DEPARTMENT OF ECONOMICS AND FINANCE

Chair in Theory of Finance

Estimating the Stochastic Discount Factor: evaluation and historical approaches

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Abstract

Financial markets have become more and more complex over the last 30 years and especially after the subprime crisis. For this reason, it is important to be able to understand the mechanisms that move the market variables by interpreting them properly.

My final dissertation aims at studying and analyzing different methodologies to estimate stochastic discount factors using the three-factor model developed by Fama-French and then testing their performance in pricing assets during the past 30 years.

Estimation techniques play a crucial role in this context because they may dramatically affect profits and losses of a portfolio. Therefore, having a good proxy of the real discount factor as a starting point could help to avoid mistakes in pricing an asset.

Not only is my goal to investigate the effectiveness of the empirical stochastic discount factor, but also my objective is to obtain a simple tool to be used in the financial job market.
1. Introduction to Discount Factor and Factor models & Connection between them

According the fundamental theorem of asset pricing, the price of any asset is equal to its expected discounted payoff. In asset pricing theory, the payoff is discounted by a factor, which depends on market parameters and on the data. Apart from asset pricing, another use of the stochastic discount factor is to evaluate the performance of actively managed portfolios. Based on Cochrane (2001), risk neutral valuation implies the existence of a positive random variable, which is called the stochastic discount factor and is used to discount the payoffs of any asset. Modern asset pricing theory assumes the existence of this discount factor $m$ so that the following equation holds:

$$ p = E(mx) $$

Where:
- $p$ is the price of any asset;
- $m$ is the stochastic discount factor (SDF);
- $x$ is the asset payoff(s).

Note that prices and returns available are nominal so that in this framework $p = E(mx)$ will refer to a nominal discount factor.

To get the real price of an asset, it is possible to divide the discount factor by a parameter to adjust for the effect of inflation:

$$ p_t = E_t \left( \frac{\Pi_t}{\Pi_{t+1}} m_{t+1} x_{t+1} \right) $$

Where $\Pi$ denotes the price level (the CPI, or Consumer Price Index).

The term *stochastic discount factor* refers to the way $m$ generalizes the standard discount factor ideas: on one hand, it can incorporate all risk-corrections by defining a single stochastic discount factor – the same for each
asset – and putting it into the expectation. On the other hand, m is defined stochastic or random because it is not known with certainty at the present time.

The conditions for the existence of the stochastic discount factor are provided by the two following theorems:

- **The law of one price** states that if two portfolios have the same payoffs (in every state of nature), then they must have the same price. As a consequence, a violation of this law would give rise to an immediate kind of arbitrage profit, as you could sell the expensive version and buy the cheap version of the same portfolio. In other words, the first theorem argues that there is a discount factor that prices all the payoffs by \( p = E(mx) \) if and only if this law of one price holds.

- **The term absence of arbitrage** is based on a stronger assumption: if the payoff of A is always at least as good as the payoff of B, and sometimes A is better, then the price of A must be greater than the price of B. To sum up, this second theorem claims that there is a positive discount factor that prices all the payoffs by using the formula \( p = E(mx) \) if and only if there are no arbitrage opportunities.

As far as the performance of actively managed portfolios is concerned, it is evaluated under the assumption that there are no arbitrage opportunities in financial markets. This assumption implies that exists at least one positive stochastic discount factor capable of price all assets. Under such condition, the price of any asset is given by the expected value of the future asset payoff adjusted by the stochastic discount factor so that the following equation is always satisfied:

\[
1 = E_t(m_{t+1}R_{i,t+1})
\]

Alternatively:

\[
0 = E_t(m_{t+1}R_{i,t+1}^e)
\]
Where

- \( mt_{t+1} \) is the stochastic discount factor at time \( t +1 \);
- \( E_t \) is the expectation conditioned on the information available up to time \( t \);
- \( R_{i,t+1} \) is the gross return;
- \( R^e_{i,t+1} \) is the excess return of any asset at time \( t+1 \) (defined as the difference between the return on the asset and the risk free rate).

### 1.1 Discount Factor model specification

There are many methods to specify the discount factor:

I. **Linear factor**: models in which \( m \) is linear in prespecified factors; in empirical asset pricing, a common approach is to specify the SDF as a linear function of a \( k \times 1 \) vector of risk factors, \( f \), with \( k < n \). The Capital Asset Pricing Model is an example of such model in which the discount factor is linearly related to the market portfolio return.

II. **Primitive-efficient**: models in which \( m \) is a linear function of the returns on the set of primitive assets (asset that have a finite number of payoff in each possible state of nature) that are weighted depending on the information available.\(^1\)

III. **Bakshi-Chen**: model in which \( m \) is constrained to be positive by cutting off the specification of the stochastic discount factor at zero, because it is assumed the no-arbitrage condition.\(^2\)

In this framework, the objective of the analysis is to estimate the discount factor by using the linear specification according to which the discount factor is defined as follows:

\[
m = a - f' b
\]

---

\(^1\) Hansen and Jagannathan (1991) shows that the discount factor derived from this specification is admissible and prices a given set of primitive assets by construction.

\(^2\) Bakshi and Chen (1998) propose a model in which the stochastic discount factor is an exponential of a linear function of the logarithmic returns on the primitive assets.
Where \( f \) is a \( k \times 1 \) vector of risk factors, \( a \) is a scalar constant and \( b \) is a \( k \times 1 \) vector of parameters.

In the unconditional case (when the investors do not consider the information set available at time \( t \)) the weight of the parameters \( a \) and \( b \) are time-constant, so it is assumed that they does not change over time.

It is possible to assume that the payoffs and discount factors are independent and identically distributed (i.i.d.) over time, so that the conditional expectations would be the same as the unconditional expectations. In this framework, this assumption will not be made, because in practice is not feasible to know the investors' information set: for this reason, it will be consider only unconditional models.

### 1.2 Validity of the model

There is no doubt that it is important to verify the legitimacy of the model. In this regard, \( m \) is considered a valid SDF if the following equality holds for at least one \((a, b)\) of the linear specification:

\[
E(R^e_m) = 0
\]

This condition, as stated before, holds if there are no arbitrage opportunities available in a set of assets.

