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Risk Parity Portfolio: Backtesting and Performance Analysis Across Different Time Horizons

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Abstract

This thesis analyzes the construction and performance evaluation of two Risk Parity portfolios across two different time horizons. Risk Parity is an investment strategy that focuses on allocating risk equally among asset classes rather than capital. The dissertation begins with a theoretical framework that examines which asset classes should be included in the portfolio, based on an analysis of the underlying risk premia—equity, interest rate, and inflation risk premia. The methodology used for portfolio construction relies on the Newton-Raphson algorithm, ensuring an equal risk contribution among assets. The empirical analysis evaluates the performance of Risk Parity portfolios with two and four asset classes over two distinct periods: 2008-2020 and 2011-2023. Backtesting is employed to assess Risk Parity strategies relative to a traditional allocation model, specifically a 60/40 portfolio. Performance is measured using three key risk-adjusted indicators: the Sharpe Ratio, Treynor Ratio, and Sortino Ratio. Additionally, the analysis examines the effects of dynamic rebalancing to determine its impact on performance when new information becomes available. However, rebalancing did not result in any significant performance improvement. Findings show that the Risk Parity portfolio outperforms the traditional allocation strategy in the first sample, for both the two-asset and four-asset portfolios. However, in the second sample, the situation reverses, with the Risk Parity portfolio underperforming relative to the benchmark. This suggests that the effectiveness of the Risk Parity approach may be influenced by specific market environments and macroeconomic conditions. This study contributes to the existing literature by conducting backtesting to assess whether a portfolio constructed according to the Risk Parity approach is a viable long-term investment strategy.

Contents

Introduction

This study focuses on the construction and performance evaluation of two Risk Parity portfolios analyzed over two distinct time horizons.

The concept of Risk Parity was first introduced in the academic literature by Qian et al. (2005) and refers to an investment strategy centered around risk allocation rather than capital allocation. Qian was among the firsts who systematically outline the key elements for constructing a Risk Parity portfolio, ensuring that risk is distributed equally among asset classes. Traditional portfolios, such as many well-known indexes and the widely recognized 60/40 portfolio, are typically capital-based. It means they are constructed by allocating a fixed percentage of capital to assets rather than based on the risk contribution of each one. As argued by Qian (2016), while the 60/40 portfolio may appear balanced from a capital allocation perspective, it is highly unbalanced in terms of risk. This imbalance arises because equities are generally more volatile than bonds, causing the portfolio's performance to be overwhelmingly driven by the equity market fluctuations. As a result, the 60/40 portfolio is heavily influenced by economic cycles and vulnerable to equity downturns.

However, a strategy belonging to the broader category of Risk Parity investments had already been introduced to the market in 1996 by Ray Dalio and his team at Bridgewater Associates, one of the largest hedge funds in the world, headquartered in the United States. The idea behind the creation of the strategy and the portfolio known as "All Weather" stems from the inherent difficulty, and sometimes impossibility, of accurately predicting the long-term evolution of the economic cycle.This led to the development of a passive investment strategy designed to perform well in all market conditions, based on the macroeconomic relationships between various asset classes (Bridgewater Associates (2012), *The All Weather Story*).

The first part of this study focuses on selecting which asset classes to include in the portfolio and in what proportion. The choice of securities is driven by an analysis of the risk premia underlying these assets, which will be further detailed in Chapter 1. The major sources of return in the space of liquid assets are derived from equity, interest rate, and inflation risk premia. This criterion is used in conjunction with studies related to the All Weather strategy. Therefore, the selection of assets is based on how asset classes respond to different macroeconomic scenarios. Specifically, these include periods of rising/falling inflation and rising/falling growth.

Following the identification of which asset classes to include or exclude, the choice of specific weights for each class adopts a more academic, mathematical, and rigorous approach. The primary contributor to this methodology is E. Qian, with his work, *The Risk Parity Fundamentals.* The weight allocation is determined using an algorithm based on the Newton-Raphson method, which aims to ensure that the risk contribution of each asset class is equal. This approach measures the extent to which each asset contributes to the overall portfolio risk, defining a Risk Parity portfolio as one where the risk contributions of all assets are equal. In practice, this means that riskier assets receive smaller allocations, while more stable ones are weighted more heavily to ensure an even distribution of risk.

To conclude, in other words, the portfolio definition is the result of the combination of two distinct yet complementary approaches, both falling under the broader category of Risk Parity investments.

At this point, after defining the portfolio, Chapters 4 and 5 focus on the empirical analysis. The first stage involves a simplified case with two assets. While theory suggests that a true Risk Parity portfolio should have an equal distribution to the three risk premiums outlined above, this initial two-asset model (comprising equities and bonds) serves as a foundational step. The objective is to evaluate whether a two-asset portfolio—constructed using Risk Parity weights—outperforms a traditional 60/40 portfolio built with the same assets. In other words, beginning with a simple two-asset portfolio allows for a step-by-step, incremental approach to the construction and testing of the Risk Parity model. This provides a clear comparison between traditional allocation methods and Risk Parity-based approaches, laying the groundwork for more complex multi-asset portfolios.

The analysis then expands to incorporate additional asset classes to capture all risk premiums, specifically: Equities, Government Bonds, Commodities and Inflation-Linked Government Bonds. For both the two-asset and multi-asset cases, the backtesting is conducted across two different time periods:

- 2008 to 2020 (first sample)
- 2011 to 2023 (second sample)

Each sample is then divided into two key sub-periods:

- Estimation Window: A segment used to calculate asset weights and construct the portfolio.
- Testing Window: The subsequent period during which the performance of the portfolio is evaluated.

After the static analysis, an annual rebalancing is applied to evaluate whether the portfolio's performance improves as new information becomes available.

The ultimate objective is to assess whether the Risk Parity portfolio consistently delivers better performance compared to conventional strategies, across various economic environments and market conditions. This thesis finally seeks to offer a comprehensive exploration of Risk Parity, from theoretical foundations to practical implementation. By testing the model across different scenarios and asset configurations, the study aims to highlight the strengths and potential limitations of the Risk Parity approach.

PART 1: Theoretical Framework

1 Portfolio construction: asset class selection

The investment landscape nowadays is far more complex compared to the past, when it was primarily divided into two main asset classes: stocks and bonds. Over time, a third category—alternatives—has emerged, adding another layer of diversification. Each of these macro asset classes presents its own internal complexities.

Equities, for instance, can be classified by style (large/mid/small or growth vs value) or by geography (US vs non-US, developed vs emerging). Similarly, bonds can be segmented by credit quality (investment grade to high yield), maturity (short, intermediate or long-term), issuer type (sovereign vs corporate), and geographic location. Alternative investments are undoubtedly the most diverse category which includes commodities, hedge funds, private equity, real estate, and infrastructure. A key question arises: which specific assets should be included in a portfolio to make it Risk Parity? The primary sources that contributed to the drafting of this chapter are *The Risk Parity Fundamentals* by E. Qian (2016), *Risk Parity: How to Invest for All Market Environments* by A. Shahidi (2021), and *The All Weather Story* by Bridgewater Associates (2012).

The selection of assets for a Risk Parity portfolio should be based on an analysis of the risk premia associated with each asset, rather than simply by categorizing them as equities, bonds, or alternatives. Assets within the same class can differ significantly in terms of their underlying risk, while assets across different categories may share similar risk characteristics. Therefore, the choice of which bonds, equities, or alternative investments to include is critical and should reflect the goal of balancing exposure across various sources of risk. This approach is crucial to prevent the construction of a portfolio that appears to be Risk Parity, because it includes diverse asset categories but it is, in reality, heavily concentrated along a single risk dimension. Such concentration often results from misclassifying asset classes into inappropriate risk categories.

To illustrate this framework, there are three primary risk premia associated with liquid asset classes (Qian, 2016):

- Equity risk premium
- Interest rate risk premium
- Inflation risk premium

The equity risk premium is the reward to investors for providing capital to businesses and, by extension, to the broader economy. Equity investors assume the risks of companies' future performance and growth, therefore they must be compensated for it.

Mathematically, the equity risk premium is measured as the expected return on equities in excess of the risk-free rate: $ERP = E[R_{\text{equity}}] - R_f$

The interest rate risk premium, on the other hand, is the compensation to bondholders for lending money to governments or corporations. Bond investors receive the principal plus interest payments at maturity, reflecting the return for providing credit. Typically, an upward-sloping yield curve indicates the presence of this risk premium where, all else being equal, bonds with a longer maturity have a higher yield than bonds with a shorter one.

Finally, the inflation risk "premium" measures the compensation for bearing inflation risk. The term "premium" is placed in quotation marks because, in equilibrium, the inflation rate should align with the risk-free rate meaning a risk premium would no longer exist. The situation is completely different in contexts of rising and volatile inflation where exposure to inflation risk (such as inflation-linked bonds or commodities) help preserve the real value of investments.

To achieve Risk Parity, therefore, a portfolio should balance exposure at least across these three key dimensions of risk. Therefore, which assets are best suited to capture the equity, interest rate, and inflation risk premia?

1.1 Equity Risk Premium

To capture the equity risk premium, equities are the most straightforward choice, as they inherently represent this type of risk. Given the broad range of options available, selecting an index that ensures balanced exposure across different market segments is essential – diversification is the key objective. For the analysis conducted later in this work, the MSCI World Index has been chosen. This index offers comprehensive coverage by mainly including large and mid-cap stocks from 23 developed markets, providing broad diversification across regions and sectors. (Emerging markets are intentionally excluded to maintain clarity and focus).

According to Shahidi (2021) and Bridgewater Associates (2012), stocks tend to perform well during periods of low inflation and rising economic growth. Numerous studies have been conducted on the relationship between stock prices, inflation, and output growth: Zhao (1999), Shahbaz et al. (2008), and Rashid (2008). Specifically, Gabrie and Devkota (2023) shows that, in the long run, the CPI is negatively related to stock prices, while output growth is positively linked. Economic growth drives up consumption and spending, benefiting companies by boosting production, sales, and ultimately profits. Similarly, in environments of declining inflation, the cost of inputs used in production decreases, leading to higher profit margins for companies.

Now, the analysis will focus on how to classify real estate investments in terms of risk premium and assess whether they should be included in a Risk Parity portfolio. As previously mentioned, commercial real estate falls within the broader category of alternative investments. Historically, investing in real estate exclusively meant directly acquiring the asset and participating in its management, development, or disposition.

However, starting in the 1960s in the United States, new publicly traded instruments were introduced, known as Real Estate Investment Trusts (REITs). These instruments provide investors with greater accessibility to real estate investments, as they are publicly traded and comparable to mutual funds.

The significant advantage of REITs, therefore, is that they provide exposure to the real estate market without sacrificing liquidity, an essential quality for building a Risk Parity portfolio. By contrast, direct investments in real assets are considerably less liquid and less affordable. The downside, however, when analyzing REITs from a Risk Parity portfolio perspective, is that they are comparable to equities in terms of risk premium. Like equities, they are biased to perform well in environments of rising economic growth. This is because investors are not only exposed to the underlying investment (the real asset) but also to the associated stock market risks. Consequently, REITs are also impacted by movements in the equity market. The study by Chaudhry et al. (2022) demonstrated that REITs tend to perform well during periods of strong economic growth.

That said, when it comes to the behavior of real assets during periods of high or rising inflation, their reaction tends to be less predictable. On one hand, investments in real assets have traditionally provided protection in high-inflation environments (Larsen and McQueen (1995)), as rental contracts are often indexed to inflation, and real estate values tend to increase over time. However, regarding REITs, studies such as that by Park et al. (1990) have shown that these instruments are not effective as a hedge against inflation. They are similarly biased to perform well under the same conditions as stocks (high growth and low inflation).

In light of this analysis, real estate investments should not be included in a Risk Parity portfolio. On one hand, direct investments in real assets would provide portfolio diversification and act as a hedge in high-inflation contexts. However, they fall under the category of private investments and do not offer the necessary liquidity. On the other hand, investing in REITs, while certainly addressing the liquidity issue, fails to provide the required diversification benefit and instead increases exposure to the equity risk premium.

1.2 Interest Rate Risk Premium

The most effective way to capture the interest rate risk premium is through government bonds.

Among the main criticisms of the Risk Parity portfolio is the claim of an excessive allocation to government bonds, which historically offer much lower returns compared to stocks. In the years following the 2008 crisis, short-term interest rates fell so low that they approached zero, leading to discussions about the 'death' of the interest rate risk premium (Qian (2016)). Forecasts for the following years predicted an inevitable rise in interest rates and a corresponding decline in bond prices. However, despite being based on valid assumptions, these predictions turned out to be inaccurate: interest rates remained low for years, contrary to expectations. Accurately predicting changes in interest rates is, therefore, more complex than it might seem, as it depends on multiple factors, primarily the macroeconomic context.

Despite offering lower average returns than stocks, government bonds are essential for building a Risk Parity portfolio because they provide diversification benefits, particularly during periods of crisis and uncertainty (Skintzi (2019)). During economic downturns and crises, the correlation between stocks and bonds tends to decrease, leading to 'flight-to-safety' dynamics. This is one of the main drivers of negative stock-bond covariance (Antolin-Diaz (2025)). Government bonds are generally considered safe assets due to their near-zero risk of default and their ability to perform well during 'bad times' for stocks.

To better understand their role, it is useful to analyze the behavior of government bonds in different macroeconomic contexts. Government bonds tend to perform well in environments characterized by low economic growth and low inflation.

When economic growth is lower than expected and this situation persists, central banks often intervene by lowering short-term interest rates to stimulate the economy. This intervention typically leads to an increase in bond prices. Additionally, during periods of low growth, investors tend to become more risk-averse and shift part of their capital towards safer assets, such as government bonds, to protect their investments.

Regarding inflation, when it falls below expectations, central banks are also likely to reduce short-term interest rates, further driving an increase in government bond prices.

In summary, although government bonds offer lower returns than stocks, their role in a Risk Parity portfolio is crucial as they reduce the overall risk and provide an essential source of diversification.

What about other types of bonds?

Corporate bonds, issued by corporations rather than governments, are generally considered riskier because are not guaranteed by government's ability to print money. Due to their higher risk of default, corporate bonds offer a higher yields, which depend on their credit rating - the lower the rating, the higher the risk premium. In theory, corporate bonds with a triple-A rating are not significantly different from government bonds in terms of default risk. They are not guaranteed by a government, but by highly creditworthy companies. However, the key distinction between corporate and government bonds lies in the underlying risk premia. As previously discussed, government bonds are the best asset to represent the interest rate risk premium. Conversely, corporate bond prices are influenced by both equity and interest rate risk premia. Bondholders finance business operations and their returns are linked to company's cash flows. However, corporate bonds are less risky than stocks because bondholders have priority over stockholders in case of bankruptcy. Nevertheless, including corporate bonds in a Risk Parity portfolio, may not provide the necessary diversification benefits as it could disproportionately increase exposure to the equity risk premium. This highlights the importance of correctly classifying each asset within its appropriate risk premium category.

To conclude, let's consider high-yield (HY) bonds and whether it is advisable to include them in the analysis. HY bonds are commonly seen as an hybrid asset between bonds and equities. One might assume that investing directly in HY, instead of both government bonds and equities, could be a good strategy to reduce investment costs. However HY bonds share similar risk characteristics with equities in term of risk premium. Both are significantly influenced by the business cycle and are residual claimants in case of default. Given the high similarity between these two asset classes, one might think they could be interchangeable. So why not to invest in HY bonds instead of equities? The analysis changes when liquidity risk is considered. HY bonds are highly illiquid assets, making them difficult to trade on the market. A critical feature of a Risk Parity portfolio is liquidity as it enables easy rebalancing as will be discussed later. In conclusion, there is no compelling reason to invest in either corporate bonds or HY bonds. These asset classes offer limited diversification benefits and unnecessarily increase the portfolio's exposure to equity-like risks.

