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Portfolio Optimization

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ACADEMIC YEAR
2023 / 2024

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1. Introduction

Portfolio optimization is a central theme in modern finance, playing a crucial role in the decision-making process of investors. Portfolio theory, introduced by Harry Markowitz in the 1950s, revolutionized the way investors perceive risk and return, laying the foundation for the quantitative analysis of investment decisions. The primary objective of this thesis is to explore and delve into portfolio optimization methods, with a particular focus on Merton's problem and its practical applications.

In 1969, Robert C Merton expanded Markowitz's model by adding time dynamics into the continuous-time portfolio optimization framework. Merton's problem considers how to allocate resources between risky and risk-free assets in light of temporal effects and uncertain market environment. This has led to more advanced models incorporating stochastic volatility, among other market dynamics and making portfolio optimization an essential tool for investment management.

The thesis takes a holistic approach towards portfolio optimization from its theoretical foundations through to practical applications. First part covers Merton's problem together with various solution techniques such as value function approach, duality principle and dynamic programming . Next I will extend the Merton model allowing choice variables in models with finite/infinite time horizons, interest rate risk, habit formation or inclusion of stochastic volatility models.

In its second part, the thesis considers some practical applications of portfolio optimization. The benefits of diversification and risk management can be shown through actual case studies in creating efficient portfolios. We look at different kinds of optimization problems under different investments scenarios: stock markets, bond markets, derivatives markets only to provide evidence on how these theories can be used for daily management of investments.

To sum up this dissertation seeks to link theory with practice where it comes to portfolio optimization thus giving investors or people from industry advanced tools they need for enhancing investment strategies. It is important that one understands well these techniques because they are crucial for success in navigating complex dynamics found within modern financial markets hence better informed risk-return management.

2. Merton's Problem

2.1. Introduction to Merton's portfolio optimization problem

People making financial decisions always have to choose among different things, ranging from average to significant, depending on whether there are better results. Though unquantifiable, this is the definition of homo economicus. Therefore, I am going to proceed from Leonhard Euler's idea about universal optimization to a more complicated scenario whereby outcomes are not deterministic but market uncertainty-dependent.

Robert C. Merton developed in 1969 what is known as Merton's Portfolio Problem. The problem lies in optimizing a stochastic value function modelled by differential equations and unpredictability inherent in financial systems. The main idea at the core of the problem is that an active investor heading toward retirement has to strike a balance between risk-free assets and risky assets. (Merton, 1969)

Post-employment and pre-retirement spending require a balanced portfolio in order to increase the returns of the portfolio after retirement.

In this section, I will also examine how dynamic programming and the Hamilton-Jacobi-Bellman equation can be used to analyze Merton's Portfolio Problem. Stochastic calculus starts my analysis with Brownian motion, stochastic integrals, differential equations, and Itô's Lemma – the mathematical tool that lies at the foundation of these methods.

After this point, I will proceed from the solution of the optimal control problem to the deterministic Hamilton-Jacobi-Bellman partial differential equation through the Verification Theorem that guarantees its validity. (Merton, 1969)

Onward moving, I will examine some approaches used to resolve the different problems presented by Merton: value function duality, dynamic equilibrium models, and the value function approach. These methods answer the above-given question while asking me about their effectiveness and under what conditions they are well-posed.

The latter part of this study will expound about the practical implications, thereby showing their importance for potential applications. I will analyse how solutions relate to their parameters by examining how each influences others through its partial derivatives or simulated trajectories. This is an academic exercise and an educational toolbox for investors who want to navigate a random portfolio space effectively because of its practical orientation. It also underlines the significance of Merton's Portfolio Problem in real life for modern portfolio management purposes.

Consequently, this chapter aims to comprehensively understand how modern portfolio theory applies Merton's foundational work in various financial contexts. Infact, profound implications can be drawn from Merton's problem regarding contemporary portfolio management. These open up several extensions and applications that have changed the face of financial decision-making.

This section of the study marks the essence of the research and discusses various ways to navigate through the random waves that flow in financial markets.

Therefore, I will examine the classical method of value function framework for solving stochastic control problems, I will delve a more detailed scrutiny of Merton’s portfolio problem. In summary, this part is concerned with different solution methods and multifaceted perspectives necessary for steering through dynamic randomness characterizing financial market operations.

In particular I will explain the main solution approaches to the problem of portfolio optimization:

- The classical value function approach. This method solves stochastic control problems by solving Hamilton-Jacobi-Bellman (HJB) equations such that you obtain a dynamic and recursive procedure for identifying optimal strategies.

- The duality approach. By shifting focus to its dual objective function, this method simplifies complex dynamics enabling easier manipulation algebraically. Simultaneously it opens new doors making certain solutions more economical coherent and more profound on economic ground than others.

- The dynamic programming approach. This method is important when thinking about making investment decisions using temporally optimizing procedures like the passage through a puzzle or maze with numerous twists. This method works breaking down complex problems into simpler subproblems that lead from local decisions towards global optimality one step at a time

These much twisted but strongly built theories provide a solid foundation for modern investment strategies. Other than being independent stances they combine analytical rigor with intuitive insights.

2.2. Solution Approaches

In delving into the domain of portfolio optimization, the solution approaches encompass a rich tapestry of methodologies, each addressing the multifaceted nature of investment decisions. The common theme is of an agent investing in one or more risky assets so as to optimize some objective. The dynamics of the agent's wealth can be characterized through the equation

$$dw_t = r_t w_t \, dt + n_t \cdot (dS_t - r_t S_t \, dt + \delta_t \, dt) + e_t \, dt - c_t \, dt \qquad (1.1)$$

$$= r_t (w_t - n_t \cdot S_t) \, dt + n_t \cdot (dS_t + \delta_t \, dt) + e_t \, dt - c_t \, dt. \qquad (1.2)$$

for some given initial wealth w_0.

In portfolio optimization, the wealth of an investor, influenced by various market factors, is described by a complex equation. The equation accounts for the dynamics of asset prices (S), represented as a multi-dimensional semimartingale, and the investor's decisions on the portfolio (n) and consumption (c), both of which are considered to be predictable processes. Additionally, dividends (δ) from the assets, any income stream (e), and the risk-free interest rate (r) are factored into the wealth calculation, with the latter dictating growth in the absence of risky asset investments. (Roger, 2013)

Investors who choose not to invest in risky assets can expect their wealth to grow risk-free, influenced by their income and expenses. However, those who opt for a fixed number of risky assets will see their wealth composition at any given time, including the market value of these assets and the cash held in a bank account. This cash appreciates at the risk-free rate r and is affected by dividends received. This underscores the role of risky assets in wealth growth and the importance of a well-balanced investment strategy.

During constructing an investment strategy, ensuring that the chosen portfolio and consumption processes are viable over time is paramount. An investment strategy is typically modulated at specified intervals, allowing the investor to adjust their portfolio in response to financial performance and other factors without causing abrupt changes in wealth. To formalize this concept, I introduce Definition 1.1:

Definition 1.1: The pair $(n_t, c_t)_{t\geq 0}$ is said to be *admissible for initial wealth* w_0 if the wealth process w_t given by equation (1.1) remains non-negative at all times. I denote this by

$$\mathscr{A}(w_0) \equiv \{(n, c) : (n, c) \text{ is admissible from initial wealth } w_0\} \qquad (1.3)$$

This definition encapsulates the idea that, for a strategy to be deemed feasible, it must maintain the investor's wealth at or above zero throughout the investment horizon. The condition of non-negativity is critical; it prevents the portfolio from reaching a deficit, which would not be sustainable. I further generalize this by introducing the set A, which encompasses all possible admissible pairs (n, c) across different initial wealth levels:

$A = \bigcup_{w>0} A(w)$

By setting these parameters, I limit my focus to strategies that promise theoretical profitability and adhere to practical constraints of risk and return. (Roger, 2013)

After establishing the prerequisites for admissible strategies, our attention shifts to the objectives steering the investor's decision-making process. At the core of the optimization challenge, the investor seeks to configure a combination of portfolio choices (n,c) within the realm of admissibility to maximize expected utility. This venture is succinctly captured by an objective function integrating utility derived from consumption over time and the terminal wealth at a specified time horizon, T.

$$\sup_{(n,c)\in\mathscr{A}(w_0)} E\left[\int_0^T u(t, c_t)\, dt + u(T, w_T)\right]. \tag{1.4}$$

The time horizon T is generally taken to be a positive constant.

Two pivotal scenarios emerge in this context: the infinite-horizon problem, which transcends temporal bounds to maximize utility indefinitely

$$\sup_{(n,c)\in\mathscr{A}(w_0)} E\left[\int_0^\infty u(t, c_t)\, dt\right], \tag{1.5}$$

and the terminal wealth problem,

$$\sup_{(n,0)\in\mathscr{A}(w_0)} E[u(w_T)], \tag{1.6}$$

that concentrates exclusively on the utility of the accumulated wealth at time T, abstracting from the consumption aspect.

This then is the problem: the agent aims to achieve (1.4) when his control variables must be chosen so that the wealth process w generated by (1.1) remains non-negative. There are methods to solve this problem, but there is a very important principle underlying many of the approaches:

Theorem 1.1 *(The Davis-Varaiya Martingale Principle of Optimal Control)*

Suppose that the objective is (1.4), and that there exists a function V : [0, T] x $R^+ \rightarrow R$ which is $C^{1,2}$, such that V(T, ·) = u(T, ·). Suppose also that for any (n, c) € A (w_0)

$$Y_t \equiv V(t, w_t) + \int_0^t u(s, c_s)\, ds \quad \text{is a supermartingale,} \tag{1.7}$$

And that for some (n*, c*) € A(w_0) the process Y is a martingale. Then (n*, c*) is optimal, and the value of the problem starting from initial wealth w_0 is

$$V(0, w_0) = \sup_{(n,c)\in\mathscr{A}(w_0)} E\left[\int_0^T u(t, c_t)\, dt + u(T, w_T)\right]. \tag{1.8}$$

Proof.

