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A Review of Merton’s Portfolio Problem

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Anno Accademico 2015/2016

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CHAPTER 1

1.1 INTRODUCTION

Investing in financial markets is a diffused activity perpetrated by different entities. Example of investors are not only individuals willing to gain a return from their saving but also banks, investment funds and insurance company can be quoted among the number of investors.

Those entities are all naturally interested in gaining as much as possible from their investment operations but at the same time they are also concerned with the risks they have to face. It is common sense to consider investors as risk averse agents, that is to consider them as reluctant to allocate their wealth on assets bearing a high level of risk.

A logical aim of an investor is as a matter of fact the allocation of their wealth in a way that maximize their returns while also not trespassing a risk level limit. It is possible to mathematically replicate this investor behaviour through the theory of stochastic control and the maximization of expected utility.

The objects taken into consideration by investors for potential financial operations can generally be differentiated into two diverse categories: risky assets, which include all the assets whose future returns are not defined and are therefore uncertain, and risk-free assets, a group containing those assets whose returns are fixed and therefore bear no risk. Examples of risky asses could be stock, real estate, commodities, derivatives and other several could come to mind. Examples of risk-free assets are instead bonds and t-bills. Based on his degree of risk aversion, an investor could compose an investment portfolio as a mixture of both risky and risk-free assets in order to match the level of risk he is comfortable with. It is then natural to question what allocation strategy should be followed by said investor during the formulation of his portfolio and what will be the result that will maximize his utility. Such must have been the question Robert C. Merton posed himself while writing his 1969 paper entitled "Lifetime portfolio selection under uncertainty: The continuous-time case". In this paper the Nobel laureate formalized what has been later regarded as the "Merton's portfolio problem".

The most basic version of the problem establishes a setting where an investor has the limited choice of allocating his wealth between a risky asset and a risk-free on. Given some additional assumptions, Merton concluded that the best allocation strategy is to maintain a constant fraction of the wealth in the risky asset and consequentially to hold fixed the part of wealth invested in risk-free assets. This setting can later be expanded and several risky assets can be incorporated in the model. However, this variation doesn't affect the first conclusion as the optimal allocation still accounts for a fixed proportion between the wealth allocated between assets bearing and not bearing risk.

If considered from a more realistic point of view, the conclusion of Merton when solving his portfolio problem is based on assumptions that don't hold with the same strength in the real world. For example, the solution assumes that the dynamics of the risky assets follow a geometric Brownian motion which implies normally distributed log returns. However, in the case of real stock prices, this assumption is hardly ever held true. An analysis of the distribution of real stock returns shows in fact that the distributions tend to have heavier or fatter tails, due to the fact that price changes are normally higher than those a normal distribution would forecast.

Another drawback of the solution proposed by Merton is the fact that said conclusion is based on a continuous mathematical framework. The investor should therefore rebalance his portfolio at the same rate of change of the stock prices in order to rigorously follow the optimal strategy. However modern financial markets are extremely liquid and price changes happens almost constantly, an event that makes impossible to stick strictly to the optimal strategy. An additional factor that could, and should, be taken into account is the fact that portfolio rebalancing happens at the price of the transaction costs, making such a behaviour an expensive one.

The aim of this thesis is to introduce the models necessary to the application of a simple model of the Merton portfolio choice where transaction costs are not considered. The selection will be made using the returns of fourteen real indexes over the past years. Chapter 1 will cover the models underlying the returns estimation, Chapter 2 will introduce the theory behind the Portfolio selection problem while Chapter 3 will focus on the application of the models to the real data.

1.2 ESTIMATING STOCK RETURNS

In order to apply portfolio theory to the previously shown data set it is necessary first to introduce the process through which said returns will be estimated. Stochastic processes are generally regarded as the best approach to such a feature and are here introduced.

First it is necessary to give a definition of a stochastic process.

1.2.1 STOCHASTIC PROCESSES

A variable whose value changes over the course of time in an uncertain way is generally said to follow a stochastic process. Stochastic processes can be divided into discrete time processes and continuous time processes.

In a discrete time stochastic process, the value of the variable can assume different value only in determined points in time while in a continuous time stochastic process those value can change at any point.
Another possible division of stochastic processes may be the one between continuous variable processes and discrete variable ones. In a continuous variable stochastic process, the used variable can assume all the values contained in a certain range, differently from the discrete variable processes where the underlying variable can take only a definite number of values.
A more rigorous definition of a stochastic process is hereby introduced.

1.2.2 PROPERTIES OF A STOCHASTIC PROCESS

A stochastic process is a mathematical model for the occurrence of a random event at each moment after the starting time. The randomness of the occurrence is represented by the introduction of a measurable space $(\Omega, \mathcal{F})$, denominated sample space, where probability measure can be located. Therefore a stochastic process is a collection of random variable X, with

$$X = \{X_t; 0 \leq t < \infty\} \text{ on } (\Omega, \mathcal{F}), \tag{1.1}$$

which take values in a second measurable space $(S, \wp)$, called the state space. The index $t \in [0, \infty)$ of the random variables X_t admits an interpretation as time.
Given a fixed sample point $\omega \in \Omega$, the function

$$t \mapsto X_t(\omega); t \geq 0 \tag{1.2}$$

is the sample path of the process X associated with ω. This sample path, or sample trajectory allows for the creation of mathematical model which can be used on a random experiments whose outcome are observable over a continuous period of time.

Stochastic processes have several properties and those relevant to successively discussed notions are hereby introduced, but it is first necessary to disclose the notions of σ-fields and filtration.

- **σ-fields**

  A σ-field $\mathcal{F}$ of subsets of X is a collection $\mathcal{F}$ of subsets of X satisfying the following conditions:

  a) $\emptyset \in \mathcal{F}$
  b) if $B \in \mathcal{F}$ then its complement B^c is also in $\mathcal{F}$
  c) if $B_1, B_2, \ldots$ is a countable collection of sets in $\mathcal{F}$ then their union is $U_{n=1}^{\infty} B_n$

- **Filtration**

  Given that $(F_t, t \geq 0)$ is a collection of σ-fields in the same probability space $(\Omega, \mathcal{F}, P)$ with $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \geq 0$, said collection is called a filtration if

$$\mathcal{F}_s \subseteq \mathcal{F}_t, \forall\, 0 \leq s \leq t \tag{1.3}$$

  In case the index t is discrete, the filtration $(\mathcal{F}_n, n = 0,1, \ldots)$ is a sequence of σ-fields on Ω with $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for all $n \geq 0$.

  The reason for the inclusion of those notions in the study of stochastic process is related to the temporal feature of the stochastic processes analysed.

  In order to include different time periods in the process of choice it is possible to equip the sample space $(\Omega, \mathcal{F})$ with a filtration, consisting in a nondecreasing family $\{\mathcal{F}_t; t \geq 0\}$ of sub-σ-fields of $\mathcal{F}$: $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s < t < \infty$. In such setting, it also holds that

$$\mathcal{F}_\infty = \sigma(\cup_{t\geq 0} \mathcal{F}_t). \tag{1.4}$$

Having introduced those two relevant notions, it is now possible to describe some relevant properties of a stochastic process.

- **Measurable processes**

  The stochastic process X is called measurable if, for every $A \in \mathcal{B}(\mathbb{R}^d)$,the set $\{(t,\omega);\ X_t(\omega) \in A\}$ belongs to the product σ-field $\mathcal{B}([0,\infty)) \times \mathcal{F}$; it means that if the mapping is such that

$$(t,\omega) \mapsto X_t(\omega): ([0,\infty)\times\Omega, \mathcal{B}([0,\infty))\times\mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \tag{1.5}$$

  then the process is measurable.

- **Adapted processes**

  A stochastic process X is defined as adapted to the filtration $\{\mathcal{F}_t\}$, if, for each $t \geq 0$, X_t is an $\mathcal{F}_t$-measurable random variable. Values of $X_t(\omega)$ can only be determined by the information available at time t.

A stochastic process useful in the determination of the stocks returns is the Markov process.

1.2.3 MARKOV PROCESS

A Markov process is a typology of stochastic process where the prediction of the future is based only on the value presently taken by the considered variable. The past history of the variable and the path through which the present situation has been generated are therefore both considered as irrelevant.

Stock prices are generally assumed to follow a Markov process. This general idea implies that the only relevant piece of information regarding a stock is the current price since the past cannot affect the future of the stock itself.

Prediction for the future movements of the price are uncertain and must therefore be expressed in terms of probability distributions. Those distributions are at any time completely independent from the path previously followed by the stock.

In order to have a mathematical definition of a Markov process, it is necessary to take into consideration a positive integer d and a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. An adapted, d-dimensional process $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ on some probability space $(\Omega, \mathcal{F}, P^{\mu})$ is therefore said to be a Markov process with intial distribution μ if the following conditions are verified:

- $P^{\mu}[X_0 \in \Gamma] = \mu(\Gamma), \forall\, \Gamma \in \mathcal{B}(\mathbb{R}^d)$;
- $P^{\mu}(X_{t+s} \in \Gamma | \mathcal{F}_s) = P^{\mu}(X_{t+s} \in \Gamma | X_s), for\ s, t \geq 0\ and\ \Gamma \in \mathcal{B}(\mathbb{R}^d)$.

A particular type of Markov stochastic process in a continuous-time setting is the Wiener Process. In order to explain what a Wiener process is, it is necessary to first introduce the concept of Brownian motion, which is itself a case of a Markov process.

1.2.4 BROWNIAN MOTION

Brownian movement is originally the name given to the irregular movement of pollen suspended in water which was observed by the botanist Robert Brown in 1828.

Such random motion, which it has been now attributed to the buffeting of the pollen by water molecules, results in the dispersal of the pollen in the water. However, this originally botanic observation has been expanded outside its original field of study.

In mathematical terms, a one-dimensional Brownian motion is a continuous and adapted process $B = \{B_t, \mathcal{F}_t; 0 \leq t < \infty\}$, defined on a probability space $(\Omega, \mathcal{F}, P)$. Properties of that process are that $B_0 = 0$ and that the increment $B_t - B_s$, given $0 \leq s < t$, is independent of $\mathcal{F}_s$ and is normally distributed with mean zero and variance $t - s$.

A process B is said to have stationary and independent increments if such a process is a Brownian motion and $0 = t_0 < t_1 < \cdots < t_n < \infty$. In that case the increments $\left\{B_{t_j} - B_{t_{j-1}}\right\}_{j=1}^{n}$ are independent and the distribution of $B_{t_j} - B_{t_{j-1}}$ depends on t_j and t_{j-1} only through the difference $t_j - t_{j-1}$, therefore it is normal with mean equal zero and variance equal to $t_j - t_{j-1}$.

While the filtration $\{\mathcal{F}_t\}$ is part of the definition of Brownian motion, it is possible to have a Brownian motion when the filtration is different from $\{\mathcal{F}_t\}$. Specifically, if no filtration is given for a process $\{B_t; 0 \leq t < \infty\}$ but it is known that $B_t = B_t - B_0$ is normal with mean zero and variance t and that B has stationary, independent increments, it is possible to define $\{B_t, \mathcal{F}_t^B; 0 \leq t < \infty\}$ as a Brownian motion.

Another case could be that $\{\mathcal{F}_t\}$ is a larger filtration, meaning that $\mathcal{F}_t^B \subseteq \mathcal{F}_t$ for $t \geq 0$. In that case, if $B_t - B_s$ is independent on $\mathcal{F}_s$ whenever $0 \leq s < t$, it is possible to consider $\{B_t, \mathcal{F}_t; 0 \leq t < \infty\}$ as a Brownian motion.

1.2.4a WIENER PROCESS

The Wiener Process, is another particular type of Markov Stochastic process and a case of standard, one-dimensional Brownian motion. A Wiener process has a mean change of zero and a variance rate of 1 per year. More formally, a variable z is said to follow a Wiener process if the following two properties hold true:

- **Property 1**

  The change Δx during a small period of time Δt is:

  $$\Delta z = \varepsilon\sqrt{\Delta t}$$

  where ε has a standard normal distribution $\phi(0,1)$.

- **Property 2**

  The values of Δz for any two different short intervals of time, Δt, are independent.

From the first property it is also possible to state that Δz has a normal distribution with the following characteristics:

  - Mean of Δz = 0
  - Standard Deviation of Δz = $\sqrt{\Delta t}$
  - Variance of Δz = Δt

The second property implies instead that x follows a Markov process.
It is now possible to take into consideration the change of value assumed by z over a relatively long period of time T. This change of value con be defined as $z(t) - z(0)$.

Another way to interpret this variation is as the sum of the changes in x during N small time intervals of length Δt, where N = $\frac{T}{\Delta t}$.
It is therefore possible to establish the following relationship:

$$z(t) - z(0) = \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t} \tag{1.6}$$

where the ϵ_i (i=1,2,....N) are distributed $\phi(0,1)$. Those various ϵ_i are independent of each other from the second property of the Wiener process. It is therefore possible to conclude that $z(t) - z(0)$ is normally distributed with

- mean of $z(t) - z(0)$=0
- variance of $z(t) - z(0)$=NΔt=T
- standard deviation of $z(t) - z(0)$=$\sqrt{T}$.

