168	-0,273
BAYER	1258	-19,78%	8,38%	-0,059%	-0,035%	2,023%	12,697	-1,267
BNP	1258	-14,53%	16,55%	0,032%	0,095%	2,266%	7,413	-0,451

Figure 59: Descriptive statistics of log returns distributions over the last five years

Regarding these data, also here the number of data observed is the same for every asset, as shown in the previous chapter, and there are three assets with a negative average return in this time series: BTP, DANONE, BAYER.

In the period analysed there have been two main events that have schocked the stock markets so the Covid-19 pandemic in march 2020 and the war between Russia and Ucraina in 2022.

For this reason some assets have been affected by these events more than others showing important changes in the prices and returns as shown in the previous graphs.

In particular DJ presents in this time series, a kurtosis higher (21,87) than the previous time series of two years (1,89); this is evidently due to the events mentioned before that have generated shocks and instability in the stock markets. Infact in the last five years the asset presents a standard deviation not high (1,315%) but the high kurtosis indicates that, while the asset generally exhibits low volatility, it is also subject to rare but extreme deviations.

Also the other index S&P presents an high kurtosis (14,53) but a low std. Dev (1,34%) like the Italian BTP (27,411 and 0,265%), BAYER (12,697 and 2,023%) and VTI ETF (12,842 and 1,359%).

This is an important indication that we are approaching log-returns distributions absolutely non-normal; their tails are sensitive heavier due to the presence of more extreme values than a normal distribution.

The two assets in this time series with a kurtosis value more similar to a normal distribution are XAU with a kurtosis of 3,104 and TESLA with 3,613; XAU presents also a low standard deviation (0,938%) while TESLA is the second riskier asset in this time series (4,075%).

APPLE(5,237), DANONE(6,168) and BNP(7,413) have a kurtosis higher than a normal distribution but not extreme like the assets shown before, it means that the risk of extreme events is moderately high.

The skewness shows that almost all assets can be approximated to a symmetric distribution; BAYER and DJ with a skewness of -1 presents a negatively skewed distribution.

2.3.3 Normality test

In this section, after have analyzed the log-returns distributions with the graphs, qqplots and with the descriptive statistics, now we are going to use an apposite test that we'll check the normality of the distribution under analysis.

To make sure the results are as reliable as possible, we'll use three normality tests: the Jarque Bera test, the Shapiro-Wilk test and the Kolmogorov Smirnov test.

Now we'll show the results of the three normality tests performed on log-returns distributions of every asset for both time series (two and five years).

All three tests will show a p-value: this value should be higher of 0,05 to not refuse the null hypothesis to consider the distribution normal. If the value is equal or lower of 0,05 we'll refuse the null hypothesis and we can consider the distribution non-normal.

We'll show first the results of normality tests on log-returns distributions of last two years.

```
Risultati per il foglio: S&P500
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 80.218, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.054511, p-value = 0.1012
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.97593, p-value = 2.365e-07
```

Figure 60: Normality tests of log-returns distribution of last two years of S&P

The Jarque Bera and Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The Kolmogorov Smirnov test show a p-value of 0,1012 that is higher of 0,05; so this test doesn't refuse the normality hypothesis but is less sensitive than other two tests

Kolmogrov Smirnov test doesn't show a deviation from normality but Jarque-Brera and Shapiro-Wilk tests, both more sensitive to deviations from normality such as kurtosis and skewness, clearly indicate a non-normal distribution of the log-returns of the last two years of S&P500.

So the distribution can have some characteristics similar to a normal distribution but doesn't fully meet normality criteria.

```
Risultati per il foglio: DJ
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 77.589, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.057087, p-value = 0.07586
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.97403, p-value = 8.947e-08
```

Figure 61: Normality tests of log-returns distribution of last two years of DJ

The Jarque Bera and Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The Kolmogorov Smirnov test show a p-value of 0,07586 that is higher of 0,05; so this test doesn't refuse the normality hypothesis but is less sensitive than other two tests

Kolmogrov Smirnov test doesn't show a deviation from normality but Jarque-Brera and Shapiro-Wilk tests, both more sensitive to deviations from normality such as kurtosis and skewness, clearly indicate a non-normal distribution of the log-returns of the last two years of DJ.

So the distribution can have some characteristics similar to a normal distribution but doesnt- fully meet normality criteria.

```
Risultati per il foglio: BTP1mz25_IT451364
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 404.21, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.11725, p-value = 2.028e-06
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.93798, p-value = 1.309e-13
```

Figure 62: Normality tests of log-returns distribution of last two years of BTP

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last two years of BTP.

```
Risultati per il foglio: VTI_ETF
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 181.07, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.086292, p-value = 0.00115
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.9552, p-value = 3.376e-11
```

Figure 63: Normality tests of log-returns distribution of last two years of VTI

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last two years of VTI.

```
Risultati per il foglio: XAU_USD
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 37.475, df = 2, p-value = 7.284e-09


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.045284, p-value = 0.2547
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.98644, p-value = 0.0001269
```

Figure 64: Normality tests of log-returns distribution of last two years of XAU/USD

The Jarque Bera and Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The Kolmogorov Smirnov test show a p-value of 0,2547 that is higher of 0,05; so this test doesn't refuse the normality hypothesis but is less sensitive than other two tests.

Kolmogrov Smirnov test doesn't show a deviation from normality but Jarque-Brera and Shapiro-Wilk tests, both more sensitive to deviations from normality such as kurtosis and skewness, clearly indicate a non-normal distribution of the log-returns of the last two years of XAU/USD.

So the distribution can have some characteristics similar to a normal distribution but doesnt- fully meet normality criteria.

```
Risultati per il foglio: AAPL
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 91.795, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.071149, p-value = 0.01241
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.97235, p-value = 3.923e-08
```

Figure 65: Normality tests of log-returns distribution of last two years of APPLE

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last two years of APPLE.

```
Risultati per il foglio: TSLA
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 33.215, df = 2, p-value = 6.129e-08


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.05948, p-value = 0.05733
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.98089, p-value = 3.688e-06
```

Figure 66: Normality tests of log-returns distribution of last two years of TESLA

The Jarque Bera and Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The Kolmogorov Smirnov test show a p-value of 0,0573 that is higher of 0,05; so this test doesn’t refuse the normality hypothesis but is less sensitive than other two tests.

Kolmogrov Smirnov test doesn’t show a deviation from normality but Jarque-Brera and Shapiro-Wilk tests, both more sensitive to deviations from normality such as kurtosis and skewness, clearly indicate a non-normal distribution of the log-returns of the last two years of TESLA.

So the distribution can have some characteristics similar to a normal distribution but doesnt- fully meet normality criteria.

```
Risultati per il foglio: DANONE
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 170.7, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.052088, p-value = 0.1312
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.97015, p-value = 1.391e-08
```

Figure 67: Normality tests of log-returns distribution of last two years of DANONE

The Jarque Bera and Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The Kolmogorov Smirnov test show a p-value of 0,1312 that is higher of 0,05; so this test doesn’t refuse the normality hypothesis but is less sensitive than other two tests.

Kolmogrov Smirnov test doesn’t show a deviation from normality but Jarque-Brera and Shapiro-Wilk tests, both more sensitive to deviations from normality such as kurtosis and skewness, clearly indicate a non-normal distribution of the log-returns of the last two years of DANONE.

So the distribution can have some characteristics similar to a normal distribution but doesnt- fully meet normality criteria.

```
Risultati per il foglio: BaYer
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 9951.9, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.098075, p-value = 0.0001279
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.86433, p-value < 2.2e-16
```

Figure 68: Normality tests of log-returns distribution of last two years of BAYER

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last two years of BAYER.

```
Risultati per il foglio: BNP
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 600.35, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.078929, p-value = 0.003843
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.94232, p-value = 4.724e-13
```

Figure 69: Normality tests of log-returns distribution of last two years of BNP

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last two years of BNP.

Final considerations: we can affirm from the normality tests on log-returns distributions of the last two years that no distribution can be considered normal but some of them (S&P500,DJ,TESLA,DANONE and XAU) can have some characteristics similar to a normal distribution but aren't enough to meet normality criteria.

The others five log returns distributions (BTP,VTI,BAYER,BNP and APPLE) from the results of all three tests are strongly non-normal.

Now we'll show the results of normality tests on log-returns distributions of last five years.

```
Risultati per il foglio: S&P500
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 11565490, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.18972, p-value < 2.2e-16
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.42704, p-value < 2.2e-16
```

Figure 70: Normality tests of log-returns distribution of last five years of S&P

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last five years of S&P.

```
Risultati per il foglio: DJ
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 25075, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.11671, p-value = 2.665e-15
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.81791, p-value < 2.2e-16
```

Figure 71: Normality tests of log-returns distribution of last five years of DJ

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last five years of DJ.

```
Risultati per il foglio: BTP1mz25_IT451364
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 39115, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.12493, p-value < 2.2e-16
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.79745, p-value < 2.2e-16
```

Figure 72: Normality tests of log-returns distribution of last five years of BTP

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last five years of BTP.

```
Risultati per il foglio: VTI_ETF
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 37607, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.28389, p-value < 2.2e-16
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.53119, p-value < 2.2e-16
```

Figure 73: Normality tests of log-returns distribution of last five years of VTI

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last five years of VTI.

```
Risultati per il foglio: XAU_USD
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 527.68, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.055232, p-value = 0.0009283
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.96852, p-value = 6.667e-16
```

Figure 74: Normality tests of log-returns distribution of last five years of XAU/USD

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last five years of XAU/USD.

```
Risultati per il foglio: AAPL
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 1429.9, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.072995, p-value = 3.012e-06
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.94427, p-value < 2.2e-16
```

Figure 75: Normality tests of log-returns distribution of last five years of APPLE

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last five years of APPLE.

```
Risultati per il foglio: TSLA
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 688.8, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.067961, p-value = 1.796e-05
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.9549, p-value < 2.2e-16
```

Figure 76: Normality tests of log-returns distribution of last five years of TESLA

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last five years of TESLA.

```
Risultati per il foglio: DANONE
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 1990.7, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.074895, p-value = 1.486e-06
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.92881, p-value < 2.2e-16
```

Figure 77: Normality tests of log-returns distribution of last five years of DANONE

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last five years of DANONE.

```
Risultati per il foglio: BaYer
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 8712.4, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.090716, p-value = 2.037e-09
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.8972, p-value < 2.2e-16
```

Figure 78: Normality tests of log-returns distribution of last five years of BAYER

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last five years of BAYER.

```
Risultati per il foglio: BNP
Jarque-Bera Test:

        Jarque Bera Test

data:  log_returns
X-squared = 2896.1, df = 2, p-value < 2.2e-16


Kolmogorov-Smirnov Test:

        Asymptotic one-sample Kolmogorov-Smirnov test

data:  log_returns
D = 0.087361, p-value = 9.156e-09
alternative hypothesis: two-sided


Shapiro-Wilk Test:

        Shapiro-Wilk normality test

data:  log_returns
W = 0.9156, p-value < 2.2e-16
```

Figure 79: Normality tests of log-returns distribution of last five years of BNP

The Jarque Bera, Kolmogrov Smirnov, Shapiro-Wilk tests show a p-value extremely low, this takes us to refuse the normality hypothesis.

The tests clearly indicate a non-normal distribution of the log-returns of the last five years of BNP.

Final considerations: we can affirm from the normality tests on log-returns distributions of the last five years that no distribution can be considered normal.

All log returns distributions from the results of all three tests are strongly non-normal.

CHAPTER 3

APPLICATION OF MARKET RISK MODELS ON DIFFERENT PORTFOLIOS

In this chapter, we are going to build five different portfolios, three for one time series (last two years) and two for the other one (last five years) with the ten assets analyzed in the previous chapter.