From the supermartingale property of Y, I have for any (n, c) ∊ A (w_0)

$$Y_0 = V(0, w_0) \geq E[Y_T] = E\left[\int_0^T u(t, c_t)\, dt + u(T, w_T)\right], \tag{1.9}$$

using the fact that $V(T,\cdot) = u(T,\cdot)$. Thus for any admissible strategy the value is no greater than V (0, w_0); when I use (n*, c*), the value is equal to V (0, w_0) since the (supermartingale) inequality in (1.9) becomes an equality. Hence (n*, c*) is optimal. (Roger, 2013)

2.2.1. The Value Function Approach

The traditional and widely used technique for addressing problems in stochastic optimal control is the value function approach, deeply anchored in the Martingale Principle of Optimal Control (MPOC). To approach this method systematically, I initially detail the dynamics of our assets; I shall suppose that

$$dS_t^i = S_t^i\Big(\sum_{j=1}^{N} \sigma^{ij} dW_t^j + \mu^i dt\Big), \tag{1.10}$$

where the σ^{ij} and the μ^i are constants, and W is an d -dimensional Brownian motion.

Additionally, I will assume that the risk-free interest rate, denoted as r, is a constant value, and that the processes representing endowment, e, and dividends, δ, are consistently zero. The previously mentioned equation (1.10) can thus be succinctly rewritten in a more compact form:

$$dS_t = S_t(\sigma \cdot dW + \mu dt). \tag{1.11}$$

Notice that the wealth equation(1.1) can be equivalently (and more usefully) expressed as

$$dw_t = rw_t\, dt + \theta_t \cdot (\sigma dW_t + (\mu - r)\, dt) - c_t\, dt. \tag{1.12}$$

To identify a function V that meets the conditions set out in Theorem (1.1), one would directly construct the process Y as delineated in (1.7) and undertake an Itô expansion, premised on the assumption that V exhibits the necessary smoothness:

$$\begin{aligned} dY_t &= V_t dt + V_w\, dw + \tfrac{1}{2} V_{ww}\, dwdw + u(t, c)\, dt \\ &= V_w \theta \cdot \sigma dW + \left\{ u(t, c) + V_t + V_w(rw + \theta \cdot (\mu - r) - c) + \tfrac{1}{2} |\sigma^T \theta|^2 V_{ww} \right\} dt. \end{aligned}$$

The stochastic integral term in the Itô expansion is a local martingale; if I could assume that it was a *martingale*, then the condition for Y to be a supermartingale whatever (θ, c) was in use would just be that the drift were non-positive. Moreover, if the supremum of the drift were equal to zero, then I should have that V was the value function, with the pointwise-optimizing (θ, c) constituting an optimal policy. Setting all the provisos aside for the moment, this would lead us to consider the equation

$$0 = \sup_{\theta, c} \left[u(t, c) + V_t + V_w(rw + \theta \cdot (\mu - r) - c) + \tfrac{1}{2} |\sigma^T \theta|^2 V_{ww} \right]. \tag{1.13}$$

This (non-linear) partial differential equation (PDE) for the unknown value function V is the *Hamilton-Jacobi-Bellman* (HJB) equation . If I have a problem with a finite horizon, then I shall have the boundary condition $V(T, \cdot) = u(T, \cdot)$; for an infinite-horizon problem, I do not have any boundary conditions to fix a solution, though in any given context, I may be able to deduce enough growth conditions to fix a solution. The point is that *if I am able to find some V which solves the HJB equation*, then it is usually possible by direct means to verify that the *V* so found is actually the value function.

If I am not able to find some *V* solving the HJB equation, then I really cannot say anything interesting about the solution.

In order to get some reasonably explicit solution, I shall have to assume a simple form for the utility *u*, such as

$$u(t,x) = e^{-\rho t}\,\frac{x^{1-R}}{1-R}, \qquad (1.14)$$

or

$$u(t,x) = -\gamma^{-1}\exp(-\rho t - \gamma x), \qquad (1.15)$$

where $\rho,\gamma,R > 0$ and $R \neq 1$. Since the derivative of the utility (1.14) is just $u'(t,x) = e^{-\rho t}x^{-R}$, in the case $R = 1$ I understand it to be

$$u(t,x) = e^{-\rho t}\log(x). \qquad (1.16)$$

All of these forms of the utility are very tractable, and if I do not assume one of these forms I will rarely be able to get very far with the solution. (Roger, 2013)

Key example: the infinite-horizon Merton problem. To illustrate the main ideas in a simple and typical example, let's assume the constant-relative-risk-aversion (CRRA) form (1.14) for *u*, which I write as

$$u(t,x) \equiv e^{-\rho t}u(x) \equiv e^{-\rho t}\,\frac{x^{1-R}}{1-R}. \qquad (1.17)$$

The aim is to solve the infinite-horizon problem; the agent's objective is to find the value function

$$V(w) = \sup_{(n,c)\in\mathscr{A}(w)} E\left[\int_0^{\infty} e^{-\rho t} \frac{c_t^{1-R}}{1-R}\, dt\right], \tag{1.18}$$

And the admissible (n, c) which attains the supremum, if possible. I shall see that this problem can be solved completely. The steps involved are:

Step 1: Using special features. What makes this problem easy is the fact that *because of scaling, I can write down the form of the solution*; indeed, I can immediately say that

$$V(w) = \gamma_M^{-R}\, u(w) \equiv \gamma_M^{-R}\, \frac{w^{1-R}}{1-R} \tag{1.19}$$

for some constant $\gamma_M > 0$. Thus finding the solution to the optimal investment/ consumption problem reduces to identifying the constant γM.

Step 2: Using the HJB equation to find the value.

$$V(t,w) = \sup_{(n,c)\in\mathscr{A}(w)} E\left[\int_t^{\infty} e^{-\rho s} \frac{c_s^{1-R}}{1-R}\, ds \,\middle|\, w_t = w\right],$$

then it is clear from the time-homgeneity of the problem that

$$V(t,w) = e^{-\rho t} V(w), \tag{1.20}$$

where V is as defined at (1.18). In view of the scaling form (1.19) of the solution, I now suspect that

$$V(t,w) = e^{-\rho t}\, \gamma_M^{-R}\, u(w), \tag{1.21}$$

and I just have to identify the constant γM . For this, I return to the HJB Eq. (1.13). The HJB equation involves an optimization over θ and c, which can be performed explicitly.

Optimization over θ.

The optimization over θ is easy :

$$(\sigma\sigma^T)\theta V_{ww} = -(\mu - r)V_w,$$

Hence

$$\theta^* = -\frac{V_w}{V_{ww}}\,(\sigma\sigma^T)^{-1}(\mu - r). \tag{1.22}$$

Using the suspected form (1.21) of the solution, this is simply

$$\boxed{\theta^* = wR^{-1}\,(\sigma\sigma^T)^{-1}(\mu - r).} \tag{1.23}$$

To interpret this solution, let us introduce the notation

$$\boxed{\pi_M \equiv R^{-1}(\sigma\sigma^T)^{-1}(\mu - r),} \tag{1.24}$$

a constant N-vector, called the *Merton portfolio*. What (1.23) tells us is that for each i, and for all $t > 0$, the cash value of the optimal holding of asset i should be

$$(\theta_t^*)^i = w_t\,\pi_M^i;$$

so *the optimal investment in asset i is proportional to current wealth wt* , with constant of proportionality π^i_M

Optimization over *c*.

For the optimization over c, if I introduce the convex dual function

$$\tilde{u}(y) \equiv \sup_x\{u(x) - xy\} \tag{1.25}$$

of u, then I have for $u(x) = x^{1-R}/(1 - R)$ that

$$\tilde{u}(y) = -\frac{y^{1-\tilde{R}}}{1-\tilde{R}}, \tag{1.26}$$

where $\tilde{R} = R^{-1}$. Thus the optimization over c develops as

$$\sup_c\{u(t,c) - cV_w\} = e^{-\rho t}\sup_c\{u(c) - ce^{\rho t}V_w\} = e^{-\rho t}\tilde{u}(e^{\rho t}V_w).$$

Substituting in the suspected form (1.21) of the solution, this gives us

$$\sup_c\{u(t,c) - cV_w\} = e^{-\rho t}\tilde{u}((\gamma_M w)^{-R}) = -e^{-\rho t}\frac{(\gamma_M w)^{1-R}}{1-\tilde{R}} = e^{-\rho t}\ \frac{R}{1-R}\ (\gamma_M w)^{1-R},$$

with optimizing c^* proportional to w:

$$\boxed{c^* = \gamma_M w.} \tag{1.27}$$

Putting it all together.

Returning the candidate value function (1.21) to the HJB Eq. (1.13), I find that

$$\begin{aligned} 0 &= e^{-\rho t}\left[\frac{R}{1-R}\ (\gamma_M w)^{1-R} - \rho\gamma_M^{-R}u(w) + rw\gamma_M^{-R}w^{-R} + \tfrac{1}{2}\gamma_M^{-R}w^{1-R}|\kappa|^2/R\right] \\ &= \frac{e^{-\rho t}w^{1-R}\gamma_M^{-R}}{1-R}\left[R\gamma_M - \rho - (R-1)(r + \tfrac{1}{2}|\kappa|^2/R)\right] \end{aligned}$$

Where:

$$\kappa \equiv \sigma^{-1}(\mu - r) \tag{1.28}$$

is the *market price of risk* vector. This gives the value of γ_M :

$$\boxed{\gamma_M = R^{-1}\left\{\rho + (R-1)(r + \tfrac{1}{2}|\kappa|^2/R)\right\},} \tag{1.29}$$

and hence the value function of the Merton problem (see (1.21)), $V_M(w) \equiv V(t, w)$, as

$$\boxed{V_M(w) = \gamma_M^{-R}\, u(w).} \tag{1.30}$$

As I draw my discussion to a close, I am presented with two critical considerations. The first pertains to the implications of the expression for γ_M becoming negative, as outlined in expression (1.29). The second is a broader, more foundational inquiry, confirming the optimality of the solution I have deduced.

