LUISS – Libera Università Internazionale degli Studi Sociali

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The economic effects of (sub)optimal fertility: impacts on median income

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1. Abstract

Human fertility has historically been far above replacement rate. However, in just over half a century, global fertility has plummeted from more than five children per woman in 1960 to 2.4 in 2024, and it is projected to fall below replacement rate (commonly set at 2.1 children per woman) sometime in the 2070s. This phenomenon is already a fact in many parts of the world (especially in the western world), overturning societal makeup and bringing about unprecedented structural changes in society, such as the ever growing burden that seniors exert on society, but also the increased productivity of adults who are less burdened by child care.

So, in order to understand the impact of this event, **the scope of this dissertation is to estimate which is the optimal fertility rate that maximizes societal wellbeing, here measured as median income.**

This is done through two methodological steps: The best dependency ratio is first estimated through the use of cross-country regression analysis. After this has been obtained, a system of ordinary differential equations is put in place in order to derive an optimal fertility function which minimizes deviations from optimal dependency (all the while accounting for the dynamics of many time sensitive factors). The model is then applied in a global scenario, and its predictions will be benchmarked against U.N. predictions.

It will be discovered that **deviations from optimal fertility will result in a constant but inevitable decrease in median income**, all else equal.

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2. Introduction

Global fertility rate has been over the replacement rate throughout most of human history. This phenomenon, which was considered the norm up until fifty years ago, started to reverse in the last half century. In fact, as of 2023, global fertility rate is at less than half of what it was in 1960 (Worldbank, 2023). Locally, the situation is quite uneven: developed countries have entered the new millennium with fertility rates already under replacement rate, while developing countries are largely above/at replacement rate (Worldbank, 2023), even though fertility rates are equating fast, and expected to fully converge by the end of the century (U.N. population division, 2023). This change in natality, paired with progressively increasing life expectancy (in relatively little time) has brought about many societal and economic challenges, oftentimes never before faced in our collective history (International Monetary Fund research dept, 2025).

Translating this phenomenon in economic terms, we have to remember that **economic activity is merely the result of** (sometimes very complicated) **exchanges between members of society**. Hence, there is evidence to believe that population makeup is a determinant of the overarching economy, both in quantitative and qualitative terms. This has to do with the fact **that children and seniors are not as economically proficient** as adults, and adults themselves are more productive in certain age cohorts more than others (Freyer, 2007). Furthermore, infants and elders require care from the active population, redirecting resources from more fruitful usage. At the same time, the workforce of today will enter old age at some point.

So, in order to maintain the same level of benefits enjoyed by current retirees and by society at large, there must be enough young individuals to at least replace today's working class (Pecchenino & Utendorf, 1999). But how much is enough, and if younger people require care, what is the maximum number that allows for future welfare maximization (here intended as median income), without excessively straining today's workforce?

For what just said, it seems to be that the nature of the forces at play implies that the **function that links fertility rate to income could be effectively non-linear**, namely showing an inverse U-shaped relationship with median income.

Hence, the research question that this dissertation aims to answer is: **which is the best possible level of fertility rate in order to maximise median income,** precisely which is the optimum point of said function.

It has in fact been shown that optimal fertility is not, as some demographers would say, exactly 2.1 children per woman (Striessnig & Lutz, 2011), and while extensive work has been done in terms of connecting optimal fertility rates in order to maximise some forms of societal welfare (which will be discussed in the next section), or in terms of how fertility rates affect income, none of the available research explores which is the global optimal fertility rate in order to maximize median income. This gap in the literature is what this dissertation is aimed at filling.

The original idea employed in order to identify this relationship was to utilize a non-linear cross sectional regression analysis which contained the median income level of different countries as a regressand, and the fertility rate of said countries as a regressor, besides other various control variables. This approach, though, proved to be cumbersome, since the actual impact of today's fertility rate is felt after a notable time span (at least 15 to 20 years). Because of this, the regression coefficients appeared to be noisy, poisoning results.

Furthermore, by employing fertility rate as a regressor , an additional issue of reverse causality would have needed to be addressed. Hence why the following approach has been chosen:

The way in which this relationship is established is through two sequential steps. First, **a cross-sectional regression analysis** is employed. The necessary dataset is constructed using Data from the last available year, covering a global sample of countries across different income levels. The analysis focuses on identifying the **dependency ratio** that maximizes median income, employing a non-linear regression model along other economic controls. The second step is using the optimal dependency ratio obtained through regression analysis to **reverse engineer the optimal global Fertility rate.** This is accomplished through the use of a system of differential equations, that treats the obtained dependency ratio as a steady state from which to derive the optimal number of yearly births. In this manner optimal births can be objectively quantified. They are then used in an age-structured cohort model, the final aim being to construct a second dependency ratio curve, dependent on the aforementioned optimal births. Differences in median income can then be obtained by comparing the evolution of said curve against a benchmark, which in this case will be based on U.N. predictions. This allows for the quantification of changes in median income in absolute and relative terms.

3. Literature Review

The relationship between fertility rates and various economic variables has been studied rather extensively.

Starting from **classical economic literature**, the causal relationship traditionally runs **from income to fertility**. This idea was pioneered by Becker (Becker, 1960), who holds that rising income leads to declining fertility rates. This is explained through the **quantity-quality trade-off**: as household income increases, parents tend to have fewer children and instead invest more heavily in each child's education, health, and other factors aimed to better the pupil's future life. In this framing, **fertility decline** is treated as a **response** to economic development, rather than a driver of it. The demographic transition theory also aligns with this logic, asserting that as societies become wealthier, fertility naturally falls due to lower infant mortality, greater access to contraception, and increased opportunity costs of childbearing (especially for women).

In recent decades, researchers have started to challenge the validity of the aforementioned paper. From an empirical standpoint, an example can be given by Luci-Greulich and Thévenon (Luci-Greulich & Thévenon, 2014). In fact, they found that GDP and fertility rate were indeed correlated, but results were not compatible with what classical theory would have suggested. In fact, the researchers identified what they called a J shaped relationship. To further explain, it was indeed demonstrated how rise in GDP per capita tended to initially suppress fertility rate. Further increase, though, did not correlate with lower fertility rates, which seemed to plateau no matter the increase in GDP per capita. Lastly, it was (although weakly) shown that, in particularly rich countries, increase in GDP per capita actually correlated with rising fertility.

More recent studies have begun to challenge the classical causal relationship by emphasizing that GDP may not be the main determinant (or even a determinant at all) of fertility. This is precisely what Doepke (Doepke, Hannusch, Kindermann, & Tertilt, 2022) has done. In their work, in fact, they potentially disproved the work of Beker, and instead linked fertility in high income countries on other factors, such as: family policies, the cooperations of fathers, social norms that support working mothers and flexibility of labour market, and other environmental factors that incentivize an healthier work-life balance. Myrskylä (Myrskylä, Kohler, & Billari, 2009) also empirically estimated how fertility rates may be affected, though not through the use of GDP, but rather the Human Development index, which is composed of several factors other than GDP. The authors found that, while fertility declines as countries move from low to medium levels of development, at very high levels of socioeconomic development the trend **is reversed**. That is, **in the most highly developed countries, further development halted the decline in birth rates and even led to modest fertility increases**, turning the previously negative development–fertility relationship **into a U-shape.**

Furthermore, researchers have also increasingly investigated the reverse direction of causality, emphasizing on **endogeneity of fertility rates and potential feedback loops** that may exist. This is what Boikos et alia (Boikos, Bucci, & Stengos, 2013) have done in their work, which followed a bifrontal theoretical and empirical approach. Their analysis was conducted on the effects of birth rates on per capita human capital investment, employing a benchmark model that followed endogeneity and monotonicity of fertility on human capital, and a revised model that instead followed an endogenous, non-monotonic approach between the two measures. Results strongly rejected the benchmark model, instead supporting the latter approach. Once again, in line with the other papers cited above, **results showed an inverted U shaped relationship between fertility and Human capital investment**.

On a purely empirical side, much of the literature shows how in high fertility scenarios, reduction of births strongly correlates with an improvement of economic conditions. This is the conclusion of Ashraf (Ashraf, Weil, & Wilde, 2013) who through the employment of a simulation model (where fertility rate is considered endogenous) which had Nigeria as the subject country, showed that an absolute reduction of birth rates by 0.5 leaded to an increase in GDP per capita by roughly 6% in 20 years and 12% in 50 years (even though they emphasize that reduction in fertility is not the key to development, rather one of many factors). Bloom and other researchers (Bloom D. , Canning, Fink, & al., 2009) **(**Employing panel data with abortion law changes as an instrument), estimated a **large negative causal effect** of fertility on female labour force participation. Fewer births led to higher women's workforce participation and persistent cohort effects. Simulations indicate that fertility reduction, via a "demographic dividend" (more workers per dependent), substantially **raises income per capita** over time.

On the other hand of the spectrum, researchers found that in sustained low fertility scenario, the opposite may hold. Specifically, research conducted by Cevik (Cevik, 2025) (employing the same strategy of abortion law as an instrument) showed that higher fertility has a positive causal effect on GDP per capita growth in Europe, which as a continent already has low, stagnant birthrates. In fact, in aging economies with ultra-low fertility, further fertility decline became a drag on growth. A simulation for Lithuania finds that its fertility drop (which has seen its total fertility rate, or TFR, decline from 2.03 to 1.27 in the span of 30 years) could lower GDP per capita by up to 17.6% over coming decades compared to a scenario with stable fertility. This highlights that in posttransition societies, **boosting fertility can mitigate workforce shrinkage and support growth**, though only to a modest degree.