Starting from an equally weighted portfolio (**Portfolio 1 and Portfolio 1.2**) for both time series, we will differently weight the assets to build other three portfolios (two for one time series and one for the other one) on the basis of the data collected so the descriptive statistics (kurtosis,standard deviation…) and normality tests (Jarque Brera, Kolomogrov-Smirnov, Shapiro Wilky).

In a second portfolio (**Portfolio 2 and Portfolio 2.2**) we'll give more weight to the assets that from the data collected have the descriptive statistics and the distribution of log-returns more similar to a normal distribution; while in a third portfolio (**Portfolio 3**) the assets more weighted will have the descriptive statistics and the distribution of log-returns strongly different from a normal one.

For every portfolio built will be computed the value of the one-day VaR and ES for three different confidence intervals (0.95, 0.975, 0.99) with two different methods: the **Historical method** and the **Montecarlo method.**

Is not possible apply the Parametric method because the distributions of log-returns of the assets taken into account don't follow a normal distribution; infact the formula ($z*\sigma-\mu$) used in the parametric method and some critical values (z) derive from a normal distribution.

If the log-returns are not normal, extreme events could be more frequent than a normal distibution and this could take to inaccurate estimation.

So the crucial assumption to apply the parametric method is not met.

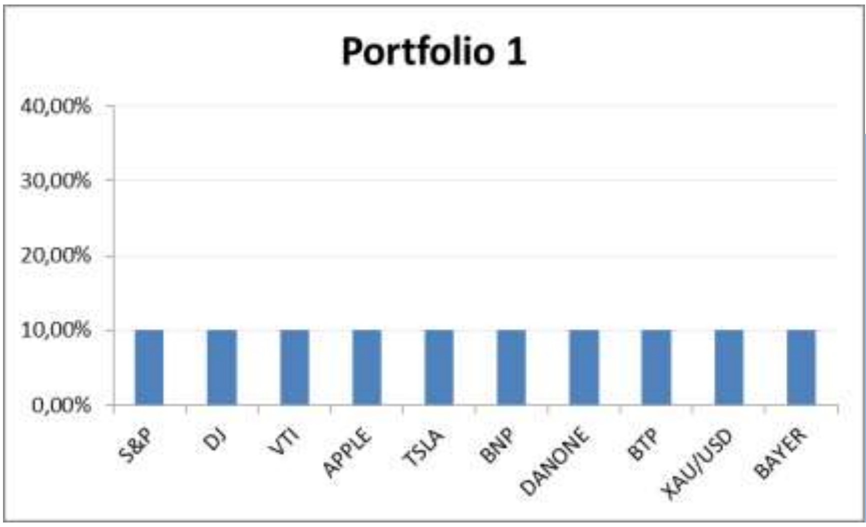
3.1 Portfolios building

Now we are going to show the portfolios' structures with the weights of the ten assets.

The following three portfolios are built considering the log-returns of the assets of the last two years.

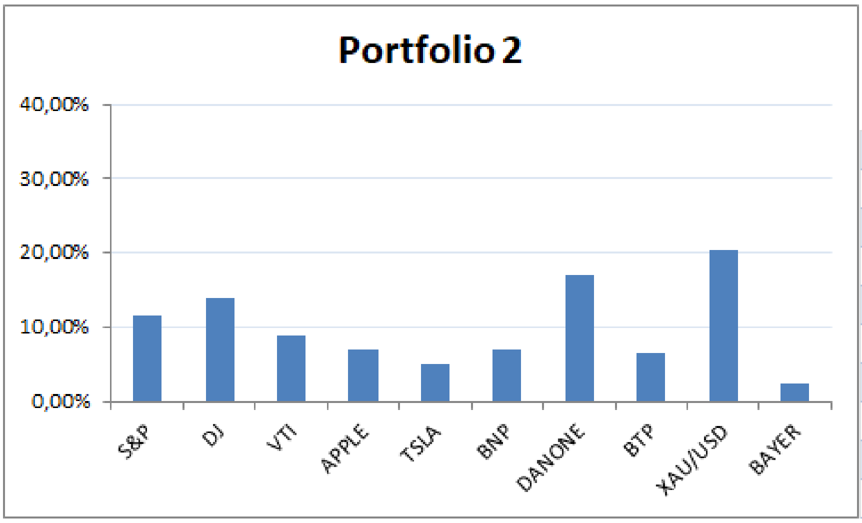
The **Portfolio 1** is the equally weighted portfolio; the **Portfolio 2** is the portfolio where are more weighted the assets with the log-returns distributions more similar to a normal one; the **Portfolio 3** is the Portfolio where are more weighted the assets with the log-returns distributions strongly non-normal.

The weights of the assets of the three portfolios are the following ones:



S&P	10,00%
DJ	10,00%
VTI	10,00%
APPLE	10,00%
TSLA	10,00%
BNP	10,00%
DANONE	10,00%
BTP	10,00%
XAU/USD	10,00%
BAYER	10,00%

Figure 80: Weights of Portfolio 1



S&P	11,50%
DJ	14,00%
VTI	9,00%
APPLE	7,00%
TSLA	5,00%
BNP	7,00%
DANONE	17,00%
BTP	6,50%
XAU/USD	20,50%
BAYER	2,50%

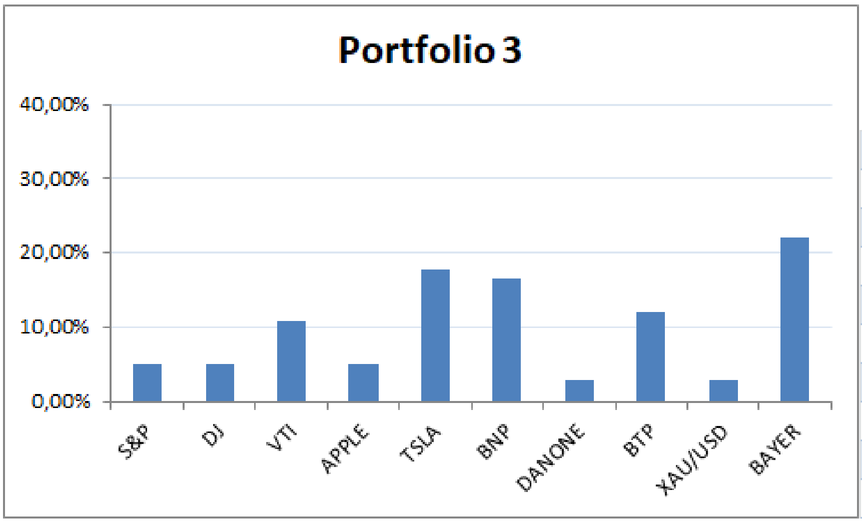
Figure 81: Weights of Portfolio 2

In the **Portfolio 2** the assets considered with the log-returns distribution of the last two years more similar to a normal are:

- S&P: the three normality tests had classified the distribution with some characteristic similar to a normal; the kurtosis of the distribution showed that the tails weren't heavy (1,97) and the standard deviation was almost equal to a normal distribution(1,1%). The weight of the asset is increased to 11,5%.

- DJ: the three normality tests had classified the distribution with some characteristic similar to a normal; the kurtosis of the distribution showed that the tails weren't heavy (1,89) and the standard deviation was was almost equal to a normal distribution (0,9%). The weight of the asset is increased to 14%.
- DANONE: the three normality tests had classified the distribution with some characteristic similar to a normal; the kurtosis of the distribution showed that the tails weren't heavy (2,6) and the standard deviation was equal to a normal distribution (1%). The weight of the asset is increased to 17%.
- XAU/USD: the three normality tests had classified the distribution with some characteristic similar to a normal; the kurtosis of the distribution showed that the tails weren't heavy (1,2) and the standard deviation was almost equal to a normal distribution (0,87%). The weight of the asset is increased to 20,5%.

Furthermore, if distribution of TESLA's log-returns from the normality tests was considered with the four previous assets having some characteristics similar to a normal one, due to the high standard deviation (3,6%) we decided to decrease the weight from 10% to 5%.



S&P	5,00%
DJ	5,00%
VTI	10,70%
APPLE	5,00%
TSLA	17,80%
BNP	16,50%
DANONE	3,00%
BTP	12,00%
XAU/USD	3,00%
BAYER	22,00%

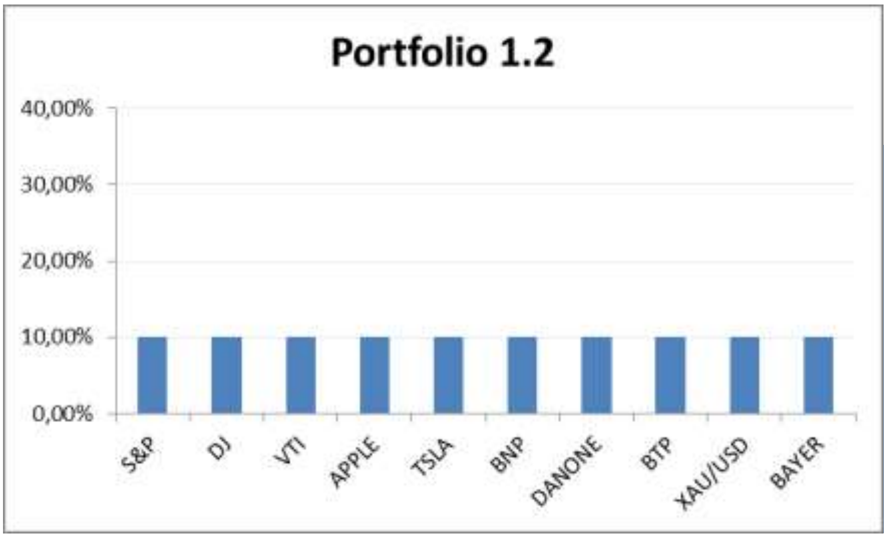
Figure 82: Weights of Portfolio 3

In the **Portfolio 3** the assets considered with the log-returns distribution of the last two years strongly non-normal are:

- VTI: the three normality tests had classified the distribution strongly non-normal; the kurtosis of the distribution showed that the tails weren't heavy (1,825) and the standard deviation was a bit higher than a normal one distribution (1,193%). The weight of the asset is increased to 10,7%.

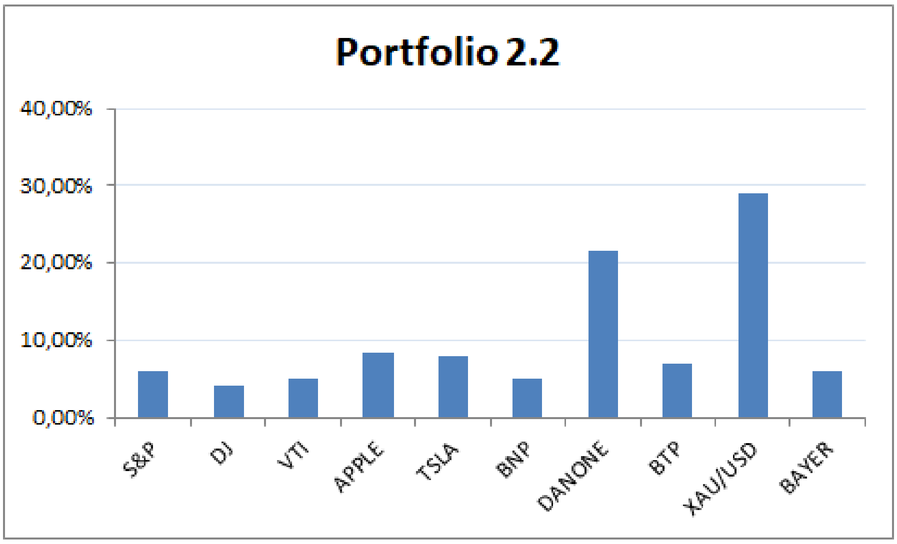
- BAYER: the three normality tests had classified the distribution strongly non-normal; the kurtosis of the distribution showed that the tails were so heavy (21,488) but the standard deviation was higher than a normal one distribution (1,9%). The weight of the asset is increased to 22%.
- BNP: the three normality tests had classified the distribution strongly non-normal; the kurtosis of the distribution showed that the tails were so heavy (4,954) and the standard deviation was higher than a normal one (1,7%). The weight of the asset is increased to 16,5%.
- BTP: the three normality tests had classified the distribution strongly non-normal; the kurtosis of the distribution showed that the tails were so heavy (4,488) but the standard deviation was lower than a normal one (0,2%). The weight of the asset is increased to 12%.
- TESLA: even if the three normality tests had classified the distribution not strongly non-normal like the previous four, due to the high standard deviation (3,6%, the second highest) we decided to increase its weight to 17,8%.