Addressing the former necessitates a tailored response specific to the problem at hand, as it speaks to the well-posedness of the problem itself. The latter inquiry, however, transcends the particularities of any single problem and is relevant across a spectrum of cases. The methodology employed in affirming the optimality of a solution is broadly applicable and, therefore, will be our initial focus. With the assumption that γ_M, as given by (1.29), is indeed positive, I will subsequently circle back to address the first point of concern.

Suppose that the initial wealth $w0$ is given, and consider the evolution of the wealth w^* under the conjectured optimal control; I see

$$\begin{aligned} dw_t^* &= w_t^* \left\{ \pi_M \cdot \sigma dW_t + (r + \pi_M \cdot (\mu - r) - \gamma_M) dt \right\} \\ &= w_t^* \left\{ R^{-1} \kappa \cdot dW_t + (r + R^{-1} |\kappa|^2 - \gamma_M) dt \right\} \end{aligned}$$

Which is solved by

$$w_t^* = w_0 \exp\left[R^{-1} \kappa \cdot W_t + (r + \tfrac{1}{2} R^{-2} |\kappa|^2 (2R - 1) - \gamma_M) t \right] \tag{1.31}$$

Step 3: Finding a simple bound.

The proof of optimality is based on the trivial inequality:

$$u(y) \leq u(x) + (y - x) u'(x) \qquad (x, y > 0), \tag{1.32}$$

which expresses the geometrically obvious fact that the tangent to the concave function u at $x > 0$ lies everywhere above the graph of u. If I consider any admissible (n, c) then, I am able to bound the objective by

$$E\left[\int_0^\infty e^{-\rho t}\, u(c_t)\, dt\right] \leq E\left[\int_0^\infty e^{-\rho t}\, \left\{u(c_t^*) + (c_t - c_t^*)u'(c_t^*)\right\} dt\right]$$
$$= E\int_0^\infty e^{-\rho t}\, u(c_t^*)\, dt + E\left[\int_0^\infty (c_t - c_t^*)\zeta_t\, dt\right], \tag{1.33}$$

where I have abbreviated

$$\zeta_t \equiv e^{-\rho t}u'(c_t^*) \propto \exp(-\kappa \cdot W_t - (r + \tfrac{1}{2}|\kappa|^2)t) \tag{1.34}$$

after some simplifications using the explicit form of w^*. Now, the key point is that ζ *is a state-price density,* also called a stochastic discount factor; and there is the property that for any admissible (n, c)

$$Y_t \equiv \zeta_t w_t + \int_0^t \zeta_s c_s\, ds \quad \text{is a local martingale.} \tag{1.35}$$

This may be verified directly by Itô calculus from the wealth equation (1.1) in this example, and I leave it to the reader to carry out this check. In general, I expect that *the marginal utility of the optimal consumption should be a state-price density.* The importance of the statement (1.35) is that since the wealth and consumption are non-negative, the process Y is in fact a non-negative supermartingale, and hence

$$w_0 = Y_0 \geq E[Y_\infty] \geq E\left[\int_0^\infty \zeta_s c_s\, ds\right]. \tag{1.36}$$

Step 4: Verifying the bound is attained for the conjectured optimum.
One last piece remains, and that is to verify the equality

$$w_0 = E\left[\int_0^\infty \zeta_s c_s^*\, ds\right] \tag{1.37}$$

For the optimal consumtion process c*, and again this can be established by direct calculation using the explicit form of c*. Combining (1.33), (1.36) and (1.37) gives me finally that for any admissible (n, c)

$$E\left[\int_0^{\infty} e^{-\rho t}\, u(c_t)\, dt\right] \leq E\left[\int_0^{\infty} e^{-\rho t}\, u(c_t^*)\, dt\right], \tag{1.38}$$

which proves optimality of the conjectured optimal solution (n^*, c^*). (Roger, 2013)

2.2.2. Duality Approach

Consider this methodology as a nuanced iteration of the value function technique. It presents an alternative modality for engaging with the HJB equation. It potentially streamlines the HJB equation. The necessity to confirm the solution's validity remains; however, the complexities are significantly reduced through this approach.

The fundamental concept involves reformulating the HJB equation as shown in (1.13), applying an appropriate transformation. As for the utility function U, the requirement is not tied to a specific form. Rather, the criteria are its concavity, a strict increase in response to the second variable, and continuity concerning the first, ensuring broad applicability of this approach.

$$\lim_{x\to\infty} u'(t,x) = 0, \tag{1.39}$$

which is necessary for the optimization to be well posed.

Since I know that the value function is concave, the derivative V_w is monotone decreasing, so I am able to define a new coordinate system

$$(t,z) = (t,\, V_w(t,w)) \tag{1.40}$$

for (t,z) in A ≡ $\{(t,z) : V_W(t,\infty) < z < V_W(t,0)\}$. Now I define a function J : A → R by

$$J(t,z) = V(t,w) - wz, \tag{1.41}$$

and I notice that by standard calculus I have the relations

$$J_z = -w, \tag{1.42}$$
$$J_t = V_t, \tag{1.43}$$
$$J_{zz} = -1/V_{ww}. \tag{1.44}$$

Now when I take the HJB equation (1.13) and optimize over θ and c I obtain

$$0 = \tilde{u}(t, V_w) + V_t + rwV_w - \tfrac{1}{2}|\kappa|^2 \frac{V_w^2}{V_{ww}} \quad (1.45)$$
$$= \tilde{u}(t, z) + J_t - rzJ_z + \tfrac{1}{2}|\kappa|^2 z^2 J_{zz} \quad (1.46)$$

which is a *linear PDE for the unknown J*. Here, $\tilde{u}(t, z) \equiv \sup\{u(t, x) - zx\}$ is the convex dual of u. (Roger, 2013)

The key example again. To see this in action, let us take the infinite-horizon Merton problem, and suppose that

$$u(t, x) = e^{-\rho t} u(x) \quad (1.47)$$

for some concave increasing non-positive u, *which I do not assume has any particular form.* In this instance, I know that $V(t, w) = e^{-\rho t} v(w)$ for some concave function v which is to be found. From the definition (1.41) of the dual value function J, I have

$$\begin{aligned} J(t, z) &= e^{-\rho t} v(w) - wz \\ &= e^{-\rho t}(v(w) - wze^{\rho t}) \\ &\equiv e^{-\rho t} j(ze^{\rho t}), \end{aligned}$$

Notice that since u is non-positive, it has to be that V is also non-positive, and that j is non-positive.

If I introduce the variable $y = ze^{\rho t}$, simple calculus gives

$$J_t = -\rho e^{-\rho t} j(y) + \rho z j'(y), \quad J_z = j'(y), \quad J_{zz} = e^{\rho t} j''(y)$$

and substituting into (1.46) gives the equation for j

$$0 = \tilde{u}(y) - \rho j(y) + (\rho - r)\, y j'(y) + \tfrac{1}{2}|\kappa|^2 y^2 j''(y) \quad (1.48)$$

I can write the solution of (1.48) as

$$j(y) = j_0(y) + Ay^{-\alpha} + By^{\beta}, \quad (1.49)$$

where $\alpha < 0$ and $\beta > 1$ are the roots of the quadratic

$$Q(t) \equiv \tfrac{1}{2}|\kappa|^2 t(t-1) + (\rho - r)t - \rho, \quad (1.50)$$

and j_0 is a particular solution. Observe that the equation (1.48) can be expressed as

$$0 = \tilde{u} - (\rho - \mathscr{G})j, \quad (1.51)$$

where $G \equiv \frac{1}{2}|\kappa|^2 y^2 D^2 + (\rho - r)yD$ is the infinitesimal generator of a log-Brownian motion

$$dY_t = Y_t\{|\kappa|\, dW + (\rho - r)dt\}, \quad (1.52)$$

So one solution would be to take

$$\boxed{j_0(y) = R_\rho \tilde{u}(y) \equiv E\left[\int_0^\infty e^{-\rho t}\tilde{u}(Y_t)\, dt \;\middle|\; Y_0 = y\right],} \quad (1.53)$$

Since $\tilde{u}$ is non-positive decreasing, it is clear that j_0 is also; moreover, since $\tilde{u}$ is convex, and the dynamics for Y are linear, it is easy to see that j_0 must also be convex. The solution j which I seek, of the form (1.49), must be convex, decreasing, and non-positive, so j_0 is a possible candidate, but what can I say about the terms $Ay{-}\alpha + By\beta$ in (1.49)? By considering the behaviour of j near zero, we see that the only way I can have j (given by (1.49)) staying decreasing and non-positive is if $A = 0$. On the other hand, since j_0 is convex non-positive, it has to be that $|j_0(y)|$ grows at most linearly for large y, and if $B \neq 0$, this would violate10 either the convexity or the non-positivity of the solution j. I conclude therefore that the only solution of (1.48) which satisfies the required properties is $j0$. (Roger, 2013)

2.2.3. Dynamic Programming

Dynamic programming is widely recognized in the area of continuous-time portfolio optimization as a powerful tool. This chapter will explore the foundations of dynamic programming and demonstrate its application through DSGE models.

A common trap in financial modeling is to model derived quantities and neglect their roots. This is what Rogers & Tehranchi discussed, leading to inconsistencies between these variables. Another similar mistake is that mathematical finance treats market clearing prices for asset prices as fundamental when they are not.

Therefore, one should pay attention to the agents' preferences, asset holdings and output processes which are the most important aspects of this problem. They represent general equilibriums which are difficult to solve. Nevertheless, it is necessary to derive models from these basic elements.

Suppose there exists an economy with a single asset that produces δt streams. The preference of the agents is specified by von Neumann-Morgenstern utility functions: being derived from fundamentals this mistake results in lack or inappropriate connections among such variables according Rogers & Tehranchi. A similar omission can be found in mathematical finance where asset prices derived from market-clearing prices in general equilibrium are often mistakenly treated as fundamentals.

This error must be rectified by focusing on the actual fundamentals such as agent's preferences, asset holding, and output process. These elements form a complete system and finding their solution may prove difficult at times but when possible, it makes sense to build models based on these components.

Imagine now that I have an economy having only one asset producing stream δt. The preferences of agents can be described by von Neumann-Morgenstern utility functions:

$$\mathscr{U}_j(c) \equiv E\left[\int_0^\infty u_j(t, c_t)\, dt\right],$$

where $u_j(t, \cdot)$ is increasing, strictly concave, and satisfies the Inada conditions.