A theoretical link that connects economic effects of both high and low fertility could be given by the work of Bloom, canning and Sevilla (Bloom, Canning, & Sevilla, The Demographic Dividend: A New Perspective on the Economic Consequences of Population Change (1st ed.)., 2003). In their paper, they describe what is known as first and second “demographic dividends”. In their work they formalize and quantify the idea that changing age structures can yield significant (but nonetheless limited) economic gains. Their reasoning boils down to the fact that, as fertility decreases, dependency (or in their case support ratios) benefit positively, since a decline of fertility brings significant additional per-capita income (as the working-age cohort as a percentage of total population rises) which they identify as the first demographic dividend. Furthermore, as the working age cohort ages, subsequent boost from higher aggregate saving (and thus investment) is then reflected in what they identify as the second demographic dividend. The authors also stress the transitory nature of this phenomenon, which (other than needing to be “fully activated” by appropriate policies) might reverse in the future due to the aging process of the general population. Lee and Mason build on this work (Mason, Lee, Abrigo, & Lee, 2017), assessing where each country stands dividend wise, and project the likely effects over the foreseeable future. Their results highlight that regions that have had continuously depressed fertility rate may soon start to face demographic drag, regions that have experienced recent fertility decline are in the process of harvesting dividends effects, while regions that have just started to see a decline in fertility have yet to see any tangible dividends.

To summarize, much of the available literature confirms that the relationship between Fertility and societal well-being of various monetary and non-monetary forms is indeed nonlinear (or more broadly speaking non monotonic), but instead supports the idea fostered in the introduction of the paper, hence that the relationship is reversely-U shaped, meaning that **both very high and very low Fertility is decremental for society**.

Important to note, most of this research was focused on average indicators (such as GDP per capita or average income) or more general indicators rather than **median income**, which better captures the reality of an individual earnings profile and provide a better picture of economic well-being for the general population. Furthermore, none of these studies provide a **direct** estimate of the **optimal fertility rate** that maximizes **median income** on a global level.

4. Methodology

In this section, the backbone of the analysis will be constructed through the use of two different techniques:

in the first part (Part I:Regression Modelling), a non-linear regression will be constructed in order to empirically understand in which way dependency ratio and median income are related; specifically which is the optimal dependency ratio in order to maximize median income. In this optic, the resulting dependency ratio can be viewed as the optimal dependency level.

In the second part (Part II: From Dependency To Births), a system of **Ordinary differential equations** will be put in place in order to decipher optimal birth rates starting from optimal dependency, both in terms of **static equilibrium** as well as the **transition states** from today to the equilibrium.

4.1 Part I: Regression Modelling

The focus of this chapter is to find the optimal cross-sectional regression specification, so to locate the model that is able to explain best the relationship between Median Income (MI) and Dependency Ratio (DR), through a set of economic and social predictors. The final goal being to estimate the median income maximizing dependency ratio (DR).

4.1.1 Variables

Dependent variable

MI (Median Income): is the income level that divides a population into two equal halves: 50% of people earn below this income level, and 50% earn above it. this is the variable under observation. It is now imperative to understand how it is related to the various economic and social forces listed below.

Independent variables:

1) **DR** (dependency Ratio): The dependency ratio is a demographic indicator, that compares the number of dependents (under 15 and over 64) to the number of working-age individuals (15-64) in a population. It measures the potential support burden on the working-age population from both children and the elderly. It is the principal regressor of this study, as higher burden on the working age population negatively impacts income. A lower burden may not always be the optimal choice, as lower burdens today tend to correspond to sensibly higher burdens tomorrow.
2) **HDI** (human development Index): Human development (that is: schooling, access to services, life expectancy etcetera) is proxied through an index that ranges from 0 to 1. It is a good indicator for our unknown since an higher HDI generally indicates greater personal prosperity, positively impacting income.
3) **GPI** (gender parity index): GPI measures the equality between females and males, ranging from 0 to 1. A GPI closer to 1 indicates gender equality. This means that an higher GPI signifies more and better participation of women in the workforce, positively impacting income.
4) **GS** (government spending per country as a percentage of GDP): Government spending on healthcare, schooling, family subsidies and pensions. It measures government spending as a percentage of GDP, typically ranging from 0 to 1. Higher governmental spending usually signifies higher financial stability for low to medium income classes, positively impacting median income.

5) **HCI** (human capital index): The skills and knowledge required to meet the job market's requirements. It is an index that ranges from 0 to 1.
6) **GDP** (gross domestic product): The primary indicator of the economic output of a country/region. It is positively correlated with median income.

4.1.2 Baseline Estimation

In order to establish a baseline, a preliminary model is implemented utilizing all the available variables in a Level-Level specification.

$$MI = \beta_0 + \beta_1 DR + \beta_2 GPI + \beta_3 HDI + \beta_4 GDP + \beta_5 HCI + \beta_6 GS + \varepsilon$$

Table 1: Summary of baseline regression results

Variable	Value
DR	123.1155 ***
GPI	60.9807
HDI	320.7439 ***
GDP	1.82×10^{-10} **
HCI	45.5826
GS	18.8676
$Constant$	-327.7323 ***
$R-squared$	0.7091

*Note. *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$*

Said specification yields the following results:

- An R^2 of 70.91%

- an increase of 1$ in the national GDP figure leads to an increase of 1.82×10^{-10} in median income, ceteris paribus. Though it appears to be statistically insignificant.
- An increase of 1 percentage point in HDI leads to an increase of 320.7$ in median income, ceteris paribus .
- An increase of 1 percentage point in DR leads to an increase of 123.1$ in median income, ceteris paribus.
- An increase of 1 percentage point in GPI leads to an increase of 61.0$ in median income, ceteris paribus. Though it appears to be statistically insignificant.
- An increase of 1 percentage point in HCI leads to an increase of 45.6$ in median income, ceteris paribus. Though it appears to be statistically insignificant.
- An increase of 1 percentage point in GS leads to an increase of 18.9$ in median income, ceteris paribus. Though it appears to be statistically insignificant.

Before diving into the interpretation of the results, it is essential to perform all the necessary diagnostic checks and make any required adjustments to this statistical tool, such as addressing potential misspecification issues, handling outliers, and ensuring overall model validity.

I. Linearity

Two-way graphs, which link the independent variable to each regressor, have been put in place in order to conduct a preliminary evaluation so to understand if the associations between predictors and the dependent variable are linear.

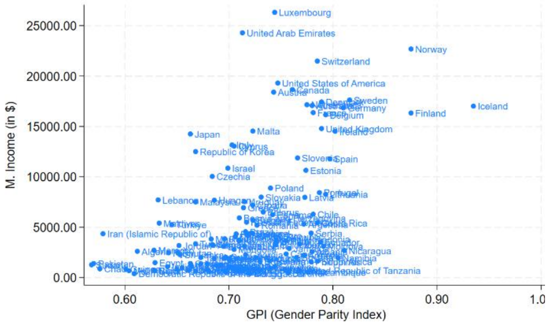

Figure 1: Scatterplot of MI against GPI

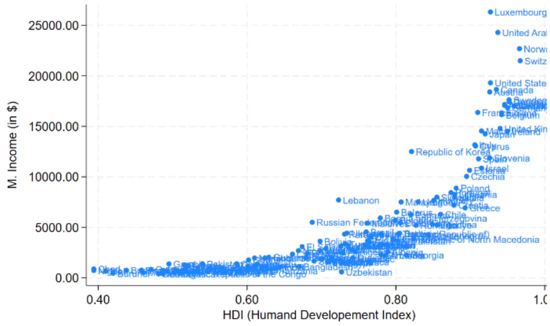

Figure 2:Scatterplot of MI against HDI

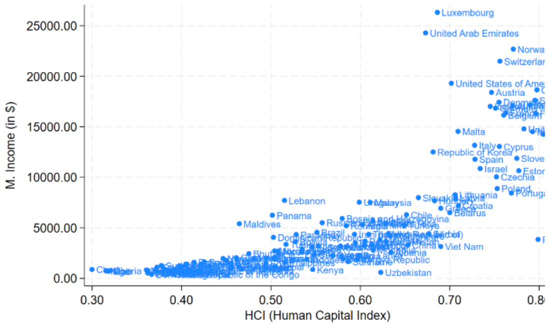

Figure 3: Scatterplot of MI against HCI

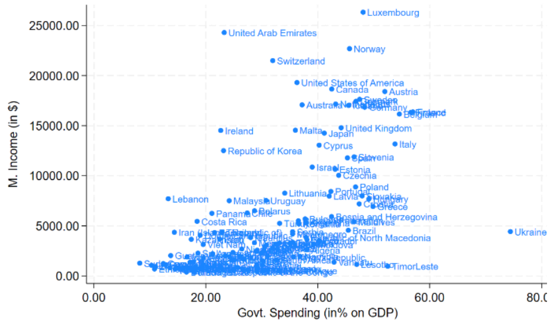

Figure 4:Scatterplot of MI against GS

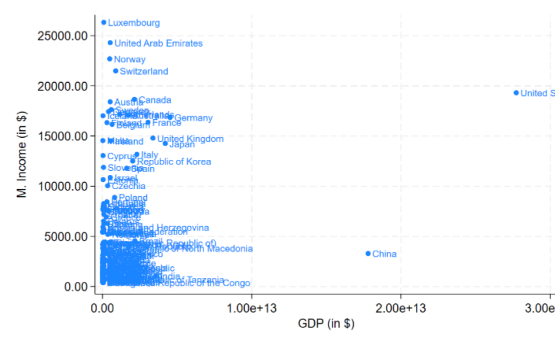

Figure 5:Scatterplot of MI against GDP

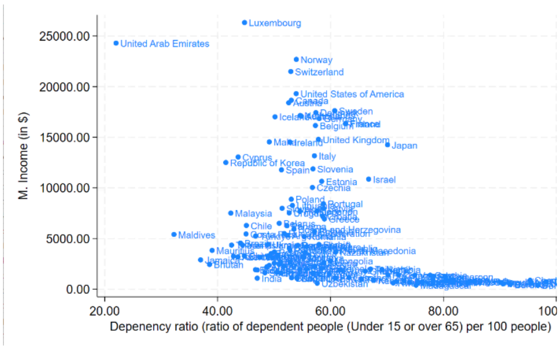

Figure 6:Scatterplot of MI against DR

While some relationships do seem to be linear indeed, other exhibit a trend that can be best approximated using a quadratic (if not exponential) relationship.