Now we are going to show the others two portfolios built considering the log-returns of the assets of the last five years



S&P	**10,00%**
DJ	**10,00%**
VTI	**10,00%**
APPLE	**10,00%**
TSLA	**10,00%**
BNP	**10,00%**
DANONE	**10,00%**
BTP	**10,00%**
XAU/USD	**10,00%**
BAYER	**10,00%**

Figure 83: Weights of Portfolio 1.2



S&P	6,00%
DJ	4,00%
VTI	5,00%
APPLE	8,50%
TSLA	8,00%
BNP	5,00%
DANONE	21,50%
BTP	7,00%
XAU/USD	29,00%
BAYER	6,00%

Figure 84: Weights of Portfolio 2.2

In the **Portfolio 2.2** (like in the Portfolio 2 in the other time series) we selected the assets considered with the log-returns distribution of the last five years more similar to a normal also if from the normality tests all ten log-returns distributions of the ten assets were considered strongly non-normal and from the descriptive statistics all the tails were heavier than a normal distribution.

- XAU/USD: the kurtosis shows tails similar to a normal distribution (3,104) and a standard deviation of 0,9%. The weight of the asset is increased to 29%.
- DANONE: the kurtosis shows heavy tails (6,168) but a standard deviation of 1,3%. The weight of the asset is increased to 21,50%.

TESLA and APPLE have kurtosis respectivley of 3,613 and 5,237: so their tails are less heay than DANONE but have a standard deviation high (TESLA 4,075% and APPLE 2,001%). Thier weights are a bit decreased to 8% and 8,5%. The other assets have kurtosis in a range 7,413-27,411 so is not possible consider their log-returns distribution similar to a normal one distribution; S&P, DJ and BTP show standard deviations respectively equal to 1,343%, 1,315% and 0,265% also lower than TESLA and APPLE but their kurtosis show too heavy tails.

3.2 VaR and ES Application

In this section we'll compute the one-day Value at Risk (Var) and Expected Shortfall (ES) of the Portfolios for three different confidence intervals (0.95, 0.975, 0.99) with two different methods: the **Historical method** and the **Montecarlo method**.

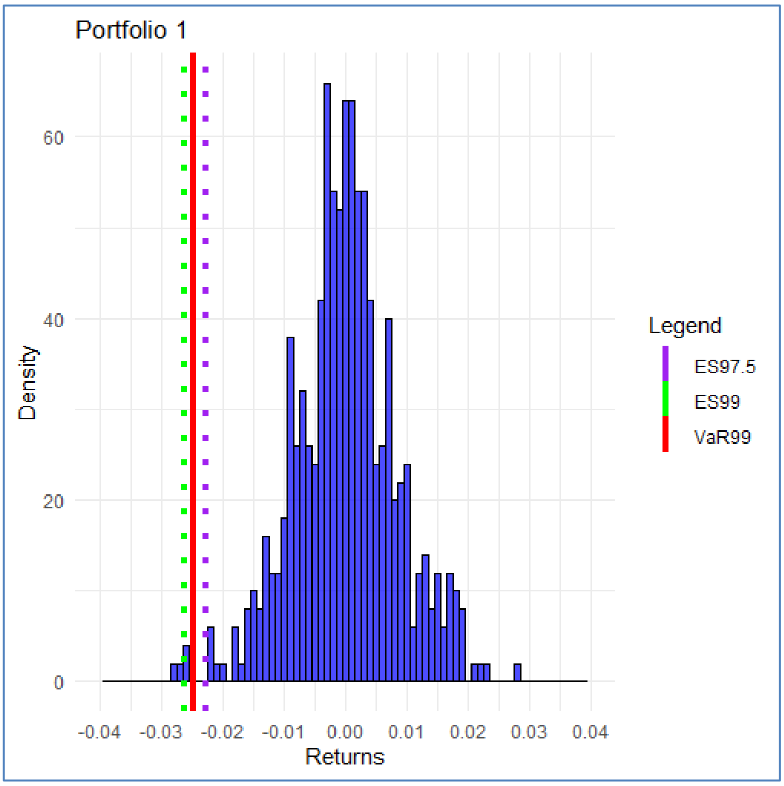
3.2.1 Historical method

In the Historical method we'll observe historical data of the portfolio log-returns, sort them from lowest to highest, and identify the number that corresponds to the desired confidence level.

We'll start computing Var and ES of portfolios built considering the log-returns of the assets of the last two years so:

- **Portfolio 1**
- **Portfolio 2**
- **Portfolio 3**

It'll be shown a distribution of every Portfolio and a section where will be reported all the results of the VaR and ES for every quantile.



Percentile	Value
ES99	-0.0262
VaR99	-0.0247
ES97.5	-0.0228
ES95	-0.0190
VaR97.5	-0.0177
VaR95	-0.0138

Figure 85: Distribution of Portfolio 1 returns with VaR and ES results

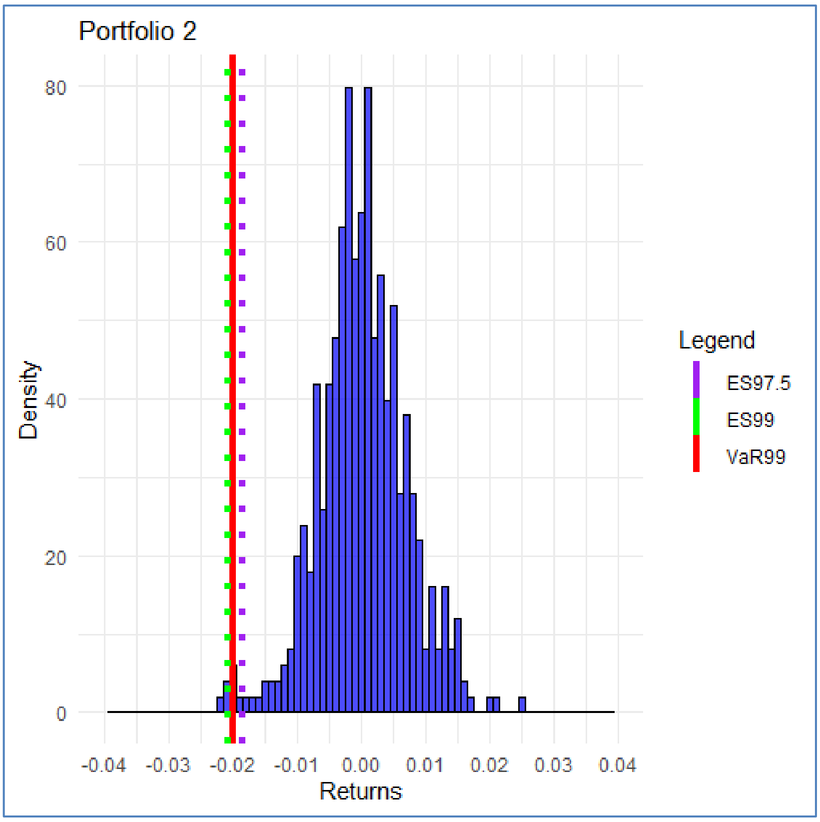
We can observe the Var99% of the Portfolio 1 equal to -2.47%.

It means there is 1% chance that Portfolio 1 will lose more than -2.47% in one day. So if I'll invest €100.000 in this portfolio in 99% of cases I'll not lose more than -2,47% x €100.000 = -2470€.

The ES99% shows the average losses in the worst 1% of cases and is equal to -2,62%, is higher because considers also the severity of losses that exceed the VaR threshold.

The ES97.5% is equal to -2.28%; if the distribution had been normally distributed, it would have been equal to Var99% (for a normally distibuted random variable ES97.5% ≈VaR99%[3]) but in this case, not being the distribution normal, the two values are close (as we can see in the Figure).

[3] V. Dendoncker and A Lebeguè , « Comparative backtesting of Expected Shortfall », 2019



Percentile	Value
ES99	-0.0208
VaR99	-0.0200
ES97.5	-0.0186
ES95	-0.0151
VaR97.5	-0.0145
VaR95	-0.0103

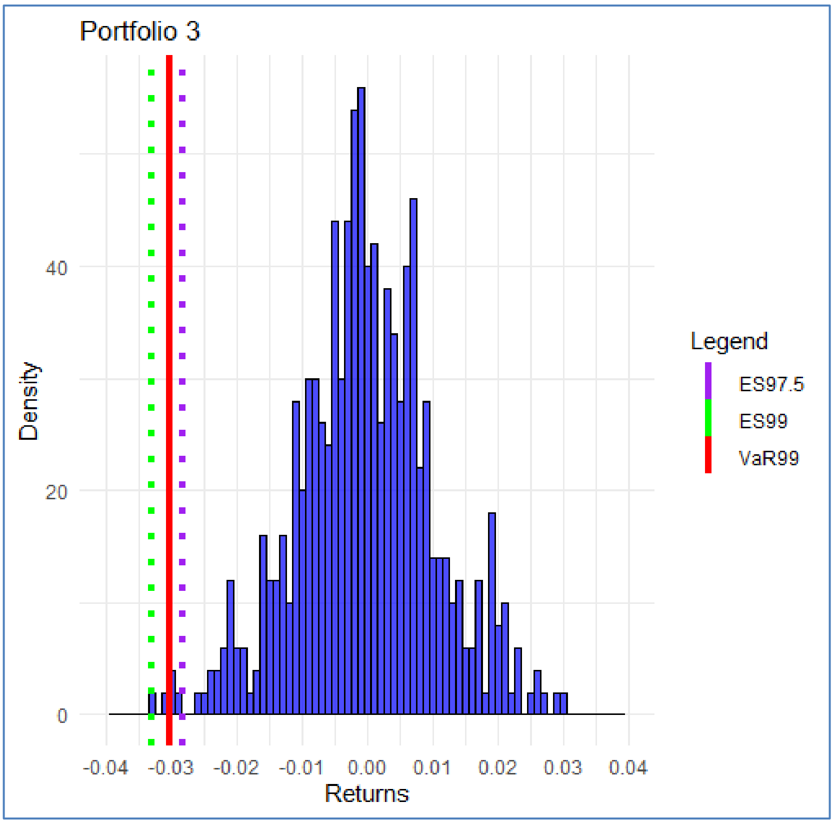
Figure 86: Distribution of Portfolio 2 returns with VaR and ES results

We can observe here that the Var99% of the **Portfolio 2** is equal to -2,00% and lower than the previous equally weighted portfolio. It means in this case that there is 1% chance that Portfolio 2 will lose more than -2,00% in one day. So if I'll invest €100.000 in this portfolio in 99% of cases I'll not lose more than -2,00% x €100.000 = -€2000.

Infact weighting more the assets with a distribution of log-returns more similar to a normal distribution, the values of VaR and ES are descreased and also the distance between Var99 % and ES99%: the ES99% is equal to -2,08%.

The ES97.5% is equal to -1,86%: also the values of Var99% and ES97.5% are even closer than before.

It means that weighting more assets with log-returns distribution more similar to normal, the tails are going to be lighter in the log-returns distribution of Portfolio than before, and in addition to the decrease of VaR and ES, also their difference (VaR99% and ES97.5%) will be smaller.