1.3 Inflation Risk Premium

Stocks and government bonds, as previously illustrated, tend to move in opposite directions during periods of rising and falling economic growth. A risk-balanced portfolio composed of these two assets is therefore created to perform well in both macroeconomic scenarios. However, stocks and bonds tend to move in the same direction in contexts of high and low inflation environments, both suffering in times of elevated inflation. For bonds, high inflation erodes the real value of future payments and typically leads to an increase in interest rates, reducing the prices of existing bonds. For stocks, rising inflation increases production costs, negatively impacting margins, and reduces consumers' purchasing power, both of which negatively affect stock prices.

Including assets capable of performing well in scenarios of high or rising inflation is therefore essential to preserve the real value of investments.

The asset classes best suited for this purpose are commodities and inflation-linked government bonds (ILGBs). Commodities, which belong to the alternative investment landscape, differ from traditional assets as they are not financial instruments and do not provide income or coupons. Purchasing physical commodities, in fact, does not offer a risk premium *per se*. In practice, commodity investments are typically made through futures contracts rather than directly purchasing the raw materials. Futures enable investors to gain exposure to commodity markets without incurring the logistical issues and storage costs of physical assets. In this context, the financial value of these contracts arises from the buyer assuming the risk of a price decline, which the seller seeks to protect himself from. Futures are also advantageous because they are traded on regulated markets, making commodity investments as liquid as stocks and bonds.

The link between inflation and commodity prices is twofold. The Consumer Price Index (CPI) is directly influenced by the prices of materials primarily including industrial metals, energy and agricultural products. Furthermore, commodities form the base of many other products included in the CPI calculation, indirectly affecting inflation measurements as well. As a result, higher inflation tends to benefit commodities because their prices generally rise. This effect occurs not only because commodities are directly included in the CPI but also because price increases in commodities often lead to higher production costs, which, in turn, affect the prices of goods and services in the broader economy.

Commodities also tend to outperform during periods of rising economic growth. In particular, since the 1970s, especially industrial commodities have shown cyclical price declines that are closely related with economic downturn. Commodities in general share common drivers and tend to be positively correlated with business cycles. Kabundi et al. (2022).

Commodity prices generally worsen in recession due to weak global economic demand. On the other hand if welfare increases, the demand for commodities rises as well, leading to higher prices. As before, the linkage is twofold: better economic conditions likely boost both direct spending on higher-quality commodities, such as precious metals, and consumption of superior products, wich require more and better raw materials for their production.

Commodities are a heterogeneous group that includes various sectors, such as agriculture, livestock, energy, and metals. It is therefore important to highlight that the various sectors can move in opposite directions within the same macroeconomic context. Gold il usually assumed to be an asset that performs very well in periods of high inflation and economic uncertainty, as occurred during the 70s (Futerman and Sarjanovic (2022)). It can be compared to a currency and for that its value today primarily comes from its status as a store of wealth rather than from jewelry production. Therefore, during times of crisis, the demand for this precious metal tends to increase.

In light of the considerations outlined and based on the investment insight suggested by Shashidi (2021), the index chosen to represent commodities will consist of 60% industrial commodities and 40% gold. Specifically, the two indexes selected are the Bloomberg Industrial Metals Subindex for industrial commodities and the LBMA Gold Price PM for gold.

The other asset capable of performing well in contexts of rising inflation is inflation-linked government bond (ILGBs). As the name suggests, these are government bonds whose principal is adjusted for inflation so that returns increases in scenarios of rising prices. They are comparable to government bonds in terms of credit risk, but they pay a lower base coupon because it is augmented by the current inflation rate. ILGBs, or TIPS (Treasury Inflation-Protected Securities) in the US market, are among the safest assets available due to these characteristics. A common concern among investors regarding these assets, in fact, is that their returns may not be sufficiently attractive to justify their inclusion in a portfolio. From a Risk Parity perspective, it is incorrect to evaluate individual assets in isolation. Instead, it is crucial to assess the benefits each asset can bring to the overall portfolio in terms of risk diversification. A narrow focus on individual assets could lead to poor allocation decisions. In light of this, while it is true that ILGBs offer lower returns compared to stocks, they are indispensable in a Risk Parity portfolio. They perform well not only in contexts of high inflation but also during periods of low economic growth, based on the same principle that applies to government bonds. Therefore, they respond in a manner diametrically opposed to stocks. For ILGBs to perform well, it is not necessary to experience extreme economic conditions with skyrocketing inflation (such as in the 1970s and 1980s); both ILGBs and commodities indeed tend to perform well when inflation exceeds prior expectations.

Having identified the asset classes to be included in a Risk Parity portfolio—equities, government bonds, commodities, and inflation-linked government bonds—it becomes evident that each asset performs well under distinct and complementary macroeconomic scenarios. A key criterion for the selection has been the analysis of their performance across different economic scenarios to determine when they are likely to perform well and when they are not. The ultimate goal is, in fact, to construct a portfolio that remains resilient regardless of future macroeconomic environments. This strategy was originally developed and implemented by Ray Dalio through his firm, Bridgewater Associates, in the creation of the so-called "All Weather" portfolio — a name that highlights its fundamental characteristic of being designed to withstand any economic season. (Bridgewater Association, 2012). For the purposes of this study, the All Weather strategy has been used as a complementary criterion alongside the risk premia analysis for the selection of asset classes in portfolio construction. This dual approach enhances the robustness of asset selection by integrating insights from macroeconomic scenario analysis with traditional risk premia considerations.

This framework identified two primary macroeconomic drivers of asset prices: inflation and economic growth, resulting in four possible environments – rising or falling growth, and rising or falling inflation. The table below summarizes the key points highlighted in the analysis conducted.

	Rising Growth	**Falling Growth**
Rising Inflation	Commodities	Inflation-linked Bonds
Falling Inflation	Stocks	Bonds

Table 1: Drivers of asset classes' performance

Now that the assets have been identified, the next step is to determine their respective weights. To achieve Risk Parity, it is crucial that each asset contributes equally to the total risk. This ensures that no single asset dominates in the portfolio, maintaining diversification and reducing the likelihood of concentrated exposures.
Risk contribution will be explained in detail in the next chapter.

2 Risk Contribution

Risk contribution is a fundamental tool in risk management, used to monitor and assess how much each asset contributes to the overall risk of an existing portfolio. However, in the Risk Parity strategy, risk contribution serves as the core criterion for portfolio construction. In other words, rather than creating a portfolio and subsequently evaluating the risk of its individual components — often resulting in imbalances — in a Risk Parity strategy the risk contribution is calculated beforehand. This ensures that every asset contributes equally to the portfolio's overall volatility, resulting in a more balanced allocation of risk. By doing so, the Risk Parity method distributes risk across various asset classes, reducing dependency on any single one and enhancing the portfolio's resilience to different macroeconomic conditions. Moreover, constructing a Risk Parity portfolio requires only the calculation of risk contribution, eliminating the need for return forecasts or assumptions about future conditions.

Risk contribution must not be confused with marginal risk. The latter measures how the total portfolio risk (dependent variable) changes as the weight of one or more constituent assets (independent variable) varies. It is calculated as the partial derivative of the portfolio risk with respect to the weight of each individual asset. In essence, marginal risk provides a dynamic analysis of risk contribution, while risk contribution itself offers a static snapshot of the current state.

This approach differs significantly from the traditional mean-variance optimization model (Markowitz, 1952). It heavily relies on forecasts of expected returns and the process aims to construct a portfolio by minimizing variance for a given level of return - or maximizing return for a given level of risk. The issue with this model is its high sensitivity to assumptions; even minor deviations from initial estimates can significantly impact the outcome. The method selects assets through an optimization technique with the objective of maximizing returns or minimizing variance. As a result, mean-variance optimization frequently leads to portfolios that are overly concentrated in specific assets or risk factors. While mathematically robust, the success of this process hinges on the accuracy and reliability of the forecasted returns: even slight deviations from the estimated values can result in significant losses. Therefore, the mean-variance optimization process is effective only if predictions regarding returns, variances, and asset correlations prove to be precise and reflect actual market conditions.

Risk Parity strategy, by contrast, starts from the assumption that accurately predicting future economic conditions and asset prices is highly challenging. Historical events such as the 2008 financial crisis, the COVID-19 pandemic, or geopolitical conflicts highlight the difficulty of forecasting market behavior. For this reason, the Risk Parity strategy emphasizes risk allocation over capital allocation, aiming to build highly diversified portfolios that are more resilient to withstand market uncertainties.

Traditional portfolios are often unbalanced and overexposed to equity risk, as equities have historically provided higher returns than bonds, but with significantly greater volatility. To verify whether a portfolio has been constructed based on Risk Parity principle, it is sufficient to compute the risk contribution of each component and ensure that all contributions are equal. Risk contribution is calculated as the covariance between the weighted return of an asset and the return of the total portfolio over the total portfolio volatility:

$$\text{Risk Contribution}_i = \frac{\text{Cov}[w_i r_i, r_p]}{\text{Var}[r_p]}$$

where:

- w_i is the weight assigned to asset i in the portfolio,
- r_i is the return of asset i,
- r_p is the total portfolio return
- $Var[r_p]$ is the total portfolio volatility.

2.1 RC calculation, case with two asset classes

We now generalize to the simpler case with two asset classes, where $i = 1, 2$. These two will have respectively rates of return $\tilde{r}_1$ and $\tilde{r}_2$, variances σ_1^2 and σ_2^2 and covariance (correlation) $\sigma_{1,2}$ ($\rho_{1,2}$).

A portfolio p invested in these two asset classes, with weights w_1 and $w_2 = 1 - w_1$ will have rate of return:

$$\tilde{r}_p = w_1 \tilde{r}_1 + w_2 \tilde{r}_2;$$

variance:

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_{1,2};$$

and covariances respectively:

$$\text{Cov}[w_1 \tilde{r}_1, \tilde{r}_p] = w_1^2 \sigma_1^2 + w_1 w_2 \sigma_{1,2};$$

$$\text{Cov}[w_2 \tilde{r}_2, \tilde{r}_p] = w_1 w_2 \sigma_{1,2} + w_2^2 \sigma_2^2.$$

1) The first step is to find the weights of each asset for constructing the portfolio. To achieve Risk Parity, the two risk contributions must be equal, so that $RC_1 = RC_2$. From the definitions above, the condition holds if:

$$\mathrm{Cov}[w_1\tilde{r}_1, \tilde{r}_p] = \mathrm{Cov}[w_2\tilde{r}_2, \tilde{r}_p],$$

hence:

$$w_1^2\sigma_1^2 + w_1w_2\sigma_{1,2} = w_1w_2\sigma_{1,2} + w_2^2\sigma_2^2,$$

which simplifies to:

$$w_1^2\sigma_1^2 = w_2^2\sigma_2^2.$$

Since $w_2 = 1 - w_1$, the formula can be rewritten as:

$$w_1^2\sigma_1^2 = (1-w_1)^2\sigma_2^2 \quad \Rightarrow \quad w_1\sigma_1 = (1-w_1)\sigma_2.$$

which holds for $w_1 \in (0,1)$.
Then solving for w_1:

$$w_1^{\mathrm{RP}} = \frac{\sigma_2}{\sigma_1+\sigma_2}, \quad \text{and consequently} \quad w_2^{\mathrm{RP}} = \frac{\sigma_1}{\sigma_1+\sigma_2}. \tag{1}$$

We have found the two weights. As shown, in the simpler case with two assets, it is sufficient to know variances of the total portfolio and of the two assets. Covariances are not necessary for the calculation.

2) Now that we have the weights, we can compute the covariances between $w_1^{\mathrm{RP}}\tilde{r}_1$ and $\tilde{r}_p$, and between $w_2^{\mathrm{RP}}\tilde{r}_2$ and $\tilde{r}_p$.

$$\mathrm{Cov}[w_1^{\mathrm{RP}}\tilde{r}_1, \tilde{r}_p] = (w_1^{\mathrm{RP}})^2\sigma_1^2 + w_1^{\mathrm{RP}}w_2^{\mathrm{RP}}\sigma_{1,2}.$$

Substituting $w_1^{\mathrm{RP}} = \frac{\sigma_2}{\sigma_1+\sigma_2}$ and $w_2^{\mathrm{RP}} = \frac{\sigma_1}{\sigma_1+\sigma_2}$, we get:

$$\mathrm{Cov}[w_1^{\mathrm{RP}}\tilde{r}_1, \tilde{r}_p] = \frac{\sigma_1^2\sigma_2^2}{(\sigma_1+\sigma_2)^2} + \frac{\sigma_1\sigma_2}{(\sigma_1+\sigma_2)^2}\rho_{1,2}\sigma_1\sigma_2{}^{1}.$$

[1] Here we made use of the fact that $\sigma_{1,2} = \rho_{1,2}\sigma_1\sigma_2$

which simplifies to:

$$\text{Cov}\left[w_1^{\text{RP}}\tilde{r}_1, \tilde{r}_p\right] = \frac{\sigma_1^2\sigma_2^2}{(\sigma_1+\sigma_2)^2}(1+\rho_{1,2}). \tag{2}$$

Similarly:

$$\text{Cov}[w_2^{\text{RP}}\tilde{r}_2, \tilde{r}_p] = w_1^{\text{RP}}w_2^{\text{RP}}\sigma_{1,2} + (w_2^{\text{RP}})^2\sigma_2^2$$

$$= \frac{\sigma_1\sigma_2}{(\sigma_1+\sigma_2)^2}\rho_{1,2}\sigma_1\sigma_2 + \frac{\sigma_1^2\sigma_2^2}{(\sigma_1+\sigma_2)^2}$$

$$\text{Cov}[w_2^{\text{RP}}\tilde{r}_2, \tilde{r}_p] = \frac{\sigma_1^2\sigma_2^2}{(\sigma_1+\sigma_2)^2}(\rho_{1,2}+1) = \text{Cov}\left[w_1^{\text{RP}}\tilde{r}_1, \tilde{r}_p\right]. \tag{3}$$

We have demonstrated that, with w_1^{RP} and w_2^{RP} (1), the risk contributions of the two assets are identical and we have therefore constructed a Risk Parity portfolio.

Finally, variance of $\tilde{r}_p$ is:

$$\text{Var}[\tilde{r}_p] = (w_1^{\text{RP}})^2\sigma_1^2 + (w_2^{\text{RP}})^2\sigma_2^2 + 2w_1^{\text{RP}}w_2^{\text{RP}}\sigma_{1,2}$$

$$= \frac{\sigma_2^2}{(\sigma_1+\sigma_2)^2}\sigma_1^2 + \frac{\sigma_1^2}{(\sigma_1+\sigma_2)^2}\sigma_2^2 + 2\frac{\sigma_2\sigma_1}{(\sigma_1+\sigma_2)^2}\sigma_{1,2}$$

$$= 2\frac{\sigma_1^2\sigma_2^2}{(\sigma_1+\sigma_2)^2} + 2\frac{\sigma_1\sigma_2}{(\sigma_1+\sigma_2)^2}\rho_{1,2}\sigma_1\sigma_2$$

$$= \frac{\sigma_1^2\sigma_2^2}{(\sigma_1+\sigma_2)^2}(2+2\rho_{1,2}) = 2\,\text{Cov}[w_1^{\text{RP}}\tilde{r}_1, \tilde{r}_p] = 2\,\text{Cov}[w_2^{\text{RP}}\tilde{r}_2, \tilde{r}_p].$$

which is double the covariances and then implies:

$$\frac{\text{Cov}[w_1^{\text{RP}}\tilde{r}_1, \tilde{r}_p]}{\text{Var}[\tilde{r}_p]} = \frac{1}{2}, \quad \frac{\text{Cov}[w_2^{\text{RP}}\tilde{r}_2, \tilde{r}_p]}{\text{Var}[\tilde{r}_p]} = \frac{1}{2}.$$

This is equivalent to saying that the betas of the two components with respect to the overall portfolio are the same and equal to $\frac{1}{2}$. The Risk Parity portfolio, in fact, ensures that the contribution to the total risk is equal for each asset. To compute the beta it is enough to multiply the previously determined beta, equal to $\frac{1}{2}$, by the reciprocal of the Risk Parity weight that is:

$$\beta_1 = \frac{\text{Cov}[\tilde{r}_1, \tilde{r}_p]}{\text{Var}[\tilde{r}_p]} = \frac{1}{2} \cdot \frac{1}{w_1^{RP}} = \frac{1}{2} \cdot \frac{\sigma_1 + \sigma_2}{\sigma_2},$$

$$\beta_2 = \frac{\text{Cov}[\tilde{r}_2, \tilde{r}_p]}{\text{Var}[\tilde{r}_p]} = \frac{1}{2} \cdot \frac{1}{w_2^{RP}} = \frac{1}{2} \cdot \frac{\sigma_1 + \sigma_2}{\sigma_1}.$$

To conclude, it might be appropriate to illustrate, using an example, how to determine the contribution of individual assets to the total variance of a 60/40 portfolio, which will serve as the benchmark for the following analysis. Consider, as such, a classic 60/40 portfolio composed of two assets, with 60% allocated to the S&P 500 index and 40% to the WGBI. We analyze monthly returns from 2000 to 2023. The annualized volatility of the S&P 500 returns is approximately 15.57%, while that of the WGBI is about 7%. Additionally, the two assets exhibit a correlation of 0.17. Based on these results, calculating each asset's contribution to the portfolio's total risk reveals a significant imbalance: the S&P 500 contributes approximately 88% to the total risk, while the WGBI contributes only 12%. A portfolio that appears 'balanced' in terms of capital allocation is, in reality, highly unbalanced in terms of risk.