Furthermore there seems to be a need to rescale variables, a logarithmic transformation of the dependent (and maybe of some independent) variable may be considered.

II. **Inspection for outliers**

Outlier presence is established by examining the distribution of regression residuals, which serve as model-based indicators of data points poorly explained by the predictor.

A preliminary Quantile-Quantile plot demonstrates that outliers may indeed be present. It in fact seems that the tails of the distribution are heavier than expected, which further reinforces the possibility of a logarithmic transformation (which should somewhat alleviate this problem).

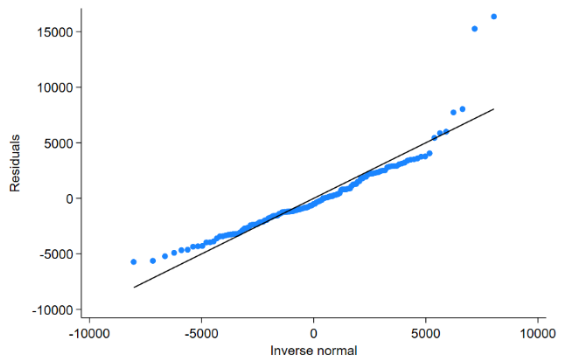

Figure 7: Quantile-Quantile distribution of residuals

Further visualization is given from a frequency table of residual distribution

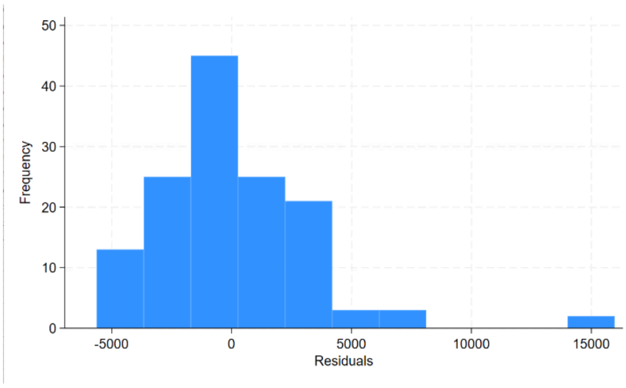

Figure 8: Distribution frequency of residuals

The graph above shows that distribution is quasi-normal, but displays skewness and (most importantly) kurtosis that renders it non-Gaussian. This phenomenon is likely a direct effect of outliers in the data, which may need to be expunged (other than of course model misspecification).

A Shapiro-Wilk test confirms the abnormality of sample residual distribution:

Table 2: Results of Shapiro-Wilk testing on baseline estimation residuals

Hypothesis	P value
H0: Residuals are normally distributed	0.0000

Note. An higher p value signifies higher probability that residuals are normally distributed

This signifies that **there is evidence to believe presence of outliers** in the data, further investigation is necessary.

III. Investigating for presence of multicollinearity

The alleged presence of multicollinearity is investigated through the use of variance inflation factor.

Table 3: Variable specific and mean variance inflation factor

Variable	VIF coefficient
DR	2.03
GPI	1.41
HDI	11.18
GDP	1.06
HCI	9.06
GS	1.75
$Mean\ VIF$	4.41

VIF analysis reveals a value of 4.41, which is no cause of concern, but reasonably close to the limit value commonly set at 5.

Consequently, another correlation matrix between predictors is used in order to asses which variables display a problematic correlation level, and the results are the following:

Table 4: Correlation matrix

Variable	DR	GPI	HDI	GDP	HCI	GS
DR	1					
GPI	-0.2638	1				
HDI	-0.6783	0.5049	1			
GDP	-0.1206	0.0417	0.2157	1		

HCI	-0.5826	0.5045	0.9387	0.2157	1	
GS	-0.3357	0.4284	0.6226	0.1198	0.6406	1

It is easy to understand that there indeed exists a multicollinearity problem between HDI and Human Capital Index, which display a correlation coefficient of 0.939.

As such, **there's reason to believe that multicollinearity is present** in the model specification.

IV. Heteroskedasticity presence analysis

A Breusch-Pagan test is conducted in order to assess the presence of heteroskedasticity. The results are the following:

Table 5: Results of Breush-Pagan on baseline estimation

Hypothesis	P value
H0: specification is homoscedastic	0.0000

Results show that **there is strong evidence for heteroskedasticity** in this model.

4.1.3 Fine Tuning

In the section above it has been established that:

- **The best model specification may not be linear**
- **Some variables may need to be rescaled**
- **There is strong evidence for multicollinearity**

- **There is strong evidence for heteroskedasticity**
- **There is evidence for presence of outliers in the dataset**

The aim of this chapter is to change and transform parts of the model in order to solve these issues. This is done through:

I. Addressing multicollinearity

The presence of multicollinearity may violate OLS assumptions, hence the need to minimize this problematic. In order to solve this, there is the need to eliminate strong correlations between regressors, which in this case are HDI and HCI.

To achieve this, the best course of action is to **drop HCI as a predictor**.

The removal of HCI has been revealed to have a minor impact in the model, as the drop in R squared is less than substantial; from 0.7091 to 0.7075.

Table 6:Summary of regression results after dropping HCI

Variable	value
DR	127.2251 ***
GPI	62.9955
HDI	359.4630 ***
GDP	1.85×10^{-10} **
GS	22.9471
$Constant$	-335.3667 ***
$R-squared$	0.7075

Note. *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$

II. Change variable specification

In this section, logarithmic transformations to both the dependent and independent variables are discussed. This is a crucial step in improving model specification, enhancing interpretability, and assisting in the satisfaction of OLS assumptions.

Based on a series of scatterplot diagnostics, namely lin-lin, log-lin, and log-log plots between median income and each explanatory variable, a clear case emerges for log rescaling several variables.

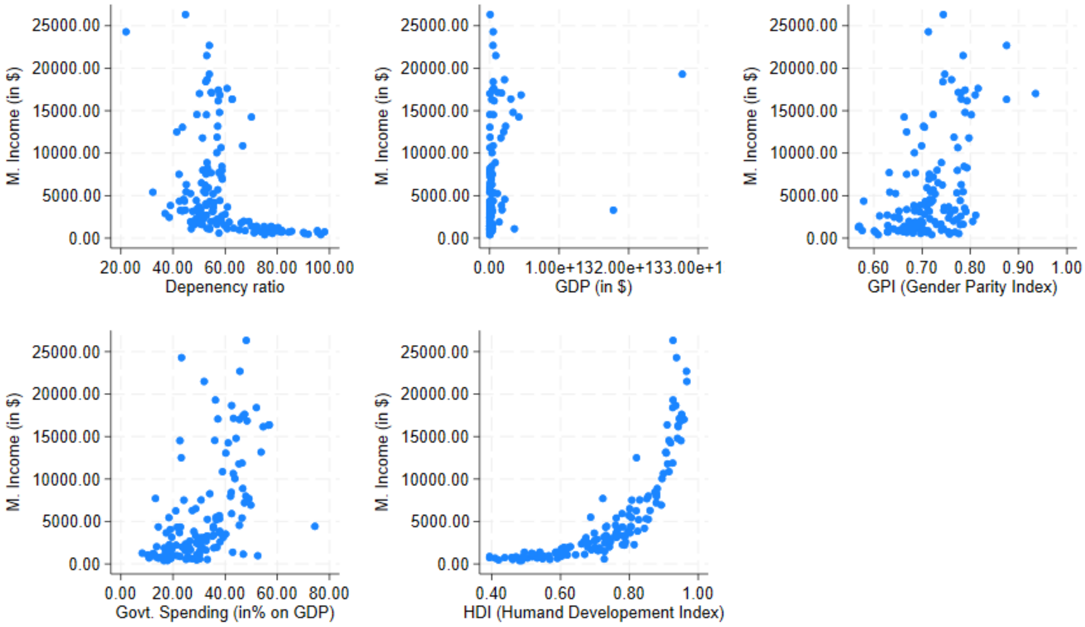

Figure 9: Lin-lin scatterplots between regressand and regressors

Starting with the dependent variable, **median income (MI)**, the visual analysis of its raw distribution reveals considerable positive skewness, with income values ranging from a few hundred dollars to well above $20,000 per capita. This variation introduces significant heteroskedasticity, inflates residuals in poorer countries, and weakens the fit of the model. Furthermore, income is typically the product of several multiplicative factors such as: productivity, capital accumulation, and demographic dynamics, further justifying the need to rescale the variable.