Percentile	Value
ES99	-0.0331
VaR99	-0.0304
ES97.5	-0.0282
ES95	-0.0245
VaR97.5	-0.0232
VaR95	-0.0193

Figure 87: Distribution of Portfolio 3 returns with VaR and ES results

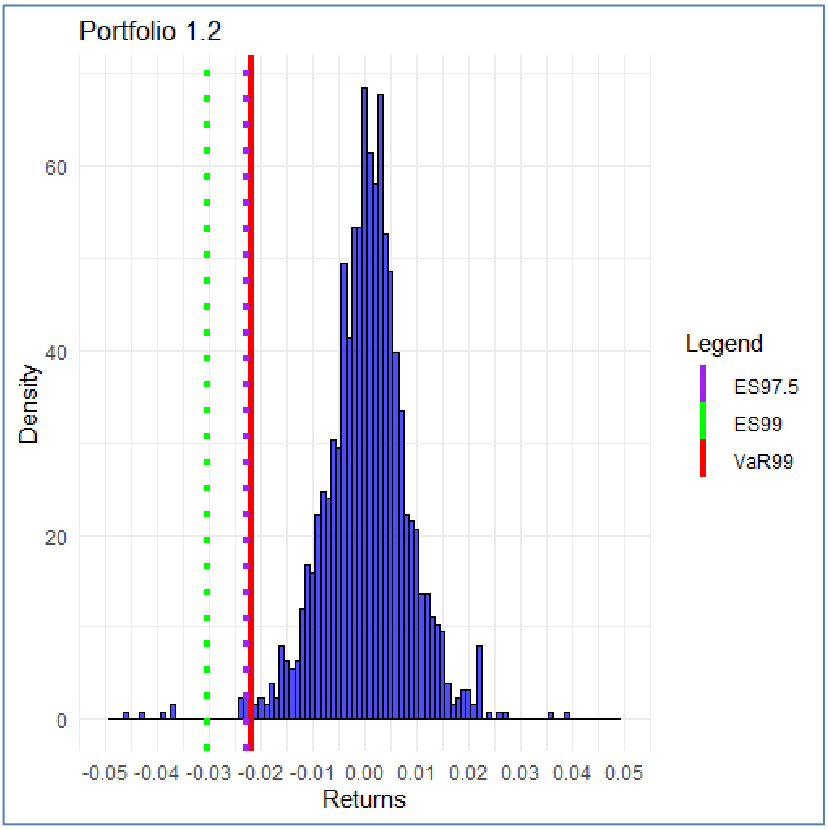
In the **Portfolio 3** the VaR99% is equal to -3,04%; weighting more the assets with a distribution of log-returns strongly non-normal, the tails are going to be heavier than the previous Portfolios and in presence of more outliers the values of VaR and ES will increase.

In this case if I'll invest €100.000 in this portfolio in 99% of cases I'll not lose more than -3,04% x €100.000 = -€3040 in one day.

Also the distance between Var99% and ES99% is higher than in previous portfolio as is increased that between VaR99% and ES97.5% indicating a distribution of log-returns of Portfolio 3 that is even further from a normal distribution.

Now we'll compute Var and ES of portfolios built considering the log-returns of the assets of the last five years so:

- **Portfolio 1.2**
- **Portfolio 2.2**



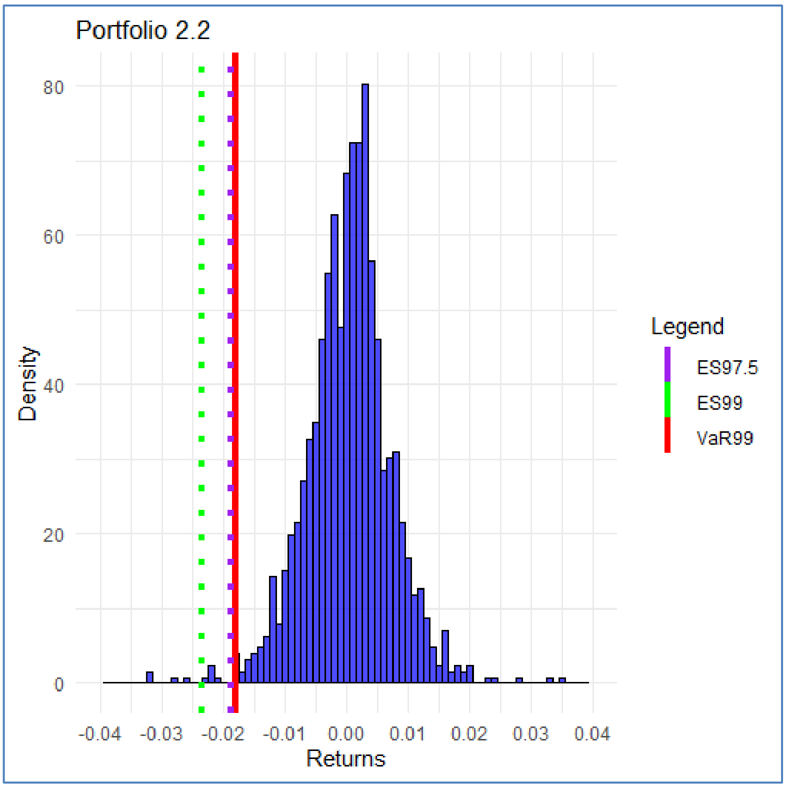
Percentile	Value
ES99	-0.0304
VaR99	-0.0217
ES97.5	-0.0229
ES95	-0.0186
VaR97.5	-0.0162
VaR95	-0.0123

Figure 88: Distribution of Portfolio 1.2 returns with VaR and ES results

If we expand the historical view on the portfolio's asset behaviour from two years (medium term) to five years (medium-long term), we can observe that the situation remains quite stable.

In the **Portfolio 1.2,** we have an equally weighted portfolio, the same of the Portfolio 1, but we are basing our analysis on the log-returns distribution of the assets of last five years; so in the period considered 2019-2024 is involved the pandemic Covid-19 and the war outbreak in Europe between Russia and Ucraine, and this inevitably has created great instability and volatility on the stock markets for some periods.

In this time series, the log-returns distributions of the assets have heavier tails and higher standard deviations than the previous time series (see Figure 58 and 59).
The ES99% of the Portfolio 1.2 is equal to -3,04%, a value higher in comparison to the Portfolio 1's ES99% of -2,62%. It means that the average losses over the VaR threshold have been higher in this period than the previous one.
Instead the VaR99% is equal to 2,17% and is lower than the VaR99% result on the equal Portfolio on the two years time series (-2,47%). This situation shows perfect how the ES99% has captured the more extreme losses occurred in the period considered in comparion to the previous one while the VaR99% not.
If we invest €100.000 on the Portfolio 1.2 in the 99% of the cases we should not lose more than -2,17% x €100.000 = €2.170. The ES97.5% with the value of – 2,29% is not so far from VaR99% revealing a log-returns distribution of the Portfolio 1.2 rather regular.



Percentile	Value
ES99	-0.0236
VaR99	-0.0180
ES97.5	-0.0186
ES95	-0.0153
VaR97.5	-0.0138
VaR95	-0.0109

Figure 89: Distribution of Portfolio 2.2 returns with VaR and ES results

We can observe that the VaR99% in the **Portfolio 2.2** is equal to -1,80% and is so lower than the VaR99% of Portfolio 1.2. It means that if we invest €100.000 in the Portfolio 2.2 in the 99% of cases we should not lose more than -1,80% x €100.000= -€1.080

As we have observed in the previous time series, weighting more the assets with a log-returns distribution more similar to a normal distribution , the VaR and ES decrease and also the distance between ES99% and Var99%: the ES99% of the Portfolio 2.2 is -2,36%.

The ES97.5% is -1,86% and is so close to Var99% value of -1,80%, confirming that the tails are lighter with new weights and the distribution of log-returns of the Portfolio is more similar to a normal one.

In a table below we'll summarize all the results obtained with the application of the historical method for the computation of VaR and ES.

	VaR99%	VaR97.5%	VaR95%	ES99%	ES97.5%	ES95%
Portfolio 1	-2,47%	-1,77%	-1,38%	-2,62%	-2,28%	-1,90%
Portfolio 2	-2,00%	-1,45%	-1,03%	-2,08%	-1,86%	-1,51%
Portfolio 3	-3,04%	-2,32%	-1,93%	-3,31%	-2,82%	-2,45%
Portfolio 1.2*	-2,17%	-1,62%	1,23%	-3,04%	-2,29%	-1,86%
Portfolio 2.2*	-1,80%	-1,38%	-1,09%	-2,36%	-1,86%	-1,53%

Figure 90: VaR and ES Historical method results

***portfolios built considering historical data of last five years**

From the results obtained we can assume that the two equally portfolios (Portfolio 1 and Portfolio 1.2), despite the different time series of log-returns distributions taken into account, have similar results. Infact the Portfolio generally is well risky diversified and owns also indexes (S&P,DJ) funds (VTI) and company's stocks rather solid and stable despite the delicate events that have occurred in the last five years. The ES99% values show more extreme values over the VaR99% threshold in the Portfolio 1.2 than Portfolio 1, infact the worst loss of Portfolio 1.2 of -5,22% is signifícatly higher than the Portfolio 1's worst loss of -2,81%.

Is also important consider that future market conditions can differ signifícatly from the past data, so past could not be a good predictor for the future as underlyed from empirical studies.

In the Portfolios built with the two years time series the -VaR99%>-ES97.5%, this happens because in the Historical method is not present the normality hypothesis in the distribution of the returns, so the Portfolio distributions could be asymmetrical, while on an higher data sample, despite the non-normality hypothesis, the distributions could have a more regular shape.

Also for this reason the graphs of Portfolio returns seems more regular in the Portfolios built with five years time series with the -ES97.5% >-VaR99%.

3.2.2. Montecarlo method

This method creates a large number of random scenarios of the portfolio returns over the desired time horizon and is based on distribution and correlation of log-returns. Differently from the Parametric method avoids any type of distribution assumption.

To make a correct estimation of VaR and ES with this method, are important the descriptive statistics of the log-returns distributions of the assets like media or standard deviation and compute the covariance and correlation matrix.

Here below there are the correlation and covariance matrix of the log-returns of the assets of the last two years.

ρ(x,y)	S&P	DJ	VTI	DANONE	APPLE	TSLA	XAU/USD	BAYER	Btp1mz25 5%	BNP
S&P	1	0,942987617	0,99489648	0,009407423	0,82168982	0,571666815	0,054547756	0,060567299	0,160019636	0,206426325
DJ	0,94298762	1,000000000	0,938562866	0,038941293	0,736160958	0,451527373	0,06700107	0,050601772	0,156691852	0,21513337
VTI	0,99489648	0,938562866	1	0,00800142	0,804065777	0,585084525	0,055628765	0,056329563	0,154366406	0,220627985
DANONE	0,00940742	0,038941293	0,00800142	1	-0,017737264	0,004503308	-0,013848239	-0,019103412	0,072703552	0,016030143
APPLE	0,82168982	0,736160958	0,804065777	-0,017737264	1	0,541066821	-0,01016675	0,048286539	0,144768619	0,107797362
TSLA	0,57166682	0,451527373	0,585084525	0,004503308	0,541066821	1	-0,02707975	0,035632084	0,030248546	0,130028082
XAU/USD	0,05454776	0,06700107	0,055628765	-0,013848239	-0,01016675	-0,01016675	1	-0,000760724	-0,009610503	0,002207394
BAYER	0,0605673	0,050601772	0,056329563	-0,019103412	0,048286539	0,035632084	-0,000760724	1	0,032866815	0,078842656
Btp1mz25 5%	0,16001964	0,156691852	0,154366406	0,072703552	0,144768619	0,030248546	-0,009610503	0,032866815	1	0,022974955
BNP	0,20642633	0,21513337	0,220627985	0,016030143	0,107797362	0,130028082	0,002207394	0,078842656	0,022974955	1

Figure 91: Correlation matrix between assets' log-returns of last two years

In the matrix we can notice very high correlation between the log-returns of the indexes S&P and DJ (0,9430) and also between the two indexes with the ETF VTI: DJ and VTI have a correlation of 0,9385 while S&P and VTI of 0,9948.