However, validity of the SDF does not uniquely determine \( a \) and \( b \): as Cochrane (2001) shows, it is not possible to separately identify the two parameters without a normalization. For this reason, it is common to adopt a standardization of the SDF that reduces the dimension of the parameters space.

Since the pricing errors do not depend on the choice of normalization, it should be the most convenient.

As far as the computational process is concerned, the linear specification should be rewritten as follows in order to calculate the SDF normalization:

\[
m = a(1 + \delta'f)
\]

Where \( \delta \) is equal to \( b/a \). In this case:
\[ m^\delta = 1 + \delta'f \]

It is referred to as the intercept-normalization of m since it is a scaled version of the discount factor with a unit intercept.

The non-arbitrage equation implies that:

\[ E(R^e\delta) = E(R^e) \]

To summarize, \( m \) is a valid SDF if the above equation holds for at least one \( \delta \).

### 1.3 Connection between the Factor model and Discount Factor

As Cochrane (2001) demonstrates, it is possible to link any linear factor model (like the CAPM, APT and the three-factor model of Fama-French) as expected return-beta model and it can be shown that Beta-pricing models are equivalent to linear models for the discount factor \( m \):

\[ E(R^i) = \alpha + \lambda'\beta_i \leftrightarrow m = a + b'f \]

One way to show the connection between the discount factor specification and the expected return/beta factor model formulation is to set up the factors in the constant, writing \( E(m) = 1 \) (so \( a=1 \)), so that the linear specification becomes:

\[ m = 1 + b'f \]

From this normalization, it is possible to find a vector of parameters \( \lambda \) such that:

\[ E(R^e) = \beta_i\lambda \]

Thus, the relationship between \( \lambda \) (the regressor of the Beta-pricing models) and \( b \) (the regressor of the linear factor model) is:

\[ \lambda = -\text{var}(f)b \]
This connection between the two models is fundamental because it states that, from an expected return/beta model with factors, it is possible to stem a discount factor m such that $p=E(mx)$ holds.

A great number of empirical works in finance are cast in terms of expected return - beta representations of linear factor pricing models. The Fama-French three factor model is the most commonly used to derived the discount factor. For this reason, in this research the tradition will continue.

Generally speaking, the SDF models are estimated by GMM (General Method of Moments), defined by Hansen\(^3\): in his paper, the author assumes some regularity conditions, such as the asymptotic distribution of $\delta$.

Deriving the discount factor with the GMM approach requires a deep knowledge of the asset pricing theory, as well as experience in econometrical and matrix computation. Consequently, these kinds of models are more suited for theoretical exercises rather than the practical use.

On the contrary, a simple OLS regression can be a better and easier method to determine the empirical SDF. It allows to simplify the assumptions, by using less calculations and theory and by deriving a good proxy of the real discount factor.

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2. Estimation of the discount factor & analysis

To start with, it is fundamental to choose the correct factor model from which to estimate the discount factor. Since the Fama-French three factor model is the most popular asset pricing model in empirical work, in this dissertation it will be the starting point for the estimation.

Fama and French (1992) show that, apart from the market risk, there are other important factors that help explain the average return in the stock market. This evidence has been demonstrated in several works for different stock markets, for example in Gaunt (2004)⁴.

2.1 Fama-French Three Factor model

The Fama-French model is characterized by the following three factors:

I. The excess return on the value-weighted U.S. stock market, Rm-Rf, calculated as the value-weighted return of all CRSP firms incorporated in the US and listed on the NYSE, AMEX, or NASDAQ minus the US one-month Treasury bill rate.

II. The return differential between a portfolio of small firms and a portfolio of large firms (SMB) is constructed in order to measure the size premium. In fact, it is designed to track the additional return that investors have historically received by investing in stocks of companies with relatively small market capitalization.

A positive SMB in a given month indicates that small cap stocks have outperformed the large cap stocks in that month. On the contrary, a negative SMB suggests that large caps have outperformed the others. It is computed as the average return on the three small portfolios minus the average return on the three big portfolios as follows:

---

\[ \textit{SMB} = \frac{1}{3} (\text{Small Value} + \text{Small Neutral} + \text{Small Growth}) \]
\[ - \frac{1}{3} (\text{Big Value} + \text{Big Neutral} + \text{Big Growth}) \]

III. The return differential between a portfolio of high-value firms and a portfolio of low-value firms (HML) is constructed to measure the premium-value provided to investors for investing in companies with high book-to-market values.

\[ \textit{HML} = \frac{1}{2} (\text{Small Value} + \text{Big Value}) - \frac{1}{2} (\text{Small Growth} + \text{Big Growth}) \]

A positive HML in a given month suggests that value stocks have outperformed growth stocks that month, whereas a negative HML indicates that growth stocks have outperformed the others. Note that the firm’s value is measured as the ratio of its book value to its market capitalization.

It is necessary to establish a risk free rate in order to calculate the excess return: in this framework, the US one-month Treasury bill will be considered the “safe” asset (the instrument with certain future return). The models are estimated by using monthly data for the period from July, 1984 to February, 2015.
Table 1 shows the factors’ summary statistics:

<table>
<thead>
<tr>
<th>Factors</th>
<th>Mkt-RF (Yt)</th>
<th>SMB</th>
<th>HML</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>0.704</td>
<td>0.072</td>
<td>0.227</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.234</td>
<td>0.163</td>
<td>0.156</td>
</tr>
<tr>
<td>Max</td>
<td>12.47</td>
<td>22.02</td>
<td>13.88</td>
</tr>
<tr>
<td>Min</td>
<td>-23.24</td>
<td>-16.40</td>
<td>-12.61</td>
</tr>
</tbody>
</table>

*Table 1 - Summary Statistics of the Fama-French risk factors*

Note that the Fama-French three factor model is based on the US market: for this reason, it is reasonable to presume that the model does not only do well in pricing US stocks, but international assets as well, due to the fact that the US financial market is the market leader worldwide.