Once log-transformed, median income exhibits (on average) a much more stable relationship with predictors. As such, the dependent variable in this study is expressed as the natural logarithm of median income (LMI) which also allows for percentage-based interpretations of coefficients.

$$lMI = \ln(MI)$$

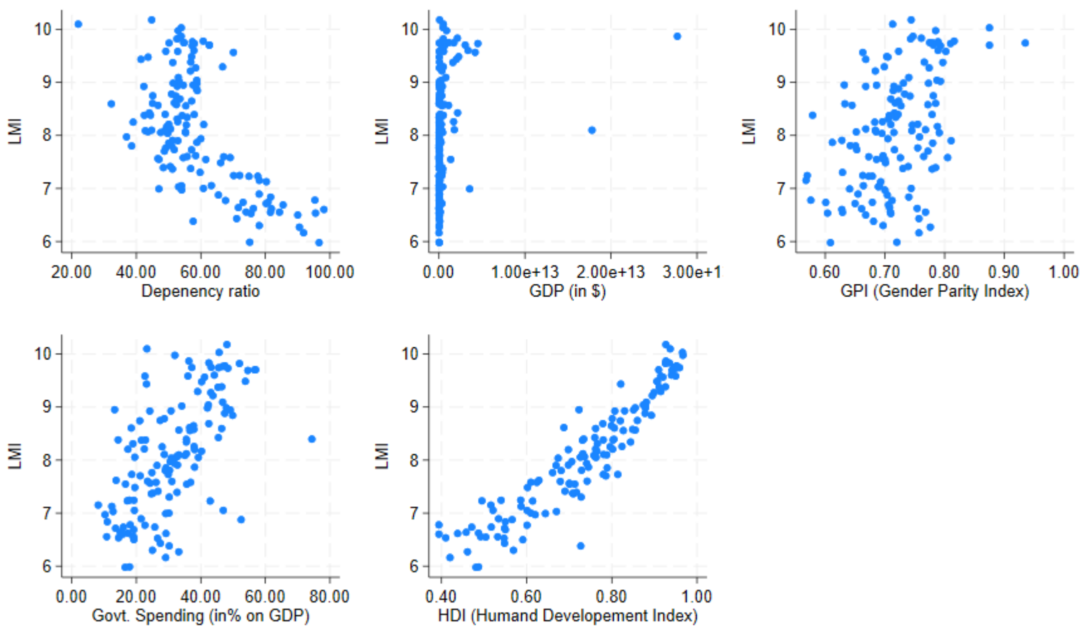

Figure 10: Log-lin scatterplots between regressand and regressor

On the side of the independent variables, several transformations are similarly performed.

GDP is notoriously right-skewed in global data, with extremely high values concentrated in a few countries. In its level form, GDP produces a vertical line in the scatterplot, hence creating high leverage among high-income nations. Once log-transformed (LGDP), the spread becomes more uniform and the functional form with LMI appears approximately linear. This justifies the use of log-GDP,

$$lGDP = \ln{(GDP)}$$

Dependency ratio is log-transformed due to the multiplicative nature of its relationship with income. As DR increases, its marginal impact on income changes nonlinearly: the burden imposed by an additional dependent is more economically disruptive in low-burden societies than in those already strained. This suggests a power-law relationship. Log transforming DR enables the model to capture relative demographic shifts, and log-log scatterplots confirm that this transformation spreads the data more evenly across the regression space.

$$lDR = \ln{(DR)}$$

Government spending is also log-transformed. Like DR, GS appears to have a **nonlinear, multiplicative effect** on income. Higher levels of public expenditure are associated with rising income, but with diminishing marginal returns. Log-transforming GS both improves the visual linearity in diagnostic plots.

$$lGS = \ln{(GS)}$$

In contrast to DR and GS, the **Human Development Index (HDI)** is **retained in its level form**, as it exhibits a clearly **exponential relationship** with income, as theory would predict. Scatterplots show that as HDI increases (particularly past the 0.7–0.8 threshold) median income rises at an accelerating rate. This pattern is best captured using a **log-lin specification**, where median income is logged but HDI remains in levels.

The **Gender Parity Index (GPI)** is **retained in level form**. Although a log transformation is technically possible, GPI exhibits low variation across countries in the sample. Applying a log transformation in this case does not improve the situation. Moreover, visual diagnostics suggest that GPI's relationship with income is weakly linear at best and not meaningfully improved by rescaling. For these reasons, GPI is modelled in levels.

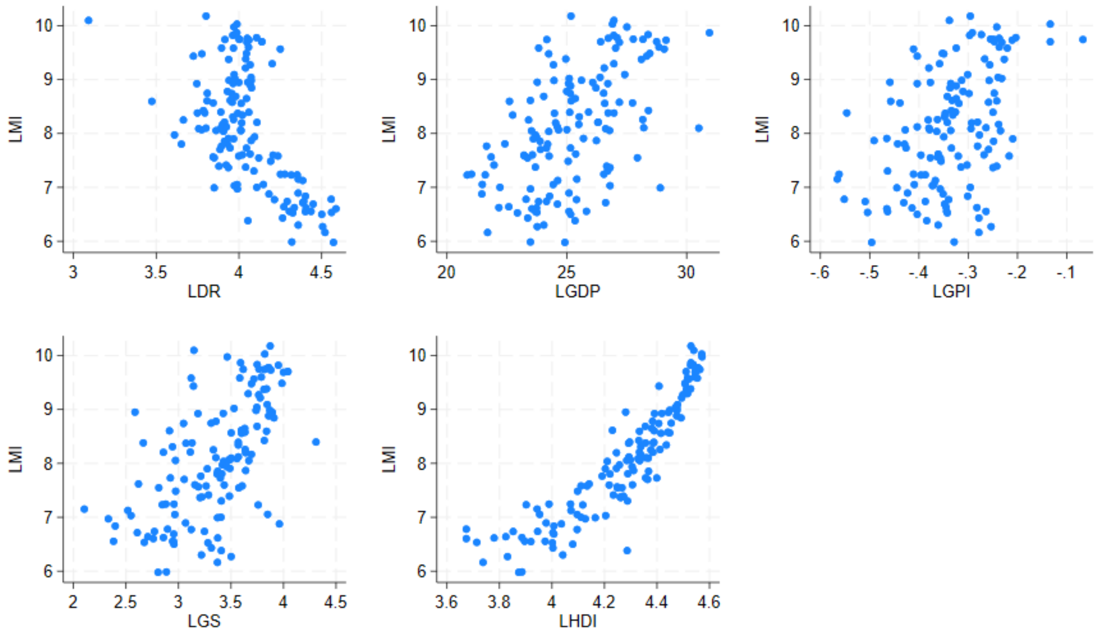

Figure 11: Log-Log scatterplots between regressand and regressors

III. Addressing heteroskedasticity

Fortunately, heteroskedasticity can be easily addressed by running a Robust standard error regression (other than having minimal impact on coefficient estimation). The Log transformations just applied are also beneficial in order to solve this problem.

IV. Modelling for (non) linearity

The current specification of the model, after applying the aforementioned changes, is the following:

$$lMI = \beta_0 + \beta_1 lDR + \beta_2 GPI + \beta_3 HDI + \beta_4 lGDP + \beta_5 lGS + \varepsilon$$

It is now time to establish if the relationship between the variables is linear, or if it can be best approximated by other families of functions. The way in which this is done is by analysing the relationship between each regressor and the regressed variable by utilizing multiple two-way graphs: a scatterplot overlaid with a fractional-polynomial fitting line, so that the best fitting curve may emerge naturally. Applying what has just been explained:

a) Dependency Ratio (lDR)

A two-way graph between lDR and lMI shows that the best family of curves to approximate the relationship is a quadratic polynomial, hence:

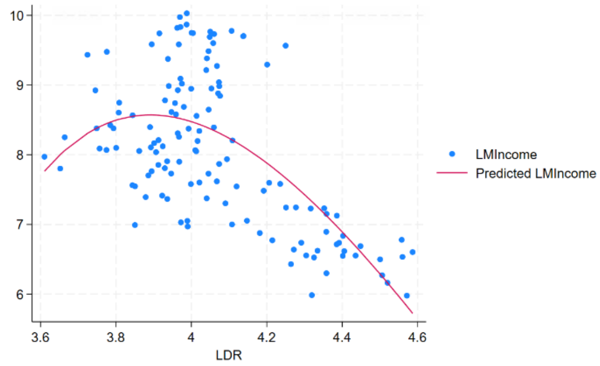

Figure 12: Best fit between LMI and LDR

$lMI = \beta_1 lDR + \beta_2 lDR^2$

This relationship confirms the theoretical hypothesis and is also in line with the literature, as such this non linearity will be accepted and included in the specification.

b) **Human Development Index ($\boldsymbol{HDI}$)**

A two-way graph between lMI and HDI shows that the best family of curves to approximate the relationship is linear:

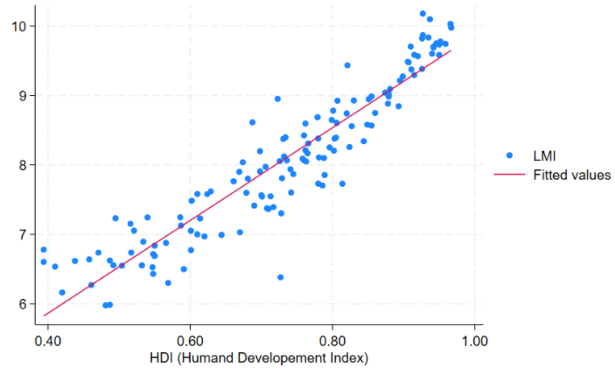

Figure 13: Best fit between LMI and HDI

$$lMI = \beta_3 HDI$$

c) **Gross Domestic Product ($\boldsymbol{lGDP}$)**

A two-way graph between lMI and $lGDP$ shows that the best family of curves to approximate the relationship is linear, hence:

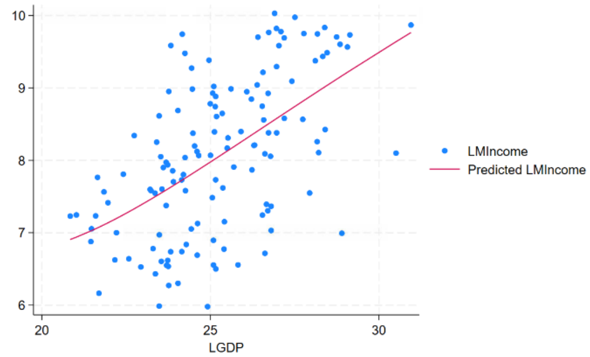

Figure 14: Best fit between LMI and LGDP

$$lMI = \beta_4 lGDP$$

In all technicality, the scatterplot suggests that there is a very weak (if none at all) relationship between the two. Nonetheless, this variable will be kept in the model for completeness.