Also the log-returns of APPLE has an high correlation with the two indexes: 0,8217 with S&P and 0,7362 with DJ.

XAU/USD and TESLA have the log-returns most negatively correlated: their correlation is equal to -0,027

COV(x,y)	S&P	DJ	VTI	DANONE	APPLE	TSLA	XAU/USD	BAYER	Btp1mz25 5%	BNP
S&P	0,00013401									
DJ	0,00010565	0,00009367								
VTI	0,00013744	0,00010841	0,00014241							
DANONE	0,00000111	0,00000385	0,00000098	0,00010440						
APPLE	0,00016827	0,00012604	0,00016974	-0,00000321	0,00031293					
TSLA	0,00024368	0,00016092	0,00025711	0,00000169	0,00035245	0,00135592				
XAU/USD	0,00000548	0,00000563	0,00000577	-0,00000123	-0,00000156	-0,00000325	0,00007544			
BAYER	0,00001362	0,00000951	0,00001306	-0,00000379	0,00001659	0,00002549	-0,00000013	0,00037732		
Btp1mz25 5%	0,00000390	0,00000319	0,00000388	0,00000156	0,00000539	0,00000235	-0,00000018	0,00000134	0,00000443	
BNP	0,00004179	0,00003641	0,00004604	0,00000286	0,00003335	0,00008373	0,00000034	0,00002678	0,00000085	0,00030582

Figure 92: Covariance matrix

To generate the random scenario on which compute the VaR and ES, we have created a code on R; we entered the following data: mean of every asset's log-return distribution, standard deviation of every asset's log-return distribution, weights of the assets, correlation and covariance matrix.

We decided to generate **10.000 random scenarios.**

The code is the following one (the data in the code are about last two years):

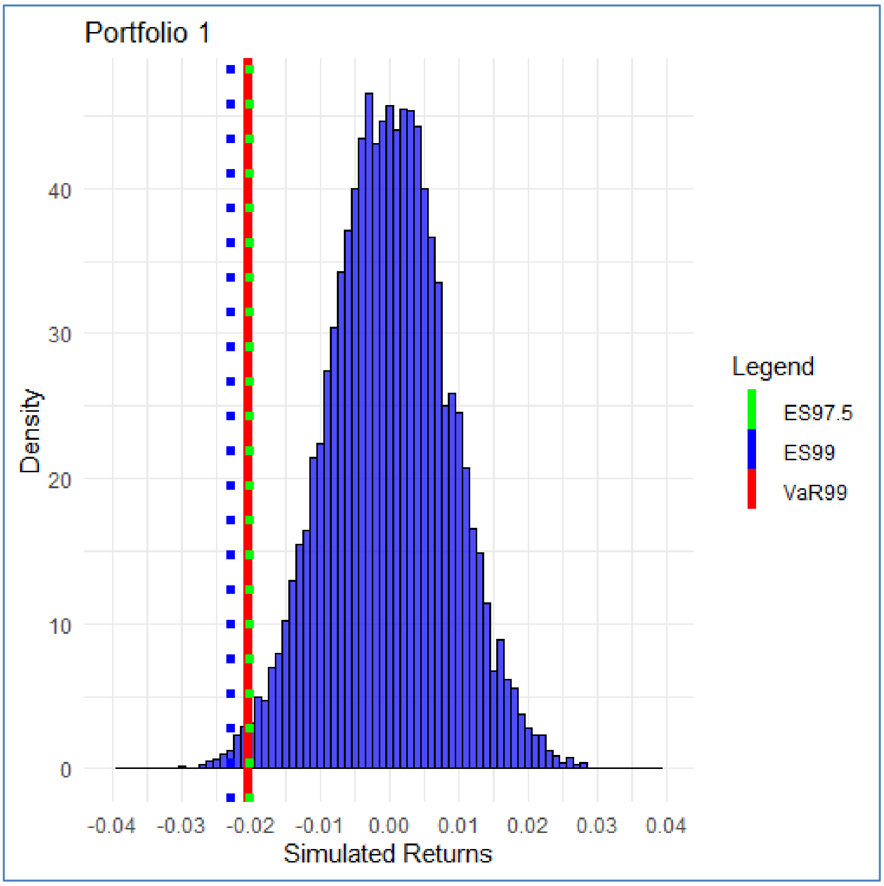
```r
library(MASS)  # For mvrnorm()
library(Matrix)  # For nearPD()
library(ggplot2)  # For plotting
library(knitr)  # For kable()
library(kableExtra)  # For kable styling

# Portfolio parameters,correlation and covariance matrix
mean_returns <- c(0.00027, 0.00024, 0.00024, 0.00032, -0.00006, -0.00155, 0.00038, -
0.00162, -0.0002, 0.00047)
std_dev <- c(0.01157, 0.00968, 0.01193, 0.01022, 0.01769, 0.03682, 0.00868, 0.01942,
0.002106, 0.01749)

correlation_matrix <- matrix(c(
  1, 0.942987617, 0.99489648, 0.009407423, 0.82168982, 0.571666815, 0.054547756,
0.060567299, 0.160019636, 0.206426325,
  0.942987617, 1.000000000, 0.938562866, 0.038941293, 0.736160958, 0.451527373,
0.06700107, 0.050601772, 0.156691852, 0.21513337,
  0.99489648, 0.938562866, 1, 0.00800142, 0.804065777, 0.585084525, 0.055628765,
0.056329563, 0.154366406, 0.220627985,
  0.009407423, 0.038941293, 0.00800142, 1, -0.017737264, 0.004503308, -0.013848239, -
0.019103412, 0.072703552, 0.016030143,
  0.82168982, 0.736160958, 0.804065777, -0.017737264, 1, 0.541066821, -0.01016675,
0.048286539, 0.144768619, 0.107797362,
  0.571666815, 0.451527373, 0.585084525, 0.004503308, 0.541066821, 1, -0.02707975,
0.035632084, 0.030248546, 0.130028082,
  0.054547756, 0.06700107, 0.055628765, -0.013848239, -0.01016675, -0.01016675, 1, -
0.000760724, -0.009610503, 0.002207394,
  0.060567299, 0.050601772, 0.056329563, -0.019103412, 0.048286539, 0.035632084, -
0.000760724, 1, 0.032866815, 0.078842656,
  0.160019636, 0.156691852, 0.154366406, 0.072703552, 0.144768619, 0.030248546, -
0.009610503, 0.032866815, 1, 0.022974955,
  0.206426325, 0.21513337, 0.220627985, 0.016030143, 0.107797362, 0.130028082,
0.002207394, 0.078842656, 0.022974955, 1
), nrow = 10, byrow = TRUE)

cov_matrix <- diag(std_dev) %*% correlation_matrix %*% diag(std_dev)

# Ensure the covariance matrix is positive definite
eigen_values <- eigen(cov_matrix)$values
if (any(eigen_values <= 0)) {
  cov_matrix <- nearPD(cov_matrix)$mat
}

# Portfolio weights (updated)
weights <- c(0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1)

# Simulate returns
set.seed(123)  # For reproducibility
n_simulations <- 10000  # Number of simulations
simulated_returns <- mvrnorm(n = n_simulations, mu = mean_returns, Sigma = cov_matrix)
```

```r
# Calculate portfolio returns
portfolio_returns <- simulated_returns %*% weights

# Calculate VaR and ES
VaR_99 <- quantile(portfolio_returns, probs = 1 - 0.99)
VaR_975 <- quantile(portfolio_returns, probs = 1 - 0.975)
VaR_95 <- quantile(portfolio_returns, probs = 1 - 0.95)

ES_99 <- mean(portfolio_returns[portfolio_returns <= VaR_99])
ES_975 <- mean(portfolio_returns[portfolio_returns <= VaR_975])
ES_95 <- mean(portfolio_returns[portfolio_returns <= VaR_95])

# Print results
cat("The VaR at the 99% confidence level is:", VaR_99, "\n")
cat("The ES at the 99% confidence level is:", ES_99, "\n")
cat("The VaR at the 97.5% confidence level is:", VaR_975, "\n")
cat("The ES at the 97.5% confidence level is:", ES_975, "\n")
cat("The VaR at the 95% confidence level is:", VaR_95, "\n")
cat("The ES at the 95% confidence level is:", ES_95, "\n")

summary_results <- data.frame(
  Percentile = c("Number of Simulations", "Mean Simulated Returns", "Median Simulated
Returns",
                 "Standard Deviation of Simulated Returns", "Worst Loss", "Percentage of
Losses",
                 "VaR (95%)", "ES (95%)", "VaR (97.5%)", "ES (97.5%)", "VaR (99%)", "ES
(99%)"),
  Value = c(n_simulations, mean(portfolio_returns), median(portfolio_returns),
            sd(portfolio_returns), min(portfolio_returns), mean(portfolio_returns < 0) *
100,
            round(VaR_95, 4), round(ES_95, 4), round(VaR_975, 4), round(ES_975, 4),
            round(VaR_99, 4), round(ES_99, 4))
)

# Print the summary table
summary_results %>%
  kable(caption = "Portfolio 1 Summary Results",
        col.names = c("Metrics", "Value"),
        digits = 4) %>%
  kable_styling(full_width = F, position = "left") %>%
  row_spec(0, bold = TRUE, background = "#D3D3D3") %>%
  column_spec(1, border_right = TRUE) %>%
  column_spec(2, border_left = TRUE)

df <- data.frame(Returns = portfolio_returns)

# Draw histogram and plot VaR and ES using ggplot2
ggplot(df, aes(x = Returns)) +
  geom_histogram(binwidth = 0.001, aes(y = ..density..), color = "black", fill = "blue",
alpha = 0.7) +
  geom_vline(aes(xintercept = VaR_99, color = "VaR99"), linetype = "solid", size = 2) +
```

```
  geom_vline(aes(xintercept = ES_99, color = "ES99"), linetype = "dotted", size = 2) +
  geom_vline(aes(xintercept = ES_975, color = "ES97.5"), linetype = "dotted", size = 2) +
  labs(title = "Portfolio 1",
       x = "Simulated Returns",
       y = "Density") +
  theme_minimal() +
  scale_x_continuous(
    breaks = seq(from = -0.04, to = 0.04, by = 0.01),  # Set x-axis ticks with 0.01
intervals
    limits = c(-0.04, 0.04)  # Limit x-axis to the desired range
  ) +
  scale_color_manual(values = c("VaR99" = "red", "ES99" = "blue", "ES97.5" = "green"),
                     labels = c("VaR99 (Red)" = paste("VaR99 =", round(VaR_99, 4)),
                                "ES99 (Blue)" = paste("ES99 =", round(ES_99, 4)),
                                "ES97.5 (Green)" = paste("ES97.5 =", round(ES_975, 4)))) +
  theme(legend.title = element_blank())
```

We'll firsty compute the VaR and ES with the Montecarlo method on the Portfolios composed with the assets' log-returns distributions of the last two years so:

- **Portfolio 1**
- **Portfolio 2**
- **Portfolio 3**



Metrics	Value
Number of Simulations	10000.0000
Mean Simulated Returns	-0.0002
Median Simulated Returns	-0.0002
Standard Deviation of Simulated Returns	0.0086
Worst Loss	-0.0347
Percentage of Losses	50.7800
VaR (95%)	-0.0143
ES (95%)	-0.0179
VaR (97.5%)	-0.0171
ES (97.5%)	-0.0204
VaR (99%)	-0.0203
ES (99%)	-0.0229