2.2 RC calculation, general case with multiple asset classes

Let's now generalize the analysis to the case with n asset classes, each identified by an index i (with i = 1, 2, ..., n). We want a portfolio p such that the sum of the weights assigned to each class is equal to 1, which is analogous to saying that $\mathbf{w}^T\mathbf{1} = 1$[2]. This requirement does not ensure that all weights will be positive, but since we assume that all asset classes are positively correlated, it should generally be satisfied.

We now define $\mathbf{\Sigma}$ as an $n \times n$ symmetric square matrix containing the variances and covariances of the n asset classes. The element in row i and column j, with $i \neq j$, represents the covariance between the two assets, which we express as $\sigma_{i,j} = \text{Cov}[\tilde{r}_i, \tilde{r}_j]$. Instead, the elements on the main diagonal, when $i = j$, represent the variance of the corresponding assets. The return on portfolio p is therefore:

$$\tilde{r}_p = \sum_{i=1}^{n} w_i \tilde{r}_i,$$

which can be expressed in matrix form as:

$$\tilde{r}_p = \mathbf{w}^T \tilde{r} = \tilde{r}^T \mathbf{w},$$

[2] T denotes a transposed vector and the formula represents an inner product from linear algebra.

where $\tilde{r}$ is an $n \times m$ matrix containing the returns of each asset class. Here, n is the number of assets and m is the number of return observations. The result is a $1 \times m$ vector of portfolio returns. The variance of the return on p is:

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{i,j}$$

which in matrix form is:

$$\sigma_p^2 = \mathbf{w}^T \mathbf{\Sigma} \mathbf{w}.$$

The covariance between the return of the part of portfolio p invested in class i and the total return of portfolio p is:

$$\mathrm{Cov}[w_i \tilde{r}_i, \tilde{r}_p] = w_i \sum_{j=1}^{n} w_j \sigma_{i,j}$$

or in matrix form:

$$\mathrm{Cov}[w_i \tilde{r}_i, \tilde{r}_p] = \mathbf{w}^T \mathbf{\Sigma}_i w_i. \tag{4}$$

1) Now, as before, we want to find the weights that satisfy the following condition:

$$\mathrm{Cov}[w_i \tilde{r}_i, \tilde{r}_p] = \mathrm{Cov}[w_j \tilde{r}_j, \tilde{r}_p] \iff \mathbf{w}^T \mathbf{\Sigma}_i w_i = \mathbf{w}^T \mathbf{\Sigma}_j w_j = \mu \qquad \forall i, j. \tag{5}$$

The constant μ is used to verify that the two covariances are actually the same.
This condition can also be written as follows:

$$w_i \,\mathrm{Cov}[\tilde{r}_i, \tilde{r}_p] = w_j \,\mathrm{Cov}[\tilde{r}_j, \tilde{r}_p]$$

$$w_i \frac{\mathrm{Cov}[\tilde{r}_i, \tilde{r}_p]}{\sigma_p^2} = w_j \frac{\mathrm{Cov}[\tilde{r}_j, \tilde{r}_p]}{\sigma_p^2}$$

$$w_i \beta_{i,p} = w_j \beta_{j,p}. \tag{6}$$

The $\beta_{i,p}$ derives from:

$$\mathrm{Cov}[\tilde{r}_i, \tilde{r}_p] = \mathrm{Cov}\left[\tilde{r}_i, \sum_{j=1}^{n} w_j \tilde{r}_j\right].$$

which, using the linearity property of covariance, can be written as:

$$\mathrm{Cov}[\tilde{r}_i, \tilde{r}_p] = \sum_{j=1}^{n} \mathrm{Cov}[\tilde{r}_i, w_j \tilde{r}_j] = \sum_{j=1}^{n} w_j \, \mathrm{Cov}[\tilde{r}_i, \tilde{r}_j] = \sum_{j=1}^{n} w_j \sigma_{i,j}.$$

This is analogous to multiplying the transpose of $\mathbf{w}$ by the column i of the matrix $\boldsymbol{\Sigma}$:

$$\mathrm{Cov}[w_i \tilde{r}_i, \tilde{r}_p] = \mathbf{w}^T \boldsymbol{\Sigma}_i.$$

Then $\beta_{i,p}$ is equal to:

$$\beta_{i,p} = \sum_{j=1}^{n} w_j \frac{\sigma_{i,j}}{\sigma_p^2} = \frac{\mathbf{w}^T \boldsymbol{\Sigma}_i}{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}.$$

From (6), it follows that the Risk Parity portfolio is invested in asset i in a way that is inversely proportional to the beta between this asset and the total portfolio risk. In other words, the riskier an asset is (in terms of volatility), the smaller its weight in the portfolio will be, as:

$$w_i \sim \frac{1}{\beta_{i,p}}, \tag{7}$$

2) We finally define λ, a constant value such as:

$$w_i \beta_{i,p} = w_j \beta_{j,p} = \lambda.$$

Given the composition of this portfolio we have that:

$$\beta_{p,p} = \sum_{i=1}^{n} w_i \beta_{i,p},$$

but the beta of a portfolio with itself is by definition equal to 1 which means that, in sum, we have

$$w_i \beta_{i,p} = \lambda,$$

$$\sum_{i=1}^{n} w_i \beta_{i,p} = 1.$$

Thus, we can derive that λ is equal to $\frac{1}{n}$ and impose the following conditions to identify the Risk Parity portfolio:

$$w_i \beta_{i,p} = \frac{1}{n} \qquad \text{with } i = 1, 2, \ldots, n.$$

3 Newton-Raphson algorithm for Risk Parity Portfolio Construction

This chapter illustrates the approach used to determine the optimal assets weights for constructing the Risk Parity portfolio. The method relies on the well-known Newton-Raphson optimization technique, a numerical procedure for solving nonlinear systems of equations. Before illustrating the specific algorithm adopted in this context, it is essential to briefly outline the rationale behind the Newton-Raphson method, how it works and why it is relevant for Risk Parity portfolio construction.

3.1 General cases with single and multiple functions

The Newton-Raphson algorithm is a powerful tool used to find the roots, or zeros, of a function – those points such that $f(x) = 0$. It is primarily used to solve systems of equations due to its ability to provide accurate results with fast convergence speed. Starting from an initial guess, as close as possible to the desired result, the algorithm iterates the process, progressively approaching the solution. The process stops when convergence is achieved. Naturally, the closer the initial guess is to the actual result, the faster the convergence.

Considering the simple case of an equation, we want to solve: $f(x) = 0$, with $f(x)$ differentiable. The iterative process is defined as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where x_n is the initial guess or, in subsequent iterations, the result of the previous step. The formula works by computing the tangent at the point x_n and determining the x-intercept of this tangent, which serves as the new updated guess. In other words, the algorithm starts from an initial point x_n, computes the value of the function at that point, $f(x_n)$, then calculates the first derivative $f'(x_n)$. By finding the point where the tangent intersects the x-axis, the next approximation x_{n+1} is determined. This process continues until the tangent intersects the x-axis exactly at the root. The algorithm stops when $|f(x^{(k)})| < \epsilon$, where ϵ is a positive tolerance error typically close to zero. The method is actually primarily used to solve systems of equations.

Let $\boldsymbol{f}(\cdot)$ be a vector of m functions defined over a vector $\boldsymbol{x}$ of m inputs. We aim to solve the system $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$. To find the root, we introduce the Jacobian matrix $\boldsymbol{J}(\boldsymbol{x})$, which contains the partial derivatives of the m functions with respect to the m elements of x.

$$\boldsymbol{J}(\boldsymbol{x}) = \frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}.$$

Assuming $\boldsymbol{J}(\boldsymbol{x})$ is invertible, we update the solution as:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \boldsymbol{J}^{-1}(\boldsymbol{x}^{(k)})\boldsymbol{f}(\boldsymbol{x}^{(k)}),$$

and the stopping condition becomes:

$$\sqrt{\frac{1}{m}\boldsymbol{f}(\boldsymbol{x}^{(k)})^\top \boldsymbol{f}(\boldsymbol{x}^{(k)})} < \epsilon$$

3.2 Newton-Raphson Method for determining Risk Parity weights

After this brief introduction to the Newton-Raphson method showing both the simple case of a single function and the more complex case of a system of equations, we apply the algorithm to the specific context of this dissertation: determining the optimal weights to assign to each asset in the construction of our Risk Parity portfolio.[3]
Going back to (5), the Risk Parity condition for $i = 1, 2, \ldots, n$ can be written compactly as

$$\boldsymbol{\Sigma w} = \boldsymbol{\mu} \begin{pmatrix} \frac{\boldsymbol{1}}{\boldsymbol{w_1}} \\ \vdots \\ \frac{\boldsymbol{1}}{\boldsymbol{w_n}} \end{pmatrix}.$$

The vector $\boldsymbol{\Sigma w}$ contains the covariances between the n asset classes and the portfolio p. Let $\boldsymbol{y} = (\boldsymbol{w}, \boldsymbol{\mu})$ be a column vector containing the portfolio weights and the constant μ.

Define $\boldsymbol{F}(\boldsymbol{y})$ as:

$$\boldsymbol{F}(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{\Sigma w} - \mu \begin{pmatrix} \frac{1}{w_1} \\ \vdots \\ \frac{1}{w_n} \end{pmatrix} \\ \boldsymbol{w}^\top \boldsymbol{1} - 1 \end{pmatrix}.$$

[3] This is just one of the possible algorithms for constructing a Risk Parity portfolio. See Appendix A.2 for an alternative approach.

Our goal is to to determine the weights w such that the first equation equals zero, which, in other words, means finding the portfolio combination that makes the risk contribution of each asset equal. The second equation represents the condition that ensures that the sum of all weights is equal to 1.

Then the Jacobian will be:

$$\boldsymbol{J(y)} = \begin{pmatrix} \boldsymbol{\Sigma} + \mu\,\mathrm{diag}\begin{pmatrix} \frac{1}{w_1^2} \\ \vdots \\ \frac{1}{w_n^2} \end{pmatrix} & -\begin{pmatrix} \frac{1}{w_1} \\ \vdots \\ \frac{1}{w_n} \end{pmatrix} \\ \mathbf{1}^\top & \mathbf{0} \end{pmatrix}^{4}.$$

The algorithm adopted follows the procedure outlined below:

1. Start from an initial guess $\boldsymbol{y^{(0)}} = (\boldsymbol{w^{(0)}}, \boldsymbol{\mu^{(0)}})$.
2. Calculate the functions $\boldsymbol{F(y^{(k)})}$ and stop if:

$$\sqrt{\frac{1}{n+1}\boldsymbol{F(y^{(k)})}^\top \boldsymbol{F(y^{(k)})}} < \epsilon$$

3. Calculate the Jacobian $\boldsymbol{J(y^{(k)})}$ and update:

$$y^{(k+1)} = y^{(k)} - J(y^{(k)})^{-1}F(y^{(k)})$$

4. Return to step 2 and iterate.

The process terminates when the value of the function at the current step is small enough (less then epsilon), indicating that the weights have been found or when the maximum number of iterations is reached without finding a proper solution. [5]

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[4] Diag is a diagonal matrix of the form $\begin{pmatrix} \frac{1}{w_1^2} & 0 & \dots & 0 \\ 0 & \frac{1}{w_2^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{w_n^2} \end{pmatrix}$

[5] See Appendix A for Matlab code.

PART 2: Empirical Analysis

4 Empirical Analysis: Overview and Methodology

The following chapters are dedicated to the empirical analysis, which constitutes the core section of this thesis. The analysis is composed of two main parts.

The first one starts with a simplified portfolio made up of only two asset classes. The choice was made to proceed incrementally, starting with this simplified version before moving to the more complex case with four asset classes. The aforementioned portfolio is then composed by:

- **FTSE World Government Bond Index (WGBI)**: The WGBI is a broad index providing exposure to the global sovereign fixed-income market, covering investment-grade government bonds from over 20 developed countries. The index has a history of more than 30 years and is diversified by maturity (ranging from 1-3 years to 10+ years) and geography. FTSE Russell (2024)
- **MSCI World Index (MXWO)**: The MSCI World Index represents large and mid-cap stocks across 23 developed markets. The index captures approximately 85% of the free-float-adjusted market capitalization in each country. As with the WGBI, the MSCI World focuses on developed markets, leaving emerging markets outside the scope of this analysis. MSCI Inc. (2024)

The second analysis with four asset classes involves the use of:

- **FTSE World Government Bond Index (WGBI)**: government bond index
- **MSCI World Index (MXWO)**: equity index
- **Bloomberg Industrial Metals Subindex (BCOMINTR)**: 60%
  The indices track exchange-traded futures on physical commodities.
  LBMA Gold Price PM (GOAULNPM): 40%
  The LBMA Gold Price PM Index measures the performance of setting price of gold. It is designed to fix a price for settling contracts between members of the London bullion market, but informally provides a recognized rate that is used as a benchmark for pricing the majority of gold products and derivatives throughout the world's markets. LBMA (2024)
- **iShares Global Inflation Linked Government Bonds**:
  It is an ETF that tracks the Bloomberg World Government Inflation-Linked Bond Index. The Bloomberg World Government Inflation-Linked Bond Index (WGILB) measures the performance of investment-grade government inflation-linked debt from 12 developed market countries. The index includes only liquid markets where a global government linker fund is likely to invest. BlackRock (2024)

The scope of the analysis it to evaluate the performance of the portfolios relatively to a benchmark. In both cases, the benchmark is a traditional 60/40 portfolio composed of the same two assets as those considered within the Risk Parity portfolio with two assets. The key difference between the two lies in the specific allocation method used to construct the portfolio. As is well known, the reference portfolio is constructed on a capital-based principle, meaning that each asset appears in the portfolio with a predetermined fixed allocation; in this case, 60% MSCI and 40% WGBI, regardless of the relationship between the assets, which evolves over time. Differently, as will be better explained in the course of the analysis, the weights assigned to assets in the Risk Parity portfolio depend on the correlation between them.