d) Gender Parity Index (GPI)

A two-way graph between lMI and GPI shows that the best family of curves to approximate the relationship is (quasi) linear, hence:

$lMI = \beta_5 GPI$

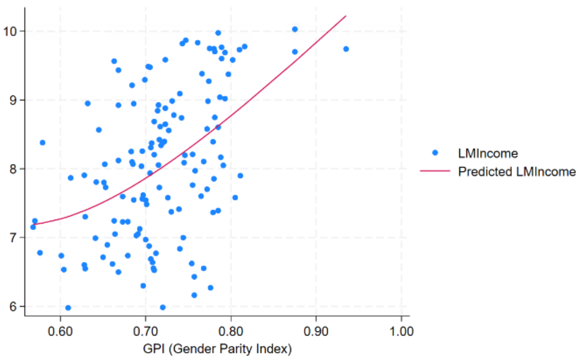

Figure 15:Best fit between LMI and GPI

e) Government Spending (lGS)

A two-way graph between lMI and lGS shows that the best family of curves to approximate the relationship is a quadratic polynomial, hence:

$lMI = \beta_6 lGS + \beta_7 lGS^2$

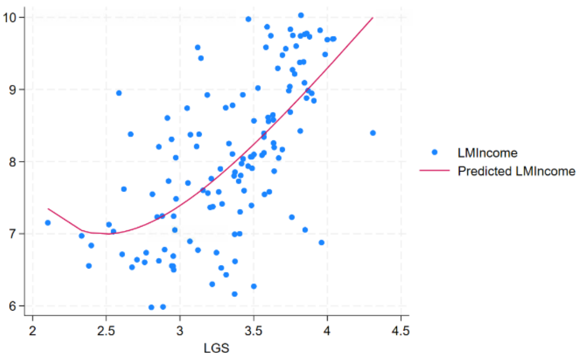

Figure 16:Best fit between LMI and LGS

There is some evidence in the literature of a U shape relationship between government investment spending and economic growth (Aznan & Goh, 2023). Hence, this relationship will be accepted and included in the model.

V. Detection of outliers

In the last chapter it has been shown that outliers may be present in the sample, for this reason this section is devoted to their detection and, if justified, expulsion from the dataset.

To determine a starting point, another QQ distribution plot is performed, this time on the residual distribution of the model that accounts for all the changes to date. This is done in order to understand how much of the deviation of the distribution of the sample error was due to model misspecification and how much is due to genuine outlier presence.

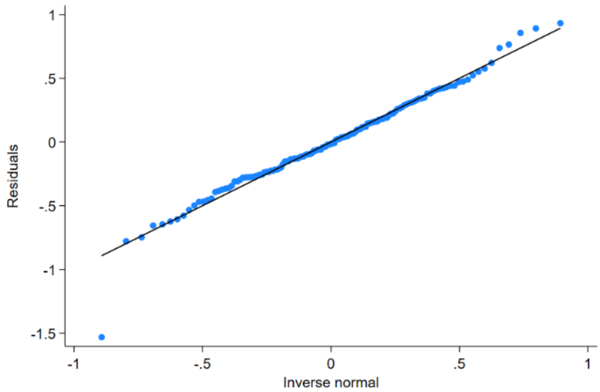

Figure 17: QQ plot after model specification change

The graph shows that the majority of the points lie along the 45-degree reference line. Minor deviations at the extremes suggest slight non-normality in the tails, but overall the plot supports the normality assumption required for regression inference.

Nonetheless there is evidence to believe that a few important outliers may be present.

For this reason, a leverage VS residual squared graph is employed in order to understand which may be the possible culprit.

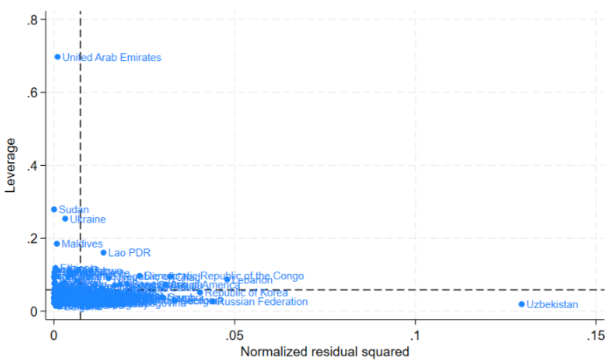

Figure 18: First Leverage vs residual squared iteration

Two clear outliers emerge from this plot:

- The **United Arab Emirates,** which has significantly more leverage than expected. The economic reason behind this phenomenon is due to its status as a global offshore economic hub, which significantly distorts standard macroeconomic indicators. The UAE exhibits abnormally high median income levels and low dependency ratios, driven largely by a concentrated influx of expatriate labour rather than organic demographic or economic dynamics. hence **it is expunged** from the dataset.
- **Uzbekistan** is extremely Ill approximated by the model, most likely because of statistical discrepancies and statistical lag. Because of this, **it is also expunged**.

A second iteration of the leverage VS residual squared is performed anew in order to control for masked outliers.

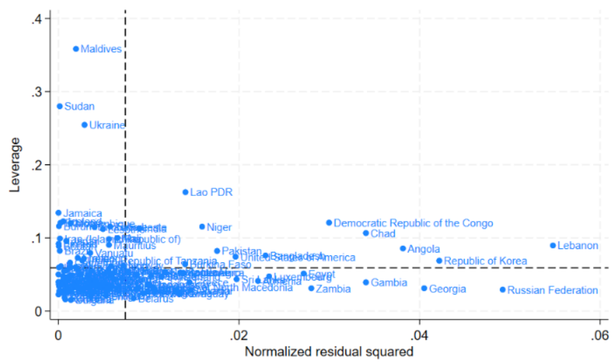

Figure 19: Second Leverage vs residual squared iteration

In the leverage-residual plot, Maldives, Sudan, and Ukraine exhibit relatively high leverage, indicating that their predictor values are atypical compared to the rest of the sample.

- **The Maldives** stands out with an exceptionally high leverage score, making it a clear outlier with potential to skew influence regression coefficients. This may have to do with the fact that the small island nation has a heavily tourism-driven economy (>60% of GDP) and unique demographic structure.
- **Sudan** and **Ukraine** show lower but still unexplainable leverage, suggesting their influence is less severe but still cause for concern. Their outlier status may have to do with the fact that both nations have experienced recent or ongoing armed conflicts, which significantly disrupt economic activity, demographic stability, institutional functioning, and provokes statistical lag.

For these reasons**, these observations are expunged** from the dataset.

A third and final Iteration of leverage vs residual squared is performed anew, so to ensure that smaller masked outliers are not present in the dataset.

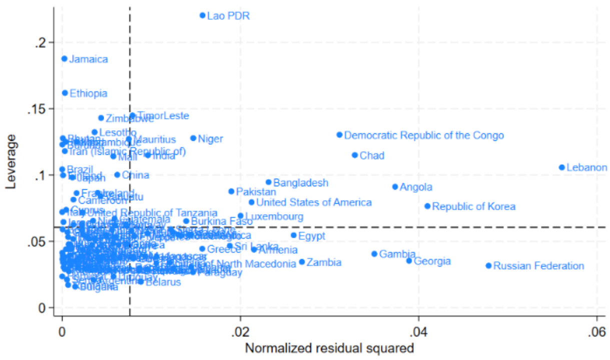

Figure 20: Third Leverage vs residual squared iteration

Although a few outliers are present, they are not cause for concern: **no single observation exhibits unreasonably high leverage or high residual error**, meaning that their influence on the estimated coefficients is limited. In other words, these points do not exert disproportionate pull on the regression, and the model remains reasonably well mannered in their presence. Furthermore, there is no socioeconomic explanation that justifies the removal of the bad mannered data points.

A second QQ distribution plot further confirms the absence of supplementary outliers.

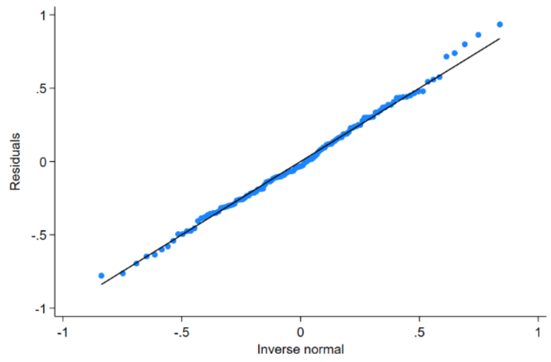

Figure 21: QQ plot after controlling for model specification and outlier detection

To conclude, while a small number of countries (e.g. Lao PDR, the DRC, Lebanon …) exhibit relatively high leverage and/or moderately large residuals, the **majority of observations are tightly pressed** in the low-leverage, low-residual region. This indicates that the regression model fits the data well overall, and that **most countries follow the general trend** captured by the specification.

4.1.4 Final Model And Optimum

In this section everything that has been built up until now will be combined into a single, refined model, which will be then used to estimate the optimal dependency ratio (thanks to which we will be able to derive steady state fertility rates).

I. Final Specification and results

For what said in the last chapters, the optimal, final cross-country regression specification is the following:

$$\boldsymbol{lMI = \beta_0 + \beta_1 lDR + \beta_2 lDR^2 + \beta_3 HDI + \beta_4 lGDP + \beta_5 GPI + \beta_6 lGS + \beta_7 lGS^2 + \varepsilon}$$

Proceeding with the sample estimation yields the following results:

Table 7:Results summary of final regression specification

Variable	value
lDR	-12.3237**
lDR^2	1.5356**
HDI	6.9066***
$lGDP$	-0.0062
GPI	0.0014
lGS	-2.9272**
lGS^2	0.4767**
$Constant$	32.3995***
$R - squared$	0.9005

*Note. *** p < 0.01, ** p < 0.05, * p < 0.1*

Commenting results, it is apparent that R squared rose significantly: from 70% in chapter 4.1, to 90% now; a difference of circa 20 percentage points. Furthermore, an R squared of 90% means that the current model is able to explain over nine tenths of the variation in the data.