Table 2 shows the correlations between the factors, calculated as follow:

\[
Corr(f_i, f_j) = \frac{Cov(f_i, f_j)}{\sigma_i \sigma_j}
\]

<table>
<thead>
<tr>
<th>Factors</th>
<th>HML</th>
<th>SMB</th>
<th>Mkt-Rf</th>
</tr>
</thead>
<tbody>
<tr>
<td>HML</td>
<td>1</td>
<td>-0.3216</td>
<td>-0.2908</td>
</tr>
<tr>
<td>SMB</td>
<td>-0.3216</td>
<td>1</td>
<td>0.2316</td>
</tr>
<tr>
<td>Mkt-Rf</td>
<td>-0.2908</td>
<td>0.2316</td>
<td>1</td>
</tr>
</tbody>
</table>

*Table 2 – Matrix of Correlations*

The Excess return factor is negatively correlated with the HML factor and whereas it is positively correlated with the Size factor. Furthermore, the HML factor is also negatively correlated with the SMB factor.
2.2 Methodology

As said earlier, in order to estimate the discount factor, the Fama-French (1993) three factor specification of the stochastic discount factor will be used. Since all the factors are excess returns, the stochastic discount factor in the unconditional version of the Fama-French three-factor model can be specified as:

\[ m = 1 - b_1YM - b_2HML - b_3SMB \]

This method is the one of the simplest models used to estimate the discount factor and it is based exclusively on the hypothesis that the discount factor price a null excess return, as Cochrane (2001) demonstates.

To estimate the discount factor, it is possible to run the following regression:

\[ 1 = b_1YM_t + b_2HML_t + b_3SMB_t + \eta_t \] (1)

Moreover, it is possible to show that solving the following moment condition is equivalent to the OLS regression above mentioned:

\[ E(R^e m) = E(mf^*) = 0 \]

Indeed, since the factors are excess returns, it is possible to write:

\[
\begin{cases}
    E[mYM] = 0 \\
    E[mSMB] = 0 \rightarrow E[mf] = 0 \\
    E[mHML] = 0
\end{cases}
\]

In addition, \( m=1-b_jf_j \). If this equivalence is substituted into the assumption, the moment condition can be written as:

\[ E[(1 - b_jf_j)f_j] = 0 \]

Therefore, \( 1-b_jf_j \) is exactly \( \eta_t \) (the error term in the regression), so that:

\[ E[\eta_tf_j] = 0 \]
which is specifically the strict exogeneity condition (the errors in the regression should have a zero conditional mean), a necessary assumption to calculate the OLS regression.

However, this computational process could lead to some problems: the estimation of the regressors could be biased due to the fact that the dependent variable is a constant.

Cochrane (2001) shows that, if the factors are excess returns, the predicted \( b_j \) of the linear model is equal to:

\[
    b_j = \frac{E[R^e]}{Cov(R^e,f_j)} = \frac{E[f_j]}{Cov(f_j,f_j)} = \frac{E[f_j]}{Var(f_j)}
\]

(2)

Therefore, the following step is to calculate the betas from our data (in table 3, the first row is the predicted beta, calculated using equation 2, while the second row shows the estimated beta, derived from regression 1):

<table>
<thead>
<tr>
<th>( b_j )</th>
<th>Mkt-RF</th>
<th>SMB</th>
<th>HML</th>
</tr>
</thead>
<tbody>
<tr>
<td>Predicted</td>
<td>0.035122</td>
<td>0.007363</td>
<td>0.025439</td>
</tr>
<tr>
<td>Estimated</td>
<td>0.041265</td>
<td>0.007153</td>
<td>0.044815</td>
</tr>
</tbody>
</table>

Table 3 – Differences between predicted and estimated betas

The prediction of the theory is quite close to the actual estimation. To establish this equivalence, it is necessary to test if:

\[ b_j^{predicted} = b_j^{estimated} \]

To do this, it is possible to run a t-test, in which the hypothesis is:

\[ H_0: b_j^{predicted} = b_j^{estimated} \]

The significance level is set at 10%. Thus, it is necessary to calculate the t-test, as follows:
From the previous equation, it is possible to get the values of \( t \); then, the \( p \)-values are calculated and shown in the next table:

<table>
<thead>
<tr>
<th>Significance test</th>
<th>Mkt-RF</th>
<th>SMB</th>
<th>HML</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>0,5155</td>
<td>-0,0119</td>
<td>1,0473</td>
</tr>
<tr>
<td>( p )-value</td>
<td>0,3032</td>
<td>0,5047</td>
<td>0,1478</td>
</tr>
</tbody>
</table>

Table 4 – \( t \) and \( p \)-value of the test

From the values above follows that the null hypothesis can’t be rejected at the predetermined confidence level so that it is statistically correct to say that the theoretical coefficients and the estimated ones are equivalent. These findings provide an evidence of the validity of the model.

2.3 Estimation results

At this point, it is possible to compute \( m_t \) from the following equation:

\[
m_t = 1 - b_{YM}Y_{Mt} - b_{HML}HML_t - b_{SMB}SMB_t
\]

Using the betas estimated in the previous section.
The table below provides the summary statistics for the time-series of the SDF:

<table>
<thead>
<tr>
<th></th>
<th>MEAN</th>
<th>STD</th>
<th>MIN</th>
<th>MAX</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount Factor (m)</td>
<td>0.98932</td>
<td>0.12860</td>
<td>0.38640</td>
<td>1.44889</td>
</tr>
</tbody>
</table>

The results are interesting because the SDF has no negative values and the mean is close to one, due to the mean-normalization assumption. Furthermore, the discount factor seems to be more volatile over the 1999-2002 and 2007-2011 period. In this regard, the higher standard deviation could be explained with the following evidences:

IV. After the so-called “Internet bubble”, caused by the rise of commercial growth of the Internet, the market participants were not prepared for such a situation, so they failed to price the stocks in the US market correctly.

V. After the Global Financial Crisis, started in 2007, the uncertainty in the financial market led to fast an adjustment of the preferences of consumers, raising the level of uncertainty in the economy.

Nevertheless, the empirical discount factor seems to be quite stable with the exception of these two periods.
3. Testing the Performance of the discount factor

In order to check the performance of the discount factor, it is relevant to test if a set of SDF candidates satisfy the law of one price, such that:

\[ E(mR_i) = 1 \]

Thus, we say that a SDF correctly "prices" the assets if this equation is satisfied.

In fact, despite the assumption, it is possible that the empirical discount factor may not respect the moment condition and some assets could have significant pricing errors.