1.2.4b GENERALIZED WIENER PROCESS

The mean change per unit time for a stochastic process is generally defined as the drift rate and the variance per time unit is known as the variance rate. The basic Wiener process previously introduced, shortly referred as dz, has a drift rate of zero and a variance rate of 1.
The drift rate of zero indicated that the expected value of z at any future time is identical to its present value. The variance rate of one instead suggests that the variance of the change in z in a time interval of duration T equals T itself. A generalized Wiener process for a variable x can be therefore defined in terms of dz as

$$dx = adt + bdz \tag{1.7}$$

where a and b are constants.

This definition is made of two essential components: the first is the term "adt" which implies that x has an expected drift rate of a per unit of time. The second term "bdz" can be instead regarded as an addition of noise and variability to the path followed by x. The quantity and magnitude of this noise is b times a Wiener process.
Since a Wiener process has a standard deviation of 1, it follows that b times a Wiener process has a standard deviation of b. If this second term wasn't to be added, the equation would have been $dx = adt$ which would have lead to $a = \frac{dx}{dt}$. The integration of this relationship with respect to time yields the following result: $x = x_0 + at$, where x_0 indicates the value of x at time 0.
In a period of time T, the variable x would then increase by an amount aT, a result which would leave out lots of possibilities which are instead integrated by the addition of the second component "bdz".
As a result, it is possible to state that in a small time interval Δt, the change Δx in the value of x is:

$$\Delta x = a\Delta t + b\varepsilon\sqrt{\Delta t} \tag{1.8}$$

As it was previously mentioned, ε has a normal standard distribution, thus it is possible to state that also Δx has a normal distribution with:

- Mean of $\Delta x = a\Delta t$
- Standard deviation of $\Delta x = b\sqrt{\Delta t}$
- Variance of $\Delta x = b^2\Delta t$

With the same reasoning it is also possible to determine that the change of value in x in any time interval T is normally distributed with:

- Mean of change in $x = aT$

- Standard deviation of change in $x \ = b\sqrt{T}$
- Variance of change in $x = b^2T$.

To sum up, the generalized Wiener process has an expected drift rate of a and a variance rate of b².

1.2.4c ITO PROCESS

The Ito process is another type of stochastic process based on the generalized Wiener process. It is in fact a generalized Wiener in which the parameters a and b are functions of the value of the considered variable x and of the time variable t. An Ito process is algebraically expressed as:

$$dx(t) = a(x(t))dt + b(x(t))dz(t) \tag{1.9}$$

Bothe the expected drift rate and the variance rate of an Ito process can change value over time. If the time interval taken into consideration is small, that is in an interval between t and $t + \Delta t$, then the variable value changes from x to $x + \Delta x$, where:

$$x(t + \Delta t) - x(t) \approx a(x(t))\Delta t + b(x(t))\varepsilon(t)\sqrt{\Delta t} \tag{1.10}$$

A small approximation is assumed in that relationship. The approximation consists in the fact that the mean and the standard deviation of x, conditional to $x(t)$, remain constant, respectively equal to $a(x(t))\,\Delta t$ and $b(x(t))\sqrt{\Delta t}$, during the time interval from t to $t + \Delta t$.

1.2.5 STOCHASTIC PROCESSES FOR STOCK PRICES

1.2.5a HISTORY OF STOCHASTIC PROCESSES FOR STOCK PRICES

The birth of the relationship between mathematics and financial model must be tracked back to Louis Bachelier's 1900 dissertation on the theory of speculation in the Paris markets. That work gave life to both the continuous time mathematics of stochastic processes and the continuous time economics of option pricing.

During his analysis of option pricing, Bachelier elaborated two different derivations of the partial differential equation for the probability density for the Wiener process. In one derivation, he worked out what is now known as the Chapman-Kolmogorov convolution probability integral. Bachelier exploited the ideas of the Central Limit Theorem and, realizing that market noise should be without memory, he reasoned that increments of stock prices should be independent and normally distributed. Combining this reasoning with the Markov property, he was able to connect Brownian motion with the heat equation in order to model the market noise. He was also able to define other processes related to the Brownian motion, in particular he computed the maximum change during a time interval for a one-dimensional Brownian motion. After Bachelier's crucial works, it took more than 60 years before a new breakthrough could happen in the field of stochastic processes for stock prices.

It was Paul Samuelson in 1965 that introduced the Geometric Brownian Motion as a good model for stock price movements. In his research Samuelson tried to improve Bachelier's model and to fix its inability to ensure positive prices for the stocks, a capacity that the Brownian motion had instead.

Samuelson contributions are mainly contained in two papers. In those, he gave, 65 years after Bachelier had stated it, his economic arguments that prices must fluctuate randomly. He postulated the idea that discounted futures prices follow a martingale and went on to prove that futures prices changes were uncorrelated across time. Moreover, his proposition could be also extended to arbitrary functions of the sport price therefore allowing for an application on options.

Another turning point was the publication 1973 if the Black-Scholes model for option pricing.

The two economists Fischer Black and Myron Scholes deduced an equation that provided the first strictly quantitative model for calculating the prices of options. The basic insight underlying the Black-Scholes model is that a dynamic portfolio trading strategy in the stock can replicate the returns from an option on that stock. That action is defined as "hedging an option" and it is the most important idea underlying the Black-Scholes approach.

Conceptual breakthroughs in finance theory in the 1980s were fewer and less crucial than in the 1960s and 1970s but the resources employed in the field were more than that used in the previous decades. The development of more powerful computing machines and the growth of the number of sophisticated mathematical models for financial practices happened almost simultaneously, allowing for more effective models than those used in the past.

1.2.5b MODEL DERIVATION

The generalized Wiener process fails to capture some key aspect of stock prices due to its constant expected drift rate and its constant variance rate. The biggest shortfall is the incapacity of the model to capture the fact that the percentage return required by investors from a stock is independent of the stock's price.

The assumption of constant expect drift rate must therefore be removed and replaced by the assumption that the expected return, represented by the expected drift divided by the stock price, is constant.

In this assumption, if S is used to represent the stock price at time t and μ is used to represent the expected rate of return on the stock expressed in decimal form, it is possible to state that the expected drift rate in S is assumed to be $\mu S\Delta t$ and that represents the increase in S during a short time period Δt.

Assuming a constant stock price volatility of zero, the presented model implies that :

$$\Delta S = \mu S \Delta t \tag{1.11}$$

which, as Δt→ 0, becomes:

$$dS = \mu S dt \tag{1.12}$$

or also

$$\frac{dS}{S} = \mu \, dt \tag{1.13}$$

Integrating this relationship between time 0 and time T it results that:

$$S_T = S_0 e^{\mu T} \tag{1.14}$$

Where S_0 and S_T are the stock price at time 0 and time T. This equation indicates that when the effect of the variance rate is absent, the stock price grows at a continuously compounded rate of μ.

It is however impossible in reality to experience an absence of volatility in the growth of stock prices. A reasonable assumption that can be made regarding volatility of stock prices is that the variability of the percentage returns in a short period of time Δt is the same regardless of the stock price. The aim of the assumption is to represent the fact that an investor is indifferently uncertain about the future of stock independently from the price of the stock itself. Moreover, the assumption suggests that the standard deviation of the change in a short period of time Δt should be proportional to the stock price and allows for the creation of the following model:

$$dS = \mu S dt + \sigma S dz \tag{1.15}$$

or

$$\frac{dS}{S} = \mu dt + \sigma dz. \tag{1.16}$$

In that equation the variable μ represents the expect rate of return of the stock and the variable σ represents the volatility of the stock prices.

This model of stock price behaviours is mostly known as geometric Brownian motion. It is also possible to elaborate a discrete time version of the model. In that version it happens that

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \varepsilon \sqrt{\Delta t} \tag{1.17}$$

or

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t} \tag{1.18}$$

The variable ΔS is the variation in the stock price S during a small time interval Δt while ε is a parameter with a standard normal distribution. The parameters μ and σ are again the expected rate of return per unit of time and the volatility of the stock price.

The right side of this equation is of particular interest. The term $\mu\Delta t$ is the one representing the expected value of the return on a short amount of time Δt while the term $\sigma\varepsilon\sqrt{\Delta t}$ stands for the stochastic component of those returns. The variance of that component, and therefore of the returns as a whole, is $\sigma^2\Delta t$.In the end it is possible to state that $\frac{\Delta S}{S}$ is normally distributed with mean $\mu\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$, or in another notation:

$$\frac{\Delta S}{S} \sim \phi(\mu\Delta t, \sigma^2\Delta t) \tag{1.19}$$

1.2.5c ITO'S LEMMA

The Ito's lemma, named after the mathematician K. Ito, is an in important process in the understanding of the behaviour of functions of stochastic variable.

Assuming that a variable x follows the Ito process

$$dx = a(x(t))dt + b(x(t))dz \tag{1.20}$$

where dz is a Wiener process and a and b are functions of x and t, it is possible to define the drift rate of the variable x as a and its variance rate as b^2. Ito's lemma shows that a function G of x and t follows the process

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}bdz \tag{1.21}$$

where dz indicates the previously mentioned Ito process. G also follows an Ito process and the drift rate is defined by

$$\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2 \tag{1.22}$$

and the variance rate is

$$\left(\frac{\partial G}{\partial x}\right)^2 b^2 \tag{1.23}$$

It is now possible to connect those results with the previous conclusion that

$$dS = \mu S dt + \sigma S dz \tag{1.15}$$

can be considered a valuable model of stock price movements with μ and σ constant.
From Ito's lemma it follows that the process undergone by a function G of S and t is

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial G}{\partial S}\sigma S dz \tag{1.24}$$

It is relevant to highlight how both S and G are affected by the underlying source of uncertainty dz.

1.2.5d LOG-NORMAL PROPERTY

It is now possible to use Ito's lemma to derive the process followed by lnS, which is the natural log of the stock price, when S follows the previously introduced equation

$$dS = \mu S dt + \sigma S dz. \tag{1.15}$$

First it is necessary to define $ln\ S$ as G. After that it is possible to state that:

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \frac{\partial G}{\partial t} = 0. \tag{1.25}$$

Ito's lemma can now be applied and the process followed by G results to be:

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz. \tag{1.26}$$

Since μ and σ are constant, the equation essentially states that the equation $G = lnS$ follows a generalized Wiener process.
The constant drift rate is $\mu - \frac{\sigma^2}{2}$ and the constant variance rate is σ^2.
Therefore the variation in lnS from time 0 to time T is normally distributed with a mean of $\left(\mu - \frac{\sigma^2}{2}\right) T$ and a variance of $\sigma^2 T$.

In a more formal way it means that

$$lnS_T - lnS_0 \sim \phi \left[\left(\mu - \frac{\sigma^2}{2}\right) T, \sigma^2 T\right] \tag{1.27}$$

or also

$$lnS_t \sim \phi \left[lnS_0 + \left(\mu - \frac{\sigma^2}{2}\right) T, \sigma^2 T\right]. \tag{1.28}$$

This equation shows how lnS_T is normally distributed.
A variable has a lognormal distribution if the natural logarithm of the variable is normally distributed as it is in this case. This model of stock price behaviour implies that stock's price at time T, given the current price, is log-normally distributed.
The standard deviation of the logarithm is $\sigma\sqrt{T}$ which implies a growth in the standard deviation as the time span of interest grows larger.

1.3 TEST OF RETURN ESTIMATION

It is now possible to test the soundness of the previous assumptions. As a matter of fact, the whole model relies on the condition that stock prices are log-normally distributed and that therefore the returns are normally distributed. It is possible to verify if those conditions hold also in reality by using statistical tests on a selected data set.

1.3.1 DATA SET

The data set taken into consideration for the application of the theories presented is composed by the variations of fourteen different indexes over the course of a 16 years' time span. The data are analysed from the 1st of January 2000 to the 4th of March 2016.
The tables below contain a summary of the most relevant characteristics of the daily returns of those samples over the considered time length.

INDEX	*SPXINDEX*	*CCMP Index*	*DAX Index*	*SASEIDX Index*
N OF OBSERVATIONS	4065	4065	4110	4357
MEAN	0.01%	0.01%	0.02%	0.03%
STANDARD DEVIATION	1.27%	1.67%	1.55%	1.46%
KURTOSIS	11.08	8.67	7.37	12.77
SKEWNESS	0.01	0.22	0.12	-0.65
MEDIAN	0.05%	0.08%	0.07%	0.09%
MAX RETURN	11.58%	14.17%	11.40%	9.85%
MIN RETURN	-9.03%	-9.67%	-8.49%	-9.81%

INDEX	*NKY Index*	*CAC Index*	*UKX Index*	*FTSE MIB Index*
N OF OBSERVATIONS	3969	4134	4086	4102
MEAN	0.01%	0.05%	0.006%	-0.007%
STANDARD DEVIATION	1.56%	1.50%	1.23%	1.56%
KURTOSIS	8.95	7.78	9.03	7.24
SKEWNESS	-0.18	0.15	-0.003	0.05
MEDIAN	0.03%	0.03%	0.03%	0.04%
MAX RETURN	14.15%	11.18%	9.84%	11.49%
MIN RETURN	-11.41%	-9.04%	-8.85%	-8.24%

INDEX	*INDEX CF Index*	*SHCOMP Index*	*SMI Index*
N OF OBSERVATIONS	4002	3909	4065
MEAN	0.08%	0.03%	0.01%
STANDARD DEVIATION	2.17%	1.66%	1.22%
KURTOSIS	18.90	7.29	9.80
SKEWNESS	0.30	-0.18	-0.002
MEDIAN	0.11%	0.06%	0.05%
MAX RETURN	28.69%	9.86%	11.39%
MIN RETURN	-18.66%	-8.84%	-8.67%

INDEX	*AS51 Index*	*SPTSX Index*	*MXBR Index*
N OF OBSERVATIONS	4091	4062	4215
MEAN	0.01%	0.01%	0.03%
STANDARD DEVIATION	1.02%	1.15%	2.21%
KURTOSIS	8.19	11.51	9.57

SKEWNESS	-0.36	-0.45	0.07
MEDIAN	0.04%	0.06%	0.05%
MAX RETURN	5.79%	9.82%	18.08%
MIN RETURN	-8.34%	-9.32%	-16.74%

It is already possible to spot some of the most essential features of the indexes that will be used for the portfolio composition.