Equilibrium and Price Processes

The goal is to find equilibrium price processes for assets and interest rates that clear the market. This implies that the total output is consumed, and the total wealth matches the asset itself. The individual agents' optimal solution links their marginal utility of consumption to their state-price density:

$$u'_j(t, c_t^j) = \zeta_t^j$$

which defines the state-price density for agent j, determining how agent j prices all assets and contingent claims.

Market-Clearing Conditions

For markets to clear:

$$\sum_j c_t^j = \delta_t, \qquad \sum_j w_t^j = S_t.$$

This signifies that all the output of the asset is exactly consumed and the total wealth equals the asset itself. In equilibrium, the asset price at time t can be valued by any agent j as:

$$S_t^j = (\zeta_t^j)^{-1} E\left[\int_t^\infty \zeta_s^j \, \delta_s \, ds \,\middle|\, \mathscr{F}_t\right],$$

Central Planner Equilibrium

A central planner equilibrium is considered as a variant of the representative agent equilibrium, where a central planner optimizes the weighted sum of individual utilities subject to the aggregate consumption constraint.

Complete Market Case

In a complete market, a unique state-price density up to constant multiples exists, and market-clearing conditions provide the relationship:

$$\delta_t = \sum_j c_t^j = \sum_j I_j(t, \eta_j \zeta_t)$$

where η_j are constants derived from the initial wealth of the agents.

Example: Single Asset Economy

Consider an economy with a single asset that provides a log-Brownian dividend process. The goal is to determine the equilibrium price of this asset and the state-price density in a complete market setting.

The dividend process δ_t follows a log-Brownian motion:

$$d\delta_t = \delta_t(\sigma dW_t + \mu dt),$$

where, σ is the constant volatility, μ is the constant drift, Wt is a standard Brownian motion.

Assume the agents have utility functions of the form:

$$u_j(t,c)=e^{-\rho t}u_j(c),$$

where ρ is the common discount rate, and $uj(c)$ is the utility derived from consumption c, which is C^2, strictly concave, increasing, and satisfies the Inada conditions.

The market-clearing condition in this economy is:

$$\delta_t=\sum_j I_j(\eta_j e^{\rho t}\zeta_t),$$

where I_j is the inverse of the marginal utility function u_j', η_j are constants derived from the agents' initial wealth, and ζt is the state-price density.

To find the state-price density ζt, we introduce the function f:

$$e^{\rho t}\zeta_t=f(\delta_t)\equiv f(x_t).$$

This function can be inverted to find δt in terms of ζt.

The stock price S_t at time t can be expressed using the state-price density and the expected discounted value of future dividends:

$$S_t = \tilde{\varphi}(\delta_t) \equiv \varphi(x_t)$$

where:

$$\begin{aligned}\tilde{\varphi}(\delta) = \varphi(x) &= \tilde{f}(\delta)^{-1}E\left[\int_0^\infty e^{-\rho t}\delta_t\tilde{f}(\delta_t)\,dt \,\middle|\, \delta_0=\delta\right]\\ &= f(x)^{-1}E\left[\int_0^\infty e^{-\rho t}e^{x_t}f(x_t)\,dt \,\middle|\, x_0=x\right].\end{aligned}$$

Given that agent j's consumption stream is $c^j{}_t=I_j(\eta_j e^{\rho t}\zeta_t)$, which is a function of x_t alone, we can deduce the wealth process $w_j(t)$ as:

$$\begin{aligned}w_j(t) = \psi_j(x_t) &= \tilde{f}(\delta_t)^{-1}E\left[\int_t^\infty e^{-\rho(s-t)}\tilde{f}(\delta_s)\,q_j(x_s)\,ds \,\middle|\, \mathscr{F}_t\right]\\ &= f(x_t)^{-1}E\left[\int_t^\infty e^{-\rho(s-t)}f(x_s)\,q_j(x_s)\,ds \,\middle|\, \mathscr{F}_t\right]\end{aligned}$$

Using the resolvent operator R_0 of the diffusion, we have:

- $f(x)\phi(x)=(R_0F)(x)$,
- $f(x)\psi j(x)=(R0Qj)(x)$,

where $F(x)=e^{\rho t}f(x)$ and $Q_j(x)=f(x)q_j(x)$.

The resolvent density for the log-Brownian motion is given by:

$$r_\rho(x,y) = r_\rho(0, y-x) = \exp\left(\frac{c(y-x) - |y-x|\sqrt{c^2+2\rho\sigma^2}}{\sigma^2}\right) / \sqrt{c^2+2\rho\sigma^2},$$

where

$$c = \mu - \tfrac{1}{2}\sigma^2.$$

Hence

$$f(x)\varphi(x) = \int r_\rho(x,y)\, e^y f(y)\, dy = \int r_\rho(0, y-x)\, e^y f(y)\, dy,$$

in which we recognize a convolution integral, which can be evaluated numerically using Fast Fourier Transform.

For the special case where all agents share a common CRRA (Constant Relative Risk Aversion) utility: $I(x)=x^{-1/R}$, $f(x)=e^{-Rx}$. In this case, from the above equations, we get that $S_t \infty \delta_t$, aligning with the Black-Scholes-Merton model for a stock paying dividends at a constant proportional rate.

The example illustrates how dynamic programming and equilibrium models provide a rigorous foundation for understanding asset pricing and wealth distribution in an economy. The solution involves finding the state-price density and using it to determine equilibrium prices and agents' optimal consumption and wealth processes.

This chapter illustrates that dynamic programming in continuous-time portfolio optimization is a robust framework that integrates economic theory with mathematical rigor. It ensures that investment strategies derived from such models are not only mathematically coherent but also economically justifiable. (Jacobsson, 2022)

2.3. Implications of the Merton Problem on Modern Portfolio Theory

The Merton Paradox, which was named after Robert C. Merton, an economist who went beyond the theories of Paul Samuelson and advanced the boundaries of portfolio planning, is one of the critical contributions to Modern Portfolio Theory (MPT). Merton’s continuous-time optimization framework has led to a far reaching impact on MPT, altering our perception about optimum investment and consumption options in an uncertain environment over time.

This paper articulates a dynamic wealth accumulation and allocation process whereby an investor seeks to maximize utility from consumption and terminal wealth over a continuous time horizon. It uses stochastic control methods for determining optimal portfolio paths that adapt to changing market conditions and personal circumstances.

2.3.1 Extending the Markowitz Approach

Conventional MPT entailed optimizing portfolios through a trade-off between risk and return based on a single period model developed by Harry Markowitz. This is augmented by Merton's continuous time model which incorporates lifetime span in a stochastic environment thereby bringing temporal element into investment decision making (Markowitz, 2008). Key Contributions to MPT

1. Optimal Consumption and Investment Strategies: In order to make wealth useful over time it takes into account the consumption aspect while embedding it in the investment problem.

2. Incorporation of Stochastic Processes: The model links investment strategies with predictable or unpredictable events throughout any investor’s lifespan by capturing randomness associated with market returns as well as other phenomena.

3. Dynamic Asset Allocation: He introduced this concept as a means where proportions of risky assets are never constant making them vary when market conditions change or situations change for investors.

4. Intertemporal Hedging Demand: This strategy mitigates uncertainty regarding future investments opportunities or preferences related to what will be consumed tomorrow, next year or in future general.

2.3.2 Modern Portfolio Management

Modern portfolio management largely relies on the approach adopted by analysts who try to apply more dynamic frameworks such as:

• Human Capital Considerations: These can involve aspects like labor income and retirement planning for an investor that can be integrated into asset allocation exercises.

• Liability Driven Investing: That takes into account future liabilities OE objectives since individual utility is unique to each person.

• Lifecycle Investing: This means a change of the nature of the risk profile over time, most often maturing from high-risk to low-risk assets with age among investors.

However, Merton's model has certain assumptions that have been criticized and modified in recent applications as follows:

• Utility Functions: The new versions of utility functions take into consideration the various behavior patterns as well as different degrees of risk aversion.

• Market Completeness: There are situations where real markets are not complete thus there is need for adjustments to this effect by looking at market frictions and constraints in such models.

• Computational Complexity: In other words, solutions to numerical problems associated with implementation of Merton's policies continue to be developed up to date under the field referred to as computational finance.

The Merton Problem has brought about significant changes in MPT. Based on this problem, this paper refers to a broad-based portfolio optimization framework which fits with the whole industry rather than just one aspect. It features wealth formation versus distribution alongside an investment strategy decision setup for an entire life range. Therefore it remains an integral part of financial economics influencing both theoretical development and practical asset management.

3. Extension of the Merton Model

3.1 Choice variables in models with finite and infinite time horizon

In investment models, choice variables in portfolio optimization signify the most important parts of the decision-making process that affect an investor's asset allocation. These components are used to develop a strategic plan for allocating assets to maximize returns, minimize risks, or attain an optimal balance between these two aims. They are so important in portfolio management because they can be used directly to create investment strategies, shaping and affecting the overall performance and healthiness of the investments made.

What makes choice variables strategically significant is their direct influence on risk and return relationships within a given portfolio. For instance, one may choose to adjust his/her investments towards more risky securities such as shares, which carry higher expected earnings but also expose him/her to a greater probability of losing significant amounts of money. Similarly, someone else might decide to increase bonds or other fixed incomes, thus reducing volatility levels within his/her investment while at the same time depressing potential rewards from it. Therefore, it becomes essential that we make precise changes to them if we want particular goals achieved through investments.

All in all, effective management of choice variables is key to realizing desired financial outcomes. Whether an individual wants to save for retirement, generate steady income or protect wealth; he/she must intelligently adjust these parameters so that they can help design appropriate strategies taking into account ones' risk appetite, time horizon among other things related with personal finance planning . By integrating deep knowledge about them into practice, people would have better chances of gaining success not just during different market cycles but also in various institutional settings where financial services are rendered. Such approach fosters more refined decision making process required when dealing with intricacies prevailing in today's money markets.

