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Some Paradoxes in Voting Theory

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CHAPTER 1 GENERAL VOTING SYSTEMS

1.1 INTRODUCTION

Every moment during every day's life is a matter of choice. Every action we make, is a consequence of a choice among a series of alternatives at disposal at the moment. When our choices do not affect the life of people around us it is a mere personal issue. But, sooner or later, during our lives, we are asked to express our preferences together with other people, on matters that can have a great relevance, or little as well, for the whole society. Even if we do not pay attention, we are required to deal with voting rules even during an evening spent with our friends, for example when we have to decide among some options, as watching a film or going to a restaurant, or going out for a walk; when everybody expresses his opinion, then, the following step, is aggregating the individual preferences and taking a definitive decision for the evening: this is a trivial example given with the extent to make the reader understand that we always deal with the matter of aggregating individual preferences. Clearly, in this paper, this study will be approached considering more serious circumstances, as elections of our political representatives, for example, or referenda where we are asked to give our opinion. The major concern is to let the reader understand how important is the way in which we aggregate individual preferences; Donald Saari, one of the most important contemporary theorists of voting systems, asserted: “For a price, I will come to your organization just prior to your next important election. You tell me who you want to win. I will talk with the voters to determine their preferences over the candidates. Then, I will design a democratic voting method which involves all candidates. In the election the specified candidate will win”. The offer was clearly a joke, Saari’s proposal was undoubtedly exaggerated, but this assertion can give the reader a measure to understand how unfair our “fair” elections can be; according to the voting procedure adopted, in fact, different outcomes for the very same election can result, and this is all but an exaggeration; it’s fundamental to understand how important is the act of choosing a voting rule that fairly reflects the will of the voters, leading to a truly democratic outcome. During past centuries, several mathematicians, political scientists and theorists, interrogated on how to design a voting system that could satisfy previous basic request. This paper will provide an overlook on two particular voting systems developed in the time, one by Marie Jean Antoine Nicolas De Caritat Condorcet, known as the Marquis De Condorcet and the other one developed by Jean Charles De Borda; Borda and Condorcet were two French contemporaries that lived during the end of the eighteenth century and the beginning of the nineteenth. Some paradoxes related to their voting systems will be analyzed, in particular this paper will give an explanation of the “Borda Paradox” and of the “Condorcet Cycle”, two very well-known paradoxes that can lead to not-fair outcome. Other versions of Condorcet paradox, partly or completely related to his system will also be analyzed, together with some actual occurrences of these paradoxes. After this, attention will be focused on “Multiple elections paradoxes”, in particular analyzing the work of Brams, Kilgour, Zwicker; this analysis will introduce the work of Scarsini, about a “Strong paradox of multiple elections.” An actual occurrence of the main paradox will be presented. It will be done in order to let the reader understand how some circumstances can lead to an undesired outcome, that translates into the victory of a non-desired candidate. I will explain the conditions that prohibit the existence of the paradoxes, and the various interpretation that theorists gave about this matter.

Theorists agree on the fact that in the case of an election with only two candidates the resolution is a very simple task: majority rule, in fact, always leads to a democratic outcome that fairly reflects the will of the voters. In such a case, the candidate that receives the greatest number of votes is elected. Rae(1969) and Taylor(1969) both justify on a mathematical basis the notion of using the majority rule in a two-candidates election.

Several problems can arise in an election, if not considering every hypothetic scenario. One of the most frequent situations occurred is the one where the “spoiler effect” turns into play: it happens when a minor candidate with very few chances of winning draws votes from another candidate with real chances of winning that is, in some aspects, equal to the former who is referred to as a clone of the major candidate.

1.2 1970 U.S. SENATE ELECTIONS IN NEW YORK STATE

This is exactly what happened during the 1970 U.S. Senate election in New York State: the three candidates for the election were Ottinger, Goodell and Buckley. The last one was supported by the political conservatives; Ottinger was endorsed by the Democrat Party while Goodell enjoyed the support both of the Liberal Party and the Republican Party. The latter was in this case defined as a clone of Ottinger: as Poundstone underlines, “both were antiwar activists. Even their biographies were similar. Both had served in the airforce. Ottinger was Harward Law School class 1953, Goodell was Yale Law School class 1951. Both were multi term New York congressmen.”