The most glaring one is the low mean of the daily changes, always close to zero, and the general symmetry of the data sets, represented by the low skewness value.

Moreover, all the indexes have a high kurtosis value, a predictable feature, indicating that the tails of the returns distribution are fatter than those of a normal distribution.

The most volatile indexes of the bunch are the Russian Index, the INDEX CF, and the Brazilian index, MXBR, whose daily return volatility is higher than the 2%.

Still, the indexes seem to fit the logical expectation an investor may have when dealing with assets replicating an index performance, that is a non excessive daily volatility and expected returns.

Having introduced the data set, it is now possible to compare how the real returns would fare when compared with a normal distribution.

In order to perform such a comparison, it is first necessary to present two useful statistical test for the normality of the returns. Those are the Kolmogorov-Smirnov test and Jarque-Bera test.

1.3.2 KOLMOGOROV-SMIRNOV TEST

The Kolmogorov-Smirnov test is a nonparametric test that can be used to compare a sample with a reference probability distribution or to compare two sample.
The two sample test may be used to test whether the two underlying one-dimensional probability distributions differ while the one-sample test will indicate whether the sample has a distribution similar to a standard normal one or not..

- **Kolmogorov-Smirnov one-sample test**
  Assuming that:

  - $x_1,...,x_m$ be observations on i.i.d. r.vs $X_1,...,X_m$ with a c.d.f. F_1,

  The aim is to test the null hypothesis:

$$H_0 : F_1(x) = F_2(x), \text{ for all x} \tag{1.29}$$

  where F_0 is a known c.d.f.
  The Kolmogorov-Smirnov test statistic D_n is defined by

$$D_n = \sup_{x \in R} \left| \hat{F}(x) - F_0(x) \right|, \tag{1.30}$$

  In that case $\hat{F}$ is an empirical cumulative distribution which can be defined as

$$\hat{F}(x) = \frac{\#(i : x_i \leq x)}{n} \tag{1.31}$$

It can be noticed that the supremum from equation (1.30) can occur only at one of the observed values x_i or to the left of x_i.

It is now possible to perform a Kolmogorov-Smirnov test and verify whether the log-normal assumption holds true for the analysed indexes by testing whether the returns appear to follow a standard normal distribution or not.
The test will be made at a significance level of 0.05, or 5% and it will be performed using the MATLAB software.
The results of the test are hereby reported, where "REJECT" will indicate the rejection of the null hypothesis of the sample following a standard normal distribution.

INDEX	**RESULT OF K-S TEST**$(\boldsymbol{\alpha = 0.05})$
SPXINDEX	REJECT
CCMP Index	REJECT
DAX Index	REJECT
SASEIDX Index	REJECT
NKY Index	REJECT
CAC Index	REJECT
UKX Index	REJECT
FTSE MIB Index	REJECT
INDEX CF Index	REJECT
SHCOMP Index	REJECT
SMI Index	REJECT
AS51 Index	REJECT
SPTSX Index	REJECT
MXBR Index	REJECT

Matlab code available in the Matlab Appendix.

As it can be seen the indexes' returns distribution seems not to be corresponding to a standard normal, indicating that the assumption that prices follow a log-normal distribution may be an incorrect one. In order to avoid the possibility of a Kolmogorov-Smirnov test failure in correctly assessing the returns distribution, it is possible to introduce and perform another normality test as the Jarque-Bera test is.

1.3.3 JARQUE-BERA TEST

The Jarque-Bera test is a statistical test capable of assessing whether a sample data has a distribution approximately normal. The test tries to determine the distribution by matching the sample skewness and kurtosis with those of a normally distributed sample.
The sample skewness is defined as:

$$S = \frac{1}{n} \cdot \frac{\sum_{i=1}^{n}(x_i - \bar{x})^3}{(\tilde{\sigma}^2)^{3/2}} \tag{1.32}$$

where

$$\tilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \tag{1.33}$$

Skewness provides a measure of how symmetric the observations are around the mean. For a distribution skewed to the right, the skewness has a positive value while the opposite holds in case the observations are skewed to the left.
The sample kurtosis is instead defined as:

$$K = \frac{1}{n} \cdot \frac{\sum_{i=1}^{n}(x_i - \bar{x})^4}{(\tilde{\sigma}^2)^2} \qquad (1.34)$$

Kurtosis gives a measure of the thickness in the tails of a probability density function. For a normal distribution the kurtosis is 3.

Additionally, the Excess Kurtosis is defined as:

$$EK = K - 3 \qquad (1.35)$$

It follows that, for a normal distribution, the excess kurtosis is 0. A fat-tailed or thick-tailed distribution has a value for kurtosis that exceeds 3. That is, excess kurtosis is positive. This is called leptokurtosis.

It is now possible to present the Jarque-Bera test for normality.

The test is set by formulating the null hypothesis:

H_0: skewness and excess kurtosis are zero

against the alternative hypothesis:

H_1:non-normal distribution.

The Jarque-Bera test statistic is:

$$JB = n \cdot \left[\frac{S^2}{6} + \frac{(EK)^2}{24}\right] \qquad (1.36)$$

This test statistic can be compared with a chi-square distribution with 2 degrees of freedom. The null hypothesis of normality is rejected if the calculated test statistic exceeds a critical value from the $\chi^2_{(2)}$ distributions.

Critical values can be chosen by stating the preferred significance level for the hypothesis test. The significance level, generally indicated by the greek letter α, represents the probability of rejecting the null hypothesis when it is indeed true.

1.3.4 APPLICATION OF THE J-B TEST

It is now possible to apply a Jarque-Bera test in order to verify whether the returns of the indexes are normally distributed. The significance level will again be of the five percent and the calculation will be performed in MATLAB.

The results are hereby summarized by the following table:

INDEX	**RESULT OF J-B TEST($\alpha = 0.05$)**
SPXINDEX	REJECT
CCMP Index	REJECT
DAX Index	REJECT
SASEIDX Index	REJECT
NKY Index	REJECT
CAC Index	REJECT
UKX Index	REJECT
FTSE MIB Index	REJECT
INDEX CF Index	REJECT
SHCOMP Index	REJECT
SMI Index	REJECT
AS51 Index	REJECT
SPTSX Index	REJECT
MXBR Index	REJECT

Matlab code available in the Matlab Appendix.

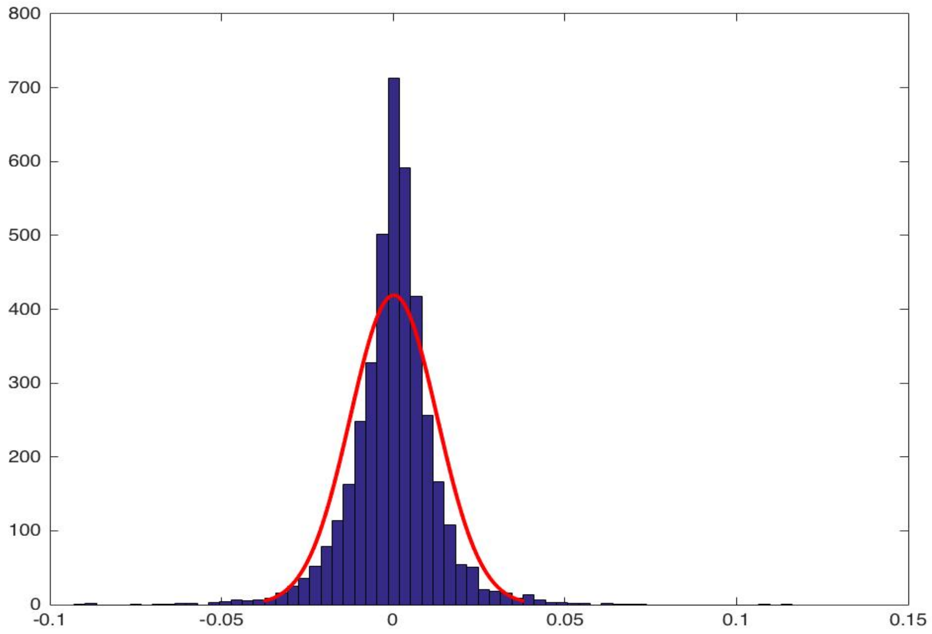
Again the statistical test refuses the hypothesis that the distribution of the indexes returns is similar to a normal distribution therefore implying that the distribution of the prices cannot be considered a log-normal one. It is possible to highlight some of the main reasons behind the failure of the log-normal model.

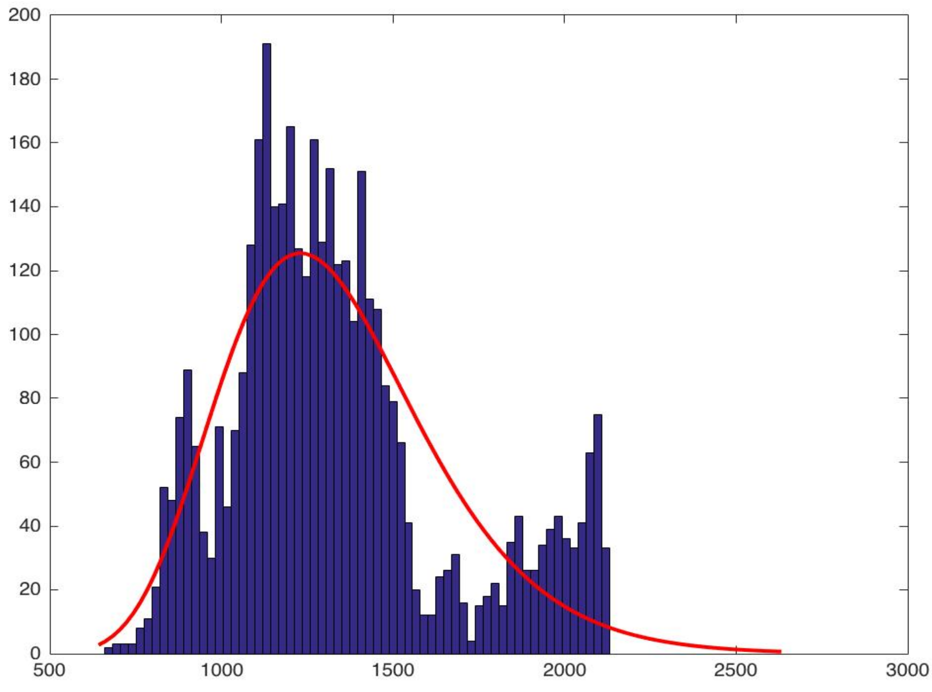
1.3.5 FAILURES OF THE LOG-NORMAL MODEL

The failure of the log-normal model in correctly predicting the distribution of the stock returns is a widely accepted notion nowadays. There are, as a matter of facts, a number of properties empirically experienced that are hardly replicable by a stochastic process. It is possible to list some of such properties in order to provide an overview of the main characteristics of returns that are hard to incorporate in a stochastic model.

- **Heavy Tails:** the distribution of returns generally displays an heavy tail with positive excess kurtosis. Such a behaviour is not correctly reproduced by a normal distribution, moreover the precise form of the tails is hard to determine at all.
- **Absence of autocorrelations:** linear autocorrelations of assets returns are mostly insignificant, exception made for small intraday time windows where instead some forms of autocorrelation can be observed.
- **Gain/Loss asymmetry:** drawdowns in stock prices and indexes values tend to be larger than upward movements therefore causing asymmetric movements in the two directions.
- **Volatility clustering:** volatility clustering is a term used to indicate the fact that large changes in stock returns tend to be followed by large changes, of either sign, while small changes tend to be followed by small changes. This situation has also a quantitative reflection. While correlation is mostly absent among returns, that is not true for absolute returns. Absolute returns, defined as $|r_t(\Delta)|$, display a positive and significant autocorrelation function.

The main result of those events consists in the impossibility of replicating the movement of the stock prices by a continuous model. Models assuming that prices move in a continuous manner neglect the abrupt movements in which most of the risk is concentrated.





The pictures represent the histograms of the returns and the prices of the index SPX plotted against a normal distribution and a lognormal distribution respectively. As it can be seen the distributions don't seem to fit well the data.