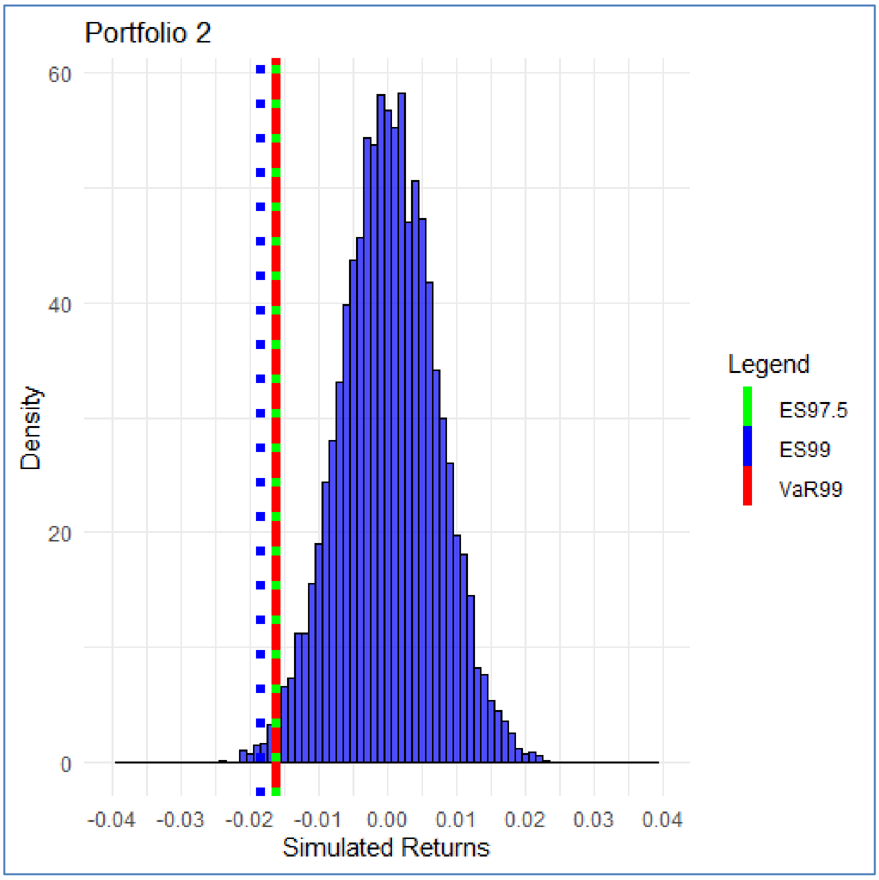
Figure 93: Distribution of Portfolio 1 simulated returns with VaR and ES results

The VaR and ES results computed using the Montecarlo method on a 10.000 simulated random scenarios are a bit different in this case from the results computed using the Historical method; we can observe that the most extreme values (VaR99%, ES99%, ES97,5%), computed using the Montecarlo method, are lower than those computed using the Historical method, this because there are more extreme values in the last two years of the **Portfolio 1** than in the 10.000 simulated returns.

The worst simulated return is equal to -3,47%, and the portfolio on 10.000 simulated returns is at a loss 50,78% of the cases.

The value of ES97,5% is equal to -2,04% and is closer to VaR99% (2,03%) value than in the Historical Method.

It happens because the Montecarlo simulation usually generates a more regular distribution having a lot of iterations and probably less influenced by extreme values.



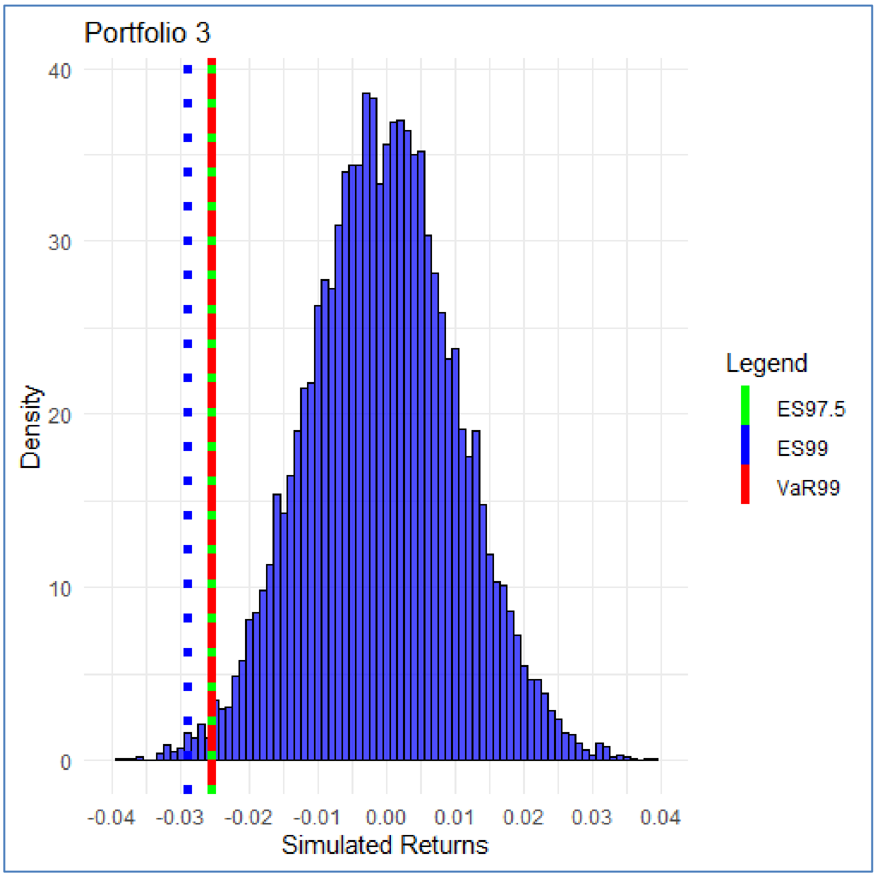
Metrics	Value
Number of Simulations	10000.0000
Mean Simulated Returns	0.0001
Median Simulated Returns	0.0002
Standard Deviation of Simulated Returns	0.0070
Worst Loss	-0.0287
Percentage of Losses	49.2500
VaR (95%)	-0.0114
ES (95%)	-0.0145
VaR (97.5%)	-0.0137
ES (97.5%)	-0.0163
VaR (99%)	-0.0162
ES (99%)	-0.0186

Figure 94: Distribution of Portfolio 2 simulated returns with VaR and ES results

In the **Portfolio 2** we can observe still difference in the VaR and ES values computed with Montecarlo method from the values computed with Historical method than the previous Portfolio 1. Also in this case the most extreme values (VaR99%, ES99%, ES97,5%), computed using Montecarlo method, are lower than those computed using Historical method.

The Worts Loss is -2,87% and the portfolio on 10.000 simulated returns is at a loss 49,25% of the cases (lower than Portfolio 1).

The values of VaR99% and ES97.5% are very similar (-1,62% and -1,63%) and are closer than in the Portfolio 1 computed with Montecarlo method and Portfolio 2 computed with Historical method: it means that the distribution of simulated returns is very similar to a normal distribution.



Metrics	Value
Number of Simulations	10000.0000
Mean Simulated Returns	-0.0005
Median Simulated Returns	-0.0005
Standard Deviation of Simulated Returns	0.0107
Worst Loss	-0.0426
Percentage of Losses	51.9000
VaR (95%)	-0.0182
ES (95%)	-0.0225
VaR (97.5%)	-0.0213
ES (97.5%)	-0.0255
VaR (99%)	-0.0253
ES (99%)	-0.0290

Figure 95: Distribution of Portfolio 3 simulated returns with VaR and ES results

In the **Portfolio 3** we can observe how the Montecarlo simulation in this case could have understimated the risk considering the Portfolio is composed weighting more assets with higher kurtosis and so non-normal: the Historical method could have reflected better the gravity of the events already happened.
The Var99% here is equal to -2,53% while the Var99% computed with the Historical method is -3,04%.
The Worst Loss equal to -4,26% is higher than the worst loss of the Portfolio in the last two years (-4,06%) and the portfolio on 10.000 simulated returns is at a loss 51,90% of the cases (higher than in the Portfolio 1 and Portfolio 2).
The distribution, like in the previous two Portfolios, probably thanks to the 10.000 iterations, is less influenced by extreme values present in the historical data and shows again values of VaR99% and ES97,5% very close (2,53% and 2,55%).

Now we'll compute VaR and ES using Montecarlo method on the Portfolios composed with the assets' log-returns distributions of the last five years so:

- **Portfolio 1.2**
- **Portfolio 2.2**

Here below there are the correlation and covariance matrix of the log-returns of the assets of the last five years.

ρ(x,y)	S&P	DJ	VTI	DANONE	APPLE	TSLA	XAU/USD	BAYER	Btp1mz25 5%	BNP
S&P	1	0,957829338	0,994787526	0,016863345	0,802822465	0,504675773	0,060447164	0,064832796	0,229448342	0,100719416
DJ	0,95782934	1,000000000	0,950802086	0,032653743	0,716451511	0,412622361	0,071925695	0,0649061	0,230326015	0,128825345
VTI	0,99478753	0,950802086	1	0,018825337	0,788539151	0,52362743	0,060899773	0,060448767	0,229072213	0,114920799
DANONE	0,01686335	0,032653743	0,018825337	1	-0,007547065	-0,02050755	-0,070245815	-0,017503208	-0,099017149	0,093815599
APPLE	0,80282246	0,716451511	0,788539151	-0,007547065	1	0,50073055	0,014657028	0,038028794	0,18905589	-0,00311655
TSLA	0,50467577	0,412622361	0,52362743	-0,02050755	0,50073055	1	0,026386356	0,022277488	0,127567715	0,07346921
XAU/USD	0,06044716	0,071925695	0,060899773	-0,070245815	0,014657028	0,014657028	1	0,013829593	0,067938016	-0,04068297
BAYER	0,0648328	0,0649061	0,060448767	-0,017503208	0,038028794	0,022277488	0,013829593	1	0,000902534	0,08442523
Btp1mz25 5%	0,22944834	0,230326015	0,229072213	-0,099017149	0,18905589	0,127567715	0,067938016	0,000902534	1	-0,01104544
BNP	0,10071942	0,128825345	0,114920799	0,093815599	-0,003116546	0,07346921	-0,040682966	0,08442523	-0,011045442	1

Figure 96: Correlation matrix between assets' log-returns of last five years

In the matrix we can still notice, like in the previous matrix, very high correlation between the two indexes S&P500 and DJ (0,9578) and between the two indexes with APPLE (0,8028 with S&P500 and 0,7164 with DJ).
The correlation of the log-returns of two indexes with the ETF VTI in the last five years is still strong like we had observed in the last two years: the correlation of VTI with S&P500 now is 0,9947 and with DJ is 0,9508.
Italian BTP and DANONE are the two assets with log-return most negatively correlated (-0,099).

COV(x,y)	S&P	DJ	VTI	DANONE	APPLE	TSLA	XAU/USD	BAYER	Btp1mz25 5%	BNP
S&P	0,00018039									
DJ	0,00016914	0,00017287								
VTI	0,00018164	0,00016995	0,00018481							
DANONE	0,00000303	0,00000574	0,00000342	0,00017896						
APPLE	0,00021574	0,00018847	0,00021448	-0,00000202	0,00040031					
TSLA	0,00027619	0,00022105	0,00029005	-0,00001118	0,00040821	0,00166025				
XAU/USD	0,00000762	0,00000887	0,00000777	-0,00000882	0,00000275	0,00000560	0,00008800			
BAYER	0,00001741	0,00001706	0,00001643	-0,00000468	0,00001521	0,00001815	0,00000259	0,00039980		
Btp1mz25 5%	0,00000816	0,00000802	0,00000825	-0,00000351	0,00001002	0,00001376	0,00000169	0,00000005	0,00000701	
BNP	0,00003066	0,00003839	0,00003541	0,00002844	-0,00000141	0,00006784	-0,00000865	0,00003826	-0,00000066	0,00051363

Figure 97: Covariance matrix

Like with the previous time serie, to generate the random scenario on which compute the VaR and ES, we have used the same code on R; we entered the following data: mean of every asset's log-return distribution, standard deviation of every asset's log-return distribution, weights of the assets, correlation and covariance matrix.