As stated before, the fact of having a couple of clones as candidates of the same election can lead to an undesired outcome; in this situation, for example, the situation after the vote was as follow: Buckley (38.8%), Goodell(24.3%) and Ottinger(36.9%). Nevertheless, according to Gerhlein, either Goodell or Ottinger would have surely beaten Buckley in a head-to-head vote. This case will be taken into considerations and better analyzed later in the paper.

1.3 PAIRWISE MAJORITY RULE

Now, in order to fully understand every following statement, some explanations are given: suppose we have a set of three candidates {A,B,C}; suppose that A>B denotes the fact that a voter prefers Candidate A to Candidate B. A voter is said to have "complete preferences" if he has a preference on every pair of candidate, so he's not indifferent between any two candidates. Moreover we also assume that every voter has "transitive preferences", i.e., if the voter prefers A to B and prefers B to C, then he can't prefer C to A: otherwise the voter is not acting rationally. Individual preferences that are complete and transitive are defined as "linear preference rankings"; there are six possible linear preference rankings in an election with three candidates {A,B,C}:

Linear Preference Rankings:

	A	A	B	C	B	C
	B	C	A	A	C	B
	C	B	C	B	A	A
Number of voters	n_1	n_2	n_3	n_4	n_5	n_6

Tab.1

Here n_i denotes the number of voters that have the associated linear preference ranking on the various candidates.

According to Condorcet's computation there are 8 different linear preference rankings a voter can have on three candidates, but he noted that 2 of these 8 combinations would lead to a "contradiction of terms" since they underline a cyclic preference structure; it leads, in turn, to the conclusion that there really are only six possible options, the ones in tab.1

Here's an extension of the majority rule in the case with more than two candidates: just as in the case with two candidates the voter casts the vote for his most preferred candidate; at the end of the election, the candidate with the greatest number of votes wins; referring to the case in fig. 1.1, A wins if both $[n_1 + n_2 > n_3 + n_5]$ and $[n_1 + n_2 > n_4 + n_6]$, so if we let A P B indicates the fact that A is preferred to B according to the Plurality Rule, we have A P B and A P C.

We can consider another extension of the majority rule to the three candidates election. Let A M B denotes the fact that A wins over B according to the majority rule. If we only consider A and B ignoring the relative position of C in this first step, we have that A M B if [$n_1 + n_2 + n_4 > n_3 + n_5 + n_6$]. If A M B, then we say that A wins over B according to the Pairwise Majority Rule. The same can apply for A and C; ignoring the relative position of B in the preference rankings, we have that A M C if [$n_1 + n_2 + n_3 > n_4 + n_5 + n_6$]. If we both have A M B and A M C, we say that A is the Pairwise Majority Rule Winner, or PMRW; at the same time, ignoring the relative position of A and given [$n_1 + n_3 + n_5 > n_2 + n_4 + n_6$] we have B M C, so, also considering that A M C we can refer to C as the PMRL-Pairwise Majority Rule Loser.

Now, let's consider again the case of the 1970 U.S. Senate election. The election was held according to the plurality rule, and it led Buckley to win the election. This example is widely cited to show that there are some occurrences in which the PMRL can win an election; as stated earlier, according to Gehrlein, in fact, there is little doubt that either Goodell or Ottinger would have beaten Buckley by PMR. Riker (1982) presents an analysis of this same election concluding that Ottinger would have been the PMRW. The same conclusion is the one reached by Poundstone, who explains how an episode may have influenced the election leading to this undesired outcome.

CHAPTER 2. THE VOTING RULE OF BORDA

2.1 BORDA'S PARADOX

Jean Charles De Borda was born a couple of centuries before Buckley won the election against Ottinger and Goodell, but he was already interested in the analysis of such a situation: he studied a voting rule in order to always avoid the situation in which the PMRL wins the election. Borda presented a very interesting example: he considers the following voting situation with three candidates and 21 voters:

Linear Preference Rankings:

	A	A	B	C
	B	C	C	B
	C	B	A	A
Number of voters:	n_1=1	n_2=7	n_3=7	n_4=6

Tab.2

Let's now analyze the results that arise from tab.2. If the winner is selected according to the plurality rule, then the candidate A results to be the winner of the election: in fact A P B (8-7), A P C (8-6) and B P C(7-6). Let's now try to elect the winner following the Pairwise Majority Rule: B M A (13-8), C M B (13-8) and C M A (13-8). In this case the situation is completely reversed: C is the PMRW, even if he was the looser of the previous election, and A, the plurality rule winner, is the Pairwise Majority Rule Loser. This situation looks quite paradoxical. When PMR completely reverse the rankings of plurality rule, we say that we are facing a “Strict Borda Paradox”, while when plurality rule elects the PMRL without necessarily having a complete reversal on the other rankings, we say that we are facing a “Strong Borda Paradox”. The French theorist gives a clear explanation of what happens in his example: “On reflection, we see that candidate A gains the advantage only because Candidate B and Candidate C have more or less split the 13 votes against him. We might compare them to two athletes who, having exhausted themselves competing against one another, are beaten by a third who is weaker than either”. Let’s consider, once again, our previous example about the Senate elections: Buckley wins because Ottinger and Goodell split between themselves the 61.2% votes against him. Buckley can be compared to the Candidate A: in fact, though he was recognized as the PMRL, he won the election because of plurality rule, while Ottinger only earned a second place despite he was the PMRW. This is an example of an actual occurrence of a “Strong Borda Paradox”.

2.2 BORDA’S RULE

Borda develops two voting systems with the precise aim of avoiding the occurrence of a Borda Paradox. The first procedure is similar to the one proposed by Condorcet, that we will analyze later: it states that the PMRW should win the election. Even if, nowadays it would be an easy and fast computation thanks to computers, at the end of the 18^{th} century, the process of checking every Pairwise Winner was extremely time consuming, so Borda proposed another procedure. He called it “Election by order of merit”, it’s now known as “Borda rule” and it has been developed as follow: every voter should rank every candidate from the most preferred to the least preferred, and then every candidate receives a score according to the following rule: in an m candidate election every candidate would earn [a+(m-x)b] points, where x is the relative position of the candidate and “a” and “b” are two weights; Borda suggests to use the particular weighting scheme with a=b=1; it follows that, for example, the most preferred candidate in a 10 candidates election would earn 1+ (10-1)1 points, the second preferred candidate would earn 1+(10-2)1 points and so on, until the least preferred candidate which will be awarded 1+(10-10)1 points. That’s intuitive that, following Borda’ s weights, the least preferred candidate always gets 1 point. Let’s now try to solve the situation in fig.2 according to the Borda Rule, using the weights suggested by Borda:

Score(A)= 8 * [1+ (3-1)1] + 13 * [1+(3-3)1] = 37

Score(B)= 7 * [1+ (3-1)1] + 7 * [1+(3-2)1] + 6 * [1+(3-3)1]=41

Score(C)= 6 * [1+ (3-1)1] + 14 * [1+(3-2)1] + 1 * [1+(3-3)1]=47

Let A B B denote the fact that Candidate A beats Candidate B by Borda Rule, then we have: C B A, C B B, B B A. Candidate C was the PMRW, but, in this situation, the situation under plurality rule is won by Candidate A, the PMRL that earns the fewest points.

2.3 MAJOR CONCERNS OF BORDA RULE

At a first glance, Borda rule seems to be a fair enough rule that reflects, democratically, the true will of the voters. But there are some concerns with this rule; the first can maybe be defined an ideological one and is related to the issue of using linearly decreasing points to obtain any given values of “a” and “b” in the election by order of merit. If we consider a three candidate election, supposing A>B>C, the points awarded to the candidates are, respectively, 3:2:1. It means that B is, in voter’s preference, exactly in the middle between A and C; in other words, B is not considered to be closer to A than it is to C: the issue here is that the” intensity of preference” is not being considered, and some theorists argue that it should always be taken into consideration. William Poundstone (Gaming the vote) exposes another issue related to this voting system: he considers an election with three candidates, 81 voters, and the distribution of votes as follows:

Linear Preference Rankings:

	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A
Number of voters:	n_1=30	n_2=1	n_3=29	n_4=10	n_5= 10	n_6=1

Tab.3

Let's now compute the Borda score for the three candidates:

Score(A)= 31 * [1+ (3-1)1] + 39 * [1+(3-2)1] + 11 * [1 + (3-3)1] = 182

Score(B)= 39 * [1+ (3-1)1] + 31 * [1+(3-2)1] + 11 * [1+(3-3)1]=190

Score(C)= 11 * [1+ (3-1)1] + 11 * [1+(3-2)1] + 59 * [1+(3-3)1]=114

It results that Candidate B is the winner by Borda rule, but according to the PMR the winner of the election is Candidate A: in fact, A M B (41-40) and A M C(60-21). The "election by order of merit" does not give the same result of the PMR. Borda Rule sounds fair in the same way as PMR, but in this example, it fails in electing the PMRW. Actually, Borda rule, does not always elect the PMRW because Borda did not drew his procedure with this aim, but he did it with the aim of always avoiding the PMRL to win the election. And it always satisfies this requirement: there are several proofs that Borda rule can never elect the PMRL as the winner of the election; according to Gehrlein, “for sufficiently large n, Borda rule is the only weighted scoring rule that can meet Borda’s criterion of not electing the PMRL as the winner”. Borda, in fact, was primarily concerned with the notion that the PMRL should not be selected as the winner. It clearly does not necessarily means that Borda cannot absolutely fail. Poundstone (Gaming the vote) again well presents a situation that could lead Borda rule to fail: suppose again there’s a political election between three candidates: the first one is endorsed by the Republican party, we will refer to him as Candidate A, the second one is supported by the Democratic party, our Candidate B, while the third one is a ruthless and violent Nazi, Candidate C: only a very little share of voters would really desire this latter to win the election, and, actually, he has not so many chances of winning the election, he actually has no chances at all, and voters know it. Trying to forecast voters’ behavior is useful in order to understand what will happen under this situation: in fact we can imagine that ones that support Candidate A will rank him as the most preferred; in order to help their favorite candidate they will rank the Candidate B as the least preferred in order to award him as less points as possible, being sure, at the same time, that the Nazi candidate has no real chances of winning. The same reasoning applies for the supporters of candidate B, who will rank Candidate A as least preferred, even if they would sincerely prefer candidate A to win instead of the Nazi candidate to win. Now, let's assume Candidate A and Candidate B enjoy a more or less equal percentage of votes, assuming both have 49% of preferences, and also assume that there's a very small part of the voters who support the Nazi candidate, the remaining 2%, who clearly rank him as the most preferred candidate: you do not need nothing more than a trivial computation to notice that Candidate C, the undesired Nazi, would be the winner of this election based on Borda rule; in this case, the "election by order of merit" leads to an undesired outcome for the most part of the voters. At the same time, it is clear that, without any doubt, the Nazi candidate would be the PMRL. In this case, Borda fails even in satisfying his prime request, i.e. not electing the PMRL as the winner. It is a consequence of a very precise behavior, i.e. it happens when voters do not vote according to their true preferences and behave strategically in order to favor their most preferred candidate. As Laplace states: "This election method would be undoubtedly the best, if considerations other than merit did not often influence the choices of even the most honest voters". Mc Lean(1995) notes Borda's response to criticism of his voting rule being vulnerable to manipulations as “My election method is only for honest men”.

CHAPTER 3 THE PARADOX OF CONDORCET

3.1 CONDORCET’S PARADOX

Marquis de Condorcet’s proposal was to solve elections by using the PMR to determine the winner of an election; for this reason the PMRW is usually referred to as the “Condorcet Candidate” or the “Condorcet Winner”, and the PMRL is known as the “Condorcet Loser”. Anyway, Gehrlein underlines how “ Borda was the first to suggest that the PMRW should win an election, but Borda was much more concerned about the undesirable possibility of electing the PMRL.” Condorcet strongly stressed for the use of the PMR for electing the winner of the election. But one of the main issue Condorcet was interested in, was the occurrence of various forms of paradoxes. The most known is the one known as “Condorcet’s paradox”: it refers to the presence of a cyclic fashion in the aggregated preferences of the voters. Condorcet assumes that a voter can only have “transitive preference”, otherwise his behavior is absolutely not rational; but the transitivity of the single “linear preference ranking” does not necessarily mean that cannot be an intransitive outcome once all preferences are aggregated, and this is exactly what is going to be showed; consider an election with 60 voters and 3 candidates that goes as follow:

Linear Preference Rankings:

	A	B	B	C	C
	B	A	C	A	B
	C	C	A	B	A
Number of Voters:	n_1=23	n_2=2	n_3=17	n_4=10	n_5=8

Tab. 4

Candidate A wins the elections given the adoption of the "plurality rule" voting procedure, since it is the most preferred candidate according to the will of voters, while respectively 19 and 18 voters absolutely prefer B and C over the other two candidates. But the situation is still to be analyzed according to the Condorcet criterion, or the PMR: A M B (33-27), B M C(42-18), and C M A (35-25), noting the occurrence of the cyclic fashion is easy: even if every voter has complete preferences, the aggregation of every voter's preference results in a non-transitive outcome. At the beginning of the paper, it was asserted that a requirement for an election to be considered valid and fair is "transitivity": the paradox exactly results in the "lack of transitivity". Donald Saari (1995a) makes a very interesting observation about the lack of transitivity in the case of the cycle preferences: he states that when a voter's preference is

A>B>C, this procedure only cares that A>C, and does not take account of the real position of B that is considered to be precisely between A and B, but it can actually be closer to A with respect to C in the preference of the voter as well. Once again, the problem of not considering the strength of preferences, becomes an important issue, that is enough to lead an election to be unfair.

3.2 CONDITIONS THAT PROHIBIT THE PARADOX OF CONDORCET

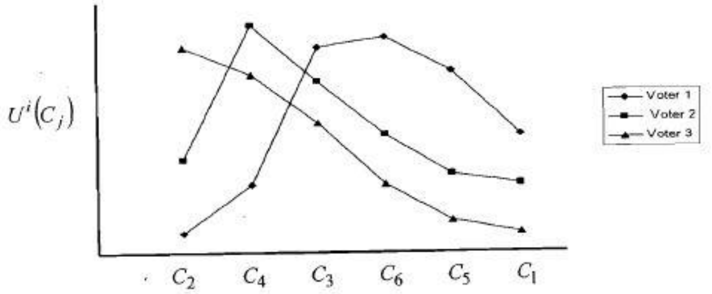
It has been proved that there are some conditions that surely lead the paradox of Condorcet not to occur: "Black (1958) found this to be the case when voter's preferences are restricted to have the property of single peaked preferences"; this passage will be explained thanks to an example. An election with 6 candidates and three voters is considered, with the following linear preference rankings:

1st voter: $C_6>C_3>C_5>C_1>C_4>C_2$

2nd voter: $C_4>C_3>C_6>C_2>C_5>C_1$

3rd voter: $C_2>C_4>C_3>C_6>C_5>C_1$

These preferences can be graphed in a very precise manner that, better than any word, shows why the definition "single peaked preferences" is used. Let C_i O C_j denote the fact that C_i is ranked as preferred to C_j in this overall reference ranking. The graph is drawn according to the following specific reference ranking: C_2 O C_4 O C_3 O C_6 O C_5 O C_1.



As the graph clearly shows, every preference curve does not change direction more than once: quoting Black (1958, pg.7) a "single peaked (preference curve) is one which changes its direction at most once, from up to down"; in other words: only one peak is possible. It is commonly agreed, among theorists, that facing "single peaked preferences" leads to a transitive result when PMR is adopted and, at the same time, no presence of Condorcet's cycle is possible in that case. There's an alternative definition stating that preferences are single peaked if at least one candidate is never ranked last (or first), for every triple of candidates. The same applies for a candidate that is never ranked in the middle, Ward(1965). Furthermore Ward adds one more condition that certainly leads the PMR cycle not to exist: he asserts that if there is not any Latin Square on any triple of candidates, than a PMR cycle cannot occur. A Condorcet triple- one in which A>B>C, B>C>A and C>A>B- this one exclusively, would in fact lead to the creation of a Perfect Latin Square.