The pricing error can be written as:

\[ \theta = E(mR_i) - 1 \]

In practice, the thetas of all assets should be zero. In fact, the moment condition on which the estimation is based states that the discount factor assigns a price equal to one to any gross return, as the theory suggests.

3.1 The Data

In order to evaluate the performance of the empirical discount factor, it will be tested on a set of reference assets. In theory, all assets available should be included in the sample. However, it is possible to employ only a subset of available assets. The decision of which assets to include is guided by the type of assets in which the fund invests, i.e., the choice of primitive assets should reflect a fund manager’s investment universe. This analysis will be based on different types of assets (indices and portfolios) in order to provide a good proxy of the market.
Firstly, the financial indices used are the following:

I. The S&P 500 (Standard & Poor's 500): it is an American stock market index based on the market capitalizations of 500 large companies having common stock listed on the NYSE or NASDAQ. S&P Dow Jones Indices determine the S&P 500 index components and their weightings. It is one of the most commonly followed equity indices, and many economists consider it one of the best representations of the U.S. stock market, and a bellwether for the U.S. economy.

II. The NASDAQ-100: this index includes 100 of the largest domestic and international non-financial securities listed on The Nasdaq Stock Market based on market capitalization. The Index reflects companies across major industry groups including computer hardware and software, telecommunications, retail/wholesale and biotechnology. It does not contain securities of financial companies including investment companies.

III. The EURO STOXX 50: it is an index that represents the performance of the 50 largest companies among the 19 economic sectors in terms of free-float market cap in 12 Eurozone countries. The index has a fixed number of components and it is part of the STOXX blue-chip index family. It captures about 60% of the free-float market cap of the EURO STOXX Total Market Index (TMI) and it is one of the most liquid indices for the Eurozone: it serves as an underlying asset for financial products (options, futures, ETFs) and is also used for benchmarking purposes. In order to compare it with the SDF, the values have been multiplied for the monthly average exchange rate EUR/USD.

The discount factor will price the following portfolios:

I. The portfolios formed based on size: they are constructed monthly by the intersections of 2 portfolios formed based on size (market equity, ME) and 3 portfolios formed on prior (2-12) returns. The monthly size
breakpoint is the median NYSE market equity. The monthly prior (2-12) return breakpoints are the 30th and 70th NYSE percentiles. The six portfolios constructed each month include NYSE, AMEX, and NASDAQ stocks with prior return data. To be included in a portfolio for month \( t \) (formed at the end of month \( t-1 \)), a stock must have a price for the end of month \( t-13 \) and a good return for \( t-2 \).

II. The portfolios formed on Momentum: they are constructed monthly by using NYSE prior (2-12) return decile breakpoints. The portfolios constructed each month include NYSE, AMEX, and NASDAQ stocks with prior return data. To be included in a portfolio for month \( t \) (formed at the end of month \( t-1 \)), a stock must have a price for the end of month \( t-13 \) and a good return for \( t-2 \).

III. Portfolios formed on Book-to-Market: the B/M ratio is calculated considering the book equity value at the last fiscal year end of the prior calendar year divided by market equity value at the end of December of the prior year.

IV. Portfolios formed on Cashflow/Price: it is computed with the cashflow (earnings before extraordinary plus deferred tax income plus common equity’s share of depreciation) at the last fiscal year end of the prior calendar year divided by the price at the end of December of the prior year. The cashflow used in June of year \( t \) is total earnings before extraordinary items, plus equity's share of depreciation, plus deferred taxes (if available) for the last fiscal year end in \( t-1 \).

V. The 10 Industry Portfolios: they represent the performance among the 10 economic sectors (consumer nondurables, consumer durables, manufacturing, energy, business equipment, telephone and television transmission, wholesale, healthcare, utilities and services) assigned each to a NYSE, AMEX, and NASDAQ stock to an industry portfolio at the end of June of year \( t \) based on its four-digit SIC code at that time. The returns are calculated from July of \( t \) to June of \( t+1 \).
3.2 Pricing errors

At this point, it can be considered the set of different test portfolios and indices in order to establish if the discount factor estimated in this paper prices other assets. Let $y_i$ be the returns of these other assets and let $p_t = m y_i$ be the price of the assets’ residuals. Then, $p_t - 1$ should not be significantly different from zero. The STD has been adjusted to be heteroscedasticity and autocorrelation consistent (HAC), using the Unweighted HAC Estimator described by Newey & West (1986)\(^5\). The last column of the following tables shows whether the pricing error of the asset is significantly different from zero at a 95% confidence level. The Matlab code is shown in Annex II.

<table>
<thead>
<tr>
<th>Index</th>
<th>Average</th>
<th>STD</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>≠ 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>-0,288</td>
<td>0,2960</td>
<td>-0,868</td>
<td>0,292</td>
<td>No</td>
</tr>
<tr>
<td>Stoxx 50</td>
<td>-0,318</td>
<td>0,3233</td>
<td>-0,952</td>
<td>1,316</td>
<td>No</td>
</tr>
<tr>
<td>Nasdaq 100</td>
<td>0,058</td>
<td>0,5490</td>
<td>-1,018</td>
<td>2,134</td>
<td>No</td>
</tr>
</tbody>
</table>

It is possible to see that the excess returns of the three are well priced: in fact, the pricing errors are not significantly different from zero. This result is logical: since the Fama-French three-factor model is based on the US stock market, it should be a good indicator of the change in the return of the most famous indices in the States. However, the discount factor seems to also discount a European index well: even if it is possible to assume that the Stoxx 50 follows the US indices, the fact that the empirical $m$ prices the European index shows the validity of the model.