Coefficientwise, it can be observed that:

- An increase of one base point (bp) of the dependency ratio signifies a decrease of 12,3% in median income, ceteris paribus

- An increase of one base point (bp) of the dependency ratio squared signifies an increase of 1,5% in median income, ceteris paribus
- An increase of one percentage point in HDI signifies an increase of 7% in median income, ceteris paribus
- An increase of 1 percent in GDP decreases Median income by 0,006% ceteris paribus, tough this variable seems to be statistically insignificant (one would argue completely uncorrelated)
- An increase of 1 percentage point in GPI decreases Median income by 0,001% ceteris paribus, tough this variable seems to be statistically insignificant
- An increase of one base point in government spending decreases median income by 2,9%, ceteris paribus
- An increase of one base point in government spending squared increases median income by 0.5%, ceteris paribus.

II. Optimality condition and optimum

For the scope of this analysis, the coefficients of interest are the ones associated with Dependency ratio, hence $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$. With the help of those values the optimal dependency ratio can be mathematically derived:

$$lMI = \beta_0 + \beta_1 lDR + \beta_2 lDR^2 + \beta_3 HDI + \beta_4 lGDP + \beta_5 GPI + \beta_6 lGS + \beta_7 lGS^2 + \varepsilon$$

Now, in order to find the maximum of this function (dependent on DR), it is necessary to satisfy the first order condition and solve for DR.

$$lMI = \beta_0 + \beta_1 lDR + \beta_2 lDR^2 + \beta_3 HDI + \beta_4 lGDP + \beta_5 GPI + \beta_6 lGS + \beta_7 lGS^2 + \varepsilon$$

$$\frac{\delta lMI}{\delta DR} = \frac{\delta}{\delta DR}(\beta_0 + \beta_1 lDR + \beta_2 lDR^2 + \beta_3 HDI + \beta_4 lGDP + \beta_5 GPI + \beta_6 lGS + \beta_7 lGS^2 + \varepsilon)$$

$$\frac{\delta lMI}{\delta DR} = 0 \rightarrow FOC$$

$$\frac{\delta}{\delta DR}(\beta_1 \ln(DR) + \beta_2 \ln(DR)^2) = 0$$

$$\frac{\beta_1 + 2\beta_2 \ln(DR)}{DR} = 0$$

$$DR^* = e^{-\frac{\beta_1}{2\beta_2}}$$

Now that the theoretical optimal value of DR has been established, it is time to compute it:

$$\beta_1 = -12.3237$$

$$\beta_2 = 1.5356$$

$$DR^* = e^{-\frac{\beta_1}{2\beta_2}} = 55.242$$

At last, the analysis shows that dependency ratio of **55.2%** is the optimal, **median income maximising, value**.

4.2 Part II: From Dependency To Births

Now that the optimal dependency ratio has been estimated, it is necessary to reverse engineer the birth rate. To do so, a system of Ordinary Differential Equations (ODEs) has been put in place, this allows for the simulation of population flows depending on different parameters. One of these "parameters" is the birth rate, which is consequently isolated and estimated, both on a static (birth rate that maintains optimum dependency) and dynamic (progressive change of birth rate to optimum, accounting for other time sensitive factors) basis.

4.2.1 Modelling Population Dependency Variables

The dependency ratio at time t is defined as such:

$$DR_t = \frac{Y_t + O_t}{A_t}$$

Where:

- Y_t is the number of people aged between $[0,15)$, referred to as juniors
- A_t is the number of people aged between $[15,65)$, referred to as actives
- O_t is the number of people aged between $[65, \infty)$, referred to as seniors
- b is the number of children per actives per year (easily convertible from the fertility rate, or TFR, since $b = \left(\frac{TFR}{50}\right)\frac{1}{2} = \frac{TFR}{100}$,assuming half of the active population is female)
- γ_Y, γ_A represent the mobility coefficients for juniors and actives, respectively
- μ_Y, μ_A, μ_O are the mortality coefficients for each cohort

These variables are modelled as follows:

1) $\dot{Y} = bA - (\gamma_Y + \mu_Y)Y$

The change, (or mathematically, the derivative) of Y is equal to: $b(t)A$, so the numbers of births per working age person in the span of 50 years (so the number of births per adult year (b), times the actives (A)) minus $\gamma_Y Y$: the coefficient of people leaving the age group due to growth (γ_Y) or premature death (μ_Y) times Y, the number of juniors.

2) $\dot{A} = -(\gamma_A + \mu_A)A + \gamma_Y Y$

The derivative of A is equal to: $\gamma_Y Y$ (γ_Y, the coefficient of people outgrowing the age group Y times Y) minus $(\gamma_A + \mu_A)A$ (γ_A, the coefficient of people leaving the age group A plus μ_A the premature death coefficient for cohort A, multiplied by A itself).

3) $\dot{O} = \gamma_A A - \mu_O O$

The derivative of O is equal to: $\gamma_A A$ (γ_A,the coefficient of people leaving the age group A times A) minus $\mu_O O$, (μ_O, the mortality rate for seniors times O, the seniors per se).

While the parameters μ need to be estimated (and U.N. mortality tables will do just fine), γ_Y and γ_A can be assumed to be the reciprocal of the width of the age cohort in years, since it is assumed that every individual spends, on average, exactly 15 years in the "child" class before turning 15, and exactly 50 years in the "working-age" class before turning 65. Hence:

- $\gamma_Y = \frac{1}{15}$
- $\gamma_A = \frac{1}{50}$

4.2.2 Backwards Induction

It will be assumed that the optimal dependency ratio is considered to be in the steady state, hence, once reached, it will not diverge. The same can be said for the optimal Y, O and A.

$$DR^* = \frac{Y^* + O^*}{A^*}$$

It follows that the derivatives of the age cohorts will equal 0, since the changes between the "entrances" and the "departures" of each cohort will be perfectly offset:

$$\begin{cases} bA^* - (\gamma_Y + \mu_Y)Y^* = 0 \\ \gamma_Y Y^* - (\gamma_A + \mu_A)A^* = 0 \\ \gamma_A A^* - \mu_o O^* = 0 \end{cases}$$

It follows that $Y^* = \frac{bA}{\gamma_Y + \mu_Y}$, while $O^* = \frac{\gamma_A A^*}{\mu_O}$.

Substituting back into the DR function and solving for b, the **equation that links births per active person (per year) is**:

$$\boldsymbol{b^* = (\gamma_Y + \mu_Y)(DR^* - \frac{\gamma_A}{\mu_O})}$$

Through the help of this equation, **the optimal number of children per person per adult year can be directly derived** from the dependency ratio.

4.2.3 Transition State

In order to demonstrate the full effects of changes in birth rates, it is necessary to model how the system transitions from the initial to the steady state

We start from the assumption that the derivative of the birth rate is equal to the difference between the optimal birth rate and the birth rate at time t, multiplied by a factor k, which signifies the speed of the adjustment of the birth rate through time. This embodies the essence of a continuous logistic equation, which will be added to the model in order to simulate the transition path.

$$b' = k\big(b^* - b(t)\big)$$

Putting it all together, the system of differential equations gets augmented in this manner:

$$\begin{cases} bA - (\gamma_Y + \mu_Y)Y = \dot{Y} \\ \gamma_Y Y - (\gamma_Y + \mu_Y)A = \dot{A} \\ \gamma_A A - \mu O = \dot{O} \\ k\big(b^* - b(t)\big) = \dot{b} \end{cases}$$

It is important to note that, after setting the initial condition to today's (as of 2023) birthrate (b_0), and solving the differential equation, the solution of the aforementioned equation becomes:

$$\boldsymbol{b(t) = b^* + (b_0 - b^*)e^{-kt}}$$

Furthermore, since this analysis will take place during a long time span, the assumption of constant mortality seems to be a tad too strong. For this reason, mortality rates will also be modelled to be decreasing functions through time, starting from an initial condition and progressively lowering to an assumed long term mortality rate. The equations through which they will be modelled are much like the equation that governs **$\boldsymbol{b(t)}$, hence:**

$$\mu_Y(t) = \mu_Y{}^* + (\mu_{Y0} - \mu_Y{}^*)e^{-\rho t}$$

$$\mu_A(t) = \mu_A{}^* + (\mu_{A0} - \mu_A{}^*)e^{-\rho t}$$

$$\mu_O(t) = \mu_O{}^* + (\mu_{O0} - \mu_O{}^*)e^{-\rho t}$$

This implies that **optimal fertility will not be constant throughout time**, but rather adapting to changes in mortality rates.

Putting it all back together, the final model yields:

$$
\begin{cases}
\boldsymbol{b(t)A - (\gamma_Y + \mu_Y(t))Y = \dot{Y}} \\
\boldsymbol{\gamma_Y Y - (\gamma_Y + \mu_Y(t))A = \dot{A}} \\
\boldsymbol{\gamma_A A - \mu_O(t)O = \dot{O}} \\
\boldsymbol{b(t) = b(t)^* + (b_0 - b(t)^*)e^{-kt}} \\
\boldsymbol{b(t)^* = \left(\gamma_Y + \mu_Y(t)\right)\left(DR^* - \frac{\gamma_A}{\mu_O(t)}\right)} \\
\boldsymbol{\mu_Y(t) = \mu_Y{}^* + (\mu_{Y0} - \mu_Y{}^*)e^{-\rho t}} \\
\boldsymbol{\mu_A(t) = \mu_A{}^* + (\mu_{A0} - \mu_A{}^*)e^{-\rho t}} \\
\boldsymbol{\mu_O(t) = \mu_O{}^* + (\mu_{O0} - \mu_O{}^*)e^{-\rho t}} \\
\boldsymbol{DR = \frac{Y + O}{A}}
\end{cases}
$$

This system of differential equations will govern **how birth rates and dependency ratios will act through time,** depending on different **initial and steady state conditions.**

In the next chapter this model will be employed and its results will be discussed, so to ascertain which are the effects of continued fertility above/below optimal rate.