3.3 OTHERS CONDORCET'S PARADOXES

There are some other paradoxes in general related to voting procedures and more in particular related to Condorcet. Recalling Borda' scheme of voting is necessary in order to explain a paradox shown by Condorcet. Borda rule, in fact, is a special case of weighted scoring rules. According to this rule, every candidate receives a certain number of points according to its position in the ballot. In a three candidate election, the first candidate is awarded 3 points, the second candidate receives λ points and only one point is awarded to the last candidate. Borda rule is a case with λ equals 2. It would not make sense, in fact, awarding to the second candidate more than 3 points or less than 1 point. Condorcet showed that, in certain condition, it may happen that no scoring rule including Borda's, can elect the PMRW. Consider a situation with 81 voters, three candidates and with the following preference rankings:

Linear Preference Rankings:

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A

Number of voters: $n_1=3$ $n_2=1$ $n_3=29$ $n_4=10$ $n_5=10$ $n_6= 1$

Tab.5

According to the PMR, A is the Condorcet winner. Using Borda rule, instead, the winner is B by 190 points over 182 points of the candidate A. This situation makes clear once more that Borda rule does not always elect the PMRW, but, once more, it has to be recalled that Borda was not primarily interested in electing the PMRW, but his major concern was never electing the PMRL. Condorcet goes farther in explaining this paradox and he approaches the issue considering a general weighted scoring rule assigning 1 and 3 respectively to the last and the first candidate and assigning λ points to the second candidate. It's trivial to show that, in order the Borda count of the PMRW-candidate A- to be greater than the Borda Count of the Borda winner- candidate B- the value of λ should be larger than 3. But this can never happen since λ is necessarily a value between 1 and 3.

CHAPTER 4 OTHER VOTING PARADOXES

4.1 THE PARADOX OF MULTIPLE ELECTIONS

Reality gives actual occurrences for the most part of the paradoxes explained by the theorists. The same applies for the paradox of multiple elections analyzed by Brams, Kilgour and Zwicker in a paper of 1998. Before explaining the theory of the paradox the actual occurrences will be given. It happened in California on November 1990, when voters were asked to give their preferences over a series of 28 propositions. They had to choose between "yes(Y)" or "no(N)" on every of the 28 proposition concerning local political issues such healthcare, alcohol drugs and so on. The proposition passed if Ys exceeded Ns for that proposition, otherwise it did not. There were about 1.8 million voters. The paradox of this election was given by the fact that none of the 1.8 million voters voted for the winning combination formed by NNNYNNYNYNNNNNNYNYYYNYYNNYNY. Actually, Brams et al. underline the fact that there were more than 250million Y-N combinations, and that nobody has to be surprised for this result. They underline that-given every voter to vote for a different combination- no more than 1% of the combinations could receive at least one vote.

After this example it should be easier to understand the concept of "Paradox of Multiple Elections". "It occurs when the winning combination under proposition aggregation receives the fewest, but not necessarily zero, votes". In a theoretic way, at least in the case in which the winning combination does not receive any support, it leads no voter to be satisfied with the result of the election.

Later in the paper some examples of the paradoxes of multiple elections will be considered; following there are some concepts that will be fundamental in order to understand the following situations of the various paradoxes. The elections that will be considered are cases in which the voter has to choose among a series of propositions and has to express his preference by choosing between Yes or No for every proposition. The definition "Proposition Aggregation" will refer to the process of aggregating the preferences separately for every proposition, while "Combination Aggregation" is a definition related to the final outcome given by the aggregation of all the preferences.

The following is the first of a series of example that will be exposed in the paper. It is referred to as a "basic example" or "minimal example" too, in the sense that no other example with fewer voters or propositions could meet the conditions stated in the paradox. Consider a situation in which there are 13 voters and 3 propositions on which a preference is to be expressed. With 3 propositions and 2 alternatives for every proposition the number of combinations is 2^3=8. The result of the ballot is as follow:

YYY:1 vote YYN:1 vote YNY:1 vote NYY:1 vote NNY:3 votes NYN:3votes YNN:3votes NNN:0

By combination aggregation, the set of the winners is given by NNN that is the only combination that did not receive neither one vote. In fact, for the first proposition the number on Ns is larger than the number of Ys, and the same applies for the second and the third combination: in all cases N wins over Y by 7 votes to 6. This one just showed, is only the basic example over a series of paradoxes related to multiple election. The reader can clearly understand how it can apply to the referenda in which we are asked to give our opinion.