CHAPTER 2

2.1 PORTFOLIO THEORY

Portfolio theory deals with the problem of building a desirable investment out of a collection of assets. The objective is to construct a portfolio which features satisfy the demand of the financial agent who is considering the investment itself.

The management of a portfolio is a fundamental aspect in modern economics and finance. The first attempt to solving a portfolio problem was the mean-variance approach introduced by H. Markowitz in a one-period decision model. The simplicity of the approach, caused by the static nature of the problem it tackles, brings many drawbacks since the investor's job is limited to the selection of the initial portfolio.

As a matter of fact, after the initial selection procedure, the investor become a passive agent as he can only watch the prices fluctuate without any intervention possibility. It is however relevant to introduce the Markowitz model before the discussion of a more advanced intertemporal model.

2.2 MARKOWITZ PORTFOLIO

The mean-variance paradigm of Markowitz is definitely the most common formulation of the portfolio choice problems. It takes into consideration N risky assets with a random return vector R_{t+1} and a singular riskfree assets with certain returns R^f_t. The excess returns, indicated as r_{t+1}, are made by the difference between R_{t+1} and R^f_t and their conditional means and covariance matrix are indicated by μ_t and $\sum_t$ respectively. In addition, the excess returns are also considered i.i.d. with constant moments.

It is now first necessary to temporarily eliminate the risk free asset from the problem environment. That is, it is necessary to suppose an investor which can only allocate his wealth to the N risky securities available. In the absence of a risk-free asset , the mean-variance problem is to choose the vector of portfolio weights x, which indicate the investor's relative allocations of wealth to the various risky assets, in order to minimize the variance of the resulting portfolio return $R_{p,t+1} = x'R_{t+1}$ while also generating the predetermined goal of expected return $R^f_t+\bar{\mu}$. That is as saying:

$$\min_x \mathrm{var}[R_{\mathrm{p,t+1}}] = x'\textstyle\sum x \tag{2.1}$$

Subject to

$$E\left[R_{p,t+1}\right] = x'(R^f + \mu) = (R^f + \bar{\mu}) \text{ and } \textstyle\sum_{i=1}^{N} x_i = 1 \tag{2.2}$$

The first constraint has the role of ensuring that the expect return of the portfolio is equal to the desired target while the second constraint ensures that all wealth is invested in the risky assets. After setting up the Lagrangian and solving the resulting first-order conditions, the optimal portfolio weights turn out to be:

$$x^* = \Lambda_1 + \Lambda_2\bar{\mu} \tag{2.3}$$

with

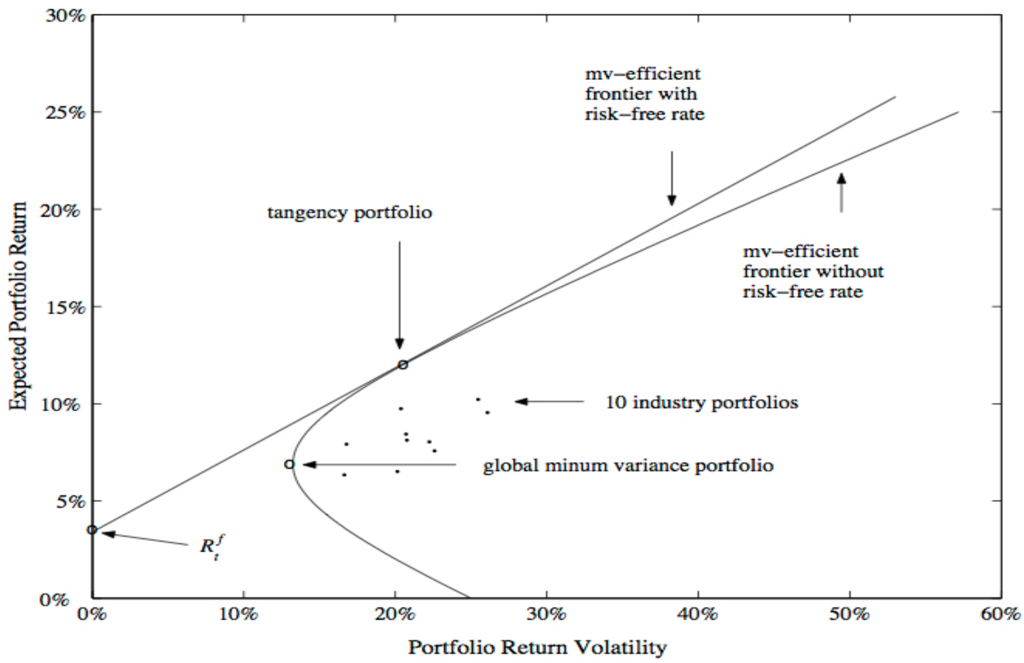
$$\Lambda_1 = \frac{1}{D}\left[B(\textstyle\sum_{.}^{-1} \iota) - A(\textstyle\sum_{.}^{-1} \mu)\right] \text{ and } \Lambda_2 = \frac{1}{D}\left[C(\textstyle\sum_{.}^{-1} \mu) - A(\textstyle\sum_{.}^{-1} \iota)\right] \tag{2.4}$$

where ι denotes an appropriately sized vector of ones and where A $=\iota' \sum^{-1} \mu$,B=$\mu' \sum^{-1} \mu$, C= $\iota' \sum^{-1} \iota$ and D= BC-A^2.

The variance of this portfolio, which is the lowest variance possible given the fixed expected return, is equal to x*'∑x*.

The Markowitz paradigm has the virtue of highlighting two important economic insights. The first is the effect of diversification, that is the possibility of packaging imperfectly correlated assets into portfolio with better expected return-risk characteristics.

The second one is the fact that once a portfolio is fully diversified, it is possible to achieve higher expected returns only at a price of the burden of a higher risk, that is by adopting more extreme allocations of the portfolio weights. These two relevant insights are also graphically visible.



The mean-variance frontier is here plotted as a hyperbola where every point represent the minimized portfolio return volatility for a predetermined expected portfolio returns. As it can be seen, portfolios of the assets generate better risk-return performances and it is necessary to move to the right, that is to increase the volatility, in order to obtain better returns.

It is now possible to reintroduce the risk-free asset in the model. Introducing the risk-free assets introduces also the possibility of unlimited risk-free borrowing and lending at the risk-free rate R^f_t.

Due to the addition of the risk-free asset, any portfolio on the mean-variance frontier generated by the risky asset and represented by the hyperbola can now be combined with the risk-free asset in order to generate a new expected return-risk profile.

The newly created profile lies on a straight line originating from the point indicated a risk-free portfolio, that is a portfolio fully composed of risk-free assets, and tangent to the efficient frontier. The optimal combination of the risky frontier portfolios with risk-free borrowing and lending is the one maximizing the Sharpe ratio of the whole portfolio. The Sharpe ratio is defined as:

$$\frac{E[r_{p,t+1}]}{std[r_{p,t+1}]} \tag{2.5}$$

and coincide with the slope of the line starting from the risk-free asset and tangent to the efficient frontier of only risky assets.

That line therefore represents the efficient frontier when borrowing and lending is allowed and is composed by combinations of the risk-free asset and the portfolio of risky asset tangent to the new frontier.

In the presence of a risky asset, the investor devoted a fraction x of his wealth to the risky assets and the remaining wealth, indicated by (1-ι'x), is allocated to the risk-free assets.

The portfolio return is therefore a weighted average of the returns of the risk-free asset and of the tangency portfolio of risky assets. In other terms, the return of the newly composed portfolio is:

$$R_{p,t+1} = x'^{R_{t+1}} + (1 - \iota' x) R_t^f = x'^{r_{t+1}} + R_t^f \tag{2.6}$$

and the mean-variance problem can be expressed in terms of excess returns:

$$\min_{x} var\left[r_p\right] = x'\Sigma x \text{ subject to } E\left[r_p\right] = x'\mu = \bar{\mu} \qquad (2.7)$$

The solution to this problem is represented by:

$$x^* = \underbrace{\frac{\bar{\mu}}{\mu'\Sigma_{.}^{-1}\mu}}_{\lambda} \times \Sigma_{.}^{-1}\mu \qquad (2.8)$$

where λ is a constant that scales proportionately all the elements in $\Sigma_{.}^{-1}\mu$ in order to achieve the preferred portfolio risk premium $\bar{\mu}$. From this expression it is possible to find the weights of the tangency portfolio by noting that their sum must be equal to one. Therefore, for the tangency portfolio:

$$\lambda_{tgc} = \frac{1}{\iota'\Sigma_{.}^{-1}\mu} \text{ and } \bar{\mu}_{tgc} = \frac{\mu'\Sigma_{.}^{-1}\mu}{i'\Sigma_{.}^{-1}\mu} \qquad (2.9)$$

It is not difficult to identify the reasons why the Markowitz paradigm is an appealing one. It captures the two essential aspects of portfolio choice, diversification and risk-reward trade-off.

However, several objections could be posed to the paradigm. The main one could be that the mean-variance problem is a myopic single-period problem in which it is impossible to rebalance the portfolio during the investment horizon.

2.3 PREFERENCES AND RISK AVERSIONS

In a financial market where investors are facing uncertainty it is relevant to include in the portfolio choice problem also the preferences of the investor. Von Neumann and Morgenstern have shown that it is possible to represent the preferences of an individual who knows the probability distribution of the random returns by and expected utility criterion. More precisely, denoting by $\succ$ the preference order on the set of random returns, it is possible to state that $\succ$ satisfies the Von Neumann-Morgenstern criterion if there exists some increasing function U fro R into R, called utility function, such that:

$$X_1 \succ X_2 \Leftrightarrow E[U(X_1)] > E[U(X_2)] \tag{2.10}$$

The increasing property of the utility function indicates that the investor prefers more wealth to less wealth. The choice of the utility function allows to apply the notions of risk aversion and risk premium stemming from the fundamental uncertainty.

- **Risk aversion and concavity of the utility function**

It is legitimate to consider an investor who dislikes risk. Therefore, in respect to a random return X, he will prefer to receive with certainty the expectation E[X] of the investment return. This means that his utility function will satisfy the Jensen's inequality:

$$U(E[X]) \geq E[U(X)] \tag{2.11}$$

which stands true only for concave functions. Indeed, if a random return X taking values x with probability $\lambda \in (0,1)$ and x' with probability 1-λ is chosen, it will results that:

$$U(\lambda x + (1-\lambda)x') \geq \lambda U(x) + (1-\lambda)U(x') \qquad (2.12)$$

a fact that shows the concavity of the utility function U.

- **Degree of risk aversion and risk premium**

For a risk-averse agent with a concave utility function U, the risk premium associated to a random portfolio return X is defined as the positive amount π=π(X) that the agent would pay in order to receive a certain gain. It is defined by the equation:

$$U(E[X] - \pi) = E[U(X)] \qquad (2.13)$$

The quantity $\mathcal{E}(X) = E[X] - \pi$ is called the certainty equivalent of X and is smaller than the expectation of X.

Denoting $\bar{X} = E[X]$ and supposing that the portfolio return X is lowly risky, the following approximation can be obtained:

$$U(X) \approx U(\bar{X}) + (X - \bar{X})U'(\bar{X}) + \frac{1}{2}(X - \bar{X})^2 U''(\bar{X}) \qquad (2.14)$$

and so by taking expectation:

$$E[U(X)] \approx U(\bar{X}) + Var(X)\frac{U''(\bar{X})}{2} \qquad (2.15)$$

it will also result that

$$U(\bar{X} - \pi) \approx U(\bar{X}) - \pi U'(\bar{X}) \tag{2.16}$$

which gives the approximation for the risk premium

$$\pi \approx -\frac{U''(\bar{X})}{2U'(\bar{X})} Var(X) \tag{2.17}$$

Hence, the certainty equivalent of X is given approximately by

$$\mathcal{E}(X) = E[X] - \frac{1}{2}\alpha(\bar{X}) Var(X) \tag{2.18}$$

where α(x) is defined as the local absolute risk aversion at the return level x :

$$\alpha(x) = -\frac{U''(x)}{U'(x)} \tag{2.19}$$

The coefficient α is also called the Arrow-Pratt coefficient of absolute risk aversion of U at level x.

$\alpha(X)$ can therefore be considered as the factor by which an economic agent with utility function U weights the risk and as the factor grows larger so must the expectation of returns in order to compensate for the risk. When applying utility theory, it is possible to consider the variance of the portfolio as a good indicator of the risk being faced by the investor.

It is also possible to write the random return X as $X = \bar{X}(1+\varepsilon)$ where ε is interpreted as the relative payoff of the return X with respect to $\bar{X}$, and define the relative risk premium ρ of X as:

$$U\big(\bar{X}(1-\rho)\big) = E[U(X)] = E\big[U\big(\bar{X}(1+\varepsilon)\big)\big] \tag{2.20}$$

The relative risk premium is interpreted as the proportion of return that the investor is ready to pay in order to receive a certain payoff. As it has happened before, the risk premium ρ can be approximated as:

$$\rho \approx \frac{1}{2}\gamma(\bar{X})Var(\varepsilon) \tag{2.21}$$

Where

$$\gamma(x) = -\frac{xU''(x)}{U'(x)} \tag{2.22}$$

Is the relative risk aversion at level x.