The primary question, therefore, is whether a Risk Parity portfolio can outperform a 60/40. The performances of both portfolios are evaluated using three key risk-adjusted performance metrics: the **Sharpe Ratio**, the **Treynor Ratio**, and the **Sortino Ratio**. It is anyway important to highlight that in the four-assets analysis, however, the primary comparison is conducted against the Risk Parity portfolio with only two assets. The rationale behind expanding the portfolio to include four asset classes lies in the expectation that this added complexity will yield improved risk-adjusted performance. Such an effort is justified only if the additional diversification results in a measurable enhancement in terms of risk-adjusted returns. Before proceeding with the illustration of the analysis conducted, it is appropriate to briefly assess how each performance metric is calculated and how to interpret the performance results that will emerge from the study.

- **The Sharpe Ratio**: It is probably the most widely used and well-known risk-adjusted measure of performance. It is defined as the ratio of the portfolio's expected excess return (R_p) over a risk-free asset (R_f) to the standard deviation of the portfolio (σ_p). Similarly this can be expressed as follows:

$$\text{SR} = \frac{\overline{R_p} - R_f}{\sigma_p}$$

  In this case, the use of the notation $\overline{R_p}$ rather than $E[R_p]$ highlights that we are dealing with historical data: therefore, we have the actual mean value instead of the expected return. A positive Sharpe Ratio indicates that the portfolio has generated wealth, providing a risk-adjusted return higher than the risk-free rate. Its use as a performance measure involves comparing the ratios obtained for the Risk Parity portfolio and the benchmark. The higher the Sharpe Ratio, the better the performance. The main weakness of this ratio is that it considers the portfolio's total risk, rather than focusing solely on the systematic risk that cannot be diversified away. This implies that portfolios heavily exposed to a single dimension of risk could still exhibit a high Sharpe Ratio, especially during a favorable backtesting period. However, this performance might be entirely random and could reverse in less favorable conditions. In other words, the Sharpe Ratio may not be the most suitable performance measure for comparing two portfolios that differ significantly in terms of diversification.

- **The Treynor Ratio**: To address the limitations of the Sharpe Ratio, a good alternative is the Treynor Ratio. It measures the expected excess return of portfolio p over a risk-free asset per unit of systematic risk:

  $$\text{Treynor Ratio} = \frac{\overline{R_p} - R_f}{\beta_p}$$

  The beta of the portfolio captures the systematic risk and it is obtained through the regression of the portfolio's excess return over the market portfolio's excess return. In other words, beta is the slope of the regression line between the excess return of the Risk Parity portfolio and that of the market one. As before, the portfolio that showed a higher ratio achieved a higher performance.

  There is a close link between the Treynor ratio and Jensen's alpha. The latter is an indicator of the additional performance achieved by the portfolio compared to the expected return. It is commonly used to evaluate the selection skill of a portfolio manager in picking securities. If Jensen's alpha is positive, it indicates that the portfolio has generated higher returns than expected given its level of systematic risk, highlighting the manager's ability to generate excess returns beyond what is explained by market movements. Graphically, Jensen's alpha is measured as the intercept of the regression of the portfolio's excess return on the market's excess return, expressed as follows:

  $$R_{p,t} = \alpha_p + \beta_p R_{m,t} + \epsilon_{p,t}$$

  Taking the average: $\overline{R_p} = \alpha_p + \beta_p \bar{R}_m$ and dividing by β_p (knowing that β_m is equal to 1), Treynor Ratio can also be expressed as:

  $$\text{TR}_p = \frac{\alpha_p}{\beta_p} + \text{TR}_m$$

- **The Sortino Ratio**: To conclude, the final performance measure utilized in this analysis is the Sortino Ratio. It is defined as the expected excess return of portfolio p over a target return, divided by the corresponding downside risk.

  The downside risk (DD) is a measure of risk which consider only the negative deviations of returns relative to a predetermined target. Mathematically, it is defined as:

$$DD = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(\min[0, \tilde{r}_t - \tilde{r}_{\text{target}}]\right)^2}$$

Where:

- T is the total number of observations,
- $\tilde{r}_t$ is the portfolio return at time t,
- $\tilde{r}_{\text{target}}$ is the predefined target return.

At this point, the Sortino Ratio can be defined:

$$SRT_p = \frac{\overline{R_p} - \overline{R_{\text{TRT}}}}{DD}$$

In this specific case, the target return is always set equal to the risk-free rate. As a result, the DD measures the instances where the portfolio's return—whether for the Risk Parity portfolio or the benchmark—falls below the risk-free rate, which is equivalent to a negative return. To conclude, as for the other performance metrics, a higher Sortino Ratio indicates better performance.

Now whether or not the Risk Parity portfolio outperform the benchmark, the following question is whether the performance is time-consistent, meaning it occurs regardless of the time horizon considered. To answer the question, the analysis is conducted across two different samples:

- The first spans from **January 2008 to December 2020** with monthly closing prices of the two assets obtained from Bloomberg. This sample is then divided into two non-overlapping time windows:
  - **Estimation Window**: January 2008 to December 2015. This period is used to compute the covariance matrix between the assets and to construct the Risk Parity portfolio.
  - **Testing Window**: January 2016 to December 2020. This period is used for backtesting the portfolio constructed using the information from the Estimation Window.
- The second spans from **January 2011 to December 2023**, also with monthly closing prices obtained from Bloomberg. The sample is similarly divided into two non-overlapping time windows:
  - **Estimation Window**: January 2011 to December 2018.

- **Testing Window**: January 2019 to December 2023.

In both cases, the same methodology is applied to evaluate whether the Risk Parity portfolio consistently outperforms the benchmark across the two different time periods.

The choice to evaluate the portfolio (both the two-asset and the four-asset version) over two different time horizons was made to assess the robustness of the Risk Parity strategy across two time framework characterized by different macroeconomic and geopolitical scenarios. The goal is to analyze how the portfolio performed during a period of solid economic growth, albeit the outbreak of the pandemic in 2020 (2016-2020), in contrast to the five-year period from 2019 to 2023. The latter was marked by the consequences of the pandemic, the strong economic recovery in 2021, which led to the highest inflation levels since the 1980s, and the subsequent restrictive monetary policies to curb the uncontrolled rise in prices.

The analysis continues with an annual rebalancing of the Risk Parity portfolio's weights. The goal is to assess whether, by incorporating new information and adjusting the portfolio accordingly, it can lead to higher returns compared to not rebalancing. The analysis follows the steps outlined below:

1. Derive the realized returns of the Risk Parity portfolio built using the original estimation window only for the first year of the testing period.
2. Shift the estimation window forward by one year (e.g., from January 2009 to December 2016) and calculate the portfolio returns for the subsequent year (e.g., January to December 2017).
3. Repeat this process until all monthly returns for each year (up to 2020) have been calculated. At this point, the vector composed of all monthly returns over the five years will represent the new return vector for the Risk Parity portfolio.

Naturally, the assessment is conducted for both portfolio compositions across the two samples.

5 Results: Risk Parity Portfolio with two asset classes

5.1 Static analysis: January 2008 - December 2020

The data, sourced from Bloomberg, represent the monthly closing prices of the assets from which the monthly returns for both the MSCI and WGBI indices are calculated.

The chart below illustrates the cumulative monthly returns of the two assets. As evident from the graph, the equity index (MSCI) exhibits much higher volatility compared to the WGBI. During the global financial crisis of 2008-2009, the MSCI experienced a sharp decline, reaching a significant low in early 2009. In contrast, the WGBI showed higher stability, highlighting its role as a safe haven during periods of economic uncertainty. Another sharp decline in the MSCI's performance is observed in 2020, a year marked by the market crisis caused by the Covid-19 pandemic. The WGBI experienced only a minor drop during this period and quickly recovered.

The trends in the returns of the two indices align with theoretical expectations: stocks tend to perform poorly during periods of low economic growth or crisis, while government bonds provide stability and act as a defensive asset in adverse economic conditions. Additionally, WGBI returns are much more stable, fluctuating between approximately 1.0 and 1.45 over the period, reflecting the lower risk associated with government bonds. The same cannot be said for MSCI returns, which exhibit a much broader range, approximately between 0.5 and 1.83, consistent with the higher risk and return profile of equities.

Overall, the cumulative return of MSCI increased by approximately 79.13% over the period, whereas WGBI grew by 41.55%, highlighting the typical risk-return trade-off between equities and government bonds.

Figure 1: Monthly Cumulative Returns of MSCI and WGBI, 2008-2020

The following tables summarize the distribution characteristics and descriptive statistics of the two assets. The means and returns confirm the trends already observed in the cumulative returns chart. The MSCI index exhibits a relatively high mean return compared to the WGBI, but this is accompanied by significantly higher volatility (StdDev).

Regarding kurtosis, a value of 3 indicates a normal distribution. A value greater than 3, as observed for both assets, suggests that the distribution has thicker tails. This implies a higher likelihood of extreme events that deviate significantly from the mean. While kurtosis is elevated for both assets, it is higher for equities, confirming the inherent nature of stocks, which tend to be more volatile and risky.

Finally, skewness measures the symmetry of a distribution. A value close to zero indicates a symmetrical bell curve. A negative skewness, as seen for both assets in this case, indicates a distribution skewed to the right, meaning that the left tail (representing extreme negative events) is heavier than the right tail (representing extreme positive events). In other words, there are more extreme negative events than positive ones.

Once again, it is not surprising to observe a more pronounced negative skewness for equities as stocks tend to experience extreme losses more frequently than extreme gains. On the other hand, the smaller negative skewness of government bonds reflects their lower likelihood of facing extreme negative events, despite remaining slightly skewed to the left.

Statistic	Mean	StdDev	Kurtosis	Skewness
Value	0.0039147	0.049034	5.337	-0.92408

Table 2: Summary Statistics for MSCI log returns - Monthly Values

Statistic	Mean	StdDev	Kurtosis	Skewness
Value	0.0023986	0.017863	4.4069	-0.19385

Table 3: Summary Statistics for WGBI log returns - Monthly Values

The final part of the preliminary analysis, before delving into the construction and evaluation of the Risk Parity portfolio, concerns the characteristics of the time series. The following charts illustrate the Autocorrelation Function for the two securities. Autocorrelation measures how much prices are influenced by their past values. In both cases, the graph goes to zero fast (values within the 95% confidence margin, delineated by the two blue bands, are statistically not different from zero). Only for the WGBI the value corresponding to lag 7 exceed the lower limit. A detailed time series analysis is beyond the scope of this dissertation, but it seems possible to exclude that this value is due to seasonality. The value at lag 14 is, in fact, again not significantly different from zero and the anomaly at lag 7 could therefore be due to statistical noise. In any case, the autocorrelation graphs indicate an absence of correlation between asset prices at different lags. In other words, prices do not follow a predefined pattern, and their variations are not predictable based on past changes. Such a characteristic is typical of liquid assets, whose prices quickly reflect new market information. This feature is essential for the construction of a Risk Parity portfolio, as it assumes an annual rebalancing of portfolio weights. If the assets are not highly liquid, this operation would be difficult and very costly.

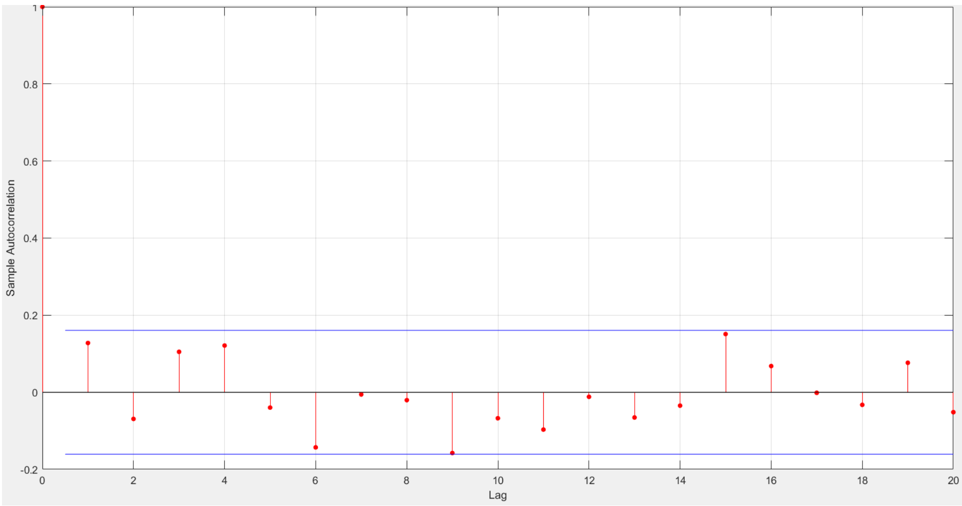

Figure 2: Autocorrelation Function for MSCI

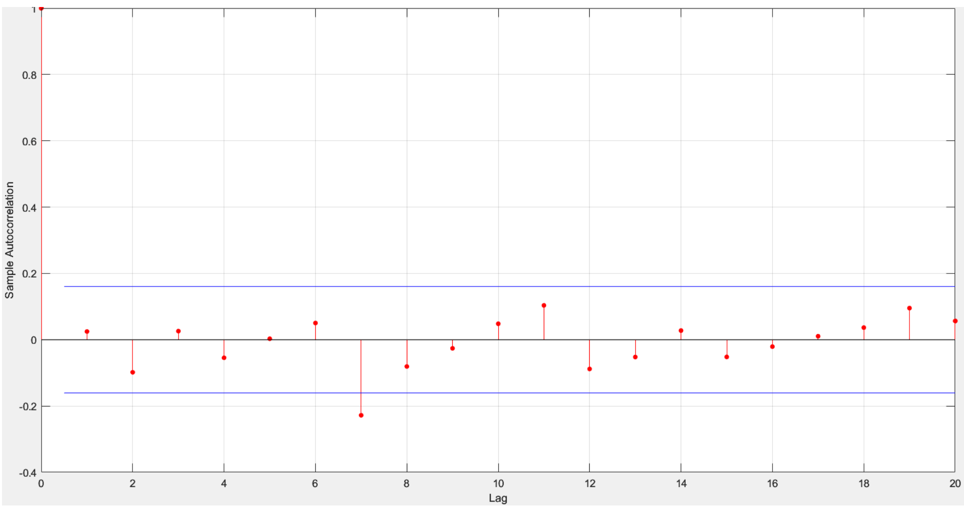

Figure 3: Autocorrelation Function for WGBI

At this stage, the focus shifts to analyzing the main exercise we have performed. Returns are calculated for the entire sample, but only those corresponding to the estimation window are selected. Based on these returns, the covariance matrix for the two assets is calculated, serving as the necessary input for determining the weights of each asset.

To provide a clearer understanding of the relationship between the assets, here it's presented the correlation matrix. This matrix is derived from the covariance matrix by dividing each of its element by the product of the standard deviations of the corresponding variables, resulting in a standardized measure of the relationship between the assets. The correlation matrix between MSCI and WGBI is:

	MSCI	WGBI
MSCI	1	0.3203
WGBI	0.3203	1

Table 4: Correlation Matrix for MSCI and WGBI, Estimation Window 2008-2015

The two assets exhibit a positive but relatively weak correlation. Correlation values range from -1 to +1, representing, respectively, perfect negative or perfect positive correlation. In the context of constructing a diversified portfolio, the goal is to select assets that move in different directions in response to the same events. Although the correlation is not particularly low or negative (which would be ideal), it is moderately weak, indicating that the two assets do not move in a strictly similar manner and then diversification benefits do exist. All the elements are now in place to construct the portfolio. Using the algorithm defined in Chapter 3 and the covariance matrix just calculated, the Risk Parity weights are determined:

Asset	Weight
MSCI	0.2700
WGBI	0.7300

Table 5: Risk Parity Portfolio Weights, MSCI and WGBI, 2008-2020

Not surprisingly, the weight assigned to MSCI is significantly lower than that assigned to WGBI. This is directly related to the riskiness of the asset, as the weight is inversely proportional to its volatility. Conversely, the less risky and less volatile asset, WGBI, has been allocated a considerably higher weight in the portfolio. It is also important to emphasize how the obtained weights are significantly different from those of the 60/40 portfolio, which, as previously illustrated, has a fixed allocation of 60% in MSCI and 40% in WGBI.