3.1.1 Fundamental Aspects of Choice Variables and Types of Choice Variables

Principles of Essential Variables of Choice

1. **Nature and Purpose:** Understanding choice variables is crucial as they are the specific, controllable actions within an investment strategy. For instance, they dictate the allocation of the portfolio across various asset classes such as stocks, bonds, commodities, or alternative investments like real estate and private equity. This understanding empowers you to make informed decisions about your investments.
2. **Strategic Implications:** The selection and manipulation of these key inputs significantly impact the expected return on investment (ROI) and riskiness measured by standard deviation (SD), among other factors. This information can guide you in building your portfolio based on your personal financial objectives, risk appetite, and current market views.
3. **Dynamic Adjustments:** It's essential that choice variables adapt over time to changes in ambient conditions such as market environments. This flexibility allows you to remain consistent with your targets while also being able to adapt if necessary, ensuring the robustness of your portfolio under various economic scenarios.

Types Of Decision-Making Factors

1. **Asset Allocation:** It is arguably the most important one, i.e., where money should be put and distributed across various forms of investments. Allocation decisions are guided by anticipated returns on each class vis-à-vis its risks plus correlations between these classes which may impact overall volatility levels within a given portfolio.
2. **Rebalancing Strategy:** How often should I adjust my holdings to match up against benchmarks? When do you decide what assets have gained too much value and those that have not performed well enough? Rebalancing becomes necessary to maintain the original balance while keeping the desired exposure level constant over time once target asset allocation has been achieved.
3. **Derivative Utilization:** Derivatives like options or futures can hedge against risks or amplify rewards, respectively. It matters whether specific derivative instruments will be included in a particular plan and how extensively they might

be employed since such decisions can significantly affect results regarding portfolios.

4. **Tax Considerations:** Tax consequences remain essential, especially regarding personal investment decisions. What matters here is maximizing pre-tax returns and after-tax ones, thereby making tax efficiency a vital goal that should be considered during the strategy formulation process.
5. **Liquidity Requirements:** Liquidity, or the ease with which assets can be converted to cash without a significant loss in value, is another important choice variable. Investors need to decide how much liquidity is necessary to meet potential cash flow needs without disrupting the investment strategy's overall effectiveness.

Application in Models

In models of portfolio optimization, the decision variables are frequently parameters which can be altered within the model's bounds to locate the greatest possible investment portfolio given some assumptions. Scenarios may be examined and results produced by using quantitative approaches such as mean-variance optimization or Monte Carlo simulations or more advanced machine learning methods based on different values for these inputs.

Setting and adjusting choice variables correctly is crucial in aligning investment strategies with personal requirements and prevailing market circumstances; thus making them key elements of successful portfolio management.

3.1.2 Choice Variables in Infinite Time Horizon Models

Infinite time horizon models is a method of managing investments in the portfolio optimization universe without any fixed endpoint. Such models are necessary for cases where the goal is to perpetuate wealth through generations or endowments which have perpetuity as their main purpose. In these scenarios, sustainability becomes more important than anything else since what is needed primarily is an investment plan that can allow for continuous withdrawals while at the same time keeping its value intact over unlimited period of time.

Choice Variables in Infinite Time Horizon Models

- **Sustainable Withdrawal Rate:** The most important thing about these models is finding out what withdrawal rate will not compromise with how long portfolio survives. The problem lies on determining what percentage should be used so that it grows together with investment; this means that there should be no withdrawal exceeding return generated from investing. This rate has to consider inflation impact, market changes and need for principal to last forever.
- **Asset Allocation:** Asset allocation under infinite time horizon models takes place throughout different phases dynamically. It requires strategic thinking which focuses not only on present income but also total return encompassing capital gains plus interest earned over time. Thus, allocations ought to respond flexibly towards shifting economic environment and include those types of assets that offer higher prospects of long run growth or stability.
- **Risk Management**: Risk management within infinite time context needs complex strategies due to various reasons involved in it. Here we are looking at risks associated with finance over extended periods such as those related to market unstability, economic recessions and high inflations. At this level diversification becomes very significant together with assets capable of hedging against different forms of risks during different times. Henceforth resilience must be built into a portfolio so that it can survive through ups & downs cycles without compromising ability to make regular withdrawals even when economy slows down.

Considerations for Long-term Growth and Stability

When optimizing the financial goals over time, there is a strong emphasis on those strategies that seek continuous development. This could involve putting money into various assets such as shares with good track records on dividends, bonds that give reliable yields, real estate, and even alternative investment vehicles expected to appreciate over long periods.

The designs tend to be conservative because they provide stability rather than speculative gains. However, this does not mean that the methods do not have inflation-beating targets; they should ensure purchasing power protection against erosion with time. Therefore, balancing short-term market downturn protection and long-term growth positioning for any given set of investments is necessary.

The optimization process also considers legal and fiscal frameworks that may impact estate planning or endowment funding within unlimited life span models. This means that individuals can structure their portfolios in ways that would enable them to benefit from tax-efficient growth. At the same time, institutions need only comply with relevant regulations governing the management of these funds.

In conclusion, success in managing portfolios with infinitely enduring targets will depend mainly on how skillfully one manipulates decision variables. It implies making wise choices that guarantee survival across different eras, catering to current financial demands alongside future needs, or aligning organizational objectives toward sustainable development over extended periods. (ReSolve Asset Management, n.d.)

3.1.3 Choice Variables in finite Time Horizon Models

Finite time horizon models in portfolio optimization are made for the investment period, which is fixed and specified. Such models are helpful when investors clearly know where they want to reach, for example, retirement, buying a house, or funding education. Judging whether the portfolio meets those objectives by a given deadline is essential in such cases.

Features Of Finite Time Horizon Models

These are characterized as models that are designed to achieve investment goals over some time frame. This means there is an upper limit on how long we can wait until we recover from potential losses, affecting risk and return management. Since this point will be reached soon enough, usually towards the end of the day, it may require changing our approach towards investing, starting from growth-oriented strategies up to capital preservation initiatives aimed at ensuring the availability of funds whenever needed.

Choice Variables In Finite Time Horizon Models

- **Asset Allocation:** This refers to deciding on initial percentages of different securities within a given portfolio, such as stocks and bonds. In the early stages of investments, one might choose more risky assets like shares. However, as time goes by, this could change so that more conservative instruments, such as government debt, become preferable due to their low volatility levels relative to equities.

- **Rebalancing Frequency:** How often should rebalancing take place? It is essential to know what frequency works best under specific circumstances. Failure to adjust allocations may expose us to too much or too little risk depending on market conditions throughout the investment period. For instance, rebalances may need to occur frequently when approaching the terminal date, enabling slow-down exposure toward volatile areas.
- **Consumption Rate:** The rate at which people spend money matters most, especially if one wants to retire comfortably without running out of cash before dying. Also known as optimal consumption rate setting, it involves balancing immediate needs for current income production against future withdrawal requirements based on maintaining adequate reserves capable of generating additional revenue streams required to survive longer life spans.

Considerations for Optimizing Terminal Wealth

Different considerations come into play when optimizing terminal wealth in managing finite time horizon portfolios:

- **Expected Returns and Risks**: The choice of assets may be influenced by their expected returns and associated risks. Understanding the risk-return profiles for various classes of assets can help us design a portfolio that will deliver desired levels of return while keeping within tolerable limits on risk.
- **Liquidity Needs**: Liquidity becomes more critical as we approach the endpoint. Therefore, it is necessary to ensure that our portfolio has enough liquidity so that we do not sell them in unfavorable market conditions, thus incurring significant losses.
- **Tax Considerations**: Taxes impact net returns from investments through purchase, holding, and sale phases. It is, therefore, essential to plan taxes efficiently and know which accounts or investments are tax-advantaged to maximize after-tax returns required towards achieving financial goals set at the endpoint.
- **Adjustment for Changing Market Conditions**: Financial markets are constantly changing, with time being a critical resource constraint. To this effect, any strategy with fixed limits must allow for adjustments whenever necessary lest gains made should be lost because opportunities were missed but still within the overall risk management strategy.

All these factors considered by investors who use finite time horizon models will enable them to increase their wealth and safeguard their accrued profits towards realizing critical points in relation to attaining personal monetary targets. In other words, such an approach ensures that our investments are contextually relevant vis-à-vis current market realities and future-oriented given individual aspirations. (ReSolve Asset Management, n.d.)

3.2 Interest rate risk and Habit formation

In this part, I focus on the influence of fluctuating rates on handling investment portfolios. In less complicated models, I work with a more practical situation than a fixed interest rate. Such a scenario assumes that the interest rate changes over time since this happens in reality, where rates can be up or down, affecting investment choices and returns significantly.

To represent these variations in interest rates, I adopt the Vasicek model, a mathematical model. This model will enable us to understand and predict how interests behave when changing. Thus, our primary objective is to find out how we can best manage the portfolio, given these unpredictable fluctuations of rates.

I shall look into some complex mathematics equations, which describe an increase in an individual's wealth over time when investing in risky assets such as savings accounts or government bonds with variable interest rates. The formulas are complex and, therefore, do not have direct solutions; hence, I will rely on computer-based methods for determining optimal investment strategy.

To sum up, my concern is about what should be done with my money, considering different possibilities brought about by changing the interest rates, hence getting better results. This means balancing consumption today and saving for tomorrow while riding along waves of these interest rates.

3.2.1 Interest rate Risk

This time I take the wealth dynamics to be

$$
\begin{aligned}
dw_t &= r_t w_t dt + \theta(\sigma dW_t + (\mu - r_t)dt) - c_t dt \\
dr_t &= \sigma_r dB_t + \beta(\bar{r} - r_t)dt,
\end{aligned}
$$

the salient difference being that the riskless rate is no longer supposed constant, but follows a Vasicek process. The parameters σ_r and $\bar{r}$ are constants, and the two Brownian motions W and B are correlated, $dWdB = \eta dt$. The objective will be

$$
V(w, r) = \sup E\left[\int_0^\infty e^{-\rho t} u(c_t)\, dt \,\middle|\, w_0 = 0, r_0 = r\right]
$$

where as usual $u(w) = w^{1-R}/(1 - R)$.