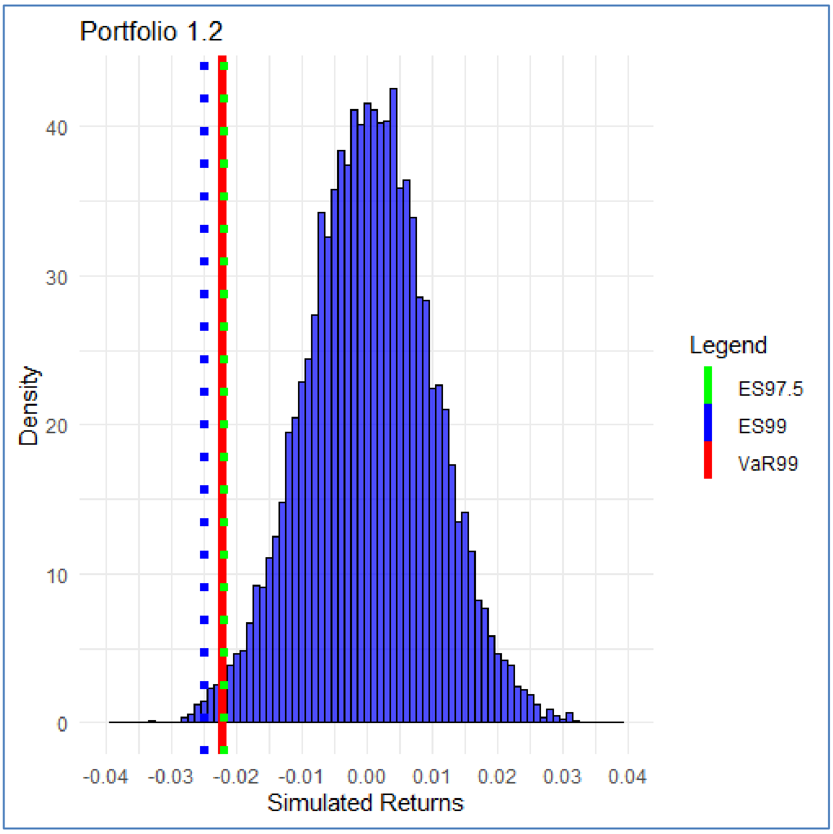
We decided to generate **10.000 random scenarios.**

The code is the following one (the data in the code are about last five years):

```
library(MASS)  # For mvrnorm()
library(Matrix)  # For nearPD()
library(ggplot2)  # For plotting
library(knitr)  # For kable()
library(kableExtra)  # For kable styling

# Portfolio parameters
mean_returns <- c(0.00047, 0.00032, 0.00045, -0.00012, 0.00099, 0.00172, 0.00045, -
0.00059, -0.00012, 0.00032)
std_dev <- c(0.01343, 0.01315, 0.01359, 0.01338, 0.02001, 0.04075, 0.00938, 0.02000,
0.00265, 0.02266)

# Updated correlation matrix
correlation_matrix <- matrix(c(
  1, 0.957829338, 0.994787526, 0.016863345, 0.802822465, 0.504675773, 0.060447164,
0.064832796, 0.229448342, 0.100719416,
  0.957829338, 1, 0.950802086, 0.032653743, 0.716451511, 0.412622361, 0.071925695,
0.0649061, 0.230326015, 0.128825345,
  0.994787526, 0.950802086, 1, 0.018825337, 0.788539151, 0.52362743, 0.060899773,
0.060448767, 0.229072213, 0.114920799,
  0.016863345, 0.032653743, 0.018825337, 1, -0.007547065, -0.02050755, -0.070245815, -
0.017503208, -0.099017149, 0.093815599,
  0.802822465, 0.716451511, 0.788539151, -0.007547065, 1, 0.50073055, 0.014657028,
0.038028794, 0.18905589, -0.003116546,
  0.504675773, 0.412622361, 0.52362743, -0.02050755, 0.50073055, 1, 0.026386356,
0.022277488, 0.127567715, 0.07346921,
  0.060447164, 0.071925695, 0.060899773, -0.070245815, 0.014657028, 0.014657028, 1,
0.013829593, 0.067938016, -0.040682966,
```

```
  0.064832796, 0.0649061, 0.060448767, -0.017503208, 0.038028794, 0.022277488, 
0.013829593, 1, 0.000902534, 0.08442523,
  0.229448342, 0.230326015, 0.229072213, -0.099017149, 0.18905589, 0.127567715, 
0.067938016, 0.000902534, 1, -0.011045442,
  0.100719416, 0.128825345, 0.114920799, 0.093815599, -0.003116546, 0.07346921, -
0.040682966, 0.08442523, -0.011045442, 1
), nrow = 10, byrow = TRUE)

# Calculate the covariance matrix
cov_matrix <- diag(std_dev) %*% correlation_matrix %*% diag(std_dev)

# Ensure the covariance matrix is positive definite
eigen_values <- eigen(cov_matrix)$values
if (any(eigen_values <= 0)) {
  cov_matrix <- nearPD(cov_matrix)$mat
}

# Portfolio weights (updated)
weights <- c(0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1)

# Simulate returns
set.seed(123)  # For reproducibility
n_simulations <- 10000  # Number of simulations
simulated_returns <- mvrnorm(n = n_simulations, mu = mean_returns, Sigma = cov_matrix)

# Calculate portfolio returns
portfolio_returns <- simulated_returns %*% weights

# Calculate VaR and ES
VaR_99 <- quantile(portfolio_returns, probs = 1 - 0.99)
VaR_975 <- quantile(portfolio_returns, probs = 1 - 0.975)
VaR_95 <- quantile(portfolio_returns, probs = 1 - 0.95)

ES_99 <- mean(portfolio_returns[portfolio_returns <= VaR_99])
ES_975 <- mean(portfolio_returns[portfolio_returns <= VaR_975])
ES_95 <- mean(portfolio_returns[portfolio_returns <= VaR_95])

# Print results
cat("The VaR at the 99% confidence level is:", VaR_99, "\n")
cat("The ES at the 99% confidence level is:", ES_99, "\n")
cat("The VaR at the 97.5% confidence level is:", VaR_975, "\n")
cat("The ES at the 97.5% confidence level is:", ES_975, "\n")
cat("The VaR at the 95% confidence level is:", VaR_95, "\n")
cat("The ES at the 95% confidence level is:", ES_95, "\n")

# Create a dataframe for summary table
summary_results <- data.frame(
  Percentile = c("Number of Simulations", "Mean Simulated Returns", "Median Simulated 
Returns",
                 "Standard Deviation of Simulated Returns", "Worst Loss", "Percentage of 
Losses",
```

```r
                "VaR (95%)", "ES (95%)", "VaR (97.5%)", "ES (97.5%)", "VaR (99%)", "ES
(99%)"),
  Value = c(n_simulations, mean(portfolio_returns), median(portfolio_returns),
            sd(portfolio_returns), min(portfolio_returns), mean(portfolio_returns < 0) *
100,
            round(VaR_95, 4), round(ES_95, 4), round(VaR_975, 4), round(ES_975, 4),
            round(VaR_99, 4), round(ES_99, 4))
)

# Print the summary table
summary_results %>%
  kable(caption = "Portfolio 1.2 Summary Results",
        col.names = c("Metrics", "Value"),
        digits = 4) %>%
  kable_styling(full_width = F, position = "left") %>%
  row_spec(0, bold = TRUE, background = "#D3D3D3") %>%
  column_spec(1, border_right = TRUE) %>%
  column_spec(2, border_left = TRUE)

# Create dataframe for ggplot
df <- data.frame(Returns = portfolio_returns)

# Draw histogram and plot VaR and ES using ggplot2
ggplot(df, aes(x = Returns)) +
  geom_histogram(binwidth = 0.001, aes(y = ..density..), color = "black", fill = "blue",
alpha = 0.7) +
  geom_vline(aes(xintercept = VaR_99, color = "VaR99"), linetype = "solid", size = 2) +
  geom_vline(aes(xintercept = ES_99, color = "ES99"), linetype = "dotted", size = 2) +
  geom_vline(aes(xintercept = ES_975, color = "ES97.5"), linetype = "dotted", size = 2) +
  labs(title = "Portfolio 2.2",
       x = "Simulated Returns",
       y = "Density") +
  theme_minimal() +
  scale_x_continuous(
    breaks = seq(from = -0.04, to = 0.04, by = 0.01),  # Set x-axis ticks with 0.01
intervals
    limits = c(-0.04, 0.04)  # Limit x-axis to the desired range
  ) +
  scale_color_manual(values = c("VaR99" = "red", "ES99" = "blue", "ES97.5" = "green"),
                     labels = c("VaR99 (Red)" = paste("VaR99 =", round(VaR_99, 4)),
                                "ES99 (Blue)" = paste("ES99 =", round(ES_99, 4)),
                                "ES97.5 (Green)" = paste("ES97.5 =", round(ES_975, 4)))) +
  guides(color = guide_legend(title = "Legend")) +
  annotate("text", x = VaR_99, y = Inf, label = paste("VaR99 =", round(VaR_99, 4)), color
= "red", vjust = -1.5, hjust = 0.5) +
  annotate("text", x = ES_99, y = Inf, label = paste("ES99 =", round(ES_99, 4)), color =
"blue", vjust = -1.5, hjust = 0.5) +
  annotate("text", x = ES_975, y = Inf, label = paste("ES97.5 =", round(ES_975, 4)), color
= "green", vjust = -1.5, hjust = 0.5)
```



Metrics	Value
Number of Simulations	10000.0000
Mean Simulated Returns	0.0004
Median Simulated Returns	0.0004
Standard Deviation of Simulated Returns	0.0096
Worst Loss	-0.0375
Percentage of Losses	48.3200
VaR (95%)	-0.0154
ES (95%)	-0.0194
VaR (97.5%)	-0.0184
ES (97.5%)	-0.0221
VaR (99%)	-0.0219
ES (99%)	-0.0251

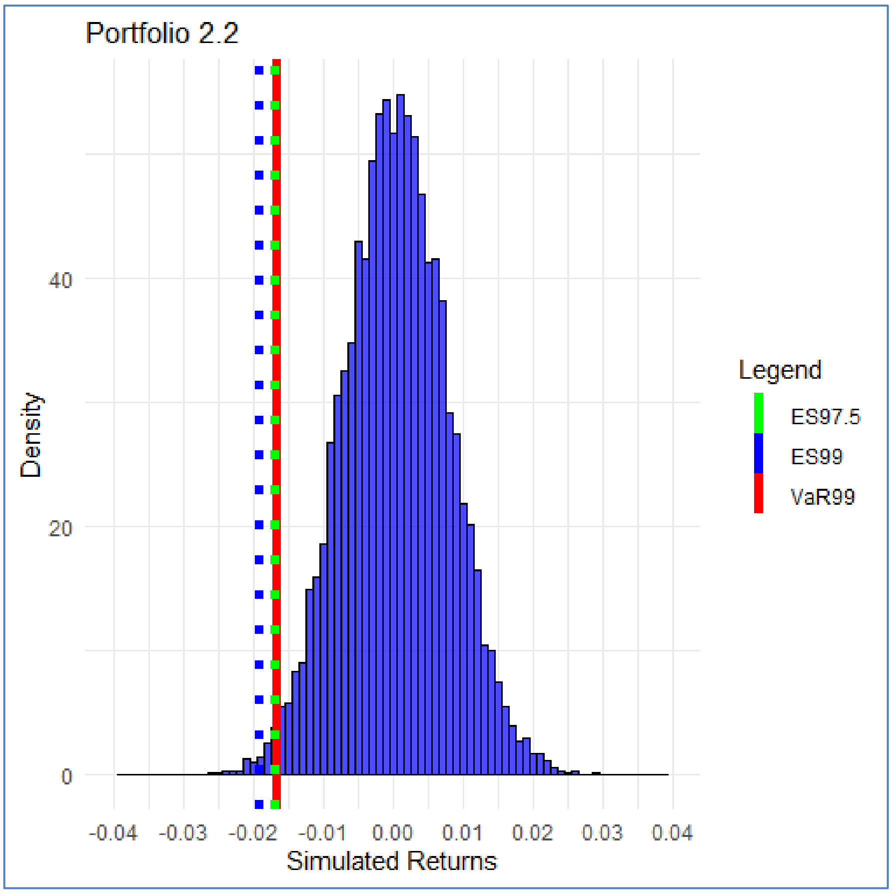
Figure 98: Distribution of Portfolio 1.2 simulated returns with VaR and ES results

The VaR and ES results in the **Portfolio (1.2),** computed with Montecarlo method, have some differences with the results of the Historical method: first of all the ES99% shows a value of 2,51%, lower than the Historical Method 3,04% while ES97,5%, ES95% are quite similar. Probably the losses happened over the VaR99% threshold in the last five years haven't been estimated with the same gravity by the simulations.