At this point, having determined the weights, it is possible to calculate the monthly returns of the Risk Parity portfolio for the testing window (2016-2020). During the same period, the returns of the benchmark portfolio (60/40 portfolio) is also calculated. Another important decision is the selection of a market portfolio as a reference which is represented by the MSCI World. The following chart compares the monthly returns of the three portfolios.

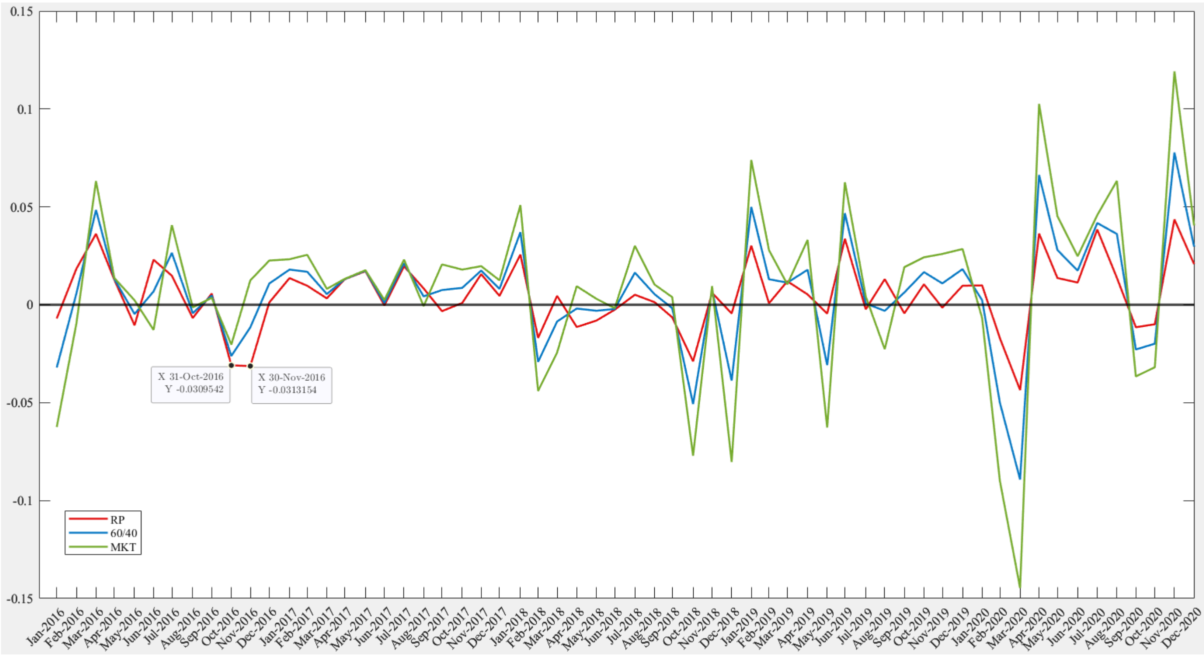

Figure 4: Monthly Returns for 2A Risk Parity, 60/40 and Market Portfolios. 2016-2020

In the graph, the red line shows the returns of the Risk Parity portfolio, the blue represents the 60/40, and the green corresponds to the market portfolio. At first glance, it is evident that, for most of the period, the market portfolio exhibited the highest variability, with wider fluctuations. In contrast, the Risk Parity portfolio was the most stable, showing moderate peaks, both positive and negative hence offering the best diversification benefits. However, there were moments when the Risk Parity portfolio exhibits the worst negative returns. One such instance occurred between October and November 2016, likely due to the political uncertainty leading up to the results of the U.S. elections. In October 2016, there was strong fear that the Federal Reserve would raise interest rates by the end of the year, which eventually happened. This expectation led to massive sell-offs in the bond market, resulting in poor performance of these securities. As the Risk Parity portfolio is heavily exposed to the WGBI, it was the most affected by the fear and the eventual rate hike.

	Risk Parity Portfolio	60/40 Portfolio	Market Portfolio
Mean	0.0041	0.0054	0.0071
Std Dev	0.0174	0.0279	0.0441
Beta	0.3005	0.6167	1

Table 6: Risk-Return Metrics for 2A RP, 60/40 and Market Portfolios. 2016-2020

Table 6 presents the metrics for the three mentioned portfolios and confirms the intuition derived from the graph. The Risk Parity portfolio exhibited both the lowest average return and the lowest risk, while the Market portfolio showed the opposite trend, with higher returns and greater risk.

These findings align with the theory behind the Risk Parity strategy, which prioritizes a significant risk reduction at the expense of lower returns. To assess which portfolio outperformed, it is necessary to evaluate the risk-adjusted return. The following table presents the findings regarding Sharpe Sortino and Traynor Ratios for the portfolio and his benchmark.

Performance Ratio	Risk Parity Portfolio	60/40 benchmark portfolio
Sharpe Ratio	0.2345	0.1950
Sortino Ratio	0.3903	0.2953
Treynor Ratio	0.0136	0.0088

Table 7: Performance Analysis, 2A Risk Parity Portfolio, 2016-2020

The table above summarizes the final results of this initial analysis. The Risk Parity portfolio outperformed the benchmark across all three performance metrics. Regarding the Sharpe Ratio, despite having a lower average return (0.0041 compared to 0.0054 for the benchmark), the Risk Parity portfolio achieved better performance due to significantly lower variability (0.0174 versus 0.0279). The performance is even more favorable when considering the Treynor Ratio, which accounts only for systematic risk. The beta of the Risk Parity portfolio is approximately half that of the benchmark (0.3005 vs 0.6167).

Based on this results, it can be concluded that, for the time horizon from January 2008 to December 2020, the portfolio constructed using the Risk Parity method outperformed the benchmark portfolio according to all the performance metrics analyzed.

5.2 Dynamic analysis: January 2008 - December 2020

The following section examines the impact of introducing an annual rebalancing of the weights assigned to each asset in a Risk Parity portfolio, using the same time window as the previous analysis. It is important to highlight that asset correlations are not constant over time; instead, they change and adapt in response to varying economic conditions. Table 8 illustrates how the correlation between the assets in question, MSCI and WGBI, has changed depending on the estimation window considered.

Estimation Window	Correlation
Correlation 2008-2015	0.01924
Correlation 2009-2016	0.01786
Correlation 2010-2017	0.01597
Correlation 2011-2018	0.01465
Correlation 2012-2019	0.01445

Table 8: Correlation Coefficients 2A Over Rolling Estimation Windows, 2008-2020

The analysis aims to determine whether a portfolio that is rebalanced annually—employing updated information and newly emerging relationships—can outperform a fully static portfolio. In this case, the portfolio is constructed only once at the outset and remains unchanged throughout the investment period.

Based on the time-varying covariance, the weights for portfolio construction were calculated for each testing window. The results are summarized in the table below:

RP weights	Weights 1 2008-2015	Weights 2 2009-2016	Weights 3 2010-2017	Weights 4 2011-2018	Weights 5 2012-2019
MSCI	0.2700	0.2936	0.2998	0.2959	0.3085
WGBI	0.7300	0.7064	0.7002	0.7041	0.6915

Table 9: 2A Risk Parity Weights for Each Testing Window, 2016-2020

Using the specified weights, a Risk Parity portfolio was constructed for each year of the testing window, specifically from 2016 to 2020. Subsequently, the overall return vector, formed by concatenating the monthly returns for each year, was used to calculate the performance indices. Results are presented below: the benchmark portfolio remains unchanged hence the comparison is primarily made between the performance indices of the static and dynamic analyses of the Risk Parity portfolio.

Performance Ratio	Static RP	Dynamic RP	60/40
Sharpe Ratio	0.2345	0.2276	0.1950
Sortino Ratio	0.3903	0.3737	0.2953
Treynor Ratio	0.0136	0.0126	0.0088

Table 10: Performance Analysis, Static vs Dynamic 2A Risk Pairty Portfolios. 2008-2020

Unfortunately, the data did not reveal any significant improvement in performance following an annual rebalancing. In fact, all three performance indices turned out to be lower than the values observed for the static analysis. On the bright side, the results remain superior to those of the benchmark and overall positive. This poor result could stem from an inadequacy in the rebalancing frequency. A higher frequency might be more suitable in a market characterized by high volatility and, consequently, by asset correlations that change over time. Or differently, the mistake could arise from the choice of a limited time span. With a longer period, it is possible to obtain a more accurate measure of the estimation of the real co-movements between the two indexes. To conclude, another hypothesis could be that the weights selected for the static analysis were already optimal, and the imposition of an annual rebalancing might have disrupted this balance, leading to slightly inferior results.

5.3 Static analysis: January 2011 - December 2023

We now proceed with the same two-step analysis, focusing on the period from 2011 to December 2023. The choice to shift the sample was made because the years between 2020 and 2023 were characterized by significant and unexpected macroeconomic events: the COVID-19 pandemic and the outbreak of Russia-Ukraine war. The objective is therefore to evaluate how the Risk Parity portfolio performed during years marked by significant uncertainty (see Section 4).

As for the previous analysis, Figure 5 illustrates the cumulative returns for both MSCI and WGBI for years 2011-2023. The equity index shows very strong growth following the drawdown in March 2020. The market quickly recovered after the COVID-19 pandemic crisis, but it has also been characterized by high volatility, likely due to the war in Ukraine, which broke out in February 2022. On the other hand, WGBI remained relatively stable but reached one of its lowest levels of the last decade between 2022 and 2023.

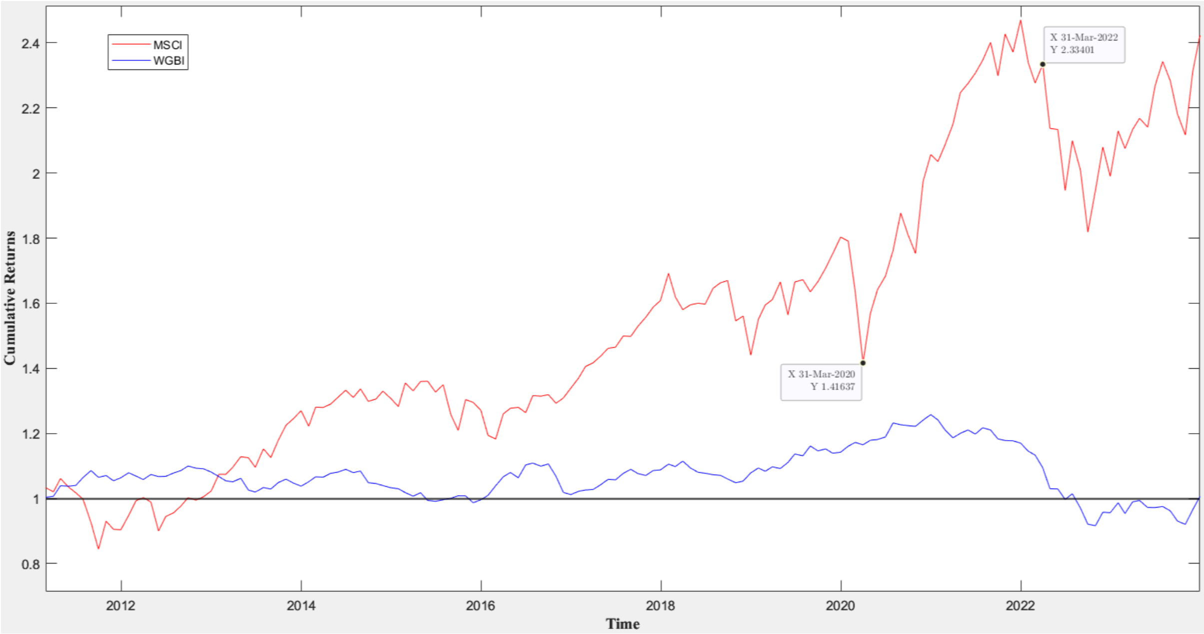

Figure 5: Cumulative Returns of MSCI and WGBI, 2011-2023

Comparing Table 2 and Table 11 — outlined below — which summarize the main statistics of the MSCI index, it is evident that the average return increased (from 0.0039 to 0.0057), aligning with the trends observed in the cumulative returns chart. When comparing the 2008 Global Financial Crisis to pandemic recession, the rebound in the latter case was significantly faster. Pushed by leading sectors such as technology, markets not only recovered quickly but also reached unprecedented picks within a relatively short period. Volatility slightly decreased from 0.049 to 0.0427. When comparing Figure 1 and Figure 5, the graph for the second period shows a strong upward trend but with fewer fluctuations compared to the previous one. While the trend appears smoother in this case, the years immediately following the 2008 crisis display much more frequent oscillations before stabilizing. The same applies to kurtosis and skewness, both of which decreased, suggesting that the second sample exhibits fewer extreme events and a more balanced distribution of returns compared to the first one.

Statistic	Mean	StdDev	Kurtosis	Skewness
Value	0.0057	0.0427	3.8530	-0.5026

Table 11: Summary Statistics for MSCI log returns - Monthly Values

Statistic	Mean	StdDev	Kurtosis	Skewness
Value	3.8628e-05	0.0180	4.0109	-0.3311

Table 12: Summary Statistics for WGBI log returns - Monthly Values

Looking now at WGBI, comparing Table 3 and Table 12, the results have worsened more significantly. The average return dropped sharply (from 0.0024 to 0.0000386), indicating a substantial decline in overall performance. Additionally, the skewness became more negative, highlighting a distribution that is more towards unfavorable outcomes, reflecting an increased prevalence of adverse events during the second sample period.

Between 2020 and 2023, government bond returns were influenced by a series of decisions taken by central banks in response to the pandemic crisis first and the Ukraine war later. In the early stages, government bonds served as a safe haven for investors seeking to protect their capital from equity market drops. Central banks implemented expansive monetary policies, first lowering short-term interest rates to the zero lower bound (or even negative in some regions) and then implementing large-scale quantitative easing (QE) programs to stimulate the economy. This initially drove bond prices to historic highs up to early 2022, as shown in Figure 5.

However, inflation had already started to rise in 2021 due to post-pandemic recovery, supply chain bottleneck, and increasing demand. The outbreak of the war in Ukraine in February 2022 exacerbated this situation, as Russia, one of the world's largest exporters of oil, natural gas, and grains, faced sanctions and disruptions in its exports. These factors drove commodity prices to record highs, further fueling inflation. In response, central banks shifted to restrictive policies, increasing interest rates to combat rising price pressures. This shift is clearly visible in the graph, where there is a sharp decline in bond returns, dropping from around 1.2 to 0.8 in the early 2022. [6]

	MSCI	WGBI
MSCI	1	0.1327
WGBI	0.1327	1

Table 13: Correlation Matrix for MSCI and WGBI, Estimation Window 2011-2018

Following now the previous scheme, the table outlined above shows the updated correlation matrix between the two assets. Comparing this matrix with Table 4, the correlation between MSCI and WGBI drops significantly from 0.3203 to 0.1237. This underscores a weaker linear relationship, suggesting greater potential benefits from diversification.

At this point, it has been possible to calculate the asset weights for the construction of the new Risk Parity portfolio. As shown in the table below, the weight allocated to the equity index has increased compared to the previous model, while, consequently, the weight of the WGBI has decreased.

[6] See Appendix A.3 for the CPI trend in both the US and the Eurozone.

Asset	Weight
MSCI	0.2959
WGBI	0.7041

Table 14: Risk Parity Portfolio Weights, MSCI and WGBI, 2011-2023

The first part of the chart below, up until early 2022, illustrates a trend similar to the previous one (Figure 4), with the market benchmark exhibiting the most volatile returns and the Risk Parity portfolio being the most balanced. Starting from the highlighted point in the chart, corresponding to 04.28.2022, the performance of the three portfolios began to follow a very similar pattern. This led to the conclusion that the implemented diversification did not actually provide any benefit to Risk Parity portfolio.