A moment's reflection shows that the solution of the Merton problem now will still scale, with the value function taking the form

$$
V(w, r) = u(w) f(r).
$$

Writing down the HJB equation for this problem, I find (with $c = qw$, $\theta = sw$)

$$
\begin{aligned}
0 &= \sup\left[u(c) - \rho V + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} + \sigma\sigma_r\eta\theta V_{wr} + \tfrac{1}{2}\sigma_r^2 V_{rr} + (rw + \theta(\mu - r) - c)V_w + \beta(\bar{r} - r)V_r\right] \\
&= \sup u(w)\left[q^{1-R} - q(1 - R)f - \rho f - \tfrac{1}{2}R(1 - R)\sigma^2 s^2 f + (1 - R)\sigma\sigma_r\eta s f' + \tfrac{1}{2}\sigma_r^2 f'' \right. \\
&\quad \left. + (r + s(\mu - r))(1 - R)f + \beta(\bar{r} - r)f'\right].
\end{aligned}
$$

Now optimising this over q and s gives us

$$
\begin{aligned}
q &= f^{-1/R}, \\
s &= \frac{(\mu - r)f + \sigma\sigma_r\eta f'}{\sigma^2 R f}
\end{aligned}
$$

and when substituted back in gives the following second-order ODE for the HJB equations:

$$
0 = Rf^{1-1/R} - \rho f + r(1 - R)f + (1 - R)\frac{\{(\mu - r)f + \sigma\sigma_r\eta f'\}^2}{2\sigma^2 R f} + \tfrac{1}{2}\sigma_r^2 f'' + \beta(\bar{r} - r)f'.
$$

The HJB ODE derived from the optimization problem does not have a closed-form solution, meaning that one cannot simply write down an explicit equation for the solution. As a result, numerical methods must be used. The method chosen for the numerical solution is based on the concept of policy improvement. This is a technique often used in dynamic programming, which iteratively improves the control policy (in this case, the investment/consumption strategy) until an optimal policy is found. To implement the numerical method, the continuous problem is discretized. This means that the continuous range of possible interest rates is broken up into a finite set of values on a grid. This grid is centered around the long-term mean of the interest rate, r_∞, and is designed to cover a wide range of interest rates by extending multiple standard deviations (specifically seven standard deviations in this example) on either side of this mean. When solving differential equations, it's important to specify boundary conditions. Here, the boundary conditions are set to be reflecting at both ends of the interval defined by the grid. Reflecting boundary conditions imply that the solution will not "pass through" these boundaries; this can be thought of as the interest rate 'bouncing back' if it reaches a value at the edge of the considered range. To test the sensitivity of the numerical solution to the choice of grid and ensure that the numerical solution is robust it is important to calculate the efficiency at r=0 using different grid widths (standard deviation grids). (Roger, 2013)

3.2.2 Habit formation

Habit formation is a crucial idea for understanding personal spending and saving. It has been noted that what people consume in the present is highly influenced by what they consume in the past. In this chapter, we look at habit formation theory related to portfolio choice and consumption-saving decisions to gain insight into how individuals change their investment strategies over time while being sensitive to habitual consumption patterns.

Under traditional utility maximization models, consumers compare current consumption levels with previous periods. The model introduced by Constantinides in this section does this by considering an exponentially smoothed moving average of historical consumption records when computing an agent’s utility.

The dynamics taken are a simple variant of the usual wealth equation:

$$
\begin{aligned}
dw_t &= rw_t dt + \theta_t(\sigma dW_t + (\mu - r)dt) - c_t dt \\
d\bar{c}_t &= \lambda(c_t - \bar{c}_t)dt.
\end{aligned}
$$

The agent's objective in Constantinides' account is

$$\sup E \int_0^\infty e^{-\rho t} u(c_t - \bar{c}_t)\, dt$$

so that present consumption is in some sense evaluated relative to the exponentially-weighted (EW) average $\bar{c}_t$ of past consumption. If I use a CRRA utility u, then what I find is that the consumption may never fall below $\bar{c}$, so the agent must keep $\bar{c}_t/r$ in the bank account to guarantee that level of consumption, and then he invests the remaining wealth $w_t - \bar{c}_t/r$. very much as before. (Roger, 2013)

What I propose to do here is to keep the dynamics of the model, but to take as the objective

$$V(w, \bar{c}) \equiv \sup E \left[\int_0^\infty e^{-\rho t} u(c_t/\bar{c}_t)\, dt \,\middle|\, w_0 = w, \bar{c}_0 = \bar{c} \right]$$

which rewards the *ratio* of current consumption to the EW average. This objective permits current consumption to fall below the EW average of past consumption at various times, again a more realistic feature.

The problem does not admit a simple closed-form solution, in contrast to the problem studied by Constantinides, but there is an obvious scaling, for any $\alpha > 0$:

$$V(\alpha w, \alpha \bar{c}) = V(w, \bar{c}),$$

which allows us to write more simply

$$V(w, \bar{c}) = V(w/\bar{c}, 1) \equiv v(w/\bar{c}).$$

The solution is a function of the scaled variable $x_t \equiv w_t/\bar{c}_t$ alone, so I must first understand how this process evolves. I introduce the notation $q_t = c_t/\bar{c}_t$ for the scaled consumption rate. Some routine calculations with Itô's formula give us the dynamics of x:

$$dx_t = rx_t\, dt + \varphi_t(\sigma dW_t + (\mu - r)dt) - (\lambda x_t + 1)q_t dt + \lambda x_t dt,$$

where $\varphi = \theta/\bar{c}$. This dynamic is interesting because, although the dependence on the portfolio variable φ is conventional, the dependence on the consumption variable q is not. One observation should be made straight away. It is always a feasible strategy to come out of the risky asset completely ($\varphi \equiv 0$), and to maintain x at its current level; from the last equation, this implies that I could maintain q at the constant value

$$q^{(0)} = \frac{(\lambda + r)x}{1 + \lambda x}$$

forever, guaranteeing that the value of the problem would be $\rho^{-1}u(q^{(0)})$. So the value is bounded below by

$$v(x) \geq \rho^{-1}\, u\left(\frac{(\lambda + r)x}{1 + \lambda x}\right).$$

For very small x , I would expect that the portfolio φ would have to be small, since x has to be kept non-negative, and if φ remained bounded away from zero as $x \downarrow 0$, the volatility arising from the investment in the risky asset would carry x below zero. This gives us the boundary condition

$$\lim_{x \downarrow 0} v(x)/u(x) = \rho^{-1}\, (\lambda + r)^{1-R}.$$

I have reduced the problem to finding

$$v(x) \equiv \sup_{\varphi, q} E\left[\int_0^\infty e^{-\rho t} u(q_t)\, dt \,\middle|\, x_0 = x\right]$$

In these terms, the HJB equations become more simply

$$\sup_{\varphi, q}\left[u(q) - \rho v + \{rx + \varphi(\mu - r) - (1 + \lambda x)q + \lambda x\}\, v' + \tfrac{1}{2}\varphi^2\sigma^2 v''\right] = 0.$$

As usual, optimal values of q and φ are found explicitly from

$$u'(q) = (1 + \lambda x)v'(x), \quad \varphi = -\kappa v'/\sigma v''.$$

I can transform to the dual equation (via $z \equiv v'(x)$, $J(z) = v(x) - xz$), but the second-order ODE which results:

$$\tilde{u}(z(1 - \lambda J')) - \rho J + (\rho - r - \lambda)zJ' + \tfrac{1}{2}\kappa^2 z^2 J'' = 0$$

no longer admits a closed-form solution, so I am forced down a numerical path.

To estimate the optimal consumption and investment policies within this framework, has been implemented two distinct numerical methods.

The first method is rooted in policy improvement. This approach assumes that the specified lower boundary condition strictly holds at the termini of the chosen numerical grid. Policy iteration refines decision-making rules step by step by turning the optimization task into a Markov decision process that stops at boundary points. The algorithm converges to an optimal policy over the discretized state space because it is iterative. (Roger, 2013)

At the same time, I have also tried another way of dealing with dual Hamilton-Jacobi-Bellman equations. After replacing $s = \log z$, where z represents a transformation of value function, the original HJB equation becomes a differential equation with constant coefficients. Next, the Newton-Raphson method was utilized, considering its ability to find the roots of the equation for solving this new formulation. I used natural boundary conditions and made λ take on different values to robustly obtain our solutions using a comparative diagnostic check.

A series of plots were used to present numerical results obtained through these approaches, consistent with expected theoretical behaviors. In particular, the wealth-consumption trajectory initially goes up rapidly but starts leveling off once it reaches higher levels of wealth. This pattern reveals the effect produced by habituation and confirms that consumption should speed up as people get richer but slow down when they become more accustomed to things.

To summarize, the part on numbers shows why we need complicated computational techniques if there are no analytic answers for models involving habits. Numerical findings back what theory predicted, proving the effectiveness of numerical methods used and showing relevant portfolio selection and savings decisions under habit formation.

3.3 Incorporating Stochastic Volatility Models

Financial markets thrive on unpredictability. It is an indicator of price changes in unceasing assets. Conventional decision-making models in investments rely on constant volatility; this assumption does not consider the disorderly nature of trading floors. However, it only works when the market is stable and fragile and frequently collapses under real-world pressure.

Sophisticated financial analysis deals with uncertainty directly. Stochastic volatility models are the most significant breakthroughs because they do not accommodate randomness or variability. Instead, they assume them, model them, and wrap their every part around these forces instead of their constant-volatility counterparts, which work best under stable market conditions alone.

We need to rethink our portfolio optimization, given stochastic volatility. Now, the decision variables – points where an investor can act on his strategy – move against a backdrop that keeps shifting. Asset allocations were optimized for static environments, so they should be revised considering changing risk profiles brought about by stochastically volatile settings.

What this chapter does is link practical mechanics of portfolio management with an advanced sense of volatility brought by these random elements into our models, which enables us to react faster, sharpening investment strategies vis-a-vis current situations and then opening up the next chapter more deeply analyzing how traditional models should adjust for capturing proper awareness about dynamicity induced into optimization landscape through stochastic volatilities affecting markets awareness.