The VaR99% is equal to -2,20% while with the Historical method is -2,17%; the ES97,5% is equal to

-2,21% while in the Historical method -2,29%. We can notice immediately, thanks tot he proximity of the two values in this distribution, that the Montecarlo simulation has still generated a more regular distribution due the numerous iterations.

The worst loss is equal to -3,75% and the percentage of losses of Portfolio 1.2 is 48,32%.



Metrics	Value
Number of Simulations	10000.0000
Mean Simulated Returns	0.0003
Median Simulated Returns	0.0003
Standard Deviation of Simulated Returns	0.0075
Worst Loss	-0.0266
Percentage of Losses	48.2200
VaR (95%)	-0.0118
ES (95%)	-0.0149
VaR (97.5%)	-0.0142
ES (97.5%)	-0.0169
VaR (99%)	-0.0168
ES (99%)	-0.0194

Figure 99: Distribution of Portfolio 2.2 simulated returns with VaR and ES results

In the **Portfolio 2.2**, like in the previous, the values computed are different than the values computed with Historical method on the same portfolio, especially the most extreme.

The VaR 99% (-1,68%) is sensibly lower than Portfolio 1.2, thanks to the higher weight given to certain assets, but is also lower from the values of the VaR99% estimated on the same portfolio but with the Historical method (-1,80%).

The values of VaR99% and ES97,5% are still very close (-1,68% and -1,69%).

The worst estimated loss is -2,66%, while the worst loss in the last five years is -3,34%.

The percentage of losses are 48,22% (lower than Portfolio 1.2).

The tails of the simulated distribution seems have less extreme values than those observed in the last five years: the worst losses over the Var99% threshold also in this case haven't the same gravity of Historical observations (ES99% is 1,94%, while in the Historical Method is 2,36%).

In the table below are summarized all the results obtained with Montecarlo method for the computation of VaR and ES:

	VaR99%	VaR97.5%	VaR95%	ES99%	ES97.5%	ES95%
Portfolio 1	-2,03%	-1,71%	-1,41%	-2,29%	-2,04%	-1,79%
Portfolio 2	-1,62%	-1,37%	-1,14%	-1,86%	-1,63%	-1,45%
Portfolio 3	-2,53%	-2,13%	-1,82%	-2,90%	-2,55%	-2,25%
Portfolio 1.2*	-2,19%	-1,84%	-1,54%	-2,51%	-2,21%	-1,94%
Portfolio 2.2*	-1,68%	-1,42%	-1,18%	-1,94%	-1,69%	-1,49%

Figure 100: VaR and ES Montecarlo method results

***portfolios built considering historical data of last five years**

It has been observed that the results of VaR and ES, in particular the losses over a certain threshold (VaR99%,ES99%, ES97.5%) are always lower in the Montecarlo Method than in the Historical method; this happens because the events from the 10.000 simulations are not always extreme as in the historical data; the simulated distribution usually generates a more regular distribution as we can observe from the greater proximity between VaR99% and ES97.5% in every Portfolio in comparison to the historical distributions.

This happens because in the Historical Method there is the non-normality hypothesis, and for this reason we have in the three Portfolios of the two years time series the -VaR99%>-ES97.5%.

Instead in the Montecarlo Method there is the assumption that the simulated returns follow a normal distribution, also for this reason the generated distribution are very symmetric, the ES97.5% is never lower than VaR99% and the values of VaR99% and ES97.5% are very close in every result obtained with the Montecarlo Method.

CONCLUSIONS

The Value at Risk (VaR) and Expected Shortfall (ES) are two risk metrics used in the assessment of the future potential losses of an investment portfolio with the main difference that is about the consideration of magnitude of losses over the VaR threshold of the ES risk metrics.

The aim of this analysis was essentially to prove the greater accuracy of the ES risk metrics in comparison to VaR in the assessment of the risk and in the computation of market risk capital, as properly affirmed by the Fundamental Review of Trading Book (FRTB) that after the GFC, taking into account the inaccuracy of the VaR estimations in a period were prices fell sharply, decided to switch from VaR to ES for the computation of market risk capital.

The analysis involves an important focus also on the characteristics of the log-returns distributions of the singular assets chosen for the building of the various portfolios and of the various portfolios built, highlighting in this last case how also the VaR and ES results can give also important indications on a distribution: is important to highlight that the log-returns of the assets have been collected on a time series of two and five years.

The assets chosen for the analysis are very diversified between them, infact belong to different asset classes (5 company's stock, 2 indexes, 1 ETF, 1 BTP, 1 commodity) , are listed on different stock markets (3 on NYSE, 2 on NASDAQ, 2 on Euronext Paris, 1 on FWB (Frankfurter Wertpapierbörse**),** 1 on MOT, 1 on forex market/gold sport market) and the company's stocks belong all to different sectors.

The ten assets have been used to build one equally-weighted portfolio for each time series, one "normal" portfolio for each time series (greater weight to assets with a distribution more similar to a normal one), and one "strongly-non normal" portfolio only for the the two years time series (greater weight to assets with a distribution strongly non-normal): to estimate the VaR and ES results have been used the Historical Method and the Montecarlo Simulation with 10.000 iterations.

The results of analysis confirm that **Expected Shortfall(ES) estimates better the risk in comparison to Value at Risk(VaR)** especially in capturing the extreme losses in the tail of the distribution: as a matter of fact, in every Portfolio the average loss in stress conditions, necessary to compute the capital charge to mitigate the market risk, provided by the ES is always higher of the VaR's maximum loss.

The values of VaR and ES computed in the "normal" Portfolios, where assets with distributions more similar to the normal distribution are greater weighted, are lower, and in these cases Var99% and ES97.5% are very close, which means that the distribution of log-returns of the portfolio is more regular and more similar to a normal one due to the greater proximity of these two values.

Instead if we consider the "Strongly Non-Normal" Portfolios, where assets with distribution strongly non-normal so fat-tailed distributions are greater weighted, VaR and ES are showing a less stable risk profile and higher potential losses: also the distance between VaR99% and ES97.5% is higher compared to Equally Weighted and "Normal" Portfolios, therefore indicating a Portfolio distribution far from a normal one.

These results show the importance to understand the characteristics of the log-returns distributions when are used risk metrics like VaR and ES, as **the shape of the distribution affects the accuracy and reliability of these two risk metrics**: knowing if the distribution follows a normal distribution or somenthing different can make a big difference in the risk assessment and extreme losses.

The VaR and ES results, computed with Montecarlo Simulation, are usually lower than those computed with Historical Method in almost every quantile, but if we focus only on the most extreme quantiles so VaR99%, ES99% and ES97.5%, we can deduce how the extreme values of the 10.000 simulations are not extreme as those of the Historical data: surely the considered historical period with great instability, high volatility (especially the five-years time series) and extreme events that probably are not been fully captured by simulations, forcely leading to differences in the results.

Moreover the higher proximity of VaR99% and ES97.5% results in Montecarlo Simulation than Historical Method let to the conclusion that **the simulations generate a more regular distribution than the Historical data**: it happens because in the Montecarlo Method there is the assumption that the returns follow a normal distribution.

We can state that VaR and ES are two very important risk metrics used in the risk management, *de facto* being the most used by portfolio managers, financial institutions, banks, investors: briefly VaR is very simple to compute but doesn’t consider the more extreme values while ES is a coherent risk measure and is recognized by regulations but can be more complex to compute.

On that regard, it is likewise paramount to consider that are both models with their weaknesses and it is important use them with the appropiate caution.

Infact both metrics rely on historical data and the the past could not always be a good predictor for the future, because **a significative difference between past and future market conditions could take to inaccurate predictions.**

Moreover VaR and ES operate under the assumption that statistical properties of the assets are stationary like mean, standard deviation and correlations and so remain constant over time but ,contrary to this, the financial markets are non-stationary and could change in every moment; considering the assumption that past mean, past standard deviation and past correlations will remain costant in the future, it could take to understimate the risk of tail events.

It is important to combine this historical-based models with forward-looking indicators and stress tests that account for potential future scenarios.

The backtesting is an important tool used to evaluate the accuracy and quality of the model comparing the losses of the model with the real losses and to check that model provides a real estimation of extreme losses.

Backtesting on VaR is easier than ES and is very important to continuosly update the models to the current market conditions to improve the accuracy of estimations.

References

Hull J, « Risk management and financial institutions », 2006

Artzner P, F Delbaen, J-M Eber and D Heath, « Coherent measures of risk », 1999

Artzner P. , Delbaen, F., Eber J. M., and Heath D. «Thinking coherently. Risk » 68–71, 1997

R. Hauser, «Coherent Risk Measures » Oxford University, 2009

A. Krause, «Exploring the Limitations of Value at Risk: How Good Is It in Practice?,» The Journal of Risk Finance , 2003

V. Dendoncker and A Lebeguè, « Comparative backtesting of Expected Shortfall », 2019

S. Benati, « The optimal portfolio problem with coherent risk measure constraints», European Journal of Operational Research, 2003

Dowd, Kevin, « Measuring Market Risk. » 1st ed. Wiley, 2005

D. Hendricks, « Evaluation of Value-at-Risk Models Using Historical Data », 1996

Acerbi C. and Tasche D. « On the coherence of expected shortfall » Journal of Banking & Finance, 1487–1503, 2002

Acerbi C. and Tasche D. « Expected Shortfall: a natural coherent alternative to Value at Risk », 2001

Hofert M., Hornik K., and McNeil A. J. « qrmtools: Tools for Quantitative Risk Management », 2021

Jorion P. «Value at Risk: The New Benchmark for Managing Financial Risk » McGraw-Hill, New York, 3 edition, 2007

Simona Riccioletti « Backtesting Value at Risk and Expected Shortfall », 2016

Chia-Lin Changa, Juan-Angel Jimenez-Martinb, Esfandiar Maasoumic, Michael McAleerb, Teodosio Perez-Amaralb « Choosing expected shortfall over VaR in Basel III using stochastic dominance » , International Review of Economics and Finance, 2019

Alexander J McNeil, Rüdiger Frey e Paul Embrechts. « Quantitative risk management: concepts, techniques and tools », 2015

Natalia Nolde e Johanna F Ziegel.. «Elicitability and backtesting: Perspectives for banking regulation » , 2017

Sitography

https://www.wallstreetmojo.com/expected-shortfall/

https://www.mathworks.com/help/risk/estimate-var-using-parametric-methods.html

https://rpc.cfainstitute.org/en/research/cfa-digest/1997/02/evaluation-of-value-at-risk-models-using-historical-data-digest-summary