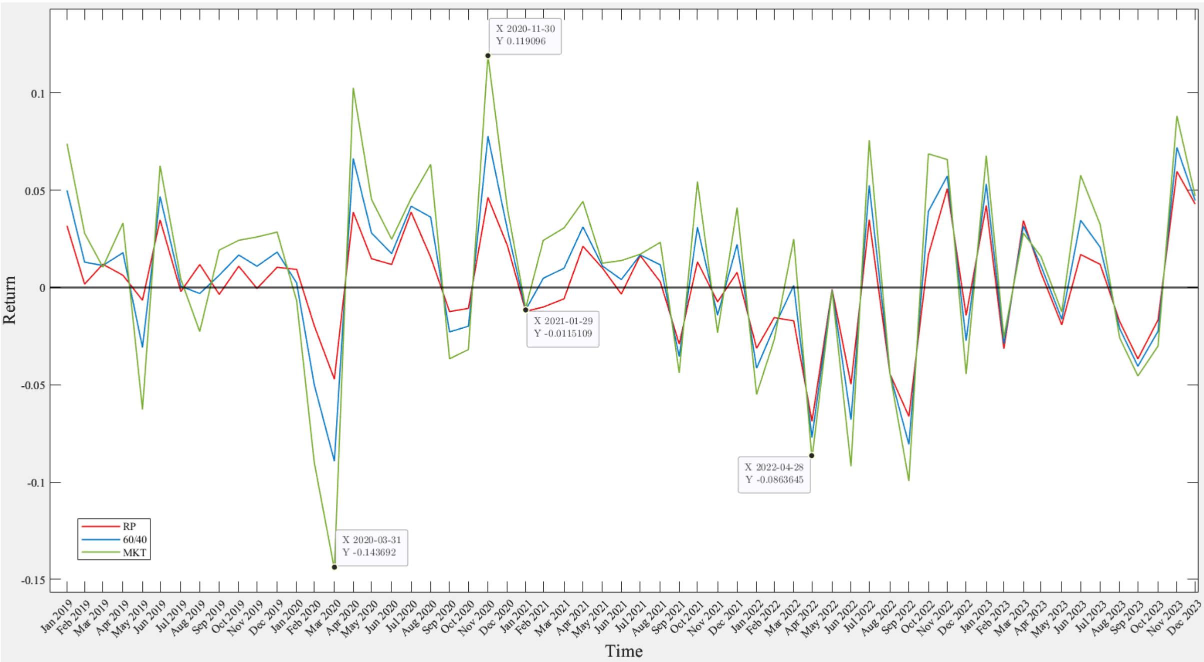

Figure 6: Monthly returns for 2A Risk Parity, 60/40 and Market portfolios. 2019-2023

	Risk Parity Portfolio	60/40 Portfolio	Market Portfolio
Mean	1.6012e-04	0.0031	0.0071
Std Dev	0.0277	0.0375	0.0530
Beta	0.4600	0.6932	1

Table 15: Risk-Return Metrics for 2A RP, 60/40 and Market Portfolios. 2019-2023

The intuition is confirmed by the data shown in Table 15. Even in this context, as observed before, the Risk Parity portfolio recorded the lowest average return, but also the lowest risk. What appears to have changed compared to Table 6 is that the average return is significantly lower than that of the other two portfolios. Additionally, the risk measures have increased relative to the previous sample, with the standard deviation rising from 0.0174 to 0.0277 and the beta increasing from 0.3005 to 0.4600.

Performance Ratio	Risk Parity Portfolio	60/40 benchmark portfolio
Sharpe Ratio	0.00578	0.08405
Sortino Ratio	0.0078	0.1178
Treynor Ratio	3.48e-04	0.0045

Table 16: Performance Analysis, 2A Risk Parity Portfolio, 2019-2023

Indeed, the summary table of the portfolio's performance ratios unfortunately confirms what has just been observed. All the indices are significantly lower than those of the benchmark and are generally low in absolute terms, highlighting the portfolio's inability to generate sufficient risk-adjusted returns. For example, a Sharpe ratio close to zero would indicate that the strategy performed similarly to the risk-free rate, failing to compensate investors for the risk taken. This result underscores the inefficiency of the strategy in the specific market conditions analyzed, making it an unsuitable investment approach during the period considered.

The causes that could explain this significant underperformance lie in the extraordinary behavior of both stocks and bonds during the 2020-2023 period. As previously observed, following an initial drop in the stock market in March 2020 due to restrictions on business activities caused by the COVID-19 pandemic, the market quickly rebounded, leading to exceptional growth. This was undoubtedly driven by the surge in sectors such as technology, which outperformed historical averages. The economic recovery was further supported by extraordinary monetary policies implemented by the Federal Reserve, the European Central Bank (ECB), and other central banks to stimulate the economy. The Risk Parity portfolio, more heavily weighted towards bonds due to their lower volatility, did not fully benefit from the boom offered by the MSCI.

On the other hand, the government bond market experienced a volatile period. Initially (2020-2021), the WGBI showed significant positive performance due to expansive monetary policies designed to foster economic recovery. However, the situation changed drastically following interest rate hikes beginning in early 2021. This monetary tightening negatively impacted government bond returns, as shown in Figure 5. Consequently, the Risk Parity portfolio, being primarily exposed to credit risk, was adversely affected by the combined performance of stocks and bonds.

To conclude, the joint performance of returns shown in Figure 6 suggests a potential shift in asset correlations compared to the earlier part of the chart. The dynamic analysis, which will be addressed in the next section, will aim to determine whether there has been an actual change in the correlation between assets and whether the use of rebalancing is sufficient to improve performance.

5.4 Dynamic analysis: January 2011 - December 2023

The table below illustrates how the correlation between the two assets has changed throughout the estimation windows.

Estimation Window	Correlation
Correlation 2011-2018	0.1327
Correlation 2012-2019	0.0821
Correlation 2013-2020	0.1525
Correlation 2014-2021	0.1285
Correlation 2015-2022	0.3274

Table 17: Correlation Coefficients 2A Over Rolling Estimation Windows, 2019-2023

It generally exhibits significantly higher correlation coefficients compared to the corresponding table for the previous sample (Table 8). In particular, the value of 0.3274 between 2015 and 2022 is notably higher, indicating stronger interdependence between the assets, which may have reduced the benefits of diversification. This rising correlation could be attributed to the spike in inflation observed between 2021 and 2022.

Prices began to rise in the US and Eurozone in the second half of 2021 following the reopening after COVID-19 lockdowns. This was primarily driven by increased household spending and supply chain bottlenecks affecting goods such as motor vehicles and appliances. Inflation then continued to rise throughout 2022, reaching levels in the United States not seen since the 1970s oil crisis.

Europe, on the other hand, faced a dual challenge of rising inflation driven by post-pandemic reopenings and the Russian invasion of Ukraine, which triggered a severe energy crisis that significantly increased production costs across all sectors of the economy. As before, Table 18 shows the weights for the Risk Parity portfolio updated based on the rolling estimation windows.

RP weights	Weights 1 2011-2018	Weights 2 2012-2019	Weights 3 2013-2020	Weights 4 2014-2021	Weights 5 2015-2022
MSCI	0.2959	0.3085	0.2724	0.2721	0.2900
WGBI	0.7041	0.6915	0.7276	0.7279	0.7100

Table 18: 2A Risk Parity Weights for Each Testing Window, 2011-2023

The annual rebalancing, as in the previous case, does not improve performance results; on the contrary, they slightly worsen.

Performance Ratio	Static RP	Dynamic RP	60/40
Sharpe Ratio	0.0058	0.0021	0.0841
Sortino Ratio	0.0078	0.0029	0.1178
Treynor Ratio	3.4805e-04	1.12729e-04	0.0045

Table 19: Performance Analysis, Static vs Dynamic 2A Risk Parity Portfolios. 2011-2023

The performance analysis of the Risk Parity portfolio composed of two assets, MSCI and WGBI, for the 2011-2023 window shows lower risk-adjusted returns than the benchmark. These results, as anticipated, may have been influenced by the unique economic conditions of recent years, particularly the 2021-2022 period, which saw inflation reach the highest peaks in at least the last 20 years. As analyzed in Chapter 1, equities and government bonds tend to move in the same direction in response to inflation and generally underperform when inflation is rising.

The next chapter will, in fact, focus on the comprehensive analysis of a Risk Parity portfolio. By including two additional assets, commodities and inflation-linked bonds, which are favored in high and rising inflation contexts, we will assess whether this will help improve the portfolio's performance.

6 Results: Risk Parity Portfolio with four asset classes

6.1 Static analysis: January 2008 - December 2020

Now, we enter the final section of this dissertation, focused on analyzing a Risk Parity portfolio composed of four asset classes: one representing equities, two for government bonds (nominal and inflation-linked), and a composite index for commodities. The primary objective, as in the previous chapter, is to evaluate its performance against a hypothetical 60/40 benchmark portfolio. Additionally, the analysis will compare the results with the two-asset Risk Parity portfolios discussed earlier. Investing in a more complex portfolio, with a larger number of assets and higher management costs, is justified only if the additional "effort" is adequately rewarded. Thus, the performance of the four-asset Risk Parity portfolio will always be assessed relative to the simpler two-asset portfolio, using the same time frame.

As before, asset closing prices were retrieved from Bloomberg, from which monthly returns were calculated. To facilitate the reading of tables and graph legends, assets are referred to by acronyms: MSCI for the MSCI World Index, WGBI for the World Global Bond Index, COMM for the commodities index, and ILGB for inflation-linked government bonds.

Figure 7: Monthly Cumulative Returns of MSCI-WGBI-COMM-ILGB, 2008-2020

Figure 7 shows the cumulative returns for the four assets during the analyzed period. For the performance of MSCI and WGBI, refer to Section 5.1. ILGB is the asset with the highest cumulative return over these years, recording an increase of + 84.76% by the end of the period while still maintaining relative stability compared to MSCI. Commodities, on the other hand, represent the worst-performing asset, experiencing a sharp decline between the second half of 2011 and the end of 2015. This trend reflects the broader downturn in the commodities market, likely driven by excess supply and weakening global demand.

Statistic	Mean	StdDev	Kurtosis	Skewness
Value	0.000118	0.0495	7.9553	-1.1497

Table 20: Summary Statistics for COMM log returns - Monthly Values

Statistic	Mean	StdDev	Kurtosis	Skewness
Value	0.0040	0.0157	7.4202	-0.7112

Table 21: Summary Statistics for ILGB log returns - Monthly Values

The tables above summarize the distribution characteristics and descriptive statistics for commodities and inflation-linked bonds. For MSCI and WGBI, refer to Tables 2 and 3.

As anticipated by the analysis of the cumulative returns graph, commodities are the worst-performing asset in this window, with a very low mean return and high variability, comparable to MSCI (0.0495 vs. 0.0490), but with significantly lower overall returns. In terms of kurtosis, COMM exhibits the highest value, indicating that the distribution has very fat tails and extreme events are more frequent relative to a normal distribution. Inflation-linked bonds (ILGB) also show a high kurtosis, but they have the highest mean return, very close to that of equities (0.0040 vs. 0.0039), while exhibiting significantly lower variability (0.0157 vs. 0.0490).

Figures 8 and 9 show the Autocorrelation Functions for COMM and ILGB. In the case of commodities, the number of lags was increased to 30 since the value at the 12th lag appeared to be significantly different from zero, suggesting the possibility of annual seasonality in the index. However, by increasing the number of lags, the graph does not seem to indicate the presence of a trend. In fact, the value at the 24th lag is again not significantly different from zero, but detailed analyses of seasonality and trends fall beyond the scope of this thesis. From the study of the two graphs, it is indeed possible to conclude that both indices are highly liquid.

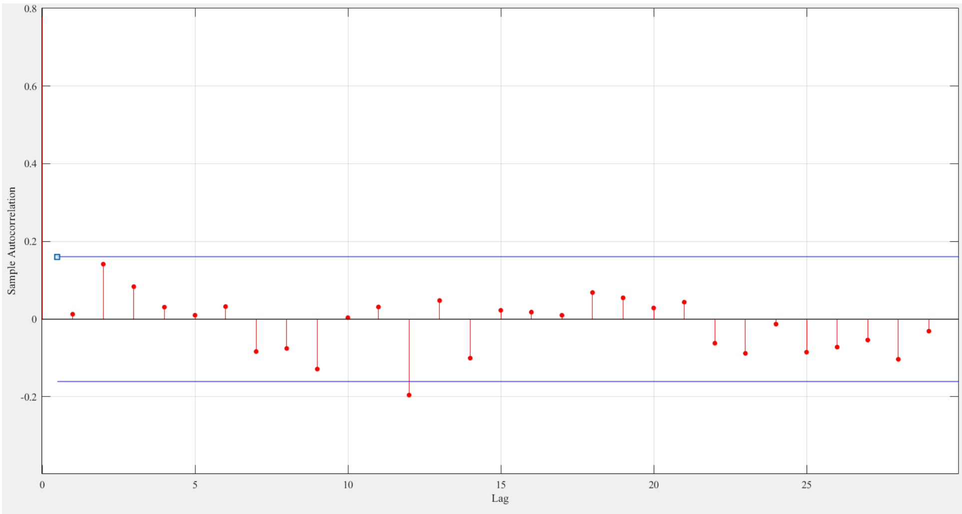

Figure 8: Autocorrelation Function for GSCI

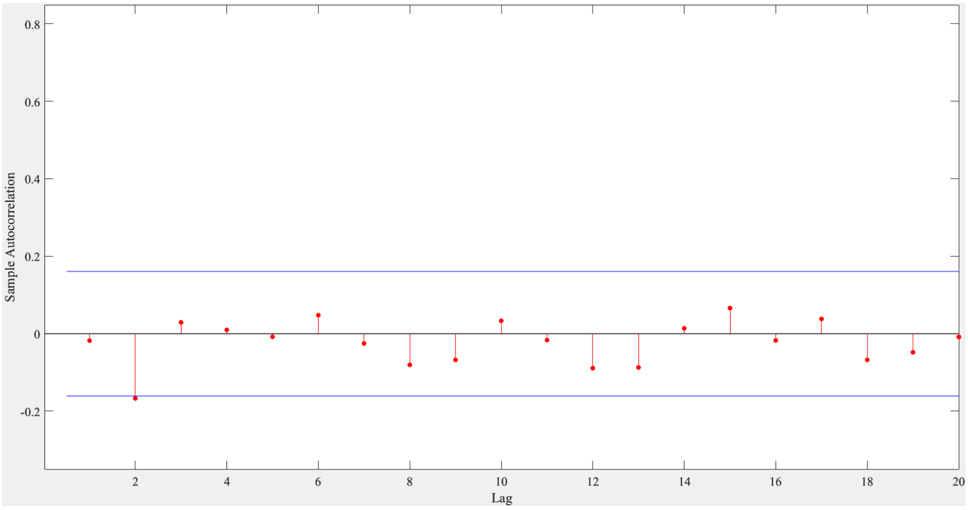

Figure 9: Autocorrelation Function for ILGB

At this point, the following table summarizes the correlation indices between the assets. As previously explained, the algorithm used to calculate the Risk Parity weights is based on the covariance matrix. Nevertheless, the correlation table is presented here for its simplicity and ease of interpretation.

The highest correlation is observed between COMM and MSCI (0.6043), indicating that these two assets tend to move in the same direction. High is also the one between WGBI and ILGB, which aligns with the nature of these two indices, as the latter is essentially the former indexed to inflation. The lowest correlation, on the other hand, is found between MSCI and WGBI, followed by that between COMM and ILGB. This analysis suggests that the inclusion of both WGBI and ILGB can provide significant benefits in terms of risk diversification, as they are the assets with the lowest correlations to more volatile assets (commodities and equities).