There are other examples regarding the multiple elections paradoxes. One of the most interesting cases among the ones chosen from the paper of Brams et al. is the one concerning the “complete reversal paradox”. This particular kind of paradox occurs when the opposite of the proposition aggregation winner receives the most votes. The situation is as follow:

31 voters and 4 propositions; clearly in this case the number of combination is higher, since we now have 2^4=16:

YYYY:0 voters YYYN:4voters YYNY:4 voters

YNYY:4voters NYYY:4voters YYNN:1voters

YNYN:1voter YNNY:1voter NYYN:1voter NYNY:1voter

NNYY:1voter YNNN:1voter NYNN:1voter NNYN:1

NNNY:1 NNNN:5 voters

In this case Y wins over N in all the proposition by 16 to 15 points, in such a way the winner by combination aggregation is YYYY. But here, not only YYYY is the least voted combination but also NNNN, that is it's opposite, is the most voted one. This is the reason why this is called the "complete reversal paradox".

Moreover there are some cases in which a multiple election paradox can occur even with a certain number of "abstention": this term means that a voter is free not to give his preference on one or more or all the propositions. The following is a basic paradox that allows ties for the fewest voted combination, with 2 propositions and 15 voters voting for the 3^2=9 combinations:

YY:0 YN:3 NY:3 YA:3 AY:3 NA:1 AN:1 NN:1

AA:0

Here again YY is the winner under combination aggregation, but once more that combination receives the least number of votes together with AA. Brams et al., actually, underline that it can be misleading to state that nobody chose AA, since, it is logical to think that nobody would go to the poll for voting Abstention on both the proposition presented.

One case more, is worthy to be analyzed. In this situation, again, the voter can face three alternatives for every proposition: Yes No or Abstention, but with three propositions and 52 voters, the number of combination goes up to 27 (3^3) and with 52 voters casting their preferences the situation look as follow:

YYY:0 YYN:4 YNY:4 NYY:4 YYA:4 YAY:4

AYY:4 YNN:1 NYN:1 NNY:1 YAA:1 AYA:1 AAY:1

NAA:1 NAN:1 NYA:1 ANY:1 YAN:1 AAA:5 ANN:1

ANA:1 AAN:1 AYN:1 NAY:1 YNA:1 NNN:5

This case is for some aspects similar to one of those observed earlier. In fact, here, the winner by combination aggregation is again YYY, that wins with by 20 votes to 16 on every proposition, but nobody cast a vote for that very precise combination, with, at the same time, the two opposite combinations, that are AAA and NNN, receive the largest number of votes,5.

4.2 A STRONG PARADOX OF MULTIPLE ELECTIONS

Marco Scarsini (1998) showed that an even stronger version of multiple elections paradox can occur: his proofs consists in showing that it could actually happen that “not only the winning combination of proposition receives 0 votes, but also all the combinations sufficiently close to it receive 0 votes”. Scarsini, in fact, found that “it is possible that all the propositions that agree on the winning one on at least [(n+1)/2] propositions receive 0 votes (here the symbol [x] indicates the smallest integer larger than x)". Supposing that there are 5 propositions, for example, it is possible that all propositions agreeing on 4 over 5 propositions receive 0 votes: i.e.: NNNNY, NNNYN, NNYNN, NYNNN,YNNNN, and NNNNN too, clearly, are not awarded any vote. In the case of a 7 proposition elections, instead, it may happen that combinations that agree on 2 over 7 combinations receive 0 votes.

These examples were given with the extent to let the reader understand how large the discrepancy can be between aggregating preferences "by combination" or "by proposition".

4.3 CONSTRUCTION OF A MULTIPLE ELECTION PARADOX

It can be clearly understood that a paradox can occur only under precise conditions. Recalling one of the previous examples is necessary in order to explain how a paradox can be constructed. The example that will be taken into considerations is the basic paradox without ties for fewest and with 13 voters and 3 Y-or-No propositions: in this example NNN resulted to be the winner under combination aggregation, even if 0 votes were awarded to it; observing the distribution of the other votes, the combination that agreed in

2 over 3 propositions result to be the most voted, while the next-to-last fewest votes were cast for the combination that agreed on only one proposition with the winning combination. It means that NNN “joined the support” of the combination that agreed with it for 2 over 3 propositions. In order to make the concept clearer every combination will be assigned a certain sum of points according to the following rule: considering NNN as the reference combination, NNY is a close combination that supports the NNN one: assigning 1 point for every NO and -1 point for every YES it is intuitive that in the case of the votes cast for NNY, the winning combination joins:

1[N] + 1[N] + (- 1)[Y] = 1 point.