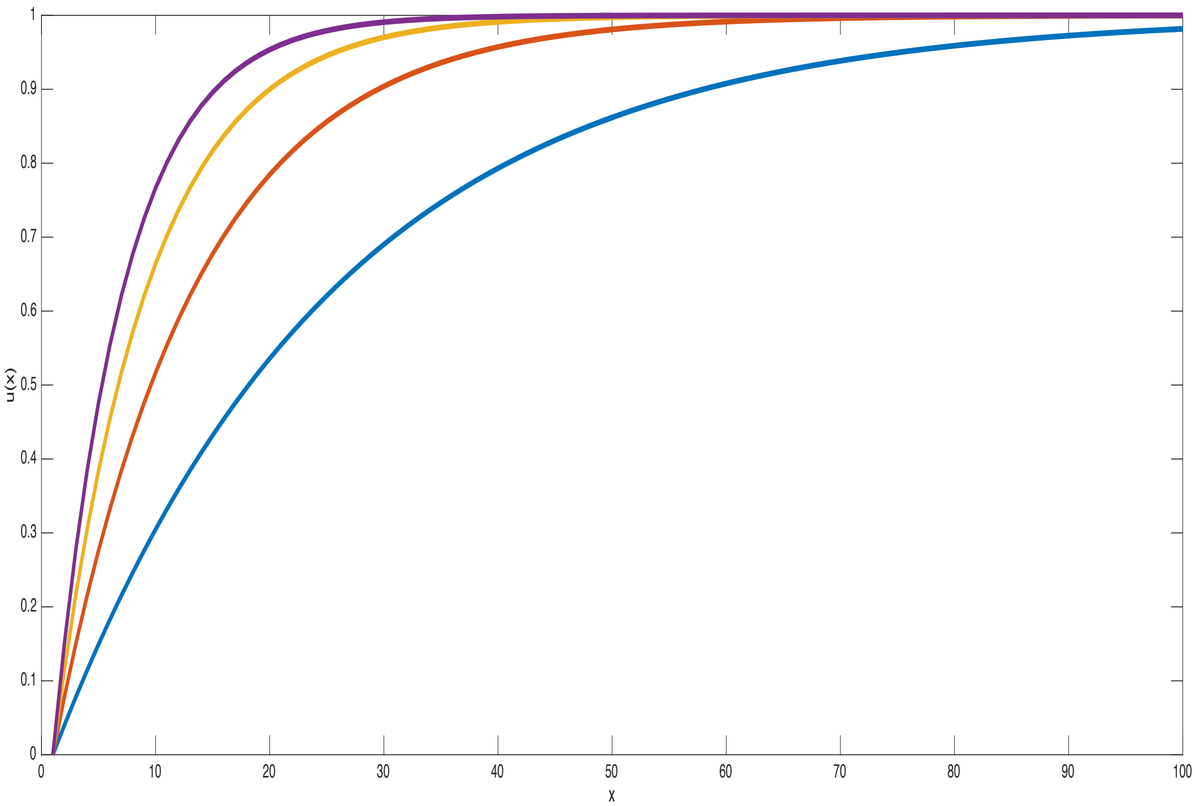
- **Common utility functions**
    - *Constant Absolute Risk Aversion (CARA):* $\alpha(x)$ equals a constant α>0.
      Since $\alpha(x) = -(\ln U')'(x)$, it follows that $U(x) = a - be^{-\alpha x}$.
      By using an affine transformation, U can be normalized to

$$U(x) = 1 - e^{-\alpha x} \tag{2.23}$$

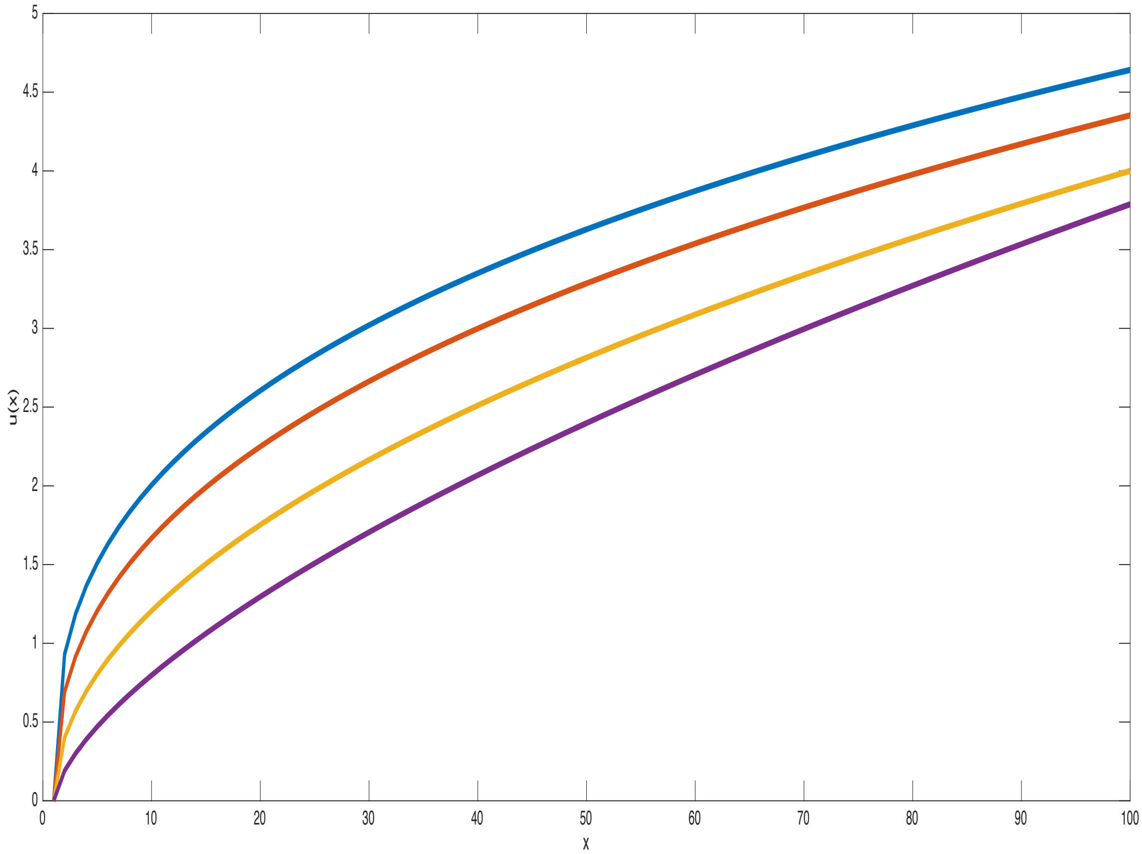
- *Constant Relative Risk Aversion (CRRA):* $\gamma(x)$ equals a constant $\gamma \in (0,1]$.

  Due to affine transformations, it is possible to obtain that

$$U(x) = \begin{cases} \ln x\,, & \text{for } \gamma = 1 \\ \frac{x^{1-\gamma}}{1-\gamma}, & for\ 0 < \gamma < 1 \end{cases} \tag{2.24}$$



<u>*Example of a CARA utility function*</u>



Example of a CRRA utility function

2.4 MERTON PORTFOLIO PROBLEM

Another approach to selection of a portfolio which includes for the risk aversion of the investor and is not subject to the static nature of the Markowitz paradigm was introduced by Robert Merton in 1969.

The scenario considered by Merton was one where an investor had the limited choice of investing his wealth in only two different assets: a risky asset and a risk-free asset. Given a limited time horizon, the goal of the investor, who is risk averse, was to maximize the expected utility of his wealth at the end of the time span taken into consideration.

Merton's goal was to determine how the investor should allocate and reallocate his wealth at each time point in order to reach the previously selected goal.

In order to explain the solution to the problem it is necessary to recall the previously introduced dynamics concerning the price of the risky asset. The price of the risky asset will therefore be denoted as S_t at time t.

The parameters μ and σ represent respectively the drift and the volatility of the risky asset. The price of the risk-free asset at time t is denoted by R_t and satisfies the following deterministic differential equation:

$$dR_t = rR_t d\mathrm{t} \qquad (2.25)$$

The parameter r stands for the risk-free continuously compounding interest rate. In this setting, it is easy to assume that $E[S_t] > E[R_t]$ which also indicates that $\mu > r$.

It is now necessary to introduce in the model the wealth of the investor at time t, denoted by V_t. At each time point t the investor must allocate a fraction u_t of his wealth in the risky asset.

The remaining wealth $1\text{-}u_t$ is invested in the risk-free asset. This means that the value of the risky investment at time t is $u_t V_t$ and that the value of the risk-free investment is $(1 - u_t)V_t$. The stochastic differential equation of the wealth or portfolio value is therefore:

$$dV_t = du_t V_t + d(1 - u_t)V_t = \mu u_t V_t dt + \sigma u_t V_t dB_t + r(1 - u_t)V_t dt =$$
$$\big(\mu u_t + r(1 - u_t)\big)V_t dt + \sigma u_t V_t dB_t.$$

The goal is now to decide the optimal allocation strategy for u_t at each time point t in order to obtain the best possible outcome at some future terminal time T for the investor.

However, as it has been previously discussed, an investor is not concerned with wealth maximization per se but with utility maximization. It is therefore possible to introduce an increasing and concave utility function U(x) representing the expected utility of a risk averse investor.

The goal of the problem is not anymore to maximize the expected portfolio value but to maximize the expected utility stemming from the wealth at the terminal time T.

If a time horizon restricted by an initial time t_0 and terminal time T and an initial portfolio value V_{t0} are assumed, then it is possible to state the maximization problem as:

$$I(t,x) = \max_{u_t} E[\,U(V_T)|t_0 = t, V_{t0} = x] \tag{2.26}$$

This constitutes an optimal control problem, where the allocation strategy u_t is the actual control function. Defining

$$\begin{aligned}\phi(t,x) &= \frac{\partial I(t,x)}{\partial t} + \big(\mu u_t + r(1-u_t)\big)\frac{\partial I(t,x)}{\partial x} + \frac{1}{2}\sigma^2 u_t^2 x^2 \frac{\partial^2 I(t,x)}{\partial x^2} \\ &= \frac{\partial I(t,x)}{\partial t} + (r + (\mu - r)u_t)\frac{\partial I(t,x)}{\partial x} + \frac{1}{2}\sigma^2 u_t^2 x^2 \frac{\partial^2 I(t,x)}{\partial x^2}\end{aligned} \tag{2.27}$$

The optimal solution must satisfy

$$\max_{u_t}[\phi(t,x)] = 0, t \in [t_0, T], \tag{2.28}$$

where ϕ indicates the cumulative distribution function of the standard normal distribution, and

$$I(T, V_T) = U(V_T). \tag{2.29}$$

This maximization problem is a continuous-time version of the Bellman-Dreyfus fundamental equation of optimality. This requirement also gives the optimal solution to the problem.

To find a solution that is compatible with the utility function $U(x)$, and that therefore is increasing and concave, it is required that

$$I_x = \frac{\partial I(t,x)}{\partial x} > 0 \text{ and } I_{xx} = \frac{\partial^2 I(t,x)}{\partial x^2} < 0. \tag{2.30}$$

Also, a first order condition for finding a maximum is

$$(\mu - r)I_x + \sigma^2 u_t x I_{xx} = 0 \tag{2.31}$$

which is equivalent to

$$u_t = -\frac{(\mu - r)I_x}{\sigma^2 x I_{xx}} \tag{2.32}$$

If this equation is substituted in the equation for $\phi(t, x)$ it happens that:

$$\begin{cases} \begin{cases} I_t + x\left(r + (\mu - r)\left(-\dfrac{(\mu - r)I_x}{\sigma^2 x I_{xx}}\right)\right) I_x \quad t < T \\ \qquad + \dfrac{1}{2}\sigma^2\left(-\dfrac{(\mu - r)I_x}{\sigma^2 x I_{xx}}\right)^2 x^2 I_{xx} = 0 \end{cases} \\ I(t,x) = U(x), \quad t = T \end{cases}$$

$$\leftrightarrow \begin{cases} I_t + rxI_x - \dfrac{(\mu - r)^2 I_x^2}{\sigma^2 I_{xx}} + \dfrac{1}{2}\dfrac{(\mu - r)^2 I_x^2}{\sigma^2 I_{xx}} = 0, \quad t < T \\ I(t,x) = U(x), \quad t = T \end{cases}$$

$$\leftrightarrow \begin{cases} I_t + rxI_x - \dfrac{(\mu - r)^2 I_x^2}{2\sigma^2 I_{xx}} = 0, \quad t < T \\ I(t,x) = U(x), \quad t = T \end{cases}$$

(2.34)

with

$$I_t = \frac{\partial I(t,x)}{\partial t}. \tag{2.35}$$

2.4.1 POWER UTILITY

In the solution to the Merton's portfolio problem utility function has been left as an unknown function U(x). It is now useful to identify a function which could be used in place of the generic notation U(x).

A classic solution to the problem is the use of the power function to indicate the utility of wealth x. In particular, it could be said that

$$U(x) = x^{\gamma}, \;\; 0 < \gamma < 1 \tag{2.36}$$

The use of this utility function is coherent with assumption previously made concerning the increasing and concave characteristics of the function. It is possible to refer to γ as the risk aversion parameter.