	MSCI	WGBI	COMM	ILGB
MSCI	1	0.3203	0.6043	0.3553
WGBI	0.3203	1	0.3793	0.5355
COMM	0.6043	0.3793	1	0.3449
ILGB	0.3553	0.5355	0.3449	1

Table 22: Correlation Matrix 4A, Estimation Window 2008-2015

Starting from the covariance matrix, the weights of the Risk Parity portfolio have been determined and are summarized in the following table. Not surprisingly, assets with lower variability have the highest weights (ILGB followed by WGBI). Commodities are the most penalized asset, with a weight of approximately 11.61%, followed by MSCI with 12.75%.

Asset	Weight
MSCI	0.1275
WGBI	0.3491
COMM	0.1161
ILGB	0.4073

Table 23: 4A Risk Parity Portfolio Weights, 2008-2020

The return pattern shown in Figure 10, comparing the newly constructed Risk Parity portfolio, the benchmark, and the market portfolio, generally follows the same trend. At first glance, there are no notable differences compared to Figure 4. Similar to that case, the Risk Parity portfolio demonstrates lower variability compared to the other two portfolios. For example, the drawdown experienced by all of them in March 2020, following the shutdown of economic activities to contain the spread of Covid-19, was significantly less pronounced for the Risk Parity portfolio than for the other two. The negative peak observed between October and November 2016, which was analyzed in the previous case, is less pronounced here, possibly due to the greater diversification of the portfolio.

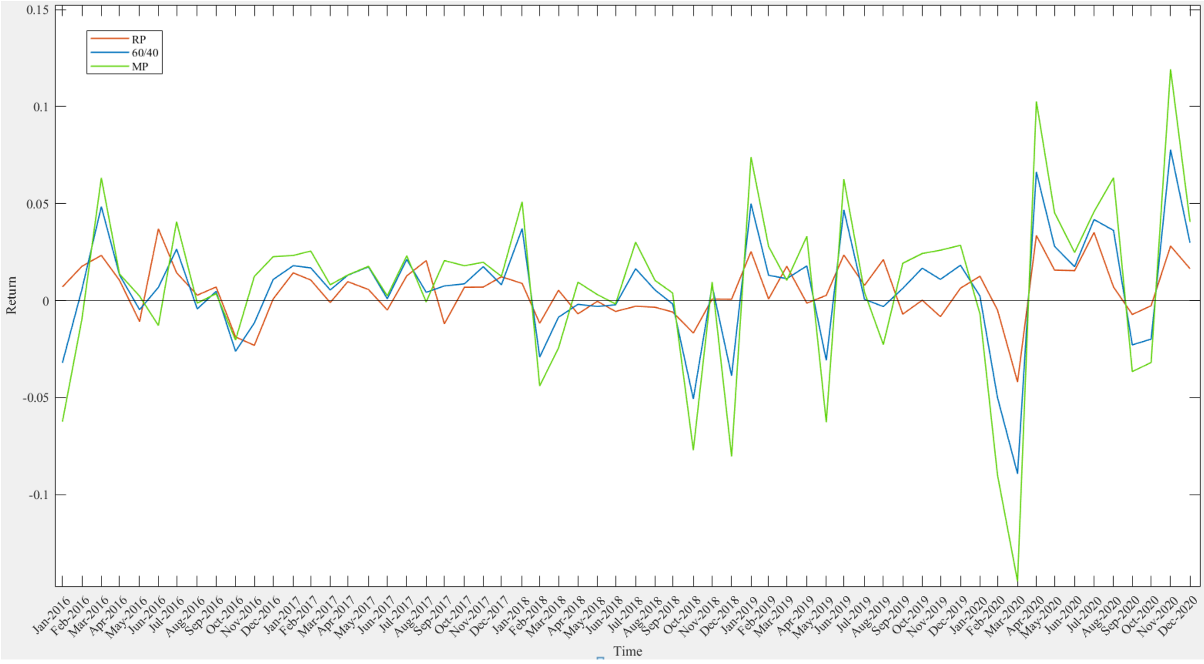

Figure 10: Monthly Returns for 4A Risk Parity, 60/40 and Market Portfolios. 2016-2020

Table 24 presents the standard risk and return metrics. Compared to Table 6, the average return of the Risk Parity portfolio has remained essentially unchanged (0.0042 vs. 0.0041). However, both total and systematic risk have declined, with the standard deviation decreasing from 0.0174 to 0.0146 and the beta from 0.3005 to 0.1945. This suggests that the inclusion of additional asset classes may have enhanced diversification benefits, leading to a more efficient risk-adjusted portfolio.

	Risk Parity Portfolio	60/40 Portfolio	Market Portfolio
Mean	0.0042	0.0054	0.0071
Std Dev	0.0146	0.0279	0.0441
Beta	0.1945	0.6167	1

Table 24: Risk-Return Metrics for 4A RP, 60/40 and Market Portfolios. 2016-2020

To conclude then, the table below summarizes the risk-adjusted performance metrics for this model. This portfolio demonstrates significantly better performance compared to the benchmark and also relative to the Risk Parity portfolio constructed with only MSCI and WGBI. As stated at the beginning of the chapter, the comparison with the previous model is essential to evaluate whether the additional effort and cost of investing in a more complex portfolio are justified by improved performance. The analysis conducted here suggests that the four-asset portfolio outperforms both the two-asset portfolio and the traditional 60/40 strategy.

Performance Ratio	Risk Parity 4 Assets	Risk Parity 2 Assets	60/40 benchmark portfolio
Sharpe Ratio	0.2881	0.2345	0.1950
Sortino Ratio	0.5079	0.3903	0.2953
Treynor Ratio	0.0216	0.0136	0.0088

Table 25: Performance Analysis, 4A Risk Parity Portfolio, 2016-2020

6.2 Dynamic analysis: January 2008 - December 2020

The following section focuses on the dynamic analysis of the portfolio with the aim of understanding whether an annual rebalancing of weights leads to superior risk-adjusted performance. The applied procedure remains unchanged even in the case of a portfolio with 4 assets: starting from the original estimation window, returns are calculated only for the first year of the testing one. From there, the process proceeds incrementally, scaling by one year, and finally, the return vector of the Risk Parity portfolio is the product of the vertical concatenation of each annual return (from 2016 to 2020).

Estimation Window	MSCI-WGBI	MSCI-COMM	MSCI-ILGB
Correlation 2008-2015	0.3203	0.6043	0.3553
Correlation 2009-2016	0.3135	0.5194	0.0909
Correlation 2010-2017	0.2297	0.4852	-0.0275
Correlation 2011-2018	0.1327	0.4052	-0.0345
Correlation 2012-2019	0.0821	0.3073	-0.0759

Table 26: Correlation Coefficients 4A RP, Rolling Estimation Windows, 2008-2020 (1)

Tables 26 and 27 illustrate how the relationships between each pair of assets have evolved as the estimation window shifted over the years. The correlation between MSCI and both WGBI and ILGB has weakened over time, becoming negative in the case of ILGB. This trend could be explained by the extraordinary measures implemented by central banks to inject liquidity into the markets and stimulate the economy in response to the financial crisis. Such an economic scenario benefits equities, which are cyclical assets, but may have penalized bond returns, which, as discussed in the opening chapter, tend to perform better in contexts of falling growth.

A similar trend is evident between MSCI and COMM, where the association has declined, although it remains relatively strong, fluctuating between approximately 0.60

and 0.30. The connection between WGBI and ILGB has been consistently high and stable, staying around 0.50. Lastly, the correlation between WGBI and COMM has increased slightly but remained steady, ranging from 0.40 to 0.42.

Estimation Window	WGBI-COMM	WGBI-ILGB	COMM-ILGB
Correlation 2008-2015	0.3793	0.5355	0.3449
Correlation 2009-2016	0.4443	0.5524	0.1379
Correlation 2010-2017	0.4357	0.4814	0.1195
Correlation 2011-2018	0.4276	0.4872	0.1172
Correlation 2012-2019	0.4186	0.4991	0.1278

Table 27: Correlation Coefficients 4A RP, Rolling Estimation Windows, 2008-2020 (2)

Here too, based on the results of the covariance between assets, the weights for the Risk Parity portfolio were determined for each testing window, and the results are listed in Table 28. MSCI and COMM are the assets with the lowest exposure. In the case of MSCI, the assigned weight was more variable and increased over time, rising from approximately 13% to 18%. For commodity index instead the exposure was more stable, ranging from 10% to 12%.

RP weights	Weights 1 2008-2015	Weights 2 2009-2016	Weights 3 2010-2017	Weights 4 2011-2018	Weights 5 2012-2019
MSCI	0.1275	0.1396	0.1530	0.1626	0.1799
WGBI	0.3491	0.2867	0.2926	0.3088	0.3048
COMM	0.1161	0.1161	0.1049	0.1052	0.1220
ILGB	0.4073	0.4575	0.4495	0.4233	0.3934

Table 28: 4A Risk Parity Weights for Each Testing Window, 2008-2020

To conclude, the performance metrics achieved are summarized in Table 29. Similarly to the previous cases involving two assets, annual rebalancing did not yield any benefits in terms of performance; on the contrary, it worsened in all three cases. The analyses conducted thus far have not been able to demonstrate the validity of annual rebalancing in the case of a Risk Parity portfolio.

Performance Ratio	Dynamic 4A RP	Static 4A RP	60/40
Sharpe Ratio	0.2765	0.2881	0.1950
Sortino Ratio	0.4767	0.5075	0.2953
Treynor Ratio	0.0186	0.0216	0.0088

Table 29: Performance Analysis, Static vs Dynamic 4A Risk Parity Portfolios. 2008-2020

6.3 Static analysis: January 2011 - December 2023

The following section represents the final part of this dissertation. Building on the analogous experiment conducted with a two-asset portfolio, it is important to note that, unfortunately, the performance in that case did not yield positive results (see paragraph 5.3). A possible explanation for the portfolio's underperformance, both in absolute terms and relative to the benchmark, can be attributed to two main factors. On one hand, the equity market experienced significant growth, primarily driven by the technology sector, which expanded considerably during the pandemic. On the other hand, the bond market, which initially acted as a safe haven during the early years of the pandemic, faced a steep decline in early 2022 due to unprecedented inflation levels over the past two decades and the subsequent rate hikes implemented by central banks. As previously discussed, this environment did not favor the Risk Parity portfolio, likely due to the effects of soaring inflation.

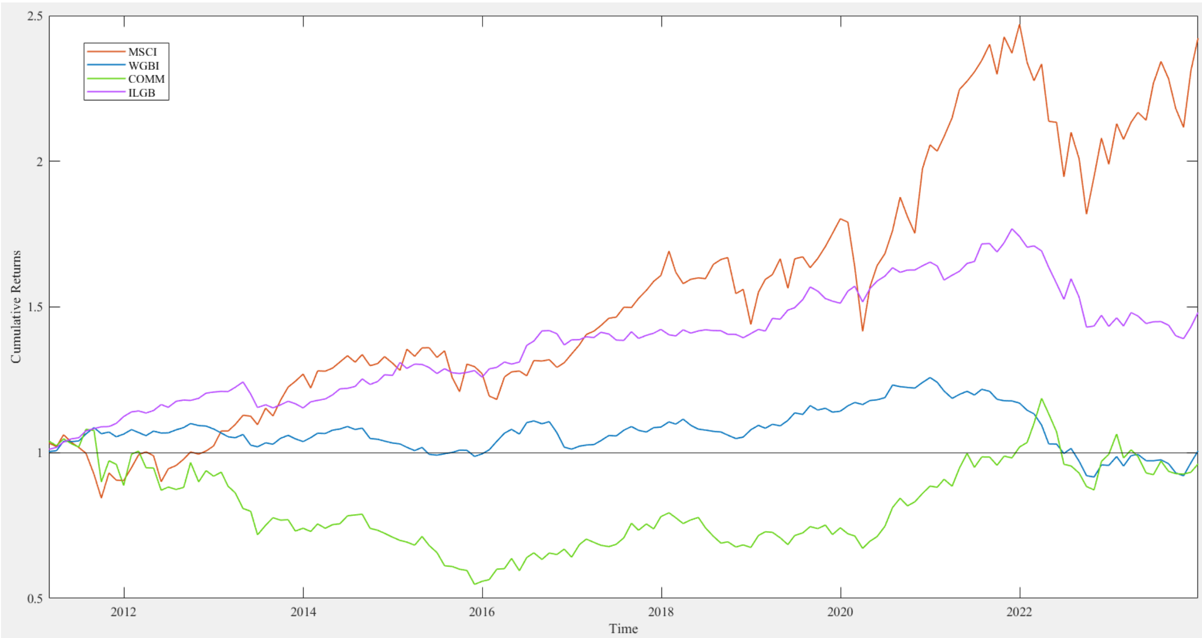

Figure 11: Cumulative Returns of MSCI-WGBI-COMM-ILGB, 2011-2023

In this context, the question arises as to whether a portfolio extended with the addition of two assets—commodities and inflation-linked bonds, which are designed to perform well in inflationary scenarios—could have achieved better results in the years from 2019 to 2023. Furthermore, this analysis wants to evaluate how the expanded portfolio performed relative to the simpler two-asset configuration, both in terms of risk-adjusted returns and its ability to adapt to changing macroeconomic conditions.

The chart in Figure 11 for the period 2011–2023, updated with data for COMM and ILGB, shows that MSCI achieved the highest cumulative return. This is due to its significant growth following the sharp decline in March 2020. Despite a setback in September 2022, it continues to exhibit strong overall returns.

MSCI is followed by ILGB, which experienced significant growth, peaking on November 30, 2021, due to expectations of rising inflation. However, similar to WGBI, ILGB also faced a downturn caused by monetary tightening and rising interest rates but it maintained a higher overall return, supported by its inflation-linked nature. Inflation peaked at the end of 2022, further supporting ILGB's performance. [7]

A more in-depth analysis may be required to fully understand the performance of commodities. The chart highlights an initially negative trend up until early 2016, likely driven by the post-financial crisis recovery, characterized by declining commodity prices, such as oil, due to low global demand and oversupply. Falling inflation during this period may further contributed to the poor performance of this index. From 2016 to around 2020, the index showed moderate growth, yet it remained the asset with the worst cumulative performance. A significant recovery occurred in March 2020, coinciding with the resumption of economic activity and increased demand for commodities, particularly oil and energy. This recovery peaked in March 2022, supported by high inflation, which surged first due to the global economic rebound and later exacerbated by the Russia-Ukraine war. The geopolitical tensions drove up prices for key commodities, such as natural gas and agricultural products, especially in the Eurozone. Despite this recovery, commodities continue to exhibit high volatility and remain less performing compared to other asset classes, reflecting their sensitivity to macroeconomic and geopolitical shocks.

Statistic	Mean	StdDev	Kurtosis	Skewness
Value	-0.00026	0.0443	4.1513	-0.3697

Table 30: Summary Statistics for COMM log returns - Monthly Values

[7] See appendix A.3 for CPI charts.

Statistic	Mean	StdDev	Kurtosis	Skewness
Value	0.0025	0.0174	4.4733	-0.6248

Table 31: Summary Statistics for ILGB log returns - Monthly Values

The two tables above confirm the findings observed in the cumulative returns chart. They should be interpreted in conjunction with Tables 11 and 12, for a more comprehensive analysis. COMM recorded a negative average return, coupled with the highest variability (0.0443 compared to 0.0427 for MSCI). Although WGBI and ILGB are the least volatile assets, they exhibited high kurtosis, indicating the presence of fat tails and a higher likelihood of extreme events. In addition, ILGB displayed the most pronounced negative skewness, suggesting a greater probability of extreme negative events compared to the other assets.

	MSCI	WGBI	COMM	ILGB
MSCI	1	0.1327	0.4052	-0.0345
WGBI	0.1327	1	0.4276	0.4872
COMM	0.4052	0.4276	1	0.1172
ILGB	-0.0345	0.4872	0.1172	1

Table 32: Correlation Matrix 4A, Estimation Window 2011-2018

The matrix reveals a strong correlation between MSCI and COMM, as well as between WGBI and ILGB. COMM and ILGB exhibit a weak positive relationship, whereas the correlation between MSCI and ILGB is notably negative. Based on these results, as in previous cases, the Risk Parity portfolio is constructed.