<table>
<thead>
<tr>
<th>Cashflow/Price</th>
<th>Average</th>
<th>STD</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>≠ 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lo 10</td>
<td>0,289</td>
<td>0,289</td>
<td>0,264</td>
<td>-0,228</td>
<td>No</td>
</tr>
<tr>
<td>Dec 2</td>
<td>0,219</td>
<td>0,219</td>
<td>0,185</td>
<td>-0,143</td>
<td>No</td>
</tr>
<tr>
<td>Dec 3</td>
<td>0,222</td>
<td>0,222</td>
<td>0,197</td>
<td>-0,164</td>
<td>No</td>
</tr>
<tr>
<td>Dec 4</td>
<td>0,121</td>
<td>0,121</td>
<td>0,209</td>
<td>-0,290</td>
<td>No</td>
</tr>
<tr>
<td>Dec 5</td>
<td>0,052</td>
<td>0,052</td>
<td>0,255</td>
<td>-0,447</td>
<td>No</td>
</tr>
</tbody>
</table>

Dec 6  0.068  0.068  0.370  -0.658  No
Dec 7  0.077  0.077  0.354  -0.616  No
Dec 8  0.110  0.110  0.394  -0.661  No
Dec 9  0.083  0.083  0.338  -0.580  No
Hi 10  0.266  0.266  0.380  -0.479  No

The summary statistics of the portfolios formed on Cashflow/Price show that all the portfolios are priced: this fact could be explained by the construction of the three-factor model.

<table>
<thead>
<tr>
<th>Book-to-Market</th>
<th>Average</th>
<th>STD</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>≠ 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lo 10</td>
<td>0.290</td>
<td>0.111</td>
<td>0.073</td>
<td>0.507</td>
<td>Yes</td>
</tr>
<tr>
<td>Dec 2</td>
<td>0.221</td>
<td>0.198</td>
<td>-0.167</td>
<td>0.608</td>
<td>No</td>
</tr>
<tr>
<td>Dec 3</td>
<td>0.237</td>
<td>0.161</td>
<td>-0.080</td>
<td>0.553</td>
<td>No</td>
</tr>
<tr>
<td>Dec 4</td>
<td>0.115</td>
<td>0.275</td>
<td>-0.425</td>
<td>0.654</td>
<td>No</td>
</tr>
<tr>
<td>Dec 5</td>
<td>0.071</td>
<td>0.303</td>
<td>-0.522</td>
<td>0.664</td>
<td>No</td>
</tr>
<tr>
<td>Dec 6</td>
<td>0.034</td>
<td>0.316</td>
<td>-0.586</td>
<td>0.654</td>
<td>No</td>
</tr>
<tr>
<td>Dec 7</td>
<td>0.067</td>
<td>0.202</td>
<td>-0.328</td>
<td>0.462</td>
<td>No</td>
</tr>
<tr>
<td>Dec 8</td>
<td>-0.080</td>
<td>0.355</td>
<td>-0.776</td>
<td>0.615</td>
<td>No</td>
</tr>
<tr>
<td>Dec 9</td>
<td>0.084</td>
<td>0.318</td>
<td>-0.539</td>
<td>0.707</td>
<td>No</td>
</tr>
<tr>
<td>Hi 10</td>
<td>0.112</td>
<td>0.349</td>
<td>-0.572</td>
<td>0.796</td>
<td>No</td>
</tr>
</tbody>
</table>

The discussion made for the C/P portfolios holds also for the B-M portfolios, except for the Low decile one: in this particular case, it seems that the portfolio has consistent positive excess returns.

<table>
<thead>
<tr>
<th>Momentum</th>
<th>Average</th>
<th>STD</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>≠ 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>low</td>
<td>-0.490</td>
<td>0.379</td>
<td>-1.234</td>
<td>0.253</td>
<td>No</td>
</tr>
<tr>
<td>2</td>
<td>-0.055</td>
<td>0.179</td>
<td>-0.405</td>
<td>0.295</td>
<td>No</td>
</tr>
<tr>
<td>3</td>
<td>0.036</td>
<td>0.411</td>
<td>-0.769</td>
<td>0.841</td>
<td>No</td>
</tr>
<tr>
<td>4</td>
<td>0.080</td>
<td>0.316</td>
<td>-0.539</td>
<td>0.699</td>
<td>No</td>
</tr>
<tr>
<td>5</td>
<td>0.025</td>
<td>0.469</td>
<td>-0.895</td>
<td>0.944</td>
<td>No</td>
</tr>
<tr>
<td>6</td>
<td>0.020</td>
<td>0.475</td>
<td>-0.911</td>
<td>0.951</td>
<td>No</td>
</tr>
<tr>
<td>7</td>
<td>0.107</td>
<td>0.327</td>
<td>-0.534</td>
<td>0.747</td>
<td>No</td>
</tr>
<tr>
<td>8</td>
<td>0.250</td>
<td>0.116</td>
<td>0.022</td>
<td>0.478</td>
<td>No</td>
</tr>
<tr>
<td>9</td>
<td>0.183</td>
<td>0.179</td>
<td>-0.167</td>
<td>0.533</td>
<td>No</td>
</tr>
<tr>
<td>high</td>
<td>0.727</td>
<td>0.331</td>
<td>0.077</td>
<td>1.377</td>
<td>Yes</td>
</tr>
</tbody>
</table>

The momentum is more interesting: even if the discount factor prices nine out of ten portfolios, the high set seems to not follow the assumption; in particular, it seems to have consistent negative pricing errors, arising from its high negative
excess returns. It is one of the few cases of pricing errors significantly different from zero, in the test sample.

<table>
<thead>
<tr>
<th>Industry 10</th>
<th>Average</th>
<th>STD</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>≠ 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>NoDur</td>
<td>0,248</td>
<td>0,137</td>
<td>-0,020</td>
<td>0,516</td>
<td>No</td>
</tr>
<tr>
<td>Durbl</td>
<td>-0,095</td>
<td>0,615</td>
<td>-1,301</td>
<td>1,110</td>
<td>No</td>
</tr>
<tr>
<td>Manuf</td>
<td>0,216</td>
<td>0,237</td>
<td>-0,249</td>
<td>0,680</td>
<td>No</td>
</tr>
<tr>
<td>Enrgy</td>
<td>0,098</td>
<td>0,298</td>
<td>-0,486</td>
<td>0,682</td>
<td>No</td>
</tr>
<tr>
<td>HiTec</td>
<td>0,509</td>
<td>0,182</td>
<td>0,152</td>
<td>0,866</td>
<td>Yes</td>
</tr>
<tr>
<td>Telcm</td>
<td>0,153</td>
<td>0,402</td>
<td>-0,635</td>
<td>0,941</td>
<td>No</td>
</tr>
<tr>
<td>Shops</td>
<td>0,216</td>
<td>0,194</td>
<td>-0,164</td>
<td>0,596</td>
<td>No</td>
</tr>
<tr>
<td>Hlth</td>
<td>0,382</td>
<td>0,204</td>
<td>-0,018</td>
<td>0,782</td>
<td>No</td>
</tr>
<tr>
<td>Utils</td>
<td>-0,097</td>
<td>0,547</td>
<td>-1,168</td>
<td>0,975</td>
<td>No</td>
</tr>
<tr>
<td>Other</td>
<td>-0,022</td>
<td>0,227</td>
<td>-0,466</td>
<td>0,422</td>
<td>No</td>
</tr>
</tbody>
</table>

In this particular set of returns, the only asset that has high pricing errors is the portfolio composed by the stocks of the “high technology” sector.