A low value of the risk aversion parameter is associated with high aversion to risk and vice versa. In order to find a solution to the problem using the newly introduced power utility function, it is necessary to insert the equation $I(t,x) = f(t)x^{\gamma}$ in the final result of the previous section.

That leads to the following calculations:

$$\begin{cases} f'(t)x^{\gamma} + rxf(t)\gamma x^{\gamma-1} - \dfrac{(\mu - r)^2 f^2(t)\gamma^2 x^{2(\gamma-1)}}{2\sigma^2 f(t)\gamma(\gamma-1)x^{\gamma-2}} = 0, \quad t < T \\ f(t)x^{\gamma} = x^{\gamma}, \quad t = T \end{cases}$$

$$\leftrightarrow \begin{cases} -\dfrac{f'(t)}{f(t)} = r\gamma + \dfrac{(\mu - r)^2\gamma}{2\sigma^2(1-\gamma)}, \quad t < T \\ f(t) = 1, \quad t = T. \end{cases}$$

(2.37)

Solving these equations with respect to $f(t)$ yields

$$f(t) = \exp\left(\left(r\gamma + \frac{(\mu-r)^2\gamma}{2\sigma^2(1-\gamma)}\right)(T-t)\right) \tag{2.38}$$

Substituting this solution into the equation $I(t,x) = f(t)x^{\gamma}$ it results that

$$I(t,x) = \exp\left(\left(r\gamma + \frac{(\mu-r)^2\gamma}{2\sigma^2(1-\gamma)}\right)(T-t)\right)x^{\gamma} \tag{2.39}$$

Finally, it is possible to find the optimal control u_t^* by solving the equation

$$u_t = -\frac{(\mu-r)I_x}{\sigma^2 x I_{xx}} \tag{2.40}$$

with respect to that last equation for $I(t,x)$.

The result of that operation is

$$u_t^* = -\frac{(\mu-r)\exp\left(\left(r\gamma+\frac{(\mu-r)^2\gamma}{2\sigma^2(1-\gamma)}\right)(T-t)\right)\gamma x^{\gamma-1}}{\sigma^2 x exp\left(\left(r\gamma+\frac{(\mu-r)^2\gamma}{2\sigma^2(1-\gamma)}\right)(T-t)\right)\gamma(\gamma-1)x^{\gamma-2}} = \frac{\mu-r}{\sigma^2(1-\gamma)} \tag{2.41}$$

which is a constant independent from the time variable. It is possible to conclude that the optimal allocation strategy is to hold a constant fraction u* of the wealth in the risky asset, and hence, a constant fraction 1-u* in the risk-free asset.

The ratio

$$\frac{\mu - r}{\sigma^2(1-\gamma)} \tag{2.42}$$

is also know by the name of Merton ratio.

The numerator of the ratio is the difference between the risky asset drift and the risk-free rate of return. In the case the numerator is negative, and therefore r is bigger than μ, the investor will allocate all his wealth to the risk-free asset, as it is logical given that it would offer a higher return without bearing a risk.

In the case instead that μ is bigger than r, the investor will invest at least a fraction of his wealth on the risky asset. This fraction is partly determined by the size of the difference between the risky asset drift and the risk-free denominator. The denominator is the product between the square of the volatility of the risky asset and one minus the risk aversion coefficient.

A particular feature of the Merton ratio is given by the fact that the ratio tends to decrease with an increase in volatility. This property of the Merton ratio is an essential one since it represents the higher reluctance in investing in the risky asset from a risk averse agent in the case of a higher volatility. One minus the risk aversion is instead a scaling parameter that has the role of adjusting the impact of the volatility on the Merton ratio.

It can be seen how a low value of the parameter γ scales up the impact of volatility in relative terms and vice versa. That is another relevant property since a low risk aversion parameter value must be associated with high risk aversion.

CHAPTER 3

3.1 TEST OF PERFORMANCES

The aim of this third, and last, chapter is to apply the theoretical frameworks introduced in the previous chapters and to test the performances that they would have had if used in the real world. In particular, this chapter will perform a confrontation between the Markowitz mean-variance model and the portfolio solution provided by Merton. The test will be performed on the same data set previously introduced and will deal specifically with the SPX index. SPX is the ticker by which the Standards & Poor's 500 is indicated. The S&P 500 is based on the capitalization of 500 large companies whose stocks are traded on the NYSE and NASDAQ. All other indexes quoted in the previous section could have been used as well. The application will also show how the composition of the portfolio in the Merton framework will vary as the risk aversion of the economic agent varies.

The problem here represented is of course a big simplification of the real world, since only one risky asset is taken into consideration. Moreover, transaction costs will be left out of the model, an assumption which can't be made in reality as transaction costs will significantly limit the amount of rebalancing the agent will do at each time point.

It is now necessary to introduce the way in which the model has been estimated. The first important assumption to make in order to run the two models is the existence of a risk free asset.

3.2 RISK-FREE ASSETS

Both the Markowitz and the Merton portfolio problem solutions rely on the existence of an asset defined as risk-free. The term risk-free may mean different things to different subjects. There may be a variety of observable rates all of which may be described by some as "risk-free". The main goal in selecting an appropriate rate should be to pick a rate such that valuations of the market turn out to be consistent. In this sense, the risk-free rate may just stand for what could be otherwise described as the "reference rate". A good reference rate may then be the yield available on government debt. One justification for doing so is that the government debt seems to fit more naturally the conventional meaning of the term 'risk-free', especially if the debt is in the same currency of the one used by the agent.

It is however noteworthy to underline how different market participants may place different interpretations on what should be defined by "risk-free" rate as the risks that they try to avoid may vary among them. However, for the matter of this application, it will be sufficient to hold to common interpretation and assume that the rate available on short-term government bonds of the same country of the index of reference is a good approximation of a risk-free rate.

Another consideration concerning risk-free rates must be made in the light of the recent economic crisis. Since 2008 the market itself is as a matter of fact questioning the level of risk behind government-backed securities. During late 2008 the Bank of England started observe same anomalous behaviour in the yield curves of Sterling-denominated UK government debt. It was noticed that long-term debt was trading at a higher rate than overnight index swap contracts, an event which indicated that the markets were attributing a non-zero market implied probability to the government altering downwards previously ruling coupon rates on some of their outstanding long-term debt. The same behaviour was spotted in other European country and signalled a growing concern about the longer-term creditworthiness of the governments of major developed countries. In order to avoid such a trend, it is useful to consider short term debt since it has been less influenced by this market behaviour due to its lower level of exposition to future political decisions.

Since the index taken into account for this chapter will be the SPX, that is the S&P 500, an index based on the market capitalizations of 500 large company having common stock listed on the NYSE or NASDAQ, the risk-free rate considered will be that of 13 weeks US government-backed t-bills. Those are short term debit securities issued by the American government.

The rates that will be later used are gathered in the following table.

DATE OF ISSUANCE	**13 WEEKS RATE**
3rd January 2012	0.02
2nd January 2013	0.06
2nd January 2014	0.06
2nd January 2015	0.03

Source: U.S. Department of the Treasury

3.3 RISK AVERSION COEFFICIENTS

The Merton portfolio choice model, as it has been previously shown, includes the utility of the economic agents into its framework. In particular, in order to have the discretization of the problem shown in equation (2.41) it is necessary to select a risk aversion coefficient. Risk aversion is a commonly accepted notion in economics and is easily recognizable in everyday life. Both investors and households would be willing to give up at least a small fraction of their potential earnings in order to have a guaranteed income. In the aforementioned utility function, it can be easily seen how risk aversion increases as the factor γ decreases.

As a matter of fact, a value of γ close to one or equal to one would therefore be an unpractical one as it would make the agent indifferent between certain incomes and expected incomes.

For the purpose of this application, different values of γ will be taken into consideration. Those values have not been inferred from populations studies, as it could be possible, since such an effort is beyond the scope. However, they should be significant enough in understanding how different relationships to risk by economic agents can influence their approach to the market.
The risk aversion coefficients used are therefore:

$$\gamma \in \{0.2; 0.3; 0.4; 0.5\}$$

3.4 PROCEDURE FOR THE MERTON PORTFOLIO

Having introduced the risk free rates and the values of risk aversion it is now possible to move on to more practical matters.
The application of the two portfolio theories will be done in a simplified way, as there will be only one risky asset considered. The risky asset of matter will be the SPX index, the index comprising the 500 American companies with the highest capitalization.
The Merton portfolio will be reallocated for different years and its evolution will be reported in order to assess what would have been the final return on a hypothetical investment. Given the length of the data set introduced in Chapter One, the first portfolio will be created on the first available day of the year 2012 and the data from year 2000 to year 2011 will be used as the background for the purpose of estimating the historic expected returns and volatility of the market. As the portfolio will be reallocated, the additional year of information will be merged into the background data and used for the historic estimations.
The same procedure will be done for the different degrees of risk aversion selected so that in the end it will be possible to perform a comparison within different time periods and different typology of investors.

The calculations will be performed on the MATLAB platform and the commands used to complete the procedure will be reported in the appendix.

3.5 APPLICATION OF MERTON PORTFOLIO THEORY

The MATLAB codes used in those calculation are available in the appendix. For a better clarity, only the findings of the applications will be reported here in the following tables.

3.5.1 $\gamma = 0.2$

Ratio of portfolio on market asset for 2012	36.73%
Return of portfolio for 2012	4.73%
Ratio of portfolio on market asset for 2013	59.38%
Return of portfolio for 2013	16.28%
Ratio of portfolio on market asset for 2014	100%
Return of portfolio for 2014	13.16%
Ratio of portfolio on market asset for 2015	100%
Return of portfolio for 2015	0.51%
Annualized Return	9.6%

3.5.2 $\gamma = 0.3$

Ratio of portfolio on market asset for 2012	41.98%
Return of portfolio for 2012	5.39%
Ratio of portfolio on market asset for	67.87%

2013	
Return of portfolio for 2013	18.57%
Ratio of portfolio on market asset for 2014	100%
Return of portfolio for 2014	13.16%
Ratio of portfolio on market asset for 2015	100%
Return of portfolio for 2015	0.51%
Annualized Return	10.31%

3.5.3 $\gamma = 0.4$

Ratio of portfolio on market asset for 2012	48.98%
Return of portfolio for 2012	6.27%
Ratio of portfolio on market asset for 2013	79.18%
Return of portfolio for 2013	21.62%
Ratio of portfolio on market asset for 2014	100%
Return of portfolio for 2014	13.16%
Ratio of portfolio on market asset for 2015	100%
Return of portfolio for 2015	0.51%
Annualized Return	11.66%

3.5.4 $\gamma = 0.5$

Ratio of portfolio on market asset for 2012	58.77%
Return of portfolio for 2012	7.51%
Ratio of portfolio on market asset for	95.01%

2013	
Return of portfolio for 2013	25.90%
Ratio of portfolio on market asset for 2014	100%
Return of portfolio for 2014	13.16%
Ratio of portfolio on market asset for 2015	100%
Return of portfolio for 2015	0.51%
Annualized Return	13.28%

3.6 PROCEDURE FOR THE MARKOWITZ PORTFOLIO

The procedure used in the application of the Markowitz portfolio theory will be slightly different from the one used in the Merton portfolio. Given the different nature of the two portfolio problem, the Markowitz portfolio won't be reallocated over the course of time but will instead be fixed on the first trading day of 2012 and held until the end of 2015. The risk free rates used here will be the same as the one used in precedence.

However, another relevant difference must be highlighted. The main investor characteristic by which the Merton portfolio choice is influenced is the investor risk aversion, a factor which affects the final choice of the portfolio. In the Markowitz case the subjectivity of the investor is incorporated through the selection of the preferred volatility rate or return rate. As the matter of fact, it is impossible to determine an exact portfolio allocation without first fixing one of those two constraints.

As it was done before, those constraints will be arbitrarily placed in order to show how the allocation of the portfolio may vary as the risk accepted to burden by the investor decreases or increases.