The same applies for the other combinations; consider, for example, YYN: summing up its points –taken NNN as reference combination- the result is:

-1[Y] + (-1)[Y] + 1[N] = -1 point.

This way of computation recalls the one suggested by Borda, in which every vote is assigned a score.

The issue can be seen from a more general point of view:

Let “Q” be the total sum of points assigned to the winning combination, composed, in turn, by “C” and “I”, where “C” defines the concept of “Coherent support”, that is given by a combination that completely agrees with the winning one and

"I" defines the concept of "Incoherent support", that is given by a combination that agrees with the winning one in 2 over 3 propositions; in practice, for what concerns "Coherent support", given NNN to be the winning combination, one vote for NNN is to be considered Coherent; with respect to "Incoherent support", given YYY to be the winning combination, one vote for YYN is to be considered Incoherent, and the same applies for YNY and NYY. A proposition that gives "Coherent Support", supports 3 times the winning combinations with respect to a "Incoherent Support" combination.

Brams et al. underline two necessary condition: there's no way to rearrange their expression to explain it in a clearer way: always considering the previous example with NNN as the winning combination, they assert that " the total margin by which voters favor combinations with more Ns than Ys over their opposites must b greater than twice the maximum of the margins corresponding to indirect support", at the same time, "the more direct support that NNN receives over its opposite, YYY, then the less uniform the tilt must be in order for NNN to prevail"

4.4 CONDITIONS THAT PROHIBIT MULTIPLE ELECTIONS PARADOX

Earlier, conditions prohibiting Condorcet's paradox to occur, were explained. There are some conditions that prohibit

"Multiple elections paradox" as well. These considerations were made by Brams et al., they intuitively understood that a paradox cannot occur if there is a large difference between the combinations that Incoherently support the winning one, and their opposite combinations: in fact, if the difference is large, then, it will not be possible to overwhelm the great direct support that is joined by the opposite combination of the winning one. Previously, it was said: " the total margin by which voters favor combinations with more Ns than Ys over their opposites must be greater than twice the maximum of the margins corresponding to indirect support"; if this requirement is not satisfied, then no paradox can happen.

CONCLUSION

Up to now, this paper showed how difficult is having a fair aggregation of preferences in case of an election with more than two candidates. This reflects into some important considerations about the level of democracy of the States: in the first chapter it was showed, through an actual occurrence, the one relative to the elections in the U.S. Senate in 1970, that a "clone" candidate can misrepresent the real will of the voters, leading, in turn, in a widespread dissatisfaction related to the election. Giving evidence of the importance of voting systems was the aim since the beginning of the paper, and a further demonstration of it can be found in the elections on November 1990 in California, an example cited in 4.1. As stated earlier, it is not difficult finding cases that led to undesired outcomes. This fact should be a warning to everybody. People are not concerned about this issue, but they should: their future could be “paradoxically” determined by a “paradox”; consider the case of a “clone”: once an undesired politician wins an election, everybody will suffer his choices for the country, or for the area he represents, and, hopefully, nobody but the “clone” itself - desires such a situation. But, at the same time, it was showed how difficult is finding the perfect voting rule. Borda’ seemed to be fair enough, but it was subsequently demonstrated that it wasn’t, since susceptible of strategic vote. Condorcet’s voting rule – PMRW - seemed fair as well, but again a problem appears, i.e. the not-so-remote possibility of a Condorcet cycle, as shown in 3.1. The paper focused particularly on giving evidence of the presence of a long series of voting paradoxes, and not to its solution: the reason for the adoption of this approach is that finding a voting rule that never fails seems to be unreasonable, even due to the fact that Arrow formulated his “Impossibility theorem”: it apparently changed the approach to this issue: since that moment, in fact, the starting point of every research, is that a fair voting system cannot be drawn. Paradoxes occur even when facing multiple elections, and it was widely demonstrated in Chapter 4. Scarsini found evidence of it even in a stronger form of “multiple elections paradox”. This matter will be object of several researches for long time in the future but a solution to it seems still to be far from reached.

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