The explanation is straightforward, since, in the sample, the technological sector has the highest abnormal returns among all the industry portfolios: it has the highest growth rate (during the period 1996-2000) and the biggest drop (during the “Internet bubble”). The portfolio starts to grow again in 2004, and the trend continues nowadays.

This trend in a short period contributes to make the stock the most volatile among the group of assets: for this reason, the pricing error is significantly different from zero.

<table>
<thead>
<tr>
<th>Size</th>
<th>Average</th>
<th>STD</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>≠ 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td>0,153</td>
<td>0,254</td>
<td>-0,344</td>
<td>0,649</td>
<td>No</td>
</tr>
<tr>
<td>2</td>
<td>0,294</td>
<td>0,161</td>
<td>-0,023</td>
<td>0,610</td>
<td>No</td>
</tr>
<tr>
<td>High</td>
<td>0,253</td>
<td>0,191</td>
<td>-0,121</td>
<td>0,626</td>
<td>No</td>
</tr>
<tr>
<td>Big</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td>0,281</td>
<td>0,158</td>
<td>-0,028</td>
<td>0,590</td>
<td>No</td>
</tr>
<tr>
<td>2</td>
<td>0,032</td>
<td>0,308</td>
<td>-0,572</td>
<td>0,636</td>
<td>No</td>
</tr>
<tr>
<td>High</td>
<td>-0,068</td>
<td>0,387</td>
<td>-0,827</td>
<td>0,692</td>
<td>No</td>
</tr>
</tbody>
</table>

In this set of assets, the discount factor performs very well.
Figure 3 shows the summary of the effectiveness of the discount factor among the portfolios.

Generally, the empirical discount factor model prices well, indeed it is consistent around the 94% of the time.
4. Comparison with other model

One way to understand the validity of the empirical discount factor is to compare the estimated $m$ with the discount factor evaluated by other researchers. Since the principal aim of the paper is to develop a simple model to estimate a discount factor that is a good proxy of the real one, if the difference between this $m$ and the ones estimated using more theoretical and complicated methodologies is small, the central objective is satisfied.

4.1 Risk Neutral case

Firstly, it is important to understand the market condition; in particular, it is necessary to establish if the state of the market leads to some uncertainty. Considering that the risk free rate at time $t+1$ is known at time $t$, using the asset pricing formula for the risk-free rate, it is possible to get:

$$1 = E_{t+1}(m_{t+1}R_{t+1}^f) = R_{t+1}^f E_{t+1}(m_{t+1})$$

The risk free rate comes out of expectations because its value is known at time $t$. Consequently, the formula for the risk free rate can be written as:

$$R^f = \frac{1}{E(m)}$$

This equation says that, in case of no uncertainty, the mean of the SDF must be close to the inverse of the mean of the gross Treasury bill return. This equivalence holds in a risk-neutral world, if the investors do not change their consumption over time (if it is constant i.e. the utility function is linear).

Using the empirical discount factor, the following result can be achieved:

$$E(m) = 0.98932 \Rightarrow \frac{1}{E(m)} = 1.0107983 \approx 1.003084239$$
From this result, it is possible to conclude that the current world is not risk-neutral: in particular, the discount factor should include a correction, known as *risk adjustment*, which covers the investors from the risk of having a volatile consumption stream. In general, payoffs that are positively correlated with consumption growth are associated with lower prices, to compensate investors for risk.

4.2 The Hansen-Jagannathan distance

In the asset pricing literature, some measures are suggested to compare competing asset pricing models. The most famous measure is the Hansen and Jagannathan distance⁶ which is a summary of the mean pricing errors across a group of assets. It may also be interpreted as the distance between the candidate SDF and one that would correctly price the primitive assets. As shown by Hansen and Jagannathan, the HJ-distance is:

\[ \xi = \min_{m \in M} \| y - m \| \]

Where \( \xi \) is the distance of the SDF model \( y \) to a family of SDFs, in the state space \( \mathbb{R}^2 \), that correctly price the assets.

In another interpretation, Hansen and Jagannathan show that the HJ-distance is the pricing error for the portfolio that is most mispriced by the underlying model. The HJ distance summarizes the relative fit of a model to the cross-section of test assets, but it provides no insight into the economic magnitudes of the pricing errors for particular assets. In this sense, even though the investigated SDF models are misspecified, in practical terms, the models with the lowest HJ distance are the most interesting.

For a given set of test asset gross returns \( R \) and a candidate stochastic discount factor \( m \), the population value of the Hansen-Jagannathan distance measure takes the quadratic form:

\[ \alpha = E(mR_i)\{E(R_iR_i')^{-1}\}E(mR_i) \]

---

It is possible to evaluate the HJ-distance of the empirical discount factor using the same portfolios that have been used before. The results, obtained by a code in matlab, are shown in the next section.

4.3 Comparison between empirical discount factors

In this section, I am going to show three alternative methods (Farnsworth et al, Bessler et al and Gutierrez & Gaglianone) of estimating the discount factor: as explained in the first section, these other approaches, more technical and challenging, have been conducted and explained by professional researchers in their publications. All the specifications apply as risk factor the three factor model of Fama-French.

i) In their paper, Farnsworth, Ferson, Jackson and Todd\(^7\) (2002) estimate the SDF by using the Generalized Method of Moments, as in the Hansen’s paper (1982), that is based on the moment conditions:

\[
E[(m_{t+1}R_{t+1} - 1) \otimes Z_t] = 0
\]

Using the linear specification:

\[
m_{t+1} = a(Z_t) + b(Z_t)'F_{t+1}
\]

Where

- \(R_{t+1}\) is an N-element vector of the gross returns on a set of primitive assets;
- \(Z_t\) is a vector of lagged instruments;
- \(a\) and \(b\) denote the parameters of the model.