The volatility rates selected for the portfolio are hereby summarized:

Annual Portfolio Accepted Variance
5%
10%
15%
17.50%

3.7 APPLICATION OF MARKOWITZ PORTFOLIO

3.7.1 $\sigma = 5\%$

Ratio invested on market asset	22.48%
Ratio invested on risk-free asset	77.52%
Portfolio return for 2012	2.92%
Portfolio return for 2013	6.31%
Portfolio return for 2014	3.15%
Portfolio return for 2015	0.21%
Annualized portfolio return	3.12%

3.7.2 $\sigma = 10\%$

Ratio invested on market asset	44.97%
Ratio invested on risk-free asset	55.03%%
Portfolio return for 2012	5.77%
Portfolio return for 2013	12.38%
Portfolio return for 2014	6.05%
Portfolio return for 2015	0.29%
Annualized portfolio return	6.04%

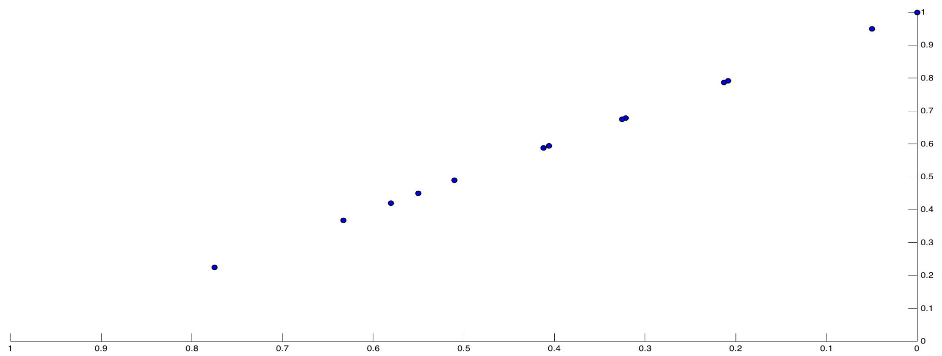
3.7.3 $\sigma = 15\%$

Ratio invested on market asset	67.45%
Ratio invested on risk-free asset	32.55%
Portfolio return for 2012	8.61%
Portfolio return for 2013	18.46%
Portfolio return for 2014	8.96%
Portfolio return for 2015	0.38%
Annualized portfolio return	8.91%

3.7.4 $\sigma = 17.5\%$

Ratio invested on market asset	78.69%
Ratio invested on risk-free asset	21.31%
Portfolio return for 2012	10.03%
Portfolio return for 2013	21.49%
Portfolio return for 2014	10.41%
Portfolio return for 2015	0.42%
Annualized portfolio return	13.28%

The different allocations of the portfolios created with the two models can now be summarized in the following graph. The percentage of the risky asset is reported on the y-axis and the percentage of risk-free asset is on the x-axis. As it can be easily seen all the portfolios relies on a straight line as they are simply linear combinations of the two assets typology.



3.8 ANALYSIS OF THE TWO RESULTS

It is now possible to highlights some of the differences that have arisen in the results of the two models.

The main difference consists in the incapacity of the Markowitz model in adapting to a well performing market. The Merton model as a matter of fact partly "recognize" the way in which the market was strongly outperforming the t-bills returns and allocates more and more resources on the market asset. It is important to also specify that in this application short-selling was not allowed. In 2014 and 2015 the Merton model suggested some amount of borrowing for all the investors and that is the reason behind the fact that the entire portfolios were allocated on the market assets.

Another differences stems from the higher variations in the allocation of the portfolio stemming from changes in the preference for volatility in the Markowitz model than those recognizable for changes in risk aversion in the Merton model.

For year 2012, the difference between the percentage of risky assets in the least averse agent and the most averse agent is around twenty percent for the Merton model. When the Markowitz model is instead taken into account, the same difference stands at more than 50%.

Some more clarifications are indeed necessaries concerning the results of the two application. In particular, it is striking how the performances of the SPX index in the analysed years are incredibly higher than the historic records for the same asset. Moreover, it is also relevant to underline how substantial in magnitude is the difference between the returns on the risk free asset and the risky asset. While those characteristics are not crucial for the consistency of this thesis, they will be briefly introduced.

It is easy to identify 2013 as the year where the performances of the S&P 500 index were high enough to increase sensibly the performance of the entire portfolios. Such a strong effect is also the product of a behaviour of the applied model which could be never experienced in reality and that is the lack of diversification.

The use of a non-diversified portfolio obviously exposes the investor, and consequently the whole portfolio, to an outlying behaviour of the market. This is essentially in the case in the previous application.

In 2013 the Standard & Poor's 500 Index posted its biggest annual advance since 1997 as an increase in consumer confidence and housing prices bolstered the American economy.

The S&P500 jumped 30 percent in 2013 and ended the year at an all-time high posting a 173 percent increase from its 12-year low reached in 2009. The fact that such a low was in the historic data used to estimate the stock average return is certainly another factor that has had an effect on the applications of the models.

Additional remarkable events were the fact that all the 10 main industries in the index concluded the year with a positive increase and a total of 460 stocks were up during 2013, a condition that hadn't happen since 1990.

The number of those companies included astonishing performances from Netflix, up 298 percent, Micron, up 243 percent and Best Buy, up 237.

In the light of these events, it is easier to understand the reason behind the unexpected results of the two portfolios in the analysed timespan.

Another condition which is strongly different from a typical setting for those kinds of portfolio problem is constituted by the extraordinarily low rates of the t-bills, which are here used as a proxy for a risk-free rate.

On the 16th of December of 2008 the Federal Reserve took the decision of cutting the funds rate to the band of 0 to 0.25%, concluding a cut of 500 basis points over the course of the precedent year. This move was a reaction to the deepening recession that the American economy was experiencing. The aim of such a move from the FED was to stimulate growth by prompting individuals and businesses to increase the level of investment and spending.

The effect of such a policy can be easily seen also in the application of the two portfolio theories reported before. Given the incredibly low interest rates, the risk premium from an investment in market stocks is incredibly higher and that is widely manifested in the portfolio allocation for Merton theory during the years 2014 and 2015. In those years, as a matter of fact, the Merton ratio suggests to short-sell risk free assets and invest on the S&P500 index, a condition not allowed in this setting.

3.9 CONCLUSION

It is now possible to briefly sum up the topics discussed before in this work.

In the first chapter the main theoretical frameworks behind returns estimation and prediction have been introduced. As it has been seen, such models hardly adapt well when applied to real world assets. However, those models are the pillars upon which modern mathematical finance is founded and are still useful if properly used. The main assumption which doesn't translate to the real financial world is the normality of the return distribution. Due to a number of reasons already explained such an assumption is not consistent with empirical data and a proof of that has been provided by the Jarque-Bera test.

The second chapter introduces the portfolio problem theories that are at the core of this thesis. The Merton portfolio problem in particular has been introduced and the mathematical framework behind has been explained and described. In order to have a discrete form of the same model, a power utility function has been adopted and plugged into the model. Due to this choice, an explicit solution to the problem can be found.

In the third and last chapter the two portfolio problem solutions introduced in the second chapter, the Merton portfolio and Markowitz portfolio models, have been applied to real world data in the form of the S&P500 index and the U.S. 13-weeks t-bills. The two applications showed how the portfolio composition could vary as the type of economic agent changed. This change was firstly represented by different degrees of risk aversion and then by different preferred portfolio volatility. The performance of the various portfolios has then been reported and analysed both on a year-by-year basis and then on an overall point of view. The main aim of this thesis was to provide a first look at more advanced form of portfolio selection theory than the basic mean-variance portfolio optimization problem. Many factors could have been added in order to further advance the theoretical framework, in particular the addition of transaction costs and stochastic volatility would have helped in the process of further aligning the theoretical world with the real one.
Even though those elements were not part of the work, the topics introduced here should still represent a valid foundation and are a solid foundation upon which further developments could be made.

MATLAB APPENDIX

KOLOMOGOROV-SMIRNOV TEST

The codes here reported have been used to perform the Kolmogorov-Smirnov test in Chapter One. The main command here used is "kstest" which is the built-in Matlab command for said test.

```
standSPX=(ReturnsSPX-nanmean(ReturnsSPX))/nanstd(ReturnsSPX)
hKSspx=kstest(standSPX)
standUKX=(ReturnsUKX-nanmean(ReturnsUKX))/nanstd(ReturnsUKX)
hKSukx=kstest(standUKX)
standAS51=(ReturnsAS51-
nanmean(ReturnsAS51))/nanstd(ReturnsAS51)
hKSas51=kstest(standAS51)
standCAC=(ReturnsCAC-nanmean(ReturnsCAC))/nanstd(ReturnsCAC)
hKScac=kstest(standCAC)
standCCMP=(ReturnsCCMP-
nanmean(ReturnsCCMP))/nanstd(ReturnsCCMP)
hKSccmp=kstest(standCCMP)
standDax=(ReturnsDax-nanmean(ReturnsDax))/nanstd(ReturnsDax)
hKSdax=kstest(standDax)
standFTSEMIB=(ReturnsFTSEMIB-
nanmean(ReturnsFTSEMIB))/nanstd(ReturnsFTSEMIB)
hKSftsemib=kstest(standFTSEMIB)
standINDEXCF=(ReturnsINDEXCF-
nanmean(ReturnsINDEXCF))/nanstd(ReturnsINDEXCF)
hKSindexcf=kstest(standINDEXCF)
standMXBR=(ReturnsMXBR-
nanmean(ReturnsMXBR))/nanstd(ReturnsMXBR)
hKSmxbr=kstest(standMXBR)
standNKY=(ReturnsNKY-nanmean(ReturnsNKY))/nanstd(ReturnsNKY)
hKSnky=kstest(standNKY)
standSASEIDX=(ReturnsSASEIDX-
nanmean(ReturnsSASEIDX))/nanstd(ReturnsSASEIDX)
hKSsaseidx=kstest(standSASEIDX)
standSHCOMP=(ReturnsSHCOMP-
nanmean(ReturnsSHCOMP))/nanstd(ReturnsSHCOMP)
hKSshcomp=kstest(standSHCOMP)
standSMI=(ReturnsSMI-nanmean(ReturnsSMI))/nanstd(ReturnsSMI)
hKSsmi=kstest(standSMI)
standSPTSX=(ReturnsSPTSX-
nanmean(ReturnsSPTSX))/nanstd(ReturnsSPTSX)
hKSsptsx=kstest(standSPTSX)
standUKX=(ReturnsUKX-nanmean(ReturnsUKX))/nanstd(ReturnsUKX)
hKSukx=kstest(standUKX)
```

JARQUE-BERA TEST

The codes here reported have been used to perform the Jarque-Bera test in Chapter One. The main command here used is “jbtest” which is the built-in Matlab command for said test.

```
hJBspx=jbtest(ReturnsAS51)
hJBcac=jbtest(ReturnsCAC)
hJBCCMP=jbtest(ReturnsCCMP)
hJBDax=jbtest(ReturnsDax)
hJBFTSEMIB=jbtest(ReturnsFTSEMIB)
hJBINDEXCF=jbtest(ReturnsINDEXCF)
hJBMXBR=jbtest(ReturnsMXBR)
hJBNKY=jbtest(ReturnsNKY)
hJBSASEIDX=jbtest(ReturnsSASEIDX)
hJBSHCOMP=jbtest(ReturnsSHCOMP)
hJBSMI=jbtest(ReturnsSMI)
hJBSPTSX=jbtest(ReturnsSPTSX)
hJBSPX=jbtest(ReturnsSPX)
hJBUKX=jbtest(ReturnsUKX)
```