4.2.4 Derivation Of Imbalance Costs

Quantifying the implicit cost that fertility and dependency imbalances generate on an individual level would be very beneficial in order to identify the economic impact of the phenomenon at hand.

Thankfully, there is a mathematical strategy that allows to do just that. It employs a combination of the regression variables utilized in PART I and the transition model constructed in PART II.

In fact, the regression built in the previous chapter is specified in the following manner:

$$\ln(MI) = \beta_1 \ln(DR) + \beta_2 \ln(DR)^2 + \cdots + \varepsilon$$

So that,

$$\frac{d\ln(MI)}{d\,DR} = \frac{d}{d\,DR}\left((\beta_1 \ln(DR) + \beta_2 \ln(DR)^2)\right) = \frac{\beta_1 + 2\beta_2 \ln(DR)}{DR}$$

The change of MI with respect to the change in DR is now known.

It is now necessary to extend the computation of MI from a general change of DR to an interval change of DR with respect to time.

To do this, the midpoint rule is used to calculate D of t, which is the average change of MI In between DR at t and DR at t+1:

$$D_t = \frac{\beta_1 + 2\beta_2 \ln\left(\frac{DR_{t+1} + DR_t}{2}\right)}{\left(\frac{DR_{t+1} + DR_t}{2}\right)}$$

This difference is then multiplied by the difference between DR at time t and DR at time t+1:

$$\Delta \ln(MI_t) = D_t(DR_{t+1} - DR_t)$$

Extending this line of reasoning for all t and calculating the average:

$$\frac{\sum_{i=1}^{T} \Delta \ln(MI_i)}{T} = \Delta \ln(MI)_{avg}$$

$$\boldsymbol{\Delta \ln(MI)_{avg} = \frac{1}{T}\sum_{i=1}^{T} D_i(DR_{i+1} - DR_i)}$$

This Riemann sum approach is the most straightforward way to account for yearly data (especially when calculating the change in MI from U.N. projections, which will be conducted later).

This final equation will allow for the **quantification of the implied cost of fertility** (and inevitably) dependency **imbalances**, enhancing findings (in the next section).

5. Findings And Recommendations

In this chapter, the assembled model **will be implemented** into real word scenarios in order to **quantify the economic effects of birth rate evolutions and deviations**. Findings will be organized in two sections:

In the first section, the economic importance of optimal birth rates and the **indirect cost of sub-optimal fertility will be assessed**.

The second section, instead, will focus on the so called “demographic dividend” and shed some light on the obtained the dependency curve.

In the third section, at last, results will be summarized and recommendations will be given.

5.1 The Implicit Cost Of Sub Optimal Fertility

It is now time to put the model to the test. The objective being to understand how **economically quantify deviations from optimal births** (which will result in deviations in dependency ratios) impact **changes in median income**. Since the optimal dependency ratios have been obtain utilizing a global perspective and the model lacks controls for migration, it only makes sense to employ this model in a global scenario.

5.1.1 Model Calibration

Before applying the model, though, **different parameters should be estimated**. Thankfully, population, mortality, and birth rates data and projections are compiled by the U.N. population division (U.N. population division, 2023) , and as such can be obtained from there. Estimation for parameter rho and k will be a little bit more difficult, but they can easily be reverse engineered by looking at the reduction speed of mortality and fertility in different years.

The estimated final parameters are the following:

Table 8:Estimated values of parameters

Parameter	value
μ_{Y0}	0.0049
μ_{A0}	0.0027
μ_{O0}	0.1280
${\mu_Y}^*$	0.0018
${\mu_A}^*$	0.0014
${\mu_O}^*$	0.1054
k	0.0402
ρ	0.0196
$b(0)$	0.0240
$b(100)$	0.0159
Y_0	2
A_0	5.2
O_0	0.8

Note. All the values with subscript 0 are to be intended at year 2023.

5.1.2 Implementation And Results

It is finally time to **apply the model** to a real world scenario. It will be **benchmarked against** (medium scenario) **U.N. predictions** (U.N. population division, 2023), both in terms of dependency and birth rates. The **initial conditions** will be **identical for both scenarios**. After that, fertility rates in the model will be allowed to **deviate from projections**, so to **adjust for optimal fertility** and consequently **the best dependency ratio** profile obtainable.

The results are the following:

Table 9: Values of optimal/projected births and respective dependency ratios

variable	2023	2030	2050	2070	2100
$b^*(t)$	0.0240	0.0279	0.0270	0.0263	0.0257
$b(t)$	0.0240	0.0213	0.0179	0.0168	0.0161
$DR(b(t)^*)$	0.5308	0.5193	0.5233	0.5342	0.5401
$DR(b(t))$	0.5308	0.5288	0.5795	0.6198	0.6759

Note. $DR(\ x(t))$ is to be intended as DR of birth function x at time t, $b^*(t)$ is to be intended as the optimal birth function, while $b(t)$ as the projected birth function through time.

The table above shows how the difference between projected and optimal fertility will impact the dependency ratio in time. It can be clearly seen that, while $\boldsymbol{b(t)}$ **decreases sharply in time, b(t)* remains relatively stable, hovering between 2.4 and 2.8 children per woman**. In fact, the behaviour of b(t)* can be explained by the model adapting to the decreasing mortality over time, at first increasing from its base level (to compensate for relatively high mortality), then decreasing again due to the long run reduction in mortality.

In fact, it seems that the relative difference between the curves will stabilize to around a child per woman. This evolution can further be visualized in the graph below.

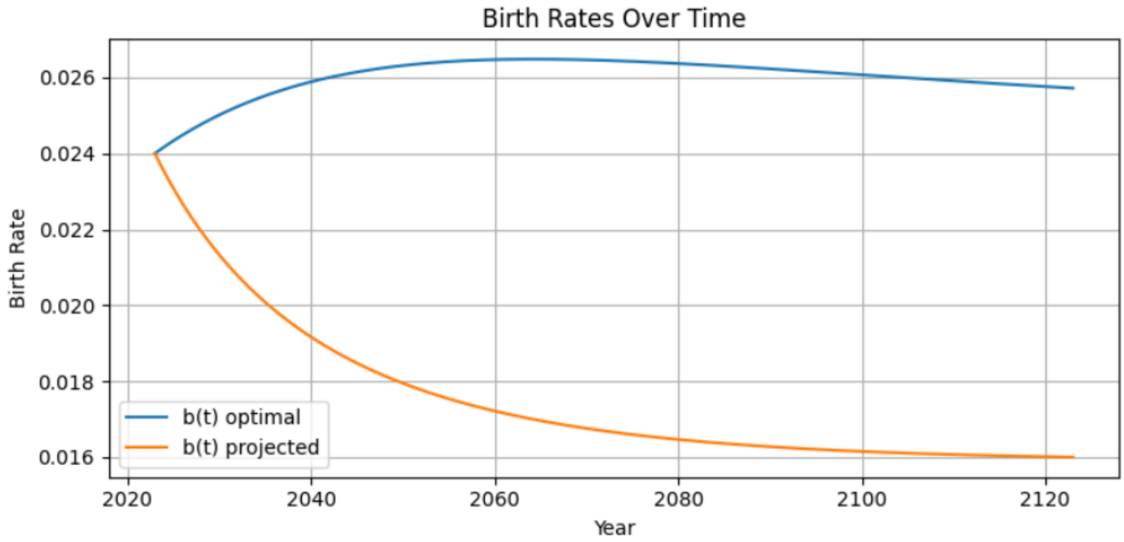

Figure 22: Evolution of optimal and projected births (b) over time

The **evolution of birth rates** will inevitably **impact the dependency ratio**. In fact, as the table shows, **dependencies will increase in both scenarios** (this behaviour will be addressed later). **The crucial difference is the velocity** at which they'll rise.

In fact, **progressive deviations from optimal births** will lead to projected dependency of the benchmark (especially old age dependency) to increase much faster than the counterpart, as aging adults are not replaced by enough youngsters.

Summarizing, while (almost) monotonic **increases are true for both** functions, this effect is more than **contained** in the mathematical design (thanks to optimal fertility adjustments), where total dependency, after a period of fall and subsequent rise, will effectively stay stable. UN projections, instead, clearly show how **suboptimal fertility has a deeply negative effect on dependency**, increasing its value by a staggering 14 percentage points from base level. This can be easily seen in the chart below:

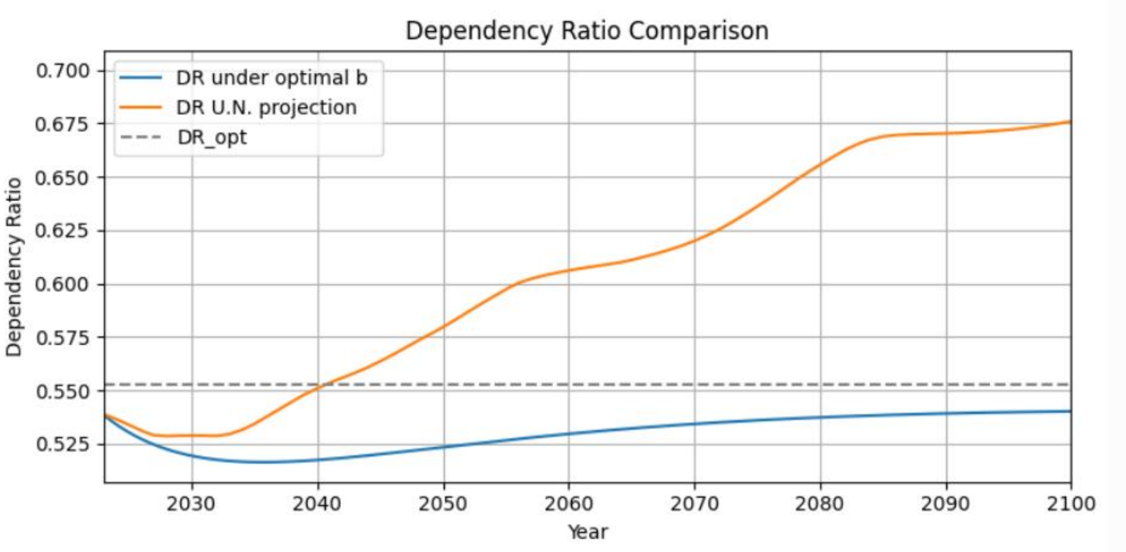

Figure 23: evolution of dependency ratios under projected and optimal births scenarios