Their sample involves monthly data from July of 1963 through December 1994 for a total of 378 observations.

\(^7\)H. Farnsworth et all, “Performance evaluation with stochastic discount factors” (2002)
ii) Bessler, Drobetz and Zimmermann\textsuperscript{8} because of their interest in the performance of a German mutual equity funds sample, adopted the same approach of the above mentioned paper with the addition of one more restriction; particularly, they regarded the moment conditions as follows:

\[
\begin{align*}
\{E[(m_{t+1}R_{t+1} - 1) \otimes Z_t] = 0
\end{align*}
\]

Using a similar specification to the one described in this paper:

\[
m_{t+1} = a + b_1 YM_{t+1} - b_2 HML_{t+1} - b_3 SMB_{t+1}
\]

Where

- \( R_{t+1} \) is an N-element vector of the gross returns on a set of primitive assets;
- \( Z_t \) is a vector of lagged instruments;
- \( \alpha_p \) is the abnormal performance of the portfolio, measured as:

\[
\alpha_p = (m_{t+1}R_{t+1}|Z_t) - 1
\]

Thus, they added the so called “manager’s performance” condition, in which the abnormal fund performance can imply some risk-adjustment to the SDF. They used the returns over the period from January 1994 to December 2003.

iii) Finally, Carrasco Gutierrez and Piazza Gaglianone\textsuperscript{9} conducted a research by running a multiple regression of returns on Fama-French factors as follows:

\[
R_{it} = R_{ft} + \beta_{iy} YM_t + \beta_{ih} HML_t + \beta_{is} SMB_t + \epsilon_{it}
\]

\textsuperscript{8} H. Bessler et al, “Conditional Performance Evaluation for German Mutual Equity Funds” (2007)

Where $R_{t+1}$ are the returns generated by the expected-beta representation of the linear factor pricing model shown just below:

$$E(R_i) = \gamma + \beta_{iy}\lambda_y + \beta_{ih}\lambda_h + \beta_{is}\lambda_s$$

The linear specification is:

$$m_{t+1} = a + b'f$$

Then, the parameters $a$ and $b$ can be computed as:

$$a = \frac{1}{\gamma}$$
$$b = -\gamma[\text{cov}(ff'')]^{-1}\lambda$$

The main feature of this model is that it is based on an iterative approach. Namely, the authors at first calibrate each parameter according to the previous estimations of Fama and French and then generate a vector of returns along the time dimension. Afterwards, they get a matrix of asset returns by repeating the former step for all the assets. In particular, the rows and the columns represent, respectively, different returns and the time dimension. Then, it is possible to get the $\gamma$ and the $\lambda$ values of the different assets by just evaluating the mean of each asset. The last step is to estimate $a$ and $b$ by using the parameters found before in order to get the SDF.

In this paper, they use artificial data instead of real and replicate the process to obtain a vector of SDFs.

Not only does this work aim at describing the approach of the previous models, but it also compares their results. Therefore, the next table provides the statistics concerning the above mentioned models in the first three rows, with
the addition of the one described in this paper in the last row. In particular, the table shows the mean, the standard deviation, the minimum and the maximum value and also the first order autocorrelation of the estimated stochastic discount factor ($\rho_1$), the number of $m<0$ in the sample and the unconditional HJ distance.

<table>
<thead>
<tr>
<th>Unconditional models</th>
<th>E(m)</th>
<th>SD(m)</th>
<th>$\rho_1$(m)</th>
<th>min(m)</th>
<th>max(m)</th>
<th>n.(m&lt;0)</th>
<th>HJ uncon</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Farnsworth et all</td>
<td>0.9890</td>
<td>0.2250</td>
<td>0.2340</td>
<td>-0.2040</td>
<td>1.6800</td>
<td>2</td>
<td>0.235</td>
<td></td>
</tr>
<tr>
<td>Bessler et all</td>
<td>0.9905</td>
<td>0.4745</td>
<td>0.0550</td>
<td>-0.7183</td>
<td>2.1711</td>
<td>3</td>
<td>0.4909</td>
<td></td>
</tr>
<tr>
<td>Gutierrez &amp; Gaglianone</td>
<td>0.9967</td>
<td>0.3346</td>
<td>N/A</td>
<td>-0.5184</td>
<td>2.1010</td>
<td>N/A</td>
<td>0.4207</td>
<td></td>
</tr>
<tr>
<td>Empirical Discount Factor$^{10}$</td>
<td>0.9893</td>
<td>0.1286</td>
<td>0.1608</td>
<td>0.3864</td>
<td>1.4489</td>
<td>0</td>
<td>0.4318</td>
<td></td>
</tr>
</tbody>
</table>

Even if the discount factors are estimated using a different set of data, it is still possible to see how close the estimation conducted in this paper is to the other ones:

- the mean and the standard deviation are in line with the other estimations;
- the first order autocorrelation shows a small value;
- it is the only model to have a minimum value higher than zero, so that it does not have any negative discount factors;
- the HJ distance is in line with the other models.

Although the discount factors have been estimated with different sets of data, it is still possible to see some similarities between the estimates of this work and the others’:

- The mean and the standard deviation are coherent with the other estimations;
- The first order autocorrelation shows a small value as well;

$^{10}$ Discount Factor estimated in Section 2.3
• The HJ distance is consistent with the other models.

Even though the model applied in this work seems to compute an efficient discount factor easily, it is important to take into account its limits and features. In fact, it is based only on one assumption (the discount factor places a value of zero on the excess return) whereas the other methods explained involve more than one moment conditions.

Furthermore, the other papers, having more assumptions, have expanded the scope of the discount factor.