MERTON PORTFOLIOS

The codes here reported are those used to estimate the composition of the portfolios under the Merton framework. First the historical moments of the asset have been estimated and then later they have been inserted in the explicit solution found in Chapter Two. The process has been done four times for each portfolio and the historical moments have been updated every time in order to incorporate the additional informations provided by the previous market year.

```
SPX2011return=price2ret(SPX2011,[],'Periodic')
MeanRet2011=((1+(nanmean(SPX2011return))).^252)-1
VAR2011=((1+var(SPX2011return)).^252)-1
RiskFree2012=((1.0002).^4)-1
MertonRatio201104=(MeanRet2011-RiskFree2012)./(VAR2011*(1-
0.4))
MarketReturn2012=price2ret(SPXONLY2012,[],'Periodic')
MeanRet2012ONLY=((1+(nanmean(MarketReturn2012))).^252)-1
```

```
PortfolioReturn201204=(MertonRatio201104*MeanRet2012ONLY)+((1-
MertonRatio201104)*RiskFree2012)
SPX2012return=price2ret(SPX2012,[],'Periodic')
MeanRet2012=((1+(nanmean(SPX2012return))).^252)-1
VAR2012=((1+var(SPX2012return)).^252)-1
RiskFree2013=((1.0006).^4)-1
MertonRatio201204=(MeanRet2012-RiskFree2013)./(VAR2012*(1-
0.4))
MarketReturn2013=price2ret(SPXONLY2013,[],'Periodic')
MeanRet2013ONLY=((1+(nanmean(MarketReturn2013))).^252)-1
PortfolioReturn201304=(MertonRatio201204*MeanRet2013ONLY)+((1-
MertonRatio201204)*RiskFree2013)
SPX2013return=price2ret(SPX2013,[],'Periodic')
MeanRet2013=((1+(nanmean(SPX2013return))).^252)-1
VAR2013=((1+var(SPX2013return)).^252)-1
RiskFree2014=((1.0006).^4)-1
MertonRatio201304=(MeanRet2013-RiskFree2014)./(VAR2013*(1-
0.4))
MarketReturn2014=price2ret(SPXONLY2014,[],'Periodic')
MeanRet2014ONLY=((1+(nanmean(MarketReturn2014))).^252)-1
PortfolioReturn201404=(1*MeanRet2014ONLY)+((1-1)*RiskFree2014)
SPX2014return=price2ret(SPX2014,[],'Periodic')
MeanRet2014=((1+(nanmean(SPX2014return))).^252)-1
VAR2014=((1+var(SPX2014return)).^252)-1
RiskFree2015=((1.0003).^4)-1
MertonRatio201404=(MeanRet2014-RiskFree2015)./(VAR2014*(1-
0.4))
MarketReturn2015=price2ret(SPXONLY2015,[],'Periodic')
MeanRet2015ONLY=((1+(nanmean(MarketReturn2015))).^252)-1
PortfolioReturn201504=(1*MeanRet2015ONLY)+((1-1)*RiskFree2015)

SPX2011return=price2ret(SPX2011,[],'Periodic')
MeanRet2011=((1+(nanmean(SPX2011return))).^252)-1
VAR2011=((1+var(SPX2011return)).^252)-1
RiskFree2012=((1.0002).^4)-1
MertonRatio201105=(MeanRet2011-RiskFree2012)./(VAR2011*(1-
0.5))
MarketReturn2012=price2ret(SPXONLY2012,[],'Periodic')
MeanRet2012ONLY=((1+(nanmean(MarketReturn2012))).^252)-1
PortfolioReturn201205=(MertonRatio201105*MeanRet2012ONLY)+((1-
MertonRatio201105)*RiskFree2012)
SPX2012return=price2ret(SPX2012,[],'Periodic')
MeanRet2012=((1+(nanmean(SPX2012return))).^252)-1
VAR2012=((1+var(SPX2012return)).^252)-1
RiskFree2013=((1.0006).^4)-1
MertonRatio201205=(MeanRet2012-RiskFree2013)./(VAR2012*(1-
0.5))
MarketReturn2013=price2ret(SPXONLY2013,[],'Periodic')
MeanRet2013ONLY=((1+(nanmean(MarketReturn2013))).^252)-1
PortfolioReturn201305=(MertonRatio201205*MeanRet2013ONLY)+((1-
MertonRatio201205)*RiskFree2013)
SPX2013return=price2ret(SPX2013,[],'Periodic')
MeanRet2013=((1+(nanmean(SPX2013return))).^252)-1
VAR2013=((1+var(SPX2013return)).^252)-1
RiskFree2014=((1.0006).^4)-1
MertonRatio201305=(MeanRet2013-RiskFree2014)./(VAR2013*(1-
0.5))
```

```
MarketReturn2014=price2ret(SPXONLY2014,[],'Periodic')
MeanRet2014ONLY=((1+(nanmean(MarketReturn2014))).^252)-1
PortfolioReturn201405=(1*MeanRet2014ONLY)+((1-1)*RiskFree2014)
SPX2014return=price2ret(SPX2014,[],'Periodic')
MeanRet2014=((1+(nanmean(SPX2014return))).^252)-1
VAR2014=((1+var(SPX2014return)).^252)-1
RiskFree2015=((1.0003).^4)-1
MertonRatio201405=(MeanRet2014-RiskFree2015)./(VAR2014*(1-
0.5))
MarketReturn2015=price2ret(SPXONLY2015,[],'Periodic')
MeanRet2015ONLY=((1+(nanmean(MarketReturn2015))).^252)-1
PortfolioReturn201505=(1*MeanRet2015ONLY)+((1-1)*RiskFree2015)

SPX2011return=price2ret(SPX2011,[],'Periodic')
MeanRet2011=((1+(nanmean(SPX2011return))).^252)-1
VAR2011=((1+var(SPX2011return)).^252)-1
RiskFree2012=((1.0002).^4)-1
MertonRatio201103=(MeanRet2011-RiskFree2012)./(VAR2011*(1-
0.3))
MarketReturn2012=price2ret(SPXONLY2012,[],'Periodic')
MeanRet2012ONLY=((1+(nanmean(MarketReturn2012))).^252)-1
PortfolioReturn201203=(MertonRatio201103*MeanRet2012ONLY)+((1-
MertonRatio201103)*RiskFree2012)
SPX2012return=price2ret(SPX2012,[],'Periodic')
MeanRet2012=((1+(nanmean(SPX2012return))).^252)-1
VAR2012=((1+var(SPX2012return)).^252)-1
RiskFree2013=((1.0006).^4)-1
MertonRatio201203=(MeanRet2012-RiskFree2013)./(VAR2012*(1-
0.3))
MarketReturn2013=price2ret(SPXONLY2013,[],'Periodic')
MeanRet2013ONLY=((1+(nanmean(MarketReturn2013))).^252)-1
PortfolioReturn201303=(MertonRatio201203*MeanRet2013ONLY)+((1-
MertonRatio201203)*RiskFree2013)
SPX2013return=price2ret(SPX2013,[],'Periodic')
MeanRet2013=((1+(nanmean(SPX2013return))).^252)-1
VAR2013=((1+var(SPX2013return)).^252)-1
RiskFree2014=((1.0006).^4)-1
MertonRatio201303=(MeanRet2013-RiskFree2014)./(VAR2013*(1-
0.3))
MarketReturn2014=price2ret(SPXONLY2014,[],'Periodic')
MeanRet2014ONLY=((1+(nanmean(MarketReturn2014))).^252)-1
PortfolioReturn201403=(1*MeanRet2014ONLY)+((1-1)*RiskFree2014)
SPX2014return=price2ret(SPX2014,[],'Periodic')
MeanRet2014=((1+(nanmean(SPX2014return))).^252)-1
VAR2014=((1+var(SPX2014return)).^252)-1
RiskFree2015=((1.0003).^4)-1
MertonRatio201403=(MeanRet2014-RiskFree2015)./(VAR2014*(1-
0.3))
MarketReturn2015=price2ret(SPXONLY2015,[],'Periodic')
MeanRet2015ONLY=((1+(nanmean(MarketReturn2015))).^252)-1
PortfolioReturn201503=(1*MeanRet2015ONLY)+((1-1)*RiskFree2015)

SPX2011return=price2ret(SPX2011,[],'Periodic')
MeanRet2011=((1+(nanmean(SPX2011return))).^252)-1
VAR2011=((1+var(SPX2011return)).^252)-1
RiskFree2012=((1.0002).^4)-1
MertonRatio201102=(MeanRet2011-RiskFree2012)./(VAR2011*(1-
```

```
0.2))
MarketReturn2012=price2ret(SPXONLY2012,[],'Periodic')
MeanRet2012ONLY=((1+(nanmean(MarketReturn2012))).^252)-1
PortfolioReturn201202=(MertonRatio201102*MeanRet2012ONLY)+((1-
MertonRatio201102)*RiskFree2012)
SPX2012return=price2ret(SPX2012,[],'Periodic')
MeanRet2012=((1+(nanmean(SPX2012return))).^252)-1
VAR2012=((1+var(SPX2012return)).^252)-1
RiskFree2013=((1.0006).^4)-1
MertonRatio201202=(MeanRet2012-RiskFree2013)./(VAR2012*(1-
0.2))
MarketReturn2013=price2ret(SPXONLY2013,[],'Periodic')
MeanRet2013ONLY=((1+(nanmean(MarketReturn2013))).^252)-1
PortfolioReturn201302=(MertonRatio201202*MeanRet2013ONLY)+((1-
MertonRatio201202)*RiskFree2013)
SPX2013return=price2ret(SPX2013,[],'Periodic')
MeanRet2013=((1+(nanmean(SPX2013return))).^252)-1
VAR2013=((1+var(SPX2013return)).^252)-1
RiskFree2014=((1.0006).^4)-1
MertonRatio201302=(MeanRet2013-RiskFree2014)./(VAR2013*(1-
0.2))
MarketReturn2014=price2ret(SPXONLY2014,[],'Periodic')
MeanRet2014ONLY=((1+(nanmean(MarketReturn2014))).^252)-1
PortfolioReturn201402=(1*MeanRet2014ONLY)+((1-1)*RiskFree2014)
SPX2014return=price2ret(SPX2014,[],'Periodic')
MeanRet2014=((1+(nanmean(SPX2014return))).^252)-1
VAR2014=((1+var(SPX2014return)).^252)-1
RiskFree2015=((1.0003).^4)-1
MertonRatio201402=(MeanRet2014-RiskFree2015)./(VAR2014*(1-
0.2))
MarketReturn2015=price2ret(SPXONLY2015,[],'Periodic')
MeanRet2015ONLY=((1+(nanmean(MarketReturn2015))).^252)-1
PortfolioReturn201502=(1*MeanRet2015ONLY)+((1-1)*RiskFree2015)
```

MARKOWITZ PORTFOLIOS

The codes here reported are those used to estimate the composition of the portfolios under the Markowitz framework. The previously estimated historical moments of the stock have been used and plugged into the portfolio composition formula. Due to the static nature of the portfolio, the initial composition has been held still and a simulation of its performance has been done by computing the performance which would have been registered on the real market for the four years considered in this application.

```
RiskFreeRatio5= (1-(0.05./(sqrt(VAR2011))))
RiskyRatio5=(1-RiskFreeRatio5)
MarkPortReturn20125=(RiskFreeRatio5*RiskFree2012)+(RiskyRatio5
*MeanRet2012ONLY)
MarkPortReturn20135=(RiskFreeRatio5*RiskFree2013)+(RiskyRatio5
*MeanRet2013ONLY)
MarkPortReturn20145=(RiskFreeRatio5*RiskFree2014)+(RiskyRatio5
*MeanRet2014ONLY)
MarkPortReturn20155=(RiskFreeRatio5*RiskFree2015)+(RiskyRatio5
*MeanRet2015ONLY)
TotRetMark5=(1+MarkPortReturn20125)*(1+MarkPortReturn20135)*(1
+MarkPortReturn20145)*(1+MarkPortReturn20155)-1
AnnualRetMark5=(1+TotRetMark5).^(1./4)-1

RiskFreeRatio10= (1-(0.1./(sqrt(VAR2011))))
RiskyRatio10=(1-RiskFreeRatio10)
MarkPortReturn201210=(RiskFreeRatio10*RiskFree2012)+(RiskyRati
o10*MeanRet2012ONLY)
MarkPortReturn201310=(RiskFreeRatio10*RiskFree2013)+(RiskyRati
o10*MeanRet2013ONLY)
MarkPortReturn201410=(RiskFreeRatio10*RiskFree2014)+(RiskyRati
o10*MeanRet2014ONLY)
MarkPortReturn201510=(RiskFreeRatio10*RiskFree2015)+(RiskyRati
o10*MeanRet2015ONLY)
TotRetMark10=(1+MarkPortReturn201210)*(1+MarkPortReturn201310)
*(1+MarkPortReturn201410)*(1+MarkPortReturn201510)-1
AnnualRetMark10=(1+TotRetMark10).^(1./4)-1

RiskFreeRatio15= (1-(0.15./(sqrt(VAR2011))))
RiskyRatio15=(1-RiskFreeRatio15)
MarkPortReturn201215=(RiskFreeRatio15*RiskFree2012)+(RiskyRati
o15*MeanRet2012ONLY)
MarkPortReturn201315=(RiskFreeRatio15*RiskFree2013)+(RiskyRati
o15*MeanRet2013ONLY)
MarkPortReturn201415=(RiskFreeRatio15*RiskFree2014)+(RiskyRati
o15*MeanRet2014ONLY)
```

```
MarkPortReturn201515=(RiskFreeRatio15*RiskFree2015)+(RiskyRati
o15*MeanRet2015ONLY)
TotRetMark15=(1+MarkPortReturn201215)*(1+MarkPortReturn201315)
*(1+MarkPortReturn201415)*(1+MarkPortReturn201515)-1
AnnualRetMark15=(1+TotRetMark15).^(1./4)-1


RiskFreeRatio175= (1-(0.175./(sqrt(VAR2011))))
RiskyRatio175=(1-RiskFreeRatio175)
MarkPortReturn2012175=(RiskFreeRatio175*RiskFree2012)+(RiskyRa
tio175*MeanRet2012ONLY)
MarkPortReturn2013175=(RiskFreeRatio175*RiskFree2013)+(RiskyRa
tio175*MeanRet2013ONLY)
MarkPortReturn2014175=(RiskFreeRatio175*RiskFree2014)+(RiskyRa
tio175*MeanRet2014ONLY)
MarkPortReturn2015175=(RiskFreeRatio175*RiskFree2015)+(RiskyRa
tio175*MeanRet2015ONLY)
TotRetMark175=(1+MarkPortReturn2012175)*(1+MarkPortReturn20131
75)*(1+MarkPortReturn2014175)*(1+MarkPortReturn2015175)-1
AnnualRetMark175=(1+TotRetMark175).^(1./4)-1


Portfoliofinalreturn02=((1+PortfolioReturn201102)*(1+Portfolio
Return201202)*(1+PortfolioReturn201302)*(1+PortfolioReturn2014
02))-1
Portfoliofinalreturn03=((1+PortfolioReturn201103)*(1+Portfolio
Return201203)*(1+PortfolioReturn201303)*(1+PortfolioReturn2014
03))-1
Portfoliofinalreturn04=((1+PortfolioReturn201104)*(1+Portfolio
Return201204)*(1+PortfolioReturn201304)*(1+PortfolioReturn2014
04))-1
Portfoliofinalreturn05=((1+PortfolioReturn201105)*(1+Portfolio
Return201205)*(1+PortfolioReturn201305)*(1+PortfolioReturn2014
05))-1
AnnualizedPortfolio02=((1+Portfoliofinalreturn02).^(1./4))-1
AnnualizedPortfolio03=((1+Portfoliofinalreturn03).^(1./4))-1
AnnualizedPortfolio04=((1+Portfoliofinalreturn04).^(1./4))-1
AnnualizedPortfolio05=((1+Portfoliofinalreturn05).^(1./4))-1


scatter(RFREERATIOS,RiskRATIOS)
```

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SOURCES OF DATA

The data used in this thesis have been taken from the following sources:

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Yahoo Finance

U.S. Department of the Treasury