The effects in terms of median income are sensible. Applying the formula described in 4.2.4, the average yearly growth (or rather decline) in median income $\Delta \ln(MI)$ for our model projection (MI_{b^*}) against the U.N. estimation(MI_b) is:

$$\Delta \ln(\boldsymbol{MI_{b^*}})_{\boldsymbol{avg}} = -0.000118$$

$$\Delta \ln(\boldsymbol{MI_{b}})_{\boldsymbol{avg}} = -0.007237$$

In a time span of 77 years (from 2023 to 2100) we can calculate the yearly change, which turns out to be a **0.012 percent decrease in median income per annum in the optimal fertility** scenario and a **0.72 percent decrease in median income per annum in the U.N. projection scenario** (all else equal). This means that in this case, **deviations** from optimal fertility will bring about an **additional .712 percent decrease in Median income** per annum (ceteris paribus). It is important to note that this is an average value. Restricting the calculation on different time spans reveals even more worrying results.

Table 10:Impact on median income from different scenarios and estimation periods

Estimation period	U.N. projection	Optimal fertility
2023-2100	-.72%	-0.012%
2040-2100	-.80%	-0.015%
2060-2100	-.62%	-0.009%
2080-2100	-.41%	-0.004%

Multiple considerations can be done. The first one is that, while decreasing effects will persist for the whole century, the bulk of the negative effect will be felt between 2030 and 2080. Secondly, optimal fertility systemically outperforms the model benchmark, yielding on average a difference of 63 base points.

In conclusion, it has been shown that **deviations from the optimal births scenario** (the fertility gap), will indeed have **long lasting negative effects on median income**, one might even call it a fertility tax.

In this case, the fertility gap turned out to be about a child per woman. The consequential effects of the fertility gap on median income (from 2023 to 2100) revealed themselves to be a **.712 percent decrease in median income per annum** (all else equal), relative to the difference between scenarios. This is an alarming figure, considering that real salary growth in the last 18 years has hovered around 2 percent (Statista, 2025), meaning that, all else equal, increase in income per capita will be cut almost by half. Another way to visualize the situation is that, under the current prediction scenario, the yearly loss (assuming continuous compounding) could **eat up more than 40% of the initial median income in the span of 77 years** (all else equal)**.**

5.2 Assessing Curve Behaviour

In the last section, it has been shown which are the economic implications of optimal and sub-optimal fertility, in terms of median income.

The model also showed that, no matter the fertility profile, both dependency curves increased in the long run. This, however, seems to be no model's flaw, but rather an unavoidable outcome under these circumstances.

In this chapter, the reasons behind this changes will be decomposed and assessed.

5.2.1 Accounting For The "Demographic Dividend"

Counter intuitively, model output seems to be in line with the literature (Bloom, Canning, & Sevilla, The Demographic Dividend: A New Perspective on the Economic Consequences of Population Change (1st ed.)., 2003), (Mason, Lee, Abrigo, & Lee, 2017), (Mason, Demographic Transition and Demographic Dividends in Developed and Developing Countries, 2005) which has hinted that the level of **dependency ratio of today** in many industrialized countries (and at which the industrializing countries are projected to converge) **is rather temporary** and no more than an effect of what is called the "**first demographic dividend**" by scholars.

This phenomenon is explained by the fact that, as births decrease, the junior cohort enters working age without fully replacing itself, yielding lower dependency ratios in the short to medium term. Hence median income improvements. The active cohort of today, though, will be the senior cohort of tomorrow, which will have less actives of tomorrow (the juniors of today) to support it. Because of this, dependency ratio will inevitably climb back up again.

The bottom line is that **current worldwide dependency does not constitute a steady state**, but rather a temporary achievement, which cannot be sustained in the very long term.

Armed with this new found knowledge, it becomes easier to explain why the optimal dependency curve rises in the long term, and most importantly why the behaviour of the function is like so.

The reasons behind the general rise in dependency are:

Firstly, the model is **calibrated around the static optimal dependency** ratio (0,55242), which is **greater than worldwide dependency at 2023** (0,5308). This inevitably results in an increase in the optimal dependency curve, above 2023 levels.

Secondly, since the focus here is trying to stabilize dependency around an optimum, the model cannot immediately converge to optimal dependency (no matter the birth rate), but rather will show the U shaped curve because of the transitory initial conditions discussed above (many adults, and few dependents overall).

Furthermore, the initial conditions of the model are heavily concentrated in the active cohort (2 billion juniors, 5.2 billion actives, 0.8 billion seniors). **In order to reach a steady state, an increase in dependency is inevitable**, confirming the effects of the demographic dividend.

5.3 Recommendations

In the last chapter, the economic cost (in terms of median income) has been assessed. Results showed that, although a **rise in dependency ratio is unavoidable, adjustments for optimality conditions will sensibly minimize the "fertility tax"**.

It is important to note that the situation under scrutiny was a global one, which can be seen as an average between many local situations. Hence, while local restrictions of the model have not been studied, there is reason to believe that the need to **address sub-optimal fertility are most urgent for developed countries**, for whom the fertility tax may be tenfold higher and, above all, the tax collector could already be knocking at their door. Most middle income countries, instead, still have decades of low dependency to enjoy.

On the other hand lower income countries may need to reduce fertility in order to experience a significant decrease in dependency, but, given the trends in natality, eventually will (U.N. population division, 2023).

Thus, the main recommendations of this dissertation is to **raise spending towards pro natality politics**, especially in developed countries.

The primary goal of these policies should be to limit the effect of the natality tax. Policymakers should in fact take into consideration that **fertility increases could save** up to/ at least (depending on the situation) **0.712% in annual median income growth**, which in the span of the 77 year period in which this analysis takes place, continuously compounds to a **loss of more than 40% of initial endowment**, all else equal.

6. Limitations

Everything has its limits, and in this chapter the ones which pertain to this dissertation will be put on the spotlight. As per usual, it will be divided into two sections: In the former, limitations belonging to regression analysis will be conveyed, while in the latter the focus will shift on the limitations of the ODE system.

6.1 Regression Analysis Limitations

Several limitations are present in the least squares estimation.

First, **there are only 132 observations per variable**. This limits the model's ability to detect subtle but meaningful effects may be limited. Consequently, the estimated median income and dependency ratios, while mostly accurate, may not truly reflect the reality of the situation.

Secondly, despite inclusion of several socioeconomic controls, the model may still suffer from **omitted variable bias**. Unobserved factors, such as regional dynamics, unmeasured aspects of income, submerged economy and/or cultural norms, could most definitely be correlated both with dependency ratios and income levels, hence **distorting the estimations**.

Finally, the chosen specification **cannot guarantee** that **all relevant nonlinearities are captured**.

6.2 Differential Model Limitations

This section critically analyses the assumptions and simplifications underlying the age-structured cohort compartment model. There are several assumptions and omissions that need to be addressed.

First and foremost, the **assumption of constant age distribution within each cohort**. The model represents each of the three age-cohorts (juniors, actives, seniors) with a single entry/exit rate. This implicitly **assumes a uniform distribution of ages within each cohort**. Hence, a Distorted cohort dynamic is inevitable. Cohort “concentrations” (for example the baby-boom generation in the western world) are not accounted for, distorting results. While this distortion presents challenges when applying the model to a country-specific level, it must be noted that as of right now the world population has a relatively uniform intra-cohort distribution (Population-pyramid.net, 2025), so the simulation does not overly distort reality.

Secondly, **static optimum fertility is used as a target of a dynamic system.** The birth-rate is obtained by inverting the static steady-state dependency formula. This line of thinking presumes the age structure is already at equilibrium. As a result, the model will only converge asymptotically (but still retaining a satisfactory level of accuracy).

In conclusion, The flat-cohort and static-optimum assumptions, paired with statistical errors make this mathematical representation of reality able to **provide an overview of the phenomenon, but in an imprecise manner**.

Future work should employ stronger statistical estimations, explore better age-structure modelling and employ fully dynamic formulations in order to capture entirely the effects of demographic and economic changes.

7. Conclusion

This dissertation demonstrated that the **relationship between the dependency ratio and societal well-being** (here proxied by median income) **takes an inverted-U form**, with an optimal dependency ratio of approximately 55.2 percent.

By combining a cross-country regression analysis with a system of ordinary differential equations, it has been shown how this **optimum dependency level can be translated into a steady-state annual birth rate**, and **how the projected global deviations in births** from said rate will **incur constant median income** losses over time, estimated to be around **0.7% per annum**, **or a 40% loss of initial endowment** in the span of 77 years.

The ODE dynamic model proves to be a useful tool, as **it provides a way to simulate the evolution of demographic structure in time and its economic consequences,** especially if compared with alternative (in this case projected) scenarios. These results underline that **very low fertility rates are most definitely undesirable**. Instead, a balanced fertility profile, which in the long run transforms into a balanced structural demographic profile, will yield the greatest improvements in median income.

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