Asset	Weight
MSCI	0.1626
WGBI	0.3088
COMM	0.1052
ILGB	0.4233

Table 33: 4A Risk Parity Portfolio Weights, 2011-2023

In general, the macrostructure of the portfolio has not changed compared to the previous case for the 2008-2020 sample (Table 23). The assets with the largest exposure remain the two government bonds (WGBI and ILGB). The exposure to MSCI has increased (from approximately 12.75% to 16.26%), and the same applies to ILGB, which rose from 40.73% to 42.33%. The opposite is true for the other two assets, with WGBI decreasing from 35% to 31% and COMM from 11.6% to 10.5%.

Before moving on to the results, it is worth taking a quick look at the chart showing the returns of the Risk Parity portfolio, the benchmark, and the market portfolio. Actually, the trend is not significantly different from what was presented in Figure 6, where the Risk Parity portfolio was unable to mitigate oscillations starting from April 28, 2022. At first glance, the performance does not appear to have improved.

Figure 12: Monthly returns for 4A Risk Parity, 60/40 and Market Portfolios. 2019-2023

Even Table 34 does not appear to yield particularly promising results. However, it is important to highlight that, compared to the two-asset version of the portfolio, the risk-adjusted performance in this case should be better. This is due to both an increase in the average return (from 0.00016 to 0.00044) and a decline in risk measures, with the standard deviation dropping from 0.0277 to 0.0249 and the beta from 0.4600 to 0.3869. While an improvement over the simpler two-asset case is expected, the results do not indicate a superior performance relative to the benchmark.

	Risk Parity Portfolio	60/40 Portfolio	Market Portoflio
Mean	4.3825e-04	0.0031	0.0071
Std Dev	0.0249	0.0375	0.0530
Beta	0.3869	0.6932	1

Table 34: Risk-Return Metrics for 4A Risk Parity, 60/40 and Market Portfolios. 2019-2023

Indeed, the results, even in this case, are not satisfactory. In line with expectations, all three ratios have substantially improved compared to the two-asset portfolio analyzed over the same period. However, the observed improvement was not sufficient to allow the portfolio to outperform the benchmark in the 2018-2023 window.

Performance Ratio	Risk Parity 4 Assets	Risk Parity 2 Assets	60/40 benchmark portfolio
Sharpe Ratio	0.0176	0.0058	0.0841
Sortino Ratio	0.0233	0.0078	0.1178
Treynor Ratio	0.0011	3.4805e-04	0.0045

Table 35: Performance Analysis, 4A Risk Parity Portfolio, 2019-2023

Once again, as this study approaches its conclusion, the impact of annual rebalancing on the performance measures has been analyzed. The subsequent section presents the results, highlighting how the rebalancing strategy influences the overall performance metrics.

6.4 Dynamic analysis: January 2011 - December 2023

Tables 36 and 37 illustrate how the correlation indices for each pair of assets have changed as the estimation window shifted. Overall, it is evident that correlations increased over time as the reference window shifted forward.

A particularly notable example is the MSCI-ILGB pair, whose correlation transitioned from negative values in the first two samples (-0.0345 and -0.0759) to a moderately positive relationship between 2012 and 2019 equal to 0.3693. Similarly, the correlation between MSCI and WGBI, as well as MSCI and ILGB, also increased, albeit to a lesser extent, stabilizing at 0.3274 and 0.3036, respectively. The one between WGBI and COMM remained relatively stable throughout the period. To conclude, the highest level of correlation was observed between WGBI and ILGB during the 2012–2019 window, reaching 0.6635. The overall increase in asset correlations could, in fact, explain the poorer performance of the portfolio compared to the benchmark during this period.

Estimation Window	MSCI-WGBI	MSCI-COMM	MSCI-ILGB
Correlation 2008-2015	0.1327	0.4052	-0.0345
Correlation 2009-2016	0.0821	0.3073	-0.0759
Correlation 2010-2017	0.1525	0.3586	0.1369
Correlation 2011-2018	0.1285	0.4031	0.1090
Correlation 2012-2019	0.3274	0.4289	0.3693

Table 36: Correlation Coefficients 4A RP, Rolling Estimation Windows, 2011-2023 (1)

Estimation Window	WGBI-COMM	WGBI-ILGB	COMM-ILGB
Correlation 2008-2015	0.4276	0.4872	0.1172
Correlation 2009-2016	0.4186	0.4991	0.1278
Correlation 2010-2017	0.4500	0.5342	0.2236
Correlation 2011-2018	0.4793	0.5110	0.1724
Correlation 2012-2019	0.4573	0.6635	0.3036

Table 37: Correlation Coefficients 4A RP, Rolling Estimation Windows, 2011-2023 (2)

RP weights	Weights 1 2011-2018	Weights 2 2012-2019	Weights 3 2013-2020	Weights 4 2014-2021	Weights 5 2015-2022
MSCI	0.1626	0.1799	0.1510	0.1482	0.1522
WGBI	0.3088	0.3048	0.3374	0.3315	0.3346
COMM	0.1052	0.1220	0.1381	0.1394	0.1667
ILGB	0.4233	0.3934	0.3735	0.3809	0.3465

Table 38: 4A Risk Parity Weights for Each Testing Window, 2011-2023

The table 38 presents the Risk Parity weights for each testing window, illustrating how the allocation across the four assets evolves over time. Overall, the macrostructure of the portfolio remains relatively stable, with ILGB consistently holding the largest weight, while the other assets, particularly COMM and MSCI, show more gradual changes.

The weight of MSCI remains quite stable, ranging between 0.1482 and 0.1799 providing a moderate but consistent contribution to the portfolio. WGBI shows a slight increase in weight across the windows, from 0.3088 to 0.3346, reflecting their low volatility and stable role within the portfolio.

COMM increases from 0.1052 in the first window to 0.1667 in the last. This rise could be attributed to lower volatility as the testing period progresses. On the other hand, ILGB, while maintaining the highest weight, shows a gradual decline from 0.4233 to 0.3465. This decrease may be driven by a reduction in the diversification benefit, as observed in Tables 36 and 37.

Performance Ratio	Dynamic RP	Static RP	60/40
Sharpe Ratio	0.0177	0.0176	0.0841
Sortino Ratio	0.0235	0.0233	0.1178
Treynor Ratio	0.0011	0.0011	0.0045

Table 39: Performance Analysis, Static vs Dynamic 4A Risk Parity Portfolios. 2011-2023

Finally, as shown in Table 39, it can be observed that, while there has been some improvement in performance, it is neglecting and not significant. In fact, rebalancing failed to deliver any meaningful enhancement. The differences between the performance metrics of the dynamic and static portfolios are negligible, with the Treynor Ratio showing no difference.

Based on the analysis conducted, it can be concluded that annual rebalancing, across all the model specifications examined, was ineffective in enhancing the returns of the Risk Parity portfolio. As previously suggested, one possible explanation for this underperformance is the inappropriate choice of the rebalancing interval. More frequent rebalancing—such as quarterly or semi-annual adjustments—might better capture shifts in asset relationships and provide a more accurate reflection of evolving market dynamics. However, it is important to acknowledge, even though it extends beyond the scope of this thesis, that increasing the frequency of rebalancing would also lead to higher transaction costs, stemming from the need to open and close positions more frequently.

Conclusions and future research suggestions

The analysis conducted involved the construction and performance evaluation of two portfolios built according to the Risk Parity principle. The first portfolio is a simplified version, consisting of only two assets: one representing global equities, proxied by the MSCI World Index, and one representing global government bonds, proxied by the WGBI Index. In the second case, the portfolio is expanded by adding two additional asset classes: Commodities and Inflation-Linked Government Bonds. Both portfolio configurations have been assessed over two different time windows: from 2008 to 2020 in the first case, and from 2011 to 2023 in the second. Performance evaluation is based on three key indicators: the Sharpe Ratio, the Sortino Ratio, and the Treynor Ratio. In all cases, performance is assessed relative to a benchmark, specifically a traditional 60/40 portfolio composed of 60% MSCI World and 40% WGBI.

The first analysis, conducted on the two-asset portfolio over the 2008-2020 period, yields satisfactory results, as the portfolio outperformes the benchmark across all performance indicators. However, a completely different outcome emerges when the same two-asset portfolio is analyzed over the 2011-2023 period. In this case, the Risk Parity portfolio underperformes the benchmark according to all performance metrics. The potential reasons identified include, on one hand, the strong growth of the equity market, primarily driven by the technology sector, which experienced significant expansion due to the COVID-19 pandemic restrictions. On the other hand, the tightening of monetary policy in response to soaring inflation initially triggered by the reopening of economic activities and later exacerbated by the energy crisis following Russia's invasion of Ukraine. That measure has a negative impact on the Risk Parity portfolio, given its higher exposure to bonds relative to equities.

The analysis is then replicated for the four-asset portfolio. In this case as well, performance during the 2008-2020 period has been positive, with all performance indicators surpassing those of the benchmark. Additionally, this portfolio is compared to the two-asset Risk Parity portfolio over the same time frame. The results is again favorable, as the expanded portfolio outperformes the simpler version. This comparison is particularly relevant because, to justify the additional costs and complexities associated with managing a broader portfolio, there needs to be a clear performance advantage. The superior performance of the four-asset portfolio emphasizes how true risk diversification—rather than mere capital allocation—delivers tangible benefits in terms of returns.

However, when the four-asset portfolio is assessed over the 2011-2023 period, the results are once again unsatisfactory. Although the performance indicators show some improvement compared to the two-asset portfolio, they remain significantly lower than those of the benchmark.

For all the analyses conducted, an annual rebalancing system is implemented to evaluate whether periodically adjusting asset weights in response to new information could enhance performance. The findings indicate no clear benefits from annual rebalancing. In the first three cases, performance deteriorates across all indicators. Only in the final case does performance show slight improvement, but not enough to justify the costs associated with annual rebalancing.

Based on the analysis conducted, it can be concluded that the Risk Parity strategy exhibits lower variability compared to both the 60/40 portfolio and the market portfolio in all the examined cases, albeit at the cost of lower average returns. In terms of risk-adjusted performance measures, the Risk Parity strategy proves to be superior for both the two-asset and four-asset portfolios in the first sample period. However, this is not the case in the second sample, where performance for both portfolios is significantly weaker than the benchmark. Nonetheless, it is important to highlight that expanding the portfolio from two to four assets results in some performance improvements, albeit insufficient to surpass the benchmark.

While it is undeniable that the 2020-2023 window was marked by major macroeconomic and geopolitical disruptions, the Risk Parity strategy should, in theory, have been able to perform better even during periods of instability. The present analysis does not support this view. Instead, the results suggest that performance is highly dependent on the specific time frame analyzed. These findings align with the conclusions of Anderson et al. (2012), which emphasize that the success of the Risk Parity strategy is heavily dependent on the prevailing macroeconomic conditions.

The analysis conducted could certainly be strengthened by first incorporating transaction costs. These costs can have a significant impact on portfolio performance, especially when periodic rebalancing is involved. Since annual rebalancing entails costs associated with closing and reopening positions, future research should include a realistic estimation of transaction costs to provide a more accurate assessment of net returns.

Regarding rebalancing, the study failed to demonstrate the effectiveness of an annual rebalancing strategy, which, as previously discussed, led to a deterioration in performance metrics in almost all cases, or at best, had a neutral effect. A first suggestion for improvement would be to explore the impact of a more frequent rebalancing strategy, evaluating the trade-off between potential benefits and associated costs. I.e. Ning et al. (2022) found that quarterly rebalancing of Risk Parity portfolios tends to yield better results compared to both annual and monthly rebalancing strategies.

Then, a key aspect of real-world Risk Parity portfolios is the use of leverage, which was not explored in this study. While the Sharpe Ratio and other performance indicators demonstrated superior risk-adjusted returns in the first time window, the absolute returns of the Risk Parity portfolio remained lower than both the 60/40 and the market portfolio. Leverage is typically employed to enhance expected returns or to bring the portfolio's risk level in line with traditional benchmarks, such as the 60/40 portfolio or an equity-dominated strategy. Future research could assess how leveraged implementations of Risk Parity perform relative to unleveraged versions. For instance, the study by Anderson et al. (2012) presents a comparison between four different investment strategies: a value-weighted portfolio, a 60/40 portfolio, un unlevered and a levered Risk Parity portfolio. Their analysis also incorporates borrowing and trading expenses providing a more realistic assessment of portfolio performance under different market conditions. The first accounts for the cost of leveraging whilst the latter is associated with the rebalancing of portfolio weights.

Appendix A

A.1 Risk Parity Weights Matlab Code

This appendix presents the MATLAB function implemented for determining the weights of the Risk Parity portfolio. It is the code for the algorithm illustrated in Chapter 3.

```

function [w_optimal, var_p, RC] = risk_parity_weights(Sigma)

% Find Risk Parity Portfolio Weights using Newton-Raphson optimization algorithm

    max_iter = 1000; %The maximum number of iterations set for the algorithm to find
    %the optimal weights. If the maximum number of iterations is  reached without
    %finding the values,the algorithm will return the best approximation found so far.

    epsilon= 0.0000001; % Epsiolon represents the tolerance margin used
    % to determine the convergence of the algorithm. It defines how close
    % the result must be to the optimal solution for the algorithm to stop.
    %In this case, epsilon is set to be a very small value near zero.

    % Number of assets
    n = size(Sigma, 1);

    % Starting weights (uniform) e mu (constant value)
    w0 = ones(n, 1) / n;
    mu0 = 1;

    y = [w0; mu0];

    %% F(y) function (sistem of equations)

    F = @(y) [Sigma * y(1:n) - y(n+1) ./ y(1:n);
        sum(y(1:n)) - 1];

    %% Jacobian matrix J(y)

    J = @(y) [Sigma + diag(y(n+1) ./ (y(1:n).^2)), -1 ./ y(1:n);
        ones(1, n), 0];

    %% Newton-Raphson algorithm

```

```
    for iter = 1:max_iter

        y = y - J(y) \ F(y);

        if sqrt(1/(n+1)*F(y)'*F(y)) < epsilon
            break;
        end

    end

    w_optimal = y(1:n); % optimal weights

%% Adding a control: if iteration is correct RCs must be equal for each asset.

var_p = w_optimal' * Sigma * w_optimal;

% Risk Contribution

RC = zeros(n, 1);

    for i = 1:n

    RC(i) = w_optimal(i) * Sigma(i, :) * w_optimal;

    end

RC = RC / var_p;


end
```

Risk Parity Weights: Matlab Code

A.2 Alternative Method for Weights Calculation

Another - simpler but less efficient - algorithm can be used to search for the Risk Parity portfolio:

1. Start from an initial guess, such as

$$\mathbf{w}^{(0)} = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right).$$

2. Calculate σ_p^2 for the current portfolio $w = w^{(k)}$. Hence, calculate $\beta_{i,p}^{(k)}$ for $w = w^{(k)}$ for $i = 1, 2, \ldots, n$.

3. If

$$\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(w_i^{(k)}\beta_{i,p}^{(k)} - \frac{1}{n}\right)^2} < \epsilon$$

   stop, where ϵ is some tolerance error, with $\epsilon > 0$.

4. Set new weights

$$w_i^{(k+1)} = \frac{\beta_{i,p}^{(k)}}{\sum_{j=1}^{n}\beta_{j,p}^{(k)}}, \quad \text{for } i = 1, 2, \ldots, n.$$

5. Return to step 2.

A.3 Consumer Price Index

The following charts display the CPI for both the US and the Eurozone.

Figure 13: United States ,CPI - All Urban Samples: All Items, Chg Y/Y. Source LSEG Workspace

Figure 14: Euro zone, Consumer Prices, All Items, 00FLASH, Total, Chg Y/Y. Source LSEG Workspace

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