5. Conclusion

The methodology proposed to estimate the stochastic discount factor as well as the other models are all based on the Fama and French factors, these well represent the main characteristics of US firms. The discount factor calculated here could not be generally applied and consistent for all the assets in the financial market due to the shortcomings concerning the computational approach. In addition, the comparison between the models could be not completely reliable because of different data sets, which can negatively affect the results. Nevertheless, it is undeniable that the m estimated could be a good proxy of the real discount factor: as stated in the introduction, it could be used as a first approximation for the real discount factor to test if the price of an asset is correct, when dealing with of new items first appearing in the financial market.
Bibliography


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C. Gaunt, *Size and book to market effects and the Fama French three factor asset pricing model: evidence from the Australian stockmarket*, 2004


E. F. Fama and K.R. French, *Common risk factors in the returns on stocks and bonds*, 1993


### Annex I

#### Summary Statistics

<table>
<thead>
<tr>
<th>Financial Index</th>
<th>MEAN</th>
<th>STD</th>
<th>MIN</th>
<th>MAX</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>0.008098</td>
<td>0.002332</td>
<td>-0.217817</td>
<td>0.131767</td>
</tr>
<tr>
<td>STOXX50E</td>
<td>0.006455</td>
<td>0.003502</td>
<td>-0.249100</td>
<td>0.193791</td>
</tr>
<tr>
<td>Nasdaq 100</td>
<td>0.011510</td>
<td>0.004208</td>
<td>-0.485123</td>
<td>0.264310</td>
</tr>
<tr>
<td>Risk-Free Asset</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>One-month bill</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dec 2</td>
<td>0.010635</td>
<td>0.002452</td>
<td>-0.248100</td>
<td>0.134900</td>
</tr>
<tr>
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<td>0.002418</td>
<td>-0.257200</td>
<td>0.147800</td>
</tr>
<tr>
<td>Dec 4</td>
<td>0.011004</td>
<td>0.002552</td>
<td>-0.234800</td>
<td>0.151600</td>
</tr>
<tr>
<td>Dec 5</td>
<td>0.010878</td>
<td>0.002393</td>
<td>-0.237400</td>
<td>0.151200</td>
</tr>
<tr>
<td>Risk-Free Asset</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Book-to-Market</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dec 2</td>
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</tr>
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<td>Dec 3</td>
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<td>0.144100</td>
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<td>Dec 4</td>
<td>0.010878</td>
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<td>-0.237400</td>
<td>0.121200</td>
</tr>
<tr>
<td>CF/P</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dec 6</td>
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<td>0.002483</td>
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<td>0.147400</td>
</tr>
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clear all
clc
load('database');

%Regression 1 = b_1[YM]_t + b_2[HML]_t + b_3[SMB]_t + \eta_t

y=ones(368,1);
Fama_French=[HML SMB ExcRetMkt];
plot(HML,Time,SMB,Time,ExcRetMkt,Time);
Corr=corr(Fama_French);
betas=LinearModel.fit(Fama_French,y,'intercept',false);

% Estimation of the predicted Betas

predicted_HML=(mean(HML)/var(HML));
predicted_SMB=(mean(SMB)/var(SMB));
predicted_ExcRetMkt=(mean(ExcRetMkt)/var(ExcRetMkt));

% t-test predicted Beta = estimated Betas

t_testHML=(betas_HML-predicted_HML)/SE_betas_HML;
t_testSMB=(betas_SMB-predicted_SMB)/SE_betas_SMB;
t_testExcRetMkt=(betas_ExcRetMkt-predicted_ExcRetMkt)/SE_betas_ExcRetMkt;
p_value_HML=1-tcdf(t_testHML,367);
p_value_SMB=1-tcdf(t_testSMB,367);
p_value_ExcRetMkt=1-tcdf(t_testExcRetMkt,367);

% Calculation of the discount factor

discount_factor=ones(368,1);
for i=1:368
discount_factor(i,1)=1-betas_HML*HML(i,1)-betas_ExcRetMkt*ExcRetMkt(i,1)-betas_SMB*SMB(i,1);
end
plot(Time,discount_factor);

Aut_Corr=autocorr(discount_factor);
% Calculation of the HAC S.E.

function [ var_adj ] = ds_truncatedHAC( u, m )
if nargin < 2
  m = length(u)-1;
end
acf = ds_ccov(u, u, m)./ds_ccov(u, u, 0);
acf = acf((length(acf)-1)/2+2:end, 1);
adj = 1+2*sum(acf, 1);
var_adj = var(u)*adj;
end

function [ cv ] = ds_ccov( x, y, lag )
T = length(x);
mx = mean(x);
my = mean(y);
cv = zeros(lag*2+1,1);
i2 = 1;
if lag == 0
  cv = ((x-mx)'*(y-my))/T;
else
  for i=-lag:lag
    if i >= 0
      x1 = x(1:end-i);
y1 = y(1+i:end);
    elseif i < 0
      y1 = y(1:end+i);
x1 = x(1-i:end);
    end
    x1 = x1-mx;
y1 = y1-my;
cv(i2) = (x1'*y1)/(T-abs(i));
i2 = i2 + 1;
  end
end
end

% Calculation of the Pricing Errors C.I.
Confidence_value=1.96
CI_portfolio = mean(Price_portfolio) + Confidence_value * (var_adj/ square(length(CI_portfolio)))

% Estimation of the HJ DISTANCE and ERRORS

function [Errors,JC_distance]=hjdistance(REturns,Discount_Factor)
[T,Nr] = size(REturns);
if nargin < 6
nlags = floor(T^(1/3));
end
if isempty(Discount_Factor)
distdum = 0;
Discount_Factor = zeros(T,1);
else
distdum = 1;
mbar = mean(Discount_Factor);
end
iota = ones(T,1);
Rin = RETURNS;
Rbar = mean(REturns)';
dbar = mean(Discount_Factor);
Sigi = (REturns'*REturns/T - Rbar*Rbar')/eye(Nr);
Rdev = REturns - repmat(Rbar',T,1);
if distdum == 0
mlop = repmat(mbar',T,1) + Rdev*Sigi*(1-Rbar*mbar');
else
lambda = ((REturns'*REturns/T)
end
Errors = Discount_Factor.^2 - (Discount_Factor - RETURNS*lambda).^2 - 2*lambda'*ones(Nr,1);

Figure 5 – the ERRORS derived from the Matlab code