3.3.1 Literature Review

The financial markets have been transformed by using models with stochastic volatility, which has changed my thinking about them. My thoughts have been influenced by important papers that were practical guides for the industry and paved new directions for research in academia. In this paper, I will discuss some implications of using these models in finance theory and practice while adding to ongoing discussions on their strengths and weaknesses.

Stochastic volatility models took a significant leap forward with Heston's model (1993). It was an extension from Black-Scholes, which introduced stochasticity into the process of pricing options – a square-root diffusion process capable of capturing clusteredness and smile effect of volatilities observed in real-world markets (Black & Scholes, 1973). The breakthrough here is that Heston made it possible to analytically price European options, something rarely done for exotic derivatives whose payoffs depend on multiple variables. (Heston, 1993)

Hull and White (1987) achieved another similar level when they proposed an autoregressive structure for modeling volatilities within financial instruments; moreover, they devised a calibration procedure against market data, making it more realistic. Hull-White served as an inspiration point where various researchers could either modify them further or even extend beyond what has been done so far toward describing how one deals with uncertainty under different situations surrounding globalizing economies driven mainly by technological changes. (HULL & WHITE, 1987)

These new contributions resulted in many academic papers that tried to examine the relations between the prices of assets and their volatility in a way that had not been done before. Such works considered different types of stochastic volatility, ranging from mean-reverting models to those allowing for jumps or levy processes, among others, to represent more faithfully what goes on in markets.

However, this was relevant to abstract research and had practical implications for finance-oriented businesses like banks, where risk management systems were greatly enhanced by these methods introduced through such models. For example, they became handy tools for derivative pricing, portfolio optimization, or strategic financial planning.

With time, other scholars continued working on these ideas, making them better by mixing them with empirical results and taking into consideration various computational advances achieved throughout history, which eventually led to models with more sophistication, such as Bates' stochastic volatility ones that incorporate jumps or Barndorff-Nielsen and Shephard's time-varying volatilities among others thus expanding our understanding about this subject matter.

The more people learned the more they could do in different situations. At one point, the literature started incorporating trader sentiment into investor behavior regulatory effect on asset prices, thus enabling a wide range of model options that could be used by traders, analysts, and portfolio managers to know what is happening in the market comprehensively. This continuous exploration of different stochastic volatility models shows that these instruments are still highly applicable in contemporary finance because they connect theoretical finance with actual market conditions.

3.3.2 Theorethical Framework

Stochastic volatility models are important in financial mathematics because they help describe complex market movements. These models are based on stochastic differential equations (SDEs)—strong mathematical tools that specify how both asset prices and their volatilities change over time.

The main idea behind stochastic volatility is that it treats volatility as a dynamic random process rather than a fixed parameter. This stands in contrast to the constant volatility assumption made in the Black-Scholes model, which is foundational but not very realistic given what we see happening in real-world markets.

Usually, SDEs for such models involve several linked equations. The first one of these describes how prices move and includes a variable factor for volatility instead of assuming it to be constant like in simpler setups. Another SDE then governs the path that volatility follows, often with mean-reverting features and other stochastic elements designed to capture empirical regularities like sudden shifts or different sensitivities to events depending on where we're coming from vis-à-vis current levels.To understand the derivation of these SDEs, I begin with the foundational model for asset price S_t, typically expressed as:

$$dS_t = \mu S_t dt + \sqrt{\mathrm{v}_t S_t dW_t^S}$$

where μ is the drift coefficient, v_t represents the stochastic volatility process, and dW_t^S is a Wiener process representing the random market movements influencing the asset price. The stochastic volatility v_t itself is modeled by an equation such as:

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t dW_t^v}$$

where κ is the rate of mean reversion, θ is the long-term mean volatility,σ is the volatility of the volatility, and dW_t^v is another Wiener process that may be correlated with dW_t^S. These equations together form a system that reflects the intertwined evolution of asset prices and volatility. The correlation between dW_t^S and dW_t^v captures the leverage effect, a key market observation that asset prices and volatility are negatively correlated.

A quintessential manifestation of stochastic volatility in financial markets is the volatility smile—an empirical observation that the implied volatility of options is a function of strike price and expiration. Traditional models, which assume constant volatility, predict a flat structure for implied volatility across strikes. The smile effect, however, reveals higher implied volatilities for deep in-the-money and out-of-the-money options. Stochastic volatility models account for this curvature and offer a theoretical foundation for understanding and quantifying the smile.

As I contemplate the inclusion of stochastic volatility into portfolio optimization, the volatility smile serves as a crucial point. It reminds us that the assumptions governing our choice of models have profound implications on the strategies I devise. In the following sections, I will integrate these stochastic volatility constructs into portfolio optimization frameworks, examining their implications for asset allocation decisions and overall portfolio management. (Samuelson, 1969)

3.3.3 Model Implementation

Integrating stochastic volatility into the Merton portfolio optimization framework is a complex yet insightful endeavor. It requires an expansion of the traditional Hamilton-Jacobi-Bellman (HJB) equations to factor in the variable nature of market volatility. This extension enables the modeling of an investor's wealth where asset prices and volatility evolve according to stochastic processes.

The starting point is to redefine the investor's wealth dynamics, considering not only a risk-free asset and a risky asset but also the uncertain movements of the volatility itself. Under this model, the investor's wealth W_t at any given time t is a function of the chosen rate of consumption c_t, the portion of wealth π_t invested in the risky asset, and a stochastic risk-free interest rate r_t:

$$dW_t = (r_t W_t + \pi_t(\mu_t - r_t) - c_t)d_t + \pi_t \sigma_t dZ_t$$

where μ_t and σ_t are the expected return and stochastic volatility of the risky asset, respectively, and dZ_t denotes the Brownian motion.

To account for these additional layers of randomness, the HJB equation—which describes the optimal control of the portfolio—must be adjusted. This equation now encapsulates the maximization of the investor's expected utility derived from both consumption and the terminal wealth, with the added complexity of a volatile market environment:

$$0 = \max_{c_t,\pi_t}\left(\frac{\partial V}{\partial t} + \max_{\pi_t}\left[(r_t W_t + \pi_t(\mu_t - r_t) - c_t)\frac{\partial V}{\partial W_t} + \frac{1}{2}\pi_t^2\sigma_t^2\frac{\partial^2 V}{\partial W_t^2}\right] - \lambda(c_t, W_t) + V_{r_t} f(r_t) + \frac{1}{2}V_{r_t r_t} g^2(r_t)\right)$$

In this equation, V is the value function representing the maximized expected utility, V_{rt} and V_{rtrt} are its first and second derivatives with respect to the stochastic rate r_t, $f(r_t)$ and $g(r_t)$ represent the dynamics of the stochastic rate, and $\lambda(c_t,W_t)$ is the utility function of consumption.

To empirically implement and solve the extended HJB equation, simulation techniques such as Monte Carlo methods or finite difference schemes are often employed. These simulations explore the efficacy of portfolio optimization under both constant and stochastic volatility scenarios, contrasting the outcomes to highlight the influence of volatility's randomness on investment decisions.

Through such a comparative analysis, one can evaluate the alterations in optimal asset allocation and consumption strategies necessitated by stochastic volatility. The findings typically reveal significant differences in risk-return profiles between portfolios optimized under constant versus stochastic volatility conditions. These insights provide an empirical testament to the practical significance of accommodating stochastic volatility in the pursuit of portfolio optimization, enhancing our understanding of risk management and strategic financial planning.

4. Practical Application and Case Study

Bearing this in mind, portfolio optimization — as discussed in this paper — is a powerful mathematical technique that can be used to construct effective investment portfolios. Optimization models are used to determine an ideal return for a given level of risk or the minimum risk for a desired return by evaluating the risk and return attributes of different assets.

However, as many have said, ‘What good is it if it does not work?’ The following section looks at how we can put these ideas into practice when dealing with real-world investments. I intend to look at case studies that will help us understand better how portfolio optimization translates from theory into practice.

During these instances, I will see how much difference optimization methods can make, showing what investors could achieve even with dissimilar aims and risk tolerance levels. I will also examine various optimization models designed around diversified investing scenarios comprising stocks, bonds — and derivatives. These types of funds consider different classes of assets, thus widening my understanding of the flexibility of portfolio optimization in reality.

This part seeks to fill in some gaps between theories and their applications. By studying examples like this one while applying them to different models, I can gain insights into how people can use Portfolio Optimization to meet their financial objectives.

4.1 Real Case Analysis of Portfolio Optimization

Case Study 1: Diversification and Risk Reduction (Connecting to Merton's Problem)

1. **Investor Situation:** An entrepreneur named Michael recently sold his startup for a significant sum (say, $5 million). He seeks to invest the proceeds to achieve long-term financial security but is concerned about the risk of a single investment downturn.
2. **Problem:** Merton's Problem highlights the risk associated with concentrating capital in a single asset. If the startup's success was heavily reliant on a specific market or technology, Michael's entire investment could be vulnerable to unforeseen circumstances.
3. **Optimization Approach:** A financial advisor recommends portfolio optimization with a focus on diversification. This involves allocating Michael's wealth across various asset classes with low correlations, such as:
    - Stocks in different sectors and geographic regions
    - Bonds with varying maturities and credit risks
    - Potentially alternative investments like real estate or commodities

4. **Results:** By implementing a diversified portfolio, Michael achieves several benefits that align with Merton's Problem's focus on managing risk:

    - Reduced overall portfolio risk: Diversification spreads the risk across different asset classes. Even if a specific sector experiences a downturn, losses would be offset by the stability of other asset classes, similar to the concept of optimal investment decisions in Merton's Problem.
    - Potentially increased returns: By including a mix of asset classes with varying risk-return profiles, the portfolio could potentially achieve higher returns compared to a single, concentrated investment.

Case Study 2: Risk Management for Retirement Planning

Investor Situation: Sarah, approaching retirement, has accumulated a significant nest egg (say, $1 million) in her retirement portfolio. She desires a steady income stream throughout retirement but worries about market volatility impacting her savings.

- **Problem:** Merton's Problem emphasizes the importance of considering risk tolerance and future liabilities when making investment decisions. Sarah's primary concern is preserving capital to ensure a comfortable retirement, highlighting the need for risk management strategies.
- **Optimization Approach:** A portfolio optimization strategy is implemented with a focus on risk management. This might involve:
    - Allocating a significant portion of the portfolio to low-risk assets like high-quality bonds and government securities.
    - Including a smaller portion of growth-oriented assets like stocks, but with a focus on lower volatility options (e.g., dividend-paying stocks).
    - Utilizing a dynamic asset allocation strategy that adjusts the portfolio composition based on market conditions, potentially reducing exposure to riskier assets during market downturns.
- **Results:** By employing a risk-managed portfolio, Sarah achieves several benefits that align with Merton's Problem's focus on optimal investment decisions considering risk:
    - Reduced portfolio volatility: The allocation prioritizes low-risk assets, ensuring a more stable income stream throughout retirement.

    - Potential for growth: The inclusion of some growth-oriented assets offers the possibility of long-term capital appreciation to maintain purchasing power in the face of inflation.
    - Alignment with risk tolerance: The portfolio prioritizes capital preservation, aligning with Sarah's risk tolerance as she approaches retirement.

Case Study 3: Targeted Risk Management with Options

- **Investor Situation:** David, a young professional with a high-risk tolerance, has a significant portion of his portfolio invested in a concentrated portfolio of technology stocks. While he enjoys the growth potential, he wants to hedge against a major market downturn in the technology sector.
- **Problem:** Merton's Problem highlights the importance of considering downside risk. While David enjoys the potential for high returns from his technology stocks, a major sector downturn could significantly impact his portfolio.
- **Optimization Approach:** A portfolio optimization strategy is implemented with targeted risk management utilizing options. This might involve:
    - Maintaining the core allocation to technology stocks for growth potential.
    - Purchasing put options on a technology sector ETF (Exchange Traded Fund). Put options provide downside protection by allowing David to sell his holdings at a predetermined price (strike price) even if the market price falls.
- **Results:** By incorporating options, David achieves several benefits:
    - Preserves capital in a downturn: If the technology sector experiences a significant decline, the put options will allow David to sell his holdings at the strike price, limiting his losses.
    - Maintains growth potential: He retains the upside potential of his technology stocks if the market performs well.
    - Manages risk within his tolerance: This strategy allows David to maintain his aggressive investment style while mitigating potential downside risk, aligning with the risk management concepts in Merton's Problem.

These case studies showcase how portfolio optimization, building upon the theoretical foundation of Merton's Problem, translates into practical applications. By considering risk tolerance and utilizing diversification or risk management strategies, investors can create portfolios that optimize returns within their acceptable level of risk.

4.2 Application of models in diversified investment scenarios: stocks, bonds, and derivatives

Portfolio optimization builds upon the theoretical foundation established by Merton's Problem. While Merton's Problem focuses on optimal investment decisions for a single decision-maker, portfolio optimization extends these concepts to create efficient portfolios for investors with diverse goals and risk tolerances. This section explores how different models handle various asset classes (stocks, bonds, and potentially derivatives) within a portfolio optimization framework, considering the importance of risk management as highlighted by Merton's Problem.

1. Mean-Variance Optimization (MVO):

MVO remains a valuable tool for portfolio optimization, especially for basic asset allocation decisions involving traditional assets. Let's revisit its functionalities:

- **Strengths:**
    - **Focuses on Risk-Return Trade-off:** MVO explicitly considers both expected return and risk (volatility) of individual assets, similar to how Merton's Problem emphasizes balancing potential gains with downside risk. MVO's framework allows investors to quantify these factors and make informed decisions about their portfolio composition, similar to the calculations involved in Merton's Problem to determine the optimal investment strategy.
    - **Provides Optimal Allocation for Risk Tolerance:** By analyzing risk and return data, MVO helps determine the portfolio allocation that achieves the desired level of return within an investor's acceptable risk tolerance. This directly aligns with the core objective of Merton's Problem - making optimal investment decisions considering both potential returns and the level of risk an investor is willing to take. MVO helps investors find the efficient frontier, a concept similar to the optimal investment strategy identified in Merton's Problem.

- **Limitations:**
    - **Relies on historical data:** MVO assumes historical returns and volatilities are reliable predictors of future performance, which might not always be the case.
    - **Ignores transaction costs:** The model doesn't account for transaction costs associated with buying and selling assets, which can impact portfolio performance.
- **Application in Diversified Scenarios:**
    - MVO is a good starting point for optimizing portfolios with traditional assets like stocks and bonds, which are often considered in Merton's Problem framework. These asset classes are well-suited for MVO's historical data analysis, and the risk-return trade-off directly addresses the decision-making process outlined in Merton's Problem.
    - By focusing on risk management through the risk-return trade-off, MVO helps investors construct portfolios that align with their risk tolerance and potential investment goals, similar to the decision-making process in Merton's Problem. Investors can utilize MVO to achieve a level of diversification that manages risk and optimizes return within their acceptable risk tolerance, similar to the optimal investment strategy identified in Merton's Problem.

2. Multi-Factor Models (Example: GARCH):

While MVO provides a solid foundation, more sophisticated models can incorporate additional factors for a more nuanced approach, particularly when considering derivatives. Here's an example:

- **GARCH Model:**
    - The GARCH model is an example of a multi-factor model that addresses a limitation of MVO.
    - It accounts for volatility clustering, a phenomenon where periods of high or low volatility tend to persist. This can be particularly relevant for assets with higher volatility swings, like some derivative instruments.
- **Strengths:**

    - Addresses volatility clustering: By capturing volatility dynamics, GARCH provides a more realistic picture of risk compared to MVO's reliance on historical averages.
    - Can incorporate additional factors: Multi-factor models can go beyond just return and volatility to consider factors like market liquidity, credit risk (for bonds), or industry trends, potentially enhancing portfolio optimization for complex scenarios.
- **Limitations:**
    - Increased complexity: Multi-factor models involve more complex calculations and require a deeper understanding of financial theory.
    - Data requirements: These models often require more data points for accurate estimation, which can be challenging with limited historical data, particularly for newer or more complex asset classes.
- **Application in Diversified Scenarios:**
    - GARCH is particularly useful for portfolios with assets prone to volatile swings, like some derivative instruments. This aligns with Merton's Problem's focus on managing risk, as derivatives can introduce additional risk factors beyond the scope of traditional asset classes considered in Merton's Problem. GARCH can help investors using derivatives to quantify and manage this additional risk.
    - Multi-factor models, in general, are suitable for investors seeking a more sophisticated approach that considers additional risk factors beyond just historical averages, especially when dealing with diversified portfolios that may include derivatives, which Merton's Problem might not explicitly consider. These models can provide a more comprehensive analysis of risk, aligning with the spirit of Merton's Problem, which emphasizes making informed investment decisions considering all relevant factors.

Challenges and Opportunities of Derivatives:

Derivatives introduce unique challenges and opportunities in portfolio optimization:

- **Challenges:**

  - Derivatives can be complex instruments with leverage, making risk assessment more intricate. GARCH-like models can be helpful in this regard.
  - Many derivatives are relatively new, with limited historical data for model estimation. This can be a limitation for multi-factor models that rely heavily on historical data.
- **Opportunities:**
  - Derivatives can be used for risk management strategies like hedging or portfolio protection. For example, options contracts can be used to hedge against downside risk in specific asset classes.
  - Options can provide targeted exposure to specific market movements for potential return enhancement. However, careful consideration of the risks involved is crucial.

Model Selection:

Model selection ultimately depends on the specific needs of the investor and the complexity of the portfolio.

- **Basic Allocation with Traditional Assets:** MVO remains a valuable tool for basic portfolio optimization with traditional asset classes like stocks and bonds.
- **Complex Scenarios with Derivatives:** Multi-factor models like GARCH can be beneficial for portfolios with more complex asset classes like derivatives, where volatility clustering and additional risk factors play a significant role.

By understanding the strengths and limitations of different models, investors and financial advisors can choose the most appropriate tool for optimizing diversified investment scenarios.

5. Conclusion

This thesis has explored the multifaceted realm of portfolio optimization, delving into both its theoretical underpinnings and practical applications. Beginning with the foundational work of Harry Markowitz and extending through the dynamic models introduced by Robert C. Merton, the study has illuminated the evolution and sophistication of investment strategies over the past decades.

Merton’s problem, with its focus on optimizing a portfolio in a continuous-time setting under conditions of uncertainty, has been a central theme. This work has examined various solution approaches to Merton’s problem, including the value function approach, duality, and dynamic programming. Each methodology has provided unique insights into the complex interplay between risk and return, and how investors can navigate this relationship to achieve optimal outcomes.

The thesis also extended the Merton model to incorporate additional dimensions such as finite and infinite time horizons, interest rate risk, habit formation, and stochastic volatility. These extensions have demonstrated the model's versatility and its ability to adapt to different investment scenarios and market conditions, providing a more comprehensive framework for portfolio optimization.

In the practical domain, real-world case studies have been instrumental in bridging theory and practice. The analysis of diversified portfolios, risk management strategies, and the application of various optimization models has shown how theoretical concepts can be effectively translated into actionable investment strategies. By examining scenarios involving stocks, bonds, and derivatives, the thesis has highlighted the importance of diversification and tailored risk management in achieving robust and resilient portfolios.

Key takeaways from this study include the critical role of diversification in mitigating risk, the utility of dynamic asset allocation in responding to changing market conditions, and the benefits of incorporating advanced models such as GARCH to address volatility clustering and other market anomalies. These insights underscore the importance of a nuanced and flexible approach to portfolio management, one that is responsive to both the investor's goals and the prevailing market environment.

In conclusion, the exploration of portfolio optimization in this thesis has underscored its fundamental importance in modern finance. By providing a rigorous analysis of both theoretical models and practical applications, this work aims to equip investors and financial professionals with the tools and knowledge necessary to enhance their investment strategies. As financial markets continue to evolve, the principles and techniques discussed herein will remain vital in guiding informed and effective decision-making, ensuring that portfolios are both optimized for return and resilient against risk.

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